Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **81**, 1–16; https://doi.org/10.14232/ejqtde.2019.1.81 www.math.u-szeged.hu/ejqtde/

Serrin-type blowup criterion of three-dimensional nonhomogeneous heat conducting magnetohydrodynamic flows with vacuum

Ling Zhou ✉

Liangjiang Primary School, Yubei District,
Chongqing 401120, People's Republic of China

Received 4 September 2019, appeared 4 November 2019
Communicated by Maria Alessandra Ragusa

Abstract. We consider an initial boundary value problem for the nonhomogeneous heat conducting magnetohydrodynamic flows. We show that for the initial density allowing vacuum, the strong solution exists globally if the velocity field satisfies Serrin's condition. Our method relies upon the delicate energy estimates and regularity properties of Stokes system and elliptic equations.

Keywords: heat conducting magnetohydrodynamic flows, Serrin-type criterion, vacuum.

2010 Mathematics Subject Classification: 35Q35, 35B65, 76N10.

1 Introduction

Magnetohydrodynamics studies the dynamics of electrically conducting fluids and the theory of the macroscopic interaction of electrically conducting fluids with a magnetic field. Due to the profound physical background and important mathematical significance, a great deal of attention has been focused on studying well-posedness of solutions to the MHD system, both from a pure mathematical point of view and for concrete applications. For more background, we refer to [6] and references therein.

Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain, the motion of a viscous, incompressible, and heat conducting magnetohydrodynamic fluid in Ω can be described by the following MHD system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla P = \mathbf{b} \cdot \nabla \mathbf{b}, \\ c_v[\partial_t(\rho\theta) + \operatorname{div}(\rho \mathbf{u} \theta)] - \kappa \Delta \theta = 2\mu |\mathfrak{D}(\mathbf{u})|^2 + \nu |\operatorname{curl} \mathbf{b}|^2, \\ \partial_t \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{b} = \nu \Delta \mathbf{b}, \\ \operatorname{div} \mathbf{u} = 0, \ \operatorname{div} \mathbf{b} = 0 \end{cases} \tag{1.1}$$

✉ Email: lzhou1204@sina.com

with the initial condition

$$(\rho, \mathbf{u}, \theta, \mathbf{b})(0, x) = (\rho_0, \mathbf{u}_0, \theta_0, \mathbf{b}_0)(x), \qquad x \in \Omega, \tag{1.2}$$

and the boundary condition

$$\mathbf{u} = \mathbf{0}, \quad \frac{\partial \theta}{\partial \mathbf{n}} = 0, \quad \mathbf{b} = \mathbf{0}, \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Here $\rho, \mathbf{u}, \theta, P, \mathbf{b}$ are the fluid density, velocity, absolute temperature, pressure, and the magnetic field, respectively. $\mathfrak{D}(\mathbf{u})$ denotes the deformation tensor given by

$$\mathfrak{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{tr}).$$

The constant $\mu > 0$ is the viscosity coefficient. Positive constants c_v and κ are respectively the heat capacity, the ratio of the heat conductivity coefficient over the heat capacity, and $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field.

When $\mathbf{b} = \mathbf{0}$, the system (1.1) reduces to the nonhomogeneous heat conducting Navier–Stokes equations and there are a lot of results on the existence in the literature. For the initial density containing vacuum states, Lions [14, Chapter 2] established the global existence of weak solutions in any space dimensions. Later on, Cho–Kim [5] proposed a compatibility condition on the initial data and investigated the local existence of strong solutions. By delicate energy estimates, Zhong [19] showed the global existence of strong solutions on three-dimensional bounded domains under some smallness assumption. There are also very interesting investigations about the existence of strong solutions to the three-dimensional nonhomogeneous heat conducting Navier–Stokes equations, please refer to [15, 17, 18, 21].

Recently, the local and global existence of strong solutions to the multi-dimensional viscous heat conducting magnetohydrodynamic flows with non-negative density were established. Inspired by [5], Wu [16] proved the local existence of strong solutions. By using the techniques in [19], the author [20] studied the global strong solutions for small initial data. At the same time, he also obtained a blowup criterion of strong solutions. By a critical Sobolev inequality of logarithmic type, Fan–Li–Nakamura [7] showed the global strong solutions with no restrictions on the initial data in two-dimensional bounded domains. Very recently, Zhu–Ou [22] obtained the global existence and algebraic decay of strong solutions to the nonhomogeneous heat-conducting magnetohydrodynamic equations with density-temperature-dependent viscosity and resistivity coefficients. At the same time, many authors studied blowup criteria and regularity criteria of incompressible magnetohydrodynamic equations and related system, please refer to [1, 2, 4, 10–12]. In this paper, motivated by [19], we aim at giving a Serrin-type blowup criterion of strong solutions of the system (1.1).

Before stating our main results, we first explain the notations and conventions used throughout this paper. We use the notation

$$\int \cdot \, dx = \int_\Omega \cdot \, dx.$$

For $1 \le p \le \infty$ and integer $k \ge 0$, the standard Sobolev spaces are denoted by:

$$\begin{cases} L^p = L^p(\Omega), \ W^{k,p} = W^{k,p}(\Omega), \ H^k = H^{k,2}(\Omega), \\ H_0^1 = \{u \in H^1 \mid u = 0 \text{ on } \partial\Omega\}, \ H_{\mathbf{n}}^2 = \{u \in H^2 \mid \nabla u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}. \end{cases}$$

Now we define precisely what we mean by strong solutions to the problem (1.1)–(1.3).

Definition 1.1. $(\rho, \mathbf{u}, \theta, \mathbf{b})$ is called a strong solution to (1.1)–(1.3) in $\Omega \times (0, T)$, if for some $q_0 > 3$,

$$\begin{cases} \rho \geq 0, \ \rho \in C([0,T]; W^{1,q_0}), \ \rho_t \in C([0,T]; L^{q_0}), \\ (\mathbf{u}, \mathbf{b}) \in C([0,T]; H_0^1 \cap H^2) \cap L^2(0,T; W^{2,q_0}), \\ \theta \geq 0, \ \theta \in C([0,T]; H^2) \cap L^2(0,T; W^{2,q_0}), \\ (\mathbf{b}_t, \mathbf{u}_t, \theta_t) \in L^2(0,T; H^1), \ (\mathbf{b}_t, \sqrt{\rho}\mathbf{u}_t, \sqrt{\rho}\theta_t) \in L^\infty(0,T; L^2). \end{cases}$$

Furthermore, both (1.1) and (1.2) hold almost everywhere in $\Omega \times (0, T)$.

Our main results read as follows.

Theorem 1.2. *For constant $q \in (3, 6]$, assume that the initial data $(\rho_0 \geq 0, \mathbf{u}_0, \theta_0 \geq 0, \mathbf{b}_0)$ satisfy*

$$\rho_0 \in W^{1,q}(\Omega), \quad (\mathbf{u}_0, \mathbf{b}_0) \in H_0^1(\Omega) \cap H^2(\Omega), \quad \theta_0 \in H^2_{\mathbf{n}}(\Omega), \quad \operatorname{div} \mathbf{u}_0 = \operatorname{div} \mathbf{b}_0 = 0, \tag{1.4}$$

and the compatibility conditions

$$\begin{cases} -\mu\Delta\mathbf{u}_0 + \nabla P_0 - \mathbf{b}_0 \cdot \nabla \mathbf{b}_0 = \sqrt{\rho_0}\mathbf{g}_1, \\ \kappa\Delta\theta_0 + 2\mu|\mathfrak{D}(\mathbf{u}_0)|^2 + \nu|\operatorname{curl} \mathbf{b}_0|^2 = \sqrt{\rho_0}\mathbf{g}_2, \end{cases} \tag{1.5}$$

for some $P_0 \in H^1(\Omega)$ and $\mathbf{g}_1, \mathbf{g}_2 \in L^2(\Omega)$. Let $(\rho, \mathbf{u}, \theta, \mathbf{b})$ be a strong solution to the problem (1.1)–(1.3). If $T^* < \infty$ is the maximal time of existence for that solution, then we have

$$\lim_{T \to T^*} \|\mathbf{u}\|_{L^s(0,T;L^r)} = \infty, \tag{1.6}$$

where r and s satisfy

$$\frac{2}{s} + \frac{3}{r} \leq 1, \quad s > 1, \quad 3 < r \leq \infty. \tag{1.7}$$

Remark 1.3. The local existence of a strong solution with initial data as in Theorem 1.2 has been established in [16]. Hence, the maximal time T^* is well-defined.

Remark 1.4. The conclusion in Theorem 1.2 is somewhat surprising since the criterion (1.6) is independent of the magnetic field. The result indicates that the magnetic field acts no significant roles on the mechanism of blowup of nonhomogeneous heat conducting magnetohydrodynamic flows. Thus we generalize [19, Theorem 1.1] to the heat conducting MHD flows.

Remark 1.5. Due to the Sobolev inequality $\|\mathbf{u}\|_{L^6} \leq C\|\nabla\mathbf{u}\|_{L^2}$, we thus improve the blowup criterion obtained in [20, Theorem 1.1].

We will prove Theorem 1.2 by contradiction in Section 3. In fact, the proof of the theorem is based on a priori estimates under the assumption that $\|\mathbf{u}\|_{L^s(0,T;L^r)}$ is bounded independent of any $T \in (0, T^*)$. The a priori estimates are then sufficient for us to apply the local existence result repeatedly to extend a local solution beyond the maximal time of existence T^*, consequently, contradicting the maximality of T^*.

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the proof of Theorem 1.2.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

First, the following Gagliardo–Nirenberg inequality (see [9]) will be useful in the next section.

Lemma 2.1 (Gagliardo–Nirenberg). *Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. Assume that $1 \le q, r \le \infty$, and j, m are arbitrary integers satisfying $0 \le j < m$. If $v \in W^{m,r}(\Omega) \cap L^q(\Omega)$, then we have*

$$\|D^j v\|_{L^p} \le C\|v\|_{L^q}^{1-a}\|v\|_{W^{m,r}}^{a},$$

where

$$-j + \frac{3}{p} = (1-a)\frac{3}{q} + a\left(-m + \frac{3}{r}\right),$$

and

$$a \in \begin{cases} [\frac{j}{m}, 1), & \text{if } m - j - \frac{3}{r} \text{ is an nonnegative integer,} \\ [\frac{j}{m}, 1], & \text{otherwise.} \end{cases}$$

The constant C depends only on m, j, q, r, a, and Ω.

Next, we give some regularity results for the following Stokes system

$$\begin{cases} -\mu\Delta \mathbf{U} + \nabla P = \mathbf{F}, & x \in \Omega, \\ \operatorname{div} \mathbf{U} = 0, & x \in \Omega, \\ \mathbf{U} = \mathbf{0}, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Lemma 2.2. *Let $m \ge 2$ be an integer, r any real number with $1 < r < \infty$ and let Ω be a bounded domain of $\mathbb{R}^3$ of class $C^{m-1,1}$. Let $\mathbf{F} \in W^{m-2,r}(\Omega)$ be given. Then the Stokes system (2.1) has a unique solution $\mathbf{U} \in W^{m,r}(\Omega)$ and $P \in W^{m-1,r}(\Omega)/\mathbb{R}$. In addition, there exists a constant $C > 0$ depending only on m, r, and Ω such that*

$$\|\mathbf{U}\|_{W^{m,r}} + \|P\|_{W^{m-1,r}/\mathbb{R}} \le C\|\mathbf{F}\|_{W^{m-2,r}}.$$

Proof. See [3, Theorem 4.8]. □

3 Proof of Theorem 1.2

Let $(\rho, \mathbf{u}, \theta, \mathbf{b})$ be a strong solution described in Theorem 1.2. Suppose that (1.6) were false, that is, there exists a constant $M_0 > 0$ such that

$$\lim_{T \to T^*} \|\mathbf{u}\|_{L^s(0,T;L^r)} \le M_0 < \infty. \tag{3.1}$$

Rewrite the system (1.1) as

$$\begin{cases} \rho_t + \mathbf{u} \cdot \nabla\rho = 0, \\ \rho\mathbf{u}_t + \rho\mathbf{u} \cdot \nabla\mathbf{u} - \mu\Delta\mathbf{u} + \nabla P = \mathbf{b} \cdot \nabla\mathbf{b}, \\ c_v[\rho\theta_t + \rho\mathbf{u} \cdot \nabla\theta] - \kappa\Delta\theta = 2\mu|\mathfrak{D}(\mathbf{u})|^2 + \nu|\operatorname{curl}\mathbf{b}|^2, \\ \mathbf{b}_t - \mathbf{b} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{b} - \nu\Delta\mathbf{b} = \mathbf{0}, \\ \operatorname{div}\mathbf{u} = 0, \ \operatorname{div}\mathbf{b} = 0. \end{cases} \tag{3.2}$$

In what follows, C stands for a generic positive constant which may depend on $M_0, \mu, c_v, \kappa, \nu, T^*$, and the initial data.

First of all, we have the following basic energy estimates and the upper bound of the density.

Lemma 3.1. *It holds that for any $T \in (0, T^*)$,*

$$\sup_{0\le t\le T} \left(\|\rho\|_{L^\infty} + c_v\|\rho\theta\|_{L^1} + \|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2\right) + \int_0^T \left(\mu\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|\nabla\mathbf{b}\|_{L^2}^2\right) dt$$
$$\le \|\rho_0\|_{L^\infty} + c_v\|\rho_0\theta_0\|_{L^1} + \|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{b}_0\|_{L^2}^2. \tag{3.3}$$

Proof. Since $\operatorname{div}\mathbf{u} = 0$, we then derive from [14, Theorem 2.1] that for any $t \in (0, T^*)$,

$$\|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty}. \tag{3.4}$$

Applying standard maximum principle to $(3.2)_3$ along with $\theta_0 \ge 0$ shows (see [8, p. 43])

$$\inf_{\Omega\times[0,T]} \theta \ge 0. \tag{3.5}$$

Multiplying $(3.2)_2$ by $\mathbf{u}$ and integrating (by parts) over Ω, we derive that

$$\frac{1}{2}\frac{d}{dt}\int \rho|\mathbf{u}|^2 dx + \mu\int |\nabla\mathbf{u}|^2 dx = \int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}dx. \tag{3.6}$$

Multiplying $(3.2)_4$ by $\mathbf{b}$ and integrating (by parts) over Ω, we get after using (1.3) that

$$\frac{1}{2}\frac{d}{dt}\int |\mathbf{b}|^2 dx + \nu\int |\nabla\mathbf{b}|^2 dx = \int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx - \int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}dx. \tag{3.7}$$

Due to $\operatorname{div}\mathbf{b} = 0$ and $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$, we have

$$\int \mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}dx = \int b^i\partial_i b^j u^j dx = -\int \mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx. \tag{3.8}$$

Similarly, one obtains

$$-\int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}dx = -\int u^i\partial_i b^j b^j dx = \int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}dx,$$

and thus

$$\int \mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}dx = 0. \tag{3.9}$$

Combining (3.6)–(3.9), we deduce that

$$\frac{1}{2}\frac{d}{dt}\int \left(\rho|\mathbf{u}|^2 + |\mathbf{b}|^2\right) dx + \int \left(\mu|\nabla\mathbf{u}|^2 + \nu|\nabla\mathbf{b}|^2\right) dx = 0. \tag{3.10}$$

Integrating $(3.2)_3$ with respect to the spatial variable gives rise to

$$c_v\frac{d}{dt}\int \rho\theta dx - 2\mu\int |\mathfrak{D}(\mathbf{u})|^2 dx - \nu\int |\nabla\mathbf{b}|^2 dx = 0. \tag{3.11}$$

Substituting (3.11) into (3.10) and noting that

$$\begin{aligned}
-2\mu\int |\mathfrak{D}(\mathbf{u})|^2 dx &= -\frac{\mu}{2}\int (\partial_i u^j + \partial_j u^i)^2 dx \\
&= -\mu\int |\nabla\mathbf{u}|^2 dx - \mu\int \partial_i u^j\partial_j u^i dx \\
&= -\mu\int |\nabla\mathbf{u}|^2 dx,
\end{aligned}$$

we derive that

$$\frac{d}{dt}\int\left(c_v\rho\theta+\frac{1}{2}\rho|\mathbf{u}|^2+\frac{1}{2}|\mathbf{b}|^2\right)dx=0, \tag{3.12}$$

which combined with (3.10) leads to

$$\frac{d}{dt}\int\left(c_v\rho\theta+\rho|\mathbf{u}|^2+|\mathbf{b}|^2\right)dx+\int\left(\mu|\nabla\mathbf{u}|^2+\nu|\nabla\mathbf{b}|^2\right)dx=0.$$

Integrating the above equality over $(0,T)$ yields

$$\begin{aligned}\sup_{0\le t\le T}\left(c_v\|\rho\theta\|_{L^1}+\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\|_{L^2}^2\right)+\int_0^T\left(\mu\|\nabla\mathbf{u}\|_{L^2}^2+\nu\|\nabla\mathbf{b}\|_{L^2}^2\right)dt\\ \le c_v\|\rho_0\theta_0\|_{L^1}+\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2+\|\mathbf{b}_0\|_{L^2}^2.\end{aligned}$$

This along with (3.4) implies the desired (3.3) and consequently completes the proof. □

The following lemma was deduced in [13], we sketch it here for completeness.

Lemma 3.2. *Under the condition (3.1), it holds that for $p\in[2,12]$ and $T\in[0,T^*)$,*

$$\sup_{0\le t\le T}\|\mathbf{b}\|_{L^p}+\int_0^T\int|\mathbf{b}|^{p-2}|\nabla\mathbf{b}|^2dxdt\le C. \tag{3.13}$$

Proof. Multiplying $(3.2)_4$ by $p|\mathbf{b}|^{p-2}\mathbf{b}$ and integrating the resulting equation over Ω, we deduce

$$\frac{d}{dt}\int|\mathbf{b}|^pdx+\nu p(p-1)\int|\mathbf{b}|^{p-2}|\nabla\mathbf{b}|^2dx\le p\int(\mathbf{b}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{b})\cdot|\mathbf{b}|^{p-2}\mathbf{b}dx. \tag{3.14}$$

Integration by parts together with $(3.2)_5$ yields

$$-p\int(\mathbf{u}\cdot\nabla)\mathbf{b}\cdot|\mathbf{b}|^{p-2}\mathbf{b}dx=\int\operatorname{div}\mathbf{u}|\mathbf{b}|^pdx=0. \tag{3.15}$$

We derive from integration by parts, Hölder's inequality, and Gagliardo–Nirenberg inequality that for r and s satisfying (1.7),

$$\begin{aligned}&p\int(\mathbf{b}\cdot\nabla)\mathbf{u}\cdot|\mathbf{b}|^{p-2}\mathbf{b}dx\\ &\le\frac{\nu p(p-1)}{4}\int|\mathbf{b}|^{p-2}|\nabla\mathbf{b}|^2dx+C(\nu,p)\int|\mathbf{u}|^2|\mathbf{b}|^pdx\\ &\le\frac{\nu p(p-1)}{4}\int|\mathbf{b}|^{p-2}|\nabla\mathbf{b}|^2dx+C\|\mathbf{u}\|_{L^r}^2\||\mathbf{b}|^{\frac{p}{2}}\|_{L^2}^{\frac{2(r-3)}{r}}\||\mathbf{b}|^{\frac{p}{2}}\|_{L^6}^{\frac{6}{r}}\\ &\le\frac{\nu p(p-1)}{4}\int|\mathbf{b}|^{p-2}|\nabla\mathbf{b}|^2dx+\delta\|\nabla|\mathbf{b}|^{\frac{p}{2}}\|_{L^2}^2+C(\delta)(1+\|\mathbf{u}\|_{L^r}^s)\|\mathbf{b}\|_{L^p}^p.\end{aligned} \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14) and choosing δ suitably small give that

$$\frac{d}{dt}\int|\mathbf{b}|^pdx+\frac{\nu p(p-1)}{2}\int|\mathbf{b}|^{p-2}|\nabla\mathbf{b}|^2dx\le C\left(1+\|\mathbf{u}\|_{L^r}^s\right)\int|\mathbf{b}|^pdx.$$

We thus obtain (3.13) directly after using Gronwall's inequality and (3.1). This finishes the proof of Lemma 3.2. □

Next, the following lemma concerns the key time-independent estimates on the $L^\infty(0,T;L^2)$-norm of the gradients of the velocity and the magnetic field.

Lemma 3.3. *Under the condition* (3.1), *it holds that for any* $T \in (0, T^*)$,

$$\sup_{0\le t\le T}\left(\|\nabla \mathbf{u}\|_{L^2}^2+\|\nabla \mathbf{b}\|_{L^2}^2\right)+\int_0^T\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2+\|\mathbf{u}\|_{H^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)dt\le C. \tag{3.17}$$

Proof. Multiplying $(3.2)_2$ by $\mathbf{u}_t$ and integrating the resulting equations over Ω, we derive from Cauchy–Schwarz inequality that

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int|\nabla\mathbf{u}|^2dx+\int\rho|\mathbf{u}_t|^2dx\\
&\quad=\int\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}_tdx-\int\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_tdx\\
&\quad=-\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\int\left[\mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t\right]dx\\
&\quad\le-\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac{1}{2}\int\left(\rho|\mathbf{u}_t|^2+|\mathbf{b}_t|^2\right)dx+\int\left(2\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+8|\mathbf{b}|^2|\nabla\mathbf{u}|^2\right)dx,
\end{aligned}$$

and thus

$$\begin{aligned}
&\frac{d}{dt}\int\left(\mu|\nabla\mathbf{u}|^2+2\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\right)dx+\int\rho|\mathbf{u}_t|^2dx\\
&\quad\le\int|\mathbf{b}_t|^2dx+\int\left(4\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+16|\mathbf{b}|^2|\nabla\mathbf{u}|^2\right)dx.
\end{aligned}\tag{3.18}$$

Multiplying $(3.2)_4$ by $\mathbf{b}_t$ and integrating by parts yield

$$\begin{aligned}
\nu\frac{d}{dt}\int|\nabla\mathbf{b}|^2dx+2\int|\mathbf{b}_t|^2dx&=2\int(\mathbf{b}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{b})\cdot\mathbf{b}_tdx\\
&\le\frac{1}{2}\int|\mathbf{b}_t|^2dx+8\int\left(|\mathbf{b}|^2|\nabla\mathbf{u}|^2+|\mathbf{u}|^2|\nabla\mathbf{b}|^2\right)dx,
\end{aligned}\tag{3.19}$$

which combined with (3.18) implies

$$\begin{aligned}
&\frac{d}{dt}\int\left(\mu|\nabla\mathbf{u}|^2+\nu|\nabla\mathbf{b}|^2+2\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\right)dx+\int\left(\rho|\mathbf{u}_t|^2+|\mathbf{b}_t|^2\right)dx\\
&\quad\le C\int\left(\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2+|\mathbf{b}|^2|\nabla\mathbf{u}|^2+|\mathbf{u}|^2|\nabla\mathbf{b}|^2\right)dx.
\end{aligned}\tag{3.20}$$

Recall that $(\mathbf{u}, P)$ satisfies the following Stokes system

$$\begin{cases}
-\mu\Delta\mathbf{u}+\nabla P=-\rho\mathbf{u}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{b}\cdot\nabla\mathbf{b}, & x\in\Omega,\\
\operatorname{div}\mathbf{u}=0, & x\in\Omega,\\
\mathbf{u}=\mathbf{0}, & x\in\partial\Omega.
\end{cases}$$

Applying Lemma 2.2 with $\mathbf{F}\triangleq-\rho\mathbf{u}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{b}\cdot\nabla\mathbf{b}$, we obtain from (3.4) that

$$\begin{aligned}
\|\mathbf{u}\|_{H^2}^2&\le C\left(\|\rho\mathbf{u}_t\|_{L^2}^2+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right)\\
&\le C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right).
\end{aligned}\tag{3.21}$$

It follows from the standard L^2-estimates of elliptic system and $(3.2)_4$ that

$$\|\nabla^2\mathbf{b}\|_{L^2}^2\le C\left(\|\mathbf{b}_t\|_{L^2}^2+\|\mathbf{u}\cdot\nabla\mathbf{b}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{u}\|_{L^2}^2\right),\tag{3.22}$$

which together with (3.21) leads to for some $K > 0$,

$$\|\mathbf{u}\|_{H^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2 \le K\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2\right) + C\left(\|\sqrt{\rho}\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right) + C\left(\|\mathbf{u}\cdot\nabla\mathbf{b}\|_{L^2}^2 + \|\mathbf{b}\cdot\nabla\mathbf{u}\|_{L^2}^2\right). \tag{3.23}$$

Adding (3.23) multiplied by $\frac{1}{2K}$ to (3.20), we get from Hölder's inequality, (3.13), and the Gagliardo–Nirenberg inequality that

$$\begin{aligned}
&A'(t) + \frac{1}{2}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2\right) + \frac{1}{2K}\left(\|\mathbf{u}\|_{H^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\right)\\
&\le C\int\left(\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2 + |\mathbf{b}|^2|\nabla\mathbf{u}|^2 + |\mathbf{u}|^2|\nabla\mathbf{b}|^2 + |\mathbf{b}|^2|\nabla\mathbf{b}|^2\right)dx\\
&\le C\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^r}^2\|\nabla\mathbf{u}\|_{L^{\frac{2r}{r-2}}}^2 + C\|\mathbf{b}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^6}\\
&\quad + C\|\mathbf{u}\|_{L^r}^2\|\nabla\mathbf{b}\|_{L^{\frac{2r}{r-2}}}^2 + C\|\mathbf{b}\|_{L^6}^2\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^6}\\
&\le C\|\mathbf{u}\|_{L^r}^2\|\nabla\mathbf{u}\|_{L^2}^{2-\frac{6}{r}}\|\nabla\mathbf{u}\|_{H^1}^{\frac{6}{r}} + C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\\
&\quad + C\|\mathbf{u}\|_{L^r}^2\|\nabla\mathbf{b}\|_{L^2}^{2-\frac{6}{r}}\|\nabla\mathbf{b}\|_{H^1}^{\frac{6}{r}} + C\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{H^1}\\
&\le \frac{1}{2}\left(\|\mathbf{u}\|_{H^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\right) + C\left(1 + \|\mathbf{u}\|_{L^r}^s + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2\right)\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2\right),
\end{aligned}$$

and thus

$$\begin{aligned}
&A'(t) + \frac{1}{2}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2\right) + \frac{1}{2K}\left(\|\mathbf{u}\|_{H^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2\right)\\
&\le \left(1 + \|\mathbf{u}\|_{L^r}^s + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2\right)\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2\right).
\end{aligned} \tag{3.24}$$

Here

$$A(t) \triangleq \int\left(\mu|\nabla\mathbf{u}|^2 + \nu|\nabla\mathbf{b}|^2 + 2\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\right)dx$$

satisfies

$$\frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|\nabla\mathbf{b}\|_{L^2}^2 - C \le A(t) \le \frac{3\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|\nabla\mathbf{b}\|_{L^2}^2 + C \tag{3.25}$$

due to

$$\left|\int 2\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx\right| \le \frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2 + \frac{8}{\mu}\|\mathbf{b}\|_{L^4}^4 \le \frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2 + C.$$

Consequently, the desired (3.17) follows from (3.24), Gronwall's inequality, (3.25), (3.3), and (3.1). This completes the proof of Lemma 3.3. □

Finally, the following lemma will deal with the higher order estimates of the solutions which are needed to guarantee the extension of the local strong solution to be a global one.

Lemma 3.4. *Under the condition* (3.1), *it holds that for any* $T \in (0, T^*)$,

$$\sup_{0\le t\le T}\left(\|\rho\|_{W^{1,q}} + \|\mathbf{u}\|_{H^2}^2 + \|\theta\|_{H^2}^2 + \|\mathbf{b}\|_{H^2}^2\right) \le C. \tag{3.26}$$

Proof. Differentiating $(3.2)_2$ with respect to t and using $(1.1)_1$, we arrive at

$$\begin{aligned}
&\rho\mathbf{u}_{tt} + \rho\mathbf{u}\cdot\nabla\mathbf{u}_t - \mu\Delta\mathbf{u}_t\\
&\quad = -\nabla P_t + \operatorname{div}(\rho\mathbf{u})\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \rho\mathbf{u}_t\cdot\nabla\mathbf{u} + \mathbf{b}_t\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{b}_t.
\end{aligned} \tag{3.27}$$

Multiplying (3.27) by $\mathbf{u}_t$ and integrating (by parts) over Ω yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \rho|\mathbf{u}_t|^2dx+\mu\int|\nabla\mathbf{u}_t|^2dx=&\int \operatorname{div}(\rho\mathbf{u})|\mathbf{u}_t|^2dx+\int \operatorname{div}(\rho\mathbf{u})\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_tdx\\&-\int\rho\mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_tdx+\int \mathbf{b}_t\cdot\nabla\mathbf{b}\cdot\mathbf{u}_tdx\\&+\int\mathbf{b}\cdot\nabla\mathbf{b}_t\cdot\mathbf{u}_tdx\triangleq\sum_{k=1}^{5}J_k.\end{aligned}\tag{3.28}$$

By virtue of Hölder's inequality, Sobolev's inequality, (3.4), and (3.17), we find that for $\delta>0$,

$$\begin{aligned}|J_1|&=\left|-\int\rho\mathbf{u}\cdot\nabla|\mathbf{u}_t|^2dx\right|\\&\le 2\|\rho\|_{L^\infty}^{\frac12}\|\mathbf{u}\|_{L^6}\|\sqrt{\rho}\mathbf{u}_t\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2}\\&\le C\|\rho\|_{L^\infty}^{\frac12}\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}\\&\le C\|\rho\|_{L^\infty}^{\frac34}\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac32}\\&\le\frac{\mu}{10}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2;\end{aligned}$$

$$\begin{aligned}|J_2|&=\left|-\int\rho\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t)dx\right|\\&\le\int\left(\rho|\mathbf{u}||\nabla\mathbf{u}|^2|\mathbf{u}_t|+\rho|\mathbf{u}|^2|\nabla^2\mathbf{u}||\mathbf{u}_t|+\rho|\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\mathbf{u}_t|\right)dx\\&\le\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^6}\|\mathbf{u}_t\|_{L^6}+\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}^2\|\nabla^2\mathbf{u}\|_{L^2}\|\mathbf{u}_t\|_{L^6}\\&\quad+\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}_t\|_{L^2}\\&\le C\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{u}\|_{H^2}\|\nabla\mathbf{u}_t\|_{L^2}\\&\le\frac{\mu}{10}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{u}\|_{H^2}^2;\end{aligned}$$

$$\begin{aligned}|J_3|&\le\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^4}^2\le\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac32}\\&\le C\|\rho\|_{L^\infty}^{\frac34}\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac32}\\&\le\frac{\mu}{10}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2;\end{aligned}$$

$$\begin{aligned}|J_4|&\le\|\mathbf{b}_t\|_{L^3}\|\nabla\mathbf{b}\|_{L^2}\|\mathbf{u}_t\|_{L^6}\le C\|\mathbf{b}_t\|_{L^2}^{\frac12}\|\mathbf{b}_t\|_{L^6}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}\\&\le\frac{\mu}{10}\|\nabla\mathbf{u}_t\|_{L^2}^2+C(\delta)\|\mathbf{b}_t\|_{L^2}^2+\frac{\delta}{2}\|\nabla\mathbf{b}_t\|_{L^2}^2;\end{aligned}$$

$$\begin{aligned}|J_5|&=\left|-\int\mathbf{b}\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}_tdx\right|\\&\le\|\mathbf{b}\|_{L^6}\|\nabla\mathbf{u}_t\|_{L^2}\|\mathbf{b}_t\|_{L^3}\le C\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}\|\mathbf{b}_t\|_{L^2}^{\frac12}\|\mathbf{b}_t\|_{L^6}^{\frac12}\\&\le\frac{\mu}{10}\|\nabla\mathbf{u}_t\|_{L^2}^2+C(\delta)\|\mathbf{b}_t\|_{L^2}^2+\frac{\delta}{2}\|\nabla\mathbf{b}_t\|_{L^2}^2.\end{aligned}$$

Substituting the above estimates into (3.28), we deduce that

$$\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\mu\|\nabla\mathbf{u}_t\|_{L^2}^2\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{u}\|_{H^2}^2+C\|\mathbf{b}_t\|_{L^2}^2+\delta\|\nabla\mathbf{b}_t\|_{L^2}^2.\tag{3.29}$$

Next, differentiating (3.2)$_4$ with respect to t and multiplying the resulting equations by $\mathbf{b}_t$, we obtain after using integrating by parts and (3.17) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\mathbf{b}_t|^2dx+\nu\int|\nabla\mathbf{b}_t|^2dx&\le C\left(\||\mathbf{u}_t||\mathbf{b}|\|_{L^2}+\||\mathbf{u}||\mathbf{b}_t|\|_{L^2}\right)\|\nabla\mathbf{b}_t\|_{L^2}\\&\le C\left(\|\mathbf{u}_t\|_{L^6}\|\mathbf{b}\|_{L^3}+\|\mathbf{u}\|_{L^6}\|\mathbf{b}_t\|_{L^2}^{\frac12}\|\mathbf{b}_t\|_{L^6}^{\frac12}\right)\|\nabla\mathbf{b}_t\|_{L^2}\\&\le C\left(\|\nabla\mathbf{u}_t\|_{L^2}+\|\mathbf{b}_t\|_{L^2}^{\frac12}\|\nabla\mathbf{b}_t\|_{L^2}^{\frac12}\right)\|\nabla\mathbf{b}_t\|_{L^2}\\&\le\frac{\nu}{2}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{b}_t\|_{L^2}^2,\end{aligned}$$

which implies that for some $C_1>0$,

$$\frac{d}{dt}\|\mathbf{b}_t\|_{L^2}^2+\nu\|\nabla\mathbf{b}_t\|_{L^2}^2\le C_1\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{b}_t\|_{L^2}^2. \tag{3.30}$$

Adding (3.29) multiplied by $\frac{2C_1}{\mu}$ to (3.30) and then choosing δ suitably small, we deduce that

$$\frac{d}{dt}\left(2C_1\mu^{-1}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+C_1\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\nu}{2}\|\nabla\mathbf{b}_t\|_{L^2}^2\le C\|\mathbf{u}\|_{H^2}^2+C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right).$$

Thus we infer from the Gronwall inequality and (3.17) that

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)+\int_0^T\left(\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\nabla\mathbf{b}_t\|_{L^2}^2\right)dt\le C. \tag{3.31}$$

As a consequence, we derive from the regularity theory of elliptic system, (3.2)$_4$, (3.31), and (3.17) that

$$\begin{aligned}\|\mathbf{b}\|_{H^2}^2&\le C\left(\|\mathbf{b}_t\|_{L^2}^2+\|\mathbf{u}\cdot\nabla\mathbf{b}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\|_{H^1}^2\right)\\&\le C+C\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{b}\|_{L^3}^2+C\|\mathbf{b}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2\\&\le C+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^6}+C\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{H^1}\|\nabla\mathbf{u}\|_{L^2}^2\\&\le C+\frac12\|\mathbf{b}\|_{H^2}^2,\end{aligned}$$

where we have used the following Sobolev's inequality

$$\|\mathbf{v}\|_{L^\infty}\le C\|\nabla\mathbf{v}\|_{L^2}^{\frac12}\|\nabla\mathbf{v}\|_{H^1}^{\frac12}\quad\text{for }\mathbf{v}\in H_0^1\cap H^2.$$

Hence we get

$$\sup_{0\le t\le T}\|\mathbf{b}\|_{H^2}^2\le C. \tag{3.32}$$

Furthermore, it follows from Lemmas 2.2 and 2.1, (3.4), (3.17), (3.31), and (3.32) that

$$\begin{aligned}\|\mathbf{u}\|_{H^2}^2&\le C\left(\|\rho\mathbf{u}_t\|_{L^2}^2+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2}^2\right)\\&\le C\|\rho\|_{L^\infty}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\rho\|_{L^\infty}^2\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^3}^2+C\|\mathbf{b}\|_{L^6}^2\|\nabla\mathbf{b}\|_{L^3}^2\\&\le C+C\|\nabla\mathbf{u}\|_{L^2}^3\|\nabla\mathbf{u}\|_{L^6}\\&\le C+\frac12\|\mathbf{u}\|_{H^2}^2,\end{aligned}$$

which yields

$$\sup_{0\le t\le T}\|\mathbf{u}\|_{H^2}^2\le C. \tag{3.33}$$

Now we estimate $\|\nabla\rho\|_{L^q}$. First of all, applying Lemma 2.2 once more, we have

$$\begin{aligned}\|\mathbf{u}\|_{W^{2,6}}^2&\le C\left(\|\rho\mathbf{u}_t\|_{L^6}^2+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^6}^2+\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^6}^2\right)\\&\le C\|\rho\|_{L^\infty}^2\|\mathbf{u}_t\|_{L^6}^2+C\|\rho\|_{L^\infty}^2\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^6}^2+C\|\mathbf{b}\|_{L^\infty}^2\|\nabla\mathbf{b}\|_{L^6}^2\\&\le C\|\nabla\mathbf{u}_t\|_{L^2}^2+C,\end{aligned}$$

which together with (3.31) implies

$$\int_0^T\|\mathbf{u}\|_{W^{2,6}}^2dt\le C. \tag{3.34}$$

Then taking spatial derivative ∇ on the transport equation $(3.2)_1$ leads to

$$\partial_t\nabla\rho+\mathbf{u}\cdot\nabla^2\rho+\nabla\mathbf{u}\cdot\nabla\rho=\mathbf{0}.$$

Thus standard energy methods yields for any $q\in(3,6]$,

$$\frac{d}{dt}\|\nabla\rho\|_{L^q}\le C(q)\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^q}\le C\|\mathbf{u}\|_{W^{2,6}}\|\nabla\rho\|_{L^q},$$

which combined with Gronwall's inequality and (3.24) gives

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^q}\le C.$$

This along with (3.4) yields

$$\sup_{0\le t\le T}\|\rho\|_{W^{1,q}}\le C. \tag{3.35}$$

Finally, we need to estimate $\|\theta\|_{H^2}$. Motivated by [19], denote by $\bar\theta\triangleq\frac{1}{|\Omega|}\int\theta dx$, the average of θ, then we obtain from (3.4), (3.3), and the Poincaré inequality that

$$|\bar\theta|\int\rho dx\le\left|\int\rho\theta dx\right|+\left|\int\rho(\theta-\bar\theta)dx\right|\le C+C\|\nabla\theta\|_{L^2},$$

which together with the fact that $\left|\int v dx\right|+\|\nabla v\|_{L^2}$ is an equivalent norm to the usual one in $H^1(\Omega)$ implies that

$$\|\theta\|_{H^1}\le C+C\|\nabla\theta\|_{L^2}. \tag{3.36}$$

Similarly, one deduces

$$\|\theta_t\|_{H^1}\le C\|\sqrt{\rho}\theta_t\|_{L^2}+C\|\nabla\theta_t\|_{L^2}. \tag{3.37}$$

Multiplying $(3.2)_3$ by θ_t and integrating the resulting equation over Ω yield that

$$\begin{aligned}\frac{\kappa}{2}\frac{d}{dt}\int|\nabla\theta|^2dx+c_v\int\rho|\theta_t|^2dx=&-c_v\int\rho(\mathbf{u}\cdot\nabla\theta)\theta_tdx+2\mu\int|\mathfrak{D}(u)|^2\theta_tdx\\&+\nu\int|\nabla\mathbf{b}|^2\theta_tdx\triangleq\sum_{i=1}^3I_i.\end{aligned} \tag{3.38}$$

By Hölder's inequality, (3.4), and (3.33), we get

$$|I_1| \le c_v \|\rho\|_{L^\infty}^{\frac{1}{2}} \|\sqrt{\rho}\theta_t\|_{L^2} \|\mathbf{u}\|_{L^\infty} \|\nabla\theta\|_{L^2} \le \frac{c_v}{2} \|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2. \tag{3.39}$$

From (3.33) and (3.36), one has

$$\begin{aligned} I_2 &= 2\mu \frac{d}{dt} \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx - 2\mu \int (|\mathfrak{D}(\mathbf{u})|^2)_t \theta dx \\ &\le 2\mu \frac{d}{dt} \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx + C \int \theta |\nabla \mathbf{u}||\nabla \mathbf{u}_t| dx \\ &\le 2\mu \frac{d}{dt} \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx + C\|\theta\|_{L^6} \|\nabla \mathbf{u}\|_{L^3} \|\nabla \mathbf{u}_t\|_{L^2} \\ &\le 2\mu \frac{d}{dt} \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx + C\|\theta\|_{H^1} \|\mathbf{u}\|_{H^2} \|\nabla \mathbf{u}_t\|_{L^2} \\ &\le 2\mu \frac{d}{dt} \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx + C\|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C. \end{aligned} \tag{3.40}$$

Moreover, one infers

$$\begin{aligned} I_3 &= \nu \frac{d}{dt} \int |\nabla \mathbf{b}|^2 \theta dx - \nu \int (|\nabla \mathbf{b}|^2)_t \theta dx \\ &\le \nu \frac{d}{dt} \int |\nabla \mathbf{b}|^2 \theta dx + C \int \theta |\nabla \mathbf{b}||\nabla \mathbf{b}_t| dx \\ &\le \nu \frac{d}{dt} \int |\nabla \mathbf{b}|^2 \theta dx + C\|\theta\|_{L^6} \|\nabla \mathbf{b}\|_{L^3} \|\nabla \mathbf{b}_t\|_{L^2} \\ &\le \nu \frac{d}{dt} \int |\nabla \mathbf{b}|^2 \theta dx + C\|\theta\|_{H^1} \|\mathbf{b}\|_{H^2} \|\nabla \mathbf{b}_t\|_{L^2} \\ &\le \nu \frac{d}{dt} \int |\nabla \mathbf{b}|^2 \theta dx + C\|\nabla \mathbf{b}_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C. \end{aligned} \tag{3.41}$$

Inserting (3.39)–(3.41) into (3.38), we get

$$\begin{aligned} &\frac{d}{dt} \int \left(\kappa|\nabla\theta|^2 - 4\mu|\mathfrak{D}(\mathbf{u})|^2\theta - 2\nu|\nabla \mathbf{b}|^2\theta\right) dx + c_v\|\sqrt{\rho}\theta_t\|_{L^2}^2 \\ &\quad \le C\|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\nabla \mathbf{b}_t\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C. \end{aligned} \tag{3.42}$$

Noting that

$$4\mu \int |\mathfrak{D}(\mathbf{u})|^2 \theta dx \le C\|\theta\|_{L^6} \|\nabla \mathbf{u}\|_{L^{\frac{12}{5}}}^2 \le C\|\theta\|_{H^1} \|\mathbf{u}\|_{H^2}^2 \le \frac{\kappa}{4} \|\nabla\theta\|_{L^2}^2 + C,$$

and

$$2\nu \int |\nabla \mathbf{b}|^2 \theta dx \le C\|\theta\|_{L^6} \|\nabla \mathbf{b}\|_{L^{\frac{12}{5}}}^2 \le C\|\theta\|_{H^1} \|\mathbf{b}\|_{H^2}^2 \le \frac{\kappa}{4} \|\nabla\theta\|_{L^2}^2 + C,$$

which combined with (3.42), Gronwall's inequality, and (3.31) leads to

$$\sup_{0\le t\le T} \|\nabla\theta\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}\theta_t\|_{L^2}^2 dt \le C.$$

This together with (3.36) yields

$$\sup_{0\le t\le T} \|\theta\|_{H^1}^2 + \int_0^T \|\sqrt{\rho}\theta_t\|_{L^2}^2 dt \le C. \tag{3.43}$$

Differentiating $(3.2)_3$ with respect to t and using $(1.1)_1$, we arrive at

$$\begin{aligned} &c_v[\rho\theta_{tt}+\rho\mathbf{u}\cdot\nabla\theta_t]-\kappa\Delta\theta_t\\ &\quad= c_v\,\mathrm{div}(\rho\mathbf{u})\,(\theta_t+\mathbf{u}\cdot\nabla\theta)-c_v\rho\mathbf{u}_t\cdot\nabla\theta+2\mu(|\mathfrak{D}(\mathbf{u})|^2)_t+\nu(|\nabla\mathbf{b}|^2)_t. \end{aligned} \tag{3.44}$$

Multiplying (3.44) by θ_t and integrating (by parts) over Ω give rise to

$$\begin{aligned} &\frac{c_v}{2}\frac{d}{dt}\int\rho|\theta_t|^2dx+\kappa\int|\nabla\theta_t|^2dx\\ &= c_v\int\mathrm{div}(\rho\mathbf{u})|\theta_t|^2dx+c_v\int\mathrm{div}(\rho\mathbf{u})(\mathbf{u}\cdot\nabla\theta)\theta_t dx\\ &\quad -c_v\int\rho(\mathbf{u}_t\cdot\nabla\theta)\theta_t dx+2\mu\int(|\mathfrak{D}(\mathbf{u})|^2)_t\theta_t dx+\nu\int(|\nabla\mathbf{b}|^2)_t\theta_t dx\triangleq\sum_{k=1}^{5}\bar{J}_k. \end{aligned} \tag{3.45}$$

Applying Hölder's inequality, Sobolev's inequality, (3.4), (3.32), (3.33), (3.37), and (3.43), we find

$$\begin{aligned} |\bar{J}_1| &= \left|-c_v\int\rho\mathbf{u}\cdot\nabla|\theta_t|^2dx\right|\\ &\le 2c_v\|\rho\|_{L^\infty}^{\frac12}\|\mathbf{u}\|_{L^\infty}\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta_t\|_{L^2}\\ &\le \frac{\kappa}{10}\|\nabla\theta_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2; \end{aligned}$$

$$\begin{aligned} |\bar{J}_2| &= \left|-c_v\int\rho\mathbf{u}\cdot\nabla[(\mathbf{u}\cdot\nabla\theta)\theta_t]dx\right|\\ &\le c_v\int\left(\rho|\mathbf{u}||\nabla\mathbf{u}||\nabla\theta||\theta_t|+\rho|\mathbf{u}|^2|\nabla^2\theta||\theta_t|+\rho|\mathbf{u}|^2|\nabla\theta||\nabla\theta_t|\right)dx\\ &\le c_v\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^3}\|\nabla\theta\|_{L^2}\|\theta_t\|_{L^6}+c_v\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}^2\|\nabla^2\theta\|_{L^2}\|\theta_t\|_{L^6}\\ &\quad +c_v\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^\infty}^2\|\nabla\theta\|_{L^2}\|\nabla\theta_t\|_{L^2}\\ &\le C(1+\|\nabla^2\theta\|_{L^2})(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\nabla\theta_t\|_{L^2})\\ &\le \frac{\kappa}{10}\|\nabla\theta_t\|_{L^2}^2+C\|\nabla^2\theta\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C; \end{aligned}$$

$$\begin{aligned} |\bar{J}_3| &\le c_v\|\rho\|_{L^\infty}^{\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\nabla\theta\|_{L^3}\|\theta_t\|_{L^6}\\ &\le C(1+\|\nabla^2\theta\|_{L^2})(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\nabla\theta_t\|_{L^2})\\ &\le \frac{\kappa}{10}\|\nabla\theta_t\|_{L^2}^2+C\|\nabla^2\theta\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C; \end{aligned}$$

$$\begin{aligned} |\bar{J}_4| &\le C\int|\nabla\mathbf{u}||\nabla\mathbf{u}_t|\theta_t dx\le C\|\nabla\mathbf{u}\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2}\|\theta_t\|_{L^6}\\ &\le C\|\nabla\mathbf{u}_t\|_{L^2}(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\nabla\theta_t\|_{L^2})\\ &\le \frac{\kappa}{10}\|\nabla\theta_t\|_{L^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2; \end{aligned}$$

$$\begin{aligned} |\bar{J}_5| &\le C\int|\nabla\mathbf{b}||\nabla\mathbf{b}_t|\theta_t dx\le C\|\nabla\mathbf{b}\|_{L^3}\|\nabla\mathbf{b}_t\|_{L^2}\|\theta_t\|_{L^6}\\ &\le C\|\nabla\mathbf{b}_t\|_{L^2}(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\nabla\theta_t\|_{L^2})\\ &\le \frac{\kappa}{10}\|\nabla\theta_t\|_{L^2}^2+C\|\nabla\mathbf{b}_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2. \end{aligned}$$

Substituting the above estimates into (3.45), we derive that

$$c_v\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\kappa\|\nabla\theta_t\|_{L^2}^2\le C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\nabla^2\theta\|_{L^2}^2+C\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\nabla\mathbf{b}_t\|_{L^2}^2+C. \tag{3.46}$$

The standard H^2-estimate of (3.2)$_3$ gives rise to

$$\begin{aligned}\|\theta\|_{H^2}^2 &\le C\left(\|\rho\theta_t\|_{L^2}^2 + \|\rho\mathbf{u}\cdot\nabla\theta\|_{L^2}^2 + \||\nabla\mathbf{u}|^2\|_{L^2}^2 + \|\theta\|_{L^2}^2\right)\\ &\le C\|\rho\|_{L^\infty}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^4}^4 + C\|\theta\|_{L^2}^2\\ &\le C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\theta\|_{H^1}^2 + C\|\mathbf{u}\|_{H^2}^4\\ &\le C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\end{aligned} \tag{3.47}$$

due to (3.33) and (3.43). Then we obtain from (3.46) and (3.47) that

$$c_v\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \kappa\|\nabla\theta_t\|_{L^2}^2 \le C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{b}_t\|_{L^2}^2 + C,$$

which combined with the Gronwall inequality and (3.31) that

$$\sup_{0\le t\le T}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \int_0^T\|\nabla\theta_t\|_{L^2}^2dt \le C. \tag{3.48}$$

Consequently, we deduce from (3.47) and (3.48) that

$$\sup_{0\le t\le T}\|\theta\|_{H^2}^2 \le C\sup_{0\le t\le T}\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C \le C. \tag{3.49}$$

So the desired (3.26) follows from (3.32), (3.33), (3.35), and (3.49). □

With Lemmas 3.1–3.4 at hand, we are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. We argue by contradiction. Suppose that (1.6) were false, that is, (3.1) holds. Note that the general constant C in Lemmas 3.1–3.4 is independent of $t < T^*$, that is, all the a priori estimates obtained in Lemmas 3.1–3.4 are uniformly bounded for any $t < T^*$. Hence, the function

$$(\rho,\mathbf{u},\theta,\mathbf{b})(T^*,x) \triangleq \lim_{t\to T^*}(\rho,\mathbf{u},\theta,\mathbf{b})(t,x)$$

satisfy the initial condition (1.4) at $t = T^*$. Furthermore, standard arguments yield that $\rho\dot{\mathbf{u}},\rho\dot{\theta} \in C([0,T];L^2)$, here $\dot{f} \triangleq f_t + \mathbf{u}\cdot\nabla f$, which implies

$$(\rho\dot{\mathbf{u}},\rho\dot{\theta})(T^*,x) \triangleq \lim_{t\to T^*}(\rho\dot{\mathbf{u}},\rho\dot{\theta})(t,x) \in L^2.$$

Hence,

$$\begin{cases}-\mu\Delta\mathbf{u} + \nabla P - \mathbf{b}\cdot\nabla\mathbf{b}|_{t=T^*} = \sqrt{\rho}(T^*,x)\mathbf{g}_1(x),\\ \kappa\Delta\theta + 2\mu|\mathfrak{D}(\mathbf{u})|^2 + \nu|\operatorname{curl}\mathbf{b}|^2|_{t=T^*} = \sqrt{\rho}(T^*,x)\mathbf{g}_2(x),\end{cases}$$

with

$$\mathbf{g}_1(x) \triangleq \begin{cases}\rho^{-\frac{1}{2}}(T^*,x)(\rho\dot{\mathbf{u}})(T^*,x), & \text{for } x\in\{x|\rho(T^*,x)>0\},\\ \mathbf{0}, & \text{for } x\in\{x|\rho(T^*,x)=0\},\end{cases}$$

and

$$\mathbf{g}_2(x) \triangleq \begin{cases}c_v\rho^{-\frac{1}{2}}(T^*,x)(\rho\dot{\theta})(T^*,x), & \text{for } x\in\{x|\rho(T^*,x)>0\},\\ \mathbf{0}, & \text{for } x\in\{x|\rho(T^*,x)=0\},\end{cases}$$

satisfying $\mathbf{g}_1,\mathbf{g}_2 \in L^2$ due to (3.31), (3.48), and (3.26). Thus, $(\rho,\mathbf{u},\theta,\mathbf{b})(T^*,x)$ also satisfies (1.5). Therefore, taking $(\rho,\mathbf{u},\theta,\mathbf{b})(T^*,x)$ as the initial data, one can extend the local strong solution beyond T^*, which contradicts the maximality of T^*. Thus we finish the proof of Theorem 1.2. □

References

[1] A. M. Alghamdi, S. Gala, M. A. Ragusa, On the blow-up criterion for incompressible Stokes-MHD equations, *Results Math.* **73**(2018), Art. 110, 6 pp. https://doi.org/10.1007/s00025-018-0874-x; MR3836182

[2] A. M. Alghamdi, S. Gala, M. A. Ragusa, New regularity criteria for the 3D Hall-MHD equations, *Ann. Polon. Math.* **121**(2018), 7–20. https://doi.org/10.4064/ap170823-23-1; MR3816169

[3] C. Amrouche, V. Girault, Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension, *Czechoslovak Math. J.* **44**(1994), 109–140. MR1257940

[4] S. Benbernou, M. A. Ragusa, M. A. Terbeche, Logarithmically improved regularity criterion for the MHD equations in terms of one directional derivative of the pressure, *Appl. Anal.* **96**(2017), 2140–2148. https://doi.org/10.1080/00036811.2016.1207246; MR3667851

[5] Y. Cho, H. Kim, Existence result for heat-conducting viscous incompressible fluids with vacuum, *J. Korean Math. Soc.* **45**(2008), 645–681. https://doi.org/10.4134/JKMS.2008.45.3.645; MR2410236

[6] P. A. Davidson, *Introduction to magnetohydrodynamics*, Second edition, Cambridge University Press, Cambridge, 2017. https://doi.org/10.1017/9781316672853; MR3699477

[7] J. Fan, F. Li, G. Nakamura, Global strong solution to the two-dimensional density-dependent magnetohydrodynamic equations with vacuum, *Commun. Pure Appl. Anal.* **13**(2014), 1481–1490. https://doi.org/10.3934/cpaa.2014.13.1481; MR3177741

[8] E. Feireisl, *Dynamics of viscous compressible fluids*, Oxford University Press, Oxford, 2004. MR2040667

[9] A. Friedman, *Partial differential equations*, corrected reprint of the original edition, Robert E. Krieger Publishing Co., Huntington, N.Y., 1976. MR0454266;

[10] S. Gala, Q. Liu, M. A. Ragusa, Logarithmically improved regularity criterion for the nematic liquid crystal flows in $\dot{B}^{-1}_{\infty,\infty}$ space, *Comput. Math. Appl.* **65**(2013), 1738–1745. https://doi.org/10.1016/j.camwa.2013.04.003; MR3055732

[11] S. Gala, M. A. Ragusa, A logarithmic regularity criterion for the two-dimensional MHD equations, *J. Math. Anal. Appl.* **444**(2016), 1752–1758. https://doi.org/10.1016/j.jmaa.2016.07.001; MR3535787

[12] S. Gala, M. A. Ragusa, On the regularity criterion of weak solutions for the 3D MHD equations, *Z. Angew. Math. Phys.* **68**(2017), Art. 140, 13 pp. MR3735569

[13] C. He, Z. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, *J. Differential Equations* **213**(2005), 235–254. https://doi.org/10.1016/j.jde.2004.07.002; MR2142366

[14] P. L. Lions, *Mathematical topics in fluid mechanics. Vol. I. incompressible models*, Oxford University Press, Oxford, 1996. MR1422251

[15] W. Wang, H. Yu, P. Zhang, Global strong solutions for 3D viscous incompressible heat conducting Navier–Stokes flows with the general external force, *Math. Methods Appl. Sci.* **41**(2018), 4589–4601. https://doi.org/10.1002/mma.4915; MR3828345

[16] H. Wu, Strong solutions to the incompressible magnetohydrodynamic equations with vacuum, *Comput. Math. Appl.* **61**(2011), 2742–2753. https://doi.org/10.1016/j.camwa.2011.03.033; MR2795397

[17] H. Xu, H. Yu, Global regularity to the Cauchy problem of the 3D heat conducting incompressible Navier–Stokes equations, *J. Math. Anal. Appl.* **464**(2018), 823–837. https://doi.org/10.1016/j.jmaa.2018.04.037; MR3794117

[18] H. Xu, H. Yu, Global strong solutions to the 3D inhomogeneous heat-conducting incompressible fluids, *Appl. Anal.* **98**(2019), 622–637. https://doi.org/10.1080/00036811.2017.1399362; MR3904564

[19] X. Zhong, Global strong solution for 3D viscous incompressible heat conducting Navier–Stokes flows with non-negative density, *J. Differential Equations* **263**(2017), 4978–4996. https://doi.org/10.1016/j.jde.2017.06.004; MR3680944

[20] X. Zhong, Global strong solutions for 3D viscous incompressible heat conducting magnetohydrodynamic flows with non-negative density, *J. Math. Anal. Appl.* **446**(2017), 707–729. https://doi.org/10.1016/j.jmaa.2016.09.012; MR3554752

[21] X. Zhong, Global strong solutions for nonhomogeneous heat conducting Navier–Stokes equations, *Math. Methods Appl. Sci.* **41**(2018), 127–139. https://doi.org/10.1002/mma.4600; MR3745361

[22] M. Zhu, M. Ou, Global strong solutions to the 3D incompressible heat-conducting magnetohydrodynamic flows, *Math. Phys. Anal. Geom.* **22**(2019), No. 1, Art. 8, 17 pp. https://doi.org/10.1007/s11040-019-9306-8; MR3918713