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Weak upper semicontinuity of pullback attractors for nonautonomous reaction-diffusion equations

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Abstract. We consider nonautonomous reaction-diffusion equations with variable exponents and large diffusion and we prove continuity of the flow and weak upper semicontinuity of a family of pullback attractors when the exponents go to 2 in $L^\infty(\Omega)$.

Keywords: ODE limit problems, nonautonomous reaction-diffusion equations, parabolic problems, variable exponents, pullback attractors, upper semicontinuity.

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1 Introduction

In this work, we will study the following problem

$$\begin{cases} \frac{\partial u_s}{\partial t}(t) - \operatorname{div}(D_s|\nabla u_s(t)|^{p_s(x)-2}\nabla u_s(t)) + C(t)|u_s(t)|^{p_s(x)-2}u_s(t) = B(t,u_s(t)), \quad t > \tau, \\ u_s(\tau) = u_{\tau s}, \end{cases} \tag{1.1}$$

under homogeneous Neumann boundary conditions, $u_{\tau s} \in H := L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a smooth bounded domain, $D_s \in [1,\infty)$, $p_s(\cdot) \in C(\bar{\Omega})$, $p_s^- := \min_{x\in\bar{\Omega}} p_s(x) > 2$, and there exists a constant $a > 2$ such that $p_s^+ := \max_{x\in\bar{\Omega}} p_s(x) \leq a$, for all $s \in \mathbb{N}$. We assume that $D_s \to \infty$ and $p_s(\cdot) \to 2$ in $L^\infty(\Omega)$ as $s \to \infty$. The terms B and C are assumed to satisfy:

Assumption B The mapping $B : \mathbb{R} \times H \to H$ is such that

(B1) there exists $L \geq 0$ such that

$$\|B(t,y_1) - B(t,y_2)\|_H \leq L\|y_1 - y_2\|_H,$$

for all $t \in \mathbb{R}$ and $y_1, y_2 \in H$.

(B2) for all $y \in H$ the mapping $t \mapsto B(t,y)$ belongs to $L^2(\tau,T;H)$.

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(B3) the function $t \mapsto \|B(t,0)\|_H$ is nondecreasing, absolutely continuous and bounded on compact subsets of $\mathbb{R}$.

Assumption C $C(\cdot) \in L^\infty([\tau,T];\mathbb{R}^+)$ is monotonically nonincreasing in time and it is bounded from above and below, let us consider $0 < \alpha \le C(t) \le M$, $\forall\, t \in \mathbb{R}$, for some positive constants α and M. The constants α and M are taken uniform on τ and T.

The aim of this work is to study the asymptotic behavior of the solutions as $s \to \infty$. We prove continuity of the flow and weak upper semicontinuity of the family of pullback attractors as s goes to infinity for the problem (1.1) with respect to the couple of parameters (D_s, p_s), where p_s is the variable exponent and D_s is the diffusion coefficient.

It is a well-known fact that reaction-diffusion systems are used for many models of chemical, biological and ecological problems. When variable exponents are included these models often appear in applications in electrorheological fluids [8, 9, 20–22] and image processing [6, 11].

Reaction-diffusion systems for which the flow is essentially determined by an ordinary differential equation have been studied by many researchers and they often appear as shadow systems. Large diffusion phenomena have application in chemical fluid flows, see for example [19]. Recently an application was given to describing algal blooms [17]. Semilinear reaction-diffusion equations for large diffusion have been considered in many works, see for example the following works and the references therein [1, 3, 4, 7, 12–14, 32]. Moreover, quasilinear reaction-diffusion equations with large diffusion have been considered in many works for p-Laplacian problems, see for example [2, 24, 25, 28] and the references therein.

The study of the continuity with respect to initial conditions and parameters is important to verify the stability of a PDE model. In [23, 26–28] the authors investigated in which way the exponent parameter $p(x)$ affects the dynamic of PDEs involving the $p(x)$-Laplacian. In [23, 26, 27] the limit problem was also a PDE and in [28] the limit problem was an ODE.

In [10] the authors considered the following nonautonomous equation

$$\begin{cases} \frac{\partial u_s}{\partial t}(t) - \operatorname{div}(D_s|\nabla u_s|^{p_s(x)-2}\nabla u_s) + C(t)|u_s|^{p_s(x)-2}u_s = B(u_s(t)), \quad t > \tau, \\ u_s(\tau) = u_{\tau s}, \end{cases} \tag{1.2}$$

under homogeneous Neumann boundary conditions, $u_{\tau s} \in H := L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ $(n \ge 1)$ is a smooth bounded domain, $B : H \to H$ is a globally Lipschitz map with Lipschitz constant $L \ge 0$, $D_s \in [1,\infty)$, $C(\cdot) \in L^\infty([\tau,T];\mathbb{R}^+)$ is bounded from above and below and is monotonically nonincreasing in time, $p_s(\cdot) \in C(\bar{\Omega})$, $p_s^- := \min_{x\in\bar{\Omega}} p_s(x) \ge p$, $p_s^+ := \max_{x\in\bar{\Omega}} p_s(x) \le a$, for all $s \in \mathbb{N}$, when $p_s(\cdot) \to p$ in $L^\infty(\Omega)$ and $D_s \to \infty$ as $s \to \infty$, with $a, p > 2$ positive constants. They proved continuity of the flows and upper semicontinuity of the family of pullback attractors.

In this paper we will give one step more and reach the linear case, i.e., $p_s(\cdot) \to 2$ in $L^\infty(\Omega)$ as $s \to \infty$. A revision of the paper [10] shows that, with the assumptions given on B, the difference of the explicit dependence on time on the reaction term $B(t, u_s(t))$ is unimportant in order to obtain all the results included in that work, in particular, the existence of solution and pullback attractors. It is worth to mention that external forcing terms satisfying **Assumption B** were already considered in the works [16, 30]. Problem (1.1) has a strong solution u_s, i.e., $u_s \in C([\tau,T];H)$ is absolutely continuous in any compact subinterval of (τ,T), $u_s(t) \in \mathcal{D}(A^s(t))$ for a.e. $t \in (\tau,T)$, and

$$\frac{du_s}{dt}(t) + A^s(t)(u_s(t)) = B(t, u_s(t)) \quad \text{for a.e. } t \in (\tau,T),$$

where $A^s(t)(u_s) := -\mathrm{div}(D_s|\nabla u_s|^{p_s(x)-2}\nabla u_s) + C(t)|u_s|^{p_s(x)-2}u_s$ and problem (1.1) has a pullback attractor $\mathcal{U}_s = \{\mathcal{A}_s(t)\}_{t\in\mathbb{R}}$ (see [10]). We will use the technique developed in [29] for autonomous problems and make the *mutatis mutandis* over it to deal with the nonautonomous problem in order to prove a weak upper semicontinuity of the family of pullback attractors $\{\mathcal{U}_s\}_{s\in\mathbb{N}}$ as s goes to infinity for the problem (1.1), weak in the sense that we control the gaps between two consecutive exponent functions for a given δ_0 (see condition $(H2)$ in Section 3) in order to obtain, for each $\ell \in \mathbb{R}$, $\mathcal{A}_s(\ell) \subset O_{\delta_0}(\mathcal{A}_\infty(\ell))$, for s large enough, where $\mathcal{U}_\infty = \{\mathcal{A}_\infty(t)\}_{t\in\mathbb{R}}$ will be the pullback attractor of the limit problem.

Considering $p_s(\cdot) \to 2$ in $L^\infty(\Omega)$ and large diffusion, a fast redistribution process of the solution occurs having homogenization, any spatial variation of the solution is reduced to zero; i.e.; the only relevant parameter at the limit of the dynamics of the problem becomes the time. In other words, the limit problem will be the nonautonomous ODE (2.2). For this reason we will consider a family (in p) of ODE's reaching the same limit problem (2.2) when p goes to 2. So, we will consider the following hypothesis

$$\begin{cases} \text{There exists } \epsilon_0 > 0 \text{ such that if } p_s \in F_{\epsilon_0}(2) := \{g; \|g-2\|_{L^\infty(\Omega)} \leq \epsilon_0\}, \\ \text{then } p_s \text{ is a constant function.} \end{cases} \tag{H}$$

The paper is organized as follows. In Section 2 we prove a uniform estimate for the solutions of nonlinear ODEs and we prove continuity of the solutions with respect to initial conditions and exponent parameters. In Section 3 we prove that the solutions $\{u_s\}$ of the PDE (1.1) converge for $s \to \infty$ to the solution u of the limit problem (2.2) which is an ODE, and, after that, we obtain a weak upper semicontinuity of the pullback attractors for the problem (1.1).

2 The family of nonautonomous ODEs and its limit problem

Now consider the following family (in p) of ODEs

$$\begin{cases} \dot{u}_p(t) + C(t)|u_p(t)|^{p-2}u_p(t) = f(t, u_p(t)), \quad t > \tau, \\ u_p(\tau) = u_{\tau p} \in \mathbb{R}, \end{cases} \tag{2.1}$$

with $p \in (2, 3]$ a constant and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

(i) $|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$, for all $t \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}$.

(ii) for all $x \in \mathbb{R}$ the mapping $t \mapsto f(t, x)$ belongs to $L^2(\tau, T; \mathbb{R})$.

(iii) the function $t \mapsto |f(t, 0)|$ is nondecreasing, absolutely continuous and bounded on compact subsets of $\mathbb{R}$.

With the assumptions given on f, the explicit dependence on time on the reaction term is unimportant in order to obtain existence of solution and pullback attractors. With the same arguments as in Sections 5 and 6 in [10] problem (2.1) has a unique strong solution u_p and has a pullback attractor $\mathcal{V}_p = \{\mathcal{M}_p(t)\}_{t\in\mathbb{R}}$. Nonautonomous ODEs had appeared in chemotherapy models, see [15].

Now, we intend to study the sensitivity of problem (2.1) when the constant exponent p goes to 2. We guess and will prove that the limit problem is

$$\begin{cases} \dot{u}(t) + C(t)u(t) = f(t, u(t)), \quad t > \tau, \\ u(\tau) = u_\tau \in \mathbb{R}. \end{cases} \tag{2.2}$$

It is straightforward to check the abstract conditions in [18] for our problem (2.2) in order to obtain the existence of a classical unique global solution u for (2.2). Moreover, given $T > \tau$ and $u_\tau \in \mathbb{R}$, there exists a constant $K_\infty = K_\infty(u_\tau, T) > 0$ such that $|u(t)| \leq K_\infty$, for all $t \in [\tau, T]$.

In the next result we prove the continuity of the solutions of (2.1) with respect to the initial data and exponent parameter.

Theorem 2.1. *Let u_p be a solution of (2.1) with $u_p(\tau) = u_{\tau p}$ and let u be the solution of (2.2) with $u(\tau) = u_\tau$. If $u_{\tau p} \to u_\tau$ in $\mathbb{R}$ as $p \to 2$, then for each $T > \tau$, $u_p \to u$ in $C([\tau, T]; \mathbb{R})$ as $p \to 2$.*

Proof. Let $T > \tau$ be fixed and suppose that $u_{\tau p} \to u_\tau$ in $\mathbb{R}$ as $p \to 2$. Subtracting the two equations in (2.1) and (2.2) and making the product with $u_p - u$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_p(t) - u(t)|^2 &+ C(t)[|u_p(t)|^{p-2}u_p(t) - u(t)][u_p(t) - u(t)] \\ &= [f(t, u_p(t)) - f(t, u(t))][u_p(t) - u(t)].\end{aligned}$$

Adding $\pm C(t)|u(t)|^{p-2}u(t)$, using that f is Lipschitz with respect to the second variable and that for any $\xi, \eta \in \mathbb{R}^n$,

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq 0, \quad M \geq C(t) \geq \alpha \quad \forall\, t \in [\tau, T]$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_p(t) - u(t)|^2 &\leq L|u_p(t) - u(t)|^2 - C(t)(|u(t)|^{p-2} - 1)u(t)(u_p(t) - u(t)) \\ &\leq L|u_p(t) - u(t)|^2 + M\left||u(t)|^{p-1} - |u(t)|\right| |u_p(t) - u(t)|,\end{aligned}$$

for all $t \in (0, T)$.

Now, let us estimate the term

$$\left||u(t)|^{p-1} - |u(t)|\right| |u_p(t) - u(t)|.$$

By the Mean Value Theorem, for each $p > 2$ there is a $q \in (2, p)$ such that

$$\left||u(t)|^{p-1} - |u(t)|\right| = \left||u(t)|^{q-1} \ln |u(t)|\right| |p - 2|$$

provided that $u(t) \neq 0$. Consider the continuous function $g_\theta : [0, K_\infty] \to \mathbb{R}$ given by

$$g_\theta(w) = \begin{cases} w^\theta \ln w & \text{if } w \in (0, K_\infty] \\ 0 & \text{if } w = 0, \end{cases}$$

where $\theta \geq 1$ is a given number. Using this continuous function defined in the compact set $[0, K_\infty]$ with $\theta = 1$ when $|u(t)| < 1$ and with $\theta = 2$ when $|u(t)| \geq 1$, there exists a positive constant R such that

$$\left||u(t)|^{q-1} \ln |u(t)|\right| \leq R,$$

for all $t \in [\tau, T]$ with $u(t) \neq 0$. So,

$$\left||u(t)|^{p-1} - |u(t)|\right| \leq R|p - 2|,$$

for all $t \in [\tau, T]$. Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_p(t)-u(t)|^2 &\le L|u_p(t)-u(t)|^2 + MR|p-2||u_p(t)-u(t)| \\ &\le L|u_p(t)-u(t)|^2 + \frac{1}{2}[MR|p-2|]^2 + \frac{1}{2}|u_p(t)-u(t)|^2,\end{aligned}$$

for all $t \in (\tau, T)$.

Integrating from τ to t, $t \le T$, we obtain

$$|u_p(t)-u(t)|^2 \le |u_{\tau p} - u_\tau|^2 + [MR|p-2|]^2(T-\tau) + \int_\tau^t (2L+1)|u_p(\tau)-u(\tau)|^2 d\tau.$$

So, by Gronwall-Bellman's Lemma we obtain

$$|u_p(t)-u(t)|^2 \le \left[|u_{\tau p}-u_\tau|^2 + (MR|p-2|)^2(T-\tau)\right] e^{(2L+1)(T-\tau)},$$

for all $t \in [\tau, T]$. Therefore, $u_p \to u$ in $C([\tau, T]; \mathbb{R})$ as $p \to 2$. □

If we restrict the initial conditions to a bounded set $\mathcal{M} \subset \mathbb{R}$ in problem (2.1) and consider $L < \alpha$ then we can obtain the following uniform estimates of the solutions of problem (2.1).

Proposition 2.2. *Consider f with Lipschitz constant $L < \alpha$, where α is from **Assumption C**. Let $\mathcal{M}$ be a bounded set and u_p be a solution of* (2.1) *with $u_p(\tau) = u_{\tau p} \in \mathcal{M}$. There exists a positive number r_0 such that $|u_p(t)| \le r_0$, for each $t \ge \tau$ and for all $p \in (2, 3]$.*

Proof. Let $t > \tau$. Multiplying the equation on (2.1) by $u_p(t)$ we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}|u_p(t)|^2 &\le -C(t)|u_p(t)|^p + |f(t,u_p(t))||u_p(t)| \\ &\le -\alpha|u_p(t)|^p + |f(t,u_p(t)) - f(t,0)||u_p(t)| + |f(t,0)||u_p(t)|.\end{aligned}$$

So,

$$\frac{1}{2}\frac{d}{d\tau}|u_p(t)|^2 \le -\alpha|u_p(t)|^p + L|u_p(t)|^2 + C_0|u_p(t)|, \tag{2.3}$$

where $C_0 := \sup_{t\in[\tau,T]} |f(t,0)| \ge 0$.

If $|u_p(t)| > 1$, $-|u_p(t)|^p \le -|u_p(t)|^2$, then from (2.3)

$$\frac{1}{2}\frac{d}{d\tau}|u_p(t)|^2 \le (L-\alpha)|u_p(t)|^2 + C_0|u_p(t)|.$$

Consider $\epsilon > 0$ arbitrary. Using Young's inequality we obtain

$$\frac{1}{2}\frac{d}{d\tau}|u_p(t)|^2 \le \left(-\alpha + L + \frac{1}{2}\epsilon^2\right)|u_p(t)|^2 + \frac{1}{2}\left(\frac{C_0}{\epsilon}\right)^2.$$

Now, choosing $\epsilon = \epsilon_1 > 0$ sufficiently small such that $0 < \epsilon_1 < (\alpha - L)^{1/2}$ we obtain

$$\frac{1}{2}\frac{d}{d\tau}|u_p(t)|^2 \le -\beta|u_p(t)|^2 + C_1,$$

where $\beta := \frac{\alpha}{2} - \frac{L}{2} > 0$ and $C_1 := \frac{1}{2}\left(\frac{C_0}{\epsilon_1}\right)^2$. Then

$$\frac{d}{d\tau}[|u_p(t)|^2]e^{2\beta t} + 2\beta|u_p(t)|^2 e^{2\beta t} \le 2C_1 e^{2\beta t}. \tag{2.4}$$

If $|u_p(t)| \leq 1$, then from (2.3),

$$\frac{d}{d\tau}|u_p(t)|^2 \leq 2(L + C_0) =: C_2.$$

Thus,

$$\frac{d}{d\tau}[|u_p(t)|^2]e^{2\beta\tau} + 2\beta|u_p(t)|^2e^{2\beta t} \leq C_2e^{2\beta t} + 2\beta|u_p(t)|^2e^{2\beta t} \leq (C_2 + 2\beta)e^{2\beta t}. \tag{2.5}$$

Considering $y_p(t) := |u_p(t)|^2$ and $C_3 := \max\{2C_1, C_2 + 2\beta\}$, we obtain from (2.4) and (2.5) that

$$\frac{d}{d\tau}[y_p(t)e^{2\beta t}] \leq C_3e^{2\beta t}, \quad \text{for all } t > \tau.$$

Integrating from τ to ℓ, we have

$$y_p(\ell)e^{2\beta\ell} \leq y_p(\tau)e^{2\beta\tau} + \frac{C_3}{2\beta}e^{2\beta\ell} - \frac{C_3}{2\beta}e^{2\beta\tau} \leq |u_{\tau p}|^2e^{2\beta\tau} + \frac{C_3}{2\beta}e^{2\beta\ell}.$$

Multiplying by $e^{-2\beta\ell}$, we obtain

$$|u_p(\ell)|^2 = y_p(\ell) \leq |u_{\tau p}|^2e^{-2\beta(\ell-\tau)} + \frac{C_3}{2\beta}e^0 \leq |u_{\tau p}|^2e^0 + \frac{C_3}{2\beta}, \quad \text{for all } \ell \geq \tau.$$

Since $u_{\tau p} \in \mathcal{M}$ and $\mathcal{M}$ is bounded, there exists $K \geq 0$ such that $|u_{\tau p}| \leq K$ for all $p \in (2, 3]$. Thus,

$$|u_p(\ell)| \leq r_0 := \left(K^2 + \frac{C_3}{2\beta}\right)^{1/2},$$

for all $\ell \geq \tau$ and $p \in (2, 3]$. □

3 Continuity of the flow and weak upper semicontinuity of attractors

Our objective in this section is to prove that the limit problem of problem (1.1) as D_s increases to infinity and $p_s(\cdot) \to 2$ in $L^\infty(\Omega)$ as $s \to \infty$ is described by the ordinary differential equation in (2.2).

The next result guarantees that (2.2) is in fact the limit problem for (1.1), as $s \to \infty$. The proof is analogous to the proof of Theorem 5.3 in [10].

Theorem 3.1. *Let u_s be a solution of* (1.1) *with $u_s(\tau) = u_{\tau s}$ and let u be the solution of* (2.2) *with $f = B_{|\mathbb{R}\times\mathbb{R}}$ and $u(\tau) = u_\tau$. If $u_{\tau s} \to u_\tau$ in H as $s \to \infty$, then for each $T > \tau$, $u_s \to u$ in $C([\tau, T]; H)$ as $s \to +\infty$.*

Let us now review some concepts and results on processes.

Definition 3.2. An evolution process in a metric space X is a family $\{S(t, \tau) : X \to X\}_{t \geq \tau}$ of continuous maps satisfying:

(i) $S(\tau, \tau) = I$ (here I denotes the identity operator);

(ii) $S(t, \tau) = S(t, s)S(s, \tau), \tau \leq s \leq t$.

Definition 3.3. Let $\{S(t,\tau)\}_{t\geq\tau}$ be an evolution process in a metric space X. Given A and B subsets of X, we say that A pullback attracts B at time t if

$$\lim_{\tau\to-\infty} \operatorname{dist}\left(S(t,\tau)B, A\right) = 0,$$

where dist denote the Hausdorff semi-distance.

Definition 3.4. A family of subsets $\{A(t) : t \in \mathbb{R}\}$ of X is invariant relatively to the evolution process $\{S(t,\tau)\}_{t\geq\tau}$ if $S(t,\tau)A(\tau) = A(t)$ for any $t \geq \tau$.

Definition 3.5. A family of subsets $\{A(t) : t \in \mathbb{R}\}$ of X is a pullback attractor for the evolution process $\{S(t,\tau)\}_{t\geq\tau}$ if it is invariant, $A(t)$ is compact for each $t \in \mathbb{R}$, pullback attracts all bounded subsets of X at time t for each $t \in \mathbb{R}$ and it is the minimal among all closed families which pullback attracts bounded sets of X.

Definition 3.6. A process $S(\cdot,\cdot)$ in a metric space X is said to be pullback asymptotically compact if, for each $t \in \mathbb{R}$, each sequence $\{s_k\} \leq t$ with $s_k \to -\infty$ as $k \to \infty$, and each bounded sequence $\{x_k\}$ in X, the sequence $\{S(t,s_k)x_k\}$ has a convergent subsequence.

Definition 3.7. We say that a process $S(\cdot,\cdot)$ is pullback bounded dissipative if there exists a family $B(\cdot)$ of bounded sets such that $B(t)$ pullback attracts bounded sets at time t, for each $t \in \mathbb{R}$.

Definition 3.8. We say that a process $S(\cdot,\cdot)$ is strongly pullback bounded dissipative if for each $t \in \mathbb{R}$ there is a bounded subset $B(t)$ of X that pullback attracts bounded subsets of X at time τ for each $\tau \leq t$; i.e., given a bounded subset D of X and $\tau \leq t$, $\lim_{s\to-\infty} \operatorname{dist}(S(\tau,s)D, B(t)) = 0$.

Theorem 3.9 ([5, Theorem 2.23]). *If a process $S(\cdot,\cdot)$ is strongly pullback bounded dissipative and pullback asymptotically compact, then $S(\cdot,\cdot)$ has a compact pullback attractor.*

Theorem 3.10. *Consider f with Lipschitz constant $L < \alpha$, where α is from **Assumption C**. The problem (2.2) defines a pullback asymptotically compact process.*

Proof. Let $t > \tau$. We define $S(t,\tau) : \mathbb{R} \to \mathbb{R}$ by $S(t,\tau)u_\tau = u(t)$ with u being the unique global solution of the problem (2.2) with $u(\tau) = u_\tau$. It is easy to see that $\{S(t,\tau) : \mathbb{R} \to \mathbb{R}, t \geq \tau\}$ verifies the process properties. Multiplying the equation in (2.2) by $u(t)$ and proceeding in a completely analogous way as in the proof of Proposition 2.2 we obtain that there exists a positive number r_0 such that $|u(t)| \leq r_0$, for each $t \geq \tau$(with the same constant r_0). Thus, we conclude that for each $t > \tau$, $S(t,\tau)$ maps bounded sets into bounded sets and the result follows. □

Observe that the process defined by the problem (2.2) is not necessarily pullback bounded dissipative. If we consider the very simple example, $C(t) \equiv 1$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(t,u) := \eta u$ with $\eta > 1$ a real number and so the solution of (2.2) is $u(t) = u_\tau e^{(\eta-1)(t-\tau)}$ and $|u(t)| \to \infty$ as $\tau \to -\infty$. In this case a pullback attractor for the problem (2.2) does not exist. There are examples that provide situations where the process defined by the limit problem (2.2) is pullback bounded dissipative. If $C(t) \equiv 1$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(t,u) := \eta u$ with $\eta < 1$ a real number then the solution of (2.2) is $u(t) = u_\tau e^{(\eta-1)(t-\tau)}$ and $u(t) \to 0$ as $\tau \to -\infty$. So, the process defined by the limit problem (2.2) is strongly pullback bounded dissipative.

Now, we suppose that $f = B_{|\mathbb{R}\times\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is such that the limit problem (2.2) has a strongly pullback bounded dissipative process. So, let $\mathcal{U}_\infty = \{\mathcal{A}_\infty(t)\}_{t\in\mathbb{R}}$ be the pullback attractor for (2.2) with $f = B_{|\mathbb{R}\times\mathbb{R}}$.

We need to use the following

Theorem 3.11 ([10]). *Let $\mathcal{U}_s = \{\mathcal{A}_s(t)\}_{t\in\mathbb{R}}$ be the pullback attractor associated with problem (1.1) and $\mathcal{V}_p = \{\mathcal{M}_p(t)\}_{t\in\mathbb{R}}$ the pullback attractor for problem (2.1) with $f = B_{|\mathbb{R}\times\mathbb{R}}$. Then, for each $\ell \in \mathbb{R}$, we have $\text{dist}(\mathcal{A}_s(\ell);\mathcal{M}_p(\ell)) \to 0$ in the topology of H, when $p_s(\cdot) \to p > 2$ in $L^\infty(\Omega)$.*

Let us consider $\ell \in \mathbb{R}$ arbitrarily fixed. The condition (H) is needed in the proof of the weak upper semicontinuity of the family of pullback attractors for problem (1.1) as $p_s \to 2$ in $L^\infty(\Omega)$. Moreover, after the functions $p_s(\cdot)$ enter into $F_{\epsilon_0}(2)$, given $\delta_0 > 0$, in order to show $\mathcal{A}_s(\tau) \subset O_{\delta_0}(\mathcal{A}_\infty(\tau))$ for $s > 0$ large enough, we have to control the gap between two consecutive functions p_s and p_{s+1} by an appropriate term which depends on s and δ_0 (see hypothesis (H2) below).

Consider $p := 2 + \epsilon_0$, where $\epsilon_0 > 0$ is from hypothesis (H). Then there exists $s_1 \in \mathbb{N}$ large enough such that $2 < p_{s_1} < p$ and $2 < p_s \le p_{s_1}$ is constant for all $s \ge s_1$. Thus, let us call, $\{p_s\}_{s\ge s_1}$ simply $\{p_j\}_{j\ge 1}$, where $p_j := p_{s_j}$.

Theorem 3.12. *There exists a compact set K_{s_1} in $\mathbb{R}$ such that $\mathcal{M}_{s_1}(t) \subset K_{s_1}$, $\forall\, t \in \mathbb{R}$.*

Proof. Multiplying the equation $\dot{u}_{p_{s_1}}(t) + C(t)|u_{p_{s_1}}(t)|^{p_{s_1}-2}u_{p_{s_1}}(t) = f(t, u_{p_{s_1}}(t))$ by $u_{p_{s_1}}(t)$ and using the Young's Inequality we obtain

$$\frac{1}{2}\frac{d}{dt}|u_{p_{s_1}}(t)|^2 \le -\frac{\alpha}{2}|u_{p_{s_1}}(t)|^{p_{s_1}} + c, \quad t \ge \tau$$

where $c > 0$ is a constant. Therefore, the map $y_{s_1}(t) := |u_{p_{s_1}}(t)|^2$ satisfies the inequality

$$\frac{d}{dt}y_{s_1}(t) \le -\alpha(y_{s_1}(t))^{p_{s_1}/2} + 2c, \quad t \ge \tau.$$

So, by Lemma 5.1 in [31],

$$|u_{p_{s_1}}(t)|^2 \le \left(\frac{2c}{\alpha}\right)^{2/p_{s_1}} + \left(\alpha\left(\frac{p_{s_1}}{2} - 1\right)(t-\tau)\right)^{-\frac{2}{p_{s_1}-2}}, \quad \forall\, t \ge \tau.$$

Let $\xi_0 > 0$ such that $\left(\alpha\left(\frac{p_{s_1}}{2} - 1\right)\xi_0\right)^{-\frac{2}{p_{s_1}-2}} \le 1$, then

$$|u_{p_{s_1}}(t)| \le \left[\left(\frac{2c}{\alpha}\right)^{2/p_{s_1}} + 1\right]^{1/2} =: \kappa_{s_1}, \quad \forall\, t \ge \xi_0 + \tau. \tag{3.1}$$

Thus, consider the compact set in $\mathbb{R}$ defined by $K_{s_1} := \overline{B(0, \kappa_{s_1})}$.

Consider now $t \in \mathbb{R}$ arbitrarily fixed and choose τ such that $t - \tau > \xi_0$. By the invariance of the pullback attractor $\mathcal{V}_{s_1}$, we have $S_{s_1}(t,\tau)\mathcal{M}_{s_1}(\tau) = \mathcal{M}_{s_1}(t)$. So, given an arbitrary element $w \in \mathcal{M}_{s_1}(t)$ we have that $w = S_{s_1}(t,\tau)u_\tau$ with $u_\tau \in \mathcal{M}_{s_1}(\tau)$. Since that κ_{s_1} and ξ_0 did not depend on the initial data we have by (3.1) that $w \in K_{s_1}$. Therefore, $\mathcal{M}_{s_1}(t) \subset K_{s_1}$. □

Consider from now on $L < \alpha$ where α is from **Assumption C** and the constant $r_0 = r_0(M)$ in Proposition 2.2 for $M = K_{s_1}$, where K_{s_1} is from Theorem 3.12. The set K_{s_1} is compact, in particular bounded, so given $\ell \in \mathbb{R}$ and δ_0 there exists $t_0 = t_0(\ell, \delta_0, K_{s_1}) < \ell$ such that

$$\text{dist}_{\mathbb{R}}(S(\ell, t_0)K_{s_1}; \mathcal{A}_\infty(\ell)) < \frac{\delta_0}{4|\Omega|^{1/2}}, \tag{3.2}$$

where $S(\ell, t_0)u_{t_0} := u(\ell, u_{t_0})$ is the solution of (2.2) and $\text{dist}_{\mathbb{R}}(S(\ell, t_0)K_{s_1}; \mathcal{A}_\infty(\ell))$ is the Hausdorff semi-distance between $S(\ell, t_0)K_{s_1}$ and $\mathcal{A}_\infty(\ell)$ in $\mathbb{R}$. Let $\psi_0 \in K_{s_1}$ be arbitrarily fixed.

Let $\{S^j(t,\tau)\}$ be the process defined by problem (2.1) with the exponent parameter p_j and consider $u_j(t) := S^j(t,t_0)\psi_0$.

Let us first prove the following three technical lemmas and then we present our main result.

Lemma 3.13. *Given $\ell \in \mathbb{R}$ and $t_0 \leq \ell$ as in (3.2), there exists a positive constant κ such that*

$$\left| |u_{j+1}(t)|^{p_j-1} - |u_{j+1}(t)|^{p_{j+1}-1} \right| \leq \kappa |p_j - p_{j+1}|,$$

for all $j \in \mathbb{N}$ and $t \geq t_0$.

Proof. By the Mean Value Theorem we conclude that

$$\left| |u_{j+1}(t)|^{p_j-1} - |u_{j+1}(t)|^{p_{j+1}-1} \right| = \left| |u_{j+1}(t)|^{\theta_j} \ln |u_{j+1}(t)| \right| |p_j - p_{j+1}|,$$

for some $\theta_j \in (p_{j+1}, p_j)$. Consider the continuous function $g_\theta : [0, r_0] \to \mathbb{R}$ given by

$$g_\theta(x) = \begin{cases} x^\theta \ln x & \text{if } x \in (0, r_0] \\ 0 & \text{if } x = 0, \end{cases}$$

where $r_0 = r_0(K_{s_1})$ is as in Proposition 2.2 and $\theta \geq 1$ is a given number. Using this continuous function defined in the compact set $[0, r_0]$ with $\theta = 2$ when $|u_{j+1}(t)| < 1$ and with $\theta = 2 + \epsilon_0$ when $|u_{j+1}(t)| \geq 1$, there exists a positive constant κ such that

$$\left| |u_{j+1}(t)|^{\theta} \ln |u_{j+1}(t)| \right| \leq \kappa,$$

for all $j \in \mathbb{N}$ and $t \geq t_0$ and the result follows. □

Now, we can establish the following hypothesis

$(H2)$ Given $\ell \in \mathbb{R}$ and $t_0 \leq \ell$ as in (3.2), for each $j \in \mathbb{N}$,

$$|p_j - p_{j+1}| < \left[\frac{\delta_0^2}{5^{2j} M \kappa^2 |\Omega| e^{(2L+1)(\ell - t_0)} (\ell - t_0)} \right]^{1/2}.$$

Lemma 3.14. *Given $\ell \in \mathbb{R}$, consider $t_0 = t_0(\ell) \leq \ell$ as in (3.2). If condition $(H2)$ is fulfilled for a given $\delta_0 > 0$, then*

$$\mathrm{dist}_{\mathbb{R}}(S^j(\ell, t_0)K_{s_1}; S^{j+1}(\ell, t_0)K_{s_1}) \leq \frac{\delta_0}{5^j |\Omega|^{1/2}},$$

for all $j \in \mathbb{N}$.

Proof. Consider $t_0 \leq \ell < T$. Subtracting the two equations in (2.1) and multiplying by $u_j(t) - u_{j+1}(t)$, $t \in [t_0, T]$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_j(t) - u_{j+1}(t)|^2 + C(t)[|u_j(t)|^{p_j-2}u_j(t) - |u_{j+1}(t)|^{p_{j+1}-2}u_{j+1}(t)][u_j(t) - u_{j+1}(t)] \\ &\quad = [f(t, u_j(t)) - f(t, u_{j+1}(t))][u_j(t) - u_{j+1}(t)]. \end{aligned}$$

Adding $\pm C(t)|u_{j+1}(t)|^{p_j-2}u_{j+1}(t)$, using that f is Lipschitz with respect to the second variable and that for any $\xi,\eta\in\mathbb{R}^n$, $(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)\geq 0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_j(t)-u_{j+1}(t)|^2 &\leq L|u_j(t)-u_{j+1}(t)|^2\\ &\quad - C(t)\left[|u_{j+1}(t)|^{p_j-2}-|u_{j+1}(t)|^{p_{j+1}-2}\right]u_{j+1}(t)\left(u_j(t)-u_{j+1}(t)\right)\\ &\leq L|u_j(t)-u_{j+1}(t)|^2+M\left||u_{j+1}(t)|^{p_j-1}-|u_{j+1}(t)|^{p_{j+1}-1}\right||u_j(t)-u_{j+1}(t)|\\ &\leq \left(L+\frac{1}{2}\right)|u_j(t)-u_{j+1}(t)|^2+\frac{M}{2}\left||u_{j+1}(t)|^{p_j-1}-|u_{j+1}(t)|^{p_{j+1}-1}\right|^2,\end{aligned}$$

for all $t\in[t_0,T]$. Using Lemma 3.13 we obtain

$$\frac{d}{dt}|u_j(t)-u_{j+1}(t)|^2\leq(2L+1)\,|u_j(t)-u_{j+1}(t)|^2+M\kappa^2|p_j-p_{j+1}|^2,$$

for all $t\in[t_0,T]$. From condition $(H2)$,

$$|p_j-p_{j+1}|^2<\frac{\delta_0^2}{5^{2j}M\kappa^2|\Omega|e^{(2L+1)(\ell-t_0)}(\ell-t_0)}.$$

Then,

$$\frac{d}{dt}|u_j(t)-u_{j+1}(t)|^2\leq(2L+1)\,|u_j(t)-u_{j+1}(t)|^2+M\kappa^2\frac{\delta_0^2}{5^{2j}M\kappa^2|\Omega|e^{(2L+1)(\ell-t_0)}(\ell-t_0)},$$

for all $t\in[t_0,T]$. Integrating from t_0 to ℓ and using that $u_j(t_0)=u_{j+1}(t_0)=\psi_0$, we obtain

$$|u_j(\ell)-u_{j+1}(\ell)|^2\leq\frac{\delta_0^2}{5^{2j}|\Omega|e^{(2L+1)(\ell-t_0)}}+\int_{t_0}^{\ell}(2L+1)\,|u_j(t)-u_{j+1}(t)|^2dt.$$

So, by the Gronwall–Bellman Lemma we obtain

$$|u_j(\ell)-u_{j+1}(\ell)|\leq\frac{\delta_0}{5^j|\Omega|^{1/2}},$$

for all $j\in\mathbb{N}$. Thus,

$$\begin{aligned}\mathrm{dist}_{\mathbb{R}}(S^j(\ell,t_0)\psi_0;S^{j+1}(\ell,t_0)K_{s_1}) &= \inf_{b\in S^{j+1}(\ell,t_0)K_{s_1}}\mathrm{dist}_{\mathbb{R}}(S^j(\ell,t_0)\psi_0;b)\\ &\leq \mathrm{dist}_{\mathbb{R}}(S^j(\ell,t_0)\psi_0;S^{j+1}(\ell,t_0)\psi_0)\\ &= |u_j(\ell)-u_{j+1}(\ell)|\leq\frac{\delta_0}{5^j|\Omega|^{1/2}}.\end{aligned}$$

Since $\psi_0\in\mathcal{M}_{s_1}$ was arbitrary, we conclude that

$$\begin{aligned}\mathrm{dist}_{\mathbb{R}}(S^j(\ell,t_0)K_{s_1};S^{j+1}(\ell,t_0)K_{s_1}) &= \sup_{\psi_0\in K_{s_1}}\mathrm{dist}_{\mathbb{R}}(S^j(\ell,t_0)\psi_0;S^{j+1}(\ell,t_0)K_{s_1})\\ &\leq\frac{\delta_0}{5^j|\Omega|^{1/2}}.\end{aligned}$$

□

Lemma 3.15. *Given $\ell\in\mathbb{R}$, consider $t_0=t_0(\ell)\leq\ell$ as in (3.2). Given $\delta_0>0$, we have*

$$\mathrm{dist}_{\mathbb{R}}(S^i(\ell,t_0)K_{s_1};S(\ell,t_0)K_{s_1})\leq\frac{\delta_0}{4|\Omega|^{1/2}},$$

for i large enough.

Proof. Let $\psi_0 \in K_{s_1}$ arbitrarily fixed. From Theorem 2.1,

$$|S^i(\ell,t_0)\psi_0 - S(\ell,t_0)\psi_0| = |u_i(\ell) - u(\ell)| < \frac{\delta_0}{4|\Omega|^{1/2}},$$

for i large enough. So,

$$\begin{aligned}\text{dist}_{\mathbb{R}}(S^i(\ell,t_0)\psi_0; S(\ell,t_0)K_{s_1}) &= \inf_{b\in S(\ell,t_0)K_{s_1}} \text{dist}_{\mathbb{R}}(S^i(\ell,t_0)\psi_0; b)\\ &\le \text{dist}_{\mathbb{R}}(S^i(\ell,t_0)\psi_0; S(\ell,t_0)\psi_0)\\ &< \frac{\delta_0}{4|\Omega|^{1/2}}.\end{aligned}$$

Since $\psi_0 \in K_{s_1}$ was arbitrary, we conclude that

$$\text{dist}_{\mathbb{R}}(S^i(\ell,t_0)K_{s_1}; S(\ell,t_0)K_{s_1}) = \sup_{\psi_0\in K_{s_1}} \text{dist}_{\mathbb{R}}(S^i(\ell,t_0)\psi_0; S(\ell,t_0)K_{s_1}) \le \frac{\delta_0}{4|\Omega|^{1/2}},$$

for i large enough. □

Now, we are in conditions to establish the main result.

Theorem 3.16. *Consider $f = B_{|\mathbb{R}\times\mathbb{R}} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ with $L < \alpha$ (α is from **Assumption C**) and such that the limit problem (2.2) has a strongly bounded dissipative process. Assume condition (H). If condition (H2) is fulfilled for a given $\delta_0 > 0$, then for each $\ell \in \mathbb{R}$,*

$$\mathcal{A}_s(\ell) \subset O_{\delta_0}(\mathcal{A}_\infty(\ell)) = \{z \in H;\ \inf_{a\in\mathcal{A}_\infty(\ell)} \|z-a\|_H < \delta_0\},$$

for s large enough.

Proof. Consider $\ell \in \mathbb{R}$ arbitrarily fixed, $t_0 = t_0(\ell) \le \ell$ as in (3.2) and the sequence of functions $\{\tilde{p}_s(\cdot)\}_{s\in\mathbb{N}}$ defined by $\tilde{p}_1(\cdot) = p_1(\cdot)$, $\tilde{p}_2(\cdot) = p_2(\cdot), \dots, \tilde{p}_{s_1-1}(\cdot) = p_{s_1-1}(\cdot)$, $\tilde{p}_{s_1}(\cdot) \equiv p_{s_1}$, $\tilde{p}_{s_1+1}(\cdot) \equiv p_{s_1}, \dots$ Applying Theorem 3.11 for this sequence of exponent functions and for the original sequence of diffusion coefficients, we have that

$$\text{dist}(\mathcal{A}_s(\ell); \mathcal{M}_{s_1}(\ell)) < \delta_0/4$$

for s large enough. Here $\text{dist}(\mathcal{A}_s(\ell); \mathcal{M}_{s_1}(\ell))$ is the Hausdorff semi-distance between $\mathcal{A}_s(\ell)$ and $\mathcal{M}_{s_1}(\ell)$ in the Hilbert space H. So,

$$\begin{aligned}\text{dist}(\mathcal{A}_s(\ell); \mathcal{A}_\infty(\ell)) &\le \text{dist}(\mathcal{A}_s(\ell); \mathcal{M}_{s_1}(\ell)) + \text{dist}(\mathcal{M}_{s_1}(\ell); \mathcal{A}_\infty(\ell))\\ &< \delta_0/4 + |\Omega|^{1/2}\, \text{dist}_{\mathbb{R}}(\mathcal{M}_{s_1}(\ell); \mathcal{A}_\infty(\ell)),\end{aligned} \tag{3.3}$$

for s large enough.

By the invariance of the pullback attractor $\mathcal{V}_{s_1}$ we have $S^1(\ell,t_0)\mathcal{M}_{s_1}(t_0)$=$\mathcal{M}_{s_1}(\ell)$. Then,

$$\begin{aligned}\text{dist}_{\mathbb{R}}(\mathcal{M}_{s_1}(\ell); \mathcal{A}_\infty(\ell)) &= \text{dist}_{\mathbb{R}}(S^1(\ell,t_0)\mathcal{M}_{s_1}(t_0); \mathcal{A}_\infty(\ell))\\ &\le \text{dist}_{\mathbb{R}}(S^1(\ell,t_0)K_{s_1}; \mathcal{A}_\infty(\ell))\\ &\le \sum_{j=1}^{i} \text{dist}_{\mathbb{R}}(S^j(\ell,t_0)K_{s_1}; S^{j+1}(\ell,t_0)K_{s_1})\\ &\quad + \text{dist}_{\mathbb{R}}(S^{i+1}(\ell,t_0)K_{s_1}; S(\ell,t_0)K_{s_1}) + \text{dist}_{\mathbb{R}}(S(\ell,t_0)K_{s_1}; \mathcal{A}_\infty(\ell)),\end{aligned} \tag{3.4}$$

for all $i \in \mathbb{N}$. Using (3.2), Lemma 3.14, Lemma 3.15 and letting $i \to +\infty$ in (3.4), we obtain

$$\text{dist}_{\mathbb{R}}(\mathcal{M}_{s_1}(\ell); \mathcal{A}_\infty(\ell)) < \sum_{j=1}^{+\infty} \frac{\delta_0}{5^j|\Omega|^{1/2}} + \frac{\delta_0}{4|\Omega|^{1/2}} + \frac{\delta_0}{4|\Omega|^{1/2}} = \frac{3\delta_0}{4|\Omega|^{1/2}}. \tag{3.5}$$

Using (3.5) in (3.3) the result follows. □

4 Final remarks

Comparing this nonautonomous problem with the previous autonomous in [29], a natural question that raise is it also possible for each given $\ell \in \mathbb{R}$, for large s, to include the section $\mathcal{A}_s(\ell)$ of the pullback attractors $\mathcal{U}_s$ of problem (1.1) into a neighborhood of an interval? Using Theorem 3.16, this will be true if it is possible to prove that $\mathcal{A}_\infty(\ell)$ is included into an interval or it is just one equilibrium point of the limit problem (2.2).

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