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Existence of nontrivial solution for fourth-order semilinear Δ_γ-Laplace equation in $\mathbb{R}^N$

Duong Trong Luyen ✉ [1, 2]

[1]Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam
[2]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

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Abstract. In this paper, we study existence of nontrivial solutions for a fourth-order semilinear Δ_γ-Laplace equation in $\mathbb{R}^N$

$$\Delta_\gamma^2 u - \Delta_\gamma u + \lambda b(x) u = f(x,u), \quad x \in \mathbb{R}^N, \quad u \in \mathbf{S}_\gamma^2(\mathbb{R}^N),$$

where $\lambda > 0$ is a parameter and Δ_γ is the subelliptic operator of the type

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right), \quad \partial_{x_j} := \frac{\partial}{\partial x_j}, \quad \gamma = (\gamma_1, \gamma_2, \dots, \gamma_N), \quad \Delta_\gamma^2 := \Delta_\gamma(\Delta_\gamma).$$

Under some suitable assumptions on $b(x)$ and $f(x,\xi)$, we obtain the existence of nontrivial solution for λ large enough.

Keywords: fourth-order semilinear degenerate elliptic equations, Δ_γ-Laplace operator, nontrivial solutions, Cerami sequences, mountain pass theorem.

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1 Introduction

In the last decades, the biharmonic elliptic equations

$$\Delta^2 u - \Delta u + \lambda b(x) u = f(x,u), \quad x \in \mathbb{R}^N, \quad u \in H^2(\mathbb{R}^N), \tag{1.1}$$

has been studied by many authors see [12, 19, 20, 26–30] and the references therein. The biharmonic equations can be used to describe some phenomena appearing in physics and engineering. For example, the problem of nonlinear oscillation in a suspension bridge [10, 14, 15] and the problem of the static deflection of an elastic plate in a fluid [1]. In the last decades, the existence and multiplicity of nontrivial solutions for biharmonic equations have begun to receive much attention.

✉ Email: duongtrongluyen@tdtu.edu.vn

In this paper, we consider the biharmonic equation as follows:

$$\Delta_\gamma^2 u - \Delta_\gamma u + \lambda b(x) u = f(x,u), \quad x \in \mathbb{R}^N, \quad u \in \mathbf{S}_\gamma^2(\mathbb{R}^N), \tag{1.2}$$

where Δ_γ is a subelliptic operator of the form

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j} \left(\gamma_j^2 \partial_{x_j} \right), \quad \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N) : \mathbb{R}^N \to \mathbb{R}^N, \quad \Delta_\gamma^2 := \Delta_\gamma(\Delta_\gamma).$$

The Δ_γ-operator was considered by B. Franchi and E. Lanconelli in [6], and recently reconsidered in [9] under the additional assumption that the operator is homogeneous of degree two with respect to a group dilation in $\mathbb{R}^N$. The Δ_γ-operator contains many degenerate elliptic operators such as the Grushin-type operator

$$G_\alpha := \Delta_x + |x|^{2\alpha} \Delta_y, \quad \alpha \geq 0,$$

where (x,y) denotes the point of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ (see [7,21,23]), and the operator of the form

$$P_{\alpha,\beta} := \Delta_x + \Delta_y + |x|^{2\alpha} |y|^{2\beta} \Delta_z, \quad (x,y,z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

where α, β are nonnegative real numbers (see [22,24]).

We assume that the potential $b(x)$ satisfies the following conditions:

(B_1) $b : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function on $\mathbb{R}^N$, there exists a constant $C_0 > 0$ such that the set $\{b < C_0\} := \{x \in \mathbb{R}^N : b(x) < C_0\}$ has finite positive Lebesgue measure for $\widetilde{N} > 4$;

(B_2) $\Omega = \operatorname{int}\{x \in \mathbb{R}^N : b(x) = 0\}$ is nonempty and has smooth boundary with $\bar{\Omega} = \{x \in \mathbb{R}^N : b(x) = 0\}$.

Under the hypotheses $(B_1), (B_2), \lambda b(x)$ is called the steep potential well whose depth is controlled by the parameter λ. Such potential is first suggested by Bartsch–Wang [3] in the scalar Schrödinger equations. Later, the steep potential well is introduced to the study of some other types of nonlinear differential equations by some researchers, such as Kirchhoff type equations [16], Schrödinger–Poisson systems [8, 18, 31] and also biharmonic equations [13,17,25].

Next, we can state the main theorem of the paper.

Theorem 1.1. *Suppose that* $\widetilde{N} > 4$ *and conditions* $(B_1), (B_2)$ *hold. In addition, we assume that a continuous function* $f(x,\xi) = \alpha(x) g(\xi)$ *satisfies:*

(g_1) $g(\xi) = o(|\xi|)$ *as* $\xi \to 0$;

(g_2) $g(\xi) = o(|\xi|)$ *as* $\xi \to \infty$;

(α_1) $0 < \alpha(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $C_1 := \|\alpha\|_{L^\infty(\mathbb{R}^N)} \max_{\xi \neq 0} \left| \frac{g(\xi)}{\xi} \right| < \frac{1}{1+C_2^2}$;

(B_3) $\operatorname{Vol}\{b < C_0\} < \left(\frac{1 - C_1\left(1 + C_2^2\right)}{C_3^2} \right)^{\frac{\widetilde{N}}{4}}$,

where $\operatorname{Vol}(\cdot)$ *denotes the Lebesgue measure of a set in* $\mathbb{R}^N$ *and where* C_2 *is the best constant in* (2.2) *below.*

Then there exists a constant $\Lambda_0 > 0$ *such that the problem* (1.2) *has only the trivial solution for all* $\lambda \geq \Lambda_0$.

Theorem 1.2. *Suppose that $\widetilde{N} > 4$ and conditions $(B_1), (B_2)$ hold. In addition, we assume that the function $f(x,\xi)$ satisfies:*

(F_1) *$f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist a constant $p \in (2, 2^{\gamma}_{*})$ and two functions $f_1(x)$, $f_2(x) \in L^{\infty}(\mathbb{R}^N)$ satisfying $\|f_1^+\|_{L^\infty(\mathbb{R}^N)} < \Theta_2^{-1}$ and $f_2(x) > 0$ on $\bar{\Omega}$ such that*

$$\lim_{\xi \to 0^+} \frac{f(x,\xi)}{|\xi|^{p-1}} = f_1(x) \quad \text{and} \quad \lim_{\xi \to \infty} \frac{f(x,\xi)}{|\xi|^{p-1}} = f_2(x) \quad \text{uniformly in} \quad x \in \mathbb{R}^N;$$

where $f_1^+ := \max\{f_1, 0\}, \Theta_2$ is given in (2.5) below;

(F_2) *there exists are constants $1 < \ell < 2, \mu > 2$ and a nonnegative function $f_3 \in L^{\frac{2}{2-\ell}}(\mathbb{R}^N)$ such that*

$$\mu F(x,\xi) - f(x,\xi) \leq f_3(x)\, |\xi|^{\ell} \quad \text{for all } x \in \mathbb{R}^N \text{ and } \xi \in \mathbb{R},$$

where $F(x,\xi) = \int_0^{\xi} f(x,\tau)\mathrm{d}\tau$.

Then there exists a constant $\Lambda_1 > 0$ such that the problem (1.2) admits at least a nontrivial solution for all $\lambda \geq \Lambda_1$.

The paper is organized as follows. In Section 2 for convenience of the readers, we recall some function spaces, embedding theorems and associated functional settings. We prove our main results by using Ekeland's variational principle and Gagliardo–Nirenberg's inequality in Section 3.

2 Preliminary results

2.1 Function spaces and embedding theorems

We recall the functional setting in [9]. We consider the operator of the form

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right), \quad \partial_{x_j} = \frac{\partial}{\partial x_j}, \quad j = 1, 2, \ldots, N.$$

Here, the functions $\gamma_j : \mathbb{R}^N \to \mathbb{R}$ are assumed to be continuous, different from zero and of class C^1 in $\mathbb{R}^N \backslash \Pi$, where

$$\Pi := \left\{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \prod_{j=1}^{N} x_j = 0 \right\}.$$

Moreover, we assume the following properties:

i) There exists a group of dilations $\{\delta_t\}_{t>0}$ such that

$$\begin{aligned} \delta_t : \mathbb{R}^N &\longrightarrow \mathbb{R}^N \\ (x_1, \ldots, x_N) &\longmapsto \delta_t(x_1, \ldots, x_N) = (t^{\varepsilon_1} x_1, \ldots, t^{\varepsilon_N} x_N), \end{aligned}$$

where $1 = \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_N$, such that γ_j is δ_t-homogeneous of degree $\varepsilon_j - 1$, i.e.,

$$\gamma_j(\delta_t(x)) = t^{\varepsilon_j - 1}\gamma_j(x), \quad \forall x \in \mathbb{R}^N, \quad \forall t > 0, \quad j = 1, \ldots, N.$$

The number
$$\widetilde{N} := \sum_{j=1}^{N} \varepsilon_j \tag{2.1}$$
is called the homogeneous dimension of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t>0}$.

ii)
$$\gamma_1 = 1, \quad \gamma_j(x) = \gamma_j(x_1, x_2, \dots, x_{j-1}), \quad j = 2, \dots, N.$$

iii) There exists a constant $\rho \geq 0$ such that
$$0 \leq x_k \partial_{x_k} \gamma_j(x) \leq \rho \gamma_j(x), \quad \forall k \in \{1, 2, \dots, j-1\}, \quad \forall j = 2, \dots, N,$$
and for every $x \in \overline{\mathbb{R}}^N_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_j \geq 0, \forall j = 1, 2, \dots, N\}$.

iv) Equalities $\gamma_j(x) = \gamma_j(x^*)$ $(j = 1, 2, \dots, N)$ are satisfied for every $x \in \mathbb{R}^N$, where
$$x^* = (|x_1|, \dots, |x_N|) \quad \text{if } x = (x_1, x_2, \dots, x_N).$$

Definition 2.1. By $\mathbf{S}^2_\gamma(\mathbb{R}^N)$ we will denote the set of all functions $u \in L^2(\mathbb{R}^N)$ such that $\gamma_j \partial_{x_j} u \in L^2(\mathbb{R}^N)$ for all $j = 1, \dots, N$ and $\Delta_\gamma u \in L^2(\mathbb{R}^N)$. We define the norm in this space as follows
$$\|u\|_{\mathbf{S}^2_\gamma(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(|\Delta_\gamma u|^2 + |\nabla_\gamma u|^2 + |u|^2 \right) \mathrm{d}x \right)^{\frac{1}{2}},$$
where $\nabla_\gamma u = (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \dots, \gamma_N \partial_{x_N} u)$.

Let
$$\mathbf{E}_\lambda = \left\{ u \in \mathbf{S}^2_\gamma(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\Delta_\gamma u|^2 + \lambda b(x) u^2 \right) \mathrm{d}x < \infty \right\}.$$
For $\lambda > 0$, the inner product and norm of $\mathbf{E}_\lambda$ are given by
$$(u, v)_{\mathbf{E}_\lambda} = \int_{\mathbb{R}^N} (\Delta_\gamma u \Delta_\gamma v + \lambda b(x) uv) \, \mathrm{d}x, \ \|u\|_{\mathbf{E}_\lambda} = (u, u)^{\frac{1}{2}}_{\mathbf{E}_\lambda}.$$

Lemma 2.2. *The following embeddings are continuous:*

i) $\mathbf{S}^2_\gamma(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ *for all* $2 \leq p < 2^\gamma_* := \frac{2\widetilde{N}}{\widetilde{N}-4}$.

ii) Assume that (B_1) *and* (B_2) *hold, for every* $\lambda \geq \Lambda$, *the embedding* $\mathbf{E}_\lambda \hookrightarrow \mathbf{S}^2_\gamma(\mathbb{R}^N)$ *and* $\mathbf{E}_\lambda \hookrightarrow L^p(\mathbb{R}^N)$, $p \in [2, 2^\gamma_*)$.

Proof. *i*) We follow the ideas in the case of bounded domains (see the proofs of Theorem 3.3, Proposition 3.2 in [9] and Lemma 2.2 in [2]). More precisely, we first embed $\mathbf{S}^2_\gamma(\mathbb{R}^N)$ into an anisotropic Sobolev-type space, and then use an embedding theorem for classical anisotropic Sobolev-type spaces of fractional orders. Because the proof is very similar to the case of bounded domains [2,9], so we omit it here.

ii) For all $u \in C^\infty_0(\mathbb{R}^N)$, with slight modification, the proof is similar to the one of Theorems 12.85 and 12.87 in [11], there exists $C_2, C_3 > 0$ such that
$$\left(\int_{\mathbb{R}^N} |\nabla_\gamma u|^2 \, \mathrm{d}x \right) \leq C_2^2 \left(\int_{\mathbb{R}^N} |\Delta_\gamma u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} u^2 \mathrm{d}x \right)^{\frac{1}{2}}, \tag{2.2}$$

$$\left(\int_{\mathbb{R}^N}|u|^{\frac{2\widetilde{N}}{\widetilde{N}-4}}\,\mathrm{d}x\right)^{\frac{\widetilde{N}-4}{\widetilde{N}}}\leq C_3\int_{\mathbb{R}^N}|\Delta_\gamma u|^2\,\mathrm{d}x. \tag{2.3}$$

This shows that

$$\int_{\mathbb{R}^N}\left(|\Delta_\gamma u|^2+u^2\right)\mathrm{d}x\leq\|u\|^2_{\mathbf{S}^2_\gamma(\mathbb{R}^N)}\leq\left(1+\frac{C_2^2}{2}\right)\int_{\mathbb{R}^N}\left(|\Delta_\gamma u|^2+u^2\right)\mathrm{d}x. \tag{2.4}$$

From (B_1), using Hölder's inequality and (2.2), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}u^2\mathrm{d}x&=\int_{\{b\geq C_0\}}u^2\mathrm{d}x+\int_{\{b<C_0\}}u^2\mathrm{d}x\\&\leq\frac{1}{C_0}\int_{\{b\geq C_0\}}b(x)u^2\mathrm{d}x+\left(\mathrm{Vol}(\{b<C_0\})\right)^{\frac{4}{\widetilde{N}}}\left(\int_{\mathbb{R}^N}|u|^{\frac{2\widetilde{N}}{\widetilde{N}-4}}\,\mathrm{d}x\right)^{\frac{\widetilde{N}-4}{\widetilde{N}}}\\&\leq\frac{1}{C_0}\int_{\mathbb{R}^N}b(x)u^2\mathrm{d}x+C_3^2\left(\mathrm{Vol}(\{b<C_0\})\right)^{\frac{4}{\widetilde{N}}}\int_{\mathbb{R}^N}|\Delta_\gamma u|^2\,\mathrm{d}x,\end{aligned}$$

where C_3 is the best constant in (2.3). Combining the above inequality with (2.4) yields

$$\|u\|_{\mathbf{S}^2_\gamma(\mathbb{R}^N)}\leq\left(1+\frac{C_2^2}{2}\right)\left(1+C_3^2(\mathrm{Vol}(\{b<C_0\}))^{\frac{4}{\widetilde{N}}}\right)\int_{\mathbb{R}^N}|\Delta_\gamma u|^2\,\mathrm{d}x+\frac{1}{C_0}\left(1+\frac{C_2^2}{2}\right)\int_{\mathbb{R}^N}b(x)u^2\mathrm{d}x.$$

Then for $\lambda\geq\left(1+C_3^2\,\mathrm{Vol}(\{b<C_0\})\right)C_0$, we have

$$\|u\|^2_{\mathbf{S}^2_\gamma(\mathbb{R}^N)}\leq\left(1+\frac{C_2^2}{2}\right)\left(1+C_3^2\left(\mathrm{Vol}(\{b<C_0\})\right)^{\frac{4}{\widetilde{N}}}\right)\|u\|^2_{\mathbf{E}_\lambda}.$$

This implies that the embedding $\mathbf{E}_\lambda\hookrightarrow\mathbf{S}^2_\gamma(\mathbb{R}^N)$ is continuous. By using Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x&\leq\left(\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x\right)^{\frac{2\widetilde{N}-p(\widetilde{N}-4)}{8}}\left(\int_{\mathbb{R}^N}|u|^{2^\gamma_*}\,\mathrm{d}x\right)^{\frac{\widetilde{N}(p-2)}{4}\frac{\widetilde{N}-4}{2\widetilde{N}}}\\&\leq\|u\|^{\frac{2\widetilde{N}-p(\widetilde{N}-4)}{8}}_{L^2(\mathbb{R}^N)}C_3^{\frac{\widetilde{N}(p-2)}{4}}\|\Delta_\gamma u\|^{\frac{\widetilde{N}(p-2)}{4}}_{L^2(\mathbb{R}^N)}\\&\leq\|u\|^{\frac{2\widetilde{N}-p(\widetilde{N}-4)}{8}}_{\mathbf{S}^2_\gamma(\mathbb{R}^N)}C_3^{\frac{\widetilde{N}(p-2)}{4}}\|u\|^{\frac{\widetilde{N}(p-2)}{4}}_{\mathbf{S}^2_\gamma(\mathbb{R}^N)}\\&\leq C_3^{\frac{\widetilde{N}(p-2)}{4}}\|u\|^p_{\mathbf{S}^2_\gamma(\mathbb{R}^N)}\\&\leq C_3^{\frac{\widetilde{N}(p-2)}{4}}\left(1+\frac{C_2^2}{2}\right)^{\frac{p}{2}}\left(1+C_3^2\left(\mathrm{Vol}(\{b<C_0\})\right)^{\frac{4}{\widetilde{N}}}\right)^{\frac{p}{2}}\|u\|^p_{\mathbf{E}_\lambda},\end{aligned}$$

where $p\in[2,2^\gamma_*)$. We get

$$\Theta_p=C_3^{\frac{\widetilde{N}(p-2)}{4}}\left(1+\frac{C_2^2}{2}\right)^{\frac{p}{2}}\left(1+C_3^2\left(\mathrm{Vol}(\{b<C_0\})\right)^{\frac{4}{\widetilde{N}}}\right)^{\frac{p}{2}}, \tag{2.5}$$

and

$$\Lambda=\left(1+C_3^2\,\mathrm{Vol}(\{b<C_0\})\right)C_0.$$

Thus, for any $p\in[2,2^\gamma_*)$ and $\lambda\geq\Lambda$, there holds

$$\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x\leq\Theta_p\,\|u\|^p_{\mathbf{E}_\lambda},$$

which implies that the embedding $\mathbf{E}_\lambda\hookrightarrow L^p(\mathbb{R}^N)$ is continuous. $\square$

Definition 2.3. A function $u \in \mathbf{S}^2_\gamma(\mathbb{R}^N)$ is called a weak solution of the problem (1.2) if $u \in \mathbf{E}_\lambda$ and

$$\int_{\mathbb{R}^N} \left(\Delta_\gamma u \Delta_\gamma \varphi + \nabla_\gamma u \cdot \nabla_\gamma \varphi + \lambda b(x) u \varphi\right) \mathrm{d}x - \int_{\mathbb{R}^N} f\left(x, u(x)\right) \varphi \mathrm{d}x = 0, \quad \forall \varphi \in \mathbf{E}_\lambda.$$

2.2 Mountain Pass Theorem

Definition 2.4. Let $\mathbb{X}$ be a real Banach space with its dual space $\mathbb{X}^*$ and $\Phi \in C^1(\mathbb{X}, \mathbb{R})$. For $c \in \mathbb{R}$ we say that Φ satisfies the $(C)_c$ condition if for any sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{X}$ with

$$\Phi(x_n) \to c \quad \text{and} \quad (1 + \|x_n\|_{\mathbb{X}}) \left\|\Phi'(x_n)\right\|_{\mathbb{X}^*} \to 0,$$

then there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ that converges strongly in $\mathbb{X}$. If Φ satisfies the $(C)_c$ condition for all $c > 0$ then we say that Φ satisfies the Cerami condition.

We will use the following version of the Mountain Pass Theorem.

Lemma 2.5 (see [4,5]). *Let $\mathbb{X}$ be an infinite dimensional Banach space and let $\Phi \in C^1(\mathbb{X}, \mathbb{R})$ satisfy the $(C)_c$ condition for all $c \in \mathbb{R}, \Phi(0) = 0$, and*

(i) *There are constants $\rho, \alpha > 0$ such that $\Phi(u) \geq \alpha$ for all $u \in \mathbb{X}$ such that $\|u\|_{\mathbb{X}} = \rho$;*

(ii) *There is an $e \in \mathbb{X}$, $\|u\|_{\mathbb{X}} > \rho$ such that $\Phi(e) \leq 0$.*

Then $\beta = \inf_{\theta \in \Gamma} \max_{0 \leq t \leq 1} \Phi(\theta(t)) \geq \alpha$ is a critical value of Φ, where

$$\Gamma = \{\theta \in C([0,1], \mathbb{X}) : \theta(0) = 0, \theta(1) = e\}.$$

3 Proofs of the main results

Define the Euler–Lagrange functional associated with the problem (1.2) as follows

$$\Phi(u) = \frac{1}{2} \int_\Omega \left(|\Delta_\gamma u|^2 + |\nabla_\gamma u|^2 + \lambda b(x) u^2\right) \mathrm{d}x - \int_\Omega F(x, u)\, \mathrm{d}x.$$

By f satisfies $(f_1), (f_2), (\alpha_1)$ or (F_1), hence its not difficult to prove that the functional Φ is of class C^1 in $\mathbf{E}_\lambda$, and that

$$\Phi'(u)(v) = \int_\Omega \left(\Delta_\gamma u \Delta_\gamma v + \nabla_\gamma u \cdot \nabla_\gamma v + \lambda b(x) u v\right) \mathrm{d}x - \int_\Omega f(x, u)\, v \mathrm{d}x$$

for all $v \in \mathbf{E}_\lambda$. One can also check that the critical points of Φ are weak solutions of the problem (1.2).

3.1 Proof of Theorem 1.1

By condition (g_1), for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, we have

$$|g(u)| \leq \varepsilon |u| \quad \text{for all } |u| < \delta(\varepsilon).$$

By condition (g_2), there exists $M > 0$, we obtain

$$|g(u)| \leq |u| \quad \text{for all } |u| > M.$$

Since is a continuous function, g achieves its maximum and minimum on $[\delta(\varepsilon), M]$, so there exists a positive number $C(\varepsilon)$, we have that

$$|g(u)| \le C(\varepsilon) \le C(\varepsilon)\frac{|u|}{\delta(\varepsilon)} \quad \text{for all } \delta(\varepsilon) \le |u| \le M.$$

Then we obtain that

$$|g(u)| \le \left(1 + \varepsilon + \frac{C(\varepsilon)}{\delta(\varepsilon)}\right)|u| \quad \text{for all } u \in \mathbb{R}.$$

Hence $\max_{\xi\neq 0}\left|\frac{g(\xi)}{\xi}\right|$ is well defined.

Let u is a nontrivial solution of the problem (1.2), we get

$$\|u\|^2_{\mathbf{E}_\lambda} = \int_{\mathbb{R}^N} \alpha(x)g(u)u\mathrm{d}x,$$

hence

$$\|u\|^2_{\mathbf{E}_\lambda} \le \|\alpha\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \left|\frac{g(u)}{u}\right| u^2\mathrm{d}x \le C_1 \int_{\mathbb{R}^N} u^2\mathrm{d}x.$$

By Lemma 2.2 and condition (B_3), we have

$$\|u\|^2_{\mathbf{E}_\lambda} < \|u\|^2_{\mathbf{E}_\lambda},$$

which is a contradiction, thus $u \equiv 0$. The proof of Theorem 1.1 is therefore complete.

3.2 Proof of Theorem 1.2

Lemma 3.1. *Assume that conditions $(B_1), (B_2)$ and (F_1) hold. Then for each $\lambda \ge \Lambda$, there exists $\rho, \beta > 0$ such that*

$$\inf\{\Phi(u) : u \in \mathbf{E}_\lambda, \|u\|_{\mathbf{E}_\lambda} = \rho\} > \alpha.$$

Proof. For any $\varepsilon > 0$, it follows from the condition (F_1) that there exists $C_\varepsilon > 0$ and $p \in (2, 2^\gamma_*)$ such that

$$f(x,\xi) \le \left(\|f_1^+\|_{L^\infty(\mathbb{R}^N)} + \varepsilon\right)\xi + C_\varepsilon \xi^{p-1} \quad \text{for all } \xi \in \mathbb{R} \tag{3.1}$$

and

$$F(x,\xi) \le \frac{\|f_1^+\|_{L^\infty(\mathbb{R}^N)} + \varepsilon}{2}\xi^2 + \frac{C_\varepsilon}{p}\xi^p \quad \text{for all } \xi \in \mathbb{R}.$$

From Lemma 2.2, we have for all $u \in \mathbf{E}_\lambda$,

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,u)\mathrm{d}x &\le \frac{\|f_1^+\|_{L^\infty(\mathbb{R}^N)} + \varepsilon}{2}\int_{\mathbb{R}^N} u^2\mathrm{d}x + \frac{C_\varepsilon}{p}\int_{\mathbb{R}^N} u^p\mathrm{d}x \\ &\le \frac{\left(\|f_1^+\|_{L^\infty(\mathbb{R}^N)} + \varepsilon\right)\Theta_2}{2}\|u\|^2_{\mathbf{E}_\lambda} + \frac{C_\varepsilon\Theta_p}{p}\|u\|^p_{\mathbf{E}_\lambda}.\end{aligned} \tag{3.2}$$

Hence

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|_{\mathbf{E}_\lambda}^2 + \frac{1}{2}\int_{\mathbb{R}^N}|\nabla_\gamma u|^2\,\mathrm{d}x - \int_{\mathbb{R}^N}F(x,u)\mathrm{d}x\\ &\geq \frac{1}{2}\|u\|_{\mathbf{E}_\lambda}^2 - \int_{\mathbb{R}^N}F(x,u)\mathrm{d}x\\ &\geq \frac{1}{2}\|u\|_{\mathbf{E}_\lambda}^2 - \frac{\left(\|f_1^+\|_{L^\infty(\mathbb{R}^N)}+\varepsilon\right)\Theta_2}{2}\|u\|_{\mathbf{E}_\lambda}^2 - \frac{C_\varepsilon\Theta_p}{p}\|u\|_{\mathbf{E}_\lambda}^p\\ &= \frac{1}{2}\left[1-\left(\|f_1^+\|_{L^\infty(\mathbb{R}^N)}+\varepsilon\right)\Theta_2\right]\|u\|_{\mathbf{E}_\lambda}^2 - \frac{C_\varepsilon\Theta_p}{p}\|u\|_{\mathbf{E}_\lambda}^p.\end{aligned}$$

So, fixing $\varepsilon \in (0, \Theta_2^{-1} - \|f_1^+\|_{L^\infty(\mathbb{R}^N)})$ and letting $\|u\|_{\mathbf{E}_\lambda} = \rho > 0$ small enough, it is easy to see that there exists $\alpha > 0$ such that this lemma holds. □

Lemma 3.2. *Assume that conditions $(B_1), (B_2)$ and (F_1) hold. Let $\rho > 0$ be as in Lemma 3.1. Then there exists $e \in \mathbf{E}_\lambda$ with $\|e\|_{\mathbf{E}_\lambda} > \rho$ such that $\Phi(e) < 0$ for $\lambda > 0$.*

Proof. Since $f_2 > 0$ on Ω, we can choose a nonnegative function $\phi \in \mathbf{E}_\lambda$ such that

$$\int_{\mathbb{R}^N} f_2(x)\phi^p(x)\mathrm{d}x > 0. \tag{3.3}$$

From (3.3), the condition (F_1) and Fatou's lemma, we get

$$\begin{aligned}\lim_{t\to\infty}\frac{\Phi(t\phi)}{t^p} &= \lim_{t\to\infty}\left(\frac{1}{2t^{p-2}}\|\phi\|_{\mathbf{E}_\lambda}^2 + \frac{1}{2t^{p-2}}\int_{\mathbb{R}^N}|\nabla_\gamma\phi|^2\,\mathrm{d}x - \int_{\mathbb{R}^N}\frac{F(x,t\phi)}{(t\phi)^p}\phi^p\mathrm{d}x\right)\\ &= -\int_{\mathbb{R}^N}\frac{F(x,t\phi)}{(t\phi)^p}\phi^p\mathrm{d}x\\ &\leq -\frac{1}{p}\int_{\mathbb{R}^N} f_2(x)\phi^p(x)\mathrm{d}x < 0.\end{aligned}$$

Let $t \to +\infty$ we have $\Phi(t\phi) \to -\infty$. The proof of Lemma 3.2 is therefore complete. □

Lemma 3.3. *Assume that the assumptions of Theorem 1.2 hold. Then there exists a constant $\Lambda_1 > 0$ such that Φ satisfies the $(C)_c$-condition in $\mathbf{E}_\lambda$ for all $c \in \mathbb{R}, \lambda \geq \Lambda_1$.*

Proof. Let $\{u_n\}$ be a sequence in $\mathbf{E}_\lambda$ such that

$$\Phi(u_n) \to c \quad \text{and} \quad \left(1+\|u_n\|_{\mathbf{E}_\lambda}\right)\|\Phi'(u_n)\|_{\mathbf{E}_\lambda^*} \to 0.$$

We first show that $\{u_n\}$ is bounded in $\mathbf{E}_\lambda$. Indeed, for n large enough, by the condition (F_2), we have

$$\begin{aligned}c+1 &\geq \Phi(u_n) - \frac{1}{\mu}\Phi'(u_n)(u_n)\\ &= \frac{\mu-2}{2\mu}\|u_n\|_{\mathbf{E}_\lambda}^2 + \frac{\mu-2}{2\mu}\int_{\mathbb{R}^N}|\nabla_\gamma u_n|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\right)\mathrm{d}x\\ &\geq \frac{\mu-2}{2\mu}\|u_n\|_{\mathbf{E}_\lambda}^2 - \frac{\|f_3\|_{L^{\frac{2}{2-\ell}}(\mathbb{R}^N)}\Theta_2^\ell}{\mu}\|u_n\|_{\mathbf{E}_\lambda}^\ell.\end{aligned}$$

Since $1 < \ell < 2$, hence $\{u_n\}$ is bounded in $\mathbf{E}_\lambda$ for every $\lambda > \Lambda$.

Because of the above result, without loss of generality, we can suppose that

$$\begin{aligned}
u_n &\rightharpoonup u_0 && \text{in } \mathbf{E}_\lambda,\\
u_n &\to u_0 && \text{strongly in } L^p_{\mathrm{loc}}(\mathbb{R}^N), \quad \text{for } 2 \le p < 2^\gamma_*,\\
u_n &\to u_0 && \text{a.e. in } \mathbb{R}^N,
\end{aligned}$$

and $\Phi'(u_0) = 0$. Now we prove that $u_n \to u_0$ strongly in $\mathbf{E}_\lambda$. Let $v_n = u_n - u_0$. Then $v_n \rightharpoonup 0$ in $\mathbf{E}_\lambda$ hence $\{v_n\}$ is bounded in $\mathbf{E}_\lambda$. By the condition (B_2), we get

$$\begin{aligned}
\int_{\mathbb{R}^N} v_n^2 \mathrm{d}x &= \int_{\{b \ge C_0\}} v_n^2 \mathrm{d}x + \int_{\{b < C_0\}} v_n^2 \mathrm{d}x\\
&\le \frac{1}{\lambda C_0} \int_{\mathbb{R}^N} \lambda b(x) v_n^2 \mathrm{d}x + \int_{\{b<C_0\}} v_n^2 \mathrm{d}x\\
&\le \frac{1}{\lambda C_0} \|v_n\|^2_{\mathbf{E}_\lambda} + o(1).
\end{aligned} \tag{3.4}$$

Using (3.4), together with Hölder's inequality and Lemma 2.2, for any $\lambda > \Lambda$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x &\le \left(\int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x\right)^{\frac{2^\gamma_*-p}{2^\gamma_*-2}} \left(\int_{\mathbb{R}^N} |u|^{2^\gamma_*} \,\mathrm{d}x\right)^{\frac{p-2}{2^\gamma_*-2}}\\
&\le \left(\frac{1}{\lambda C_0}\|v_n\|^2_{\mathbf{E}_\lambda}\right)^{\frac{2^\gamma_*-p}{2^\gamma_*-2}} \left(C_3^{2^\gamma_*} \left(\int_{\mathbb{R}^N} |\Delta_\gamma v(n)|^{2^\gamma_*} \,\mathrm{d}x\right)^{\frac{2^\gamma_*}{2}}\right)^{\frac{p-2}{2^\gamma_*-2}} + o(1)\\
&\le C_3^{\frac{2^\gamma_*(p-2)}{2^\gamma_*-2}} \left(\frac{1}{\lambda C_0}\right)^{\frac{2^\gamma_*-p}{2^\gamma_*-2}} \|v_n\|^p_{\mathbf{E}_\lambda} + o(1).
\end{aligned} \tag{3.5}$$

Set

$$\Pi_\lambda = C_3^{\frac{2^\gamma_*(p-2)}{2^\gamma_*-2}} \left(\frac{1}{\lambda C_0}\right)^{\frac{2^\gamma_*-p}{2^\gamma_*-2}}.$$

By the condition (F_1) and (3.4) and (3.5), we get

$$\begin{aligned}
o(1) = \Phi'(v_n)(v_n) &= \|v_n\|^2_{\mathbf{E}_\lambda} + \int_{\mathbb{R}^N} |\nabla_\gamma v_n|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} f(x, v_n) v_n \mathrm{d}x\\
&\ge \|v_n\|^2_{\mathbf{E}_\lambda} - \varepsilon \int_{\mathbb{R}^N} v_n^2 \mathrm{d}x - C_\varepsilon \int_{\mathbb{R}^N} |v_n|^p \,\mathrm{d}x\\
&\le \|v_n\|^2_{\mathbf{E}_\lambda} - \frac{\varepsilon}{\lambda C_0} \|v_n\|^2_{\mathbf{E}_\lambda} - C_\varepsilon \Pi_\lambda \|v_n\|^p_{\mathbf{E}_\lambda} + o(1).
\end{aligned} \tag{3.6}$$

Since $\Pi_\lambda \to 0$ as $\lambda \to \infty$, by (3.6), there exists $\Lambda_1 \ge \Lambda$ such that for $\lambda > \Lambda_1$,

$$v_n \to 0 \quad \text{strongly in } \mathbf{E}_\lambda.$$

This completes the proof. □

Proof of Theorem 1.2. Combining Lemmas 3.1–3.3, we deduce that the problem (1.2) has a nontrivial weak solution. □

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