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Local null controllability for a parabolic-elliptic system with local and nonlocal nonlinearities

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Abstract. This work deals with the null controllability of an initial boundary value problem for a parabolic-elliptic coupled system with nonlinear terms of local and nonlocal kinds. The control is distributed, locally in space and appears only in one PDE. We first prove that, if the initial data is sufficiently small and the linearized system at zero satisfies an appropriate condition, the equations can be driven to zero.

Keywords: null controllability, parabolic-elliptic systems, nonlocal nonlinearities, Carleman inequalities.

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1 Introduction and main results

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$, with boundary $\Gamma = \partial\Omega$ of class C^2. We fix $T > 0$ and we denote by Q the cylinder $Q = \Omega \times (0,T)$, with lateral boundary $\Sigma = \Gamma \times (0,T)$. We also consider a non-empty (small) open set $\mathcal{O} \subset \Omega$; as usual, $1_{\mathcal{O}}$ denotes the characteristic function of $\mathcal{O}$.

Throughout this paper, C (and sometimes $C_0, K, K_0, \dots$) denotes various positive constants.

The inner product and norm in $L^2(\Omega)$ will be denoted, respectively, by $(\cdot\,,\cdot)$ and $\|\cdot\|$. On the other hand, $\|\cdot\|_\infty$ will stand for the norm in $L^\infty(Q)$. We will also denote $\vec{0} = (0,\dots,0) \in \mathbb{R}^n$.

We will be concerned with the null consider the following parabolic-elliptic coupled nonlinear systems

$$\begin{cases} y_t - \beta_1\left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx\right)\Delta y + F(y,z) = v1_{\mathcal{O}} & \text{in } Q,\\ -\beta_2\left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx\right)\Delta z + f(y,z) = 0 & \text{in } Q,\\ y(x,t) = 0,\ z(x,t) = 0 & \text{on } \Sigma,\\ y(x,0) = y_0(x) & \text{in } \Omega \end{cases} \tag{1.1}$$

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and

$$\begin{cases} y_t - \beta_1\left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx\right)\Delta y + F(y,z) = 0 & \text{in } Q, \\ -\beta_2\left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx\right)\Delta z + f(y,z) = w1_{\mathcal{O}} & \text{in } Q, \\ y(x,t) = 0,\ z(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega \end{cases} \tag{1.2}$$

where v is the control for the parabolic equation in (1.1), w is the control for the elliptic equation in (1.2) and (y,z) is the state for both systems.

Here $1_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$ and $y_0 = y_0(x)$ is the initial state; the nonlinearities $\beta_1 = \beta_1(r,s,l_1,\ldots,l_n,u_1,\ldots,u_n)$, $\beta_2 = \beta_2(r,s,l_1,\ldots,l_n,u_1,\ldots,u_n)$, $F = F(r,s)$ and $f = f(r,s)$ are C^1 functions (defined in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ and $\mathbb{R}\times\mathbb{R}$, resp.) that possess bounded derivatives and satisfy

$$0 < c_0 \le \beta_1(r,s,l,u),\ \beta_2(r,s,l,u) \le c_1, \quad \forall (r,s,l,u) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$$

and

$$F(0,0) = f(0,0) = 0, \quad \left|D_2 f(0,0)\right| < c_0\mu_1,$$

where μ_1 the first eigenvalue of the Dirichlet Laplacian in Ω.

If $y_0 \in L^2(\Omega)$, $v \in L^2(\mathcal{O}\times(0,T))$ (resp. $w \in L^2(\mathcal{O}\times(0,T))$) and the functions β_1, β_2, F and f satisfy the previous conditions, then (1.1) (resp. (1.2)) possesses exactly one weak solution (y,z) with

$$y \in L^2(0,T;H_0^1(\Omega)), \quad y_t \in L^2(0,T;H^{-1}(\Omega)), \quad z \in L^2(0,T;D(-\Delta)).$$

In this paper we will analyze some controllability properties of (1.1) and (1.2).

Definition 1.1. It will be said that (1.1) (resp. (1.2)) is locally null-controllable at time T if there exists $\epsilon > 0$ such that for any given $y_0 \in H_0^1(\Omega)$, with

$$\|y_0\|_{H_0^1(\Omega)} < \epsilon$$

there exist controls $v \in L^2(\mathcal{O}\times(0,T))$ (resp. controls $w \in L^2(\mathcal{O}\times(0,T))$) such that the associated states (y,z) satisfy

$$y(x,T) = 0 \text{ in } \Omega, \quad \limsup_{t\to T^-}\|z(\cdot,t)\| = 0. \tag{1.3}$$

The analysis of systems of the kind (1.1) and (1.2) can be justified by several applications. Let us indicate two of them:

- Reaction-diffusion systems with origin in physics, chemistry, biology, etc. where two scalar "populations" interact and the natural time scale of the growth rate is much smaller for one of them than for the other one. Precise examples can be found in the study of prey-predator interaction, chemical heating, tumor growth therapy, etc.
- Semiconductor modeling, where one of the state variables is (for example) the density of holes and the other one is the electrical potential of the device; see for instance [17]. Other problems with this motivation will be analyzed with more detail by the authors in the next future.

The nonlocal terms in (1.1) and (1.2) have important physical motivations, for an example: in the case of migration of populations, for instance the bacteria in a container, the diffusion coefficients may depend on the total amount of individuals.

Let us recall other two examples of real-world models where the nonlocal terms appear naturally:

- In the context of reaction-diffusion systems, it is also frequent to find terms of this kind; the particular case
$$\beta(\langle p, y(\cdot,t)\rangle, \langle q, z(\cdot,t)\rangle)$$
where $\beta(s,r)$ is a positive continuous function and l and m are continuous linear forms on $L^2(\Omega)$, has been investigated for instance by Chang and Chipot [3]. We refer to this paper for more details.

- Let us also mention that, in the context of hyperbolic systems, terms of the form
$$\beta\left(\int_\Omega |\nabla y(x.t)|^2_{\mathbb{R}^n}\,dx,\ \int_\Omega |\nabla z(x.t)|^2_{\mathbb{R}^n}\,dx\right)$$
appear in the Kirchhoff equation, which arises in nonlinear vibration theory; see for instance [22].

The control of PDEs equations and systems has been the subject of a lot of papers the last years. In particular, important progress has been made recently in the controllability analysis of semi-linear parabolic equations. We refer to the works [5,6,8,9,12–14,24,25] and the references therein. Consequently, it is natural to try to extend the known results to systems of the kind (1.1) and (1.2).

Note that if β_1 and β_2 are constants, we get, as a particular case, the results of [11] and when $\beta_1 = \beta_1\big(\int_\Omega y\,dx, \int_\Omega z\,dx\big)$ and $\beta_2 = \beta_2\left(\int_\Omega y\,dx, \int_\Omega z\,dx\right)$, we have the parabolic-parabolic system of [5].

Moreover, with the techniques of [5] based on Lemma 3.2 from the same article, it is not possible to solve the parabolic-parabolic system with
$$\beta_j = \beta_j\left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx\right), \quad j = 1,2.$$
Thus, we have a real improvement over the parabolic-elliptic works of [11] and the work [5] (even though the latter is a parabolic-parabolic system).

The main results are the following.

Theorem 1.2. *Under the previous assumptions on F, f, β_j, $j = 1,2$, if we assume that $D_1 f(0,0) \neq 0$, then the nonlinear system (1.1) is locally null-controllable at any time $T > 0$. In other words, there exists $\epsilon > 0$ such that, whenever $y_0 \in H^1_0(\Omega)$ and*
$$\|y_0\|_{H^1_0(\Omega)} < \epsilon,$$
there exists controls $v \in L^2(\mathcal{O}\times(0,T))$ and associated states (y,z) satisfying (1.3).

Theorem 1.3. *Under the previous assumptions on F, f, β_j, $j = 1,2$, if we assume that $D_2 F(0,0) \neq 0$, then the nonlinear system (1.2) is locally null-controllable at any time $T > 0$, i.e there exists $\epsilon > 0$ such that, whenever $y_0 \in H^1_0(\Omega)$ and*
$$\|y_0\|_{H^1_0(\Omega)} < \epsilon,$$
there exists controls $w \in L^2(\mathcal{O}\times(0,T))$ and associated states (y,z) satisfying (1.3).

The main difficulties found in the proof are that: (a) nonlinear terms appear in the main part ofthe partial derivative operators: (b) only one scalar control is used in the system (or in the parabolic equation or in the elliptic one). We will employ a technique relying on the so called *Liusternik's Inverse Mapping Theorem* in Hilbert spaces, see [1]. The arguments are inspired by the works of Fursikov and Imanuvilov [13] and Imanuvilov and Yamamoto [16] and rely on some estimates already used by these authors for other similar problems.

More precisely, in a first step, we will first consider similar *linearized* systems at zero

$$\begin{cases} y_t - \beta_1(0,0,\vec{0},\vec{0})\Delta y + ay + bz = v1_{\mathcal{O}} + h & \text{in } Q, \\ -\beta_2(0,0,\vec{0},\vec{0})\Delta z + cy + dz = k & \text{in } Q, \\ y = 0, \; z = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega \end{cases} \tag{1.4}$$

and

$$\begin{cases} y_t - \beta_1(0,0,\vec{0},\vec{0})\Delta y + ay + bz = h & \text{in } Q, \\ -\beta_2(0,0,\vec{0},\vec{0})\Delta z + cy + dz = w1_{\mathcal{O}} + k & \text{in } Q, \\ y = 0, \; z = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1.5}$$

where the coefficients a, b, c, d are obtained from the partial derivatives of F and f at $(0,0)$ and, in particular, $c \neq 0$ in (1.4) and $b \neq 0$ in (1.5). The adjoint of (1.4) and (1.5) is given by

$$\begin{cases} -\varphi_t - \beta_1(0,0,\vec{0},\vec{0})\Delta\varphi + a\varphi + c\psi = G_1 & \text{in } Q, \\ -\beta_2(0,0,\vec{0},\vec{0})\Delta\psi + b\varphi + d\psi = G_2 & \text{in } Q, \\ \varphi = 0, \; \psi = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi_T(x) & \text{in } \Omega. \end{cases} \tag{1.6}$$

Following well known ideas, the null controllability of (1.4) and (1.5) (for appropriate h and k) will obtained below as a consequence of suitable Carleman estimates for the solutions to (1.6). Then, in a second step, we will rewrite the null controllability property of (1.1) and (1.2) as an equation for (y, z) in a well chosen space of "admissible" state-control triplets:

$$H(y,z,v) = (0,0,y_0), \quad (y,z,v) \in Y; \quad (\text{resp. } H(y,z,w) = (0,0,y_0))$$

see the precise definitions of Y and H at the beginning of Section 3. In fact, the choice of Y is nontrivial, motivates some preliminary estimates of the null controls ans associated solutions to (1.4) and (1.5) and deserves some additional work. We will apply Liusternik's Theorem to (1.6) and we deduce the (local) desired result from a similar (global) property for the linear system (1.4) and (1.5).

This paper is organized as follows. In Section 2, we prove some technical results and we establish the null controllability of (1.4) and (1.5). Section 3 deals with the proofs of Theorems 1.2 and 1.3. Finally, some additional comments and questions are presented in Section 4.

2 Carleman estimates and the null controllability of (1.4) and (1.5)

We will first consider the general linear backwards in time system

$$\begin{cases} -\varphi_t - \beta_1(0,0,\vec{0},\vec{0})\Delta\varphi + a\varphi + c\psi = 0 & \text{in } Q, \\ -\beta_2(0,0,\vec{0},\vec{0})\Delta\psi + b\varphi + d\psi = 0 & \text{in } Q, \\ \varphi = 0,\ \psi = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi_T(x) & \text{in } \Omega \end{cases} \tag{2.1}$$

where $\varphi_T \in L^2(\Omega)$ and we assume that $|d| < c_0\mu_1$.

We will need some (well known) results from Fursikov and Immanuvilov [13]; see also [10]. Also, it will be convenient to introduce a new non-empty open set $\mathcal{O}_0$, with $\mathcal{O}_0 \Subset \mathcal{O}$. We will need the following fundamental result, due to Fursikov and Imanuvilov [13]:

Lemma 2.1. *There exists a function $\alpha_0 \in C^2(\overline{\Omega})$ satisfying:*

$$\begin{cases} \alpha_0(x) > 0 \quad \forall x \in \Omega, \quad \alpha_0(x) = 0 \quad \forall x \in \partial\Omega, \\ |\nabla\alpha_0(x)| > 0 \quad \forall x \in \overline{\Omega} \setminus \mathcal{O}_0. \end{cases}$$

Let us introduce the functions

$$\beta(t) := t(T-t), \quad \phi(x,t) := \frac{e^{\lambda\alpha_0(x)}}{\beta(t)}, \quad \alpha(x,t) := \frac{e^{R\lambda} - e^{\lambda\alpha_0(x)}}{\beta(t)},$$

where $R > \|\alpha_0\|_{L^\infty} + \ln(4)$ and $\lambda > 0$.

Also, let us set

$$\widehat{\alpha}(t) := \min_{x\in\overline{\Omega}} \alpha(x,t), \qquad \alpha^*(t) := \max_{x\in\overline{\Omega}} \alpha(x,t),$$
$$\widehat{\phi}(t) := \min_{x\in\overline{\Omega}} \phi(x,t), \qquad \phi^*(t) := \max_{x\in\overline{\Omega}} \alpha(x,t).$$

Then the following Carleman estimates hold.

Proposition 2.2. *Assume that $|d| < c_0\mu_1$ holds. There exist positive constants λ_0, s_0 and C_0 such that, for any $s \geq s_0$ and $\lambda \geq \lambda_0$ and any $\varphi_T \in L^2(\Omega)$, the associated solution to (2.1) satisfies*

$$\begin{aligned} &\iint_Q e^{-2s\alpha}\left[(s\phi)^{-1}\left(|\varphi_t|^2 + |\Delta\varphi|^2\right) + \lambda^2(s\phi)|\nabla\varphi|^2 + \lambda^4(s\phi)^3|\varphi|^2\right] dxdt \\ &\quad \leq C_0\left(\iint_Q e^{-2s\alpha}|\psi|^2 + \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\varphi|^2\right) dxdt \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\iint_Q e^{-2s\alpha}\left[(s\phi)^{-1}|\Delta\psi|^2 + \lambda^2(s\phi)|\nabla\psi|^2 + \lambda^4(s\phi)^3|\psi|^2\right] dxdt \\ &\quad \leq C_0\left(\iint_Q e^{-2s\alpha}|\varphi|^2 + \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\psi|^2\right) dxdt. \end{aligned} \tag{2.3}$$

Furthermore, C_0 and λ_0 only depend on Ω and $\mathcal{O}$ and s_0 can be chosen of the form

$$s_0 = \sigma_0(T + T^2), \tag{2.4}$$

where σ_0 only depends on Ω, $\mathcal{O}$, $|a|$, $|b|$, $|c|$ and $|d|$.

This result is proved in [13]. In fact, similar Carleman inequalities are established there for more general linear parabolic equations. The explicit dependence in time of the constants is not given in [13]. We refer to [10], where the above formula for s_0 is obtained.

For further purpose, we introduce the following notation:

$$I(s,\lambda;\varphi) = \iint_Q e^{-2s\alpha}\left[(s\phi)^{-1}(|\varphi_t|^2+|\Delta\varphi|^2)+\lambda^2(s\phi)|\nabla\varphi|^2+\lambda^4(s\phi)^3|\varphi|^2\right]dxdt$$

and

$$\widetilde{I}(s,\lambda;\psi) = \iint_Q e^{-2s\alpha}\left[(s\phi)^{-1}|\Delta\psi|^2+\lambda^2(s\phi)|\nabla\psi|^2+\lambda^4(s\phi)^3|\psi|^2\right]dxdt.$$

2.1 Some Carleman inequalities for the solutions to (1.6)

Now, from Proposition 2.2 it is deduced a Carleman estimate for the solutions to (1.6) under particular hypotheses on the coefficients.

Proposition 2.3. *Let us assume that $G_1, G_2 \in L^2(Q)$ and the coefficients in (1.6) satisfy*

$$a,b,c,d \in \mathbb{R}, \quad c \neq 0, \quad |d| < c_0\mu_1.$$

There exist positive constants λ_0, s_0 and C_1 such that, for any $s \geq s_0$ and $\lambda \geq \lambda_0$ and any $\varphi_T \in L^2(\Omega)$, the associated solution to (1.4) satisfies

$$\begin{aligned} I(s,\lambda;\varphi)+\widetilde{I}(s,\lambda;\psi) \leq C_1\left(\iint_Q e^{-2s\alpha}\left[\lambda^4(s\phi)^3|G_1|^2+|G_2|^2\right]\,dxdt\right) \\ +\ C_1\left(\iint_{\mathcal{O}\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*}\lambda^8(s\phi^*)^7|\varphi|^2dxdt\right). \end{aligned} \tag{2.5}$$

Furthermore, C_1 and λ_0 only depend on Ω and $\mathcal{O}$ and s_0 can be chosen of the form

$$s_1 = \sigma_1(T+T^2), \tag{2.6}$$

where σ_1 only depends on Ω, $\mathcal{O}$, $\beta_i(0,0,\vec{0},\vec{0})$, $|a|$, $|b|$, $|c|$ and $|d|$.

Proof. It will be sufficient to show that there exist λ_0, s_0 and C_1 such that, for any small $\varepsilon > 0$, any $s \geq s_0$ and $\lambda \geq \lambda_0$, one has:

$$I(s,\lambda;\varphi)+\widetilde{I}(s,\lambda;\psi) \leq C\varepsilon I(s,\lambda;\varphi)+C\varepsilon\widetilde{I}(s,\lambda;\psi)+\ c_\epsilon S(s,\lambda;G_1,G_2,\varphi), \tag{2.7}$$

where $S(s,\lambda;G_1,G_2,\varphi)$ is the right-hand side in (2.5).

We start from (2.2) and (2.3) for φ and for ψ separately. After addition, by taking σ_1 sufficiently large and $s \geq \sigma_1(T+T^2)$ and $\lambda \geq \lambda_0$, we easily obtain:

$$\begin{aligned} I(s,\lambda;\varphi)+\widetilde{I}(s,\lambda;\psi) &\leq C\left(\iint_Q e^{-2s\alpha}\left[\lambda^4(s\phi)^3|G_1|^2+|G_2|^2\right]\,dxdt\right) \\ &\quad +C\left(\iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\left(|\varphi|^2+|\psi|^2\right)dxdt\right) \\ &\leq C\left(\iint_Q e^{-2s\alpha}\left[\lambda^4(s\phi)^3|G_1|^2+|G_2|^2\right]\,dxdt\right) \\ &\quad +C\left(\iint_{\mathcal{O}_0\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*}\lambda^8(s\phi^*)^7|\varphi|^2\,dxdt\right) \\ &\quad +C\left(\iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\psi|^2\,dxdt\right) \end{aligned} \tag{2.8}$$

Let us now introduce a function $\xi \in \mathcal{D}(\mathcal{O})$ satisfying $0 < \xi \leq 1$ and $\xi \equiv 1$ in $\mathcal{O}_0$. Then

$$\begin{aligned}
&\iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\psi|^2 dxdt \\
&\quad\leq \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\xi|\psi|^2 dxdt \\
&\quad= \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\xi(x)\psi\left(-\frac{1}{c}(\varphi_t+\Delta\varphi+a(x,t)\varphi-G_1)\right)dxdt \\
&\quad= -\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\frac{\xi(x)}{c}\,\psi\,\varphi_t\,dxdt \\
&\qquad-\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\frac{\xi(x)}{c}\,\psi\,\Delta\varphi\,dxdt \\
&\qquad-\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\frac{\xi(x)}{c}\,a(x,t)\,\psi\,\varphi\,dxdt \\
&\qquad+\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\frac{\xi(x)}{c}\,\psi\,G_1\,dxdt \\
&\quad=: M_1+M_2+M_3+M_4.
\end{aligned} \tag{2.9}$$

Let us compute and estimate the M_i. First,

$$\begin{aligned}
M_1 = &-\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\frac{2\xi(x)}{c}\lambda^4 s^4\phi^3\alpha_t\psi\varphi\,dxdt \\
&+\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\frac{3\xi(x)}{c}\lambda^4 s^3\phi^2\phi_t\psi\varphi\,dxdt \\
&+\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\frac{\xi(x)}{c}\lambda^4(s\phi)^3\psi_t\varphi\,dxdt.
\end{aligned} \tag{2.10}$$

Using that $|\alpha_t| \leq C\phi^2$ and $|\phi_t| \leq C\phi^2$ for some $C > 0$, we get:

$$\begin{aligned}
M_1 &\leq C\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4 s^4\phi^5|\psi|\,|\varphi|\,dxdt + \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\psi_t|\,|\varphi|\,dxdt \\
&\leq \varepsilon\widetilde{I}(s,\lambda;\psi) + C_\varepsilon\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4 s^5\phi^7|\varphi|^2 dxdt \\
&\quad+\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\psi_t|\,|\varphi|\,dxdt.
\end{aligned} \tag{2.11}$$

The last integral in this inequality can be bounded as follows:

$$\begin{aligned}
&\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\psi_t|\,|\varphi|\,dxdt \\
&\quad\leq \iint_{\mathcal{O}\times(0,T)} e^{-2s\widehat{\alpha}}\lambda^4(s\phi^*)^3|\psi_t|\,|\varphi|\,dxdt \\
&\quad= \int_0^T e^{-2s\widehat{\alpha}(t)}\lambda^4(s\phi^*(t))^3\|\psi_t(\cdot,t)\|_{L^2(\mathcal{O})}\|\varphi(\cdot,t)\|_{L^2(\mathcal{O})}\,dt \\
&\quad\leq C\int_0^T e^{-2s\widehat{\alpha}(t)}\lambda^4(s\phi^*(t))^3\|\varphi_t(\cdot,t)\|\|\varphi(\cdot,t)\|_{L^2(\mathcal{O})}\,dt \\
&\quad= C\int_0^T e^{-s\alpha^*}(s\phi^*(t))^{-1/2}\|\varphi_t(\cdot,t)\|\cdot e^{-2s\widehat{\alpha}+s\alpha^*}\lambda^4(s\phi^*)^{7/2}\|\varphi(\cdot,t)\|_{L^2(\mathcal{O})}\,dt \\
&\quad\leq \varepsilon I(s,\lambda;\varphi) + C_\varepsilon\iint_{\mathcal{O}\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*}\lambda^8(s\phi^*)^7|\varphi|^2 dxdt.
\end{aligned} \tag{2.12}$$

Thus, the following is found:

$$M_1 \leq \varepsilon I(s,\lambda;\varphi) + \varepsilon \widetilde{I}(s,\lambda;\psi) + C_\varepsilon \iint_{\mathcal{O}\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*} \lambda^8 (s\phi^*)^7 |\varphi|^2 dxdt. \tag{2.13}$$

Secondly, we see that

$$\begin{aligned} M_2 &= -\iint_{\mathcal{O}\times(0,T)} \Delta\left(e^{-2s\alpha}\lambda^4(s\phi)^3 \frac{\xi(x)}{c}\psi\right) \varphi \, dxdt \\ &\leq C \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\left[\lambda^6(s\phi)^5|\psi| + \lambda^5(s\phi)^4|\nabla\psi| + \lambda^4(s\phi)^3|\Delta\psi|\right] \varphi \, dxdt \\ &\leq \varepsilon \widetilde{I}(s,\lambda;\psi) + C_\varepsilon \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^8(s\phi)^7|\varphi|^2 dxdt. \end{aligned} \tag{2.14}$$

Here, we have used the identity

$$\Delta\left(e^{-2s\alpha}\phi^3\frac{\xi(x)}{c}\psi\right) = \Delta\left(e^{-2s\alpha}\phi^3\frac{\xi(x)}{c}\right)\psi + 2\nabla\left(e^{-2s\alpha}\phi^3\frac{\xi(x)}{c}\right)\cdot\nabla\psi + e^{-2s\alpha}\phi^3\frac{\xi(x)}{c}\Delta\psi$$

and the estimates

$$\left|\Delta\left(e^{-2s\alpha}\phi^3\frac{\xi(x)}{c}\right)\right| \leq Ce^{-2s\alpha}\lambda^2 s^2\phi^5 \quad \text{and} \quad \left|\nabla\left(e^{-2s\alpha}\phi^3\frac{\xi(x)}{c}\right)\right| \leq Ce^{-2s\alpha}\lambda s\phi^4.$$

Finally, it is immediate that

$$M_3 \leq \varepsilon \widetilde{I}(s,\lambda;\psi) + C_\varepsilon \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\varphi|^2 dxdt, \tag{2.15}$$

and

$$M_4 \leq \varepsilon \widetilde{I}(s,\lambda;\psi) + C_\varepsilon \iint_Q e^{-2s\alpha}\lambda^4(s\phi)^3|G_1|^2 dxdt. \tag{2.16}$$

From (2.8), (2.9) and (2.13)–(2.16), we directly obtain (2.7) for all small $\varepsilon > 0$. This ends the proof. □

Now, we will assume that b is a non-zero constant:

$$b \in \mathbb{R}, \quad b \neq 0, \ |d| < c_0\mu_1. \tag{2.17}$$

Proposition 2.4. *Assume that (2.17) holds. There exist positive constants λ_2, s_2 and C_2 such that, for any $s \geq s_2$ and $\lambda \geq \lambda_2$ and any $\varphi^T \in L^2(\Omega)$, the associated solution to (1.6) satisfies*

$$\begin{aligned} I(s,\lambda;\varphi) + \widetilde{I}(s,\lambda;\psi) \leq C_2 &\left(\iint_Q e^{-2s\alpha}\left[\lambda^4(s\phi)^3|G_1|^2 + |G_2|^2\right] dxdt\right) \\ &+ C_2\left(\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^8(s\phi)^7|\psi|^2 dxdt\right). \end{aligned} \tag{2.18}$$

Furthermore, C_2 and λ_2 only depend on Ω and $\mathcal{O}$ and s_2 can be chosen of the form

$$s_2 = \sigma_2(T + T^2),$$

where σ_2 only depends on Ω, $\mathcal{O}$, $\beta_i(0,0,\vec{0},\vec{0})$, $|a|$, $|b|$, $|c|$ and $|d|$.

Proof. We start again from (2.8). Recalling that $\xi \in \mathcal{D}(\mathcal{O})$, $0 < \xi \leq 1$ and $\xi \equiv 1$ in $\mathcal{O}_0$, we see that

$$\begin{aligned}
&\iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\varphi|^2dxdt \\
&\quad\leq \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\xi|\varphi|^2dxdt \\
&\quad= \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\xi(x)\varphi\left(-\frac{1}{b}(\Delta\psi + d(x,t)\psi - G_2)\right)dxdt \\
&\quad= -\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\frac{\xi(x)}{b}\varphi\Delta\psi\, dxdt \\
&\qquad -\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\frac{\xi(x)}{b}d(x,t)\varphi\psi\, dxdt \\
&\qquad +\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3\frac{\xi(x)}{b}\,\varphi\, G_2\, dxdt \\
&\quad=: M_1' + M_2' + M_3'.
\end{aligned} \tag{2.19}$$

As in the proof of Proposition 2.3, it is not difficult to compute and estimate the M_i'. Indeed,

$$\begin{aligned}
M_1' &= -\iint_{\mathcal{O}\times(0,T)} \Delta\left(e^{-2s\alpha}\lambda^4(s\phi)^3\frac{\xi(x)}{b}\varphi\right)\psi\, dxdt \\
&\leq C\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\left[\lambda^6(s\phi)^5|\varphi| + \lambda^5(s\phi)^4|\nabla\varphi| + \lambda^4(s\phi)^3|\Delta\varphi|\right]|\psi|\, dxdt \\
&\leq \varepsilon I(s,\lambda;\varphi) + C_\varepsilon\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^8(s\phi)^7|\psi|^2dxdt.
\end{aligned} \tag{2.20}$$

On the other hand,

$$M_2' \leq \varepsilon I(s,\lambda;\varphi) + C_\varepsilon\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\psi|^2dxdt. \tag{2.21}$$

and

$$M_3' \leq \varepsilon\widetilde{I}(s,\lambda;\varphi) + C_\varepsilon\iint_Q e^{-2s\alpha}\lambda^4(s\phi)^3|G_2|^2dxdt. \tag{2.22}$$

From (2.8), (2.19) and (2.20)–(2.22), we find that

$$I(s,\lambda;\varphi) + \widetilde{I}(s,\lambda;\psi) \leq C\varepsilon I(s,\lambda;\varphi) + C\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\psi|^2dxdt,$$

for all small $\varepsilon > 0$. □

We will also need some Carleman inequalities for the solutions to (1.4) and (1.5) with weights not vanishing at zero. To this end, let m be a function satisfying

$$m \in C^\infty([0,T]), \qquad m(t) \geq \frac{T^2}{8} \quad \text{in } [0,T/2], \qquad m(t) = t(T-t) \quad \text{in } [T/2,T],$$

let us set $\lambda > 0$, $R > \|\alpha_0\|_{L^\infty} + \ln(4)$ and

$$\theta(x,t) := \frac{e^{\lambda\alpha_0(x)}}{m(t)}, \qquad A(x,t) := \frac{\bar{A}(x)}{m(t)}, \quad \text{with} \quad \bar{A}(x) = e^{R\lambda} - e^{\lambda\alpha_0(x)} \quad \text{and,}$$

$$\widehat{A} := \min_{x\in\overline{\Omega}} \bar{A}(x), \qquad A^* := \max_{x\in\overline{\Omega}} \bar{A}(x),$$
$$\widehat{\theta}(t) := \min_{x\in\overline{\Omega}} \theta(x,t), \qquad \theta^*(t) := \max_{x\in\overline{\Omega}} \theta(x,t),$$

and let us introduce the notation

$$\Gamma(s,\lambda;\varphi) = \iint_Q e^{-2sA}\Big[(s\theta)^{-1}\big(|\varphi_t|^2+|\Delta\varphi|^2\big)+\lambda^2(s\theta)|\nabla\varphi|^2+\lambda^4(s\theta)^3|\varphi|^2\Big]\,dxdt$$

and

$$\widetilde{\Gamma}(s,\lambda;\psi) = \iint_Q e^{-2sA}\Big[(s\theta)^{-1}|\Delta\psi|^2+\lambda^2(s\theta)|\nabla\psi|^2+\lambda^4(s\theta)^3|\psi|^2\Big]\,dxdt.$$

One has the following.

Proposition 2.5. *Let the assumptions of Proposition 2.3 be satisfied. There exist positive constants λ_3, s_3 such that, for any $s \geq s_3$ and $\lambda \geq \lambda_3$, there exists $C_3(s,\lambda)$ with the following property: for and any $\varphi_T \in L^2(\Omega)$ and any $\psi_T \in L^2(\Omega)$, the associated solution to (1.4) satisfies*

$$\begin{aligned}\Gamma(s,\lambda;\varphi)+\widetilde{\Gamma}(s,\lambda;\psi) &\leq C_3(s,\lambda)\left(\iint_Q e^{-2sA}\left[\theta^3|G_1|^2+|G_2|^2\right]\,dxdt\right)\\ &\quad + C_3(s,\lambda)\left(\iint_{\mathcal{O}\times(0,T)} e^{(-4s\widehat{A}+2sA^*)/m}(\theta^*)^7|\varphi|^2dxdt\right).\end{aligned} \tag{2.23}$$

Furthermore, s_3 and λ_3 only depend on Ω, $\mathcal{O}$, $\beta_i(0,0,\vec{0},\vec{0})$, $|a|$, $|b|$, $|c|$ and $|d|$ and $C_3(s,\lambda)$ only depend on these data, s and λ.

Proof. We can decompose all the integrals in $\Gamma(s,\lambda;\varphi)$ and $\widetilde{\Gamma}(s,\lambda;\psi)$ in the form:

$$\iint_Q = \iint_{\Omega\times(0,T/2)} + \iint_{\Omega\times(T/2,T)}.$$

Let us gather together all the integrals in $\Omega\times(0,T/2)$ (resp., $\Omega\times(T/2,T)$) in $\Gamma_1(s,\lambda;\varphi)$ and $\widetilde{\Gamma}_1(s,\lambda;\psi)$ (resp., $\Gamma_2(s,\lambda;\varphi)$ and $\widetilde{\Gamma}_2(s,\lambda;\psi)$). Then,

$$\Gamma(s,\lambda;\varphi) = \Gamma_1(s,\lambda;\varphi)+\Gamma_2(s,\lambda;\varphi)$$
$$\widetilde{\Gamma}(s,\lambda;\varphi) = \widetilde{\Gamma}_1(s,\lambda;\varphi)+\widetilde{\Gamma}_2(s,\lambda;\varphi).$$

Let us start again from the Carleman inequality in Proposition 2.3, with $s \geq s_0$ and $\lambda \geq \lambda_0$. We obviously have

$$\begin{aligned}\Gamma_2(s,\lambda;\varphi)+\widetilde{\Gamma}_2(s,\lambda;\varphi) &\leq C_1\iint_Q e^{-2s\alpha}\left[\lambda^4(s\phi)^3|G_1|^2+|G_2|^2\right]\,dxdt\\ &\quad + C_1\iint_{\mathcal{O}\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*}\lambda^8(s\phi^*)^7|\varphi|^2\,dxdt\end{aligned} \tag{2.24}$$

Now, let us come back to the energy estimate for φ and ψ. We have the following for all $t \in (0,T/2)$:

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\|\varphi\|^2+\beta_1(0,0,\vec{0},\vec{0})\|\nabla\varphi\|^2+\beta_2(0,0,\vec{0},\vec{0})\|\nabla\psi\|^2\\ &\quad\leq C(\|\varphi\|^2+\|\psi\|^2+\|G_1\|^2+\|G_2\|^2).\end{aligned} \tag{2.25}$$

Knowing that $\|\psi(\cdot,t)\|^2_{H^1_0(\Omega)} \leq M(\|\varphi(\cdot,t)\|^2_{L^2(\Omega)} + \|G_2\|^2_{L^2(\Omega)})$, we obtain from (2.25),

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 - M\|\varphi\|^2 + \beta_1(0,0,\vec{0},\vec{0})\|\nabla\varphi\|^2 + \beta_2(0,0,\vec{0},\vec{0})\|\nabla\psi\|^2 \\
&\qquad \leq C(\|\varphi\|^2 + \|G_1\|^2 + \|G_2\|^2).
\end{aligned}
\tag{2.26}
$$

From (2.26), it is easy to deduce that

$$
\begin{aligned}
&\iint_{\Omega\times(0,T/2)} (|\varphi|^2 + |\nabla\varphi|^2)\,dxdt \\
&\qquad \leq C\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dxdt + C\iint_{\Omega\times(0,3T/4)} (|G_1|^2 + |G_2|^2)\,dxdt.
\end{aligned}
\tag{2.27}
$$

Using only the first equation of the adjoint-state (1.6), a second-order energy estimate can also be deduced for φ:

$$
-\frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|^2 + \frac{\beta_1(0,0,\vec{0},\vec{0})}{2}\|\Delta\varphi\|^2 \leq C(\|\varphi\|^2 + \|G_1\|^2),
\tag{2.28}
$$

for all $t \in (0, T/2)$. This leads to the following:

$$
\iint_{\Omega\times(0,T/2)} |\Delta\varphi|^2\,dxdt \leq C\iint_{\Omega\times(T/4,3T/4)} |\nabla\varphi|^2\,dxdt + C\iint_{\Omega\times(0,3T/4)} |G_1|^2\,dxdt.
\tag{2.29}
$$

Finally, from the PDEs in (1.6), the inequalities (2.27) and (2.29) yield:

$$
\begin{aligned}
&\iint_{\Omega\times(0,T/2)} |\varphi_t|^2\,dxdt \\
&\qquad \leq C\iint_{\Omega\times(T/4,3T/4)} (|\varphi|^2 + |\nabla\varphi|^2)\,dxdt + C\iint_{\Omega\times(0,3T/4)} |G_1|^2\,dxdt.
\end{aligned}
\tag{2.30}
$$

From (2.27)–(2.30) and knowing that $\|\Delta\psi(.,t)\|^2_{L^2(\Omega)} \leq C(\|\varphi(.,t)\|^2_{L^2(\Omega)} + \|G_2\|^2_{L^2(\Omega)})$, we deduce that

$$
\begin{aligned}
&\Gamma_1(s,\lambda;\varphi) + \widetilde{\Gamma}_1(s,\lambda;\psi) \\
&\quad \leq C\iint_{\Omega\times(0,T/2)} (|\varphi_t|^2 + |\Delta\varphi|^2 + |\nabla\varphi|^2 + |\varphi|^2)\,dxdt + C\iint_{\Omega\times(0,T/2)} |G_2|^2\,dxdt \\
&\quad \leq C\iint_{\Omega\times(T/4,3T/4)} (|\varphi|^2 + |\nabla\varphi|^2)\,dxdt + C\iint_{\Omega\times(0,3T/4)} (|G_1|^2 + |G_2|^2)\,dxdt,
\end{aligned}
\tag{2.31}
$$

whence

$$
\begin{aligned}
&\Gamma_1(s,\lambda;\varphi) + \widetilde{\Gamma}_1(s,\lambda;\psi) \\
&\quad \leq C(s,\lambda)\left[I(s,\lambda;\varphi) + \widetilde{I}(s,\lambda;\psi) + \iint_{\Omega\times(0,3T/4)} (|G_1|^2 + |G_2|^2)\,dxdt\right] \\
&\quad \leq C(s,\lambda)\left(\iint_{\mathcal{O}\times(0,T)} e^{-4s\widehat{A}+2sA^*}(\theta^*)^7|\varphi|^2\,dxdt + \iint_{\Omega\times(0,3T/4)} (|G_1|^2 + |G_2|^2)\,dxdt\right)
\end{aligned}
\tag{2.32}
$$

Combining (2.24) with these inequalities, we obtain at once (2.23). □

We also have the following estimate for the solutions of (1.5).

Proposition 2.6. *Let the assumptions of Proposition 2.4 be satisfied. There exist positive constants λ_4, s_4 such that, for any $s \geq s_4$ and $\lambda \geq \lambda_4$, there exists $C_4(s,\lambda)$ with the following property: for and any $\varphi_T \in L^2(\Omega)$ and any $\psi_T \in L^2(\Omega)$, the associated solution to (1.5) satisfies*

$$\Gamma(s,\lambda;\varphi)+\widetilde{\Gamma}(s,\lambda;\psi) \leq C_4(s,\lambda)\left(\iint_Q e^{-2sA}\left[\theta^3|G_1|^2+|G_2|^2\right]\,dxdt\right) + C_4(s,\lambda)\left(\iint_{\mathcal{O}\times(0,T)} e^{-2sA}\theta^7|\psi|^2 dxdt\right). \tag{2.33}$$

Furthermore, s_4 and λ_4 only depend on Ω, $\mathcal{O}$, $\beta_i(0,0,\vec{0},\vec{0})$, $|a|$, $|b|$, $|c|$ and $|d|$ and $C_4(s,\lambda)$ only depend on these data, s and λ.

Proof. As in the proof of Proposition 2.5, we decompose all the integrals in $\Gamma(s,\lambda;\varphi)$ and $\widetilde{\Gamma}(s,\lambda;\psi)$ in the form:

$$\iint_Q = \iint_{\Omega\times(0,T/2)} + \iint_{\Omega\times(T/2,T)},$$

where

$$\Gamma(s,\lambda;\varphi) = \Gamma_1(s,\lambda;\varphi)+\Gamma_2(s,\lambda;\varphi)$$
$$\widetilde{\Gamma}(s,\lambda;\varphi) = \widetilde{\Gamma}_1(s,\lambda;\varphi)+\widetilde{\Gamma}_2(s,\lambda;\varphi).$$

From the Carleman inequality in Proposition 2.4, with $s \geq s_2$ and $\lambda \geq \lambda_2$, we have

$$\Gamma_2(s,\lambda;\varphi)+\widetilde{\Gamma}_2(s,\lambda;\varphi) \leq C_1\iint_Q e^{-2s\alpha}\left[\lambda^4(s\phi)^3|G_1|^2+|G_2|^2\right]\,dxdt + C_1\iint_{\mathcal{O}\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*}\lambda^8(s\phi^*)^7|\varphi|^2\,dxdt. \tag{2.34}$$

Using the same ideas from Proposition 2.5, we easily deduce that

$$\iint_{\Omega\times(0,T/2)}(|\varphi|^2+|\nabla\varphi|^2)\,dxdt \leq C\iint_{\Omega\times(T/4,3T/4)}|\varphi|^2\,dxdt + C\iint_{\Omega\times(0,3T/4)}(|G_1|^2+|G_2|^2)\,dxdt \tag{2.35}$$

and

$$\iint_{\Omega\times(0,T/2)}|\Delta\varphi|^2\,dxdt \leq C\iint_{\Omega\times(T/4,3T/4)}|\nabla\varphi|^2\,dxdt + C\iint_{\Omega\times(0,3T/4)}|G_1|^2\,dxdt. \tag{2.36}$$

From the PDEs in (1.6), the inequalities (2.35) and (2.36) yield:

$$\iint_{\Omega\times(0,T/2)}|\varphi_t|^2\,dxdt \leq C\iint_{\Omega\times(T/4,3T/4)}(|\varphi|^2+|\nabla\varphi|^2)\,dxdt + C\iint_{\Omega\times(0,3T/4)}|G_1|^2\,dxdt. \tag{2.37}$$

Then, from Proposition 2.4 and (2.35)–(2.37), we have

$$\begin{aligned}
&\Gamma_1(s,\lambda;\varphi)+\widetilde{\Gamma}_1(s,\lambda;\psi)\\
&\quad\le C\iint_{\Omega\times(T/4,3T/4)}(|\varphi|^2+|\nabla\varphi|^2)\,dxdt+C\iint_{\Omega\times(0,3T/4)}(|G_1|^2+|G_2|^2)\,dxdt\\
&\quad\le C(s,\lambda)\left[I(s,\lambda;\varphi)+\widetilde{I}(s,\lambda;\psi)+\iint_{\Omega\times(0,3T/4)}(|G_1|^2+|G_2|^2)\,dxdt\right]\\
&\quad\le C(s,\lambda)\left(\iint_{\mathcal{O}\times(0,T)}e^{-2sA}\theta^7|\psi|^2\,dxdt+\iint_{\Omega\times(0,3T/4)}(|G_1|^2+|G_2|^2)\,dxdt\right).
\end{aligned}\tag{2.38}$$

Combining (2.34) with the inequality (2.38), we obtain (2.33). □

In the sequel, when $\lambda=\lambda_3$ and $s=s_3$, we set

$$\rho:=e^{sA},\qquad \rho_0:=\theta^{-3/2}e^{sA},$$
$$\widehat{\rho}:=e^{(sA)/2}e^{(2s\widehat{A}-sA^*)/2m}\theta^{-3/4}(\theta^*)^{-7/4},\qquad \rho_*:=e^{(2s\widehat{A}-sA^*)/m}(\theta^*)^{-7/2}.$$

Then, we deduce from (2.23) that the solution to (1.4) satisfies:

$$\Gamma(s,\lambda;\varphi)+\widetilde{\Gamma}(s,\lambda;\psi)\le K\left(\iint_Q e^{-2sA}\left[\theta^3|G_1|^2+|G_2|^2\right]\,dxdt+\iint_{\mathcal{O}\times(0,T)}\rho_*^{-2}|\varphi|^2dxdt\right).\tag{2.39}$$

For the case where $\lambda=\lambda_4$ and $s=s_4$, we set

$$\rho:=e^{sA},\qquad \rho_0:=\theta^{-3/2}e^{sA},\qquad \widehat{\rho}:=\theta^{-5/2}e^{sA},\qquad \rho_*:=\theta^{-7/2}e^{sA},$$

whence we obtain from (2.33) that the solution to (1.5) satisfies:

$$\Gamma(s,\lambda;\varphi)+\widetilde{\Gamma}(s,\lambda;\psi)\le K\left(\iint_Q e^{-2sA}\left[\theta^3|G_1|^2+|G_2|^2\right]\,dxdt+\iint_{\mathcal{O}\times(0,T)}\rho_*^{-2}|\psi|^2dxdt\right).\tag{2.40}$$

2.2 The null controllability of the linearized systems (1.4) and (1.5)

As a consequence of Proposition 2.5, we obtain the null controllability of (1.4) for "small" right-hand sides h and k:

Proposition 2.7. *Assume that $c\neq 0$ and the functions h and k satisfy*

$$\iint_Q \rho^2\theta^{-3}(|h|^2+|k|^2)\,dxdt<+\infty.$$

Then (1.4) is null-controllable at any time $T>0$. More precisely, for any $y_0\in L^2(\Omega)$ and any $T>0$, there exist controls $v\in L^2(\mathcal{O}\times(0,T))$ and associated states (y,z) satisfying

$$\iint_{\mathcal{O}\times(0,T)}\rho_*^2|v|^2\,dxdt<+\infty,\qquad \iint_Q(\rho_0^2|y|^2+\rho^2|z|^2)\,dxdt<+\infty,\tag{2.41}$$

whence, in particular,

$$y(x,T)=0\quad in\ \Omega,\qquad \limsup_{t\to T^-}\|z(\cdot,t)\|=0.\tag{2.42}$$

Proof. Here we will use well known ideas from the work by Fursikov and Imanuvilov [13].

For each $n \geq 1$, let us introduce the functions

$$A_n := \frac{A(T-t)}{(T-t)+1/n}, \qquad \theta_n := \frac{\theta(T-t)}{(T-t)+1/n}, \qquad \rho_n := e^{sA_n}, \qquad \rho_{0,n} := \rho_n \theta^{-3/2}$$

and

$$\rho_{*,n} = \rho_* \cdot m_n = e^{(2s\widehat{A}-sA^*)/m}(\theta^*)^{-7/2} \cdot m_n, \quad \text{where } m_n = \begin{cases} 1, & \text{in } \mathcal{O} \\ n, & \text{in } \Omega - \overline{\mathcal{O}} \end{cases}$$

and the functional $J_n : L^2(Q) \times L^2(Q) \times L^2(\mathcal{O} \times (0,T)) \mapsto \mathbb{R}$, with

$$J_n(y,z,v) := \frac{1}{2} \iint_Q \left[\rho_{0,n}^2 |y|^2 + \rho_n^2 |z|^2 + \rho_{*,n}^2 |v|^2\right] \, dxdt.$$

Let us consider the following extremal problem:

$$\begin{cases} \text{Minimize } J_n(y,z,v), \\ \text{Subject to } v \in L^2(\mathcal{O} \times (0,T)), \ (y,z,v) \text{ satisfies (1.4)}. \end{cases}$$

This problem has a unique solution (y_n, z_n, v_n). Furthermore, in view of *Lagrange's Principle*, there exists (p_n, q_n) such that (y_n, z_n), (p_n, q_n) and v_n satisfy:

$$\begin{cases} y_{n,t} - \beta_1(0,0,\vec{0},\vec{0})\Delta y_n + a y_n + b z_n = v_n 1_{\mathcal{O}} + h & \text{in } Q, \\ -\beta_2(0,0,\vec{0},\vec{0})\Delta z_n + c y_n + d z_n = k & \text{in } Q, \\ y_n = 0, \ z_n = 0 & \text{on } \Sigma, \\ y_n(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{2.43}$$

$$\begin{cases} -p_{n,t} - \beta_1(0,0,\vec{0},\vec{0})\Delta p_n + a p_n + c q_n = -\rho_{0,n}^2 y_n & \text{in } Q, \\ -\beta_2(0,0,\vec{0},\vec{0})\Delta q_n + b p_n + d q_n = -\rho_n^2 z_n & \text{in } Q, \\ p_n = 0, \ q_n = 0 & \text{on } \Sigma, \\ p_n(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{2.44}$$

$$p_n = -\rho_{*,n}^2 v_n \quad \text{in } \mathcal{O} \times (0,T). \tag{2.45}$$

Multiplying the PDEs in (2.45) by y_n and z_n and integrating in Q, we get:

$$\begin{aligned} 0 = & \iint_Q \left[-p_{n,t} - \beta_1(0,0,\vec{0},\vec{0})\Delta p_n + a p_n + c q_n + \rho_{0,n}^2 y_n\right] y_n \, dxdt \\ & + \iint_Q \left[-\beta_2(0,0,\vec{0},\vec{0})\Delta q_n + b p_n + d q_n + \rho_n^2 z_n\right] z_n \, dxdt. \end{aligned} \tag{2.46}$$

Integrating by parts, we see that

$$\begin{aligned} & \iint_Q \left(\rho_{0,n}^2 |y_n|^2 + \rho_n^2 |z_n|^2\right) \, dxdt \\ & \quad = \iint_Q \left[y_{n,t} - \beta_1(0,0,\vec{0},\vec{0})\Delta y_n + a y_n + b z_n\right] p_n \, dxdt \\ & \quad + \iint_Q \left[-\beta_2(0,0,\vec{0},\vec{0})\Delta z_n + c y_n + d z_n\right] q_n \, dxdt \\ & \quad + \int_\Omega p_n(x,0) y_0(x) \, dx. \end{aligned} \tag{2.47}$$

From (2.47), taking into account (2.45) and the two PDEs from (2.43), and recalling the definition of J_n, we find that

$$J_n(y_n, z_n, v_n) = \frac{1}{2}\iint_Q (p_n h + q_n k)\, dxdt + \frac{1}{2}\int_\Omega p_n(x,0) y_0(x)\, dx.$$

Consequently,

$$\begin{aligned} J_n(y_n, z_n, v_n) \le C \left[\|p_n(\cdot,0)\|^2 + \iint_Q \rho^{-2}\theta^3 (|p_n|^2 + |q_n|^2)\, dxdt \right]^{1/2} \\ \times \left[\|y_0\|^2 + \iint_Q \rho^2\theta^{-3}(|h|^2 + |k|^2)\, dxdt \right]^{1/2} \end{aligned} \tag{2.48}$$

Let us now apply the Carleman inequality (2.23) to the solution (p_n, q_n) to (2.44). The following holds:

$$\begin{aligned} &\iint_Q \rho^{-2}\theta^3(|p_n|^2 + |q_n|^2)\, dxdt \\ &\le C_0(s,\lambda)\left(\iint_Q \left[\rho_0^{-2}\rho_{0,n}^4|y_n|^2 + \rho^{-2}\rho_n^4|z_n|^2\right]\, dxdt + \iint_{\mathcal{O}\times(0,T)} e^{-4s\widehat{A}+2sA^*}(\theta^*)^7|p_n|^2\, dxdt \right) \\ &\le C_0(s,\lambda)\left(\iint_Q \left[\rho_{0,n}^2|y_n|^2 + \rho_n^2|z_n|^2\right]\, dxdt + \iint_{\mathcal{O}\times(0,T)} \rho_*^{-2}\rho_*^4|v_n|^2\, dxdt \right) \\ &\le C_0(s,\lambda)\iint_Q \left[\rho_{0,n}^2|y_n|^2 + \rho_n^2|z_n|^2 + \rho_{*,n}^2|v_n|^2\right]\, dxdt \\ &\le C J_n(y_n, z_n, v_n), \end{aligned} \tag{2.49}$$

where we have used that $\rho_n \le \rho$, $\rho_{0,n} \le C\rho_0$ and $\rho_{*,n} = \rho_* \cdot m_n = \theta^{-7/2}\rho \cdot m_n$.

We also have

$$\|p_n(\cdot,0)\|^2 \le C J_n(y_n, z_n, v_n). \tag{2.50}$$

Indeed, let us multiply only the first PDE in (2.44) by p_n and the second one by q_n and let us integrate in Ω. Therefore, following holds:

$$-\frac{1}{2}\frac{d}{dt}\|p_n\|^2 \le \int_\Omega (\rho_{0,n}^4 y_n^2 + \rho_n^4 z_n^2)\, dx + \frac{1}{2}\int_\Omega (p_n^2 + q_n^2)\, dx + C(\|p_n\|^2 + \|q_n\|^2)$$

As $\|q_n(\cdot,t)\|^2_{H_0^1(\Omega)} \le C(\|p_n(\cdot,t)\|^2 + \|\rho_n^2 z_n(\cdot,t)\|^2)$, then

$$-\frac{1}{2}\frac{d}{dt}\|p_n\|^2 \le M\|p_n\|^2 + M\int_\Omega (\rho_{0,n}^4 y_n^2 + \rho_n^4 z_n^2)\, dx$$

and consequently,

$$-\frac{d}{dt}\left(e^{2Mt}\|p_n\|^2\right) \le 2Me^{2Mt}\int_\Omega (\rho_{0,n}^4 y_n^2 + \rho_n^4 z_n^2)\, dx.$$

Integrating the last inequality from 0 to t, with $t \in [0, 3T/4]$, we obtain

$$\|p_n(\cdot,0)\|^2 \le e^{2Mt}\|p_n(\cdot,t)\|^2 + 2Me^{3MT/2}\int_0^{3T/4}\int_\Omega (\rho_{0,n}^4 y_n^2 + \rho_n^4 z_n^2)\, dxdt. \tag{2.51}$$

From (2.51), we get that

$$\begin{aligned}
\|p_n(\cdot,0)\|^2 &= \frac{4}{3T}\int_0^{3T/4}\|p_n(\cdot,0)\|^2\,dx \\
&\le C\left(\int_0^{3T/4}\|p_n(\cdot,t)\|^2\,dx+\int_0^{3T/4}\int_\Omega(\rho_{0,n}^4y_n^2+\rho_n^4z_n^2)\,dxdt\right)\\
&\le C\iint_Q\left[\rho_0^{-2}\rho_{0,n}^4|y_n|^2+\rho^{-2}\rho_n^4|z_n|^2\right]\,dxdt+C\iint_{\mathcal{O}\times(0,T)}\rho_*^2|v_n|^2\,dxdt\\
&\quad+C\int_0^{3T/4}\int_\Omega(\rho_{0,n}^4y_n^2+\rho_n^4z_n^2)\,dxdt\\
&\le C\iint_Q\left[\rho_{0,n}^2|y_n|^2+\rho_n^2|z_n|^2+\rho_{*,n}^2|v_n|^2\right]\,dxdt\\
&\le CJ_n(y_n,z_n,v_n).
\end{aligned} \tag{2.52}$$

Then, from (2.49)–(2.50)

$$\|p_n(\cdot,0)\|^2+\iint_Q\rho^{-2}\theta^3(|p_n|^2+|q_n|^2)\,dxdt\le CJ_n(y_n,z_n,v_n). \tag{2.53}$$

From (2.48) and (2.53), we see that

$$J_n(y_n,z_n,v_n)\le C\left[\|y_0\|^2+\iint_Q\rho^2\theta^{-3}(|h|^2+|k|^2)\,dxdt\right].$$

Therefore, we get the estimates

$$\iint_Q\left(\rho_{0,n}^2|y_n|^2+\rho_n^2|z_n|^2\right)\,dxdt+\iint_{\mathcal{O}\times(0,T)}\rho_{*,n}^2|v_n|^2\,dxdt\le C,$$

whence we can extract suitable subsequences (again indexed by n) satisfying

$$\begin{aligned}
&\rho_{0,n}y_n\rightharpoonup\xi_1 \text{ and } \rho_nz_n\rightharpoonup\xi_2 \text{ in } L^2(Q),\\
&\rho_{*,n}v_n\rightharpoonup\chi \text{ in } L^2(Q).
\end{aligned} \tag{2.54}$$

From the definitions of ρ_n, $\rho_{0,n}$ and $\rho_{*,n}$ and (2.54), we have

$$\xi_1=\rho_0y,\quad \xi_2=\rho z \ \text{ and } \ \chi=\rho_*v1_{\mathcal{O}}.$$

Taking limits in the linear system (2.43), we deduce that

$$\begin{aligned}
\iint_Q\left(\rho_0^2|y|^2+\rho^2|z|^2\right)\,dxdt&\le\liminf\iint_Q\left(\rho_{0,n}^2|y_n|^2+\rho_n^2|z_n|^2\right)\,dxdt\le C\\
\iint_{\mathcal{O}\times(0,T)}\rho_*^2|v|^2\,dxdt&\le\liminf\iint_{\mathcal{O}\times(0,T)}\rho_{*,n}^2|v_n|^2\,dxdt\le C.
\end{aligned} \tag{2.55}$$

□

Similarly, we obtain the null controllability of (1.5), as a consequence of Proposition 2.6.

Proposition 2.8. *Assume that $b\neq 0$ and the functions h and k satisfy*

$$\iint_Q\rho^2\theta^{-3}(|h|^2+|k|^2)\,dxdt<+\infty.$$

Then (1.5) *is null-controllable at any time* $T>0$. *More precisely, for any* $y_0\in L^2(\Omega)$ *and any* $T>0$, *there exist controls* $w\in L^2(\mathcal{O}\times(0,T))$ *and associated states* (y,z) *satisfying*

$$\iint_{\mathcal{O}\times(0,T)}\rho_*^2|w|^2\,dxdt<+\infty,\quad \iint_Q(\rho_0^2|y|^2+\rho^2|z|^2)\,dxdt<+\infty, \tag{2.56}$$

whence, in particular,

$$y(x,T)=0\ \ in\ \ \Omega,\quad \limsup_{t\to T^-}\|z(\cdot\,,t)\|=0. \tag{2.57}$$

Proof. Analogous to Proposition 2.7. □

2.3 Some additional estimates

The state found in Proposition 2.7 satisfies some additional properties, that will be needed below, in Section 4. They have been first deduced in [15] and [16] in similar contexts. For clarity and completeness, their proofs will be recalled here.

Let us be more precise.

Proposition 2.9. *Let the hypotheses in Proposition* 2.7 *be satisfied and let* v *and* (y,z) *satisfy* (1.4) *and* (2.41). *Then one has*

$$\begin{aligned}&\iint_Q\widehat{\rho}^2(|\nabla y|^2+|\nabla z|^2)\,dxdt\\&\quad\le C\iint_Q(\rho_0^2|y|^2+\rho^2|z|^2)\,dxdt+C\iint_{\mathcal{O}\times(0,T)}\rho_*^2|v|^2\,dxdt\\&\qquad+C\|y_0\|_{L^2(\Omega)}+C\iint_Q\rho_0^2\left(|h|^2+|k|^2\right)\,dxdt\end{aligned} \tag{2.58}$$

Proof. Let us multiply the first PDE in (1.4) by $\widehat{\rho}^2y$ and the second one $\widehat{\rho}^2z$ and let us integrate in Ω. We obtain:

$$\begin{aligned}\int_\Omega\widehat{\rho}^2(y_t-\beta_1(0,0,\vec{0},\vec{0})\Delta y)y\,dx&=-\int_\Omega\widehat{\rho}^2(ay+bz-v1_{\mathcal{O}}-h)y\,dx,\\ \int_\Omega\widehat{\rho}^2(-\beta_2(0,0,\vec{0},\vec{0})\Delta z)z\,dx&=-\int_\Omega\widehat{\rho}^2(cy+dz-k)z\,dx.\end{aligned}$$

Notice that

$$\left|\int_\Omega\widehat{\rho}^2\left[(ay+bz)y+(cy+dz)z\right]\,dx\right|\le C\int_\Omega\widehat{\rho}^2\left(|y|^2+|z|^2\right),$$

$$\int_\Omega\widehat{\rho}^2y_ty\,dx=\frac{1}{2}\frac{d}{dt}\int_\Omega\widehat{\rho}^2|y|^2\,dx-C\int_\Omega\widehat{\rho}\widehat{\rho}_t|y|^2\,dx,$$

$$\int_\Omega\widehat{\rho}^2v1_{\mathcal{O}}y\,dx\le\frac{1}{2}\int_{\mathcal{O}}\rho_0^2|y|^2\,dx+\frac{1}{2}\int_{\mathcal{O}}\rho_*^2|v|^2\,dx,$$

$$\begin{aligned}&-\int_\Omega\widehat{\rho}^2\Big(\beta_1(0,0,\vec{0},\vec{0})(\Delta y)y+\beta_2(0,0,\vec{0},\vec{0})(\Delta z)z\Big)\,dx\\&\quad=\int_\Omega\widehat{\rho}^2\Big(\beta_1(0,0,\vec{0},\vec{0})|\nabla y|^2+\beta_2(0,0,\vec{0},\vec{0})|\nabla z|^2\Big)\,dx\\&\qquad-\frac{1}{2}\int_\Omega\Delta(\widehat{\rho}^2)\Big(\beta_1(0,0,\vec{0},\vec{0})|y|^2+\beta_2(0,0,\vec{0},\vec{0})|z|^2\Big)\,dx,\end{aligned} \tag{2.59}$$

and

$$\int_\Omega \widehat{\rho}^2(hy+kz)\,dx \;\le\; \frac{1}{2}\int_\Omega (\widehat{\rho}^4\rho_0^{-2})(|y|^2+|z|^2)\,dx + \frac{1}{2}\int_\Omega \rho_0^2(|h|^2+|k|^2)\,dx.$$

Therefore, from (2.47), the following is deduced:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \widehat{\rho}^2|y|^2\,dx + \int_\Omega \widehat{\rho}^2\Big(\beta_1(0,0,\vec{0},\vec{0})|\nabla y|^2 + \beta_2(0,0,\vec{0},\vec{0})|\nabla z|^2\Big)\,dx\\
&\quad\le C\int_\Omega \Big[\widehat{\rho}^2+\widehat{\rho}|\widehat{\rho}_t| + |\Delta(\widehat{\rho}^2)| + \widehat{\rho}^4\rho_0^{-2}\Big](|y|^2+|z|^2)\,dx \qquad (2.60)\\
&\qquad + \frac{1}{2}\int_{\mathcal{O}} \rho_0^2|y|^2\,dx + \frac{1}{2}\int_{\mathcal{O}} \rho_*^2|v|^2\,dx + \frac{1}{2}\int_{\mathcal{O}} \rho_0^2(|h|^2+|k|^2)\,dx.
\end{aligned}$$

From the definition of the weights ρ, ρ_0 and $\widehat{\rho}$, it is immediate that the function into brackets in the first integral in the right-hand side is bounded by $C\rho_0^2$. As a consequence, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \widehat{\rho}^2|y|^2\,dx + \int_\Omega \widehat{\rho}^2\Big(\beta_1(0,0,\vec{0},\vec{0})|\nabla y|^2 + \beta_2(0,0,\vec{0},\vec{0})|\nabla z|^2\Big)\,dx\\
&\quad\le C\int_\Omega (\rho_0^2|y|^2+\rho^2|z|^2)\,dx + \frac{1}{2}\int_{\mathcal{O}} \rho_*^2|v|^2\,dx + \frac{1}{2}\int_{\mathcal{O}} \rho_0^2(|h|^2+|k|^2)\,dx. \qquad (2.61)
\end{aligned}$$

Integrating the last estimate in time, we get the desired result. □

Proposition 2.10. *Let the hypotheses in Proposition 2.7 be satisfied and let v and (y,z) be the control and the associated state furnished by this result and let us assume that*

$$y_0 \in H_0^1(\Omega). \qquad (2.62)$$

Then one has,

$$\begin{aligned}
&\iint_Q \rho_*^2(|y_t|^2+|\Delta y|^2+|\Delta z|^2)\,dxdt + \sup_{t\in[0,T]}\int_\Omega \rho_*^2|\nabla y|^2\,dx\\
&\quad\le C\iint_Q (\rho_0^2|y|^2+\rho^2|z|^2)\,dxdt + C\iint_{\mathcal{O}\times(0,T)} \rho_*^2|v|^2\,dxdt \qquad (2.63)\\
&\qquad + C\|y_0\|_{H_0^1(\Omega)} + C\iint_Q \rho_0^2\left(|h|^2+|k|^2\right)\,dxdt.
\end{aligned}$$

Proof. Let us multiply only the first PDE in (1.4) by $\rho_*^2 y_t$ and let us integrate in Ω. The following holds:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \rho_*^2|y_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \rho_*^2\beta_1(0,0,\vec{0},\vec{0})|\nabla y|^2\,dx\\
&\quad\le C\int_\Omega |(\rho_*^2)_t||\nabla y|^2\,dx + C\int_\Omega \rho_*^2|y|^2\,dx \qquad (2.64)\\
&\qquad + C\int_{\mathcal{O}} \rho_*^2|v|^2\,dx + C\int_\Omega \rho_*^2|h|^2\,dx.
\end{aligned}$$

From the definition of the weight ρ_*, it is clear that $\rho_* \le c\widehat{\rho} \le C\rho_0 \le C\rho$ and it is easy to check that the function into parentheses in the first integral in the right-hand side is bounded by $C\widehat{\rho}^2$. Consequently, one has

$$\begin{aligned}\frac{1}{2}\int_\Omega \rho_*^2|y_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \rho_*^2\beta_1(0,0,\vec{0},\vec{0})|\nabla y|^2\,dx \\ \leq C\int_\Omega \widehat{\rho}^2|\nabla y|^2\,dx + C\int_\Omega (\rho_0^2|y|^2+\rho^2|z|^2)\,dx \\ + C\int_{\mathcal{O}} \rho_*^2|v|^2\,dx + C\int_{\mathcal{O}} \rho_0^2(|h|^2+|k|^2)\,dx.\end{aligned} \tag{2.65}$$

Integrating in time and recalling (2.62) and (2.58), we get the desired estimate for $|y_t|^2$.

In order to prove the same estimate for $|\Delta y|^2$ and $|\Delta z|^2$, let us multiply the first PDE in (1.4) by $-\rho_*^2\Delta y$ and the second one by $-\rho_*^2\Delta z$. After integration in Ω, we have,

$$\begin{aligned}\frac{1}{2}\int_\Omega \rho_*^2(\beta_1(0,0,\vec{0},\vec{0})|\Delta y|^2 + \beta_2(0,0,\vec{0},\vec{0})|\Delta z|^2)\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \rho_*^2|\nabla y|^2\,dx \\ \leq \frac{1}{2}\int_\Omega (\rho_*^2)_t|\nabla y|^2\,dx + C\int_\Omega \rho_*^2(|y|^2+|z|^2)\,dx \\ + C\int_{\mathcal{O}} \rho_*^2|v|^2\,dx + C\int_\Omega \rho_*^2(|h|^2+|k|^2)\,dx.\end{aligned} \tag{2.66}$$

From the definitions of $\widehat{\rho}$ and ρ_*, it is clear that the function between parentheses in the first integral in the right-hand side is bounded by $C\widehat{\rho}^2$. Consequently,

$$\begin{aligned}\frac{1}{2}\int_\Omega \rho_*^2(\beta_1(0,0,\vec{0},\vec{0})|\Delta y|^2 + \beta_2(0,0,\vec{0},\vec{0})|\Delta z|^2)\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \rho_*^2|\nabla y|^2\,dx \\ \leq C\int_\Omega \widehat{\rho}^2(|\nabla y|^2+|\nabla z|^2)\,dx + C\int_\Omega \rho_*^2(|y|^2+|z|^2)\,dx \\ + C\int_{\mathcal{O}} \rho_*^2|v|^2\,dx + C\int_\Omega \rho_*^2(|h|^2+|k|^2)\,dx.\end{aligned} \tag{2.67}$$

Integrating in time and recalling again (2.62) and (2.58) , we get the desired estimates for $|\Delta y|^2$ and $|\Delta z|^2$. □

We also have additional estimates for the state found in Proposition 2.8. Their proofs are similar to those of Propositions 2.9 and 2.10.

Proposition 2.11. *Let the hypotheses in Proposition 2.8 be satisfied and let w and (y,z) satisfy (1.5) and (2.56). Then one has*

$$\begin{aligned}\iint_Q \widehat{\rho}^2(|\nabla y|^2+|\nabla z|^2)\,dxdt \\ \leq C\iint_Q (\rho_0^2|y|^2+\rho^2|z|^2)\,dxdt + C\iint_{\mathcal{O}\times(0,T)} \rho_*^2|w|^2\,dxdt \\ + C||y_0||_{L^2(\Omega)} + C\iint_Q \rho_0^2\left(|h|^2+|k|^2\right)\,dxdt\end{aligned} \tag{2.68}$$

Proposition 2.12. *Let the hypotheses in Proposition 2.8 be satisfied and let w and (y,z) be the control and the associated state furnished by this result and let us assume that*

$$y_0 \in H_0^1(\Omega). \tag{2.69}$$

Then one has,

$$\begin{aligned}&\iint_Q \rho_*^2(|y_t|^2+|\Delta y|^2+|\Delta z|^2)\,dxdt+\sup_{t\in[0,T]}\int_\Omega \rho_*^2|\nabla y|^2\,dx\\&\leq C\iint_Q(\rho_0^2|y|^2+\rho^2|z|^2)\,dxdt+C\iint_{\mathcal{O}\times(0,T)}\rho_*^2|w|^2\,dxdt\\&+C\|y_0\|_{H_0^1(\Omega)}+C\iint_Q\rho_0^2\left(|h|^2+|k|^2\right)\,dxdt.\end{aligned}\tag{2.70}$$

3 The null controllability of the nonlinear systems (1.1) and (1.2)

In this Section, we present the proofs of the main results in this paper, namely Theorems 1.2 and 1.3.

3.1 Proof of Theorem 1.2

Let Y, G and Z be the functions spaces:

$$\begin{aligned}Y=\Big\{&(y,z,v):\ v\in L^2(\mathcal{O}\times(0,T)),\ \iint_{\mathcal{O}\times(0,T)}\rho_*^2|v|^2\,dxdt<+\infty\ ,\\&y,\ z,\ \partial_i y,\ \partial_i z,\ y_t-\beta_1(0,0,\vec{0},\vec{0})\Delta y,\ \beta_2(0,0,\vec{0},\vec{0})\Delta z\in L^2(Q),\\&\iint_Q(\rho_0^2|y|^2+\rho^2|z|^2)\,dxdt<+\infty,\\&\iint_Q\rho_0^2\Big[|y_t-\beta_1(0,0,\vec{0},\vec{0})\Delta y-v1_{\mathcal{O}}|^2+|\beta_2(0,0,\vec{0},\vec{0})\Delta z|^2\Big]\,dxdt<+\infty,\\&y(\cdot,0)\in H_0^1(\Omega)\Big\},\\G=\Big\{&g\in L^2(Q)\ \Big|\ \iint_Q\rho_0^2|g|^2\,dxdt<+\infty\Big\}\end{aligned}$$

and

$$Z=G\times G\times H_0^1(\Omega)$$

We introduce the Hilbertian norms:

$$\begin{aligned}\|(y,z,v)\|_Y^2:=&\iint_Q(\rho_0^2|y|^2+\rho^2|z|^2)\,dxdt+\iint_{\mathcal{O}\times(0,T)}\rho_*^2|v|^2\,dxdt\\&+\iint_Q\rho_0^2\Big[|y_t-\beta_1(0,0,\vec{0},\vec{0})\Delta y-v1_{\mathcal{O}}|^2+|\beta_2(0,0,\vec{0},\vec{0})\Delta z|^2\Big]\,dxdt\\&+\|y(\cdot,0)\|_{H_0^1(\Omega)},\\\|g\|_G^2=&\iint_Q\rho_0^2|g|^2\,dxdt\end{aligned}$$

and

$$\|(g_1,g_2,z_1)\|_Z^2:=\|g_1\|_G^2+\|g_2\|_G^2+\|z_1\|_{H_0^1(\Omega)}^2.$$

Let us consider the mapping $H:Y\to Z$ with

$$H(y,z,v)=(H_1,H_2,H_3)(y,z,v),$$

$$H_1(y,z,v) = y_t - \beta_1\left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx\right)\Delta y + F(y,z) - v1_{\mathcal{O}}, \tag{3.1}$$

$$H_2(y,z,v) = -\beta_2\left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx\right)\Delta z + f(y,z), \tag{3.2}$$

$$H_3(y,z,v) = y(\cdot,0). \tag{3.3}$$

We will prove that there exist $\epsilon > 0$ such that, if $(h,k,y_0) \in Z$ and $\|(h,k,y_0)\|_Z < \epsilon$, then the equation

$$H(y,z,v) = (h,k,y_0), \quad (y,z,v) \in Y$$

possesses at least one solution.

In particular, this shows that (1.1) is locally null controllable and, furthermore, the state-control triplets can be chosen in Y.

We will apply the following version of *Liusternik's Inverse Mapping Theorem* in infinite dimensional spaces, that can be found for instance in [1]. In the following statement, $B_r(0)$ and $B_\epsilon(\xi_0)$ are the open balls respectively of radius r and ϵ.

Theorem 3.1. *Let Y and Z be Banach spaces and let $H : B_r(0) \subset Y \to Z$ be a C^1 mapping. Let us assume that the derivative $H'(0) : Y \to Z$ is onto and let us set $\xi_0 = H(0)$. Then there exist a $\epsilon > 0$, a mapping $W : B_\epsilon(\xi_0) \subset Z \to Y$ and a constant $K > 0$ satisfying:*

$$\begin{cases} W(z) \in B_r(0) \text{ and } H(W(z)) = z, \ \forall z \in B_\epsilon(\xi_0), \\ \|W(z)\|_Y \le K\|z - H(0)\|_Z, \quad \forall z \in B_\epsilon(\xi_0). \end{cases}$$

Notice that in this theorem, W is the *inverse-to-the-right* of H.

To show that Theorem 3.1 can be applied in this setting, we will use several lemmas.

First, let us prove that the definition of H is correct.

Lemma 3.2. *Let $H : Y \to Z$ be the mapping defined by (3.1)–(3.3). Then H is well defined and continuous.*

Proof. For $(y,z,v) \in Y$, let us see that $H_i(y,z,v)$ makes sense and belongs to F, for $i = 1,2$, and, also, that $H_3(y,z,v)$ makes sense and belongs to $H_0^1(\Omega)$.

Since F is Lipschitz, for any $(y,z,v) \in Y$, we have:

$$\begin{aligned}
&\iint_Q \rho_0^2 |H_1(y,z,v)|^2\,dxdt \\
&\quad = \iint_Q \rho_0^2 \left| y_t - \beta_1\left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx\right)\Delta y + F(y,z) - v1_{\mathcal{O}}\right|^2 dxdt \\
&\quad \le C\iint_Q \rho_0^2 |y_t - \beta_1(0,0,\vec{0},\vec{0})\Delta y - v1_{\mathcal{O}}|^2\,dxdt \\
&\qquad + C\iint_Q \rho_0^2 \left|\beta_1\left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx\right) - \beta_1(0,0,\vec{0},\vec{0})\right|^2 |\Delta y|^2\,dxdt \\
&\qquad + C\iint_Q \rho_0^2 (|y|^2 + |z|^2)\,dxdt \\
&\quad = A_1 + A_2 + A_3.
\end{aligned} \tag{3.4}$$

From the definition of the space Y,

$$A_1 = C\iint_Q \rho_0^2 |y_t - \beta_1(0,0,\vec{0},\vec{0})\Delta y - v1_{\mathcal{O}}|^2\,dxdt \le C\|(y,z,v)\|_Y^2,$$

and
$$A_3 \leq C \iint_Q \rho_0^2|y|^2 + \rho^2|z|^2 \, dxdt \leq C\|(y,z,v)\|_Y^2.$$

Now, let us analyse A_2. Since β_1 is C^1 and globally Lipschitz continuous, one has:

$$\begin{aligned} A_2 &\leq C \iint_Q \rho_0^2 \left[\left(\int_\Omega y \, dx \right)^2 + \left(\int_\Omega z \, dx \right)^2 + \left(\Big| \int_\Omega \nabla y \, dx \Big|_{\mathbb{R}^n} \right)^2 + \left(\Big| \int_\Omega \nabla z \, dx \Big|_{\mathbb{R}^n} \right)^2 \right] |\Delta y|^2 \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{3.5}$$

From Proposition 2.10, we know that

$$\iint_Q \rho_*^2 |\Delta y|^2 \, dxdt < +\infty \quad \text{and} \quad \sup_{t\in[0,T]} \int_\Omega \rho_*^2 |\nabla y|^2 \, dx < +\infty,$$

where $\rho_* = \bar{c}_0 e^{(2s\widehat{A}-sA^*)/m} m^{7/2}$, $\bar{c}_0 = e^{-(7/2)\lambda\|\alpha_0\|_\infty}$.

Then, in order to prove that H_1 is well defined, we must have to obtain that $A_2 < +\infty$. In this case, from (3.5), we just have to demonstrate that $J_3 < +\infty$ (the others J_i are similar).

In fact,

$$\begin{aligned} J_3 &= \int_0^T\!\!\int_\Omega e^{2s\overline{A}/m} \left(\frac{e^{\lambda\alpha_0}}{m} \right)^{-3} \left(\Big| \int_\Omega \nabla y \, dx \Big|_{\mathbb{R}^n} \right)^2 |\Delta y|^2 \, dxdt \\ &\leq \int_0^T \left[e^{2sA^*/m} m^3 \left(\Big| \int_\Omega \nabla y \, dx \Big|_{\mathbb{R}^n} \right)^2 \int_\Omega |\Delta y|^2 \, dx \right] dt \\ &\leq m(\Omega) \int_0^T \left[e^{2sA^*/m} m^3 \left(\int_\Omega |\nabla y|^2 \, dx \right) \int_\Omega |\Delta y|^2 \, dx \right] dt \\ &\leq m(\Omega) \int_0^T \left[e^{2sA^*/m} m^3 e^{(-8s\widehat{A}+4sA^*)/m} m^{-14} \left(\int_\Omega \rho_*^2 |\nabla y|^2 \, dx \right) \left(\int_\Omega \rho_*^2 |\Delta y|^2 \, dx \right) \right] dt, \end{aligned}$$

where $m(\Omega)$ is the measure of the set Ω.

To achieve our goal, we have to prove that

$$I = e^{2sA^*/m} m^3 e^{(-8s\widehat{A}+4sA^*)/m} m^{-14} \left(\int_\Omega |\nabla y|^2 \, dx \right) < +\infty$$

and for this objective, we must have

$$e^{2sA^*/m} m^3 e^{(-8s\widehat{A}+4sA^*)/m} m^{-14} < +\infty,$$

that is,
$$e^{(6sA^*-8s\widehat{A})/m} m^{-12} \leq c,$$

which is true since $6sA^* - 8s\widehat{A} < 0$.

Then $H_1(y,z,v)$ is well defined.

That H_2 is well defined can be proved in a very similar way. That H_3 is also well defined is obvious.

Furthermore, that the three mappings H_i are continuous is very easy to prove using similar arguments. □

Lemma 3.3. *The mapping $H : Y \to Z$ is continuously differentiable.*

Proof. Let us first prove that H is G-differentiable at any $(y,z,v) \in Y$ and let us compute the G-derivative $H'(y,z,v)$.

Thus, let us fix $(y,z,v) \in Y$ and let us take $(y',z',v') \in Y$ and $\sigma > 0$. d For simplicity, we will use the notation

$$\begin{aligned}
&\beta_{j\sigma} := \beta_j \left(\int_\Omega (y+\sigma y')\,dx, \int_\Omega (z+\sigma z')\,dx, \int_\Omega (\nabla y + \sigma \nabla y')\,dx, \int_\Omega (\nabla z + \sigma \nabla z')\,dx \right),\ j=1,2\\
&\overline{\beta}_j := \beta_j \left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx \right), \quad j=1,2\\
&\beta_j^\eta := \beta_j \left(\int_\Omega y^\eta\,dx, \int_\Omega z^\eta\,dx, \int_\Omega \nabla y^\eta\,dx, \int_\Omega \nabla z^\eta\,dx \right), \quad j=1,2\\
&\overline{\beta}_{i,j} := D_i\beta_j \left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx \right), \quad j=1,2, \text{ and } i=1,2,\dots,2n+2,\\
&\beta_{i,j}^\eta := D_i\beta_j \left(\int_\Omega y^\eta\,dx, \int_\Omega z^\eta\,dx, \int_\Omega \nabla y^\eta\,dx, \int_\Omega \nabla z^\eta\,dx \right), \quad j=1,2, \text{ and } i=1,2,\dots,2n+2,\\
&F_\sigma := F(y+\sigma y', z+\sigma z'), \quad \overline{F} := F(y,z), \quad F^\eta := F(y^\eta, z^\eta),\\
&\overline{F}_j := D_jF(y,z), \quad F_j^\eta := D_jF(y^\eta,z^\eta), \quad j=1,2
\end{aligned}$$

and similar abridged symbols for f.

We have

$$\begin{aligned}
&\frac{1}{\sigma}[H_1((y,z,v)+\sigma(y',z',v')) - H_1(y,z,v)]\\
&\qquad = y'_t - \beta_{1\sigma}\Delta y' - \frac{1}{\sigma}[\beta_{1\sigma} - \overline{\beta_1}]\Delta y + \frac{1}{\sigma}[F_\sigma - \overline{F}] - v' 1_{\mathcal{O}}.
\end{aligned}$$

Also

$$\frac{1}{\sigma}[H_2((y,z,v)+\sigma(y',z',v')) - H_2(y,z,v)] == -\beta_{2\sigma}\Delta z' - \frac{1}{\sigma}[\beta_{2\sigma} - \overline{\beta_2}]\Delta z + \frac{1}{\sigma}[f_\sigma - \overline{f}].$$

Let us introduce the linear mapping $DH \in \mathcal{L}(Y,Z)$, with

$$DH = (DH_1, DH_2, DH_3), \tag{3.6}$$

$$\begin{aligned}
DH_1(y',z',v') = &\, y'_t - \overline{\beta_1}\Delta y' - \Delta y \left(\overline{\beta}_{1,1}\int_\Omega y'\,dx + \overline{\beta}_{2,1}\int_\Omega z'\,dx + \gamma_{1,1}\cdot\int_\Omega \nabla y'\,dx + \gamma_{2,1}\cdot\int_\Omega \nabla z'\,dx \right)\\
&+ \overline{F}_1 y' + \overline{F}_2 z' - v' 1_{\mathcal{O}}, \qquad (3.7)\\
DH_2(y',z',v') = &- \overline{\beta_2}\Delta z' - \Delta z \left(\overline{\beta}_{1,2}\int_\Omega y'\,dx + \overline{\beta}_{2,2}\int_\Omega z'\,dx + \gamma_{1,2}\cdot\int_\Omega \nabla y'\,dx + \gamma_{2,2}\cdot\int_\Omega \nabla z'\,dx \right)\\
&+ \overline{f}_1 y' + \overline{f}_2 z', \qquad (3.8)\\
DH_3(y',z',v') = &\, y'(\cdot,0), \qquad (3.9)
\end{aligned}$$

where $\gamma_{1,j} = (\overline{\beta}_{3,j}, \dots, \overline{\beta}_{n+2,j}) \in \mathbb{R}^n$, $j=1,2$ and $\gamma_{2,j} = (\overline{\beta}_{n+3,j}, \dots, \overline{\beta}_{2n+2,j}) \in \mathbb{R}^n$, $j=1,2$.

For all $(y',z',v') \in Y$, one has

$$\frac{1}{\sigma}[H_1((y,v)+\sigma(y',v')) - H_1(y,v)] \to DH_1(y',v') \quad \text{strongly in } G \tag{3.10}$$

as $\sigma \to 0$.

Indeed, we have:

$$
\begin{aligned}
&\left\| \frac{1}{\sigma}(H_1((y,z,v)+\sigma(y',z',v')) - H_1(y,z,v)) - DH_1(y',z',v') \right\|_G \\
&\quad \le \left\| (\beta_{1\sigma} - \overline{\beta_1})\Delta y' \right\|_G \\
&\qquad + \left\| \left[\frac{1}{\sigma}[\beta_{1\sigma} - \overline{\beta_1}] - \left(\overline{\beta}_{1,1} \int_\Omega y'\,dx + \overline{\beta}_{2,1} \int_\Omega z'\,dx + \gamma_{1,1}\cdot \int_\Omega \nabla y'\,dx + \gamma_{2,1}\cdot \int_\Omega \nabla z'\,dx \right) \right] \Delta y \right\|_G \\
&\qquad + \left\| \frac{1}{\sigma}[F_\sigma - \overline{F}] - (\overline{F}_1 y' + \overline{F}_2 z') \right\|_G \\
&\quad = B_1 + B_2 + B_3.
\end{aligned}
$$

Arguing as in the proof of (3.5) and using Proposition 2.10, we obtain the following result, as a consequence of Lebesgue's Theorem:

$$
B_1^2 = \iint_Q \rho_0^2 \left(\beta_{1\sigma} - \overline{\beta_1}\right)^2 |\Delta y'|^2\,dxdt \to 0 \quad \text{as } \sigma \to 0.
$$

Once again, arguing as in the proof of (3.5) and using Proposition 2.10, one has from Lebesgue's Theorem that

$$
\begin{aligned}
B_2^2 &= \iint_Q \rho_0^2 \left[\frac{1}{\sigma}[\beta_{1\sigma} - \overline{\beta_1}] \right. \\
&\qquad \left. - \left(\overline{\beta}_{1,1} \int_\Omega y'\,dx + \overline{\beta}_{2,1} \int_\Omega z'\,dx + \gamma_{1,1}\cdot \int_\Omega \nabla y'\,dx + \gamma_{2,1}\cdot \int_\Omega \nabla z'\,dx \right) \right]^2 |\Delta y|^2\,dxdt \\
&= \iint_Q \rho_0^2 \left[(D_1\beta_1^* - \overline{\beta}_{1,1}) \int_\Omega y'\,dx + (D_2\beta_1^* - \overline{\beta}_{2,1}) \int_\Omega z'\,dx \right]^2 |\Delta y|^2\,dxdt \\
&\quad + \iint_Q \rho_0^2 \left[(\overline{D}_3\beta_1^* - \gamma_{1,1})\cdot \int_\Omega \nabla y'\,dx + (\overline{D}_4\beta_1^* - \gamma_{2,1})\cdot \int_\Omega \nabla z'\,dx \right]^2 |\Delta y|^2\,dxdt, \\
&\to 0,
\end{aligned}
$$

as $\sigma \to 0$, where the $D_i\beta_1^*$ are the partial derivatives of β_1 at some intermediate points, in particular $D_1\beta_1^*, D_2\beta_1^* \in \mathbb{R}$ and $\overline{D}_3\beta_1^*, \overline{D}_4\beta_1^* \in \mathbb{R}^n$.

For B_3, the argument is very similar. Indeed, we have

$$
\begin{aligned}
B_3^2 &= \iint_Q \rho_0^2 \left[\frac{1}{\sigma}[F_\sigma - \overline{F}] - (\overline{F}_1 y' + \overline{F}_2 z') \right]^2 dxdt \\
&= \iint_Q \rho_0^2 \left[(D_1F^* - \overline{F}_1)y' + (D_2F^* - \overline{F}_2)z' \right]^2 dxdt \\
&= \iint_Q \rho_0^2 \left[|D_1F^* - \overline{F}_1|^2 |y'|^2 + |D_2F^* - \overline{F}_2|^2 |z'|^2 \right] dxdt,
\end{aligned}
$$

where the D_iF^* also stand for the partial derivatives of F at some intermediate points. As $F \in C_b^1(\mathbb{R}\times\mathbb{R})$, then, arguing as the proof of (3.5) and using Proposition 2.10 and Lebesgue's Theorem, once more we also find that $B_3 \to 0$.

Taking into account the behaviour of B_1, B_2 and B_3, we deduce that (3.10) is true.

In a similar way, it can be shown that

$$
\frac{1}{\sigma}[H_2((y,z,v)+\sigma(y',z',v')) - H_2(y,z,v)] \to DH_2(y',z',v') \quad \text{strongly in } G.
$$

Consequently

$$\lim_{\sigma\to 0}\frac{1}{\sigma}(H((y,z,v)+\sigma(y',z',v'))-H(y,z,v)) = DH(y',z',v') \quad \text{strongly in } G,$$

whence we have that H is *Gâteaux* differentiable at any $(y,z,v)\in Y$ with a *Gâteaux* derivative given by DH.

As usual, let us denote by $H'(y,z,v)$ the linear mapping defined by (3.6)–(3.9). Now, we shall prove that the mapping $(y,z,v)\to H'(y,z,v)$ is continuous from Y to $\mathcal{L}(Y,Z)$. In other words, we will show that , whenever $(y^\eta,z^\eta,v^\eta)\to(y,z,v)$ in Y, one has

$$\|(DH(y^\eta,z^\eta,v^\eta)-DH(y,z,v))(y',z',v')\|_Z \le \epsilon_\eta\|(y',z',v')\|_Y \quad \text{for some } \epsilon_\eta\to 0. \tag{3.11}$$

Then, we have just to prove that

$$\|(DH_1(y^\eta,z^\eta,v^\eta)-DH_1(y,z,v))(y',z',v')\|_G \le \epsilon_\eta\|(y',z',v')\|_Y \quad \text{for some } \epsilon_\eta\to 0. \tag{3.12}$$

In effect,

$$\begin{aligned}
&\|(DH_1(y^\eta,z^\eta,v^\eta)-DH_1(y,z,v))(y',z',v')\|_G\\
&\le C\iint_Q \rho_0^2\left[\left(\beta_{1,1}^\eta\int_\Omega y'\,dx\right)\Delta y^\eta-\left(\overline{\beta}_{1,1}\int_\Omega y'\,dx\right)\Delta y\right]^2 dxdt\\
&\quad + C\iint_Q \rho_0^2\left[\left(\beta_{2,1}^\eta\int_\Omega z'\,dx\right)\Delta y^\eta-\left(\overline{\beta}_{2,1}\int_\Omega z'\,dx\right)\Delta y\right]^2 dxdt\\
&\quad + C\iint_Q \rho_0^2\left[\left(\gamma_{1,1}^\eta\cdot\int_\Omega \nabla y'\,dx\right)\Delta y^\eta-\left(\gamma_{1,1}\cdot\int_\Omega \nabla y'\,dx\right)\Delta y\right]^2 dxdt\\
&\quad + C\iint_Q \rho_0^2\left[\left(\gamma_{2,1}^\eta\cdot\int_\Omega \nabla z'\,dx\right)\Delta y^\eta-\left(\gamma_{2,1}\cdot\int_\Omega \nabla z'\,dx\right)\Delta y\right]^2 dxdt\\
&\quad + C\iint_Q \rho_0^2|\beta_1^\eta-\overline{\beta}_1|^2|\Delta y'|^2\,dxdt\\
&\quad + C\iint_Q \rho_0^2|F_1^\eta-\overline{F}_1|^2|y'|^2\,dxdt + C\iint_Q \rho_0^2|F_2^\eta-\overline{F}_2|^2|z'|^2\,dxdt\\
&= E_1+E_2+E_3+E_4+E_5+E_6+E_7,
\end{aligned}$$

where $\gamma_{1,j}^\eta=(\beta_{3,j}^\eta,\ldots,\beta_{n+2,j}^\eta)\in\mathbb{R}^n$, $j=1,2$ and $\gamma_{2,j}^\eta=(\beta_{n+3,j}^\eta,\ldots,\beta_{2n+2,j}^\eta)\in\mathbb{R}^n$, $j=1,2$.

Now, we will check that each E_i can be bounded as in (3.11). For instance, we have

$$\begin{aligned}
E_1 &= C\iint_Q \rho_0^2|\beta_{11}^\eta-\overline{\beta}_{11}|^2\left(\int_\Omega y'\,dx\right)^2|\Delta y|^2\,dxdt\\
&\quad + C\iint_Q \rho_0^2|\beta_{11}^\eta|^2\left(\int_\Omega y'\,dx\right)^2|\Delta y^l-\Delta y|^2\,dxdt.
\end{aligned}$$

The first and second integrals in the right-hand side can be bounded as follows:

$$\begin{aligned}
&\iint_Q \rho_0^2|\beta_{11}^\eta-\overline{\beta}_{11}|^2\left(\int_\Omega y'\,dx\right)^2|\Delta y|^2\,dxdt\\
&\qquad \le C\left(\iint_Q \rho_*^2|\Delta y|^2|\beta_{11}^\eta-\overline{\beta}_{11}|^2\,dxdt\right)\|(y',z',v')\|_Y^2,
\end{aligned}$$

$$\iint_Q \rho_0^2 |\beta_{11}^\eta|^2 \left(\int_\Omega y' \, dx \right)^2 |\Delta y^\eta - \Delta y|^2 \, dxdt$$
$$\leq C \left(\iint_Q \rho_*^2 |\Delta y^\eta - \Delta y|^2 \, dxdt \right) \|(y', z', v')\|_Y^2.$$

Taking into account the adopted procedure in (3.5) and using Proposition 2.10, consequently, using Lebesgue's Theorem together with the fact that $\beta_1 \in C_b^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ give us the desired estimate for E_1.

Similarly, we obtain the same conclusion for the other E_i. This shows that (3.11) is satisfied and ends the proof. □

Lemma 3.4. *Let H be the mapping defined by (3.1)–(3.3). Then $H'(0,0,0) \in \mathcal{L}(Y, Z)$ is onto.*

Proof. First notice that

$$H'(0,0,0)(y', z', v') = (K_1, K_2, K_3),$$

where

$$\begin{aligned} K_1(y', z', v') &= y'_t - \beta_1(0,0,\vec{0},\vec{0})\Delta y' + A_1 y' + A_2 z' - v' 1_\omega, \\ K_2(y', z', v') &= -\beta_2(0,0,\vec{0},\vec{0})\Delta z' + B_1 y' + B_2 z', \\ K_3(y', z', v') &= y'(\cdot, 0) \end{aligned}$$

for all $(y', z', v') \in Y$. Here the coefficients A_i and B_i are given by

$$A_i = D_i F(0,0) \quad \text{and} \quad B_i = D_i f(0,0) \quad \text{for} \quad i = 1,\ 2.$$

Consequently $H'(0,0,0)$ is onto if and only if for each $(h, k, y_0) \in Z$, there exists $(y, z, v) \in Y$ satisfying

$$\begin{cases} y_t - \beta_1(0,0,\vec{0},\vec{0})\Delta y + A_1 y + A_2 z = v 1_{\mathcal{O}} + h, & \text{in } Q, \\ -\beta_2(0,0,\vec{0},\vec{0})\Delta z' + B_1 y' + B_2 z' = k, & \text{in } Q, \\ y = z = 0, & \text{on } \Sigma, \\ y(x,0) = y_0(x), & \text{in } \Omega \end{cases}$$

By hypothesis, $B_1 \neq 0$. Hence the existence of (y, z, v) with these properties is ensured by Proposition 2.7. This shows that $H'(0,0,0)$ is surjective and this ends the proof. □

Thus, the proof of Theorem 1.2 is a consequence of Lemmas 3.2, 3.3 and 3.4.

3.2 Proof of Theorem 1.3

Here, we have similar results and proofs to Subsection 3.1 to establish the local null controllability of (1.3) .

In this case, let $\widetilde{Y}$, G and Z be the following functions spaces:

$$\begin{aligned}\widetilde{Y} = \Big\{&(y,z,w):\ w \in L^2(\mathcal{O}\times(0,T)),\ \iint_{\mathcal{O}\times(0,T)} \rho_*^2|w|^2\,dxdt < +\infty,\\ &y,\ z,\ \partial_i y,\ \partial_i z,\ y_t - \beta_1(0,0,\vec{0},\vec{0})\Delta y,\ \beta_2(0,0,\vec{0},\vec{0})\Delta z \in L^2(Q),\\ &\iint_Q (\rho_0^2|y|^2 + \rho^2|z|^2)\,dxdt < +\infty,\\ &\iint_Q \rho_0^2\Big[|y_t - \beta_1(0,0,\vec{0},\vec{0})\Delta y|^2 + |-\beta_2(0,0,\vec{0},\vec{0})\Delta z - w1_{\mathcal{O}}|^2\Big]\,dxdt < +\infty,\\ &y(\cdot,0) \in H_0^1(\Omega)\Big\},\end{aligned}$$

$$G = \Big\{ g \in L^2(Q) \Big|\ \iint_Q \rho_0^2|g|^2\,dxdt < +\infty \Big\}$$

and

$$Z = G \times G \times H_0^1(\Omega).$$

We introduce the Hilbertian norms:

$$\begin{aligned}\|(y,z,w)\|_{\widetilde{Y}}^2 := &\iint_Q (\rho_0^2|y|^2 + \rho^2|z|^2)\,dxdt + \iint_{\mathcal{O}\times(0,T)} \rho_*^2|w|^2\,dxdt\\ &+ \iint_Q \rho_0^2\Big[|y_t - \beta_1(0,0,\vec{0},\vec{0})\Delta y|^2 + |-\beta_2(0,0,\vec{0},\vec{0})\Delta z - w1_{\mathcal{O}}|^2\Big]\,dxdt\\ &+ \|y(\cdot,0)\|_{H_0^1(\Omega)},\\ \|g\|_G^2 = &\iint_Q \rho_0^2|g|^2\,dxdt\end{aligned}$$

and

$$\|(g_1,g_2,z_1)\|_Z^2 := \|g_1\|_G^2 + \|g_2\|_G^2 + \|z_1\|_{H_0^1(\Omega)}^2.$$

Let us consider the mapping $\widetilde{H}: \widetilde{Y} \to Z$ with

$$\widetilde{H}(y,z,w) = (\widetilde{H}_1, \widetilde{H}_2, \widetilde{H}_3)(y,z,w),$$

$$\widetilde{H}_1(y,z,w) = y_t - \beta_1\left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx\right)\Delta y + F(y,z), \tag{3.13}$$

$$\widetilde{H}_2(y,z,w) = -\beta_2\left(\int_\Omega y\,dx, \int_\Omega z\,dx, \int_\Omega \nabla y\,dx, \int_\Omega \nabla z\,dx\right)\Delta z + f(y,z) - w1_{\mathcal{O}}, \tag{3.14}$$

$$\widetilde{H}_3(y,z,w) = y(\cdot,0). \tag{3.15}$$

Once again, we will apply *Liusternik's Inverse Mapping Theorem* in infinite dimensional spaces to the mapping $\widetilde{H}$, given by (3.13)–(3.15), to show that (1.2) is locally null controllable, where the state-control triplets can be chosen in $\widetilde{Y}$.

For this end, we will use the following lemmas, which their proofs are similar than Subsection 3.1.

Lemma 3.5. *Let $\widetilde{H}: \widetilde{Y} \to Z$ be the mapping defined by* (3.13)–(3.15)*. Then $\widetilde{H}$ is well defined and continuous.*

Lemma 3.6. *The mapping $\widetilde{H}: \widetilde{Y} \to Z$ is continuously differentiable.*

Lemma 3.7. *Let $\widetilde{H}$ be the mapping defined by* (3.13)–(3.15)*. Then $\widetilde{H}'(0,0,0) \in \mathcal{L}(\widetilde{Y}, Z)$ is onto.*

In view of Lemmas 3.5, 3.6 and 3.7, we can apply Liusternik's Theorem to the mapping $\widetilde{H}: \widetilde{Y} \to Z$ and (1.2) is locally null controllable, with $(y,z,w) \in \widetilde{Y}$.

4 Additional comments and open questions

As a first comment, an interesting question is concerned with global null controllability to (1.1) and (1.2), which does not seem to be simple. Perhaps, this kind of result relies on a global inverse mapping theorem, see [7], but much more refined estimates are necessary.

Other important topics arise from our current research:

- In the system (1.1) and (1.2), we can replace the local nonlinearities $F(y,z)$ and $f(y,z)$ by $F(y,z,\nabla y,\nabla z)$ and $f(y,z,\nabla y,\nabla z)$, in order to analyze whether it is possible to prove results about null controllability.

- When $F(y,z)$ and $f(y,z)$ are weakly superlinear nonlinearities, that is,

$$\begin{cases} \lim_{\|(s,p)\|\to+\infty} \dfrac{\left|\int_0^1 \frac{\partial F}{\partial s}(\lambda s,\lambda p)\,d\lambda\right| + \left|\int_0^1 \frac{\partial F}{\partial p}(\lambda s,\lambda p)\,d\lambda\right|}{\ln^{3/2}(1+|s|+|p|)} = 0, \\ \lim_{\|(s,p)\|\to+\infty} \dfrac{\left|\int_0^1 \frac{\partial f}{\partial s}(\lambda s,\lambda p)\,d\lambda\right| + \left|\int_0^1 \frac{\partial f}{\partial p}(\lambda s,\lambda p)\,d\lambda\right|}{\ln^{3/2}(1+|s|+|p|)} = 0, \\ F(0,0) = f(0,0) = 0, \end{cases}$$

  then we deduce that $D_iF(0,0) = D_if(0,0) = 0$, for $i = 1,2$. Then, the linearized system given by $H'(0,0,0)$ studied in Propositions 2.7 and 2.8 has no coupling, that is, $c = 0$ and $b = 0$, respectively. Thus, it is not possible to solve (1.4) and (1.5) with only one control. What would be possible when $F(y,z)$ and $f(y,z)$ are weakly superlinear nonlinearities, and it is an open problem, is to obtain the local exact controllability to the trajectories at time T for the problems (1.1) and (1.2).

- Open questions concerning the exact controllability to the trajectories:

  It is said that (1.1) (resp. (1.2)) is locally exactly controllable to the trajectories at time T if, for any solution $(\widehat{y},\widehat{z})$ corresponding to the control $\widehat{v}$ (resp. $\widehat{w}$), there exists $\epsilon > 0$ such that, if

$$\|y_0 - \widehat{y}(\cdot,0)\|_{H_0^1(\Omega)} \le \epsilon,$$

  there exists controls $v \in L^2(\mathcal{O}\times(0,T))$ (resp. $w \in L^2(\mathcal{O}\times(0,T))$) such that the associated states (y,z) satisfy

$$y(x,T) = \widehat{y}(x,T), \quad \limsup_{t\to T^-} \|z(\cdot,t)\| = \limsup_{t\to T^-} \|\widehat{z}(\cdot,t)\| \text{ in } \Omega. \tag{4.1}$$

  The analysis of this property for (1.1) and (1.2) and other similar systems will be the objective of a forthcoming paper.

- It would be very nice to obtain some local null boundary controllability results for the systems (1.1) and (1.2), that is, instead of applying a distributed control in the interior of the domain Ω, one could consider the question of solving the controllability problems with the control acting on a portion γ of the boundary $\Gamma := \partial\Omega$ of the domain. However, these facts can not be directly deduced for systems with a reduced number of controls, see [2]. In other words, the boundary controllability of

$$\begin{cases} y_t - \beta_1 \left(\int_\Omega y\, dx, \int_\Omega z\, dx, \int_\Omega \nabla y\, dx, \int_\Omega \nabla z\, dx \right) \Delta y + F(y,z) = 0 & \text{in } Q, \\ -\beta_2 \left(\int_\Omega y\, dx, \int_\Omega z\, dx, \int_\Omega \nabla y\, dx, \int_\Omega \nabla z\, dx \right) \Delta z + f(y,z) = 0 & \text{in } Q, \\ y(x,t) = v 1_\gamma,\ z(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega \end{cases} \tag{4.2}$$

and

$$\begin{cases} y_t - \beta_1 \left(\int_\Omega y\, dx, \int_\Omega z\, dx, \int_\Omega \nabla y\, dx, \int_\Omega \nabla z\, dx \right) \Delta y + F(y,z) = 0 & \text{in } Q, \\ -\beta_2 \left(\int_\Omega y\, dx, \int_\Omega z\, dx, \int_\Omega \nabla y\, dx, \int_\Omega \nabla z\, dx \right) \Delta z + f(y,z) = 0 & \text{in } Q, \\ y(x,t) = 0,\ z(x,t) = w 1_\gamma & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega \end{cases} \tag{4.3}$$

are very interesting unknown issues.

- The controllability of hyperbolic-elliptic systems of (1.1) and (1.2) is also an open problem. Notice that the linearized systems (1.4) and (1.5), for the hyperbolic-elliptic case, can be solved with boundary controls following the works of Lasiecka–Miara [20] and Miara–Münch [23]. The greatest difficulty is to extend the works [20] and [23] to the nonlinear case, because they are not solved in spaces with weights, which prevents us from using Liusternik's Theorem. The application of fixed point theorems would be an interesting problem to study.

References

[1] V. M. Alekseev, V. M. Tikhomorov, S. V. Formin, *Optimal control*, Consultant Bureau, New York, 1987. https://doi.org/10.1007/978-1-4615-7551-1; MR924574.

[2] F. Ammar-Khodja, A. Benabdallah, M. González Burgos, L. de Teresa, Recent results on the controllability of linear coupled parabolic problems: a survey, *Math. Control Relat. Fields* **1**(2011), No. 3, 267–306. https://doi.org/10.3934/mcrf.2011.1.267; MR2846087

[3] N. H. Chang, M. Chipot, On some model diffusion problems with a nonlocal lower order term, *Chin. Ann. Math. Ser. B* **24**(2003), 147–166. https://doi.org/10.1142/S0252959903000153; MR1982061

[4] M. Chipot, V. Valente, G. Caffarelli, Remarks on a nonlocal problem involving the Dirichlet energy, *Rend. Sem. Mat. Univ. Padova* **110**(2003), 199–220. MR2033009

[5] H. R. Clark, E. Fernández-Cara, J. Límaco, L. A. Medeiros, Theoretical and numerical local null controllability for a parabolic system with local and nonlocal nonlinearities, *Appl. Math. Comput.* **223**(2013), 483–505. https://doi.org/10.1016/j.amc.2013.08.035; MR3116282

[6] J.-M. Coron, *Control and nonlinearity*, Mathematical Surveys and Monographs, Vol. 136. American Mathematical Society, Providence, RI, 2007. MR2302744

[7] G. De Marco, G. Gorni G. Zampieri, Global inversion of functions: an introduction, *NoDEA Nonlinear Differential Equations Appl.* **1**(1994), No. 3, 229–248. https://doi.org/10.1007/BF01197748; MR1289855

[8] A. Doubova, E. Fernández-Cara, M. González-Burgos, E. Zuazua, On the controllability of parabolic systems with a nonlinear term involving the state and the gradient, *SIAM J. Control Optim.* **41**(2002), No. 3, 798–819. https://doi.org/10.1137/S0363012901386465; MR1939871

[9] C. Fabre, J.-P. Puel, E. Zuazua, Approximate controllability of the semilinear heat equation, *Proc. Royal Soc. Edinburgh Sect. A* **125**(1995), 31–61. https://doi.org/10.1017/S0308210500030742; MR1318622

[10] E. Fernández-Cara, S. Guerrero, Global Carleman inequalities for parabolic systems and applications to controllability, *SIAM J. Control Optim.* **45**(2006), No. 4, 1395–1446. https://doi.org/10.1137/S0363012904439696; MR2257228

[11] E. Fernández-Cara, J. Límaco, S. B. Menezes, Null controllability for a parabolic-elliptic coupled system, *Bull. Braz. Math. Soc. (N.S.)* **44**(2013), No. 2, 1–24. https://doi.org/10.1007/s00574-013-0014-x; MR3077645

[12] E. Fernández-Cara, E. Zuazua, Null and approximate controllability of weakly blowing-up semilinear heat equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **17**(2000), No. 5, 583–616. https://doi.org/10.1016/S0294-1449(00)00117-7; MR1791879

[13] A. Fursikov, O. Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series, Vol. 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996. MR1406566

[14] O. Yu. Imanuvilov, Controllability of parabolic equations (in Russian) *Mat. Sb.* **186**(1995), 109–132. https://doi.org/10.1070/SM1995v186n06ABEH000047; MR1349016

[15] O. Yu. Imanuvilov, Remarks on exact controllability for the Navier–Stokes equations, *ESAIM Control Optim. Calc. Var.* **6**(2001), 39–72. https://doi.org/10.1051/cocv:2001103; MR1804497

[16] O. Yu. Imanuvilov, M. Yamamoto, Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations, *Publ. Res. Inst. Math. Sci.* **39**(2003), No. 2, 227–274. MR1987865

[17] A. Jüngel, *Transport equations for semiconductors*, Lecture Notes in Physics, Vol. 773, Springer, Berlin, 2009. https://doi.org/10.1007/978-3-540-89526-8; MR2723324

[18] I. Lasiecka, R. Triggiani, *Control theory for partial differential equations*, Encyclopedia of Mathematics and Its Applications, Vol. 1, Cambridge University Press, 2000. MR1745475

[19] I. Lasiecka, R. Triggiani, *Control theory for partial differential equations*, Encyclopedia of Mathematics and Its Applications, Vol. 2, Cambridge University Press, 2000. MR1745476

[20] I. Lasiecka, B. Miara, Exact controllability of a 3D piezoelectric body, *C. R. Math. Acad. Sci. Paris* **347**(2009), No. 3–4, 167–172. https://doi.org/10.1016/j.crma.2008.12.007; MR2538106

[21] J-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non-linéaires* (in French), Dunod, Paris, 1960. MR0259693

[22] L. A. Medeiros, J. Límaco, S. B. Menezes, Vibrations of elastic strings: mathematical aspects. I., *J. Comput. Anal. Appl.* **4**(2002), No. 2, 91–127. https://doi.org/10.1023/A:1012934900316; MR1875347

[23] B. Miara, A. Münch, Exact controllability of a piezoelectric body. Theory and Numerical Simulation, *Appl. Math. Optim.* **59**(2009), No. 3, 383–412. https://doi.org/10.1007/s00245-008-9059-4; MR2491704

[24] E. Zuazua, Exact boundary controllability for the semilinear wave equation, in: *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. X (Paris, 1987–1988)*, Pitman Res. Notes Math. Ser., Vol. 220, Longman Sci. Tech., Harlow, 1989, pp. 357–391. MR1131832

[25] E. Zuazua, Controllability and observability of partial differential equations: some results and open problems, in: *Handbook of differential equations: evolutionary equations*, Vol. III, 527–621, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2007, pp. 527–621. https://doi.org/10.1016/S1874-5717(07)80010-7; MR2549374