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Stable solitary waves for a class of nonlinear Schrödinger system with quadratic interaction

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Abstract. We consider the existence and orbital stability of bound state solitary waves and ground state solitary waves for a class of nonlinear Schrödinger system with quadratic interaction in $\mathbb{R}^n$ $(n = 2, 3)$. The existence of bound state and ground state solitary waves are studied by variational arguments and Concentration-compactness Lemma. In additional, we also prove the orbital stability of bound state and ground state solitary waves.

Keywords: bound (ground) state solitary waves, quadratic interaction, variational arguments.

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1 Introduction

In this paper, we consider the following system of nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u + \dfrac{1}{2m}\Delta u = \lambda v \overline{u}, & (x,t) \in \mathbb{R}^{n+1}, \\ i\partial_t v + \dfrac{1}{2M}\Delta v = \mu u^2, & (x,t) \in \mathbb{R}^{n+1}, \end{cases} \tag{1.1}$$

where u and v are complex-valued wave fields, m and M are positive constants, λ and μ are complex constants, and $\overline{u}$ is the complex conjugate of u.

Such systems have interesting applications in several branches of physics, such as in the study of interactions of waves with different polarizations [1, 11]. The Cauchy problem for System 1.1 has been studied from the point of view of small data scattering [6, 7]. In 2013, Hayashi, Ozawa and Tanaka [8] studied the well-posedness of Cauchy problem for System 1.1 with large data. In particular, System 1.1 is regarded as a non-relativistic limit of the system of nonlinear Klein–Gordon equations

$$\begin{cases} \dfrac{1}{2c^2 m}\partial_t^2 u - \dfrac{1}{2m}\Delta u + \dfrac{mc^2}{2}u = -\lambda v \overline{u}, & (x,t) \in \mathbb{R}^{n+1}, \\ \dfrac{1}{2c^2 M}\partial_t^2 v - \dfrac{1}{2M}\Delta v + \dfrac{Mc^2}{2}v = -\mu u^2, & (x,t) \in \mathbb{R}^{n+1}, \end{cases} \tag{1.2}$$

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under the mass resonance condition $M = 2m$, where c is the speed of light.

Assume $\lambda = c\overline{\mu}$, $c > 0$, $\lambda \neq 0$ and $\mu \neq 0$, we introduce new functions $(\widetilde{u}, \widetilde{v})$ defined by

$$\widetilde{u}(x,t) = \sqrt{\frac{c}{2}}|\mu| u\left(\sqrt{\frac{1}{2m}}x, t\right), \quad \widetilde{v}(x,t) = -\frac{\lambda}{2} v\left(\sqrt{\frac{1}{2m}}x, t\right),$$

and System (1.1) satisfies

$$\begin{cases} i\partial_t \widetilde{u} + \Delta \widetilde{u} = -2\widetilde{v}\overline{\widetilde{u}}, & (x,t) \in \mathbb{R}^{n+1}, \\ i\partial_t \widetilde{v} + \dfrac{m}{M}\Delta \widetilde{v} = -\widetilde{u}^2, & (x,t) \in \mathbb{R}^{n+1}, \end{cases} \tag{1.3}$$

Using the ansatz $(\widetilde{u}(x,t), \widetilde{v}(x,t)) = (e^{i\omega t}\phi(x), e^{i2\omega t}\psi(x))$, $\phi(x), \psi(x) \not\equiv 0$ with $\omega > 0$, System (1.3) becomes

$$\begin{cases} -\Delta\phi + \omega\phi = 2\phi\psi, & x \in \mathbb{R}^n, \\ -\kappa\Delta\psi + 2\omega\psi = \phi^2, & x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

where $\kappa = \frac{m}{M}$.

Let $L^p(\mathbb{R}^n)$ denote the usual Lebesgue space with the norm $|u|_p = (\int_{\mathbb{R}^n} |u|^p dx)^{\frac{1}{p}}$. The space $H^1(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n), \nabla u \in L^2(\mathbb{R}^n)\}$ with the corresponding norm $\|u\| = (\int_{\mathbb{R}^n} |\nabla u|^2 + |u|^2 dx)^{\frac{1}{2}}$, and $H^1_r(\mathbb{R}^n) := \{u \in H^1(\mathbb{R}^n); u \text{ is radially symmetric}\}$.

Recently, as $2 \leq n \leq 5$, Hayashi, Ozawa and Tanaka [8] obtained the existence of radially symmetric ground states for System (1.4) by using rearrangement method, Pohozaev identity and the Sobolev compact embedding $H^1_r(\mathbb{R}^n) \subset L^3(\mathbb{R}^n)$.

In this paper, firstly, we prove the existence of bound states for System (1.4) by using the Concentration-compactness Lemma and direct methods in the critical points theory. Secondly, we discuss the general case for System (1.4), i.e.,

$$\begin{cases} -\Delta\phi + \lambda_1\phi = 2\phi\psi, & x \in \mathbb{R}^n, \\ -\kappa\Delta\psi + \lambda_2\psi = \phi^2, & x \in \mathbb{R}^n, \end{cases} \tag{1.5}$$

where $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. By using the Concentration-compactness Lemma, variational arguments and rearrangement result of Shibata [13], we obtain the existence of ground states for System (1.5). In particular, if $\lambda_1 = \frac{1}{2}\lambda_2 > 0$, then System (1.5) can be reduced to System (1.4) and the existence of ground states for System (1.4) is obtained in [8]. Furthermore, we also prove the orbital stability of bound states and ground states.

Remark 1.1. In contrast to results in [8], we obtain the existence of bound states in the whole space $H^1(\mathbb{R}^n)$. Since the embedding $H^1(\mathbb{R}^n) \subset L^3(\mathbb{R}^n)$ is only continuous, we apply the Concentration-compactness Lemma and variational arguments to obtain the existence of bound states.

2 Preliminaries and main results

In this section, we state our main results in this paper.

Now, we define the functionals I, J and $Q : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{R}$ by

$$I(\phi,\psi) = \frac{1}{2}\int_{\mathbb{R}^n} (|\nabla\phi|^2 + \kappa|\nabla\psi|^2)dx - \int_{\mathbb{R}^n} \phi^2\psi dx, \qquad \forall(\phi,\psi) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n),$$

$$J(\phi,\psi) = \frac{1}{2}\int_{\mathbb{R}^n} (|\nabla\phi|^2 + \kappa|\nabla\psi|^2)dx, \qquad \forall(\phi,\psi) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n),$$

and

$$Q(\phi,\psi)=\frac{\omega}{2}\left(\int_{\mathbb{R}^n}|\phi|^2dx+2\int_{\mathbb{R}^n}|\psi|^2dx\right),\qquad \forall(\phi,\psi)\in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n).$$

It is obvious that I, J and $Q\in C^1(H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n),\mathbb{R})$. Hence, (ϕ,ψ) is a weak solution of System (1.4) if and only if (ϕ,ψ) is a critical point of the functional $S:=I+Q$.

Let $M_N=\{(\phi,\psi)\in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n): Q(\phi,\psi)=N,\ |\phi|_2^2,|\psi|_2^2>0\}$ for some $N>0$, and the minimizing problem

$$I_N=\inf\{I(\phi,\psi);(\phi,\psi)\in M_N\}. \tag{2.1}$$

Besides, for every $N>0$, let P_N denote the set of bound states of System (1.4), that is,

$$P_N=\{(\phi,\psi)\in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n); I(\phi,\psi)=I_N \text{ and } (\phi,\psi)\in M_N\},$$

which generates the solitary waves of System (1.1).

Theorem 2.1. *Let $n=2,3$. Then we have:*

(1) For all $N>0$, there exists $(\phi_N,\psi_N)\in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ a solution of

$$\begin{aligned}(\phi_N,\psi_N)\in M_N,\\ I(\phi_N,\psi_N)=\min\{I(\phi,\psi);(\phi,\psi)\in M_N\}.\end{aligned} \tag{2.2}$$

(2) If (ϕ_N,ψ_N) is a solution of the minimizing problem (2.2), then there exists a Lagrange multiplier $\sigma_N>0$ such that

$$\begin{cases}-\Delta\phi+\sigma_N\omega\phi=2\phi\psi, & x\in\mathbb{R}^n,\\ -\kappa\Delta\psi+2\sigma_N\omega\psi=\phi^2, & x\in\mathbb{R}^n,\end{cases} \tag{2.3}$$

where σ_N is given by

$$\sigma_N=\frac{\frac{2}{n}J(\phi_N,\psi_N)-I_N}{N}. \tag{2.4}$$

(3) The set

$$\Sigma:=\{(N,\sigma_N);N>0,\sigma_N \text{ is a Lagrange multiplier of the minimizing problem (2.2)}\}$$

is a closed graph in $(0,+\infty)\times(0,+\infty)$. In particular, if Σ is a function, then it is continuous and there exists $N_0>0$ such that $\sigma_{N_0}=1$. So, (ϕ_{N_0},ψ_{N_0}) is a bound state of System (1.4).

Next, we define the set

$$M_{\alpha,\beta}=\{(\phi,\psi)\in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n):\ |\phi|_2^2=\alpha,\ |\psi|_2^2=\beta\}$$

for any $\alpha,\beta>0$, and the minimizing problem

$$I_{\alpha,\beta}=\inf\{I(\phi,\psi);(\phi,\psi)\in M_{\alpha,\beta}\}.$$

Besides, for any $\alpha,\beta>0$, let

$$G_{\alpha,\beta}=\{(\phi,\psi)\in M_{\alpha,\beta}; I(\phi,\psi)=I_{\alpha,\beta}\},$$

which denotes the set of ground states of System (1.5).

Theorem 2.2.

(1) *For any $\alpha, \beta > 0$, any minimizing sequence $\{(\phi_n, \psi_n)\}_{n\geq 1} \subset H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ with respect to $I_{\alpha,\beta}$ is pre-compact. That is, taking a subsequence, there exist $(\phi, \psi) \in M_{\alpha,\beta}$ and $\{y_n\}_{n\geq 1} \subset \mathbb{R}^n$ such that $\phi_n(\cdot - y_n) \to \phi$, $\psi_n(\cdot - y_n) \to \psi$ in $H^1(\mathbb{R}^n)$ as $n \to \infty$.*

(2) *Let (λ_1, λ_2) be the Lagrange multiplier associated with (ϕ, ψ) on $M_{\alpha,\beta}$, we have $\lambda_1 > 0$.*

(3) *If $(\phi, \psi) \in G_{\alpha,\beta}$, we have $(|\phi|, |\psi|) \in G_{\alpha,\beta}$. One also has $(\phi^*, \psi^*) \in G_{\alpha,\beta}$ whenever $(\phi, \psi) \in G_{\alpha,\beta}$ and $\phi^*, \psi^* > 0$, where f^* represents the symmetric decreasing rearrangement of the function f.*

Definition 2.3. For any $N > 0$, the set P_N is stable if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $(\phi_0, \psi_0) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ verifies

$$\inf_{(\phi_N,\psi_N)\in P_N} \|(\phi_0, \psi_0) - (\phi_N, \psi_N)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} < \delta(\varepsilon),$$

then the solution $(\phi(t), \psi(t))$ of the System (1.1) with $\phi(0) = \phi_0,\ \psi(0) = \psi_0$ satisfies

$$\sup_{t\in\mathbb{R}} \inf_{(\phi_N,\psi_N)\in P_N} \|(\phi(t), \psi(t)) - (\phi_N, \psi_N)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} < \varepsilon.$$

Besides, we can also define the set $G_{\alpha,\beta}$ is stable in the same way.

Theorem 2.4. *Let $n = 2, 3$, the sets P_N and $G_{\alpha,\beta}$ are stable.*

Now, we recall the rearrangement results of Shibata [13] as presented in [9]. Let u be a Borel measureable function on $\mathbb{R}^n$. Then u is said to vanish at infinity if $|\{x \in \mathbb{R}^n; |u(x)| > s\}| < \infty$ for every $s > 0$. Here $|\cdot|$ stands for the n-dimensional Lebesgue measure. Considering two Borel functions u, v which vanish at infinity in $\mathbb{R}^n$, we define for $s > 0$, set $A^\star(u, v; s) := \{x \in \mathbb{R}^n; |x| < r\}$ where $r \geq 0$ is chosen so that

$$|B_r(0)| = |\{x \in \mathbb{R}^n; |u(x)| > s\}| + |x \in \mathbb{R}^n; |v(x)| > s\}|,$$

and $\{u, v\}^\star$ by

$$\{u, v\}^\star(x) := \int_0^\infty \chi_{A^\star(u,v;s)}(x) ds,$$

where $\chi_A(x)$ is a characteristic function of the set $A \subset \mathbb{R}^n$.

Lemma 2.5 ([9, Lemma A.1])**.**

(1) *The function $\{u, v\}^\star(x)$ is radially symmetric, non-increasing and lower semi-continuous. Moreover, for each $s > 0$ there holds $\{x \in \mathbb{R}^n; \{u, v\}^\star > s\} = A^\star(u, v; s)$.*

(2) *Let $\Phi : [0, \infty) \to [0, \infty)$ be non-decreasing, lower semi-continuous, continuous at 0 and $\Phi(0) = 0$. Then $\{\Phi(u), \Phi(v)\}^\star = \Phi(\{u, v\}^\star)$.*

(3) *$|\{u, v\}^\star|_p^p = |u|_p^p + |v|_p^p$ for $1 \leq p < \infty$.*

(4) *If $u, v \in H^1(\mathbb{R}^n)$, then $\{u, v\}^\star \in H^1(\mathbb{R}^n)$ and $|\nabla\{u, v\}^\star|_2^2 \leq |\nabla u|_2^2 + |\nabla v|_2^2$. In addition, if $u, v \in (H^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n))\backslash\{0\}$ are radially symmetric, positive and non-increasing, then we have*

$$\int_{\mathbb{R}^n} |\nabla\{u, v\}^\star|^2\, dx < \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} |\nabla v|^2 dx.$$

(5) *Let $u_1, u_2, v_1, v_2 \geq 0$ be Borel measurable functions which vanish at infinity, then we have*

$$\int_{\mathbb{R}^n} (u_1 u_2 + v_1 v_2) dx \leq \int_{\mathbb{R}^n} \{u_1, v_1\}^\star \{u_2, v_2\}^\star dx.$$

3 Bound states

Let $\{(\phi_n,\psi_n)\}_{n\geq 1}$ be a minimizing sequence for the minimizing problem (2.1), that is, the sequence $\{(\phi_n,\psi_n)\}_{n\geq 1}\in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ satisfies $Q(\phi_n,\psi_n)\to N$ and $I(\phi_n,\psi_n)\to I_N$, as $n\to\infty$. Then, we have

Lemma 3.1. *As $n=2,3$, there exists $B>0$ such that $\|(\phi_n,\psi_n)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}\leq B$ for all n, and the functional I is bounded below on M_N.*

Proof. By the Gagliardo–Nirenberg inequality, we have

$$\left(\int_{\mathbb{R}^n}|\phi|^3dx\right)^{\frac{1}{3}}\leq C\left(\int_{\mathbb{R}^n}|\nabla\phi|^2dx\right)^{\frac{n}{12}}\left(\int_{\mathbb{R}^n}|\phi|^2dx\right)^{\frac{1}{2}-\frac{n}{12}}.$$

Hence, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\phi^2\psi dx&\leq\left(\int_{\mathbb{R}^n}(\phi^2)^{\frac{3}{2}}dx\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^n}|\psi|^3dx\right)^{\frac{1}{3}}=\left(\int_{\mathbb{R}^n}|\phi|^3dx\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^n}|\psi|^3dx\right)^{\frac{1}{3}}\\&\leq C\left(\int_{\mathbb{R}^n}|\nabla\phi|^2dx\right)^{\frac{n}{6}}\left(\int_{\mathbb{R}^n}|\nabla\psi|^2dx\right)^{\frac{n}{12}}.\end{aligned}$$

Since $n=2,3$, we have $\frac{n}{6}+\frac{n}{12}<1$. Thus, I is coercive and in particular $I_N>-\infty$. By the coerciveness of I on M_N, the sequence $\{(\phi_n,\psi_n)\}_{n\geq 1}$ is bounded in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$. Thus, there exists $B>0$ such that $\|(\phi_n,\psi_n)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}\leq B$ for all n. □

Lemma 3.2. *For any $N>0$, $I_N<0$ and I_N is continuous with respect to N.*

Proof. Let $A(\phi)=\frac{1}{2}\int_{\mathbb{R}^n}|\nabla\phi|^2dx$, $B(\psi)=\frac{\kappa}{2}\int_{\mathbb{R}^n}|\nabla\psi|^2dx$, and $C(\phi,\psi)=\int_{\mathbb{R}^n}\phi^2\psi dx$, hence,

$$I(\phi,\psi)=A(\phi)+B(\psi)-C(\phi,\psi).$$

Now let $(\phi(x),\psi(x))\in M_N$ be fixed. For any $b>0$, we define $\phi_\theta(x)=\theta^{\frac{bn}{2}}\phi(\theta^bx)$, $\psi_\theta(x)=\theta^{\frac{bn}{2}}\psi(\theta^bx)$, then $(\phi_\theta(x),\psi_\theta(x))\in M_N$ as well. We have the following scaling laws:

$$\begin{aligned}A(\phi_\theta(x))&=\frac{1}{2}\int_{\mathbb{R}^n}|\theta^{\frac{bn}{2}}\nabla\phi(\theta^bx)|^2dx=\theta^{2b}A(\phi(x)),\\B(\psi_\theta(x))&=\frac{\kappa}{2}\int_{\mathbb{R}^n}|\theta^{\frac{bn}{2}}\nabla\psi(\theta^bx)|^2dx=\theta^{2b}B(\psi(x)),\end{aligned}$$

and

$$C(\phi_\theta(x),\psi_\theta(x))=\int_{\mathbb{R}^n}\theta^{bn}\phi^2(\theta^bx)\theta^{\frac{bn}{2}}\psi(\theta^bx)dx=\theta^{\frac{bn}{2}}C(\phi(x),\psi(x)).$$

So, we get

$$I(\phi_\theta(x),\psi_\theta(x))=\theta^{2b}A+\theta^{2b}B-\theta^{\frac{bn}{2}}C.$$

Since $n=2,3$, we have $\frac{bn}{2}<2b$. Letting $\theta\to 0$, then $I(\phi_\theta(x),\psi_\theta(x))\to 0^-$. Hence, we prove $I_N<0$.

In order to prove that I_N is a continuous function, we assume $N_n=N+o(1)$. From the definition of I_{N_n}, for any $\varepsilon>0$, there exists $(\phi_n,\psi_n)\in M_{N_n}$ such that

$$I(\phi_n,\psi_n)\leq I_{N_n}+\varepsilon. \tag{3.1}$$

Setting

$$(u_n, v_n) := \left(\sqrt{\frac{N}{N_n}} \phi_n, \sqrt{\frac{N}{N_n}} \psi_n \right),$$

we have that $(u_n, v_n) \in M_N$ and

$$I_N \leq I(u_n, v_n) = I(\phi_n, \psi_n) + o(1). \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$I_N \leq I_{N_n} + \varepsilon + o(1).$$

Reversing the argument, we obtain similarly that

$$I_{N_n} \leq I_N + \varepsilon + o(1).$$

Therefore, since $\varepsilon > 0$ is arbitrary, we deduce that $I_{N_n} = I_N + o(1)$. □

Lemma 3.3. *$\frac{I_N}{N}$ is decreasing in $(0, +\infty)$.*

Proof. For $(\phi, \psi) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, we define $(\phi_\theta(x), \psi_\theta(x)) := (\theta^b \phi(\theta^a x), \theta^b \psi(\theta^a x))$, $\forall \theta > 0$. Choosing $a, b > 0$, such that $2b - na = 1$, it follows that $Q(\phi_\theta(x), \psi_\theta(x)) = \theta Q(\phi, \psi)$ and we can write

$$I(\phi_\theta(x), \psi_\theta(x)) = \theta^{2a+1} I(\phi, \psi) + \theta^{2a+1} \int_{\mathbb{R}^n} \phi^2 \psi dx - \theta^{b+1} \int_{\mathbb{R}^n} \phi^2 \psi dx. \tag{3.3}$$

We can choose $a, b > 0$ such that $2b - na = 1$, $b > 2a$ and it follows from (3.3) that

$$I(\phi_\theta(x), \psi_\theta(x)) < \theta^{2a+1} I(\phi, \psi), \qquad \forall \theta > 1.$$

Since $(\phi(x), \psi(x)) \in M_N \Leftrightarrow (\phi_\theta(x), \psi_\theta(x)) \in M_{\theta N}$, $\forall \theta, N > 0$, it follows that

$$I_{\theta N} < \theta^{2a+1} I_N < \theta I_N, \qquad \forall \theta > 1.$$

Thus,

$$\frac{I_{\theta N}}{\theta N} < \frac{I_N}{N}, \qquad \forall \theta > 1.$$

□

Lemma 3.4. *For any $N > 0$ and $\lambda \in (0, N)$, we have $I_N < I_\lambda + I_{N-\lambda}$.*

Proof. Thanks to the following well-known inequality: $\forall a,\ b,\ A,\ B > 0$,

$$\min\left\{\frac{a}{A}, \frac{b}{B}\right\} \leq \frac{a+b}{A+B} \leq \max\left\{\frac{a}{A}, \frac{b}{B}\right\},$$

where the equalities hold if and only if $\frac{a}{A} = \frac{b}{B}$, we get

$$\frac{(-I_\lambda) + (-I_{N-\lambda})}{\lambda + N - \lambda} \leq \max\left\{\frac{-I_\lambda}{\lambda}, \frac{-I_{N-\lambda}}{N-\lambda}\right\}.$$

Without loss of generality, we assume $\frac{-I_\lambda}{\lambda}$ is larger than $\frac{-I_{N-\lambda}}{N-\lambda}$, then

$$\frac{(-I_\lambda) + (-I_{N-\lambda})}{N} \leq \frac{-I_\lambda}{\lambda}.$$

By Lemma 3.3, we have

$$I_\lambda + I_{N-\lambda} \geq \frac{N}{\lambda} I_\lambda > I_N.$$

□

Proof of Theorem 2.1. Our proof is divided into five steps:

Step 1. The minimizing problem (2.2) has a solution. By Lemma 3.1, the sequence $\{(\phi_n, \psi_n)\}$ is bounded in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. If

$$\sup_{y\in\mathbb{R}^n} \int_{B_R(y)} (|\phi_n|^2 + |\psi_n|^2)dx = o(1),$$

for some $R > 0$, the $\phi_n \to 0$, $\psi_n \to 0$ in $L^p(\mathbb{R}^n)$ for $2 < p < 2^*$, see [11, 12]. This is incompatible with the fact that $I_N < 0$, see Lemma 3.2. Thus, the vanishing of minimizing sequence $\{(\phi_n, \psi_n)\}$ does not exist. Besides, Lemma 3.4 prevents their dichotomy. According to Concentration-compactness Lemma, only concentration exists, and we get a solution (ϕ_N, ψ_N) of the minimizing problem (2.2).

Step 2. There exists a positive Lagrange multiplier σ_N. Let (ϕ_N, ψ_N) a solution of the minimizing problem (2.2). From the Lagrange Multiplier Theorem, there exists $\theta \in \mathbb{R}$ such that $I'(\phi_N, \psi_N) = \theta Q'(\phi_N, \psi_N)$, that means

$$\begin{aligned} -\Delta\phi_N - 2\phi_N\psi_N &= \theta\omega\phi_N, \\ -\kappa\Delta\psi_N - \phi_N^2 &= 2\theta\omega\psi_N. \end{aligned} \tag{3.4}$$

By multiply the above equations respectively by ϕ_N, ψ_N and integrating on $\mathbb{R}^n$, we get

$$I_N - \frac{1}{2}\int_{\mathbb{R}^n} \phi_N^2\psi_N dx = \theta N. \tag{3.5}$$

Since $I_N < 0$, $\forall N > 0$, we obtain easily from (3.5) that $\theta < 0$.

For any $\lambda, c > 0$, we consider

$$(\phi_\lambda(x), \psi_\lambda(x)) := \left(\lambda^{\frac{cn}{2}}\phi_N(\lambda^c x), \lambda^{\frac{cn}{2}}\psi_N(\lambda^c x)\right),$$

then $(\phi_\lambda(x), \psi_\lambda(x)) \in M_N$ and $I(\phi_N, \psi_N) = \min_{\lambda>0} I(\phi_\lambda(x), \psi_\lambda(x))$. In particular,

$$0 = \frac{d}{d\lambda} I(\phi_\lambda(x), \psi_\lambda(x))\bigg|_{\lambda=1} = 2cJ(\phi_N, \psi_N) - \frac{cn}{2}\int_{\mathbb{R}^n} \phi_N^2\psi_N dx. \tag{3.6}$$

Merging (3.5) and (3.6), we get

$$I_N - \frac{2}{n}J(\phi_N, \psi_N) = \theta N,$$

which implies that $\theta < 0$ and the Lagrange multiplier

$$\sigma_N = -\theta = \frac{\frac{2}{n}J(\phi_N, \psi_N) - I_N}{N} > 0. \tag{3.7}$$

Step 3. There exist $\gamma(n) > 0$ such that

$$-\frac{I_N}{N} < \sigma_N < \gamma(n) - \frac{I_N}{N}. \tag{3.8}$$

Since $I(\phi_N, \psi_N) < 0$, we get from Hölder's inequality and the Gagliardo–Nirenberg inequality that

$$\begin{aligned} J(\phi_N, \psi_N) &< \int_{\mathbb{R}^n} \phi_N^2\psi_N dx \le \frac{1}{2}\left(|\phi_N|_3^4 + |\psi_N|_2^3\right) \\ &\le C\left(|\nabla\phi_N|_2^{\frac{2n}{3}}|\phi_N|_2^{4-\frac{2n}{3}} + |\nabla\psi_N|_2^{\frac{n}{3}}|\psi_N|_2^{2-\frac{n}{3}}\right) \\ &\le C\left(J(\phi_N, \psi_N)^{\frac{n}{3}} + J(\phi_N, \psi_N)^{\frac{n}{6}}\right)\rho(N), \end{aligned} \tag{3.9}$$

where $C > 0$ and $\rho(N) := \max\left\{N^{(2-\frac{n}{3})},\ N^{(1-\frac{n}{6})}\right\}$.

Let $f : (0, \infty) \to \mathbb{R}$ the function defined by

$$f(s) := \frac{s}{s^{\frac{n}{3}} + s^{\frac{n}{6}}},$$

and we know $f'(s) > 0, \forall s > 0$ and $\lim_{s\to 0^+} f(s) = 0$. So, we can rewrite (3.9) as

$$J(\phi_N, \psi_N) < f^{-1}(C\rho(N)). \tag{3.10}$$

Note that

$$\rho(s) = s^{(1-\frac{n}{6})} \quad \text{if } s \le 1, \quad \text{and} \quad f(s) \ge \frac{1}{2}s^{(1-\frac{n}{6})} \quad \text{if } s \le 1,$$
$$\rho(s) = s^{(2-\frac{n}{3})} \quad \text{if } s \ge 1, \quad \text{and} \quad f(s) \ge \frac{1}{2}s^{(1-\frac{n}{3})} \quad \text{if } s \ge 1.$$

By a straightforward calculation we see that there exists $C_1 > 0$ such that

$$f^{-1}(C\rho(N)) \le C_1 N \quad \text{if } N \le 1,$$
$$f^{-1}(C\rho(N)) \le C_1 N^{(\frac{6-n}{3-n})} \quad \text{if } N \ge 1.$$

Hence, we obtain from (3.10) that

$$J(\phi_N, \psi_N) < C_1 N, \qquad \forall N > 0.$$

Let $\gamma(n) = \frac{2C_1}{n}$, (3.8) holds.

Step 4. Σ is closed in $(0, +\infty) \times (0, +\infty)$. For all (ϕ_N, ψ_N) solution of the minimizing problem (2.2), we define

$$\sigma(\phi_N, \psi_N) := \frac{1}{N}\left(\frac{2}{n}J(\phi_N, \psi_N) - I_N\right),$$

$$\Sigma_N := \{\sigma(\phi_N, \psi_N);\ (\phi_N, \psi_N) \text{ solution of the minimizing problem (2.2)}\}.$$

Then it is easy to see that $\Sigma = \{(N, \sigma_N); N > 0, \sigma_N \in \Sigma_N\}$.

Let $(N_n, \sigma_n) \in \Sigma$ such that $(N_n, \sigma_n) \to (N, \sigma)$, $N > 0$. By definition, there exists $(\phi_n, \psi_n) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ such that $Q(\phi_n, \psi_n) = N_n$, $I(\phi_n, \psi_n) = I_{N_n}$ and

$$\sigma_n = \frac{1}{N_n}\left(\frac{2}{n}J(\phi_n, \psi_n) - I_{N_n}\right).$$

By Lemmas 3.1 and 3.2, $\{(\phi_n, \psi_n)\}$ is bounded in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. If we define

$$(u_n, v_n) := \left(\sqrt{\frac{N}{N_n}}\phi_n, \sqrt{\frac{N}{N_n}}\psi_n\right),$$

then $\{(u_n, v_n)\}$ is also bounded in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ and $Q(u_n, v_n) = N$. By using the Concentration-compactness Lemma, there exists a subsequence satisfying only one of the following three cases: 1) concentration; 2) vanishing; 3) dichotomy.

By using the argument as in step 1, only concentration exists. Therefore, there exists $\{y_n\}_{n\ge 1} \subset \mathbb{R}^n$ and $(\phi, \psi) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ such that

$$\phi_n(\cdot - y_n) \rightharpoonup \phi, \quad \psi_n(\cdot - y_n) \rightharpoonup \psi \ \text{ weakly in } H^1(\mathbb{R}^n),$$
$$\phi_n(\cdot - y_n) \to \phi, \quad \psi_n(\cdot - y_n) \to \psi \ \text{ in } L^2(\mathbb{R}^n),$$
$$\int_{\mathbb{R}^n} \phi_n^2(\cdot - y_n)\psi_n(\cdot - y_n)dx = \int_{\mathbb{R}^n} \phi_n^2\psi_n dx \to \int_{\mathbb{R}^n} \phi^2\psi dx.$$

In particular, $Q(\phi,\psi)=N$ and $I(\phi,\psi)\geq I_N$. On the other hand,

$$I(\phi_N,\psi_N)\leq \liminf_{n\to\infty} I\left(\phi_n(\cdot-y^n),\psi_n(\cdot-y^n)\right)=\lim_{n\to\infty} I(\phi_n,\psi_n)=I_N.$$

So, $I(\phi_N,\psi_N)=I_N$ and (ϕ_N,ψ_N) is a solution of the minimizing problem (2.2). Moreover, since

$$J(\phi_n,\psi_n)=I(\phi_n,\psi_n)+\int_{\mathbb{R}^n}\phi_n^2\psi_n dx\to I(\phi_N,\psi_N)+\int_{\mathbb{R}^n}\phi_N^2\psi_N=J(\phi_N,\psi_N),$$

we conclude that

$$\sigma=\frac{1}{N}\left(\frac{2}{n}J(\phi_N,\psi_N)-I_N\right)\in\Sigma_N.$$

Step 5. If Σ is a function, then it is continuous and there exists $N_0>0$ such that $\sigma_{N_0}=1$. In particular, (ϕ_{N_0},ψ_{N_0}) is a bound state of System (1.4). This follows easily from Step 4, (3.8) and Lemma 3.3. □

4 Ground states

Lemma 4.1. *The energy $I_{\alpha,\beta}$ satisfies that*

(i) For any $\alpha,\beta>0$, $-\infty<I_{\alpha,\beta}<0$.

(ii) $I_{\alpha,\beta}$ is continuous with respect to $\alpha,\beta\geq 0$.

(iii) $I_{\alpha+\alpha',\beta+\beta'}\leq I_{\alpha,\beta}+I_{\alpha',\beta'}$ for $\alpha,\alpha',\beta,\beta'\geq 0$.

Proof. The proofs of (i) and (ii) use the same arguments as in Lemmas 3.1 and 3.2. Next, we prove (iii). Indeed, for $\varepsilon>0$, there exists $(u,v)\in M_{\alpha,\beta}\cap C_0^\infty(\mathbb{R}^n)$ and $(\phi,\psi)\in M_{\alpha',\beta'}\cap C_0^\infty(\mathbb{R}^n)$. By using parallel transformation, we can assume that $(\operatorname{supp} u\cup\operatorname{supp} v)\cap(\operatorname{supp}\phi\cup\operatorname{supp}\psi)=\emptyset$. Therefore $(u+\phi,v+\psi)\in M_{\alpha+\alpha',\beta+\beta'}$ and

$$I_{\alpha+\alpha',\beta+\beta'}\leq I(u+\phi,v+\psi)=I(u,v)+I(\phi,\psi)\leq I_{\alpha,\beta}+I_{\alpha',\beta'}+2\varepsilon.$$

Since $\varepsilon>0$ is arbitrarily, it asserts (iii). □

Lemma 4.2. *For any minimizing sequence $\{(\phi_n,\psi_n)\}_{n\geq 1}$ of $I_{\alpha,\beta}$, if $(\phi_n,\psi_n)\rightharpoonup(\phi,\psi)$ weakly in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n}\phi_n^2\psi_n-(\phi_n-\phi)^2(\psi_n-\psi)dx=\int_{\mathbb{R}^n}\phi^2\psi dx+o(1).$$

Proof. The idea of its proof comes from [5] (see also Lemma 2.3 of [4]). For any a_1, a_2, b_1, $b_2\in\mathbb{R}$ and $\varepsilon>0$, we deduce from the mean value theorem and Young's inequality that

$$|(a_1+a_2)^2(b_1+b_2)-a_1^2b_1|\leq C\varepsilon(|a_1|^3+|a_2|^3+|b_1|^3+|b_2|^3)+C_\varepsilon(|a_2|^3+|b_2|^3).$$

Denote $a_1:=\phi_n-\phi$, $b_1:=\psi_n-\psi$, $a_2:=\phi$, $b_2:=\psi$. Then

$$\begin{aligned}f_n^\varepsilon &:= \left[|\phi_n^2\psi_n-(\phi_n-\phi)^2(\psi_n-\psi)-\phi^2\psi|-C\varepsilon(|\phi_n-\phi|^3+|\phi|^3+|\psi_n-\psi|^3+|\psi|^3|)\right]_+\\ &\leq|\phi^2\psi|+C_\varepsilon(|\phi|^3+|\psi|^3),\end{aligned}$$

and the dominated convergence theorem yields

$$\int_{\mathbb{R}^n}f_n^\varepsilon dx\to 0,\quad \text{as } n\to\infty. \tag{4.1}$$

Since

$$|\phi_n^2\psi_n - (\phi_n-\phi)^2(\psi_n-\psi) - \phi^2\psi| \le f_n^\varepsilon + C\varepsilon(|\phi_n-\phi|^3 + |\psi_n-\psi|^3 + |\phi|^3 + |\psi|^3|),$$

by the boundedness of $\{(\phi_n,\psi_n)\}_{n\ge1}$ in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ and (4.1), it follows that

$$\int_{\mathbb{R}^n} \phi_n^2\psi_n - (\phi_n-\phi)^2(\psi_n-\psi)dx = \int_{\mathbb{R}^n} \phi^2\psi dx + o(1). \qquad \square$$

Lemma 4.3. *Any minimizing sequence $\{(\phi_n,\psi_n)\}_{n\ge1} \subset H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ with respect to $I_{\alpha,\beta}$ is, up to translation, strongly convergent in $L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n)$ for $2<p<2^*$.*

Proof. Similar to the Step 1 of the proof of Theorem 2.1, we can know that there exists a $\beta_0>0$ and a sequence $\{y_n\}\subset\mathbb{R}^n$ such that

$$\sup_{y\in\mathbb{R}^n}\int_{B_R(y_n)}(|\phi_n|^2+|\psi_n|^2)dx \ge \beta_0 > 0,$$

and we deduce from the weak convergence in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ and the local compactness in $L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n)$ that $(\phi_n(x-y_n),\psi_n(x-y_n)) \rightharpoonup (\phi,\psi)\ne(0,0)$ weakly in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$. In order to prove that $u_n(x):=\phi_n(x)-\phi(x+y_n)\to 0$, $v_n(x):=\psi_n(x)-\psi(x+y_n)\to 0$ in $L^p(\mathbb{R}^n)$ for $2<p<2^*$, we suppose that there exists a $2<q<2^*$ such that $(u_n,v_n)\nrightarrow(0,0)$ in $L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n)$. Note that under this assumption by contradiction there exists a sequence $\{z_n\}\subset\mathbb{R}^n$ such that

$$(u_n(x-z_n),v_n(x-z_n)) \rightharpoonup (u,v)\ne(0,0)$$

weakly in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$.

Now, combining the Brézis–Lieb Lemma ([10]), Lemma 4.2 and the translational invariance, we conclude

$$\begin{aligned} I(\phi_n,\psi_n) &= I(u_n(x-y_n),v_n(x-y_n)) + I(\phi,\psi) + o(1) \\ &= I(u_n(x-z_n)-u, v_n(x-z_n)-v) + I(u,v) + I(\phi,\psi) + o(1), \end{aligned} \tag{4.2}$$

$$|\phi_n(x-y_n)|_2^2 = |u_n(x-z_n)-u|_2^2 + |u|_2^2 + |\phi|_2^2 + o(1),$$

and

$$|\psi_n(x-y_n)|_2^2 = |v_n(x-z_n)-v|_2^2 + |v|_2^2 + |\psi|_2^2 + o(1).$$

Let $\alpha' := \alpha - |u|_2^2 - |\phi|_2^2$, $\beta' := \alpha - |v|_2^2 - |\psi|_2^2$, then

$$|u_n(x-z_n)-u|_2^2 = \alpha' + o(1), \qquad |v_n(x-z_n)-v|_2^2 = \beta' + o(1). \tag{4.3}$$

Noting that

$$|u|_2^2 \le \liminf_{n\to\infty}|u_n(x-z_n)|_2^2 = \liminf_{n\to\infty}|\phi_n(x-y_n)-\phi|_2^2 = \alpha - |\phi|_2^2,$$

then $\alpha'\ge0$. Similarly, $\beta'\ge0$. Recording that $I(\phi_n,\psi_n)\to I_{\alpha,\beta}$, in consideration of (4.3), Lemma 4.1 (ii) and (4.2), we get

$$I_{\alpha,\beta} \ge I_{\alpha',\beta'} + I(u,v) + I(\phi,\psi). \tag{4.4}$$

We know from the front that $(\phi,\psi)\ne(0,0)$ and $(u,v)\ne(0,0)$. As for ϕ, ψ, u, v, if one of them is identically zero, we have

$$I_{\alpha,\beta} \ge I_{\alpha',\beta'} + I(u,v) + I(\phi,\psi) > I_{\alpha',\beta'} + I_{|u|_2^2,|v|_2^2} + I_{|\phi|_2^2,|\psi|_2^2} \ge I_{\alpha,\beta},$$

which is impossible. So, ϕ, ψ, u, $v \not\equiv 0$. If $I(u,v) > I_{|u|_2^2,|v|_2^2}$ or $I(\phi,\psi) > I_{|\phi|_2^2,|\psi|_2^2}$, we also have a contradiction. Hence $I(u,v) = I_{|u|_2^2,|v|_2^2}$ and $I(\phi,\psi) = I_{|\phi|_2^2,|\psi|_2^2}$. We denote by ϕ^*, ψ^*, u^*, v^* the classical Schwarz symmetric-decreasing rearrangement of ϕ, ψ, u, v. Since

$$|\phi^*|_2^2 = |\phi|_2^2, \qquad |\psi^*|_2^2 = |\psi|_2^2, \qquad |u^*|_2^2 = |u|_2^2, \ |v^*|_2^2 = |v|_2^2,$$
$$I(\phi^*,\psi^*) \le I(\phi,\psi), \qquad I(u^*,v^*) \le I(u,v)$$

see [10], we conclude that

$$I(\phi^*,\psi^*) = I_{|\phi|_2^2,|\psi|_2^2}, \qquad I(u^*,v^*) = I_{|u|_2^2,|v|_2^2}.$$

Therefore, (ϕ^*,ψ^*), (u^*,v^*) are solutions of the System (1.1) and from standard regularity results we have that ϕ^*, ψ^*, u^*, $v^* \in C^2(\mathbb{R}^n)$.

By Lemma 2.5, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\nabla \{\phi^*,u^*\}^\star\right|^2 dx &< \int_{\mathbb{R}^n} \left(|\nabla\phi^*|^2 + |\nabla u^*|^2\right) dx \le \int_{\mathbb{R}^n} \left(|\nabla\phi|^2 + |\nabla u|^2\right) dx,\\ \int_{\mathbb{R}^n} \left|\nabla \{\psi^*,v^*\}^\star\right|^2 dx &< \int_{\mathbb{R}^n} \left(|\nabla\psi^*|^2 + |\nabla v^*|^2\right) dx \le \int_{\mathbb{R}^n} \left(|\nabla\psi|^2 + |\nabla v|^2\right) dx,\end{aligned}$$

and

$$\int_{\mathbb{R}^n} \left(\{\phi^*,u^*\}^\star\right)^2 \{\psi^*,v^*\}^\star dx \ge \int_{\mathbb{R}^n} \left((\phi^*)^2 \psi^* + (u^*)^2 v^*\right) dx \ge \int_{\mathbb{R}^n} \left(\phi^2\psi + u^2 v\right) dx.$$

Thus,

$$I(\phi,\psi) + I(u,v) > I\left(\{\phi^*,u^*\}^\star, \{\psi^*,v^*\}^\star\right), \tag{4.5}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\{\phi^*,u^*\}^\star\right|^2 dx &= \int_{\mathbb{R}^n} \left(|\phi^*|^2 + |u^*|^2\right) dx = \int_{\mathbb{R}^n} \left(|\phi|^2 + |u|^2\right) dx = \alpha - \alpha',\\ \int_{\mathbb{R}^n} \left|\{\psi^*,v^*\}^\star\right|^2 dx &= \int_{\mathbb{R}^n} \left(|\psi^*|^2 + |v^*|^2\right) dx = \int_{\mathbb{R}^n} \left(|\psi|^2 + |v|^2\right) dx = \beta - \beta'.\end{aligned} \tag{4.6}$$

Taking (4.4)–(4.6) and Lemma 4.1 (iii) into consideration, one obtains the contradiction

$$I_{\alpha,\beta} > I_{\alpha',\beta'} + I_{\alpha-\alpha',\beta-\beta'} \ge I_{\alpha,\beta}.$$

The contradiction indicates that $u_n(x) := \phi_n(x) - \phi(x+y_n) \to 0$ and $v_n(x) := \psi_n(x) - \psi(x+y_n) \to 0$ in $L^p(\mathbb{R}^n)$ for $2 < p < 2^*$. □

Proof of Theorem 2.2. (1) Let $\{(\phi_n,\psi_n)\}$ be a minimizing sequence for the functional I on $M_{\alpha,\beta}$. In light of Lemma 4.3, we know that there exists $\{y_n\} \subset \mathbb{R}^n$ such that $\phi_n(x-y_n) \to \phi$, $\psi_n(x-y_n) \to \psi$ in $L^p(\mathbb{R}^n)$ for $2 < p < 2^*$. Hence, by weak convergence, we get

$$I(\phi,\psi) \le I_{\alpha,\beta}. \tag{4.7}$$

Now, we let $|\phi|_2^2 = \alpha'$, $|\psi|_2^2 = \beta'$. To show that $|\phi|_2^2 = \alpha$ and $|\psi|_2^2 = \beta$, we assume by contradiction that $\alpha' < \alpha$ or $\beta' < \beta$. We consider the following three cases: (1) $0 \le \alpha' < \alpha$, $0 \le \beta' < \beta$ and $\alpha' + \beta' \ne 0$; (2) $0 \le \alpha' < \alpha$, $\beta' = \beta$; and (3) $0 \le \beta' < \beta$, $\alpha' = \alpha$.

Case 1. $0 \le \alpha' < \alpha$, $0 \le \beta' < \beta$ and $\alpha' + \beta' \ne 0$. By definition $I(\phi,\psi) \ge I_{\alpha',\beta'}$ and thus it results from (4.7) that $I_{\alpha',\beta'} \le I_{\alpha,\beta}$. From Lemma 4.1 (iii), $I_{\alpha,\beta} \le I_{\alpha',\beta'} + I_{\alpha-\alpha',\beta-\beta'}$ and by Lemma 4.1 (i), $I_{\alpha-\alpha',\beta-\beta'} < 0$, we obtain $I_{\alpha,\beta} < I_{\alpha',\beta'}$ and it is a contradiction.

Case 2. $0 \le \alpha' < \alpha$, $\beta' = \beta$. By definition $I(\phi,\psi) \ge I_{\alpha',\beta}$, we get $I_{\alpha',\beta} \le I_{\alpha,\beta}$. From Lemma 4.1 (iii) $I_{\alpha,\beta} \le I_{\alpha',\beta} + I_{\alpha-\alpha',0}$, we have $I_{\alpha',\beta} \le I_{\alpha,\beta} \le I_{\alpha',\beta}$. Thus $I_{\alpha',\beta} = I_{\alpha,\beta}$. Let $|\psi|_2^2 = \beta$, and β is fixed. From the above, we know that $N = \frac{\omega}{2}(|\phi|_2^2 + 2\beta)$, then N is only related to $|\phi|_2^2$. By Lemma 3.3, $\frac{I_{N(|\phi|_2^2)}}{N(|\phi|_2^2)}$ is decreasing in $(0,+\infty)$, when $|\phi|_2^2$ gradually increases. If $|\phi|_2^2 = \alpha'$, we have $I_{N(\alpha')} = I_{\alpha',\beta}$. Similarly, $I_{N(\alpha)} = I_{\alpha,\beta}$. Since $\frac{I_{N(\alpha')}}{N(\alpha')} > \frac{I_{N(\alpha)}}{N(\alpha)}$, we have $I_{N(\alpha')} > \frac{N(\alpha)}{N(\alpha')} I_{N(\alpha')} > I_{N(\alpha)}$. So, we obtain that $I_{\alpha',\beta} > I_{\alpha,\beta}$, and it is a contradiction. As for the case (3), we can prove by the same argument.

Now we have $u_n(x) = \phi_n(x) - \phi(x+y_n) \to 0$, $v_n(x) = \psi_n(x) - \psi(x+y_n) \to 0$ in $L^2(\mathbb{R}^n)$. By using the P.-L. Lions Lemma, $u_n(x), v_n(x) \to 0$ in $L^3(\mathbb{R}^n)$. According to Hölder inequality, we have $\left|\int_{\mathbb{R}^n} u_n^2 v_n dx\right| \le |u_n|_3^2 |v_n|_3$. Hence $\int_{\mathbb{R}^n} u_n^2 v_n dx \to 0$. By the Brézis–Lieb Lemma,

$$\begin{aligned} I(\phi_n,\psi_n) &= I(\phi,\psi) + I(u_n,v_n) + o(1) \\ &= I_{\alpha,\beta} + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_n|^2 + \kappa|\nabla v_n|^2 dx + o(1) \quad \text{as } n \to \infty. \end{aligned}$$

Taking $n \to \infty$, we obtain $\lim_{n\to\infty} \int_{\mathbb{R}^n} |\nabla u_n|^2 + \kappa|\nabla v_n|^2 dx = 0$. Thus we get $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = 0$ in $H^1(\mathbb{R}^n)$.

(2) Let $(\phi,\psi) \in G_{\alpha,\beta}$ for any $\alpha,\beta > 0$. By the Lagrange multiplier method, there exists a pair $(\lambda_1,\lambda_2) \in \mathbb{R}^2$ such that $(\lambda_1,\lambda_2,\phi,\psi)$ satisfies System (1.5). By multiply the first equation of (1.5) by ϕ, we get

$$\int_{\mathbb{R}^n} |\nabla\phi|^2 dx - 2\int_{\mathbb{R}^n} \phi^2\psi dx = -\lambda_1|\phi|_2^2.$$

Since $I(\phi,\psi) < 0$ (see Lemma 4.1 (i)), we get

$$\int_{\mathbb{R}^n} |\nabla\phi|^2 dx - 2\int_{\mathbb{R}^n} \phi^2\psi dx < 2I(\phi,\psi) < 0.$$

Then $\lambda_1 > 0$.

(3) Using the fact

$$|\nabla|\phi||_2 \le |\nabla\phi|_2, \qquad |\nabla|\psi||_2 \le |\nabla\psi|_2 \quad \text{and} \quad \int_{\mathbb{R}^n} |\phi|^2|\psi| dx \ge \int_{\mathbb{R}^n} \phi^2\psi dx$$

it follows that $(\phi,\psi) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \Rightarrow (|\phi|,|\psi|) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ and $I(|\phi|,|\psi|) \le I(\phi,\psi)$. Thus, $G_{\alpha,\beta}$ contains $(|\phi|,|\psi|)$ and hence, the minimizer (ϕ,ψ) can be chosen to be $\mathbb{R}$-valued.

To prove $(\phi^*,\psi^*) \in G_{\alpha,\beta}$, we need the following fact

$$|\nabla\phi^*|_2 \le |\nabla\phi|_2, \qquad |\nabla\psi^*|_2 \le |\nabla\psi|_2 \tag{4.8}$$

see [10, Theorem 7.17]. Moreover, it is well-know that the symmetric decreasing rearrangement preserves the L^p norm, that is,

$$|\phi^*|_p = |\phi|_p, \qquad |\psi^*|_p = |\psi|_p, \qquad 1 \le p \le \infty. \tag{4.9}$$

Furthermore, we have

$$\int_{\mathbb{R}^n} (\phi^*)^2\psi^* dx \ge \int_{\mathbb{R}^n} \phi^2\psi dx \tag{4.10}$$

(see for example, Theorem 3.4 of [10]). Taking into account of (4.8), (4.9) and (4.10), it follows that

$$|\phi^*|_2^2 = |\phi|_2^2, \qquad |\psi^*|_2^2 = |\psi|_2^2 \quad \text{and} \quad I(\phi^*,\psi^*) \le I(\phi,\psi), \qquad \forall(\phi,\psi) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n),$$

which shows that $G_{\alpha,\beta}$ contains (ϕ^*, ψ^*) whenever it does (ϕ, ψ).

To show that $\phi^* > 0$ on $\mathbb{R}^n$, observe that $(|\phi|, |\psi|) \in G_{\alpha,\beta}$ satisfies the Euler–Lagrange differential equations

$$\begin{cases} -\Delta|\phi| + \lambda_1|\phi| = 2|\phi||\psi|, & x \in \mathbb{R}^n, \\ -\kappa\Delta|\psi| + \lambda_2|\psi| = |\phi|^2, & x \in \mathbb{R}^n, \end{cases}$$

where (λ_1, λ_2) is the same pair of numbers as in System (1.5). Letting $f_1(|\phi|, |\psi|) = 2|\phi||\psi|$. Since $\lambda_1 > 0$, we have

$$|\phi| = G^{\sqrt{\lambda_1}}(x) * f_1(|\phi|, |\psi|) = \int_{\mathbb{R}^n} G^{\sqrt{\lambda_1}}(x-y) f_1(|\phi|, |\psi|)(y) dy,$$

where $G^\mu(x)$ is defined by

$$G^\mu(x) = \int_0^\infty (4\pi\tau)^{-\frac{n}{2}} \exp\left\{-\frac{|x|^2}{4\tau} - \mu^2\tau\right\} d\tau,$$

for $x \in \mathbb{R}^n$, $\mu > 0$. Since the function f_1 is everywhere nonnegative and not identically zero, it follows that $|\phi| > 0$. So, we obtain $\phi^* > 0$. Besides, by the maximum principle, we get $\psi^* > 0$. This concludes the proof of statement (3). □

5 Orbital stability

In this section, we proceed as in [3] to prove the orbital stability of bound state and ground state solitary waves.

Proof of Theorem 2.4. We assume that the set P_N is not stable, then there is a $\varepsilon_0 > 0$, $\{(\phi_n(0), \psi_n(0))\} \subset H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ and $\{t_n\} \subset \mathbb{R}^+$ such that

$$\inf_{(\phi_N,\psi_N)\in P_N} \|(\phi_n(0), \psi_n(0)) - (\phi_N, \psi_N)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} \to 0 \quad \text{as } n \to \infty, \tag{5.1}$$

and

$$\inf_{(\phi_N,\psi_N)\in P_N} \|(\phi_n(t_n), \psi_n(t_n)) - (\phi_N, \psi_N)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} \geq \varepsilon_0, \tag{5.2}$$

Since by the conservation laws, we have

$$|\phi_n(t_n)|_2^2 = |\phi_n(0)|_2^2, \qquad |\psi_n(t_n)|_2^2 = |\psi_n(0)|_2^2,$$

and

$$I(\phi_n(t_n), \psi_n(t_n)) = I(\phi_n(0), \psi_n(0)).$$

If we define

$$(\hat{\phi}_n, \hat{\psi}_n) = \left(\frac{\phi_n(t_n)}{|\phi_n(t_n)|_2}\sqrt{\eta}, \frac{\psi_n(t_n)}{|\psi_n(t_n)|_2}\sqrt{\frac{2N-\omega\eta}{2\omega}}\right),$$

where $0 < \eta < \frac{2N}{\omega}$, we get that

$$Q(\hat{\phi}_n, \hat{\psi}_n) = N \quad \text{and} \quad I(\hat{\phi}_n, \hat{\psi}_n) = I_N + o(1).$$

Namely $\{(\hat{\phi}_n, \hat{\psi}_n)\}$ is a minimizing sequence for the minimizing problem (2.1). From Theorem 2.1 (1), it follows that it is precompact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ thus (5.2) fails.

The proof of the orbital stability of $G_{\alpha,\beta}$ is similar to the above proof. □

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