Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **12**, 1–13; https://doi.org/10.14232/ejqtde.2019.1.12 www.math.u-szeged.hu/ejqtde/

Lower bounds for the finite-time blow-up of solutions of a cancer invasion model

Govindharaju Sathishkumar[✉][1], **Lingeshwaran Shangerganesh**[2] and **Shanmugasundaram Karthikeyan**[1]

[1]Department of Mathematics, Periyar University, Salem, 636 011, India
[2]Department of Applied Sciences, National Institute of Technology, Goa, 403 401, India

Received 3 December 2018, appeared 23 February 2019
Communicated by Maria Alessandra Ragusa

Abstract. In this article, we consider non-negative solutions of the nonlinear cancer invasion mathematical model involving proliferation and growth functions with homogeneous Neumann and Robin type boundary conditions. We first obtain lower bounds for the finite time blow-up of solutions in $\mathbb{R}^3$ with assumed boundary conditions. Finally, we extend the blow-up results of the given system in $\mathbb{R}^2$ using first-order differential inequality techniques and under appropriate assumptions on data.

Keywords: blow up, lower bounds, cancer invasion, reaction-diffusion.

2010 Mathematics Subject Classification: 35B44, 35P15, 35K57.

1 Introduction

Cancer is the most threatening disease to the society due its mortality rate among affected patients. In the past few years, many works presented for the acid-mediated invasion hypothesis and it is proposing that tumour acidification confers an advantage to the tumor cells by producing a harsh environment. Further this process facilitates invasion of tumor cells into the normal cells by producing matrix degrading enzymes. Partial differential equation (PDE) is one of the best modelling tool to study acid mediated cancer dynamics. PDEs have been used for many cancer invasion mathematical models, for example, see [2, 4, 5, 7, 10, 15, 24–28] and the references therein. This paper investigate the properties of non-negative solutions of the following nonlinear coupled cancer invasion mathematical model in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$:

$$\left.\begin{array}{ll} u_t - d_1\Delta u = \mu u(1-u-v) & \text{in } \Omega \times (0, I), \\ v_t = -kvw + \rho v(1-u-v) & \text{in } \Omega \times (0, I), \\ w_t - d_2\Delta w = \zeta u(1-w) - \nu w & \text{in } \Omega \times (0, I), \\ \frac{\partial u}{\partial n} + h_1(t)u = 0,\ \frac{\partial v}{\partial n} = 0,\ \frac{\partial w}{\partial n} + h_2(t)w = 0 & \text{on } \partial\Omega \times (0, I), \\ u = u_0(x),\ v = v_0(x),\ w = w_0(x) & \text{in } \Omega. \end{array}\right\} \tag{1.1}$$

✉Corresponding author. Email: gskmathspu@gmail.com

The mathematical model consists of three unknown variables namely cancer cell density $u(x,t)$, extra cellular matrix (ECM) density $v(x,t)$ and matrix degrading enzymes (MDE) concentration $w(x,t)$. The proliferation rate of cancer cells are assumed to have a logistic growth and is given by $\mu u(1-u-v)$. Here $\mu > 0$ is a growth rate constant. Further, MDEs produced by the cancer cells degrade most of the components of ECM. Here the degradation processes is modeled by kvw, k is a positive constant. We also assumed that the remodeling growth of ECM follows a logistic growth, that is, $\rho v(1-u-v)$, where ρ is a positive constant. Moreover the decay and growth rates of MDEs are respectively modeled by νw and $\zeta u(1-w)$, where ν, ζ are positive constants. Further, d_1 and d_2 are positive constant diffusion coefficients of cancer cell density and MDE concentration respectively. Finally, $u_0(x), v_0(x)$ and $w_0(x)$ are non-negative functions and represent the initial conditions of u, v and w respectively. Here, we have considered the natural boundary conditions for u, v and w where $h_i(t),\ i = 1, 2$ are non-negative functions.

Due to the wide range of applications of nonlinear parabolic partial differential equations in many branches of engineering, physics, biology and other sciences, the study of nonlinear parabolic system has became an important field in mathematical analysis. In particular, the study of blow-up for nonlinear parabolic systems received much attention in the last few decades, for instance, see [1, 3, 6, 8, 17, 21, 29] and the references cited in these papers. In the above mentioned papers, various methods were developed and used to study the global existence of solutions, blow-up of solutions, asymptotic behaviours of solutions, upper bound and lower bounds for finite time blow-up of solutions. We refer the interested readers to [9, 13, 14, 18–20, 22, 23] and and the references therein.

Existence of global solutions for a similar reaction-diffusion system with nonlinear boundary condition is proved in [11, 12]. Further existence and uniqueness of classical solutions of a similar kind of cancer invasion model as (1.1) with taxis effect is studied in [16, 30]. However, in biological applications, study on lower bound for the finite-time blow-up of solutions is important due to the explosive and diffusive nature of solutions. Further there are some important physical phenomena formulated for biological models with nonlinear boundary conditions rather than the standard Dirichlet boundary conditions. Therefore, in line with these motivations, in this work, we estimate the lower bounds for the finite time blow-up of solutions in $\mathbb{R}^N,\ N = 2, 3$ with Neumann and Robin type boundary conditions for cancer invasion reaction-diffusion system (1.1) using first-order differential inequality techniques.

The paper is organized as follows. In Section 2, we estimate the lower bound for the finite-time blow-up of solutions of (1.1) with suitable auxiliary function in $\mathbb{R}^3$ under Neumann and Robin type boundary conditions. Further, in Section 3, we extend the same results in $\mathbb{R}^2$ by changing certain inequalities.

2 Lower bounds for finite time blow-up of solutions in $\mathbb{R}^3$

In this section, we consider a parabolic system (1.1) and seek a lower bound on blow-up time for a non-negative solution if it is occur at some finite time t^*. In order to obtain the desire result, we first define the suitable auxiliary function for the problem (1.1). Under the assumptions of the Neumann boundary conditions ($h_i(t) = 0$) and Robin boundary conditions ($h_i(t) > 0$) in (1.1), we attain the lower bounds for finite-time blow-up of solutions with help of certain inequalities and the considered auxiliary function.

We define the following auxiliary function to obtain the lower bounds of (u, v, w) for finite time t^*:

$$\varphi(t) = \alpha(t) \int_\Omega u^2 dx + \beta(t) \int_\Omega v^2 dx + \gamma(t) \int_\Omega w^2 dx, \tag{2.1}$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are suitable time dependent positive functions. Further, we define φ_0 as

$$\varphi_0 = \varphi(0) = \alpha(0) \int_\Omega u_0^2 dx + \beta(0) \int_\Omega v_0^2 dx + \gamma(0) \int_\Omega w_0^2 dx. \tag{2.2}$$

Definition 2.1. We say that the triple solution (u, v, w) of (1.1) blows-up in φ-measure at time t^* if

$$\lim_{t \to t^*} \varphi(t) = \infty. \tag{2.3}$$

Theorem 2.2 (with Neumann boundary condition)**.** *Suppose that (u, v, w) is a non-negative classical solution of (1.1) in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with origin inside. If the triple solution (u, v, w) becomes unbounded in $L^2(\Omega)$-norm at $t = t^*$, then t^* satisfies the lower bound*

$$t^* \geq \widetilde{H}^{-1}\left(\frac{1}{2\varphi_0^2}\right), \qquad \varphi_0 = \varphi(0), \tag{2.4}$$

where $\widetilde{H}^{-1}$ is the inverse of $\widetilde{H}(t) := \int_0^t H(\tau)d\tau$ for positive function $H := H(\alpha(t), \beta(t), \gamma(t), \Omega, \varphi_0)$ and $\varphi(t), \varphi_0$ are defined as in (2.1)–(2.2).

Proof. Differentiating (2.1), we have

$$\begin{aligned}
\varphi'(t) &= \alpha'(t) \int_\Omega u^2 dx + 2\alpha(t) \int_\Omega u u_t dx + \beta'(t) \int_\Omega v^2 dx + 2\beta(t) \int_\Omega v v_t dx + \gamma'(t) \int_\Omega w^2 dx \\
&\quad + 2\gamma(t) \int_\Omega w w_t dx \\
&= \alpha'(t) \int_\Omega u^2 dx + 2\alpha(t) \int_\Omega u(d_1 \Delta u + \mu u(1 - u - v)) dx \\
&\quad + \beta'(t) \int_\Omega v^2 dx + \gamma'(t) \int_\Omega w^2 dx \\
&\quad + 2\beta(t) \int_\Omega v(-kvw + \rho v(1 - u - v)) dx \\
&\quad + 2\gamma(t) \int_\Omega w(d_2 \Delta w + \zeta u(1 - w) - \nu w) dx \\
&= \alpha'(t) \int_\Omega u^2 dx + 2\alpha(t) d_1 \int_\Omega u \Delta u dx + 2\alpha(t)\mu \int_\Omega u^2 dx - 2\alpha(t)\mu \int_\Omega u^3 dx \\
&\quad - 2\alpha(t)\mu \int_\Omega u^2 v dx + \beta'(t) \int_\Omega v^2 dx - 2\beta(t)k \int_\Omega v^2 w dx + 2\beta(t)\rho \int_\Omega v^2 dx \\
&\quad - 2\beta(t)\rho \int_\Omega v^2 u dx - 2\beta(t)\rho \int_\Omega v^3 dx + \gamma'(t) \int_\Omega w^2 dx + 2\gamma(t) d_2 \int_\Omega w \Delta w dx \\
&\quad + 2\gamma(t)\zeta \int_\Omega u w dx - 2\gamma(t)\zeta \int_\Omega u w^2 dx - 2\gamma(t)\nu \int_\Omega w^2 dx.
\end{aligned} \tag{2.5}$$

Using zero flux boundary condition and divergence theorem, we get

$$\int_\Omega u \Delta u dx = -\int_\Omega |\nabla u|^2 dx, \tag{2.6}$$

$$\int_\Omega w \Delta w dx = -\int_\Omega |\nabla w|^2 dx. \tag{2.7}$$

Using Hölder's inequality and the standard inequality $a^r b^s \leq ra + sb, a > 0, b > 0, s + r = 1$, we get

$$\begin{aligned}\int_\Omega u^2 v dx &\leq \left(\int_\Omega u^3 dx\right)^{\frac{2}{3}} \left(\int_\Omega v^3 dx\right)^{\frac{1}{3}} \\ &\leq \frac{2}{3\epsilon_0(t)^{\frac{1}{2}}} \int_\Omega u^3 dx + \frac{\epsilon_0(t)}{3} \int_\Omega v^3 dx.\end{aligned} \tag{2.8}$$

Similarly we get

$$\begin{aligned}\int_\Omega v^2 w dx &\leq \frac{2\epsilon_0(t)}{3} \int_\Omega v^3 dx + \frac{1}{3\epsilon_0(t)^2} \int_\Omega w^3 dx, \\ \int_\Omega v^2 u dx &\leq \frac{2\epsilon_0(t)}{3} \int_\Omega v^3 dx + \frac{1}{3\epsilon_0(t)^2} \int_\Omega u^3 dx, \\ \int_\Omega w^2 u dx &\leq \frac{2}{3} \int_\Omega w^3 dx + \frac{1}{3} \int_\Omega u^3 dx, \\ \int_\Omega uw dx &\leq \frac{1}{2} \int_\Omega u^2 dx + \frac{1}{2} \int_\Omega w^2 dx.\end{aligned} \tag{2.9}$$

Substituting (2.6)–(2.9) in (2.5), we get

$$\begin{aligned}\varphi'(t) \leq \alpha'(t) \int_\Omega u^2 dx &- 2\alpha(t) d_1 \int_\Omega |\nabla u|^2 dx + 2\alpha(t)\mu \int_\Omega u^2 dx + \beta'(t) \int_\Omega v^2 dx \\ &+ 2\beta(t)\rho \int_\Omega v^2 dx + \gamma'(t) \int_\Omega w^2 dx - 2\gamma(t) d_2 \int_\Omega |\nabla w|^2 dx + \gamma(t)\zeta \int_\Omega u^2 dx \\ &+ \gamma(t)\zeta \int_\Omega w^2 dx + 2\gamma(t)\nu \int_\Omega w^2 dx + A_1(t) \int_\Omega u^3 dx + A_2(t) \int_\Omega v^3 dx \\ &+ A_3(t) \int_\Omega w^3 dx,\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}A_1(t) &= \frac{4\alpha(t)\mu}{3\epsilon_0(t)^{\frac{1}{2}}} + \frac{2\beta(t)\rho}{3\epsilon_0(t)^2} + \frac{2\gamma(t)\zeta}{3} + 2\alpha(t)\mu, \\ A_2(t) &= \left[\frac{2\alpha(t)\mu}{3} + \frac{4\beta(t)k}{3} + \frac{4\beta(t)\rho}{3}\right] \epsilon_0(t) - 2\beta(t)\rho, \\ A_3(t) &= \frac{2\beta(t)k}{3\epsilon_0(t)^2} + \frac{4\gamma(t)\zeta}{3}.\end{aligned}$$

Using the Sobolev-type inequality (see Lemma A2 in [18]) and standard inequality $(a+b)^s \leq 2^{s-1}(a^s + b^s), a, b > 0, s \geq 1$, we can estimate the terms of (2.10) as follows:

$$\begin{aligned}\int_\Omega u^3 dx &\leq \left\{P_1 \int_\Omega u^2 dx + P_2 \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}\right\}^{\frac{3}{2}} \\ &\leq 2^{\frac{1}{2}} \left\{P_1^{\frac{3}{2}} \left(\int_\Omega u^2 dx\right)^{\frac{3}{2}} + P_2^{\frac{3}{2}} \left(\int_\Omega u^2 dx\right)^{\frac{3}{4}} \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{3}{4}}\right\} \\ &\leq 2^{\frac{1}{2}} \left\{P_1^{\frac{3}{2}} \left(\int_\Omega u^2 dx\right)^{\frac{3}{2}} + \frac{P_2^{\frac{3}{2}}}{4\epsilon_1(t)^3} \left(\int_\Omega u^2 dx\right)^3 + \frac{3P_2^{\frac{3}{2}}\epsilon_1(t)}{4} \left(\int_\Omega |\nabla u|^2 dx\right)\right\},\end{aligned} \tag{2.11}$$

where

$$P_1 = \frac{3}{2\rho_0}, \quad P_2 = 1 + \frac{d}{\rho_0}, \quad \rho_0 = \min_{\partial\Omega}(x \cdot n), \quad \text{and} \quad d = \max_{\overline{\Omega}} |x| \quad \text{are positive constants.}$$

Similarly we get

$$\int_\Omega w^3 dx \le 2^{\frac{1}{2}} \left\{ P_1^{\frac{3}{2}} \left(\int_\Omega w^2 dx \right)^{\frac{3}{2}} + \frac{P_2^{\frac{3}{2}}}{4\epsilon_2(t)^3} \left(\int_\Omega w^2 dx \right)^3 + \frac{3P_2^{\frac{3}{2}}\epsilon_2(t)}{4} \left(\int_\Omega |\nabla w|^2 dx \right) \right\}, \tag{2.12}$$

where the constants are defined as before. Substituting (2.11)–(2.12) in (2.10), we get

$$\begin{aligned} \varphi'(t) \le & \left\{ \frac{3A_1(t)P_2^{\frac{3}{2}}\epsilon_1(t)}{4} - 2\alpha(t)d_1 \right\} \int_\Omega |\nabla u|^2 dx + \left\{ \frac{3A_3(t)P_2^{\frac{3}{2}}\epsilon_2(t)}{4} - 2\gamma(t)d_2 \right\} \\ & \times \int_\Omega |\nabla w|^2 dx + A_2(t) \int_\Omega v^3 dx + \frac{\alpha'(t) + 2\alpha(t)\mu + \gamma(t)\zeta}{\alpha(t)} \left(\alpha(t) \int_\Omega u^2 dx \right) \\ & + \frac{\beta'(t) + 2\beta(t)\rho}{\beta(t)} \left(\beta(t) \int_\Omega v^2 dx \right) + \frac{A_1(t)\sqrt{2}P_1^{\frac{3}{2}}}{\alpha(t)^{\frac{3}{2}}} \left(\alpha(t) \int_\Omega u^2 dx \right)^{\frac{3}{2}} \\ & + \frac{\gamma'(t) + \gamma(t)\zeta + 2\gamma(t)\nu}{\gamma(t)} \left(\gamma(t) \int_\Omega w^2 dx \right) + \frac{A_3(t)\sqrt{2}P_1^{\frac{3}{2}}}{\gamma(t)^{\frac{3}{2}}} \left(\gamma(t) \int_\Omega w^2 dx \right)^{\frac{3}{2}} \\ & + \frac{A_1(t)P_2^{\frac{3}{2}}}{4(\epsilon_1(t)\alpha(t))^3} \left(\alpha(t) \int_\Omega u^2 dx \right)^3 + \frac{A_3(t)P_2^{\frac{3}{2}}}{4(\epsilon_2(t)\gamma(t))^3} \left(\gamma(t) \int_\Omega w^2 dx \right)^3. \end{aligned} \tag{2.13}$$

Choosing $\alpha(t), \beta(t), \gamma(t), \epsilon_0(t), \epsilon_1(t)$ and $\epsilon_2(t)$ as follows:

$$\begin{aligned} &\alpha(t) = e^{2\mu t}, && \beta(t) = e^{2\rho t}, && \gamma(t) = e^{(\zeta+2\nu)t}, \\ &\epsilon_0(t) = \frac{3\beta(t)\rho}{\alpha(t)\mu + 2\beta(t)(k+\rho)}, && \epsilon_1(t) = \frac{8\alpha(t)d_1}{3A_1(t)P_2^{\frac{3}{2}}}, && \epsilon_2(t) = \frac{8\gamma(t)d_2}{3A_3(t)P_2^{\frac{3}{2}}}, \end{aligned}$$

we obtain the following first order differential inequality

$$\varphi'(t) \le B_1(t)\varphi(t) + B_2(t)\varphi^{\frac{3}{2}}(t) + B_3(t)\varphi^3(t), \tag{2.14}$$

where

$$\begin{aligned} B_1(t) &= \max\left\{ \frac{\alpha'(t) + 2\alpha(t)\mu + \gamma(t)\zeta}{\alpha(t)}, \frac{\beta'(t) + 2\beta(t)\rho}{\beta(t)}, \frac{\gamma'(t) + \gamma(t)\zeta + 2\gamma(t)\nu}{\gamma(t)} \right\}, \\ B_2(t) &= \sqrt{2}P_1^{\frac{3}{2}} \left(\frac{A_1(t)}{\alpha(t)^{\frac{3}{2}}} + \frac{A_3(t)}{\gamma(t)^{\frac{3}{2}}} \right), \\ B_3(t) &= \frac{P_2^{\frac{3}{2}}}{4} \left(\frac{A_1(t)}{(\epsilon_1(t)\alpha(t))^3} + \frac{A_3(t)}{(\epsilon_2(t)\gamma(t))^3} \right). \end{aligned}$$

If the solution blows up at t^*, then there exists a time $t_1 \ge 0$ such that $\varphi(t) \ge \varphi_0, \ t \ge t_1$ and

$$\begin{aligned} \varphi(t) &\le \varphi_0^{-2}\varphi^3(t), \\ \varphi^{\frac{3}{2}}(t) &\le \varphi_0^{-\frac{3}{2}}\varphi^3(t). \end{aligned} \tag{2.15}$$

Replacing (2.15) in (2.14), we get

$$\varphi'(t) \le H(t)\varphi^3(t), \qquad t \in [t_1, t^*), \tag{2.16}$$

where

$$H(t) = B_1(t)\varphi_0^{-2} + B_2(t)\varphi_0^{-\frac{3}{2}} + B_3(t).$$

Integrating (2.16) over t_1 to t^*, we get

$$\frac{1}{2\varphi_0^2} \le \int_{t_1}^{t^*} H(\tau)d\tau \le \int_0^{t^*} H(\tau)d\tau = \widetilde{H}(t^*). \qquad \square$$

Theorem 2.3 (with Robin boundary condition)**.** *Suppose that (u, v, w) is a non-negative classical solution of (1.1) in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with origin inside. Further assume that $0 \le \alpha'(t) < 2\alpha(t)d_1\eta_1(t)$ and $0 \le \gamma'(t) < 2\gamma(t)d_2\eta_1(t)$,*

$$\begin{aligned} &\Delta f + \eta(t)f = 0, \qquad f > 0 \text{ in } \Omega, \\ &\frac{\partial f}{\partial n} + h(t)f = 0 \quad \text{on } \partial\Omega,\ h(t) > 0, \end{aligned} \tag{2.17}$$

where $\eta_1(t)$ is the first eigenvalue of (2.17) and n is the unit normal vector. If the triple solution (u, v, w) becomes unbounded in $L^2(\Omega)$-norm at $t = t^*$, then t^* satisfies the lower bound

$$t^* \ge \widetilde{R}^{-1}\left(\frac{1}{2\varphi_0^2}\right), \qquad \varphi_0 = \varphi(0), \tag{2.18}$$

where $\widetilde{R}^{-1}$ is the inverse function of $\widetilde{R}(t) := \int_0^t R(\tau)d\tau$ and $R := R(\alpha(t), \beta(t), \gamma(t), \Omega, \varphi_0)$ is a positive function.

Proof. In order to prove the theorem, we use similar arguments as in Theorem 2.2. However, we consider the following inequality for u and w in place of (2.6) and (2.7).

$$\int_\Omega z\Delta z dx = -h_i(t)\int_{\partial\Omega} z^2 ds - \int_\Omega |\nabla z|^2 dx, \qquad i = 1, 2. \tag{2.19}$$

Substituting (2.19) and (2.8)–(2.9) in (2.5), we get

$$\begin{aligned} \varphi'(t) \le{}& \alpha'(t)\int_\Omega u^2 dx - 2\alpha(t)d_1\int_\Omega |\nabla u|^2 dx - 2\alpha(t)d_1h_1(t)\int_{\partial\Omega} u^2 ds + 2\alpha(t)\mu\int_\Omega u^2 dx \\ &+ \beta'(t)\int_\Omega v^2 dx + 2\beta(t)\rho\int_\Omega v^2 dx + \gamma'(t)\int_\Omega w^2 dx - 2\gamma(t)d_2\int_\Omega |\nabla w|^2 dx \\ &- 2\gamma(t)d_2h_2(t)\int_{\partial\Omega} w^2 ds + \gamma(t)\zeta\int_\Omega u^2 dx + \gamma(t)\zeta\int_\Omega w^2 dx + 2\gamma(t)\nu\int_\Omega w^2 dx \\ &+ A_1(t)\int_\Omega u^3 dx + A_2(t)\int_\Omega v^3 dx + A_3(t)\int_\Omega w^3 dx, \end{aligned} \tag{2.20}$$

where $A_1(t), A_2(t)$ and $A_3(t)$ are defined as before. From the variational definition of $\eta_1(t)$ and (2.17), we get

$$\eta_1(t)\int_\Omega f^2 dx \le \int_\Omega |\nabla f|^2 dx + h(t)\int_{\partial\Omega} f^2 ds, \tag{2.21}$$

and therefore we have the following inequalities for u and w

$$\begin{aligned} \alpha'(t)\int_\Omega u^2 dx &\le \frac{\alpha'(t)}{\eta_1(t)}\int_\Omega |\nabla u|^2 dx + \frac{\alpha'(t)}{\eta_1(t)}h_1(t)\int_{\partial\Omega} u^2 ds, \\ \gamma'(t)\int_\Omega w^2 dx &\le \frac{\gamma'(t)}{\eta_1(t)}\int_\Omega |\nabla w|^2 dx + \frac{\gamma'(t)}{\eta_1(t)}h_2(t)\int_{\partial\Omega} w^2 ds. \end{aligned} \tag{2.22}$$

Substituting (2.11)–(2.12) and (2.22) in (2.20), we get

$$\begin{aligned}
\varphi'(t) \le & \left\{ \frac{3A_1(t)P_2^{\frac{3}{2}}\epsilon_1(t)}{4} + \frac{\alpha'(t)}{\eta_1(t)} - 2\alpha(t)d_1 \right\} \int_\Omega |\nabla u|^2 \, dx \\
& + \frac{2\alpha(t)\mu + \gamma(t)\zeta}{\alpha(t)} \left(\alpha(t) \int_\Omega u^2 dx \right) \\
& + \left\{ \frac{3A_3(t)P_2^{\frac{3}{2}}\epsilon_2(t)}{4} + \frac{\gamma'(t)}{\eta_1(t)} - 2\gamma(t)d_2 \right\} \int_\Omega |\nabla w|^2 \, dx \\
& + \frac{\beta'(t) + 2\beta(t)\rho}{\beta(t)} \left(\beta(t) \int_\Omega v^2 dx \right) \\
& + A_2(t) \int_\Omega v^3 dx + (\zeta + 2\nu) \left(\gamma(t) \int_\Omega w^2 dx \right) + \frac{A_1(t)\sqrt{2}P_1^{\frac{3}{2}}}{\alpha(t)^{\frac{3}{2}}} \left(\alpha(t) \int_\Omega u^2 dx \right)^{\frac{3}{2}} \\
& + \frac{A_3(t)\sqrt{2}P_1^{\frac{3}{2}}}{\gamma(t)^{\frac{3}{2}}} \left(\gamma(t) \int_\Omega w^2 dx \right)^{\frac{3}{2}} + \frac{A_1(t)P_2^{\frac{3}{2}}}{4(\epsilon_1(t)\alpha(t))^3} \left(\alpha(t) \int_\Omega u^2 dx \right)^3 \\
& + \frac{A_3(t)P_2^{\frac{3}{2}}}{4(\epsilon_2(t)\gamma(t))^3} \left(\gamma(t) \int_\Omega w^2 dx \right)^3 + \frac{h_1(t)}{\eta_1(t)} \left(\alpha'(t) - 2\alpha(t)d_1\eta_1(t) \right) \int_{\partial\Omega} u^2 ds \\
& + \frac{h_2(t)}{\eta_1(t)} \left(\gamma'(t) - 2\gamma(t)d_2\eta_1(t) \right) \int_{\partial\Omega} w^2 ds.
\end{aligned} \tag{2.23}$$

Choosing $\alpha(t), \beta(t), \gamma(t), \epsilon_0(t), \epsilon_1(t)$ and $\epsilon_2(t)$ as follows:

$$\alpha(t) = e^{d_1 \int_0^t \eta_1(\tau) d\tau}, \qquad \beta(t) = e^{2\rho t}, \qquad \gamma(t) = e^{d_2 \int_0^t \eta_1(\tau) d\tau},$$

$$\epsilon_0(t) = \frac{3\beta(t)\rho}{\alpha(t)\mu + 2\beta(t)(k+\rho)}, \quad \epsilon_1(t) = \frac{8\alpha(t)d_1\eta_1(t) - 4\alpha'(t)}{3\eta_1(t)A_1(t)P_2^{\frac{3}{2}}}, \quad \epsilon_2(t) = \frac{8\gamma(t)d_2\eta_1(t) - 4\gamma'(t)}{3\eta_1(t)A_2(t)P_2^{\frac{3}{2}}},$$

we obtain the following first order differential inequality

$$\varphi'(t) \le \widetilde{B_1(t)}\varphi(t) + \widetilde{B_2(t)}\varphi^{\frac{3}{2}}(t) + \widetilde{B_3(t)}\varphi^3(t), \tag{2.24}$$

where

$$\widetilde{B_1(t)} = \max\left\{ 2\mu + \frac{\gamma(t)\zeta}{\alpha(t)}, \frac{\beta'(t)}{\beta(t)} + 2\rho, \zeta + 2\nu \right\},$$

$$\widetilde{B_2(t)} = \sqrt{2}P_1^{\frac{3}{2}} \left(\frac{A_1(t)}{\alpha(t)^{\frac{3}{2}}} + \frac{A_3(t)}{\gamma(t)^{\frac{3}{2}}} \right),$$

$$\widetilde{B_3(t)} = \frac{P_2^{\frac{3}{2}}}{4} \left(\frac{A_1(t)}{(\epsilon_1(t)\alpha(t))^3} + \frac{A_3(t)}{(\epsilon_2(t)\gamma(t))^3} \right).$$

Then similar arguments as in Theorem 2.2 leads to

$$\varphi'(t) \le R(t)\varphi^3(t), \qquad t \in [t_1, t^*), \tag{2.25}$$

where

$$R(t) = \widetilde{B_1(t)}\varphi_0^{-2} + \widetilde{B_2(t)}\varphi_0^{-\frac{3}{2}} + \widetilde{B_3(t)}.$$

Integrating (2.25) over t_1 to t^*, we get

$$\frac{1}{2\varphi_0^2} \le \int_{t_1}^{t^*} R(\tau) d\tau \le \int_0^{t^*} R(\tau) d\tau = \widetilde{R}(t^*).$$

□

3 Lower bounds for finite time blow-up of solutions in $\mathbb{R}^2$

In this section, we prove a lower bound for the finite-time blow-up of solutions of the cancer invasion parabolic system (1.1) in a bounded domain $\Omega \subset \mathbb{R}^2$. As in the previous section, we consider the two cases, the Neumann boundary condition and the Robin type boundary for the parabolic system (1.1) to prove the blow-up of solutions (u, v, w) for some time t^*.

Theorem 3.1 (with Neumann boundary condition)**.** *Suppose that (u, v, w) is a non-negative classical solution of (1.1) in a bounded convex domain $\Omega \subset \mathbb{R}^2$ with origin inside. Assume further that $h_i(t) = 0,\ i = 1, 2$, in (1.1). If the triple solution (u, v, w) becomes unbounded in $L^2(\Omega)$-norm at $t = t^*$, then t^* satisfies the lower bound*

$$t^* \geq \widetilde{N}^{-1}\left(\frac{1}{\varphi_0}\right), \qquad \varphi_0 = \varphi(0), \tag{3.1}$$

where $\widetilde{N}^{-1}$ is the inverse function of $\widetilde{N}(t) := \int_0^t N(\tau)d\tau$ and $N := N(\alpha(t), \beta(t), \gamma(t), \Omega, \varphi_0)$ is a positive function.

Proof. The proof of the theorem relies on evaluating the integrals $\int_\Omega u^3 dx$, $\int_\Omega v^3 dx$ and $\int_\Omega w^3 dx$ in (2.10). We use the following two inequalities (see [18]) in order to achieve our goal. For any $f \in C^1(\Omega)$ where Ω is a convex domain in $\mathbb{R}^2$ and ρ_0, d are defined as before,

$$\left(\int_\Omega f^4 dx\right)^{\frac{1}{2}} \leq \frac{\sqrt{2}}{4}\int_{\partial\Omega} f^2 ds + \frac{\sqrt{2}}{2}\left(\int_\Omega f^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla f|^2 dx\right)^{\frac{1}{2}}, \tag{3.2}$$

$$\left(\int_{\partial\Omega} f^2 dx\right) \leq \frac{2}{\rho_0}\int_\Omega f^2 dx + \frac{2d}{\rho_0}\left(\int_\Omega f^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla f|^2 dx\right)^{\frac{1}{2}}. \tag{3.3}$$

Substituting (3.3) in (3.2), we get

$$\left(\int_\Omega f^4 dx\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2}}\left\{\frac{1}{\rho_0}\int_\Omega f^2 dx + \left(1 + \frac{d}{\rho_0}\right)\left(\int_\Omega f^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla f|^2 dx\right)^{\frac{1}{2}}\right\}. \tag{3.4}$$

Using (3.4) and Cauchy's inequality, we get

$$\begin{aligned}\int_\Omega u^3 dx &\leq \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega u^4 dx\right)^{\frac{1}{2}}\\ &\leq \frac{1}{\sqrt{2}}\left\{\frac{2P_1}{3}\left(\int_\Omega u^2 dx\right)^{\frac{3}{2}} + \frac{P_2}{4\epsilon_1(t)}\left(\int_\Omega u^2 dx\right)^2 + P_2\epsilon_1(t)\left(\int_\Omega |\nabla u|^2 dx\right)\right\}.\end{aligned} \tag{3.5}$$

Similarly we get

$$\int_\Omega w^3 dx \leq \frac{1}{\sqrt{2}}\left\{\frac{2P_1}{3}\left(\int_\Omega w^2 dx\right)^{\frac{3}{2}} + \frac{P_2}{4\epsilon_2(t)}\left(\int_\Omega w^2 dx\right)^2 + P_2\epsilon_2(t)\left(\int_\Omega |\nabla w|^2 dx\right)\right\}, \tag{3.6}$$

where P_1 and P_2 are defined as before. Substitute (3.5)–(3.6) in (2.10), we get

$$\begin{aligned}\varphi'(t) \le & \left\{\frac{A_1(t)P_2\epsilon_1(t)}{\sqrt{2}} - 2\alpha(t)d_1\right\}\int_\Omega |\nabla u|^2\,dx \\ & + \left\{\frac{A_3(t)P_2\epsilon_2(t)}{\sqrt{2}} - 2\gamma(t)d_2\right\}\int_\Omega |\nabla w|^2\,dx \\ & + \frac{\alpha'(t)+2\alpha(t)\mu+\gamma(t)\zeta}{\alpha(t)}\left(\alpha(t)\int_\Omega u^2dx\right) + \frac{\beta'(t)+2\beta(t)\rho}{\beta(t)}\left(\beta(t)\int_\Omega v^2dx\right) \\ & + \frac{\gamma'(t)+\gamma(t)\zeta+2\gamma(t)\nu}{\gamma(t)}\left(\gamma(t)\int_\Omega w^2dx\right) + \frac{A_1(t)\sqrt{2}P_1}{3\alpha(t)^{\frac{3}{2}}}\left(\alpha(t)\int_\Omega u^2dx\right)^{\frac{3}{2}} \\ & + \frac{A_3(t)\sqrt{2}P_1}{3\gamma(t)^{\frac{3}{2}}}\left(\gamma(t)\int_\Omega w^2dx\right)^{\frac{3}{2}} + \frac{A_1(t)P_2}{4\sqrt{2}\epsilon_1(t)(\alpha(t))^3}\left(\alpha(t)\int_\Omega u^2dx\right)^2 \\ & + \frac{A_3(t)P_2}{4\sqrt{2}\epsilon_2(t)(\gamma(t))^3}\left(\gamma(t)\int_\Omega w^2dx\right)^2 + A_2(t)\int_\Omega v^3dx.\end{aligned} \tag{3.7}$$

Choosing $\alpha(t), \beta(t), \gamma(t), \epsilon_0(t), \epsilon_1(t)$ and $\epsilon_2(t)$ as follows:

$$\alpha(t) = e^{2\mu t}, \qquad \beta(t) = e^{2\rho t}, \qquad \gamma(t) = e^{(\zeta+2\nu)t},$$

$$\epsilon_0(t) = \frac{3\beta(t)\rho}{\alpha(t)\mu + 2\beta(t)(k+\rho)}, \qquad \epsilon_1(t) = \frac{2\sqrt{2}\alpha(t)d_1}{A_1(t)P_2}, \qquad \epsilon_2(t) = \frac{2\sqrt{2}\gamma(t)d_2}{A_3(t)P_2},$$

we obtain the following first order differential inequality

$$\varphi'(t) \le \overline{B_1(t)}\varphi(t) + \overline{B_2(t)}\varphi^{\frac{3}{2}}(t) + \overline{B_3(t)}\varphi^2(t), \tag{3.8}$$

where

$$\begin{aligned}\overline{B_1(t)} &= \max\left\{\frac{\alpha'(t)+2\alpha(t)\mu+\gamma(t)\zeta}{\alpha(t)}, \frac{\beta'(t)+2\beta(t)\rho}{\beta(t)}, \frac{\gamma'(t)+\gamma(t)\zeta+2\gamma(t)\nu}{\gamma(t)}\right\}, \\ \overline{B_2(t)} &= \frac{\sqrt{2}P_1}{3}\left(\frac{A_1(t)}{\alpha(t)^{\frac{3}{2}}} + \frac{A_3(t)}{\gamma(t)^{\frac{3}{2}}}\right), \\ \overline{B_3(t)} &= \frac{P_2}{4\sqrt{2}}\left(\frac{A_1(t)}{\epsilon_1(t)(\alpha(t))^3} + \frac{A_3(t)}{\epsilon_2(t)(\gamma(t))^3}\right).\end{aligned}$$

If the solution blows up at t^*, then there exists a time $t_1 \ge 0$ such that $\varphi(t) \ge \varphi_0,\ t \ge t_1$ and

$$\begin{aligned}\varphi(t) &\le \varphi_0^{-1}\varphi^2(t), \\ \varphi^{\frac{3}{2}}(t) &\le \varphi_0^{-\frac{1}{2}}\varphi^2(t).\end{aligned} \tag{3.9}$$

Replacing (3.9) in (3.8), we get

$$\varphi'(t) \le N(t)\varphi^2(t), \qquad t \in [t_1, t^*), \tag{3.10}$$

where

$$N(t) = \overline{B_1(t)}\varphi_0^{-1} + \overline{B_2(t)}\varphi_0^{-\frac{1}{2}} + \overline{B_3(t)}.$$

Integrating (3.10) over t_1 to t^*, we get

$$\frac{1}{\varphi_0} \le \int_{t_1}^{t^*} N(\tau)d\tau \le \int_0^{t^*} N(\tau)d\tau = \widetilde{N}(t^*). \qquad \square$$

Theorem 3.2 (with Robin boundary condition)**.** *Suppose that (u, v, w) is a non-negative classical solution of (1.1) in a bounded convex domain $\Omega \subset \mathbb{R}^2$ with origin inside. Further assume that $0 \leq \alpha'(t) < 2\alpha(t)d_1\eta_1(t)$ and $0 \leq \gamma'(t) < 2\gamma(t)d_2\eta_1(t)$, where $\eta_1(t)$ is the first eigenvalue of (2.17). If the triple solution (u, v, w) becomes unbounded in $L^2(\Omega)$-norm at $t = t^*$, then t^* satisfies the lower bound*

$$t^* \geq \tilde{M}^{-1}\left(\frac{1}{\varphi_0}\right), \qquad \varphi_0 = \varphi(0), \tag{3.11}$$

where $\tilde{M}^{-1}$ is a inverse function of $\tilde{M}(t) := \int_0^t M(\tau)d\tau$ and $M := M(\alpha(t), \beta(t), \gamma(t), \Omega, \varphi_0)$ is a positive function.

Proof. Substituting (2.22), (3.5)–(3.6) in (2.20), we get

$$\begin{aligned}
\varphi'(t) \leq& \left\{\frac{A_1(t)P_2\epsilon_1(t)}{\sqrt{2}} + \frac{\alpha'(t)}{\eta_1(t)} - 2\alpha(t)d_1\right\}\int_\Omega |\nabla u|^2\,dx \\
&+ \frac{2\alpha(t)\mu + \gamma(t)\zeta}{\alpha(t)}\left(\alpha(t)\int_\Omega u^2dx\right) \\
&+ \left\{\frac{A_3(t)P_2\epsilon_2(t)}{\sqrt{2}} + \frac{\gamma'(t)}{\eta_1(t)} - 2\gamma(t)d_2\right\}\int_\Omega |\nabla w|^2\,dx \\
&+ \frac{\beta'(t) + 2\beta(t)\rho}{\beta(t)}\left(\beta(t)\int_\Omega v^2dx\right) \\
&+ A_2(t)\int_\Omega v^3dx + (\zeta + 2\nu)\left(\gamma(t)\int_\Omega w^2dx\right) + \frac{A_1(t)\sqrt{2}P_1}{3\alpha(t)^{\frac{3}{2}}}\left(\alpha(t)\int_\Omega u^2dx\right)^{\frac{3}{2}} \\
&+ \frac{A_3(t)\sqrt{2}P_1}{3\gamma(t)^{\frac{3}{2}}}\left(\gamma(t)\int_\Omega w^2dx\right)^{\frac{3}{2}} + \frac{A_1(t)P_2}{4\sqrt{2}\epsilon_1(t)(\alpha(t))^3}\left(\alpha(t)\int_\Omega u^2dx\right)^2 \\
&+ \frac{A_3(t)P_2}{4\sqrt{2}\epsilon_2(t)(\gamma(t))^3}\left(\gamma(t)\int_\Omega w^2dx\right)^2 + \frac{h_1(t)}{\eta_1(t)}\left(\alpha'(t) - 2\alpha(t)d_1\eta_1(t)\right)\int_{\partial\Omega} u^2ds \\
&+ \frac{h_2(t)}{\eta_1(t)}\left(\gamma'(t) - 2\gamma(t)d_2\eta_1(t)\right)\int_{\partial\Omega} w^2ds.
\end{aligned} \tag{3.12}$$

Choosing $\alpha(t), \beta(t), \gamma(t), \epsilon_0(t), \epsilon_1(t)$ and $\epsilon_2(t)$ as follows:

$$\alpha(t) = e^{d_1\int_0^t \eta_1(\tau)d\tau}, \qquad \beta(t) = e^{2\rho t},$$

$$\gamma(t) = e^{d_2\int_0^t \eta_1(\tau)d\tau}, \qquad \epsilon_0(t) = \frac{3\beta(t)\rho}{\alpha(t)\mu + 2\beta(t)(k+\rho)},$$

$$\epsilon_1(t) = \frac{\sqrt{2}\,(2\alpha(t)d_1\eta_1(t) - \alpha'(t))}{\eta_1(t)A_1(t)P_2}, \qquad \epsilon_2(t) = \frac{\sqrt{2}\,(2\gamma(t)d_2\eta_1(t) - \gamma'(t))}{\eta_1(t)A_2(t)P_2},$$

leads to the following first order differential inequality

$$\varphi'(t) \leq \widehat{B_1(t)}\varphi(t) + \widehat{B_2(t)}\varphi^{\frac{3}{2}}(t) + \widehat{B_3(t)}\varphi^2(t), \tag{3.13}$$

where

$$\begin{aligned}
\widehat{B_1(t)} &= \max\left\{\frac{\gamma(t)\zeta}{\alpha(t)} + 2\mu, \frac{\beta'(t)}{\beta(t)} + 2\rho, \zeta + 2\nu\right\}, \\
\widehat{B_2(t)} &= \frac{\sqrt{2}P_1}{3}\left(\frac{A_1(t)}{\alpha(t)^{\frac{3}{2}}} + \frac{A_3(t)}{\gamma(t)^{\frac{3}{2}}}\right), \\
\widehat{B_3(t)} &= \frac{P_2}{4\sqrt{2}}\left(\frac{A_1(t)}{\epsilon_1(t)(\alpha(t))^3} + \frac{A_3(t)}{\epsilon_2(t)(\gamma(t))^3}\right).
\end{aligned}$$

If the solution blows up at t^*, then there exists a time $t_1 \geq 0$ such that $\varphi(t) \geq \varphi_0$, $t \geq t_1$ and

$$\begin{aligned} \varphi(t) &\leq \varphi_0^{-1}\varphi^2(t), \\ \varphi^{\frac{3}{2}}(t) &\leq \varphi_0^{-\frac{1}{2}}\varphi^2(t). \end{aligned} \tag{3.14}$$

Replacing (3.14) in (3.13), we get

$$\varphi'(t) \leq M(t)\varphi^2(t), \qquad t \in [t_1, t^*), \tag{3.15}$$

where

$$M(t) = \widehat{B_1(t)}\varphi_0^{-1} + \widehat{B_2(t)}\varphi_0^{-\frac{1}{2}} + \widehat{B_3(t)}.$$

Integrating (3.15) over t_1 to t^*, we get

$$\frac{1}{\varphi_0} \leq \int_{t_1}^{t^*} M(\tau)d\tau \leq \int_0^{t^*} M(\tau)d\tau = \widetilde{M}(t^*). \qquad \square$$

Acknowledgment

The authors would like to thank the reviewer for his comments and suggestions which improved the quality of the article. The work of the first author is supported by the University Research Fellowship of Periyar University and the work of third author is supported by DST-FIST (SR/FST/MSI-115/2016).

References

[1] I. Ahmed, C. Mu, P. Zheng, F. Zhang, Blow-up and global existence for the nonlocal reaction diffusion problem with time dependent coefficient, *Bound. Value Probl.* **2013**, 2013:239, 6 pp. https://doi.org/10.1186/1687-2770-2013-239; MR3341395; Zbl 1295.35120

[2] A. R. A. Anderson, A hybrid mathematical model of solid tumor invasion: The importance of cell adhesion, *Math. Med. Biol.* **22**(2005), No. 2, 163–186. https://doi.org/10.1093/imammb/dqi005;

[3] C. Bandle, H. Brunner, Blow-up in diffusion equations: a survey, *J. Comput. Appl. Math.* **97**(1998), No. 1–2, 3–22. https://doi.org/10.1016/S0377-0427(98)00100-9; MR1651764; Zbl 0932.65098

[4] C. Bianca, F. Pappalardo, S. Motta, M. A. Ragusa, Persistence analysis in a Kolmogorov-type model for cancer-immune system competition, in: *AIP Conference Proceedings, Vol. 1558*, 2013, No. 1, pp. 1797–1800. https://doi.org/10.1063/1.4825874;

[5] C. Bianca, M. Pennisi, S. Motta, M. A. Ragusa, Immune system network and cancer vaccine, *Numerical Analysis and Applied Mathematics (AIP Conference Proceedings, Vol. 1389)*, 2011, No. 1, pp. 945–948. https://doi.org/10.1063/1.3637764;

[6] V. Bhuvaneswari, L. Shangerganesh, K. Balachandran, Global existence and blow up of solutions of quasilinear chemotaxis system, *Math. Methods Appl. Sci.* **38**(2015), No. 17, 3738–3746. https://doi.org/10.1002/mma.3313; MR3434319;

[7] M. A. J. Chaplain, G. Lolas, Mathematical modelling of cancer cell invasion of tissue: The role of the urokinase plasminogen activation system, *Math. Model. Methods Appl. Sci.* **15**(2005), No. 11, 1685–1734. https://doi.org/10.1142/S0218202505000947; MR2180714; Zbl 1094.92039

[8] W. Chen, Y. Liu, Lower bound for the blow-up time for some nonlinear parabolic equations, *Bound. Value Probl.* **2016**, 2016:161, 6 pp. https://doi.org/10.1186/s13661-016-0669-5; MR3543237

[9] J. Ding, H. Hu, Blow-up solutions for nonlinear reaction diffusion equations under Neumann boundary conditions, *Appl. Anal.* **96**(2017), No. 4, 549–562. https://doi.org/10.1080/00036811.2016.1143933; MR3598965; Zbl 1361.35096

[10] R. A. Gatenby, E. T. Gawlinski, A reaction-diffusion model of cancer invasion, *Cancer Res.* **56**(1996), No. 24, 5745–5753.

[11] S. Kouachi, Existence of global solutions to reaction-diffusion systems with nonhomogeneous boundary conditions via a Lyapunov functional, *Electron. J. Differential Equations* **2002**, No. 88, 1–13. MR1938384; Zbl 1012.35041

[12] J. López-Gómez, R. Pardo San Gil, Coexistence in a simple food chain with diffusion, *J. Math. Biol.* **30**(1992), No. 7, 655–668. https://doi.org/10.1007/BF00173262; MR1175097; Zbl 0763.92010;

[13] M. Marras, S. Vernier-Piro, G. Viglialoro, Blow-up phenomena in chemotaxis systems with a source term, *Math. Methods Appl. Sci.* **39**(2016), No. 11, 2787–2798. https://doi.org/10.1002/mma.3728; MR3512730; Zbl 1342.35137

[14] M. Marras, S. Vernier-Piro, Blow-up time estimates in nonlocal reaction-diffusion systems under various boundary conditions, *Bound. Value Probl.* **2017**, 2017:2, 16 pp. https://doi.org/10.1186/s13661-016-0732-2; MR3591512; Zbl 1357.35061

[15] J. B. McGillen, E. A. Gaffney, N. K. Martin, P. K. Maini, A general reaction-diffusion model of acidity in cancer invasion, *J. Math. Biol.* **68**(2014), No. 5, 1199–1224. https://doi.org/10.1007/s00285-013-0665-7; MR3175202; Zbl 1288.35186

[16] M. Mizukami, H. Otsuka, T. Yokota, Global existence and boundedness in a chemotaxis-haptotaxis system with signal-dependent sensitivity, *J. Math. Anal. Appl.* **464**(2018), No. 1, 354–369. https://doi.org/10.1016/j.jmaa.2018.04.002; MR3794093; Zbl 06867351

[17] L. E. Payne, G. A. Philippin, P. W. Schaefer, Blow-up phenomena for some nonlinear parabolic problems, *Nonlinear Anal.* **69**(2008), No. 10, 3495–3502. https://doi.org/10.1016/j.na.2007.09.035; MR2450554;

[18] L. E. Payne, G. A. Philippin, S. Vernier-Piro, Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition II, *Nonlinear Anal.* **73**(2010), No. 4, 971–978. https://doi.org/10.1016/j.na.2010.04.023; MR2653764; Zbl 1198.35131

[19] L. E. Payne, G. A. Philippin, Blow-up phenomena in parabolic problems with time dependent coefficients under Dirichlet boundary conditions, *Proc. Amer. Math. Soc.* **141**(2013), No. 7, 2309–2318. https://doi.org/10.1090/S0002-9939-2013-11493-0; MR3043012; Zbl 1277.35075

[20] L. E. Payne, P. W. Schaefer, Lower bounds for blow-up time in parabolic problems under Neumann conditions, *Appl. Anal.* **85**(2006), No. 10, 1301–1311. https://doi.org/10.1080/00036810600915730; MR2263927;

[21] L. E. Payne, P. W. Schaefer, Lower bounds for blow-up time in parabolic problems under Dirichlet conditions, *J. Math. Anal. Appl.* **328**(2007), No. 2, 1196–1205. https://doi.org/10.1016/j.jmaa.2006.06.015;

[22] L. E. Payne, P. W. Schaefer, Blow-up phenomena for some nonlinear parabolic systems, *Int. J. Pure Appl. Math.* **48**(2008), No. 2, 193–202. https://doi.org/10.4236/am.2012.34049; MR2463922;

[23] L. E. Payne, J. C. Song, Lower bounds for blow-up time in a nonlinear parabolic problem, *J. Math. Anal. Appl.* **354**(2009), No. 1, 394–396. https://doi.org/10.1016/j.jmaa.2009.01.010; MR2510449;

[24] L. Preziosi, *Cancer modelling and simulation*, Mathematical and Computational Biology, Chapman & Hall/CRC Press, New York, 2003. MR2005054;

[25] I. Ramis-Conde, M. A. J. Chaplain, A. R. A. Anderson, Mathematical modelling of cancer cell invasion of tissue, *Math. Comp. Model.* **47**(2008), No. 5–6, 533–545. https://doi.org/10.1016/j.mcm.2007.02.034; MR2396789; Zbl 1148.92021

[26] G. Sashikumaar, L. Shangerganesh, A biophysical model of tumor invasion, *Commun. Nonlinear Sci. Numer. Simul.* **46**(2017), 135–152. https://doi.org/10.1016/j.cnsns.2016.10.013; MR3573862;

[27] L. Shangerganesh, V. N. Deiva Mani, S. Karthikeyan, Existence of global weak solutions for cancer invasion parabolic system with nonlinear diffusion, *Commun. Appl. Anal.* **21**(2017), No. 4, 607–629. https://doi.org/10.12732/caa.v21i4.8;

[28] T. L. Stepien, E. M. Rutter, Y. Kuang, A data-motivated density-dependent diffusion model of in vitro glioblastoma growth, *Math. Biosci. Eng.* **12**(2015), No. 6, 1157–1172. https://doi.org/10.3934/mbe.2015.12.1157; MR3394082; Zbl 1326.92043

[29] G. Tang, Y. Li, X. Yang, Lower bounds for the blow-up time of the nonlinear non-local reaction diffusion problems in $\mathbb{R}^N (N \geq 3)$, *Bound. Value Probl.* **2014**, 2014:265, 5 pp. https://doi.org/10.1186/s13661-014-0265-5; MR3292578; Zbl 06416617

[30] Y. Tao, C. Cui, A density-dependent chemotaxis-haptotaxis system modeling cancer invasion, *J. Math. Anal. Appl.* **367**(2010), No. 2, 612–624. https://doi.org/10.1016/j.jmaa.2010.02.015; MR2607285; Zbl 1186.92026