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Light Hyperweak New Gauge Bosons From Kinetic Mixing in String Models

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Abstract

String theory is at the moment our best candidate for a unified quantum theory of gravity, aiming to reconcile all the known (and unknown) interactions with gravity as well as provide insights for currently mysterious phenomena that the Standard Model and the modern Cosmology are not able to explain. In fact, it is believed that most of the problems associated to the Standard Model can indeed be resolved in string theory. Supersymmetry is supposed to be an elegant solution to the Hierarchy problem (even though more and more stringent bounds in this direction are being placed by the fact that we have been unable to experimentally find supersymmetry yet), while all the axions that compactifications bring into play can be used to resolve the strong CP problem as well as provide good candidates for Dark Matter. Inflationary models can also be constructed in string theory, providing, then, the most diffused solution to the Horizon problem. This work, in particular, is formulated in type IIB string theory compactified on an orientifolded Calabi-Yau three-fold in LARGE Volume Scenario (LVS) and focuses on the stabilisation of all the moduli in play compatible with the construction of a hidden gauge sector whose gauge boson kinetically mixes to the visible sector U(1), acquiring a mass via a completely stringy process resulting in the Stückelberg mechanism. The "compatibility" regards the fact that certain experimental bounds should be respected combined with recent data extrapolated by Coherent Elastic Neutrino-Nucleus Scattering (CEνNS) events at the Spallation Neutron Source at Oak Ridge National Laboratory. We are going to see that in this context we will be able to fix all the moduli as well as present a brane and fluxes set-up reproducing the correct mass and coupling of the hidden gauge boson. We also get a TeV scale supersymmetry, since the gravitino in this model will be of order O(TeV), with an uplifted vacuum to reproduce a de Sitter universe as well.

Abstract (in Italiano)

La Teoria delle Stringhe al momento rappresenta la nostra migliore canditata per una teoria unificata della gravità quantistica, cerca infatti di riconciliare tutte le interazioni conosciute (e non) con la gravità, così come fornire qualche intuizione per fenomeni al momento misteriosi che il Modello Standard e la Cosmologia moderna non sono in grado di spiegare. Si pensa infatti che la maggior parte dei problemi del Modello Standard possano essere risolti nel contesto della teoria delle stringhe. La supersimmetria dovrebbe essere una soluzione elegante al problema della gerarchia (anche se sempre più stringenti vincoli sperimentali in questa direzione si stanno venendo a fissare dato che ad oggi la supersimmetria non è ancora stata verificata sperimentalmente), mentre gli assioni che compaioni a seguito di compattificazioni possono essere usati per risolvere il problema della CP forte, così come fornire dei possibili candidati per la Materia Oscura. Modelli inflazionari possono essere costruiti in teoria della stringhe, fornenddo in questo modo la più diffusa soluzione al problema dell'orizzonte. Il seguente lavoro, in particolare, è formulato nel contesto di teoria delle stringhe di tipo IIB compattificata su un orientifolded Calabi-Yau three-fold in "Large Volume Scenario" (LVS) e si focalizza sulla stabilizzazione di tutti i moduli presenti in modo compatibile con la costruzione di un settore di gauge nascosto, il quale bosone di gauge si mescola cinematicamente con il settore visible U(1), acquistando una massa tramite un processo completamente frutto della teoria delle stringhe e risultante nel meccanismo di Stückelberg. La "compatibilità" riguarda il fatto che certi vincoli sperimentali devono essere rispettati, combinati con dei recenti dati estrapoli da eventi di Diffusione Coerente di Neutrini da parte di Nuclei (CEνNS che sta per Coherent Elastic Neutrino-Nucleus Scattering) alla Spallation Neutron Source all'Oak Ridge National Laboratory. Vedremo come in questo contesto riusciremo a fissare tutti i moduli ed a presentare una configurazione di flussi e brane riproducendo la corretta massa e accoppiamento del bosone di gauge nascosto. Avremo inoltre una scala di supersimmetria del TeV, dato che il gravitino in questo modello sarà dell'ordine O(TeV), con, in aggiunta, un vuoto "uplifted" per riprodurre un universo di de Sitter.

Contents

Introduction

When David Hilbert, in his 1930's retirement speech, declared that for him there was no *Ignorabimus*[1] in natural sciences, namely that there was no unsolvable problem, he was unaware of the *Incompleteness Theorem* that Gödel was about to announce in the same conference. Gödel's work shook the scientific community, since the positivism pervading scientists starts to be undermined by the fact that if even formal systems cannot be proved within themselves to be consistent, how can one be sure if the context within they are working is consistent or not if mathematical logic should be indeed at the heart of science and of the scientific reasoning? One could argue that physics, biology and chemistry are all experimental based sciences, and our knowledge is indeed constructed starting from experiments. However, we clearly have to build a consistent model, namely a theory or framework, in order to make predictions and to make use of the knowledge we assimilate from experimental data. Thus, as physicists, we are trying to find a theory which could explain all the natural phenomena within itself, with no overlying structures. However, as Freeman Dyson pointed out, referring to the Gödel's incompleteness theorems, in a review of the book "The Fabric Of Cosmos" written by Brian Greene, "*Pure mathematics is inexhaustible, (because of this) physics is inexhaustible too since the laws of physics are a finite set of rules and include the rules for doing mathematics*". If the physical laws are expressed in an axiomatic system S including the axioms of arithmetic and physical notions, then there will be undecidable propositions (of higher arithmetic) in S if S is consistent. As Solomon Feferman pointed out, however, one does not need Gödel's theorem to infer that physics is inexhaustible and his reasoning can be summarised as follows. All applied mathematics is formulated in the context of Zermelo-Fraenkel axiomatic set theory, as a matter of fact one can also employ an even weaker system in scientific applications. Regardless the axiomatic system S one employs as the foundation of their work, there will be a potential infinity of propositions which could be demonstrated in S, and at any given time only a finite set will be established. In Feferman's words: "*Experience shows that significant progress at each such point depends to an enormous extent on creative ingenuity in the exploitation of accepted principles rather than essentially new principles*". This means that the journey in the search of physical laws may never come to an end, for every new theory we construct, more questions and possibilities arise and we may never be able to achieve a real Theory of Everything. At the moment, the best candidate that we have is *String Theory*. As was originally formulated in the 1970s, string theory was a theory of propagating strings, even though initially presented to describe strong interactions, now it has become our best framework to incorporate gravity with the other interactions well described by the *Standard Model*. In our modern understanding of string theory, the fundamental objects are no more only strings, but the presence of higher dimensional extended objects known as D-branes are also allowed and, in fact, essential to construct phenomenological viable models, since they can support gauge theories

[1] *Ignoramus et Ignorabimus* was a Latin expression meaning "we do not know and we will not know" used in this sense for the first time around 1870s to express the attitude towards the scientific knowledge.

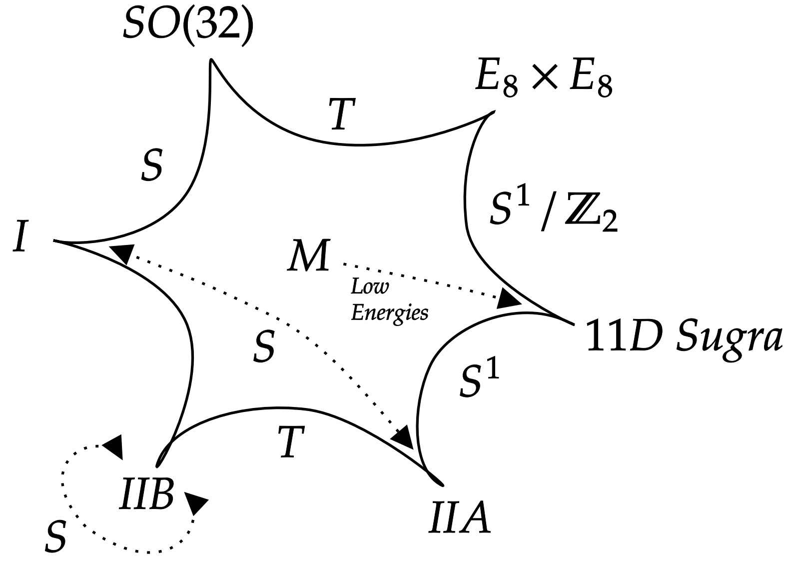


Figure 1: Pictorial representation of the dualities between string theories. The vertices are the type I, type IIB, type IIA, $E_8 \times E_8$ heterotic and $SO(32)$ heterotic string theories. At the same level we put the eleven dimensional supergravity theory thought to be the low-energy effective field theory of the M-theory. The dualities are T-duality, S-duality, compactification on S^1 and compactification on $S^1/\mathbb{Z}_2$. We see that all the different string theories are connected, and starting from a chosen one, we can reach whichever other theory we want by means of the above dualities.

(necessary to implement the interactions in the Standard Model). In order to have a consistent theory, supersymmetry must be incorporated in the description, leaving a theory of superstrings propagating in a ten dimensional spacetime. In the early years, five different superstring theories were constructed and at first they seemed to be disconnected from one another. What was found in the early '90s [Wit95], however, is that all different string theories are indeed part of the same web, interlaced by dualities as can be appreciated in figure (1). All these theories of strings are then thought to be derived from an overlying eleven dimensional theory, named *M-theory*. This awareness set off the second superstring revolution (the first happened a decade earlier after recognising an anomaly cancellation in type I string theory), reanimating physicist' souls in believing that string theory could be indeed a very promising candidate of quantum gravity. The study of string theory has gone a long way, and thanks to the work of many brilliant minds in the communities of physics and mathematics, these two disciplines have begun (again) to closely talk to each other. In fact, in order to construct vacua in string theory, algebraic geometry has been demonstrated to be a very powerful tool and thanks to physical intuitions, mathematical results have acquired a new flavour and some of them have also been proved due to the profound intimacy with physics related concepts. During the last decades lots of seemingly consistent vacua have been constructed, and the order of magnitude of the believed number of these is thought to be something like 10^{hundreds}. Physicists have then started to wonder what kind of arguments one could use to see if a low-energy effective supergravity theory could be UV completed to a theory of quantum gravity. This is known as the *Swampland Program* (for a review see [Pal19]), where the effective field theories which cannot be completed are said to be in the Swampland, while those that can be completed are said to be in the Landscape. Various conjectures have been laid down, for example it is believed that effective field theories with global symmetries belong to the swampland (when coupled to gravity), or that for every gauge theory all possible charged states must appear, so no free Maxwell-like theory belongs to the landscape. There are lots of these type of conjectures which, as the name suggests, have not been proved yet and they are guided mostly by considerations in string theory and on black holes physics. In this thesis we are not going to analyse the swampland program, rather we are going to focus on the phenomenology of string theory, namely the construction of models which could explain experimental data that the Standard Model is not able to describe. These ranges from cosmological observations, like the presence of Dark Matter or the horizon problem (solved by inflationary models which can be represented in string phenomenology), to more Standard Model's related phenomena, like the hierarchy problem, the strong CP problem, the anomalous magnetic moment of the muon, etcetera. Since there are still no experimental evidences supporting string theory (in particular there is still no experimental evidence of supersymmetry which is essential for string theory), all the works that are being (and have been) done in the string phenomenology direction are indeed trying to construct models compatible with current data and with maybe some kind of prediction testable in the near-future experiments. In this regard we should mention that a context where it could be possible to make testable string models is in cosmology, that because of all the newly precision measurements that are indeed being made in this direction. However, here we will concentrate on a context closer to Standard Model building in string theory rather than cosmological models. This thesis has been divided in four chapters, and in the following we are going to summarise the main topics covered in each of these:

1. The first chapter is devoted to present some of the current lacks in our understanding of nature (namely the problems that the Standard Model faces), as well as to introduce the foundations of string theory. All superstring theories live in ten spacetime dimensions and in order to meet the phenomenological requirement of a visible four dimensional "world", as well as having no more than one supersymmetry (a greater number of supersymmetries would not allow a chiral spectrum, in contrast with the Standard Model), six dimensions must be compactified on a special compact manifold which goes under the name of *Calabi-Yau* manifold [Can+85].

2. As we shall see in the second chapter, these Calabi-Yau manifolds are Kähler manifolds[2] with vanishing first Chern class (namely they are Ricci flat, of $SU(3)$ holonomy and there exists a nowhere vanishing covariantly constant spinor). The second chapter is also devoted to study the moduli space of these manifolds [CO91]. This is the space of all possible deformations of the metric leaving unchanged the Ricci-flatness property, and, as we are going to show, these deformations are closely related to Dolbeault cohomology classes (which themselves are related to harmonic forms thanks to the Hodge decomposition theorem), in particular there will be two kind of deformations, one of the Kähler structure (shape deformations) and one of the complex structure. The arrival into the scene of these algebraic/topological/geometric related concepts, motivated the presence of some mathematical sections in chapter 2 in which these ideas are presented in a more formal way. Even though in the first chapter we present all string theories, in the next chapters we concen-

[2]Kähler manifolds are complex and symplectic manifolds with the symplectic structure compatible with complex structure, namely it is possible to define a Riemannian metric on the manifold using these structures.

trate only on type IIB, since the phenomenological models are better understood in this context. In the compactification of type IIB on a Calabi-Yau, the number of supersymmetries in the resulting four dimensional supergravity will be two, and in order to break them down to one, an orientifold projection is employed [GL04; Gri05], cutting (clearly) also the spectrum in which there will be some bosonic fields, called moduli fields, for which no potential term will be generated, leaving these directions completely flat. If left unfixed, these moduli fields would give rise to unwanted fifth forces due to their coupling to matter particles. We should mention here that the imaginary part of the Kähler moduli are called axions, and indeed as their name suggests, they are good candidates to represent the Peccei-Quinn axion proposed to resolve the strong CP problem, also they could serve as ALPs (axion-like particles) providing also possible dark matter candidates [CGR12].

3. Chapter three is then devoted to explore how in type IIB string theory it is possible to generate a potential for all the moduli allowing to fix them at their vacuum expectation values. The complex structure moduli can be fixed by turning on gauge fluxes for some of the fields in the spectrum, this in fact generates a superpotential for these moduli [GVW00]. However, due to the no-scale property of the tree-level Kähler potential, the Kähler moduli will still be unfixed, since no potential for them is generated by the above mentioned superpotential. This will bring us to introduce non-perturbative corrections to the superpotential [Gor+04] and higher order corrections of the Kähler potential [Bec+02], the latter coming from the reduction of the ten dimensional action. These, in the context of LARGE Volume Scenario (LVS)[3] [Bal+05], allows to fix some of the Kähler moduli. For many models, however, some Kähler moduli directions will still be left flat and other perturbative corrections (string loop corrections) [BHP07] must be taken into account. In LVS models, one inevitably lands on an Anti-de Sitter vacuum, this means that an uplift method must be employed to get a de Sitter (which will be presented in chapter four) [Gal+17].

4. Once the stabilisation techniques have been presented, the chapter four will be dedicated to explicit models in which we will stabilise the moduli at some required values to match some experimental data. In particular we take into account a recent paper [Dut+19] in which the authors analysed the data extracted by the COHERENT collaboration [Aki+17] and observed a $\sim 2\sigma$ deviation from the Standard Model prediction. A possible NSI (non-standard interaction) between neutrinos and a hidden gauge boson could explain this deviation and in chapter four, after a brief introduction on the COHERENT experiment and the data that we would like to reproduce, we are going to see that these kind of interactions in string models can arise via kinetic mixing between the hidden gauge boson and the visible $U(1)$ [Abe+08; Goo+09]. As we have said, gauge theories are supported on higher dimensional extended objects called Dp-branes (where the p stands for the spatial dimensions of the brane), in particular a single Dp-brane support a $U(1)$ gauge theory and when a stack given by N of them is put together, then non-abelian gauge theories $SU(N)$ arise. In order to give a mass to the hidden gauge boson we will be using a completely stringy process (turning on appropriate fluxes) which will result in the Sückelberg mechanism thanks to which the hidden gauge boson will acquire a new degree of freedom becoming massive (it will "eat" an axion). Then, In order to give a mass to the hidden gauge boson, some

[3]The LARGE Volume Scenario is a stabilisation framework in which one is able to stabilise the volume of the Calabi-Yau to an exponentially large value, in particular as the exponential of the blow-up moduli which are required by the LVS claim, the other moduli are also fixed by using different perturbative and non-perturbative corrections.

fluxes must be turned on, and this will also result in the appearance of a D-term potential named *Fayet-Iliopoulos term* [JL05]. This can destabilise the process we outlined in chapter three, in this regard some considerations will be needed. After that, we will present a model with a Calabi-Yau given by an extension of a 2-parameter K3-fibred Calabi-Yau given as an hypersurface in the weighted projective space $\mathbb{P}^4_{(1,1,2,2,6)}$ defined as the vanishing locus of a 12-degree homogeneous polynomial. After discovering that this model will not allow to obtain values compatible with the searched data, we will consider a different Calabi-Yau embedded on a toric ambient space with a triple K3 fibration and a small blow-up mode. This, will have the right form and enough freedom to allow a nice stabilisation of the moduli with an uplift to a de Sitter reproducing the data we were seeking, as well as TeV scale supersymmetry.

Chapter 1

Foundations of String Theory

1.1 An Outstanding Century for Physics

The last century has been the centre of various revolutionary ideas which changed the way we should think about our physical world. In the 1900 **Max Planck**, seeking a mathematical way which would have led to a fit of the black body's radiation spectrum, introduced the idea that energy might be exchanged only in quantised form and its implementation was put forward by making these quanta only multiple of a constant h, which has taken its name and became the well known *Planck constant*. He himself didn't believe that it had some physical meaning, stating that it was just *"a purely formal assumption..."*, which, however, led to a spectacular fit between the theoretical prediction and the experimental data. As is well known, this gave birth to *Quantum Mechanics* which nowadays is still an astonishing "world" to explore and still hard to become used to, because of all its counter-intuitive predictions and results, which, nevertheless, are experimentally tested with high precision. In developing the structure of Quantum Mechanics new mathematical ideas were brought into physics and its accepted axiomatic framework has a rather formal structure and implements *Measure theory*, *Functional analysis*, *Complex analysis*, *Probability theory* and much more.

In the same years a new theory was being developed, namely *Special Relativity*. Making use of the work of **Handrik Lorentz** and **Hermann Minkowski**, **Albert Einstein** in 1905 managed to revolutionise our thinking. Space and time were no more distinct entities but somewhat intertwined and related to one another defining a new structure: the spacetime. Postulating the invariance of the speed of light in inertial reference frames, it turned out that observers might be viewed in different inertial frames by making use of a Lorentz transformation and fancy thinks happen when the limiting speed (which is found to be the light speed) is approached (like time dilatation and length contraction). But this was only the tip of the iceberg since some years later, with an incredibly simple but at the same time remarkable assumption, Einstein came up with *General Relativity*. By analysing the Newton's equation of an object in a gravitational field, he realised that its acceleration might be independent of the inertial and gravitational masses provided that these were equal. Its famous *Equivalence Principle* in fact states that the inertial mass is exactly the same as the gravitational mass, in such a way that a gravitational force is indistinguishable from an apparent force in a suitable non-inertial reference frame. This equivalence between gravity and non-inertial reference frames led to a new conception of our spacetime, it being a geometrical object (differentiable manifold) in principle flat. Masses, however, curve the spacetime and their motion is driven along geodesics (provided there are no other forces in play) obeying the Einstein's field equations (in natural units $c = \hbar = 1$):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.1.1}$$

where $R_{\mu\nu\rho\sigma}$ is the *Riemann curvature tensor* defined in function of the metric g of the spacetime manifold, $R_{\mu\nu}$ is the *Ricci curvature*, R is the *Ricci scalar*, Λ is the *Cosmological constant*, $T_{\mu\nu}$ is the *Stress-Energy tensor* and G is the Newton's constant. The Einstein's field equations can be interpreted in two ways: (i) The object with energy-momentum $T_{\mu\nu}$ deforms the spacetime and the resulting spacetime metric g will be given by the solution of the differential equations above; (ii) how the mass-energy $T_{\mu\nu}$ moves in presence of a gravitational field given by the metric g. These interpretations are perfectly captured in a quote by **John Wheeler**: "*Spacetime grips mass, telling it how to move... Mass grips spacetime, telling it how to curve*".

1.1.1 Dawn of Quantum Field Theory

Along the development of General Relativity, in the physics community some problems regarding nuclear physics were debated. After the discovery of radioactivity by **Henri Becquerel** in 1896 and the discovery of the electron in 1897 by **J. J. Thomson** - once it was realised that beta radiation was nothing but the particles discovered by Thomson - soon it became experimentally evident (in 1911) that the spectrum of radioactive decays was not constant but varied continuously. Moreover as **Ernest Rutherford** pointed out in 1920 it seemed that some nuclei had the wrong *statistics*, which in modern words, meant that the atomic number of some nuclei suggested that these should have obeyed a *Fermi-Dirac statistics* while experimentally they behaved as *Bose-Einstein particles*. These problems held tight until the remarkable and ingenious idea that came up in **Wolfgang Pauli**'s mind in 1930, which was expressed in a letter addressed to **Lise Meitner**. In order to make sense of the continuous spectrum of beta radioactive decays, he proposed that maybe there could be another neutral particle with spin 1/2 inside nuclei and emitted during a beta decay. His insight led **Enrico Fermi** in 1934 to propose a Hamiltonian which would have described the radioactive process:

$$\mathcal{H}_F = \frac{G_F}{\sqrt{2}} (\overline{p}(x) \gamma^\mu n(x))(\bar{\nu}(x)\gamma_\mu e(x)) + c.c. \tag{1.1.2}$$

where G_F is the *Fermi's constant*, γ^μ the Dirac matrices and $p(x), n(x), \nu(x)$ and $e(x)$ the spinor fields of the proton, neutron, neutrino and electron respectively (with the bar over them meaning the Dirac conjugation). Nowadays we know that the Fermi's theory is an *Effective theory*, in particular it is a low-energy approximation of the well-established *Glashow-Weinberg-Salam electroweak theory*. A lot of work had to be made by remarkable physicists in order for **Steven Weinberg** and **Abdus Salam** in 1967 to be to able to apply the *Englert-Brout-Higgs-Guralnik-Hagen-Kible* mechanism of spontaneous symmetry breaking (1964) to the model of **Sheldon Lee Glashow** (1961). The accomplishment of a unified gauge theory of electromagnetism and weak interactions has been a mile stone in the history of modern physics. By the subsequent implementation of the *Strong interaction*, the last century gave birth to the *Standard Model of Particle Physics* which is a $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory and describes all the known forces (except gravity) within the formalism of *Quantum Field Theory* (QFT). This is a framework which aims to unify Quantum Mechanics and Special Relativity and the first pioneer in this regard was

Paul Dirac in 1928 when he wrote down the well-known *Dirac equation* which describes the dynamical evolution of a spin 1/2 fermion compatible with Special Relativity.

Quantum Field Theory has a very rich structure, but it is also very constrained, everything fit almost in a perfect way and since the experiments confirm it with an incredible precision, not much can be changed in the theory without spoiling it.

1.1.2 A Long Standing Dichotomy

Einstein's Gravity was formulated in a classical and geometrical way and when QFT was being developed, it was fair to wonder what kind of bosonic particle should have carried the gravity force in trying to transpose Einstein's work in QFT's language. Spin 0 particles, i.e. scalar particles, cannot be the gravity mediators since they couple to the trace of the stress-energy tensor $T_{\mu\nu}$ (since scalar particles do not carry any Lorentz-index) of the matter fields, but since the energy-momentum's trace of relativistic matter vanishes, it would mean that gravity couldn't couple to light for example, which is clearly in contrast with experimental evidences. For spin 1 particles we already know the behaviour of the resulting theory following *Quantum Electrodynamics* and it would result in the existence of two kinds of "mass charges" making objects repel or attracting each other based on their "mass charge", and this too is clearly phenomenologically ruled out. The last[1] possibility is spin 2 particles and this theory was indeed proposed by **Markus Fierz** and **Wolfgang Pauli** in 1939 for a massless spin 2 particle:

$$S=\int d^4x\left(-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}+\partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda}-\partial_\mu h^{\mu\nu}\partial_\nu h+\frac{1}{2}\partial_\lambda h\partial^\lambda h\right) \tag{1.1.3}$$

where $h=\mathrm{tr}(h_{\mu\nu})$ and $h_{\mu\nu}$ should be thought of as the metric fluctuation, i.e. a small perturbation of the Minkowski metric $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$. The above action is the unique action describing the low-energy effective theory of the *Einstein-Hilbert action* (with $k=8\pi G$):

$$S=\frac{1}{2k}\int d^4x\sqrt{g}\,R \tag{1.1.4}$$

which describes the Einstein's gravity from an action-principle point of view since the equations of motion for the metric g are precisely the Einstein's field equations (1.1.1) (without the cosmological constant term which can be added in the action: $-\int d^4x\,\sqrt{g}\,2\Lambda$). The Einstein-Hilbert action is, however, non-renormalisable as can be seen by the *naïve power counting criterion* since the dimension in mass of the coupling constant is $[k]=eV^{-2}$ (in natural units $\hbar=c=1$). The fact that the coupling constant of the theory is dimensionful leads to the appearance of more and more new divergent loop integrals in the perturbative expansion in such a way that it would require an infinite amount of counterterms to renormalise the theory, making it in turn non-predictive.

The lesson that should be learnt is that the Einstein-Hilbert action has to be seen as a low-energy effective field theory of a UV completed theory of gravity. If one believes that gravity should be, at the end, treated on the same footing as the other interactions in the Standard Model, then it can be conveyed that a Quantum Theory of Gravity is indeed needed. Otherwise it can be believed that spacetime is classical, i.e. continuous and one can embed the Standard Model in a

[1]No spin $n\geq 3$ particles are able to describe interacting theories. This no-go theorem was due to **Steven Weinberg**.

general curved spacetime, which means in a semi-Riemannian manifold $(\mathcal{M}, g)$. This framework is known as *Quantum Field Theory on Curved Spacetimes* (QFTCS) and it is the generalisation of QFT in a general curved background spacetime. Even if our spacetime is quantised, we live in a curved one (even though our local curvature is quite low) and hence the best approximation that we should use in order to treat matter as quantum fields is to consider QFTCS. This framework gives rise to very interesting results and its insights lead also to a better understanding of almost automated procedures one uses in Quantum Field Theory without maybe wondering why. The best example is the existence of a unique *vacuum state* in Minkowski spacetime which every observer can agree upon, and in turn this leads to a well-posed definition of particle states. Considering a Klein-Gordon field expanded in normal modes $\phi(x) = \int d\vec{k}\,(\hat{a}_{\vec{k}} u_{\vec{k}}(x) + h.c.)$ with $\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}}$ annihilation/creation operators and $u_{\vec{k}}(x)$ normalised plane-waves, in QFT we define the Poincaré-invariant vacuum state $|0\rangle$ by requiring $\hat{a}_{\vec{k}}|0\rangle = 0$ and particle states are defined by the creation operators $|k_1 k_2 \cdots k_n\rangle = \hat{a}^{\dagger}_{\vec{k}_n} \cdots \hat{a}^{\dagger}_{\vec{k}_2} \hat{a}^{\dagger}_{\vec{k}_1} |0\rangle$. In a curved spacetime we are not allowed to define a unique vacuum state since Poincaré invariance is lost. Every observer will see a different vacuum state and this ambiguity is what in turn gives rise to the *Hawking effect*.

The Einstein-Hilbert action is then a low-energy effective field theory of a more complete and universal quantum theory of gravity. Here is where *String Theory* comes in play. Its task is to reconcile the Standard Model with the Einstein's gravity. Moreover in particle physics there are some mysterious experimental facts that the Standard Model cannot account for. Therefore there are various reasons to look for a more complete "theory of everything" and they will be explained in the following, after a very concise review of the Standard Model.

1.1.3 The Standard Model

The Standard Model is a $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory were C stands for *colour*, L stands for *left* and Y for *Hypercharge* and they are the quantum numbers conserved under these symmetries. Fermionic particles are divided among *quarks* and *leptons* depending on whether they strongly-interact (quarks) or not (leptons). The classification of particles is based on representations of these gauge groups, there are 3 families of quarks and leptons described as 2-components Weyl spinors and the classification based on the electroweak gauge group is as follows:

$$Q_L = \begin{pmatrix} U^i_L \\ \\ D^i_L \end{pmatrix} \qquad U^i_R, D^i_R \qquad L = \begin{pmatrix} \nu^i_L \\ \\ E^i_L \end{pmatrix} \qquad E^i_R \tag{1.1.5}$$

where $U^i = (u, c, t)$ are up-quarks, $D^i = (d, s, b)$ are down-quarks, $\nu^i = (\nu_e, \nu_\mu, \nu_\tau)$ are neutrinos and $E^i = (e, \mu, \tau)$ are electron, muon and tauon. The subscript L and R stands for left and right and identifies fermions which can interact electroweakly (left fermions), while right fermions are in fact singlets with respect to $SU(2)_L$. Including the strong force we can exemplify the classification of particles in the following table:

$$\begin{array}{lccc} & SU(3)_C & SU(2)_L & U(1)_Y \\ \\ Q_L^i = (U^i, D^i)_L & 3 & 2 & 1/6 \\ U_R^i & \bar{3} & 1 & -2/3 \\ D_R^i & \bar{3} & 1 & 1/3 \\ L^i = (\nu^i, E^i)_L & 1 & 2 & -1/2 \\ E_R^i & 1 & 1 & 1 \\ \phi = (\phi^-, \phi^0) & 1 & 2 & -1/2 \end{array} \tag{1.1.6}$$

where it has been added the Higgs doublet ϕ and the numbers are indeed the representations the matter fields belong to.

The Lagrangian of the Standard Model can be compactly written as follows:

$$\mathscr{L} = -\frac{1}{4}\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu} + \overline{\Psi}\not{D}\Psi + (\mathscr{Y}_{ij}\Psi_i\Psi_j\phi + h.c.) + (D_\mu\phi)^* D^\mu\phi - V(\phi) \tag{1.1.7}$$

where $\mathscr{F}_{\mu\nu}$ contains the kinetic terms of the gauge bosons, Ψ contains the matter fields of leptons and quarks, D is the covariant derivative, $\mathscr{Y}$ contains the Yukawa couplings, ϕ is the Higgs field and $V(\phi)$ is the potential of the Higgs which allows for the symmetry breaking effect, generating the masses for the matter fields and for the weak gauge bosons. Matter fields in the Standard Model are in fact massless until the symmetry breaking effect takes place and $SU(2)_L \times U(1)_Y$ is broken into $U(1)_{EM}$. The number of gauge bosons is given by the dimension of the various gauge groups. There are 8 strong gauge bosons called *gluons*, $3+1$ electroweak gauge bosons $(Z_0, W_\pm) + A$ and at low energies, when the symmetry is broken, it appears the electromagnetic mediator: the photon A_γ as the "new" gauge boson of $U(1)_{EM}$.

1.1.4 Lacks In Our Current Understanding Of Nature

The Standard Model is undoubtedly the highest intellectual achievement that mankind has ever attained. The spectacular matching between predictions and experiments and the final experimental discovery of the Higgs boson after roughly 50 years of its first proposal, contribute to make the Standard Model our best theory of particle physics. The assembling of the Standard Model in the years from 1940s to 1980s is described by people who worked on it as a thrilling process. In those years more and more particles were discovered continuously and finding the classification principles was not a simple, but surely engaging, task. The final theory has for sure a nice appeal. Thinking of how our physical laws are able to describe processes in so small spacetime regions is really astonishing and breathtaking. However, despite all these good feelings, the Standard Model presents some unattractive peculiarities. First of all, every calculation is made perturbatively, Feynman diagrams are very powerful tools but still perturbative in nature and then not fundamental. Quantum Chromodynamics (QCD) is non-perturbative in the large scale (low energy) limit because of its underlying asymptotic freedom character. Even though there are methods for studying this limit, only recently the AdS/CFT duality seems to provide some help in the study of QCD at intermediate scales between the strong coupling limit and the perturbative limit (using integrability). Secondly, the renormalisation procedure has still a somewhat unclear physical meaning. Infinities arise everywhere in perturbative Quantum Field Theory, and in renormalisable theories, new Feynman diagrams can be introduced to cancel the divergent quantities, leading to renormalised coupling constants, which will be, ultimately, the real physical parameters measurable in experiments. Carrying out this process leads to lots of arbitrary constants, various procedures can be employed to get rid of them and all of these should at the end give the same physical result. The renormalisation group equations allow to inspect the relations between all possible renormalisation prescriptions. Coupling constants are found to be dependent on the scale of the process in consideration and this "running" is parametrised by so-called β-functions. In order to get a physical intuition of the renormalisation procedure one can exploit Statistical Mechanics and its close connection to Quantum Field Theories. A renormalisable QFT can, in fact, always be represented as a Many Body System at a critical point. The critical point is a fixed point for the renormalisation group, since in a critical point the correlation length grows to infinity, making the system scale-invariant. The idea behind the renormalisation group in Statistical Mechanics is that in correspondence to a change of the length-scale, it corresponds a change of the coupling constants, and the continuous family of transformations providing this correspondence goes under the name of, indeed, Renormalisation Group. In the vicinity of a critical point, it is natural to use field theory and the action will be of the form $\mathscr{S}[\phi] = \int d^d x [1/2(\partial_i \phi)^2 + g_1 \phi + (g_2/2)\phi^2 + \cdots (g_n/n!)\phi^n + \cdots]$. The couplings $\{g\} = (g_1, ..., g_n, ...)$ define a space over which the renormalisation group acts. Under an infinitesimal rescaling $x \to x' = x/b \simeq x/(1+\delta\ell)$, the couplings will transform as: $g_a \to g'_a = g_a + (d g_a/d\ell)\delta\ell + o(\delta\ell)$. The first derivatives $d g_a/d\ell$ will be the β-functions, and they can be interpreted as the vector fields that fix the renormalisation group flow.

Even though Statistical Mechanics may help in getting some physical justification, it is just a formal correspondence and all the renormalisation machinery in QFT seems to be quite inelegant also because finite couplings result from infinite Lagrangians (not well defined in our, apparently 4-dimensional, spacetime).

Thirdly, in the Standard Model there are roughly 20 free parameters which can be determined only through experiments. This huge arbitrariness in our description of particle physics, which can be fixed only empirically, does not seem to meet the elegance and beauty that a physicists would hope for a theory describing Nature. Even though it can be argued whether beauty does or does not provide a good guidance in the intrigant realm of physical laws, still 20 free parameters seems to be too much.

Fourthly, the biggest puzzle that since the mid of the last century leaves physicists baffled, is that gravity cannot be implemented in the Standard Model. At the Plank scale general relativity breaks down, and the behaviour of physical laws in this realm are completely obscure to us (at least from a general relativity point of view).

Despite the above arguments, which regarding them as problems or not could be a matter of taste, there are some unresolved questions which plague the Standard Model and things that it does not manage to explain. They can be summarised as follows:

- **Cosmological Constant Problem**

  In order to account for an accelerating expanding universe, the cosmological constant term in the Einstein's field equations is undoubtedly needed but everything that contributes to the vacuum energy act as a cosmological constant. Following Weinberg [Wei89] Lorentz invariance tells us that in vacuum the stress-energy tensor of matter fields must take the form $\langle T_{\mu\nu}\rangle = \langle\rho\rangle g_{\mu\nu}$. In this way, the effective cosmological constant term can be defined as $\Lambda_{\text{eff}} = \Lambda + 8\pi G\langle\rho\rangle$ and in the same way it can be thought that the Einstein cosmological constant term contributes to the vacuum energy as $\rho_V = \langle\rho\rangle + \Lambda/8\pi G$. Experimental data tell us that $\rho_V \sim 10^{-47} GeV^4$ but the vacuum energy due to matter fields is extremely huge if we believe general relativity up to the Planck scale $\Lambda_{\text{cut-off}} \simeq (8\pi G)^{-1/2}$. In fact:

$$\langle \rho \rangle = \int_0^{\Lambda_{\text{cut-off}}} dk \, \frac{4\pi k^2}{(2\pi)^3} \frac{1}{2} \sqrt{k^2 + m^2} \simeq \frac{\Lambda^4_{\text{cut-off}}}{16\pi^2} \simeq 10^{71} \text{GeV}^4 \tag{1.1.8}$$

and this means that $|\langle \rho \rangle + \Lambda/8\pi G|$ should cancel with a precision of more than 118 decimal places.

- **Strong CP Problem**

  The QCD Lagrangian admits a violation of the CP discrete symmetry, which, however, isn't found experimentally. The CP violation would lead to an electric dipole moment of the neutron comparable to $10^{-18} e \cdot m$ while the current upper bound is roughly 10^{-9} times lower. One of the candidate solutions are due to **Peccei** and **Quinn**. In order to explain this phenomenon they introduce pseudo-scalar particles called *axions* which come up as *Goldstone bosons* of a spontaneous symmetry breaking of a $U(1)_{PQ}$ symmetry called *Peccei-Quinn symmetry*. The resulting mass of these axions in order for them to explain the absence of the CP violation in the strong sector should be of the order $\lesssim 10^{-2} eV$ and their coupling with normal matter should be very small $\lesssim 10^{-11}$. These characteristics make axions good candidates to be identified as possible *Dark Matter* particles (as we shall see soon).

- **Electroweak Hierarchy Problem**

  Electroweak scale is fixed by the Higgs mechanism, in particular by its *Vacuum Expectation Value* (VEV): $\langle \phi \rangle = \frac{1}{\sqrt{2}} \binom{0}{v}$ with $v = \mu^2/\lambda$ and the Higgs potential given by $V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$. The mass term, however, receives divergent quantum loop corrections from fermions, leading the parameter μ towards the cut-off of the theory: $\delta \mu^2 \simeq \frac{\alpha}{4\pi} \Lambda^2_{\text{cut-off}}$. This cut-off could be the Planck scale, the string scale or the GUT (Grand Unified Theory) scale. In any case it should be quite big and in order to trigger the electroweak spontaneous symmetry breaking, those corrections should cancel within an enormous precision.

- **Dark Matter**

  Dark matter constitutes roughly 26% of the mass-energy of the Universe and its existence can be inferred from the deviation of the galaxies rotation curves from their theoretical prediction. The adjective "dark" refers to the fact that this kind of matter seems to interact very weakly with the Standard Model, making it still undetected (directly) at the present day. Dark Matter (DM) is also required to be very cold in order to allow for galaxies and structure formation. Its abundance (5 times more than baryonic matter), made it dominate the evolution of the Universe from the end of radiation era ($7 \cdot 10^4$ years after the Big-Bang) until something like $9.4 \cdot 10^9$ years. Currently there are lots of possible candidates proposed to be DM, some of them have also been ruled out thanks to cosmological measurements with high precision. The following is a list of some these candidates and their currently state of affairs:

  (i) Massive Compact Halo Objects (MACHO). These are any kind of astronomical body (brown dwarfs, lonely planets, black holes, etc...) which could explain the presence of Dark Matter in galaxies halo. The not enough microlensing effects found by experiments suggests, however, that they cannot account for all the Dark Matter in the Universe.

(ii) Primordial Black Holes (PBH). From small density fluctuations and inhomogeneities in the early Universe, some regions could very well have undergone a gravitational collapse forming these PBH. Even though there is no actual evidence for their existence, the fact that some of them could have evaporated (those with masses $\sim 10^{15}\,g$) through Hawking effect, resulted in some speculation regarding their possible role in the Dark Matter problem. Whether they provide Dark Matter or not is still unknown (like every other candidates though). It seems [CKS16] that some ranges of PBH masses are excluded by cosmological observations while others remain still in play.

(iii) Weakly Interacting Massive Particles (WIMPs). These particles are one of the most promising candidates to Dark Matter even though experiments have been trying to find them for thirty years. A good review of the current status of WIMPs can be found in [RST18]. In short, these particles arise naturally in some Beyond Standard Model (BSM) physics, as for example, *lightest supersymmetric particles* (LSP) or *lightest Kaluza-Klein particles* (LKP) in suitable models. These particles would have been thermally created in the early Universe and their large mass would have allowed for a progressing cooling making them non-relativistic and cold at the present day.

(iv) Sterile Neutrinos. In the Standard Model, neutrinos are massless even after the spontaneous symmetry breaking effect caused by the Higgs mechanism and this is due to the fact that there are no right-handed neutrinos, only left-handed neutrinos and right-handed anti-neutrinos (their anti-particles). However, after the discovery of neutrino oscillation, it is now clear that they are indeed massive and to take this into account one has to introduce right-handed neutrinos but the fact that they would be neutral under every kind of interaction in the Standard Model (for this reason are called "sterile"), makes them still somewhat speculative since the only way in which they could be observed directly is via gravitational interaction. As was argued by Shaposhnikov and Tkachev [ST06], the introduction of three right-handed neutrinos and the implementation of inflation could explain at once the Dark Matter problem, the Baryon asymmetry and generate masses for neutrino oscillation to take place. In their work the right-handed neutrinos are of Majorana type, however it is still to be decided whether these would be Dirac spinor or Majorana spinor, experiments searching for neutrinoless double β-decays are trying to figure out exactly this fact. A recent review of the status of Sterile neutrinos as DM candidates can be found in [Boy+18].

(v) Axions and Axion-Like Particles (ALP). The possible solution of the strong CP problem, namely the axions, are also potential candidates for Dark Matter. In contrast to WIMPs, they would have small masses but would still be non-relativistic as a cold axion population could results from vacuum realignment [DB09] (also called misalignment mechanism [NS11]). This process can be summarised as follows. In the early universe, the scalar axionic field would be stuck, with no quantum oscillations due to the fact that they would be effectively massless, with a Compton wavelength greater than the horizon (this can be inferred from its equation of motion in a Friedmann-Robertson-Walker metric). As the Hubble constant starts to decrease (after the end of inflation), the field starts to oscillate and its quantisation would lead to particles.

The left-over energy density could be thought of as a coherent state of extremely cold and non-relativistic particles. Given that string compactifications naturally produce many axions, the investigation of their possible existence has started to flourish (even because WIMPs are still undetected after many years). A good account of axions in type IIB string theory can be found in [CGR12].

- **Anomalous Magnetic Moment of the Muon**

  In contrast with the spectacular agreement between experimental data and theoretical prediction of the anomalous magnetic moment of the electron (which represents the most accurate prediction in the history of physics and a precision test of QED), the anomalous magnetic moment of the muon measured by experiments deviates from the theoretical prediction by a factor of $\sim 3.5\sigma$ [Her16]. This could be very well a signal of Beyond Standard Model physics which has to be investigated. There has been a recent paper by Morishima, Futamase, and Shimizu [MFS18], claiming that the deviation of the anomalous magnetic moment of the muon from the Standard Model prediction could be caused by the interaction with the gravitational field of the Earth. However, a subsequent paper by Visser [Vis18] argues that in their work, Morishima et al., implemented the principles of General Relativity in a wrong way. Another work of László and Zimborás [LZ18], through a fully general-relativistic calculation, points out that the General Relativity corrections should be too small for our current experimental apparatuses to measure. Proposals to solve the problem involve non-standard interactions (NSI) carried by possible hidden gauge bosons and experiments like PADME (Positron Annihilation into Dark Matter Experiment) or COHERENT are indeed trying to figure out if these kind of NSI are realised or not.

That being said, string theory tries to furnish a framework in which all these problems could be somehow resolved. Though it was initially developed to describe strong interactions, string theory was found to be suitable, instead, in unifying particle physics with gravity. In string theory, the graviton emerges as a massless field in the string spectrum and the background metric in which strings are supposed to be propagating can be viewed as a coherent superposition of these gravitons. In the context of string compactification, the Standard Model and/or eventually other gauge sectors (GUT-models, hidden gauge sectors, etc...) can be constructed by means of Dp-branes, which are submanifolds embedded in spacetime where open strings could end. In the next chapter these topics will be analysed, while in the following we are giving a brief historic account of the birth of string theory and then present the various possible theories of strings.

1.2 Fundamentals of String Theory

The theory of strings was born thanks to the pioneering work of Nambu [Nam70] (which was never published, however a selected collection of his papers can be found in [NEN95]) in an attempt to study the dynamical properties of hadrons induced by the conjectured $s-t$ duality proposed by Dolen, Horn, and Schmid [DHS68] and mathematically implemented by Veneziano [Ven68]. String theory was then firstly proposed to explain behaviours connected to strong interactions. In his paper, Nambu proposed an action for the relativistic string, as a generalisation of the relativistic and manifestly Lorentz covariant action for a massless point-like particle. Quarks in hadrons were thought to be connected by such strings. The theory was further refined by the works of Susskind [Sus70] and Nielsen and Olesen [NO73], however, as experiments were

carried over, it started to be clear that this view had to be abandoned in favour of a new emergent theory: Quantum Chromodynamics (QCD), strongly supported by the discovery of asymptotic freedom in non-abelian gauge theories by Gross and Wilczek [GW73b; GW73a] and Politzer [Pol73] (they were awarded the Nobel prize in physics in 2004). In fact, the Veneziano model was accurate in the *Regge region* (large s, fixed t), but failed in the *high-energy scattering (at fixed angle) region* ($s \to \infty$, $t \to \infty$) in which the behaviour was instead well understood in *parton model* terms. In 1973-1974, thanks to QCD which did implement parton-like behaviours, the formulation in terms of strings slowly fell into oblivion.
In those years (and nowadays too) another problem was also in physicists' mind: the unification of gravity with other interactions. The first attempt was put forward by Kaluza [Kal21] inspired by the "courageous attack" (with his words) of Hermann Weyl [Wey18] in trying to reconcile Einstein's gravity with Maxwell's electromagnetism (the dawn of gauge theories). Kaluza proposed a unification of these two by formulating general relativity not in four dimensions but in five. This hint was taken up by Klein [Kle26], which managed to reproduce the first dimensional reduction obtaining a tower of massive four-dimensional states starting from a massless five dimensional one. This wonderful theory (called from there on Kaluza-Klein theory) predicted the existence of a new massless scalar field and at the time this was quite an inconvenient, since there were not experimental results pointing to the existence of such a particle. However around the 1950s, thanks to the work of Jordan, Brans and Dicke these scalars started to be seen as interesting theoretical prediction that should be experimentally tested. Even with this new interpretation those particles were rather evanescent and there seemed to be no way to reveal them. These particles were connected to the "dual model" (Veneziano model), since even there a massless scalar field was present (dilaton field).
As we have already said, the Veneziano model was presented to explain the Regge trajectories and it didn't received much credibility since it was not able to include fermions and a tachyonic particle was also predicted. The former inconvenience was bypassed by Ramond [Ram71] and Neveu and Schwarz [NS71] which managed to create a two-dimensional world-sheet supersymmetric string model. This model was further generalised to four-dimensional space-time supersymmetry by Wess and Zumino [WZ74], putting the foundation for all the works that has been done in this sector of theoretical physics. With superstrings in hand the tachyon existence problem was finally solved by Gliozzi, Scherk, and Olive [GSO77] by employing a parity projection under which the tachyon was finally projected out. The real break through was realising that this theory of strings should have been used for a more aulic purpose, namely unifying gravity with all other interactions. This awareness was reached by Scherk and Schwarz [SS74] and this led to the development of bosonic string theory (and all subsequent work in string theory). The theory which started to come out was free of ultraviolet divergences at one loop, but in string theory problems come from anomalies and other inconsistencies which have to be carefully checked in order to make a sensible theory.

1.2.1 Bosonic strings

Bosonic string theory is a theory of 1-dimensional objects, called strings, propagating in a general D-dimensional spacetime (actually higher dimensional extended objects called D-branes are also part of the theory as we shall see). The action describing such an object is the *Polyakov action* (an extension of the Nambu-Goto action), which is the generalisation of the relativistic and manifestly Lorentz covariant action of a pointlike particle, by extending it to be a string which sweeps out a surface in the spacetime called *Worldsheet* (WS). In D spacetime dimensions, we can embed the worldsheet (Σ, h) (endowed with an intrinsic metric h) into the Minkowski spacetime $(\mathbb{R}^{1,D-1}, \eta)$ with $X : \Sigma \to \mathbb{R}^{1,D-1}$ and write the Polyakov action as:

$$S[X^\mu, G_{\mu\nu}, h_{\alpha\beta}] = -\frac{T}{2}\int_\Sigma d^2\xi\, \sqrt{-h}\,\eta_{MN}\partial_\alpha X^M h^{\alpha\beta}\partial_\beta X^N \tag{1.2.1}$$

where T is the tension of the string, α, β runs over the worldsheet's coordinates $\xi = (\tau, \sigma)$, while $M, N = 0, 1, ..., D-1$ are spacetime indices. This action presents various symmetries, namely *diffeomorphisms invariance*, *Weyl invariance* (implying conformal symmetry of the worldsheet) and *Poincaré invariance*, and after a suitable gauge fixing (conformal gauge) the equations of motion are nothing but d'Alembert wave equations for left/right bosonic modes propagating in the worldsheet. Since these are differential equations, boundary conditions need to be chosen and open or closed strings can indeed be considered, leading to a different spectrum of massless states when the quantisation procedure is taken into account. In order to have full Lorentz invariance, the critical dimension D of the spacetime in which bosonic strings can propagate is 26. Another way to retrieve this critical dimension is by means of the conformal anomaly: in order to not have negative norm states (ghost fields) in the spectrum, the conformal anomaly must be equals to 26 and for bosonic strings the anomaly is exactly equals to the spacetime dimension in which the string is embedded.

The massless spectrum of a closed bosonic string consists of a tachyon T, a traceless symmetric 2-tensor G_{MN} (graviton), an antisymmetric 2-tensor B_{MN} (B-field) and a scalar particle ϕ called dilaton. Massive modes can be discarded when working at energies below the string mass M_s:

$$M_s = \frac{1}{\ell_s} = \frac{1}{2\pi\sqrt{\alpha'}} \tag{1.2.2}$$

where ℓ_s is the string length and α' is the Regge slope, and it is equals to $T = 1/2\pi\alpha'$. In fact, when energies are below M_s, it is not possible to perceive the extra spatial extent of strings and thus they behave as pointlike particles. This limit is in fact called the *Pointlike limit*. The scale M_s can be as big as the Planck scale M_P, but it depends on the model considered and can also be as low as TeV scale for particular models. The $D = 26$ effective action of the closed bosonic string involves the above massless fields and can be written as [IU12]:

$$S_{26D} = \frac{1}{2k_{26}^2}\int d^{26}x\,\sqrt{-G}\,e^{-2\phi}\left(R - \frac{1}{12}H_{MNP}H^{MNP} + 4\partial_M\phi\,\partial^M\phi\right) \tag{1.2.3}$$

where G is the determinant of G_{MN}, R is its Ricci scalar and $H_{MNP} = \partial_{[M}B_{NP]}$ is the field strength of B_{MN} (in general, the B-field is written as a 2-form $B = B_{MN}dx^M \wedge dx^N$, in this way its curvature will be $H = dB$, with d the exterior derivative). The constant k_{26} is related to the 26-dimensional Newton's constant G_{26} and string length ℓ_s as follows:

$$16\pi G_{26} = 2k_{26}^2 = \frac{1}{2\pi}(2\pi\ell_p)^{24} \tag{1.2.4}$$

The action (1.2.3) is said to be written in the *string frame*, since its fields are naturally related to string excitations. From spacetime point of view, it is however convenient to redefine the fields and write the action in the so-called *Einstein frame* by making an appropriate Weyl rescaling of the metric $G_{MN} \to \tilde{G}_{MN} = e^{(\phi_0-\phi)/6}G_{MN}$ and $\phi \to \tilde{\phi} = \phi - \phi_0$:

$$S_{26D} = \frac{1}{2k^2}\int d^{26}x\,\sqrt{-\tilde{G}}\left(\tilde{R} - \frac{1}{12}e^{\tilde{\phi}/2}H_{MNP}H^{MNP} - \frac{1}{6}\partial_M\tilde{\phi}\,\partial^M\tilde{\phi}\right) \tag{1.2.5}$$

where $k = k_0 e^{\phi_0}$.

The role of the dilaton field ϕ is very important in string theory. Its Vacuum Expectation Value (VEV) will give the string coupling constant:

$$g_s = e^{\langle\phi\rangle} \tag{1.2.6}$$

This means that in string theory the only free parameter is the string tension T (or Regge slope α', or string length $\ell_s = \sqrt{\alpha'}$ or string mass $M_s = 1/\ell_s$). The fact that the dilaton gives the string coupling constant can be seen as follows. The low-energy effective action (1.2.3) contains 3 fields G_{MN}, B_{MN} and ϕ. We can think to couple the string action (1.2.1) with these background fields. What is obtained is an action describing strings propagating in a curved spacetime given by the metric G_{MN} and two additional pieces:

$$S_{B,\phi} = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \left(\epsilon^{\alpha\beta} B_{MN}\partial_\alpha X^M \partial_\beta X^N + \sqrt{-h}\,\alpha' R\,\phi\right) \tag{1.2.7}$$

where $\epsilon^{\alpha\beta}$ is the Levi-Civita symbol in 2 dimensions. These two terms are *Topological Invariants*. The first, if we write $B_2 = 1/2\, B_{MN} DX^M \wedge DX^N$, can be recast as:

$$\frac{1}{2\pi\alpha'}\int_\Sigma B_2 \tag{1.2.8}$$

and this is a generalisation for a 2-dimensional worldsheet of the minimal coupling of a charged particle worldline to an electromagnetic potential. This means that strings are electrically charged under the field B_2. The other term, taking ϕ to be constant (its VEV), can be seen to be a topological invariant by the fact that under a Weyl rescaling of the metric $h_{\alpha\beta} \to e^{2\omega(\xi)} h_{\alpha\beta}$, ω can always be chosen in such a way to bring $R \to 0$. This can be translated in the fact that $2D$ gravity is always non-dynamical. In mathematical language this fact is the result of the *Gauss-Bonet* theorem, which says that the integration over the whole manifold of its *Euler class* is given by its *Euler characteristic* (see appendix A for a review of Characteristic classes and more specific for Euler classes):

$$\int_\Sigma e(R) = \chi(M) \tag{1.2.9}$$

For a closed Riemann surface Σ (our worldsheet), with Riemannian (semi-Riemannian) metric $h_{\alpha\beta}$, we have that (1.2.9) is translated exactly in:

$$\int_\Sigma e(R) = \frac{1}{2\pi}\int_\Sigma d^2\xi\, \sqrt{-h}\, R = \chi(\Sigma) \tag{1.2.10}$$

Given that (1.2.10) is a constant, the coupling to the dilaton will then be:

$$S_\phi = \phi\,\chi(\Sigma) \tag{1.2.11}$$

In the Euclidean functional integral, the background field ϕ will weight each worldsheet diagram by a factor of $e^{-\phi\chi}$, justifying our claim (1.2.6).

1.2.2 Superstrings

If we want to add worldsheet supersymmetry, we can introduce fermionic degrees of freedom as D Majorana fermions $\psi = \binom{\psi_-}{\psi_+}$ belonging to the vector representation of the Lorentz group $SO(1,D-1)$. In the conformal gauge ($h_{\alpha\beta} = \eta_{\alpha\beta} = \text{diag}(-1,1)$) we obtain:

$$S[X^\mu, \psi^\mu] = -\frac{T}{2}\int d^2\xi \left(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) \tag{1.2.12}$$

where ρ^α are the Dirac matrices in 2 dimensions and fermionic world sheet fields ψ^μ satisfy anti-commutation relations $\{\psi^\mu, \psi^\nu\} = 0$. The action is invariant under a supersymmetry transformation:

$$\begin{cases} \delta_\epsilon X^\mu = \bar{\epsilon}\,\psi^\mu \\ \delta_\epsilon \psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon \end{cases} \tag{1.2.13}$$

where ϵ is the infinitesimal Majorana spinor (anticommuting) parametrising the supersymmetry transformation.

The action (1.2.12) is, however, not manifestly supersymmetric. In order to resolve that, it can be employed the *Superspace* formalism at worldsheet level. Taking (ξ^α, θ_A) with $\xi^0 = \tau$, $\xi^1 = \sigma$ and $A = 1,2$ be the coordinates of the worldsheet superspace (with θ_A anticommuting), we can define the *Superfields* $Y^\mu(\xi^\alpha, \theta_A)$ and expand it with respect to the Grassmann variables θ_A obtaining (considering $\bar{\theta}\psi = \bar{\psi}\theta$ and $\theta_A\theta_B = 0$):

$$Y^\mu(\xi^\alpha, \theta_A) = X^\mu(\xi^\alpha) + \bar{\theta}\psi^\mu(\xi^\alpha) + \frac{1}{2}\bar{\theta}\theta B(\xi^\alpha) \tag{1.2.14}$$

where B is an auxiliary field which do not change the content of the theory (its equations of motion will be $B^\mu = 0$) but makes it possible to have a manifestly supersymmetric formulation. The generators of supersymmetry transformations are the *Supercharges*:

$$Q_A = \frac{\partial}{\partial \bar{\theta}^A} - (\rho^\alpha\theta)_A \partial_\alpha \tag{1.2.15}$$

and these acts on worldsheet coordinates as:

$$\begin{cases} \delta_\epsilon \theta_A = [\bar{\epsilon}Q, \theta_A] = \epsilon_A \\ \delta_\epsilon \xi^\alpha = [\bar{\epsilon}Q, \xi^\alpha] = \bar{\theta}\rho^\alpha \epsilon \end{cases} \tag{1.2.16}$$

By inspecting the action of the supercharge on the superfield $\delta_\epsilon Y^\mu = [\bar{\epsilon}Q, Y^\mu]$ and matching powers of θ_A with the variations of X^μ, ψ^μ and B we get:

$$\begin{cases} \delta_\epsilon X^\mu = \bar{\epsilon}\,\psi^\mu \\ \delta_\epsilon \psi^\mu = \bar{\theta}\rho^\alpha \partial_\alpha X^\mu \epsilon + \bar{\theta} B^\mu \epsilon \\ \delta_\epsilon B^\mu = \bar{\epsilon}\rho^\alpha \partial_\alpha \psi^\mu \end{cases} \tag{1.2.17}$$

As we can see, these transformations match those (1.2.13), with the addition of the field B^μ, which disappears when employing its equations of motion. In the superspace language, we can define a supersymmetric covariant derivative D_A, in such a way that $D_A Y^\mu$ will transform exactly as

Y^μ, making combinations of the type $\overline{D}_A Y^\mu D^A Y_\mu$ manifestly supersymmetric. The derivative is defined as:

$$D_A = \frac{\partial}{\partial \overline{\theta}^A} + (\rho^\alpha \theta)_A \partial_\alpha \tag{1.2.18}$$

and in this way the action (1.2.12) is rewritten as:

$$S[Y^\mu] = iT \int d^2\xi \, d^2\theta \, \overline{D}_A Y^\mu D^A Y_\mu \tag{1.2.19}$$

(the $-i$ comes from the fact that $\int d^2\theta \, \overline{\theta}\theta = -2i$).

When inspecting the equations of motion, we must impose the vanishing of the boundary term. In this regard we have to distinguish two cases:

- Open String:

  In this case, on one end of the string (in units of the string length ℓ_s, we consider $\sigma \in [0, \pi]$) it can always be chosen:

  $$\psi_+^\mu|_{\sigma=0} = \psi_-^\mu|_{\sigma=0} \tag{1.2.20}$$

  while on the other end we have actually two choices:

  (i) *Ramond boundary conditions*:

  $$\psi_+^\mu|_{\sigma=\pi} = \psi_-^\mu|_{\sigma=\pi} \tag{1.2.21}$$

  (ii) *Neveu-Schwarz boundary conditions:*

  $$\psi_+^\mu|_{\sigma=\pi} = -\psi_-^\mu|_{\sigma=\pi} \tag{1.2.22}$$

- Closed String:

  When the strings are closed, there are two possible choices too, which are called again *Ramond* and *Neveu-Schwarz* boundary conditions, but this time, the Ramond (R) refers to the periodic condition:

  $$\psi_\pm^\mu(\tau, \sigma) = \psi_\pm^\mu(\tau, \sigma + \pi) \tag{1.2.23}$$

  while the Neveu-Schwarz (NS) refers to the anti-periodic condition:

  $$\psi_\pm^\mu(\tau, \sigma) = -\psi_\pm^\mu(\tau, \sigma + \pi) \tag{1.2.24}$$

  Since the (R) and (NS) conditions can be imposed separately on left/right movers ($\psi_\pm$), there will be 4 possible sectors given by the combinations R-R, NS-R, R-NS and NS-NS. States coming from the R-R and NS-NS sectors will contain bosons, while states coming from NS-R and R-NS will contain fermions.

The spectrum of the superstrings can be inspected by making use of *Superconformal Field Theory* (SCFT) because the worldsheet action (1.2.12) presents superconformal gauge symmetry. After fixing the gauge, the action can be written in complex coordinates and the *Operator Product Expansion* (OPE) of the energy-momentum tensor will enable us to infer the *critical dimension* of the spacetime in which superstrings could live. The *Supeconformal Anomaly* (central charge of the superconformal theory) can be seen to be exactly equals to the spacetime dimension D, and in order to not have negative norm states, the critical dimension must be equals to 10.

Aside (Conformal Field Theory)

Conformal transformations in 2-dimensions can be easily seen to be generated by holomorphic functions:

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x^\nu) \tag{1.2.25}$$

in fact, the ϵ^μ (with $\mu = 1,2$) must satisfy the *Cauchy-Riemann equations*:

$$\begin{cases} \partial_1 \epsilon^2 = -\partial_2 \epsilon^1 \\ \partial_1 \epsilon^1 = \partial_2 \epsilon^2 \end{cases} \tag{1.2.26}$$

and in complex coordinates $z = x^1 + ix^2$ and $\epsilon = \epsilon^1 + i\epsilon^2$ (and respectively $\bar{z} = x^1 - ix^2$ and $\bar{\epsilon} = \epsilon^1 - i\epsilon^2$) we simply have:

$$\begin{cases} \partial_z \bar{\epsilon}(z,\bar{z}) = 0 \\ \partial_{\bar{z}} \epsilon(z,\bar{z}) = 0 \end{cases} \tag{1.2.27}$$

which means indeed that $\epsilon = \epsilon(z)$ is a holomorphic function and $\bar{\epsilon} = \bar{\epsilon}(\bar{z})$ is an anti-holomorphic function. It is known that if we consider a complex plane $\mathbb{C}$ compactified by adding the point at infinity, it becomes the *Riemann Sphere* $\mathbb{C} \cup \{\infty\}$ (or $\mathbb{C}_\infty$ or $\mathbb{P}$). On a Riemann sphere, the only possible global and invertible holomorphic transformations are given by the *Möbius Group* $\mathrm{SL}(2,\mathbb{C})$ (more precisely it must be quotientized by $\mathbb{Z}_2 = \{-1, 1\}$, namely $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$). Locally, the holomorphic transformations are generated by the Lie algebra given by the *Witt Algebra*. In the physics community it is said that the local conformal group in 2 dimensions is infinite dimensional, which is, however, incorrect. First there exists no infinite dimensional group of conformal transformations in the Euclidean plane ($\mathbb{R}^2 \simeq \mathbb{C}$) and second, when physicists talk about local conformal transformations, they are actually referring to the space of *Conformal Killing vector fields*. These vector fields generate an infinite dimensional Lie Algebra called *Witt Algebra* and it is not possible to find an infinite dimensional Lie group with the Witt algebra as its Lie Algebra (for more information see [Sch08]). Mathematically speaking a conformal transformation is a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}'$ from two semi-Riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$ such that there exists a smooth function $\Omega : \mathcal{M} \to \mathbb{R}$ such that $\phi^* g' = \Omega^2 \cdot g$ (where ϕ^* is the pull-back of g' with respect to ϕ). Now, for semi-Riemannian manifolds $\mathcal{M}, \mathcal{M}' = \mathbb{R}^{p,q}$, a conformal killing vector field is a vector field X, such that its flow φ_t^X is conformal for all t in a neighbourhood of 0. In a local chart (U, x), we can expand the vector field as $X = X^\mu \partial_\mu$ and it will be a conformal killing vector field provided that there exists a smooth function $k : \mathcal{M} \to \mathbb{R}$ such that $\partial_\nu X_\mu + \partial_\mu X_\nu = k \cdot g_{\mu\nu}$. Now, this X^μ is what we called in (1.2.25) ϵ^μ for the case of $M = \mathbb{R}^{2,0} \simeq \mathbb{C}$. In fact, in view of the fact that the Lie algebra of the group $Diff_+(\mathcal{M})$ is the algebra of smooth vector fields $\mathrm{Lie}(Diff_+(\mathcal{M})) = \mathfrak{X}(\mathcal{M})$ and that for a Lie algebra the flux of an element is exactly the exponential map giving an element of the group (we have that $\varphi_t^X = \exp(tX)$), we can, roughly speaking, say that conformal killing vector fields generate conformal local transformations. In this regard, if we consider an infinitesimal, holomorphic (analytic) transformation: $z \mapsto z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n z^n$, then we can informally define the vector field generating this transformation as $X = \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \partial_z$. The elements $L_n := z^{1-n}\partial_z$ span a subgroup of $\mathfrak{X}(\mathcal{M})$ which is exactly the Witt Algebra, with commutation relations given by: $[L_n, L_m] = (n-m)L_{m+n}$ (due to a natural automorphism of the Witt Algebra, the L_n can also be written as $L_n = -z^{n+1}\partial_z$, which is more used in the physics community). For the Euclidean plane, only those infinitesimal transformations L_n with $n = -1, 0, 1$ can actually be extended to global ones, giving the right conformal group $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ (considering also $\bar{L}_n := -\bar{z}^{n+1}\partial_{\bar{z}}$ and $\bar{L}_{-1}, \bar{L}_0, \bar{L}_1$).

A conformal killing vector field $X = X^\alpha \partial_\alpha$ on the worldsheet Σ brings with it a conserved current $J^\alpha = T^{\alpha\beta}X_\beta$ (with $\alpha, \beta = \tau, \sigma$) where $T^{\alpha\beta}$ is the energy-momentum tensor given by the variation of the Polyakov action with respect to the metric $T^{\alpha\beta} = -(2/\sqrt{-h})\delta S/\delta h_{\alpha\beta}$. The fact that J^α is conserved can be easily seen by using the fact that $\partial_\alpha X_\beta + \partial_\beta X_\alpha = k \cdot g_{\alpha\beta}$, and also that $T^{\alpha\beta}$ is symmetric and traceless (due to Weyl invariance). The charge associated to J^α is given by the integration $Q = \int d\sigma J^\tau$, and the action of X on fields ϕ is given by the commutator $\delta_X \phi = [Q, \phi]$. Because X can be thought of as generating local conformal transformations, the charge Q generates local conformal transformations on fields ϕ.

In order quantise the theory on the Euclidean plane it used the *Radial Quantisation*. One direction must be compactified and the resultant space will be a cylinder with coordinates $(\tau, \sigma) = (\tau, \sigma + 2\pi L)$ (the compact direction is the space direction and L is the radius of the cylinder). We can now map the cylinder to the complex plane by identifying $z = e^{(\tau + i\sigma)2\pi/L}$. In order to give well-defined quantities and in analogy to the time ordering in QFT, if we consider that the time direction in the $\mathbb{C}$-plane is given by the radial direction, then *Radial Ordering* is defined as:

$$\mathcal{R}(\phi(z)\phi(w)) = \begin{cases} \phi(z)\phi(w) & \text{if } |z| > |w| \\ \phi(w)\phi(z) & \text{if } |z| < |w| \end{cases} \tag{1.2.28}$$

(if $\tau_1 < \tau_2$ then $|z_1| = e^{2\pi\tau_1/L} < |z_2| = e^{2\pi\tau_2/L}$).

With this prescription it is possible to define commutators of operators $A = \oint (dz/2\pi i)a(z)$ and $B = \oint (dw/2\pi i)b(w)$ (with $a(z)$ and $b(w)$ holomorphic fields) as:

$$[A, B] = \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} \mathcal{R}(a(z)b(w)) \tag{1.2.29}$$

and also:

$$[A, b(w)] = \oint_w \frac{dz}{2\pi i} \mathcal{R}(a(z)b(w)) \tag{1.2.30}$$

(the w in the integration sign $\oint_w$ means that we are integrating over a closed path containing w).

Now, the Energy-Momentum tensor will have two components T_{zz} and $T_{\bar{z}\bar{z}}$, which will be holomorphic and anti-holomorphic respectively and usually they are denoted as $T_{zz}(z) := T$ and $T_{\bar{z}\bar{z}}(\bar{z}) := \bar{T}$. The current will, therefore, be given by $J := J^z(z) = T(z)X(z)$ and $\bar{J} := J^{\bar{z}} = \bar{T}(\bar{z})\bar{X}(\bar{z})$. Focusing only on the holomorphic part of the theory (since the anti-holomorphic part is completely analogous), the holomorphic part of the charge will then be given by $Q = \oint (dz/2\pi i)T(z)X(z)$. If we expand in Laurent series $X = \sum_{n\in\mathbb{Z}} X_n z^{n+1}$, then we can see that the charge will be given by:

$$Q = \oint \frac{dz}{2\pi i} \sum_{n\in\mathbb{Z}} X_n z^{n+1} T(z) = \sum_{n\in\mathbb{Z}} X_n L_n \tag{1.2.31}$$

where the modes L_n have been defined as:

$$L_n = \oint \frac{dz}{2\pi i} T(z) z^{n+1} \tag{1.2.32}$$

and it can be seen that L_n are the modes of the Energy-Momentum tensor as $T = \sum_{n\in\mathbb{Z}} = L_n/z^{n+2}$. Using the definition of the commutator (1.2.29), and also the *Operator Product Expansion* (OPE):

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular terms} \tag{1.2.33}$$

a direct calculation (using Residue theorem), shows that the L_n satisfies the *Virasoro Algebra*, which is nothing but the *central extension* of the Witt algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{1.2.34}$$

The constant c is called *Central Charge* or *Conformal Anomaly*. The term "anomaly" comes from the fact that it appears as a consequence of a breaking by quantum corrections of the classical conformal invariance. Classically, in fact, the generators of the local conformal symmetry satisfies the Witt algebra, but when one quantise the system, the symmetry is broken and it appears the conformal anomaly c as the result of an introduction of a scale in the system.

There are various superstring theories which can be constructed given these ingredients. Closed strings are called *Type II Strings*, while a theory with unoriented open and closed strings is called of *Type I*. Due to RR-tadpole cancellations, it is not possible to construct a theory with only unoriented open strings or only unoriented closed strings. Their combination can, however, produce a consistent theory which is indeed the type I string theory. As we shall see, type II strings are divided in type IIA and type IIB after a suitable projection (GSO-projection) which allows to project out the tachyonic particle (which will be otherwise present in the theory) and to reproduce a supersymmetric spectrum, which is required given that the NS-R and R-NS sectors contain each a gravitino ψ^{μ}_{α}. It can also be defined a theory with different left and right degrees of freedom, namely it can be considered 8 bosonic + 8 fermionic right movers and 24 bosonic left movers. This is the *Heterotic String* and its peculiarity is a "miraculous" anomaly cancellation. Heterotic string theory comes with two possible gauge symmetries at the perturbative level (within the massless spectrum): $E_8 \times E_8$ or $SO(32)$. The number of possible string theories rises thus to five. This proliferation of string theories made physicists wonder which one was the correct one since at first all these theories seemed to be completely unrelated. During the early '90s, the *Second Superstring Revolution*[2] took place, it was realised that those string theories are all related to one another through so-called *dualities*, and, in fact, derive from an 11-dimensional theory called M-theory, firstly proposed by Witten [Wit95]. In the same years the discovery by Dai, Leigh, and Polchinski [DLP89] of Dp-branes (Dirichlet surfaces with p-spatial dimensions, thus with a world-volume (p+1)-dimensional) as submanifolds onto which open strings could end and the subsequent realisation of Polchinski [Pol95] that Dp-branes actually carry Ramond-Ramond charges, gave a strong encouragement to the physics community in continuing to pursue the path of string theory since these charges were (are) required by string dualities.

[2]The first superstring revolution happened a decade earlier, when it was discovered an anomaly cancellation in type I strings via the *Green-Schwarz Mechanism* and the subsequent year it was discovered the Heterotic string.

1.3 Dirichlet Branes

After employing the conformal gauge in the Polyakov action, the metric of the worldsheet is brought into a Minkowski form $\eta_{\alpha\beta} = \text{diag}(-1, 1)$ and the resultant action is:

$$S = -\frac{T}{2} \int d^2\xi \, \partial_\alpha X^M \partial^\alpha X_M \tag{1.3.1}$$

When varying the above action in trying to find the equations of motion, a boundary term of the form:

$$T \int d\tau \delta X^M \partial_\sigma X_M \big|_{\sigma=0,\pi} \tag{1.3.2}$$

appears. If we want this term to vanish, we can impose $\delta X^M|_{\sigma=0,\pi} = 0$ and/or $\partial_\sigma X^M|_{\sigma=0,\pi} = 0$. The former are called *Dirichlet Boundary Conditions* while the latter are *Neumann Boundary Conditions*. In principle there 4 possible combinations that one can choose, however, mixed boundary conditions are also a possibility, for example, for $\sigma = 0$ choosing $\partial_\sigma X^i = 0$ with $i = 0, ..., p$ and $\delta X^j = 0$ for $j = p+1, ..., D-1$ is possible. The latter Dirichlet conditions imply that the position of the string ends are fixed in the $j = p+1, ..., D-1$ spacetime directions. The end points are then allowed to move on a $(p+1)$-dimensional surface. These surfaces are called *Dp*-branes, which are dynamical extended objects with a $(p+1)$-worldvolume which clearly break Poincaré invariance $SO(1, D-1) \to SO(1, p) \times SO(D-1-p)$. When the spacetime $\mathbb{R}^{1,D-1}$ is compactified into $\mathbb{R}^{1,3} \times \mathscr{Y}$ (with $\mathscr{Y}$ a suitable 6-dimensional compact manifold), then Dp-branes can be defined provided their world-volumes extend to all 4-dimensional spacetime manifold $\mathbb{R}^{1,3}$ and eventually wrap some cycles in the compact dimensions.

Under *T-duality*, Neumann boundary conditions are mapped into Dirichlet boundary conditions, and if the D-brane was wrapping a circle, in the dual theory it will not and vice versa.

The open string spectrum, when employing mixed boundary conditions, i.e. when strings end on Dp-branes, should be retrieved considering that states must fall into representations of $SO(1, p) \times SO(D-p-1)$. Massless states in the light-cone gauge will be given by a massless vector field A_α ($\alpha = 0, 1, ..., p$) and $(D-p-1)$-scalar fields ϕ^a ($a = p+1, ..., D-1$). The low-energy effective action is in fact given by:

$$S_{Dp} \propto \int d^{p+1}x \left(-\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + \frac{1}{2} \partial_\alpha \phi^a \partial^\alpha \phi^a + ... \right) \tag{1.3.3}$$

The scalars ϕ^a are in a number equals to the number of transverse directions of the Dp-brane and in this regard they can be interpreted as the fluctuations of the Dp-brane in transverse directions. This is translated in the fact that these membranes are actually dynamical objects rather than just being static boundary conditions when quantisation is employed.

As is well known, the electromagnetic field encoded in the 4-potential A_μ couples to charge particles with an action given by:

$$S_{\text{int}} = e \int A_\mu \frac{dx^\mu}{d\tau} d\tau \tag{1.3.4}$$

This means that a 0-dimensional object (pointlike particle) which sweeps out a 1-dimensional worldline, couples to a 1-form $A = A_\mu dx^\mu$. Taking into consideration now a $p+1$-form, this will couple to an object with a $(p+1)$-dimensional worldvolume. These objects are indeed Dp-branes:

$$S_{\text{int}} = \mu_p \int_W \phi^* A_{p+1} \tag{1.3.5}$$

where $\phi^* A_{p+1}$ stands for the pull-back of the $(p+1)$-form A_{p+1} to the worldvolume W of the Dp-brane by means of the embedding $\phi : W \to \mathscr{M}$ of the Dp-brane into the spacetime manifold $\mathscr{M}$. This is translated in the fact that Dp-branes are electrically charged under A_{p+1} gauge fields as can be seen by using Gauss's law: $\mu_p = \int_{S^{D-p-2}} \star F_{p+2}$, where $F_{p+2} = dA_{p+1}$ is the field strength of the gauge field A_{p+1} and we are integrating over a $(D-2-p)$-dimensional sphere S^{D-2-p}. The Hodge star-operator $\star$ (we shall describe it in §2.1.4) appears in analogy with the electromagnetic case, where, integrating the Maxwell equation $d \star F = \star J$, with $F = dA$ (where $A = A_\mu dx^\mu$) and $J = J_\mu dx^\mu$ (where $J^\mu = (\rho, \vec{J})$ and $\rho = e\delta^{(3)}(\xi)$ for a pointlike particle of charge e), over a volume Ω bounded by the 2-sphere $\partial\Omega = S^2$ with the particle inside it, we get

$$\int_\Omega d \star F = \int_\Omega J \overbrace{\Longleftrightarrow}^{\text{Stokes}} \int_{S^2} \star F = e \tag{1.3.6}$$

If a magnetic charge g is brought into play, it will carry a magnetic 4-current J_m, the first pair of Maxwell equations $dF = 0$ is then modified to $dF = \star J_m$ and as before we will get:

$$\int_{S^2} F = g \tag{1.3.7}$$

As was first pointed out by Dirac [Dir31], the existence of a magnetic charge, i.e. a magnetic monopole, would result in a quantisation of both the electric and magnetic charges, they would, in fact, obey the following *Dirac quantisation condition*:

$$e \cdot g = 2\pi n \quad n \in \mathbb{Z} \tag{1.3.8}$$

All these facts from electromagnetism are transposed and generalised to arbitrary gauge fields, namely for p-forms fields. A (p+1)-form A_{p+1} will couple electrically to a $p+1$-dimensional object (a Dp-brane), and the resulting interaction will be:

$$S_{\text{int}} = \int_{\mathscr{W}} \phi^* A_{p+1} \tag{1.3.9}$$

where $\mathscr{W}$ is the world-volume a Dp-brane (i.e. it is (p+1)-dimensional) and $\phi : \mathscr{W} \to \mathscr{M}$ is the embedding of the world-volume into the spacetime. The electric charges will be given by an integration over the spheres S^{D-p-2} (surrounding the Dp-brane) for a D-dimensional spacetime $\mathscr{M}$:

$$e = \int_{S^{D-p-2}} \star F_{p+2} \tag{1.3.10}$$

The magnetic dual branes that couple to a p-form field are given by $(D-p-4)$-branes, since these branes can be surrounded by S^{p+2} spheres so that the magnetic charge will be correctly given by:

$$g = \int_{S^{p+2}} F_{p+2} \tag{1.3.11}$$

To sum up, in $D = 10$ spacetime dimensions, p-form gauge fields A_p couple electrically to $(p-1)$-branes and magnetically to $(7-p)$-branes.

1.3.1 World-Volume Action for Dp-Branes

Until 1988, membranes - which were found as static solutions to supersymmetric gauge theories - weren't connected to string theory, and the real revolution was started by Dai, Leigh, and Polchinski [DLP89], where the authors recognized for the first time that *Dirichlet boundary conditions* on open strings could give rise to static membranes, which, after quantisation, can become dynamical. Immediately after that, Leigh [Lei89] realised that the Dirac-Born-Infeld (DBI) action for the D-brane reproduces correctly its equations of motion. After these pioneering works, Cederwall et al. [Ced+97] constructed a full supersymmetric and k-symmetric[3] action for D3-branes, including their coupling to background superfields of $10D$ type IIB supergravity. A completion of this work was done by almost the same authors [Ced+96] for general Dp-branes, with p even or odd for type IIA or type IIB backgrounds respectively.

The dynamics of the bosonic part of a Dp-brane is then governed by a low-energy effective action given by the *Dirac-Born-Infled* action plus a *Wess-Zumino* (or Chern-Simons) term:

$$S_{Dp} = S_{DBI} + S_{WS} \tag{1.3.12}$$

As we mentioned in §1.2.1, the low-energy effective field theory of a bosonic string contains a graviton G, a dilaton ϕ and a 2-form field B_2. As we shall see in the next chapter, also the superstring theories contains these fields. In particular they appear in the NS-NS sector. This means that the action we are constructing for Dp-branes must include these background fields and also, as we pointed out before, it will contains the $U(1)$ world-volume gauge field A_α (with $\alpha = 0, ..., p$). The Dirac-Born-Infled action will then take the following form:

$$S_{DBI} = -T_p e^{-\phi} \int_{\mathscr{W}} d^{p+1}\xi \sqrt{-\det\left(i^*(G+B_2) + \frac{\ell_s^2}{2\pi} F_2\right)} \tag{1.3.13}$$

where T_p is the tension of the Dp-brane, $\mathscr{W}$ is the world-volume, i is the embedding of the Dp-brane into the spacetime and ℓ_s is the string length (1.2.2). The Wess-Zumino part is needed in order to have a supersymmetric action and it is defined by means of other fields which appear in the R-R sectors of the various string theories. These fields are p-forms and are denoted as C_p. The action is then given by:

$$S_{WS} = \mu_p e^{-\phi} \int_{\mathscr{W}} \sum_p i^* C_p \wedge e^{i^* B_2 + \frac{\ell_s^2}{2\pi} F_2} \tag{1.3.14}$$

Where μ_p is the R-R charge of the brane, given that Dp-branes couple to the R-R forms C_p of the bulk. Since stable[4] Dp-branes are BPS objects, the R-R charge μ_p must be equals to the tension T_p.

[3] This is a symmetry that supermembrane presents. When the action for a supermembrane was firstly constructed in [HLP86] the requirement of the complete action to posses a local fermionic symmetry, introduces a fermionic local SUSY parameter: k.

[4] In type II string theories the stable Dp-branes have p even for type IIA and p odd for type IIB. This stability condition is obtained by considering the fact that in type IIA string theory (as we shall see later on) the R-R sector contains n-form gauge fields with $n = 1$ and $n = 3$. These fields couple electrically to D0-branes and D2-branes respectively, while they couple magnetically to D4-branes and D6-branes respectively. For type IIB string theory, the R-R sector is given by n-form gauge fields with $n = 0$, $n = 2$ and $n = 4$. In this way, the stable Dp-branes should be those with $p = -1$ (interpreted as a D-instanton), $p = 3$, $p = 5$ and $p = 7$.

Dp-branes then support gauge theories and the crucial fact is that a stack of coincident N branes support a $U(N)$ gauge theory. Including D-branes in string theory then allows to implement gauge theories, meaning that the Standard Model can be constructed in this context. The construction of the Standard Model is, however, not a simple task. The Standard Model building requires lots of consistency checks in order for it to be well-defined. In our construction of §4 we will not worry about it, we will suppose that it could be constructed without incurring in inconsistencies and anomalies.

1.4 Type II Superstrings

As we said at the beginning of §1.2, the formulation of superstrings in terms of the action (1.2.12) is attributed to Ramond [Ram71] and Neveu and Schwarz [NS71], and for this reason they are generally called then *RNS strings*. As it is, the ground state is tachyonic and the spectrum is not supersymmetric. To get around these problems, Gliozzi, Scherk, and Olive [GSO77] came up with a solution which enabled to get rid of the tachyonic state and to achieve a spacetime supersymmetric spectrum by means of a suitable projection, now called GSO-projection. This truncation is done by keeping in the NS sector only states with positive *G-parity*, where G is the operator $G = (-1)^{F+1}$ and F is the fermionic number, i.e. the number of fermionic excitations. In the R sector, the ground state is a fermion, more precisely it is a Majorana-Weyl spinor and it can have positive or negative chirality, in particular, the left and right moving modes in the R sector can have the same or opposite chirality. Based on this concordance, two different types of theories arise: *Type IIA* and *Type IIB* superstrings.

1.4.1 Type IIA Strings

The massless spectrum of $D = 10$ type IIA superstring theory after the GSO projection is given by:

Sector	Representation of SO(8)	10-dimensional field
NS-NS	$1 + 28_V + 35_V$	ϕ, B_2, G
NS-R	$8_C + 56_C$	λ^1, ψ^1
R-NS	$8_S + 56_S$	λ^2, ψ^2
R-R	$8_V + 56_V$	C_1, C_3

Where 8_S and 8_C represents two representations of opposite chirality. The fields in NS-NS sector are the dilaton ϕ, the 2-form field B_2 and the graviton G. In the NS-R and R-NS sectors there are two dilatinos λ^i and two Rarita-Schwinger gravitinos ψ^i. Finally, the R-R sector contains a 1-form field C_1 and a 3-form field C_3. The low-energy effective supergravity theory (SUGRA) in 10 spacetime dimensions is then encoded in a bosonic action composed of three pieces:

$$S_{IIA} = S_{NS} + S_R + S_{CS} \tag{1.4.1}$$

The first piece refers to the NS-NS sector and is given by (in string frame):

$$S_{NS} = \frac{1}{2k_{10}^2} \int e^{-2\phi} \left(\mathcal{R} \star \mathbb{1} + 4 d\phi \wedge \star d\phi - \frac{1}{2} H_3 \wedge \star H_3 \right) \tag{1.4.2}$$

where we are employing the p-form formalism and $H_3 = dB_2$ while the constant k_{10} is given by the obvious generalisation of (1.2.4) to 10 spacetime dimensions. The other two pieces S_R and S_{CS} contain the R-R fields and are given by:

$$S_R = -\frac{1}{4k_{10}^2} \int \left(F_2 \wedge \star F_2 + \tilde{F}_4 \wedge \star \tilde{F}_4 \right) \tag{1.4.3}$$

$$S_{CS} = -\frac{1}{4k_{10}^2} \int B_2 \wedge F_4 \wedge F_4 \tag{1.4.4}$$

where $F_2 = dC_1$, $F_4 = dC_3$ and $\tilde{F}_4 = F_3 + A_1 \wedge H_3$ is a gauge-invariant combination. Recalling from §1.2.1 that the dilaton field weights each worldsheet diagrams with Euler Characteristics χ by $e^{-\chi\phi}$ and that for a sphere the Euler Characteristic is $\chi = 2$, then we see that the terms in the S_{NS} action describe the leading order of the expansion in g_s (spherical world-sheet). It can also be noted that in S_R and S_{CS} there is not the dilaton-dependent term. This is because of the definition of our R-R fields and, by a proper rescaling, a factor of $e^{-2\phi}$ could be factorised out. However this rescaling is never employed in practice and we will leave the actions above as they are.

1.4.2 Type IIB Strings

The massless spectrum of $D = 10$ type IIB superstring theory after the GSO projection is given by:

Sector	Representation of SO(8)	10-dimensional field
NS-NS	$1 + 28_V + 35_V$	ϕ, B_2, G
NS-R	$8_S + 56_S$	λ^1, ψ^1
R-NS	$8_S + 56_S$	λ^2, ψ^2
R-R	$1 + 28_C + 35_C$	C_0, C_2, C_4

We can see that the NS-NS sector is exactly the same as that of type IIA. For NS-R and R-NS sectors the unique difference is that the fields have the same chirality instead of the opposite, type IIB superstring theory is, in fact, a theory with a chiral spectrum. The R-R sector is composed of a 0-form field C_0, a 2-form field C_2 and a 4-form field C_4. The peculiarity of C_4 is that it has a *self-dual field strength* (actually a gauge-invariant combination of it is self-dual as we are going to see shortly). In principle this wouldn't allow to define an action in a manifestly covariant form. However, it is still possible to write down an action which, when supplemented with the self-duality constraint, reproduce the correct equations of motion (for details see §8 of [BBS07]). The action will then be again composed of three pieces:

$$S_{IIB} = S_{NS} + S_R + S_{CS} \tag{1.4.5}$$

where S_{NS} is the same as that of type IIA, namely (1.4.2). The other pieces are given by:

$$S_R = -\frac{1}{4k_{10}^2} \int \left(F_1 \wedge \star F_1 + \tilde{F}_3 \wedge \star \tilde{F}_3 + \frac{1}{2} \tilde{F}_5 \wedge \star \tilde{F}_5 \right) \tag{1.4.6}$$

$$S_{CS} = -\frac{1}{4k_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \tag{1.4.7}$$

where $F_1 = dC_0$, $F_3 = dC_2$, $F_5 = dC_4$ and the gauge-invariant combinations have been defined: $\tilde{F}_3 = F_3 - C_0 H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. The self-duality condition is satisfied by the field $\tilde{F}_5$:

$$\tilde{F}_5 = \star\tilde{F}_5 \tag{1.4.8}$$

and this condition has to be imposed as a constraint on the equations of motion.

A very useful formulation of the type IIB action is given by noticing that this supergravity theory has a global $SL(2,\mathbb{R})$ symmetry. The manifestly $SL(2,\mathbb{R})$ invariant action is particularly useful when considering compactifications from F-theory. Let's define the axio-dilation as the complex field defined by the following combination of the dilaton ϕ and the R-R 0-form C_0 (which is called *axion* due to its shift-symmetry in the SUGRA approximation):

$$\tau = C_0 + ie^{-\phi} \tag{1.4.9}$$

and under $SL(2,\mathbb{R})$:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \tag{1.4.10}$$

Let's also define the combined three-flux:

$$G_3 = F_3 - \tau H_3 \tag{1.4.11}$$

which transforms as:

$$G_3 \mapsto \frac{G_3}{c\tau + d} \tag{1.4.12}$$

Using these quantities, the type IIB supergravity action can be rewritten in the following manifestly $SL(2,\mathbb{R})$ invariant form (in the Einstein frame $G^E = e^{-\phi/2}G^S$):

$$S_{IIB} = \frac{1}{2k_{10}^2}\int\left(\mathscr{R}\star\mathbb{1} - \frac{d\tau\wedge\star d\overline{\tau}}{2(\mathfrak{Im}(\tau))^2} - \frac{G_3\wedge\star\overline{G_3}}{2\mathfrak{Im}(\tau)} - \frac{\tilde{F}_5\wedge\star\tilde{F}_5}{4}\right) + \frac{1}{8k_{10}^2}\int\frac{C_4\wedge G_3\wedge\overline{G}_3}{\mathfrak{Im}(\tau)} \tag{1.4.13}$$

This formulation and the transformation of the axio-dilaton τ, recall the modular invariance of the modular parameter of a torus. This is not an accident since compactification of *M-theory* on a torus leads, through a series of dualities, to type IIB theory. Also, from a different, but equivalent, point of view, type IIB supergravity compactified on a complex n-dimensional space B_n as: $\mathscr{M} = \mathbb{R}^{1,9-2n} \times B_n$ and in the presence of 7-branes, leads to a structure of an *elliptic fibration* over the compactification space [Wei18]. The axio-dilaton field τ, in fact, is not a constant, but varies in the direction normal to the 7-brane and this is because of the backreaction of the 7-branes on the geometry and on the supergravity background. This variation of the axio-dilaton field, in turn, gives rise to an elliptic fibration over the compactification space. This is the principle of *F-Theory*, firstly proposed by Vafa [Vaf96], and then studied by both physicists and mathematicians because of all the relations between different branches of mathematics that it brings with. In the following sections we are going to point out the main features of M-theory and then how F-theory can be seen as arising form M-theory compactified on a torus T^2.

1.5 M-Theory

M-theory was firstly proposed by Witten [Wit95] while he was analysing the strong coupling limit of type IIA string theory. What he found was that the strong coupling behaviour of type

IIA theories was an eleven dimensional supergravity. The M-theory would then be the hypothetical eleven dimensional complete theory which would have, as its low-energy effective field theory, the eleven dimensional supergravity dual to the strong coupling limit of type IIA/IIB supergravity. His reasoning was roughly the following. In type IIA/IIB string theories the R-R sector gives rise to some anti-symmetric fields. Focusing on the type IIA there will be a 1-form A and a 3-form A_3. Considering the charge of the field A to be W, then a state of mass M charged under the gauge field A will obey the following condition:

$$M \geq \frac{C}{\lambda}|W|$$

which is saturated for the so-called BPS-states: $M = C/\lambda|W|$, where C is a constant and λ is the string coupling constant i.e. the dilaton VEV: $\langle e^{\phi}\rangle = \lambda$. However in the elementary string spectrum the charge W is identically zero because only massless states are considered. Nevertheless BPS-black hole solutions can be constructed. It can be assumed that in the theory these sates do indeed exist, and there will be a tower of states given by $M = c|n|/\lambda$ where n is an arbitrary integer. What is found is that an 11-dimensional supergravity theory will reproduce this spectrum, which consists of the spectrum of type IIA and these massive BPS-sates. This 11-dimensional supergravity is the low-energy effective action of a "new theory": the M-theory, which is not a theory of strings, but, however, is connected to string theories. In fact, type IIA supergravity is retrieved from 11-dimensional supergravity (which means from M-theory) by compactification on a circle:

$$\Big(\textit{11-dim SUGRA on } S^1 \textit{ with radius } R\Big) \simeq \Big(\textit{Type IIA with coupling } g_s = R/\sqrt{\alpha'}\Big)$$

And considering the T-duality between type IIA and type IIB:

$$\Big(\textit{Type IIA strings on } S^1 \textit{ with radius } R\Big) \simeq \Big(\textit{Type IIB strings on } S^1 \textit{ with radius } \tilde{R} = \alpha'/R\Big)$$

we can also get from M-theory to type IIB supergravity.

What is interesting is then the 11-dimensional supergravity theory, which, remarkably, is unique. Its construction process goes as follows. First, it must contains gravity, namely a graviton which will be a symmetric traceless tensor of $SO(D-2)$ (the little group for a massless particle). This means that it will have $\frac{(D-2)(D-1)}{2} = 44$ degrees of freedom (polarisation states). By supersymmetry, the graviton should possess a superpartner, i.e. the *gravitino* Ψ^M_α of spin 3/2 (Rarita-Schwinger field with one spacetime index M and one spinor index α). The gravitino will have a total of 128 degrees of freedom. Since the bosonic degrees of freedom should match the fermionic ones because of supersymmetry, there is a gap of 84 degrees of freedom that must be filled. This is done by introducing a gauge invariant 3-form field A_3. In this way the eleven dimensional supergravity action can be readily written down:

$$S_{11\mathrm{D}} = \frac{1}{2k_{11}^2}\int \Big(\mathscr{R}_{11} \star \mathbb{1} - \frac{1}{2}F_4 \wedge \star F_4\Big) - \frac{1}{12k_{11}^2}\int A_3 \wedge F_4 \wedge F_4 \tag{1.5.1}$$

where $\mathscr{R}_{11}$ is the Ricci scalar of the eleven dimensional graviton G_{11}, F_4 is the field strength of A_3 and $4\pi k_{11}^2 = (2\pi\ell_p)^9$.

The above action is the only one that can be written down given the following constraints (requirements) that the supergravity theory must satisfy:

- Invariance under gauge transformation $A_3 \mapsto A_3 + d\Lambda$;
- General coordinates invariance;
- Local Lorentz invariance;
- Local Supersymmetry.

In the period when it was firstly constructed (late '70s), the 11-dimensional supergravity wasn't thought to be really representing the fundamental theory of nature due to the fact that it is not renormalisable and it is non-chiral. However, within the context of string theory, if the 11-dimensional supergravity is viewed as an effective field theory, we do not need the renormalisability condition and the chiral spectrum can be recovered by either introducing branes or compactify on appropriate manifolds.

1.5.1 Kaluza-Klein Reduction of 11-Dimensional SUGRA

As we mentioned a the beginning of this section, the 11-dimensional supergravity is related to the 10-dimensional type IIA supergravity by means of a compactification of a single direction into a circle and by Kaluza-Klein reducing the fields content into the lower dimensional theory. Let's denote by Latin letters $M,N,O,P,...$ the eleven dimensional spacetime indices, while with Greek letters $\mu,\nu,\rho,\sigma,...$ the ten dimensional indices. The compactified direction will be the $M=10$, so that $\mu = 0,1,2,...,9,11$. The metric in $\mathscr{M}_{11}$ is given by (in local coordinates):

$$G_{11} = G_{MN} dx^M \otimes dx^N \tag{1.5.2}$$

By employing the Kaluza-Klein dimensional reduction the metric can be written in term of 10-dimensional fields in the following form:

$$G_{MN} = e^{-2\phi/3} \begin{pmatrix} g_{\mu\nu} + e^{2\phi} A_\mu A_\nu & e^{2\phi} A_\mu \\ e^{2\phi} A_\nu & e^{2\phi} \end{pmatrix} \tag{1.5.3}$$

so that we get a 10-dimensional metric $g_{\mu\nu}$, a $U(1)$ gauge field A_μ and the dilaton ϕ. The line element take the form:

$$ds^2 = G_{MN} dx^\mu dx^\nu = e^{-2\phi/3} g_{\mu\nu} dx^\mu dx^\nu + e^{4\phi/3}(dx^{11} + A_\mu dx^\mu)^2 \tag{1.5.4}$$

The three-form $A_3 = A^{(11)}_{MNP} dx^M \wedge dx^N \wedge dx^P$ in the eleven dimensional theory will be decomposed into a 3-form and a 2-form:

$$A^{(11)}_{\mu\nu\rho} = A_{\mu\nu\rho} \qquad A^{(11)}_{\mu\nu 11} = B_{\mu\nu} \tag{1.5.5}$$

And the field strengths will be F_4 and H_3 for $A^{(10)}_3$ and B_2 respectively.

Once the integration over the compact coordinate of the circle is performed, the 10-dimensional type IIA supergravity action (1.4.1) is recovered.

1.6 F-Theory

As we mentioned in §1.4.2, F-theory can be thought of as the non-perturbative formulation of type IIB theory when the backreaction of 7-branes on the geometry is taken into account. The construction of F-Theory is made by recognising that the type IIB theory possesses a global $SL(2,\mathbb{R})$ symmetry, and in particular it is believed that the full theory (not only the effective field theory given by the action (1.4.5), but the full non-perturbative type IIB string theory) is left invariant by the subgroup $SL(2,\mathbb{Z})$. In this way, one can see that the action of $SL(2,\mathbb{Z})$ on the axio-dilaton $\tau = C_0 + e^{-\phi}$ can be interpreted as an $SL(2,\mathbb{Z})$ monodromy. In fact, let's consider a compactification of the 10-dimensional background spacetime into: $\mathscr{M} = \mathbb{R}^{1,9-2n} \times B_n$ with B_n a compact and complex n-dimensional manifold. Take also a D7-brane extended onto the Minkowski $\mathbb{R}^{1,9-2n}$ (in order to preserve Poincaré invariance) and wrapping an holomorphic cycle in the compact dimensions, so that its world-volume will be given by: $\mathscr{W} = \mathbb{R}^{1,9-2n} \times \Sigma_{n-1}$. If we take the compact dimension to be the Riemann sphere, namely the complex plane compactified by the addition of the point at infinity, we can write $\mathscr{M} = \mathbb{R}^{1,7} \times \mathbb{C}$, so that the D7-brane will have a world-volume $\mathscr{W} = \mathbb{R}^{1,7}$. The complex plane is perpendicular to the directions into which the D7-brane extends. If we recall that Dp-branes couple electrically to $(p-1)$-forms and magnetically to $(7-p)$-forms, then we can immediately recognise that D7-branes couple to the R-R 0-form C_0 (i.e. the axio-dilaton τ). If we consider the flux of the 0-form out of a sphere S^1 in the complex plane $\mathbb{C}$ around the D7-brane, then because the brane is a magnetic source of C_0, the integral:

$$\int_{S^1} dC_0 = 1 \tag{1.6.1}$$

is not zero, and can be non-vanishing only in presence of a brunch cut. If the brane sits at $z = z_0$, then by supersymmetric requirements the only possible form that $\tau(z)$ can take is:

$$\tau(z) = \frac{1}{2\pi i}\ln(z - z_0) + \text{ regular terms at } z_0 \tag{1.6.2}$$

and this brunch cut will induce a monodromy:

$$\tau \to \tau + 1 \tag{1.6.3}$$

This means that a D7-brane induces a monodromy:

$$M_{[1,0]} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{1.6.4}$$

This set-up can be appreciated in figure (1.1). Given these considerations, the natural thing to do is to try to generalise branes and strings accommodating general monodromies given by $SL(2,\mathbb{Z})$ transformations.

1.6.1 (p,q) Strings and (p,q) 7-branes

Under a transformation of the axio-dilaton τ of the type (1.4.10) but with $a,b,c,d \in \mathbb{Z}$, the doublet of fields $\Phi^a = \binom{C_2}{B_2}$ must transform as $M\Phi$ with $M \in SL(2,\mathbb{Z})$ defined clearly by: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which by a simple computation can be proved to be equivalent to the transformation

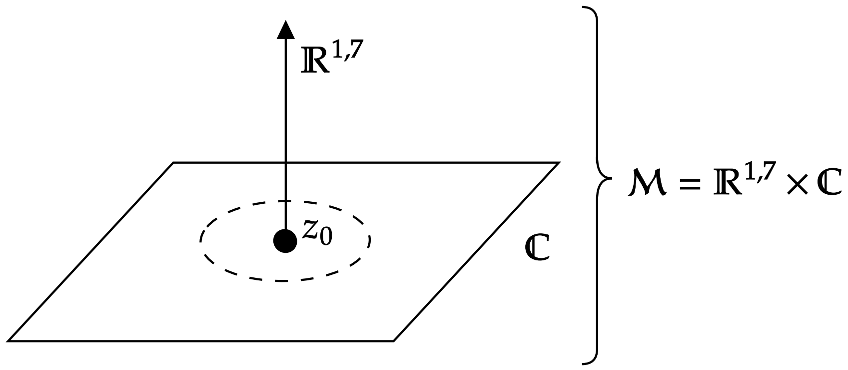


Figure 1.1: Representation of the factorisation of the spacetime manifold $\mathscr{M}$ into $\mathbb{R}^{1,7} \times \mathbb{C}$, with the brane sitting at z_0 and seen as a point in the complex plane $\mathbb{C}$, while extending in the non-compact directions $\mathbb{R}^{1,7}$.

(1.4.12). Considering these facts, we can define a new kind of strings and branes, mainly following [Wei18]. The fundamental string couples to the 2-form B_2 (as we have seen in §1.2.1, more precisely the coupling was given by (1.2.8)), i.e. it is electrically charged with respect to B_2. These strings can end on D7-branes, which are magnetically charged under C_0. But in the type IIB supergravity action, there are two scalar fields which can be rearranged in a doublet: (C_0, ϕ). A D7-brane is then called a $(1,0)$ 7-brane and the fundamental strings which end on these branes are called $(1,0)$ strings. More generally we can construct (p,q) 7-branes where (p,q) strings could end. These strings couple to $pB_2 + qC_2$ with p and q coprime integers, in fact, defined the row charge doublet $Q_a = (q,p)$, we can construct the $SL(2,\mathbb{Z})$-invariant combination $Q_a\Phi^a := \epsilon_{ab}Q^b\Phi^a$ with ϵ_{ab} the invariant anti-symmetric tensor of $SL(2,\mathbb{Z})$ given by: $\epsilon_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In this way, we see that a (p,q) string, with charge column vector given by $Q^a = \binom{p}{-q}$ (we raise/lower indices with $\epsilon^{ab}/\epsilon_{ab}$), is retrieved from an $SL(2,\mathbb{Z})$ transformation of the $(1,0)$ string:

$$Q^a = \begin{pmatrix} p \\ -q \end{pmatrix} = \begin{pmatrix} p & r \\ -q & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = g_{(p,q)} \begin{pmatrix} p \\ -q \end{pmatrix} \tag{1.6.5}$$

In this way, the $SL(2,\mathbb{Z})$ monodromy generated by the backreaction of a (p,q) 7-brane can be found by calculating the action of $g_{(p,q)}$ onto the the monodromy (1.6.4) of the $(1,0)$ 7-brane:

$$M_{[p,q]} = g_{(p,q)} M_{[1,0]} g_{(p,q)}^{-1} = \begin{pmatrix} p & r \\ -q & s \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & r \\ -q & s \end{pmatrix}^{-1} = \begin{pmatrix} 1+pq & p^2 \\ -q^2 & 1-pq \end{pmatrix} \tag{1.6.6}$$

Every (p,q) 7-brane can also be brought back to a D7-brane by the inverse transformation of the above one. This means that locally every single 7-brane can be thought of as a D7-brane. However, when more 7-branes with different p and q are considered, then it may be the case that these cannot be brought simultaneously into the $(1,0)$ form. Nevertheless there are some bound states that can be constructed and that reproduce the ADE Lie groups (see [Wei18]).

As we will see in the next chapter, the structure that is brought by the introduction of 7-branes which backreact on the supergravity geometry is that of an *Elliptic Fibration*. The ten dimensional spacetime is in general compactified on a Calabi-Yau manifold (see §2), however, the compact space B_n appearing in $\mathcal{M} = \mathbb{R}^{1,9-2n} \times B_n$ cannot be Ricci-flat (we are going to show it in §2.2.2) and therefore cannot be a Calabi-Yau. Nevertheless, it is seen that the structure that one obtains when considering the variation of τ is a holomorphic line bundle $\mathcal{L}$ over B_n. Now, the choice of sections of $\mathcal{L}^4$ and $\mathcal{L}^6$ uniquely defines an elliptic fibration over B_n with varying elliptic parameter τ:

$$\mathbb{E}_\tau \to \mathcal{Y}_{n+1} \to B_n \tag{1.6.7}$$

where the above is a *fibration sequence* with $\mathbb{E}_\tau$ an elliptic curve[5] with elliptic parameter τ, $\mathcal{Y}_{n+1}$ is the total space and B_n is the base space of the fibration. We shall come back to these topics in more details in chapter §2.

1.6.2 F-Theory from M-theory

F-theory can be interpreted geometrically by making use of the duality between M-theory on a torus and type IIB string theory on a circle. When this context is analysed, the structure of an elliptic fibration is seen to arise. In order to do that, we consider the low-energy effective supergravity action of M-theory given by (1.5.1) plus an additional topological higher curvature term [Den08; Wei18], giving:

$$S_{11\mathrm{D}} = \frac{1}{2k_{11}^2}\left(\int \mathcal{R}_{11} \star \mathbb{1} - \frac{1}{2}\int F_4 \wedge \star F_4 - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4 + \ell_s^6 \int C_3 \wedge I_8\right) \tag{1.6.8}$$

with:

$$I_8 = \frac{1}{(2\pi)^4}\left(-\frac{1}{768}(\mathrm{tr}(R^2)^2) + \frac{1}{192}\mathrm{tr}(R^4)\right) \tag{1.6.9}$$

(for details of where this term comes from see [DLM95]).

We can now compactify the spacetime into $\mathcal{M}_9 \times T^2$ and the metric will split:

$$ds^2 = \frac{v}{\tau_2}((dx + \tau_1 dy)^2 + \tau_2^2 dy^2) + ds_9^2 \tag{1.6.10}$$

where x, y are periodic coordinates on the torus with periodicity 1 and correspond to a T^2 with complex structure modulus $\tau = \tau_1 + i\tau_2$ and area of v. If we let τ and v vary in the non-compact directions of $\mathcal{M}_9$, we obtain a fibration: $T^2 \to \mathcal{Y} \to \mathcal{M}_9$. Now, the torus is homeomorphic to $S^1 \times S^1$ and the idea is to reduce along one circle to get type IIA string theory and then T-dualise along the other circle to get type IIB. Following [Den08] we see that in general the relation between M-theory compactified on a circle and type IIA is encoded in the metric:

$$ds_M^2 = L^2 e^{4\chi/3}(dx + C_1)^2 + e^{-2\chi/3} ds_{IIA}^2 \tag{1.6.11}$$

where the C_1 is the 1-form in type IIA, L is the length of the circle and x is the periodic coordinate of the circle. Comparing (1.6.10) and (1.6.11), we get the following identifications:

$$C_1 = \tau_1 dy \qquad e^{4\chi/3} = \frac{v}{L^2\tau_2} \qquad ds_{IIA}^2 = \frac{\sqrt{v}}{L\sqrt{\tau_2}}(v\tau_2 dy^2 + ds_9^2) \tag{1.6.12}$$

[5]An elliptic curve is a "punctured torus". If we consider in the upper-half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathfrak{Im}(z) \geq 0\}$ a parameter $\tau \in \mathbb{H}$, the lattice $\Lambda := \{z \in \mathbb{H} \mid z = a + b\tau, \text{ with } a, b \in \mathbb{Z}\}$ is used to generate the elliptic curve as the quotient space: $\mathbb{E}_\tau = \mathbb{C}/\Lambda$. The origin is identified with the point $z = 0$.

We want now to T-dualise along the other circle. T-duality maps the circle in type IIA of length L_A to a circle in type IIB of length $L_B = \ell_s/L_A$, the R-R 0-form will be given by $C_0 = (C_1)_y$ and the string coupling $g_s^{IIB} = \frac{\ell_s}{L} g_s^{IIA}$. The type IIB metric and axio-dilaton in terms of v, τ and ℓ_M are given by:

$$\tau = C_0 + \frac{i}{g_s^{IIB}} \qquad ds^2_{IIB} = \frac{\sqrt{v g_s^{IIB}}}{L}\left(\frac{\ell_M^6}{v^2}dy^2 + ds_9^2\right) \tag{1.6.13}$$

As we can see, this duality explain why the axio-dilaton in type IIB presents an $SL(2,\mathbb{Z})$ symmetry, in fact, it arises as a complex structure modulus of a torus which enjoys the same symmetry.

Considering M-theory on $\mathcal{M} = \mathbb{R}^{1,8-2n} \times \mathcal{Y}_{n+1}$ with $\mathcal{Y}_{n+1}$ a torus fibration with base space B_n, supersymmetry requires $\mathcal{Y}_{n+1}$ to be Calabi-Yau (see §2) and the dual type IIB will be compactified on a locally $B_n \times S^1$ with metric:

$$ds^2 = ds^2_{\mathbb{R}^{1,8-2n}} + ds^2_{B_n} + \frac{\ell_s^4}{v}dy^2 \tag{1.6.14}$$

with y periodic coordinate of S^1 with periodicity 1 and in the limit in which the area of the torus vanishes $v \to 0$, we recover type IIB compactified on B_n.

1.7 Type I Strings

When one tries to construct a theory of open and oriented superstrings coupled to closed ones, they will inevitably incur in an inconsistency called *RR Tadpole Cancellation Condition*. The RR tadpole can be thought of as a charge of the R-R fields, in particular of the 10-form C_{10} present in the spectrum and is given by the emission of a closed string out of the vacuum. The same is true if one tries to construct a theory of closed and unoriented strings. However, the key idea is to try to construct a theory of unoriented open and closed strings such that their RR tadpoles cancel out. The massless spectrum that is found is the following:

Closed Sector	Representation of SO(8)	10-dimensional field
NS-NS	$1 + 35_V$	ϕ, G
NS-R + R-NS	$8_S + 56_S$	λ, ψ
R-R	28_C	B_2
Open Sector	**Representation of SO(8)**	**10-dimensional field**
NS	8_V	A_1
R	8_C	$\overline{\lambda}$

The closed spectrum contains a dilaton ϕ, a graviton G, a 2-form field B_2, a dilatino λ and a gravitino ψ. In open sector, instead, there are a 1-form field A_1 and a fermion $\overline{\lambda}$ which are gauge boson and gaugino of the $SO(32)$ spacetime gauge symmetry. The theory possesses various local symmetries, namely the diffeomorphism invariance with G as the graviton, gauge invariance of the R-R 2-form B_2, the $SO(32)$ gauge symmetry under which A_1 and $\overline{\psi}$ transform in the adjoint representation and local $\mathcal{N} = 1$ 10-dimensional supersymmetry (there is in fact a single gravitino ψ).

The low-energy effective action is given by:

$$S_I = \frac{1}{2k_{10}^2}\int e^{-2\phi}\left(\mathscr{R}\star\mathbb{1} + 4d\phi\wedge\star d\phi - \frac{1}{2}\tilde{F}_3\wedge\star\tilde{F}_3\right) - \frac{1}{2g_{10}^2}\int e^{-\phi}\mathrm{tr}(F_2\wedge\star F_2) \tag{1.7.1}$$

where:

$$g_{10}^2 = k_{10}^2(2\pi)^{7/2}\alpha' \qquad F_2 = dA_1 \qquad \tilde{F}_3 = dB_2 - \frac{k_{10}^2}{g_{10}^2}(\omega_3 - \omega_{\mathrm{grav}}) \tag{1.7.2}$$

and ω_3, ω_{grav} the *Chern-Simons forms* for the gauge field A_1 and the gravitational field:

$$\omega_3 = \mathrm{tr}(A_1\wedge dA_1 - \frac{2}{3}A_1\wedge A_1\wedge A_1) \qquad \omega_{\mathrm{grav}} = \mathrm{tr}(\omega\wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega) \tag{1.7.3}$$

with ω the spin-connection[6].

1.8 Heterotic Strings

Heterotic string theory is the theory of closed strings which is characterised by different right and left moving degrees of freedom. The left ones are of the bosonic theory and the right are of superstrings. In the light-cone gauge, the right sector will contain 8 bosons $X_R^i(\tau-\sigma)$ and 8 fermions $\psi_R^i(\tau-\sigma)$ for $i=2,...,9$, while the left sector will contain 24 bosons split as $X_L^i(\tau+\sigma)$ and $X_L^I(\tau+\sigma)$ with $i=2,...,9$ and $I=1,...,16$. The physical dimensions are only 10 and are given by the degrees of freedom:

$$X^i(\tau,\sigma) = X_L^i(\tau+\sigma) + X_R^i(\tau-\sigma) \tag{1.8.1}$$

while the other 16 degrees of freedom should be thought as compactified on a 16-dimensional torus. How and why this can be made possible? When inspecting the modes expansion for the bosonic theory compactified on a circle, one can see that the right sector can be frozen by imposing the radius of the circle to take the critical value of $R=\sqrt{\alpha'}$. In this way, the right sector will not have a dynamics, while the left sector is left with a non-trivial dynamics. Glueing the left and right sector's spectrum, GSO-projecting the right states and employing the level matching condition $M_L^2 = M_R^2$, we obtain the following massless spectrum [IU12]:

[6]The spin connection is introduced when considering spinors in a gravitational field. Taking the Dirac equation on a general curved background spacetime, the partial derivative will no longer be a tensor. One can introduce the covariant derivative:

$$\nabla_\mu\psi = \left(\partial_\mu - \frac{i}{4}\omega_\mu^{ab}\sigma_{ab}\right)\psi \tag{1.7.4}$$

where σ_{ab} is given by the commutator of gamma matrices, a,b are "flat" spacetime indices, while μ is a "curved" spacetime index and ω_μ^{ab} is the spin connection. Everything is defined with respect to *vierbeins* (or tetrads), which are defined as:

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab} \tag{1.7.5}$$

where $g_{\mu\nu}$ is the spacetime metric and η_{ab} is the Minkowski metric. This means that g is locally flat when written in the basis e_μ^a. The vierbeins, from a geometrical point of view, are nothing but *local reference frame* of the cotangent bundle and the spin connection is the connection induced by the Levi-Civita connection on the *Spin Bundle*.

Sector	Representation of SO(8)	10-dimensional field
NS	$1+28_V+35_V$	ϕ, B_2, G
R	8_S+56_S	λ, ψ
NS	8_V	$A^{(I)}$
NS	8_V	$A^{(M)}$
R	8_C	$\overline{\lambda}^{(I)}$
R	8_C	$\overline{\lambda}^{(P)}$

In the upper part of the table we see the dilaton ϕ, the graviton G, the 2-form field B_2, the gravitino ψ and the dilatino λ. In the bottom part, instead, we have NS and R fields A and $\overline{\lambda}$ which corresponds to gauge bosons with respect to a spacetime non-abelian gauge symmetry (with the corresponding gauginos). These gauge bosons realise the algebra $E_8 \times E_8$ or $SO(32)$ depending on which roots of these Lie algebras the momentum vectors I and P corresponds to. This means that there are two possible type of heterotic strings, the one with gauge group $E_8 \times E_8$ and the one with $SO(32)$. This difference can also be grasped when analysing modular invariance of the theory where two different lattices emerge.

The theory enjoys various symmetries, the first is clearly the local change of coordinates in spacetime with G the graviton, then there is the gauge transformations of B_2, the $SO(32)$ or $E_8 \times E_8$ gauge symmetry and the local 10-dimensional $\mathcal{N}=1$ supersymmetry corresponding to a single gravitino field.

The low-energy effective action is given by:

$$S_{\text{het}} = \frac{1}{2^2_{10}} \int e^{-2\phi} \left(\mathscr{R} \star \mathbb{1} + 4\, d\phi \wedge \star d\phi - \frac{1}{2} H_3 \wedge H_3 - \frac{\alpha'}{4} \mathrm{tr}(F \wedge \star F) \right) \tag{1.8.2}$$

where tr is the gauge trace of $SO(32)$ or $SO(16)$ subgroup of E_8 in the $E_8 \times E_8$ theory. Also, as for type I theory, we have defined:

$$H_3 = dB_2 - \frac{\alpha'}{4}(\omega_3 - \omega_{\text{grav}}) \tag{1.8.3}$$

where ω_3 and ω_{grav} are given by (1.7.3).

After its proposal, heterotic string theory became shortly the physicists' favourite theory of strings thanks its wonderful anomaly cancellation which was baptised as "miraculous". In fact, from one-loop hexagon diagrams (namely for a closed string which closes into itself, with a worldsheet homeomorphic to a torus, and six leg out of the worldsheet), gravitational, gauge and mixed anomalies could arise since the theory is chiral. Because no anomaly could be possible, the contribution of this diagram should cancel against another one with the same topology. Indeed this is what Green and Schwarz [GS84] found in their work, triggering also what is called the *first superstring revolution*. Their work, in fact, contributed to make string theory one of the most active research field at that time.

Chapter 2

String Compactification

All superstring theories live in 10 space-time dimensions and have a different number of supersymmetries. In order to make contact with our world and with the *Standard Model of Particle Physics*, the vacuum state of these theories must be of the form:

$$\mathcal{M} = \mathcal{M}_4 \times \mathcal{M}_6 \tag{2.0.1}$$

Which means that some of these dimensions must be compactified, in the sense that the background manifold $\mathcal{M}$ over which the superstrings are supposed to be propagating, is assumed to be decomposable into a product of a 4-dimensional non-compact manifold $\mathcal{M}_4$ and a 6-dimensional internal compact manifold $\mathcal{M}_6$. Taking the compactification manifold $\mathcal{M}_6$ to be the 6-torus T^6, for example, it implies that no supersymmetries are broken, and the resulting 4-dimensional theory will have $\mathcal{N} = 4$ or $N = 8$ supersymmetries for heterotic/type I and type IIB/IIA string theories respectively. From a phenomenological point of view a compactification which preserves all supersymmetries does not seem to be that promising for reproducing the current spectrum of particles. In 1985, Candelas et al. [Can+85] studied for the case of heterotic strings (which seemed to be the most promising theory of strings at that time due to the anomaly cancellation) what kind of manifold should $\mathcal{M}_6$ be in order to preserve only $\mathcal{N} = 1$ supersymmetry. The result is that such $\mathcal{M}_6$ should be a manifold of $SU(3)$-holonomy which in turn means that it should be Ricci-flat and Kähler. This kind of manifolds are known as *Calabi-Yau manifolds*. In order to describe in details these spaces, a good understanding of some mathematical structures, like Kähler manifolds, Hodge theory and Cohomology is needed. In the next section we are going to present the main features of these mathematical theories.

2.1 Mathematical Preliminaries

2.1.1 Complex Manifolds

Definition 2.1.1 (Almost Complex Structure). *Let M be a $2n$-dimensional manifold and $J : TM \to TM$ a smooth tensor field of degree $\binom{1}{1}$, where TM is the tangent bundle on M. The pair (M, J) is said to be an Almost Complex Manifold if*

$$J^2 = -\mathbb{1}$$

In other words, $\forall\, p \in M$ we have that $J_p : T_pM \to T_pM$ and $\forall\, v_p \in T_pM$ it must be true that:

$$(J_p \circ J_p)(v_p) = -v_p$$

This almost-complex structure can be made a complex structure by requiring that J is *Integrable*. So that we have:

Definition 2.1.2 (Complex Structure)**.** *Let (M,J) be an almost complex manifold. If J is integrable then it is called a Complex Structure and (M,J) is said to be a Complex Manifold. The integrability of the almost complex structure J is defined as:*

$$P_-[P_+X, P_+Y] = 0$$

where $P_\pm = \frac{1}{2}(\mathbb{1} \mp iJ)$ are the projectors onto the holomorphic/anti-holomorphic bundles [1] *and the brackets $[\cdot,\cdot]$ are the Lie brackets between two vector fields $X, Y \in \mathfrak{X}(M)$.*

The integrability condition means that the Lie brackets between two holomorphic vectors must be again an holomorphic vector. This condition can be restated in terms of the *Nijenhuis Tensor* which is required to vanish in order for an almost complex structure to be a complex structure:

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \tag{2.1.1}$$

Along with this definition of complex manifold, there is an equivalent classical definition in terms of transition functions:

Definition 2.1.3 (Complex Manifold)**.** *An Hausdorff and connected topological space (M,τ) - where M is a set and τ is a topology on it - is a n-dimensional complex manifold if $\forall\, p \in M$ exists an open neighbourhood U of p and a function $\phi : U \to \phi(U) \subseteq \mathbb{C}^n$ such that ϕ is an homeomorphism, moreover the pair (U,ϕ) is called a chart and taken a collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ (A is an index set) with $\{U_\alpha\}$ an open cover, the transition functions between two overlapping charts:*

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\beta \cap U_\alpha) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

must be biholomorphic.

These two definitions are equivalent, in fact it can be proved that the requirement of biholomorphicity of transition functions induces on the tangent spaces of the manifold an almost complex structure J which is integrable. Conversely a complex structure allows to define charts with biholomorphic transition functions. These two definitions emphasize two different point of views which can be taken when looking at complex manifolds.

[1] These bundles are obtained after the complexification of the tangent bundle $TM \otimes \mathbb{C}$ by tensoring it with the complex plane $\mathbb{C}$. This space splits into $(\pm i)$-eigenspaces of J as:

$$TM \otimes \mathbb{C} = TM^{1,0} \oplus TM^{0,1}$$

which are the holomorphic and anti-holomorphic vector bundles defined as:

$$TM^{1,0} = \{v \in TM \otimes \mathbb{C} \,|\, Jv = iv\}$$

$$TM^{0,1} = \{v \in TM \otimes \mathbb{C} \,|\, Jv = -iv\}$$

More will be said later.

2.1.2 Kähler Manifolds

It is known that classical mechanics can be rephrased in a geometrical way by employing *Symplectic Manifolds*. The natural symplectic structure that the *Tangent Bundle* of every real differentiable manifold admits, makes it possible to express Lagrangian and Hamiltonian mechanics in a geometrical way. If X is a real n-dimensional differentiable manifold, then the cotangent bundle $TX^* =: M$ together with the natural symplectic form ω (i.e. a non-degenerate and closed 2-form on M) defined locally in each chart with *Darboux coordinates* (x, ξ) as $\omega = \sum_i dx^i \wedge d\xi^i$ form a real $2n$-dimensional symplectic manifold (M, ω). An *Hamiltonian* is a function $H \in C^\infty(M)$ such that the 1-form dH, with d the external derivative, is given by the contraction of the symplectic form with respect to a *Vector Field* X_H:

$$i_{X_H}\omega = dH$$

where $i_{X_H}\omega = \omega(X_H, \cdot)$ is the contraction. Now, every Hamiltonian vector field X_H can be expressed in a local chart (q, p) of the tangent bundle $TM = T(TX^*)$ as:

$$X_H = \sum \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_j} \right)$$

in this way, every integral curve $(q(t), p(t))$ must satisfy the *Hamilton equations*:

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial p_i} \end{cases} \tag{2.1.2}$$

Moreover, *Poisson brackets* can be defined for each $f, g \in C^\infty(M)$ as $\{f, g\} = \omega(X_f, X_g)$.

Symplectic manifolds are essential in describing classical mechanics from a geometrical point of view and in generalising them. The same applies to *Quantum Mechanics* where here instead of considering real manifolds, complex ones are taken into consideration. In this way, if a manifold possesses a symplectic structure and a complex structure, then it is natural to ask whether these two are compatible (in some sense) or not. The symplectic form ω on a symplectic manifold M resembles a lot a metric and in fact if M admits also a complex structure J on it, then it can be constructed a bilinear form using ω and J in the following way, $\forall X, Y \in \mathfrak{X}(M)$:

$$g(X, Y) = \omega(X, JY)$$

More correctly we should say that the 2-form ω induces a $\binom{0}{2}$ tensor field $g \in \mathfrak{X}^0_2(M)$ on the manifold by means of the above mentioned formula. If at each point p in the manifold M the bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ is *symmetric* and *positive definite*, we say that the symplectic form and the almost complex structure are compatible and the triple (M, ω, J) is called a *Kähler Manifold*. We then have:

Definition 2.1.4 (Kähler Manifold (1))**.** *A Kähler manifold is a triple (M, ω, J) where M is a manifold, ω is a symplectic form and J a complex structure where these are compatible in the sense that the bilinear form:*

$$g_p(\cdot, \cdot) = \omega_p(\cdot, J_p \cdot)$$

is symmetric and positive definite for all p in M.

If we consider a complex Riemannian manifold (M,g), then in a local patch with holomorphic coordinates $(z^a,\bar{z}^b)$, the metric tensor will be given by:

$$g = g_{ab}dz^a \otimes dz^b + g_{\bar{a}\bar{b}}d\bar{z}^a \otimes d\bar{z}^b + g_{a\bar{b}}dz^a \otimes d\bar{z}^b + g_{\bar{a}b}d\bar{z}^a \otimes dz^b$$

When the pure parts vanish: $g_{ab} = g_{\bar{a}\bar{b}} = 0$, the metric is said to be an *Hermitian Metric*. For a Kähler manifold the metric induced by the symplectic and complex structure is Hermitian and in fact a Kähler manifold can also be defined as a complex Riemannian manifold with an hermitian metric $g = g_{a\bar{b}}dz^a \otimes d\bar{z}^b$ such that the induced *Kähler Form*:

$$\omega = \frac{\sqrt{-1}}{2}\sum g_{a\bar{b}}dz^a \wedge d\bar{z}^b \tag{2.1.3}$$

is closed:

$$d\,\omega = 0 \tag{2.1.4}$$

This means that indeed a Kähler manifold defined in this way is also a symplectic manifold. An alternative definition of a Kähler manifold can then be given:

Definition 2.1.5 (Kähler Manifold (2)). *Let (M,J) be a complex manifold with an Hermitian metric g expressed locally as $g = \sum g_{a\bar{b}}dz^a \otimes d\bar{z}^b$. Then if the associated 2-form $\omega = \sqrt{-1}/2\sum g_{a\bar{b}}dz^a \wedge d\bar{z}^b$ is closed, then (M,J,ω) is called a Kähler manifold and the metric g is called a Kähler metric.*

Thanks to the closeness of the Kähler form ω, the metric satisfies $\partial_a g_{b\bar{c}} = \partial_b g_{a\bar{c}}$ and its complex conjugate relation, in this way, in a local patch, it is always possible to express this metric in terms of a scalar function $K(z,\bar{z})$:

$$g_{a\bar{b}} = \frac{\partial^2 K(z,\bar{z})}{\partial z^a \partial \bar{z}^b} \tag{2.1.5}$$

The function K is called *Kähler Potential*. This function is not unique, in fact if we consider $K'(z,\bar{z}) = K(z,\bar{z}) + f(z) + g(\bar{z})$, where f, g are two holomorphic and anti-holomorphic function respectively, then this Kähler potential gives the same metric. This entails the fact that it may be the case that different potentials are needed in different charts to reproduce the Kähler metric.

2.1.3 De Rham Cohomology

What can we say about the topology of a complex manifold? We know that for a real differentiable manifold M the study of its Cohomology groups gives some clue regarding its topology. The same study can be made for complex manifolds.

Following Cattani [Cat10], a complex structure J on a manifold M induces a decomposition of the complexification[2] of the tangent bundle into two orthogonal spaces, at each point $p \in M$:

$$(T_pM)_{\mathbb{C}} \simeq T^{1,0}_pM \oplus T^{0,1}_pM \tag{2.1.6}$$

[2] Taken a real n-dimensional vector space V we can define its complexification as $V_{\mathbb{C}} := V \otimes \mathbb{C}$ by $v \otimes (a+ib) = av + b(v \otimes i)$ and formally identify $v \otimes i =: iv$ in such a way that $V \oplus iV := V \otimes \mathbb{C}$. If (V,J) is a complex vector space with J the complex structure (i.e. a linear map such that $J^2 = -\mathbb{1}$) it can be seen that $V_{\mathbb{C}}$ can be decomposed into two orthogonal components which are nothing but the eigenspaces of J with respect to the eigenvalues $\pm i$. Written the decomposition as $V_{\mathbb{C}} = W_+ \oplus W_-$, there will also be a decomposition of its dual $V^*_{\mathbb{C}}$ ($(V_{\mathbb{C}})^* = (V^*)_{\mathbb{C}} =: V^*_{\mathbb{C}}$) induced by the dual complex structure J^* as $V^*_{\mathbb{C}} = W^{\perp}_+ \oplus W^{\perp}_-$.

Also its dual space, namely the cotangent bundle, decomposes into two pieces and (since for a vector space $(V_{\mathbb{C}})^* = (V^*)_{\mathbb{C}}$ these parenthesis can be dropped by writing $V^*_{\mathbb{C}}$) we obtain:

$$(T_pM)^*_{\mathbb{C}} \simeq T^{1,0}_p M^* \oplus T^{0,1}_p M^* \tag{2.1.7}$$

This decomposition induces an *Hodge Structure* on the k-th exterior power of the cotangent space at a given point, namely:

$$\Lambda^k((T_pM)^*_{\mathbb{C}}) = \bigoplus_{a+b=k} \Lambda^{a,b}_p M \tag{2.1.8}$$

with:

$$\Lambda^{a,b}_p M := \underbrace{T^{1,0}_p M^* \wedge \cdots \wedge T^{1,0}_p M^*}_{\text{a times}} \wedge \overbrace{T^{0,1}_p M^* \wedge \cdots \wedge T^{0,1}_p M^*}^{\text{b times}} \tag{2.1.9}$$

By making the disjoint union of these spaces, as it is customary in defining the tangent and cotangent bundle, the smooth vector bundle $\Lambda^k TM^*_{\mathbb{C}} = \bigcup_p \Lambda^k((T_pM)^*_{\mathbb{C}})$ decomposes into:

$$\Lambda^k TM^*_{\mathbb{C}} = \bigoplus_{a+b=k} \Lambda^{a,b} M \tag{2.1.10}$$

If we take a section $\omega : M \to \Lambda^k TM^*_{\mathbb{C}}$ then in a local chart with holomorphic coordinates $(U, \{z_1, ..., z_n\})$ it will be given by:

$$\omega = \sum_{I,J} \omega_{I\bar{J}} dz^I \wedge d\bar{z}^J \tag{2.1.11}$$

where $I = (i_1, ..., i_a)$ and $J = (j_1, ..., j_b)$ are two multi-indices for some $a, b \in \mathbb{N}$ such that $a + b = k$ and $dz^I \wedge d\bar{z}^J = dz^{i_1} \wedge \cdots \wedge dz^{i_a} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_b}$ are a local frame for the bundle $\Lambda^{a,b}M$.

We can now define the module of local sections as $\mathscr{A}^k(M)$ which, due to (2.1.10), will split into:

$$\mathscr{A}^k(M) = \bigoplus_{a+b=k} \mathscr{A}^{a,b}(M) \tag{2.1.12}$$

Taking $\Lambda^{a,0}M$ we note that this is an holomorphic vector bundle of dimension $\binom{n}{a}$ and we will denote the module of holomorphic sections as $\Omega^a(M)$. If we define:

$$\mathscr{A}(M) = \bigoplus_{k=0}^{\infty} \mathscr{A}^k(M) \tag{2.1.13}$$

then this is a $C^\infty(M)$—*Graded Algebra* with multiplication given by the wedge product $\wedge$ between different forms. Upon $\mathscr{A}(M)$ it acts a unique differential operator d called *Exterior derivative*, which takes an element of $\mathscr{A}^k(M)$ and returns an element of $\mathscr{A}^{k+1}(M)$:

$$d : \mathscr{A}^k(M) \to \mathscr{A}^{k+1}(M) \tag{2.1.14}$$

This operator is defined uniquely by 4 properties:

(i) d is $\mathbb{C}$-linear

(ii) $\forall f \in C^\infty(M, \mathbb{C}) \equiv \mathscr{A}^0(M)$ then df is the usual differential, i.e. $\forall v_p \in T_pM$ it is true that $(df)v_p = v_p(f)$

(iii) d satisfies the Leibniz rule, namely $\forall\omega \in \mathscr{A}^k(M)$ and $\forall\eta \in \mathscr{A}^l(M)$ it is true that: $d(\omega \wedge \eta) = (d\omega)\wedge\eta + (-1)^k \omega\wedge(d\eta)$

(iv) $d \circ d = 0$

The second property tells us that if M is a complex manifold, in a local patch $(U, \{z_1, ..., z_n\})$, then:

$$df = \sum\left(\frac{\partial f}{\partial x^i}dx^i + \frac{\partial f}{\partial y^i}dy^i\right) = \sum\left(\frac{\partial f}{\partial z^i}dz^i + \frac{\partial f}{\partial \bar{z}^i}d\bar{z}^i\right) \tag{2.1.15}$$

The last property defines the so-called *de Rham Cohomology* in the following way: in the ring $\mathscr{A}(M)$ a k-form ω is said to be *exact* if there exists a $(k-1)$-form η such that $d\eta = \omega$, while a k-form μ is called *closed* if $d\mu = 0$. It is clear that an exact form is also closed by means of the (iv) property above, while it is not always true that a closed form is also exact. This allows to define a quotient space called the *k-th de Rham Cohomology group*:

$$H^k_{\mathrm{dR}}(M, \mathbb{C}) := \frac{\text{Closed } k\text{-forms on M}}{\text{Exact } k\text{-forms on M}} \tag{2.1.16}$$

Alternatively, considering the induced *de Rham Complex* by the exterior derivative:

$$\mathbb{C} \hookrightarrow \mathscr{A}^0(M) \xrightarrow{d} \mathscr{A}^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{A}^{2n}(M) \tag{2.1.17}$$

(with n the complex dimension of M) then it is possible to re-write the k-th de Rham Cohomology group:

$$H^k_{\mathrm{dR}}(M, \mathbb{C}) := \frac{\ker\{d : \mathscr{A}^k(M) \to \mathscr{A}^{k+1}(M)\}}{\mathrm{Im}\{d : \mathscr{A}^{k-1}(M) \to \mathscr{A}^k(M)\}} \tag{2.1.18}$$

where *ker* is the kernel and *Im* is the image of the underlying linear map.

Now, from (2.1.15), we see that the exterior derivative d is not of a pure bidegree but can be split into its $(1,0)$ and $(0,1)$ components by noticing that:

$$d(\mathscr{A}^{a,b}(M)) \subset \mathscr{A}^{a+1,b}(M) \oplus \mathscr{A}^{a,b+1}(M) \tag{2.1.19}$$

In this way we might write $d = \partial + \overline{\partial}$ and from $d \circ d = 0$ we obtain: $\partial^2 = \overline{\partial}^2 = 0$ and $\partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0$ so that the *Dolbeault Complex* can be defined for all p such that $0 \le p \le n$:

$$\mathscr{A}^{p,0}(M) \xrightarrow{\overline{\partial}} \mathscr{A}^{p,1}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{A}^{p,n}(M) \tag{2.1.20}$$

As we did for the de Rham cohomology we can define the *Dolbeault Cohomology*:

$$H^{p,q}_{\overline{\partial}}(M) := \frac{\ker\left\{\overline{\partial} : \mathscr{A}^{p,q}(M) \to \mathscr{A}^{p,q+1}(M)\right\}}{\mathrm{Im}\left\{\overline{\partial} : \mathscr{A}^{p,q-1}(M) \to \mathscr{A}^{p,q}(M)\right\}} \tag{2.1.21}$$

and can easily be proved that $H^{p,0}_{\overline{\partial}}(M)$ is isomorphic to the space of Holomorphic p-forms $\Omega^p(M)$.

2.1.4 Hodge Theory

Hodge Theory on Compact Real Manifolds

Consider now a real and compact Riemannian manifold (M,g). The metric g induces a dual metric on the cotangent bundle TM^* viewed as an inner product between 1-forms $\langle\cdot,\cdot\rangle : \mathscr{A}^1(M)\times\mathscr{A}^1(M)\to\mathbb{R}$. On an *Orientable* Riemannian manifold (M,g) of dimension n, there can be defined a volume form $vol\in\mathscr{A}^n(M)$ as a normalised top form and in local coordinates (U,x) it can be written as $vol=\sqrt{g}dx^1\wedge\cdots\wedge dx^n$ with $g=\det(g_{ij})$ and $g_{ij}=g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})$. With these in hand we can define:

Definition 2.1.6 (Hodge $\star$-operator)**.** *Let (M,g) be an Orientable Riemannian manifold of dimension n. For each $p\in M$, the map $\star:\bigwedge^k T_pM^*\to\bigwedge^{n-k}T_pM^*$ defined by $\forall\alpha,\beta\in\bigwedge^k T_pM^*$:*

$$\alpha\wedge\star\beta=\langle\alpha,\beta\rangle_p vol_p \tag{2.1.22}$$

is called the Hodge $\star$-operator.

Theorem 2.1.1

The $\star$-operator is an isomorphism of vector spaces and $\star^2=(-1)^{k(n-k)}id$.

With the $\star$ map we can define an inner product on the space of k-forms:

Definition 2.1.7. *Let (M,g) be an orientable and compact Riemannian manifold of dimension n. For each $\alpha,\beta\in\mathscr{A}^k(M)$ we define:*

$$(\alpha,\beta):=\int_M\alpha\wedge\star\beta=\int\langle\alpha_p,\beta_p\rangle_p vol_p \tag{2.1.23}$$

This bilinear form is positive and symmetric, making it indeed an inner product on $\mathscr{A}^k(M)$. A very useful operator that can be defined now is the formal adjoint of the exterior derivative with respect to this inner product. We define:

$$\delta:\mathscr{A}^{k+1}(M)\to\mathscr{A}^k(M) \tag{2.1.24}$$

such that $\forall\alpha,\beta\in\mathscr{A}^k(M)$:

$$(d\alpha,\beta)=(\alpha,\delta\beta) \tag{2.1.25}$$

By direct inspection and making use of theorem (2.1.1) it can be seen that δ is given by:

$$\delta=(-1)^{nk+n+1}\star d\star \tag{2.1.26}$$

We can now define the *Laplace-Beltrami operator* which is an elliptic operator generalising the Laplace operator $\triangle=\partial_x^2+\partial_y^2+\partial_z^2$ in 3D Euclidean space to a general orientable and compact Riemannian manifold:

$$\begin{aligned}\triangle:\mathscr{A}^k(M)&\to\mathscr{A}^k(M)\\ \alpha&\mapsto\triangle\alpha:=(d\circ\delta+\delta\circ d)\alpha\end{aligned} \tag{2.1.27}$$

The Laplace-Beltrami operator is formally self-adjoint and commutes with the exterior derivative d and with δ.

Also $\triangle\alpha = 0$ if and only if $d\alpha = \delta\alpha = 0$. A k-form α such that $\triangle\alpha = 0$ is called *Harmonic* and the space of all harmonic forms is defined as:

$$\mathscr{H}^k(M) := \left\{\alpha \in \mathscr{A}^k(M) \mid \triangle\alpha = 0\right\} \tag{2.1.28}$$

The Laplace-Beltrami operator allows to proceed from the geometrical to the topological since it is also a source of topological information thanks to the celebrated *Hodge-de Rham theory*. In particular the famous *Hodge Decomposition theorem* can be used to identify the de Rham cohomology group with the group of harmonic functions.

Theorem 2.1.2 (*Hodge Decomposition*)

Let (M,g) be an orientable and compact Riemannian manifold, then:

(i) $\mathscr{H}^k(M)$ is finite dimensional;

(ii) we have the following orthogonal decomposition:

$$\begin{aligned} \mathscr{A}^k(M) &= \triangle(\mathscr{A}^k(M)) \oplus \mathscr{H}^k(M) \\ &= d\delta(\mathscr{A}^k(M)) \oplus \delta d(\mathscr{A}^k(M)) \oplus \mathscr{H}^k(M) \\ &= d(\mathscr{A}^{k-1}(M)) \oplus \delta(\mathscr{A}^{k+1}(M)) \oplus \mathscr{H}^k(M) \end{aligned} \tag{2.1.29}$$

The proof of this theorem can be found in [War10]. As we already said, using this theorem, it can be proved that each de Rham cohomology class contains a unique harmonic representative, making the group of harmonic k-forms and the k-th de Rham cohomology actually isomorphic:

Corollary 2.1.1

$$H^k_{dR}(M,\mathbb{R}) \simeq \mathscr{H}^k(M) \tag{2.1.30}$$

Hodge Theory on Compact Complex Manifolds

Consider now a compact and complex n-dimensional manifold M with a Hermitian metric h. In this context we are going to consider complex-valued k-forms $\mathscr{A}^k(M,\mathbb{C})$ and we can extend the Hodge $\star$-operator by linearity to $\bigwedge^k T_pM^*_{\mathbb{C}}$ for each $p \in M$ and the inner product $\langle\cdot,\cdot\rangle$ to this complexification denoting it as $\langle\cdot,\cdot\rangle^h$, obtaining $\forall \alpha,\beta \in \bigwedge^k T_pM^*_{\mathbb{C}}$:

$$\alpha \wedge \star\overline{\beta} = \langle\alpha,\beta\rangle^h vol \tag{2.1.31}$$

which gives a positive definite Hermitian inner product on the complex-valued k-forms $\mathscr{A}^k(M,\mathbb{C})$:

$$(\alpha,\beta)^h = \int_M \alpha \wedge \star\overline{\beta} \tag{2.1.32}$$

The formal adjoint of $\partial : \mathscr{A}^{p,q}(M) \to \mathscr{A}^{p+1,q}(M)$ and $\overline{\partial} : \mathscr{A}^{p,q}(M) \to \mathscr{A}^{p,q+1}(M)$ can be defined as $\partial^* := -\star\overline{\partial}\star$ and $\overline{\partial}^* := -\star\partial\star$ respectively. These will satisfy $(\partial\alpha,\beta)^h = (\alpha,\partial^*\beta)^h$ and $(\overline{\partial}\alpha,\beta)^h = (\alpha,\overline{\partial}^*\beta)^h$.

As for real manifolds, we can define the *Laplace-Beltrami operators*:

$$\triangle_{\partial} := \partial \circ \partial^* + \partial^* \circ \partial \tag{2.1.33}$$

$$\triangle_{\overline{\partial}} := \overline{\partial} \circ \overline{\partial}^* + \overline{\partial}^* \circ \overline{\partial} \tag{2.1.34}$$

and the kernel of $\triangle_{\overline{\partial}}$ will be as before the set of harmonic (p,q)-forms:

$$\mathscr{H}^{p,q}_{\overline{\partial}}(M) := \left\{ \alpha \in \mathscr{A}^{p,q}(M) \mid \triangle_{\overline{\partial}} \alpha = 0 \right\} \tag{2.1.35}$$

The Hodge decomposition theorem now reads:

Theorem 2.1.3

Let (M,g,J) be a compact Kähler manifold, then:

(i) $\mathscr{H}^{p,q}_{\overline{\partial}}(M)$ is finite dimensional;

(ii) we have the following orthogonal decomposition:

$$\begin{aligned} \mathscr{A}^{p,q}(M) &= \triangle_{\overline{\partial}}(\mathscr{A}^{p,q}(M)) \oplus \mathscr{H}^{p,q}_{\overline{\partial}}(M) \\ &= \overline{\partial}(\mathscr{A}^{p,q-1}(M)) \oplus \overline{\partial}^*(\mathscr{A}^{p-1,q}(M)) \oplus \mathscr{H}^{p,q}_{\overline{\partial}}(M) \end{aligned} \tag{2.1.36}$$

As a corollary, we have an isomorphism between the Dolbeault cohomology group (2.1.21) and the set of harmonic (p,q)-forms:

Corollary 2.1.2

$$H^{p,q}_{\overline{\partial}}(M) \simeq \mathscr{H}^{p,q}_{\overline{\partial}}(M) \tag{2.1.37}$$

Moreover the k-th cohomology with complex coefficients get decomposed as:

$$H^k_{dR}(M,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M) \simeq \bigoplus_{p+q=k} \mathscr{H}^{p,q}(M) \tag{2.1.38}$$

If we define the *Hodge numbers* to be the dimensions of the Dolbeault cohomologies (or analogously the number of harmonic (p,q)-forms):

$$\dim(H^{p,q}_{\overline{\partial}}(M)) = h^{(p,q)} = \dim(\mathscr{H}^{p,q}(M)) \tag{2.1.39}$$

and the *Betti numbers* to be the dimensions of the de Rham cohomologies:

$$\dim_{\mathbb{C}}(H^k_{dR}(M,\mathbb{C})) = b_k \tag{2.1.40}$$

then by (2.1.38) we get:

$$b_k = \sum_{p+q=k} h^{(p,q)} \tag{2.1.41}$$

For a Kähler manifold, the Hodge and Betti numbers satisfy the following relations. Let (M,g,J) be a n-dimensional compact Kähler manifold, then:

(i) $h^{(p,q)} = h^{(q,p)}$;

(ii) the betti numbers b_k for k odd are even;

(iii) $h^{(p,q)} = h^{(n-p,n-q)}$.

The first is a consequence of the fact that $\overline{\mathscr{H}^{p,q}}(M) = \mathscr{H}^{(p,q)}(M)$ since $\Delta_{\bar{\partial}}$ is a self-adjoint operator, the second is a consequence of the first and (2.1.40) while the third is due to the Hodge star $\star$ duality (remember that $\star : \bigwedge^k V \to \bigwedge^{n-k} V$ is an isomorphism and this in turn induces an isomorphism between cohomology groups).

All these cohomologies can be realised in a more general context, namely in *Sheaf Cohomology*, where the fundamental mathematical objects are *Sheaves*. We shall show that suitable sheaf cohomologies will be isomorphic to the de Rham and Dolbeault cohomologies. This means that on one hand we have an object completely defined in terms of the differentiable structure, while on the other hand we have a term completely defined in terms of the topological structure. This profound link will be given by *Dolbeault theorem* and *de Rham theorem*. A very exhaustive treatment can be found in the spectacular book of Kodaira [Kod08]. In what follows we are going to outline the structure of sheaf theory, pointing out the main results, leaving the details to [Kod08][GQ19].

2.1.5 Sheaves

De Rham and Dolbeault cohomology groups can also be realised as *Sheaf Cohomology Groups*. In mathematics, sheaves are very general and abstract objects defined in order to inspect and keep track of locally defined data. Their use varies from algebraic geometry to differential geometry and provides the natural context which allows to deal with the problem of how to pass from local data on a given space to global data on that space. The non-triviality of this passage is usually measured by a sheaf cohomology group. Sheaves are indeed the natural framework which allows to generalise cohomology theory and it provides important links between topological and geometrical properties of complex manifolds.

Presheaves formalise the situation in which one wants to localise or restrict themselves to open subset of a given space, so that one can restrict, for example, the space of functions to an open set of that space.

Definition 2.1.8. *A Presheaf $\mathscr{P}$ on a topological space M is an assignment to each open set $U \subset M$ of a group of so called sections $\mathscr{P}(U)$, such that to each open subset V of U it is given a map called restriction map $r^U_V : \mathscr{P}(U) \to \mathscr{P}(V)$ such that $r^U_U = id_U$ and for $U \subset V \subset W$ we have that $r^U_W = r^U_V \circ r^V_W$.*

The latter property can be represented graphically, demanding that the following diagram commutes:

$$\begin{tikzcd} \mathscr{P}(U) \arrow[r, "r^U_V"] \arrow[dr, "r^U_W"'] & \mathscr{P}(V) \arrow[d, "r^V_W"] \\ & \mathscr{P}(W) \end{tikzcd}$$

As an example, consider a topological space M, we can define the presheaf of continuous functions $\mathscr{C}$ by considering the group of sections as the group of continuous functions, for $U \subset M$:

$\mathscr{C}(U) := \{f : U \to \mathbb{R} \mid f \text{ is continuous}\}$ and the restriction maps $r_V^U : \mathscr{C}(U) \to \mathscr{C}(V)$ will be the usual restriction of functions[3] and will, therefore, satisfy the properties above.

Definition 2.1.9. *A Sheaf $\mathscr{F}$ on a topological space M is a Presheaf such that for each $U \in M$ and an open covering of $U = \bigcup U_i$, then given two sections $s_1, s_2 \in \mathscr{F}(\mathscr{U})$ such that $\forall i : r_{U_i}^U(s_1) = r_{U_i}^U(s_2)$ then $s_1 = s_2$. Moreover if $\forall i, j : r_{U_i \cap U_j}^U(s_i) = r_{U_i \cap U_j}^U(s_j)$ then there is a unique section $s \in \mathscr{F}$ such that $r_{U_i}^U(s) = s_i$.*

The first property asserts that if two sections agree on each open subset of the open cover, then these must be equal. The second property, instead, says that if sections agree on the overlap of their domains, then it is possible to assemble them in a unique section defined on the covered open set. This last property allows the passage from local data to global data, while the first says that every section is determined by its restriction, allowing the passage from global to local. These are clearly the property possessed by Sheaves that we pointed out before. Roughly speaking, sheaves can be seen as some sort of parametrised family of functions.

Remark

The definition given above of a sheaf is given in the context of *Category Theory* even if we didn't mention any category[4]. Reformulating it in this language, a presheaf is a *Contravariant Functor* between the category **O(M)** whose objects are open sets of M and whose morphisms are inclusions and the general category **C** which in our case is the *category of groups* **Grp**. A presheaf will then be a map $\mathscr{P} : \mathbf{O(M)} \to \mathbf{Grp}$ such that to each object U in **O(M)** it associates an object $\mathscr{P}(U)$ in **Grp** and to each morphism $f : U \to V$ in **O(M)** it associates a morphism $\mathscr{P}(f) : \mathscr{P}(U) \to \mathscr{P}(V)$ in **Grp** such that $\mathscr{P}(\mathrm{id}_U) = \mathrm{id}_{\mathscr{P}(U)}$ and $\mathscr{P}(f \circ g) = \mathscr{P}(g) \circ \mathscr{P}(f)$ (contravariant functors reverse the order of composition). This $\mathscr{P}(f)$ is the restriction map r_V^U defined before and everything is retrieved in a natural way.

[3]For $f : A \to B$ and $C \subset A$, the usual restriction is denoted as $f|_C$ and considering formally a function as a univalent and total relation and its graph defined by:

$$G(f) := \{(x,y) \in A \times B \mid y = f(x)\} \tag{2.1.42}$$

then the restriction is defined as:

$$G(f|_C) := \{(x,y) \in G(f) \mid x \in C\} \tag{2.1.43}$$

[4]Category theory is a branch of mathematics where its fundamental constituents are collection of objects linked by a collection of morphisms. A category **C** consists of a class[5]of objects $\mathfrak{ob}(\mathbf{C})$ and a class of morphisms $\mathfrak{hom}(\mathbf{C})$ between the objects. Each morphism $f \in \mathfrak{hom}(\mathbf{C})$ has a source object, say A, and a target object, say B and it is written as $f : A \to B$. The class of all morphisms between A and B is defined as $\mathfrak{hom}(A,B)$. For every three objects A, B and C, there exists a composition map $\circ : \mathfrak{hom}(A,B) \times \mathfrak{hom}(B,C) \to \mathfrak{hom}(A,C)$ such that it is associative and for every object X there exists a morphism $1_X : X \to X$ called the identity morphism such that for each $f : A \to X$ and $g : X \to B$ it is true that $1_X \circ f = f$ and $g \circ 1_X = g$.

[5]In the context of *Zermelo-Fraenkel-Choice* axiomatic set theory (ZFC), which is the nowadays accepted collection of axioms properly defining sets, classes are not defined in a formal way. They are, however, brought into play in an informal manner [Jec06] by defining them as consisting of all the sets satisfying a certain property:

$$C = \{x : \phi(x, p_1, ..., p_n)\} \tag{2.1.44}$$

where $\phi(x, p_1, ..., p_n)$ is a formula.

Classes are, instead, properly defined in the *Neumann-Bernays-Gödel* (NBG) system. In this system, classes are the basic objects and when a class is an element of another class, it is called a *set*. A *proper class* is, instead, a class that is not a set [Men15].

From a maybe more practical point of view, a sheaf can be defined (following Kodaira [Kod08]) as a triple $(\mathscr{F}, \varpi, M)$ where $\mathscr{F}$ is a topological space, M is a differentiable manifold , $\varpi : \mathscr{F} \to M$ is a local homeomorphism (called projection) such that:

(i) $\forall x \in M, \varphi^{-1}(x) =: \mathscr{F}_x$ is a $\mathbb{K}$-Module (with $\mathbb{K}$ equals to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}$);

(ii) $\forall c_1, c_2 \in \mathbb{K}$ and $\forall \phi, \psi \in \mathscr{F}$, it is true that $c_1\phi + c_2\psi$ and $\varpi(\phi) = \varpi(\psi)$ depends continuously on ϕ, ψ.

The module $\mathscr{F}_x$ is called the *Stalk* of $\mathscr{F}$ over $x \in M$. The stalk will be fundamental in defining cohomology of sheaves. Later we are going to define it in the context of our first definition of sheaves and it will require some additional mathematical concepts.

Taking M to be a differentiable manifold, we can then define, following the example treated before, the sheaf of *smooth functions* as $\mathscr{E}$. Sections on $U \subset M$ are C^∞ functions on U and the group structure is given by the pointwise addition of functions $\forall f, g \in \mathscr{F}(U), \forall x \in U$: $(f+g)(x) = f(x) + g(x)$. Similarly, by substituting the smoothness with real-analyticity we obtain the sheaf of *Analytic functions* $\mathscr{A}$ and by requiring M to be a complex manifold of complex dimension n, the sheaf of *Holomorphic functions* $\mathscr{O}$ is obtained by taking as sections the Holomorphic functions on M.

The evocative name "section" is not a coincidence, if we consider a bundle (fibre bundle, vector bundle, principal bundle, etc...) $E \xrightarrow{\pi} M$ then a map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$ is called *section* and the set of sections is defined as $\Gamma(M,E)$. On a bundle, it is then naturally defined a sheaf $\mathscr{F}$ by setting $\forall U \subset M : \mathscr{F}(U) := \Gamma(U,E)$. If we now let E be the tangent TM/cotangent TM^* bundle of M (provided that M is a differentiable manifold), then the sections $\Gamma(M,TM)/\Gamma(M,TM^*)$ will be the vector fields/1-forms on M. We get naturally a sheaf of vector fields/1-forms. By taking the exterior product of TM^* as usual we obtain the bundle $\bigwedge^k TM^*$ whose sections are k-forms $\Gamma(M, \bigwedge^k TM^*)$ and we get a sheaf denoted by $\mathscr{E}^k$ if we impose $\mathscr{E}^k(U) := \Gamma(U, \bigwedge^k TM^*)$.

Before continuing with sheaf cohomology and as it was hinted in the remark above, we have to define the *Stalk* in the context of the definition (2.1.9).

Definition 2.1.10. *Let $\mathscr{F}$ be a sheaf over a topological space (M,τ) (with τ its topology) and let $x \in M$. The Stalk of $\mathscr{F}$ over x is5 given by:*

$$\mathscr{F}_x := \varinjlim_{U \ni \mathscr{U}} \mathscr{F}(U) \tag{2.1.45}$$

where $\mathscr{U} := \{U \in \tau \mid x \in U\}$ (the set of all open neighbourhoods of x) and $\varinjlim$ is the direct limit.

Aside - Direct Limit

The direct limit can be defined in the context of category theory, where it takes the most general form. Here we are going to present its form when applied to algebraic structures, since if $\mathscr{F}$ is a sheaf over M and $U \subset M$, then $\mathscr{F}(U)$ is a group as required by the definition (2.1.8). Before doing that, we need some preparatory mathematical concepts:

Definition 2.1.11 (Directed Set). *We will say that a pair $(I, \leq)$ is a Directed Set provided that I is a set and $\leq$ is a binary relation on I satisfying:*

(i) Reflexivity: $\forall i \in I : i \leq i$;

(ii) Transitivity: $\forall i, j, k \in I$ if $i \leq j$ and $j \leq k$ then $i \leq k$;

(iii) "Boundness": $\forall i, j \in I, \exists k \in I$ such that $i \leq k$ and $j \leq k$.

With a directed set we can construct a *Direct System* in the following way:

Definition 2.1.12 (Direct System). *Let $(I, \leq)$ be a directed set, $\{A_i \mid i \in I\}$ a family of objects indexed by I and $f_{ij} : A_i \to A_j$ a family of homomorphisms. We will say that (A_i, f_{ij}) is a Direct System if:*

(i) $\forall i \in I : f_{ii} = id_{A_i}$;

(ii) $\forall i \leq j \leq k : f_{ik} = f_{jk} \circ f_{ij}$.

We are now ready to define the *Direct Limit* of a direct system:

Definition 2.1.13 (Direct Limit). *Let (A_i, f_{ij}) be a direct system with respect to the directed set $(I, \leq)$. The Direct Limit of (A_i, f_{ij}) is given by:*

$$\varinjlim A_i := \frac{\mathring{\bigcup} A_i}{\sim} \tag{2.1.46}$$

where $\mathring{\cup}$ is the disjoint union and the equivalence relation $\sim$ is defined as follows: $\forall a_i \in A_i, a_j \in A_j$ we have that:

$$a_i \sim a_j \iff \exists k \in I \text{ with } i \leq k \text{ and } j \leq \text{ such that } f_{ik}(a_i) = f_{jk}(a_j) \tag{2.1.47}$$

Let's apply this formalism to the definition of the Stalk (2.1.10). The directed set which we are considering is the pair $(\mathscr{U}, \tilde{\subseteq})$ where $\mathscr{U}$ is the set of all open neighbourhoods of $x \in M$ and $\tilde{\subseteq}$ is given by the reversed inclusion, i.e. for two open sets A, B we have that $A \tilde{\subseteq} B \iff B \subseteq A$. All the properties of a directed set can be proved to be satisfied. Our direct system will then be the pair of families $(\mathscr{F}(U), r_V^U)$ indexed by the directed set $(\mathscr{U}, \tilde{\subseteq})$. To see that this is the case, by definition r_U^U are the identities on $\mathscr{F}(U)$ and the composition $r_W^U = r_W^V \circ r_V^U$ holds also by definition of restriction maps. In this context we can then define the stalk to be the direct limit:

$$\mathscr{F}_x = \varinjlim_{U \ni \mathscr{U}} \mathscr{F}(U) := \frac{\mathring{\bigcup} \mathscr{F}(U)}{\sim} \tag{2.1.48}$$

and the equivalence relation is translated in the fact that taken two elements of the disjoint union $(s, U), (s', U') \in \mathring{\bigcup} \mathscr{F}(U)$ then these are equivalent $(s, U) \sim (s', U')$ provided that there exists $W \in \mathscr{U}$ with $W \subseteq U$ and $W \subseteq U'$ (which is $U \tilde{\subseteq} W$ and $U' \tilde{\subseteq} W$) such that the sections coincide in W: $r_W^U(s) = r_W^{U'}(s')$. Requiring $W \subseteq U$ and $W \subseteq U'$ is the same as requiring $W \subseteq (U \cap U')$. That being said, the stalk of a sheaf $\mathscr{F}$ over $x \in M$, will be given by equivalence classes of sections (indexed sections due to the disjoint union) $[(s,U)]$ where two sections in an equivalence class are equivalent if they coincide in some open neighbourhood of $x \in M$. Given the quotient space, there is a natural map $\mathscr{F}(U) \to \mathscr{F}_x$ associating to each section its equivalence class. This is a generalisation of the concept of *germs of functions*. If we take the sheaf of continuous function, the stalk at a point will exactly coincide with the germs of functions passing through that point.

Sheaf Cohomology

Cohomology is a measure of how far from trivial some sequences are. We must introduce exact sequences of sheaves. In order to do that we need the notion of morphisms between sheaves.

Definition 2.1.14 (Sheaves Morphism)**.** *Let $\mathscr{E}$ and $\mathscr{F}$ be two sheaves over the topological space M. A Sheaves Morphism is a collection of maps $m : \mathscr{E} \to \mathscr{F}$ between sections $m_U : \mathscr{E}(U) \to \mathscr{F}(U)$ with m_U homomorphism of groups, such that:*

$$\tilde{r}^U_V \circ m_U = m_V \circ r^U_V \tag{2.1.49}$$

where $r^U_V : \mathscr{E}(U) \to \mathscr{E}(V)$ and $\tilde{r}^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$ are the restriction maps.

This can be restated by requiring the commutativity of the following diagram:

$$\begin{array}{ccc} \mathscr{E}(U) & \xrightarrow{m_U} & \mathscr{F}(U) \\ {\scriptstyle r^U_V}\downarrow & & \downarrow{\scriptstyle \tilde{r}^U_V} \\ \mathscr{E}(V) & \xrightarrow{m_V} & \mathscr{F}(V) \end{array}$$

Let's now take $\mathscr{E}, \mathscr{F}, \mathscr{G}$ to be sheaves over the same topological manifold M and e, f to be sheaves morphisms:

$$\mathscr{E} \xrightarrow{e} \mathscr{F} \xrightarrow{f} \mathscr{G} \tag{2.1.50}$$

The sequence above is called *exact* if e, f induces maps e_x, f_x (with $x \in M$) between the Stalks such that the following sequence is exact[6]:

$$\mathscr{E}_x \xrightarrow{e_x} \mathscr{F}_x \xrightarrow{f_x} \mathscr{G}_x \tag{2.1.52}$$

As an example consider the following sheaves: $\mathscr{Z}$ (constant sheaf of integers), $\mathscr{O}$ (sheaf of holomorphic functions) and $\mathscr{O}^*$ which is the sheaf of non-zero holomorphic functions. In order for $\mathscr{O}^*(U)$ to be a group, we have to take the group operation to be the point-wise multiplication of functions $(f \diamond g)(x) = f(x)g(x)$, in this way the neutral element will be the identity map and

[6]In the context of algebraic structures (groups, algebras, vector spaces, etc...) a sequence:

$$G_0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} \to \cdots \xrightarrow{f_n} G_n \tag{2.1.51}$$

is said to be *exact* if $\forall i : \ker(f_{i+1}) = \mathrm{Im}(f_i)$.

$(\mathcal{O}^*, \diamond)$ will be a well-defined group. Consider the following *Short Exact Sequence*:[7]

$$0 \to \mathscr{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0 \tag{2.1.54}$$

where i is the inclusion map and exp takes $f(z) \in \mathcal{O}(U)$ (for some U) into $\exp(2\pi i f(z))$. This sequence is indeed exact since $(\exp \circ i)(n) = \exp(2\pi i n) = 1$ and 1 is the identity element in $\mathcal{O}^*$ (i.e. its "zero" since $\mathcal{O}^*$ is a multiplicative sheaf).

Remark

The exactness of a sequence:

$$\mathscr{E} \xrightarrow{e} \mathscr{F} \xrightarrow{f} \mathscr{G} \tag{2.1.55}$$

does not imply the exactness of the sequence of global sections:

$$\mathscr{E}(M) \xrightarrow{e_M} \mathscr{F}(M) \xrightarrow{f_M} \mathscr{G}(M) \tag{2.1.56}$$

However, if the sheaf $\mathscr{E}$ is a *Soft Sheaf*, namely a local section on U can always be extended to a global section (which means that the restriction maps $r^M_U : \mathscr{E}(M) \to \mathscr{E}(U)$ are surjective), then the sequence (2.1.56) is exact.

Sheaf cohomology can be defined in the context of category theory and it is the most general construction that one can provide. Another cohomology that can be constructed in sheaf theory is *Čech Cohomology* and since these two cohomologies are actually isomorphic for paracompact spaces [GQ19], we won't go into cohomology theory in the context of categories, in fact we will remain on the more "intuitive" land of *chains* and *cochains* in Čech Cohomology.

Čech Cohomology

Given a sheaf $\mathscr{F}$ on M and an open covering $\mathscr{U} = \{U_\alpha\}_{\alpha\in\mathfrak{A}}$ (with $\mathfrak{A}$ index set) of M, we can take the collection $\triangle := (U_1, ..., U_{q+1})$ with $\overline{\triangle} := \bigcap_{i=1}^{q+1} U_i \neq \emptyset$ and define the set of all (Čech) *q-cochains*:

$$\check{C}^q(\mathscr{U}, \mathscr{F}) := \mathscr{F}\left(\overline{\triangle}\right) \tag{2.1.57}$$

and such a collection $\triangle$ of open sets is referred to as a *q-simplex* of $\mathscr{U}$.

On the set of q-cochains we can define a *Coboundary* operator:

$$\begin{aligned} \delta_q : \check{C}^q(\mathscr{U}, \mathscr{F}) &\to \check{C}^{q+1}(\mathscr{U}, \mathscr{F}) \\ f(\triangle) \mapsto \delta_q(f(\triangle)) &:= \sum_{i=1}^{q+2} (-1)^i r^{\overline{\triangle}_i}_{\overline{\triangle}} f(\triangle_i) \end{aligned} \tag{2.1.58}$$

where $\triangle_i = (U_1, ..., \hat{U}_i, ... U_{q+1})$ and the "hat" means that the element is missing.

[7] A short exact sequence is a sequence of the form:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{2.1.53}$$

such that it is exact. In this kind of sequences, the map f should be injective while the map g surjective. Given these properties, A can be thought of as "embedded" in B and C be isomorphic to the quotient $B/\mathrm{Im}(f)$.

Since for all q the composition $\delta_{q+1} \circ \delta_q$ vanishes, we can build a complex $\{\check{C}^\bullet(\mathscr{U},\mathscr{F}),\delta\}$ (where $\check{C}^\bullet(\mathscr{U},\mathscr{F}) = \bigoplus_q \check{C}^q(\mathscr{U},\mathscr{F})$):

$$\cdots \xrightarrow{\delta_{q-1}} \check{C}^q(\mathscr{U},\mathscr{F}) \xrightarrow{\delta_q} \check{C}^{q+1}(\mathscr{U},\mathscr{F}) \xrightarrow{\delta_{q+1}} \cdots \tag{2.1.59}$$

and therefore a cohomology:

$$\check{H}^q(\mathscr{U},\mathscr{F}) := \frac{\ker(\delta_q)}{\mathrm{Im}(\delta_{q-1})} \tag{2.1.60}$$

Notice that the cohomology is defined with respect to the open cover $\mathscr{U}$ and not over all the topological base space M. In order to get rid of the covering we employ again the direct limit and the cohomology over the hole space M will be the *Čech Cohomology*:

$$\check{H}^q(M,\mathscr{F}) = \varinjlim_{\mathscr{U}} \check{H}^q(\mathscr{U},\mathscr{F}) \tag{2.1.61}$$

The direct limit is taken with respect to all open coverings and $\check{H}^q(M,\mathscr{F})$ is usually referred to as the q-cohomology group of M with coefficients in the sheaf $\mathscr{F}$. The 0-th Čech cohomology will just be the set of global sections:

$$\check{H}^0(M,\mathscr{F}) = \mathscr{F}(M) =: \Gamma(M,\mathscr{F}) \tag{2.1.62}$$

where the last definition clearly reminds the notation used in defining sections of bundles.

An important fact about short exact sequences of sheaves is that they induce *long exact sequences* in cohomology. Consider the following short exact sequence of sheaves over M:

$$0 \longrightarrow \mathscr{E} \xrightarrow{e} \mathscr{F} \xrightarrow{f} \mathscr{G} \longrightarrow 0 \tag{2.1.63}$$

this will induce a short exact sequence in the associated complexes $\check{C}^\bullet(\mathscr{U},\mathscr{E}), \check{C}^\bullet(\mathscr{U},\mathscr{F})$ and $\check{C}^\bullet(\mathscr{U},\mathscr{G})$ [GQ19]:

$$0 \longrightarrow \check{C}^\bullet(\mathscr{U},\mathscr{E}) \longrightarrow \check{C}^\bullet(\mathscr{U},\mathscr{F}) \longrightarrow \check{C}^\bullet(\mathscr{U},\mathscr{G}) \longrightarrow 0 \tag{2.1.64}$$

That is because (2.1.63) by definition is true if for all $U \in M$ the following sequence is exact $0 \longrightarrow \mathscr{E}(U) \xrightarrow{e_U} \mathscr{F}(U) \xrightarrow{f_U} \mathscr{G}(U) \longrightarrow 0$ and considering that the space of Čhech q-cochains (2.1.57) can also be written as a direct product:

$$\check{C}^q(\mathscr{U},\mathscr{E}) = \prod_{i_1<i_2<\cdots i_q} \mathscr{E}(U_{i_1} \cap \cdots \cap U_{i_q}) \tag{2.1.65}$$

then the following sequence is exact:

$$0 \longrightarrow \mathscr{E}(U_{i_1} \cap \cdots \cap U_{i_q}) \longrightarrow \mathscr{F}(U_{i_1} \cap \cdots \cap U_{i_q}) \longrightarrow \mathscr{G}(U_{i_1} \cap \cdots \cap U_{i_q}) \longrightarrow 0 \tag{2.1.66}$$

and the direct product preserves this exactness, recovering (2.1.64). When there is a short exact sequence of complexes, one can construct a long exact sequence in the cohomologies by the fact that the following exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{C}^\bullet(\mathscr{U},\mathscr{E}) & \xrightarrow{e} & \check{C}^\bullet(\mathscr{U},\mathscr{F}) & \xrightarrow{f} & \check{C}^\bullet(\mathscr{U},\mathscr{G}) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \delta^{\mathscr{E}}} & & \big\downarrow{\scriptstyle \delta^{\mathscr{F}}} & & \big\downarrow{\scriptstyle \delta^{\mathscr{G}}} & & \\
0 & \longrightarrow & \check{C}^\bullet(\mathscr{U},\mathscr{E}) & \xrightarrow{e} & \check{C}^\bullet(\mathscr{U},\mathscr{F}) & \xrightarrow{f} & \check{C}^\bullet(\mathscr{U},\mathscr{G}) & \longrightarrow & 0
\end{array}$$

induces maps called *Connecting Morphisms* between $\check{H}^q(M,\mathscr{G})$ and $\check{H}^{q+1}(M,\mathscr{E})$, yielding the following long exact sequence

$$\begin{aligned}
&0 \longrightarrow \check{H}^0(M,\mathscr{E}) \longrightarrow \check{H}^0(M,\mathscr{F}) \longrightarrow \check{H}^0(M,\mathscr{G}) \longrightarrow \\
&\longrightarrow \check{H}^1(M,\mathscr{E}) \longrightarrow \check{H}^1(M,\mathscr{F}) \longrightarrow \check{H}^1(M,\mathscr{G}) \longrightarrow \cdots \\
&\vdots \\
&\longrightarrow \check{H}^q(M,\mathscr{E}) \longrightarrow \check{H}^q(M,\mathscr{F}) \longrightarrow \check{H}^q(M,\mathscr{G}) \longrightarrow \cdots
\end{aligned} \tag{2.1.67}$$

This result is very important in order to make computations in sheaf cohomology. For a soft sheaf $\mathscr{E}$, its higher cohomology groups vanish $\check{H}^q(M,\mathscr{E}) = 0$ for $q > 0$. Taking $\mathscr{E}^p$ to be the sheaf of p-forms on a differentiable manifold M, since it is a soft sheaf its higher cohomology groups vanish $\check{H}^q(M,\mathscr{E}^p) = 0$ for $q > 0$. Taking $\mathscr{Z}^p$ to be the sheaf of closed p-forms, i to be the inclusion map, d the exterior derivative ($\mathscr{Z}^0 = \mathscr{R}$ the constant sheaf of real numbers) and noting that $\mathscr{E}^0 = \mathscr{C}$ (sheaf of smooth functions), we can consider the following short exact sequences:

$$\begin{aligned}
&0 \longrightarrow \mathscr{R} \xrightarrow{i} \mathscr{E}^0 \xrightarrow{d} \mathscr{Z}^1 \longrightarrow 0 \\
&0 \longrightarrow \mathscr{Z}^1 \xrightarrow{i} \mathscr{E}^1 \xrightarrow{d} \mathscr{Z}^2 \longrightarrow 0 \\
&\qquad\vdots \\
&0 \longrightarrow \mathscr{Z}^p \xrightarrow{i} \mathscr{E}^p \xrightarrow{d} \mathscr{Z}^{p+1} \longrightarrow 0
\end{aligned} \tag{2.1.68}$$

These will induce long exact sequences in cohomologies:

$$\begin{aligned}
&0 \longrightarrow \check{H}^0(M,\mathscr{Z}^0) \longrightarrow \check{H}^0(M,\mathscr{E}^0) \longrightarrow \check{H}^0(M,\mathscr{Z}^1) \longrightarrow \check{H}^1(M,\mathscr{Z}^0) \longrightarrow \cdots \\
&0 \longrightarrow \check{H}^0(M,\mathscr{Z}^1) \longrightarrow \check{H}^0(M,\mathscr{E}^1) \longrightarrow \check{H}^0(M,\mathscr{Z}^2) \longrightarrow \check{H}^1(M,\mathscr{Z}^1) \longrightarrow \cdots \\
&\quad\vdots \\
&0 \longrightarrow \check{H}^0(M,\mathscr{Z}^p) \longrightarrow \check{H}^0(M,\mathscr{E}^p) \longrightarrow \check{H}^0(M,\mathscr{Z}^{p+1}) \longrightarrow \check{H}^1(M,\mathscr{Z}^p) \longrightarrow \cdots
\end{aligned} \tag{2.1.69}$$

However $\check{H}^i(M,\mathscr{E}^j) = 0$ for $i,j > 0$ and this means that out of every cohomology sequence, we can take out shorter exact sequences. For the first long exact sequence, we will have:

$$\begin{aligned}
&0 \longrightarrow \check{H}^0(M,\mathscr{Z}^0) \longrightarrow \check{H}^0(M,\mathscr{E}^0) \longrightarrow \check{H}^0(M,\mathscr{Z}^1) \longrightarrow \check{H}^1(M,\mathscr{Z}^0) \longrightarrow 0 \\
&0 \longrightarrow \check{H}^{q-1}(M,\mathscr{Z}^1) \longrightarrow \check{H}^q(M,\mathscr{Z}^0) \longrightarrow 0 \quad \text{for } q = 2,3,\ldots
\end{aligned} \tag{2.1.70}$$

And the last sequence simply tells us that the two groups in the sequence are isomorphic:

$$\check{H}^{q-1}(M,\mathscr{Z}^1) \simeq \check{H}^q(M,\mathscr{Z}^0) \tag{2.1.71}$$

Repeating the process for all the long exact sequences in (2.1.69), we eventually obtain:

$$\begin{aligned} \check{H}^{q-1}(M,\mathscr{Z}^1) &\simeq \check{H}^q(M,\mathscr{Z}^0) \\ \check{H}^{q-1}(M,\mathscr{Z}^2) &\simeq \check{H}^q(M,\mathscr{Z}^1) \\ &\vdots \\ \check{H}^{q-1}(M,\mathscr{Z}^{p+1}) &\simeq \check{H}^q(M,\mathscr{Z}^p) \end{aligned} \tag{2.1.72}$$

Following the chain of isomorphisms, inductively it can be obtained:

$$\check{H}^q(M,\mathscr{Z}^0) \simeq \check{H}^{q-1}(M,\mathscr{Z}^1) \simeq \cdots \simeq \check{H}^1(\mathscr{M},\mathscr{Z}^{q-1}) \tag{2.1.73}$$

Also, taking the first part of the penultimate sequence of (2.1.69), namely:

$$0 \longrightarrow \check{H}^0(M,\mathscr{Z}^{p-1}) \longrightarrow \check{H}^0(M,\mathscr{E}^{p-1}) \longrightarrow \check{H}^0(M,\mathscr{Z}^p) \longrightarrow \check{H}^1(M,\mathscr{Z}^{p-1}) \longrightarrow 0 \tag{2.1.74}$$

it will be true that:

$$\check{H}^1(M,\mathscr{Z}^{p-1}) \simeq \frac{\check{H}^0(M,\mathscr{Z}^p)}{\mathrm{Im}(d)} \simeq \frac{\mathscr{Z}^p(M)}{\mathrm{Im}(d:\mathscr{E}^{p-1}(M)\to\mathscr{E}^p(M))} = H^p_{dR}(M) \tag{2.1.75}$$

where we have indicated with the same symbol d the map $\mathscr{E}^p \xrightarrow{d} \mathscr{Z}^{p+1}$ and its associated map between cohomologies $\check{H}^q(M,\mathscr{E}^p) \xrightarrow{d} \check{H}^q(M,\mathscr{Z}^{p+1})$. We see that, by using the results (2.1.75) and (2.1.73), we get the famous *de Rham theorem*, which we have already mentioned previously, linking differentiable data to topological data:

Theorem 2.1.4 (*de Rham*)

$$\check{H}^q(M,\mathscr{R}) \simeq H^q_{dR}(M) \tag{2.1.76}$$

Of course the same construction can be replicated for the sheaf of (p,q)-forms $\mathscr{E}^{p,q}$ on a complex topological space M. $\mathscr{E}^{p,q}$ is defined by imposing on every open set $U \subset M$ its sections to be the (p,q)-forms $\mathscr{A}^{p,q}(U) = \mathscr{E}^{p,q}(U)$. Taking i to be the inclusion $\mathscr{Z}^{p,0} \xrightarrow{i} \mathscr{E}^{p,0}$ and the $\overline{\partial}$-differential operator $\mathscr{E}^{p,q} \xrightarrow{\overline{\partial}} \mathscr{E}^{p,q+1}$, we have the following exact sequences:

$$\begin{aligned} &0 \longrightarrow \mathscr{Z}^{p,0} \xrightarrow{i} \mathscr{E}^{p,0} \xrightarrow{\overline{\partial}} \mathscr{Z}^{p,1} \longrightarrow 0 \\ &0 \longrightarrow \mathscr{Z}^{p,1} \xrightarrow{i} \mathscr{E}^{p,1} \xrightarrow{\overline{\partial}} \mathscr{Z}^{p,2} \longrightarrow 0 \\ &\qquad\vdots \\ &0 \longrightarrow \mathscr{Z}^{p,q} \xrightarrow{i} \mathscr{E}^{p,q} \xrightarrow{\overline{\partial}} \mathscr{Z}^{p,q+1} \longrightarrow 0 \end{aligned} \tag{2.1.77}$$

Making the same reasoning as before, the associated long exact sequences, together with the fact that the sheaf $\mathscr{E}^{p,q}$ is soft for each p and q, we get:

$$\check{H}^q(M,\mathscr{Z}^{p,0}) \simeq \check{H}^{q-1}(M,\mathscr{Z}^{p,1}) \simeq \cdots \simeq \check{H}^1(M,\mathscr{Z}^{p,q-1}) \tag{2.1.78}$$

Because $\mathscr{Z}^{p,0} = \Omega^p$ (the sheaf of holomorphic p-forms) and analogously to before:

$$\check{H}^1(M,\mathscr{Z}^{p,q-1}) \simeq \frac{\check{H}^0(M,\mathscr{Z}^{p,q})}{\mathrm{Im}(\overline{\partial})} \simeq \frac{\mathscr{Z}^{p,q}(M)}{\mathrm{Im}(\overline{\partial}:\mathscr{E}^{p,q-1}\to\mathscr{E}^{p,q})} = H^{p,q}_{\overline{\partial}}(M) \tag{2.1.79}$$

we can state the *Dolbeault theorem*:

Theorem 2.1.5 (*Dolbeault*)

$$\check{H}^q(M,\Omega^p) \simeq H^{p,q}_{\bar{\partial}}(M) \tag{2.1.80}$$

2.1.6 Poincaré Duality

An important tool in algebraic topology is the Poincaré duality, which relates elements in the cohomology to elements in the homology. Homology and Cohomology are duals to one another, constructed the Homology of a given topological space, one can construct its Cohomology by an appropriate process of dualisation. Poincaré duality will play a crucial role in the study of Calabi-Yau manifolds since their moduli space is parametrised by elements of their cohomology and it will be useful to have basis for cohomologies and their dual basis of cycles in homology.

Homology and Cohomology

Homology was firstly developed in order to have a tool which could define and classify "holes" in a manifold. *Homotopy Theory* seems to do exactly this task. By studying loops in topological spaces it is able, through the fundamental groups $\pi_1(X)$ and higher homotopy groups $\pi_i(X)$, to give a good classification of topological spaces. The problem with homotopy theory is that higher homotopy groups are quite difficult to compute, even for spheres, the calculation of $\pi_i(S^n)$ turns out to be a very hard problem. Homology saves the day, since even if its definition may seems to be more obscure, at the end all the computational methods developed, allow for useful calculations and insights.

There are various homology theories that can be developed. Examples are: *Simplicial Homology*, *Singular Homology*, *Cellular Homology* and others. The most general context in which it can be defined is clearly category theory where it will take the form of a covariant functor (in contrast with cohomology which will be a contravariant functor) from the category of topological spaces to the category of abelian groups (for singular homology).

In the following we are going to consider Singular Homology, where, roughly speaking, the n-th singular homology group counts the n-dimensional holes in a given space. In order to construct it, we need some preliminary concepts. If we consider a 2-dimensional polygon, it can be decomposed into smaller building blocks, in particular into triangles in a process called *Triangulation*. In Euclidean $\mathbb{R}^n$, we can try to generalise the concept of a triangle by defining a *k-simplex*. This is the smallest convex set in $\mathbb{R}^n$ containing $k+1$ points $v_0,...,v_k$:

Definition 2.1.15. *Given $v_0,...,v_k \in \mathbb{R}^n$ such that $v_1-v_0,...,v_k-v_0$ are linearly independent, then the simplex $C \subset \mathbb{R}^n$ defined by them is the convex combination:*

$$s := \left\{ a = \sum_{i=0}^{k} \lambda_i v_i \,\middle|\, \sum_{i=0}^{k} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \; \forall i \right\} \tag{2.1.81}$$

and it is generally indicated by $s =: [v_0,...,v_k]$.

A 0-simplex is simply a point, 1-simplex is a segment joining the 2 points, a 2-simplex is a triangle, a 3-simplex is a tetrahedron and so on.

There is also the *Standard Simplex* given by vertices which are the unit vectors in the coordinates axis:

Definition 2.1.16. *A standard k-simplex is defined as:*

$$\triangle^k := \left\{ (a_0, ..., a_k) \in \mathbb{R}^{k+1} \,\middle|\, \sum_{i=0}^{k} a_i = 1 \text{ and } a_i \geq 0 \; \forall i \right\} \tag{2.1.82}$$

A continuous map between a standard k-simplex and a topological space M:

$$\sigma : \triangle^k \to M \tag{2.1.83}$$

is called *Singular k-simplex*. Consider a group G and the formal sums of singular k-simplices with respect to G:

$$c = \sum_{\sigma} c_\sigma \sigma \tag{2.1.84}$$

these are called *Singular k-chains* and the *Free Abelian Group* generated by taking as a basis the set of all singular k-simplices:

$$C_k(M,G) := \text{span}_G\{\sigma \mid \sigma \text{ is a singualar k-simplex}\} \tag{2.1.85}$$

is the group of all singular k-chains. On this space we can construct a boundary map $\partial : C_k(M,G) \to C_{k-1}(M,G)$ which maps a singular k-chain in its boundary components. The boundary operation is defined for a singular k-simplex by:

$$\partial_k \sigma = \sum_{i=0}^{k+1} (-1)^i \sigma_i \tag{2.1.86}$$

where σ_i is the i-th face of the singular k-simplex σ, namely the map $\sigma_i : \triangle^{k-1} \to M$ where $\triangle^{k-1}$ is the standard $(k-1)$-simplex given by the removal of the i-th vertex from the starting k-simplex $\triangle^k$. As an example, consider the 1-simplex $s = [v_0, v_1]$ (segment joining v_0 and v_1). Its boundary will be given by $\partial s = (-1)^0[\hat{v}_0, v_1] + (-1)^1[v_0, \hat{v}_1] = [v_1] - [v_0]$ (the hat signifies the missing element), which are the two boundary points with an eventual $\pm$ sign following from the orientation of the simplex.

Theorem 2.1.6

$$\partial_{k-1} \circ \partial_k = 0 \tag{2.1.87}$$

The proof of the theorem follows very easily from the definition of the boundary map, in fact if we represent the i-th face of a singular k-simplex by $\sigma|_{[v_0,...,\hat{v}_i,...,v_k]}$, then:

$$\begin{aligned}
\partial_{k-1}(\partial_k \sigma) &= \sum_{i=0}^{k} (-1)^i \partial_{k-1}(\sigma|_{[v_0,...,\hat{v}_i,...,v_k]}) \\
&= \sum_{i=0}^{k} (-1)^i \left(\sum_{j<i} (-1)^j \sigma|_{[v_0,...,\hat{v}_j,...,\hat{v}_i,...,v_k]} + \sum_{j>i} (-1)^{j-1} \sigma|_{[v_0,...,\hat{v}_i,...,\hat{v}_j,...,v_k]} \right) \\
&= 0
\end{aligned} \tag{2.1.88}$$

where the last equality is retrieved by noting that the second summation is the opposite of the first when $i \longleftrightarrow j$ are exchanged. Thanks to the nilpotency (2.1.6) of the boundary map, the exact singular chain complex:

$$\cdots \xrightarrow{\partial} C_k(M,G) \xrightarrow{\partial} C_{k-1}(M,G) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(M,G) \longrightarrow 0 \tag{2.1.89}$$

gives rise to the *Singular Homology Groups*:

$$H_k(M,G) := \frac{\ker(\partial : C_k(M,G) \to C_{k-1}(M,G)}{\mathrm{Im}(\partial : C_{k-1}(M,G) \to C_k(M,G))} \tag{2.1.90}$$

Elements in $\ker(\partial)$ are called *Cycles* while the elements in $\mathrm{Im}(\partial)$ are *Boundaries*. The nomenclature is quite clear, since a cycle is a singular chain which do not have boundaries ($\partial\sigma = 0$) and a boundary can be viewed as, indeed, the boundary of another singular chain $\alpha = \partial\beta$.

Following Hatcher [Hat01], consider now for each chain group $C_k(M,G)$ its dual *cochain group* $\mathscr{C}^k(M,G) := \mathrm{Hom}(C_k(M,G),G)$, namely the group of homomorphisms from $C_k(M,G)$ to G. Replace the boundary map with its dual *coboundary map* $\delta = \partial^* : \mathscr{C}^k(M,G) \to \mathscr{C}^k(M,G)$ and notice that the direction of δ is reversed with respect to the direction of ∂ and this is because the dual map of $\alpha : A \to B$ is $\alpha^* : \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$ and it is defined by $\alpha^*(\phi) = \phi(\alpha)$. Since $\partial \circ \partial = 0$ even its dual map will satisfy $\delta \circ \delta = 0$ and there will be an associated exact sequence:

$$\cdots \xrightarrow{\delta} \mathscr{C}^k(M,G) \xrightarrow{\delta} \mathscr{C}^{k+1}(M,G) \xrightarrow{\delta} \cdots \tag{2.1.91}$$

which gives rise to the *Cohomology Groups*:

$$H^k(M,G) := \frac{\ker(\delta : \mathscr{C}^k(M,G) \to C^{k+1}(M,G))}{\mathrm{Im}(\delta : \mathscr{C}^{k-1}(M,G) \to \mathscr{C}^k(M,G))} \tag{2.1.92}$$

Similarly to the homology case, the elements of $\ker(\partial)$ are called *Cocycles* whereas those of $\mathrm{Im}(\delta)$ are *Coboundaries*.

Cup and Cap Products

On the singular cohomology groups $H^k(M,G)$, it can be defined a product giving it a structure of a graded-commutative Ring and it is called *Cup Product*. This kind of product can also be given to de Rham cohomology groups and it will be inherited from the wedge product between k-forms. Here, the product on cohomology is inherited from a cup product between cochains defined on a given ring R (instead of a group G):

Definition 2.1.17 (Cup Product). *The Cup Product between cochains is the map:*

$$\begin{aligned} \smile : \mathscr{C}^k(M,R) \times \mathscr{C}^l(M,R) &\to \mathscr{C}^{k+l}(M,R) \\ (\phi,\psi) &\mapsto [\sigma \mapsto (\phi \smile \psi)(\sigma) := \phi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+l}]})] \end{aligned} \tag{2.1.93}$$

where $\sigma : \triangle^{k+l} \to M$ is a singular $(k+l)$-chain, the restrictions are defined as before and the product between ϕ and ψ is clearly the ring-product.

If we recall the action of exterior differentiation on a wedge product of k-forms, then an analogous property is satisfied by the coboundary map δ with respect to the cup product, namely $\forall \phi \in \mathscr{C}^k(M,R)$ and $\psi \in \mathscr{C}^l(M,R)$ it is true that:

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi \tag{2.1.94}$$

Now, given this property, it is immediate to see that the cup product between cocycles is again a cocycle, in fact taken $\phi \in \mathscr{C}^k(M,R) : \delta\phi = 0$ and $\psi \in \mathscr{C}^l(M,R) : \delta\psi = 0$ we simply have:

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi = 0 \smile \psi + (-1)^k \phi \smile 0 = 0 \tag{2.1.95}$$

Moreover the cup product between a cocycle and a coboundary is a coboundary. The cup product (2.1.17) then define a product on the singular cohomology groups as:

$$\begin{aligned} \smile : H^k(M,R) \times H^l(M,R) &\to H^{k+l}(M,R) \\ ([\phi],[\psi]) &\mapsto [\phi] \smile [\psi] := [\phi \smile \psi] \end{aligned} \tag{2.1.96}$$

and because of the properties mentioned above, this is indeed well defined. We then have the graded-commutative ring $(H^\bullet(M,R),+,\smile)$ with $H^\bullet(M,R) = \bigoplus_k H^k(M,R)$ and the graded-commutativity comes from the fact that the cup product is graded-commutative, namely for $\phi \in \mathscr{C}^k(M,R)$ and $\psi \in \mathscr{C}^l(M,R)$:

$$\phi \smile \psi = (-1)^{kl}(\psi \smile \phi) \tag{2.1.97}$$

which is valid for commutative rings R (for a proof see [Hat01]).

Along with the cup product, and in a close connection with it, a pairing between chains and cochains can be defined. This pairing takes the form of *Cap Product*:

Definition 2.1.18 (Cap Product)**.** *For a topological space M and a ring R, the R-bilinear pairing for $k \geq l$:*

$$\begin{aligned} \frown : C_k(M,R) \times \mathscr{C}^l(M,R) &\to C_{k-l}(M,R) \\ (\sigma,\phi) &\mapsto \sigma \frown \phi := \phi(\sigma|_{[v_0,\dots,v_l]})\sigma|_{[v_l,\dots,v_k]} \end{aligned} \tag{2.1.98}$$

is called Cap Product.

Since $\phi \in \mathscr{C}^l(M,R) := \mathrm{Hom}(M,R)$ the product is well defined, because $\phi(\sigma)\sigma$ will be an element of the free abelian group $C_{k-l}(M,R)$. Also, this will induce a *Cap Product* between homology and cohomology:

$$\begin{aligned} \frown : H_k(M,R) \times H^l(M,R) &\to H_{k-l}(M,R) \\ ([\sigma],[\phi]) &\mapsto [\sigma] \frown [\phi] := [\sigma \frown \phi] \end{aligned} \tag{2.1.99}$$

Finally we can state the Poincaré duality theorem, which, thanks to the cap product, allows to identify k-chains with $(n-k)$-cochains on a suitable manifold. The first instance of the theorem was due to Poincaré itself who realised that the k-th and $(n-k)$-th Betti numbers (2.1.40) of a closed and orientable manifold were actually the same. In modern times Poincaré duality is formulated in terms of homology and cohomology:

Theorem 2.1.7 (*Poincaré Duality*)

Let M be a n-dimensional orientable and closed manifold with fundamental class $[M] \in H_n(M,R)$ and R a ring. Then the map:

$$\begin{aligned} D : H^k(M,R) &\to H_{n-k}(M,R) \\ [\phi] &\mapsto D([\psi]) := [M] \frown [\phi] \end{aligned} \tag{2.1.100}$$

is an isomorphism $\forall k$.

The fundamental class of a n-dimensional orientable and closed manifold is an element of the homology group $H_n(M,R)$ which, roughly speaking, characterises its orientability. We then have a way to pair elements in the cohomology with elements in the homology (pair cycles with cocycles). If M is a compact n-(complex)-dimensional Kähler manifold, we can consider its de Rham cohomology $H^k_{dR}(M,\mathbb{C})$ and its homology $H_k(M,\mathbb{C})$ groups. Taking a basis for the cohomology $[\omega_i]$ there will be an associated basis in the homology $[\Omega_i]$ via Poincaré duality, namely for each $[\omega_i] \in H^k_{dR}(M,\mathbb{C})$ there will be a $(n-k)$-cycle $[\Omega_i] \in H_{n-k}(M,\mathbb{C})$ such that $\forall [\alpha] \in H^k_{dR}(M,\mathbb{C})$:

$$\int_{\Omega_j} [\alpha] = \int_M \alpha \wedge \omega_i \tag{2.1.101}$$

And this will re-appear in the next section in the study of moduli space of Calabi-Yau manifolds.

2.2 Calabi-Yau Manifolds

The study of complex manifold and in particular Kähler manifolds were thoroughly developed in the early/mid decades of the last century, but until 1977 there was an unresolved conjecture stated by Calabi [Cal55] in 1955 which can be presented as follows:

Conjecture 1 (*Calabi 1955*)

Let M be a compact Kähler manifold with metric tensor given by g and in local coordinates $g = g_{i\bar{j}} dx^i \otimes d\bar{z}^j$, with first Chern class $c_1(TM) = \frac{\sqrt{-1}}{2\pi} \tilde{R}_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Then it is possible to find a Kähler metric $\tilde{g} = \tilde{g}_{i\bar{j}} dz^i \otimes d\bar{z}^j$ such that $\tilde{R}_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is its Ricci tensor and such that the corresponding Kähler forms $\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and $\tilde{\omega} = \frac{\sqrt{-1}}{2} \tilde{g}_{i\bar{j}} dz^i \wedge d\bar{z}^j$ are in the same Cohomology class.

In particular, if the first Chern class vanishes, then this conjecture would imply that the underlying Kähler manifold should be Ricci-flat. The proof of this statement was announced in [Yau77] and given by Yau [Yau78]. Nowadays a Kähler manifold with vanishing first Chern class is called a Calabi-Yau manifold, so we can define:

Definition 2.2.1 (Calabi-Yau manifold)**.** *Let $\mathscr{Y}$ be a compact Kähler manifold. It is said to be a Calabi-Yau manifold if it has a vanishing first Chern class:*

$$c_1(T\mathscr{Y}) = 0$$

For an account of what Chern classes are, and more in general Characteristic classes, we refer the reader to the appendix A. As we have hinted before, the compactification manifold of superstring theories should be a Calabi-Yau manifold and it brings within itself a complex structure inasmuch as it is a Kähler manifold. But at this point the interesting question is how do, indeed, Calabi-Yau manifolds emerge in string theory and what are the consequences of it.

2.2.1 Emergence of Calabi-Yau Manifolds in String Theory

As we mentioned earlier, Calabi-Yau manifolds arise in the compactification of superstring theories in order to break some supersymmetries and leaving just one or at least two unbroken supersymmetries. In order to see that a Calabi-Yau manifold (as internal manifold) is needed to obtain this reduction of supersymmetries, we first have to make an assumption regarding the compactification ansatz (2.0.1). We shall, in fact, require the 4-dimensional manifold to be *Maximally Symmetric* which means *Homogeneous* and *Isotropic*. This requirement restrict the form of the Riemann tensor on $\mathcal{M}_4$ to be [BBS07]:

$$R_{\mu\nu\rho\sigma} = \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$

where $R = g^{\mu\rho}g^{\nu\sigma}R_{\mu\nu\rho\sigma}$ is the constant Ricci scalar curvature. The space-time $\mathcal{M}_4$ is then constrained to be *Minkowski* ($R = 0$), *Anti-de Sitter* ($R < 0$) or *de Sitter* ($R > 0$). Let's now restrict ourselves to Heterotic string theory. Since our task is to obtain a vacuum state of the form (2.0.1) in which $\mathcal{N} = 1$ supersymmetry in $\mathcal{M}_4$ is preserved, then we must have that the SUSY charges Q_ϵ generates a SUSY transformation with parameter ϵ in such a way that the variation of the fermionic fields vanishes:

$$\delta_\epsilon \psi = 0 \tag{2.2.1}$$

The invariance of the bosonic fields in the 10-dimensional action is trivial, since a supersymmetry transformation acting on them will generate fermions but - in a classical background - these vanish, i.e. the vacuum expectation value of a fermion must be zero in order to preserve Lorentz-invariance. The fermionic spectrum of Heterotic strings contains one gravitino ψ_M, one dilatino λ and the adjoint fermion χ (in the super Yang-Mills multiplet). By looking at the supersymmetry transformations of these fields, it is found that a non-trivial *covariantly constant spinor η* must exists on the internal manifold $\mathcal{M}_6$ once H_{mnl} is put equals to zero (for a complete account of this fact see [Can+85]):

$$\nabla_m \eta = 0 \tag{2.2.2}$$

Now, this condition is very restrictive on the form of the internal manifold which in fact will be a Ricci-flat and Kähler manifold, but also on the space-time manifold, which is restricted to be *Minkowski*. As far as the internal manifold is concerned, the Ricci-flatness is seen by considering the following (which is a direct computational result considering the covariant derivative in terms of the spin connection $\nabla_m = \partial_m + 1/4\omega_{mpq}\gamma^{pq}$):

$$[\nabla_m, \nabla_n]\eta = \frac{1}{4}R_{mnpq}\gamma^{pq}\eta = 0$$

Contracting with γ^n and using the identity $\gamma^n\gamma^{pq} = \gamma^{npq} + g^{np}\gamma^q - g^{nq}\gamma^p$ we obtain:

$$R_{mn} = 0 \tag{2.2.3}$$

So that $\mathcal{M}_6$ is indeed a Ricci-flat manifold. Our task now is to prove that it also admits a complex structure, in particular a Kähler structure, by constructing the Kähler form from the covariantly constant spinor η. This spinor can be decomposed into two parts of opposite chirality η_- and η_+ and taking the following bilinears to be normalised: $\eta_+^\dagger\eta_+ = \eta_-^\dagger\eta_- = 1$, we can use them to construct an *Almost Complex Structure*:

$$J_m{}^n = i\eta_+^\dagger\gamma_m{}^n\eta_+ = -i\eta_-^\dagger\gamma_m{}^n\eta_- \tag{2.2.4}$$

in fact:

$$J_m{}^pJ_p{}^n = -\delta_m^n$$

as it is required for J to be an almost complex structure. Moreover it can be seen that J is indeed a *Complex Structure* by noting that $\nabla_m J_n{}^p = 0$ (from the fact that η is a covariantly constant spinor) and that the *Nijenhuis tensor*[8] vanishes, making it possible to introduce local complex coordinates z^a and $\bar{z}^a$ in such a way that we can define the Kähler form:

$$J = J_{a\bar{b}} dz^a \wedge d\bar{z}^b = i g_{a\bar{b}} dz^a \wedge d\bar{z}^b \tag{2.2.5}$$

which indeed can be demonstrated to be closed: $dJ = 0$.

Remark

For a compact Kähler manifold with Kähler form $J = i g_{a\bar{b}} dz^a \wedge d\bar{z}^b$, its volume will be given by:

$$\mathcal{V} = \frac{1}{6} \int J \wedge J \wedge J \tag{2.2.6}$$

This can be seen by noticing that for complex coordinates expressed as $z^a = x^a + i y^a$, the volume form is just:

$$vol = \sqrt{g} dx^1 \wedge \cdots \wedge dy^3 = \frac{1}{6} J \wedge J \wedge J \tag{2.2.7}$$

The requirement of a $\mathcal{N} = 1$ supersymmetry on a maximally symmetric space-time manifold $\mathcal{M}_4$ has led to the fact that $\mathcal{M}_4$ must be Minkowski and a covariantly constant spinor must exists on the internal compact manifold $\mathcal{M}_6$ which in turn implies that this manifold should be Ricci-flat and Kähler. The existence of a non-trivial covariantly constant spinor, implies that the spin connection is a $SU(3)$ gauge field instead of $SU(4)$ (which is isomorphic to $O(6)$ and would be the general holonomy group of a 6-dimensional manifold). This means that $\mathcal{M}_6$ should be a manifold of $SU(3)$ holonomy.

Now, the catch of the Calabi's conjecture stated at the beginning of this section is that in general it is very hard to compute/find a metric of $SU(3)$ holonomy. Nevertheless thanks to Yau [Yau78] we now know that given a compact Kähler manifold with vanishing first Chern class then it is indeed Ricci-flat and of $SU(3)$ holonomy.

Using the spinor η it can be constructed a no-where vanishing holomorphic $(3,0)$-form whose components will be the unique spinor bilinear that can be constructed (because of chirality and symmetry constraints):

$$\Omega_{abc} = \eta_-^T \gamma_{abc} \eta_- \tag{2.2.8}$$

and the $(3,0)$-form will be:

$$\Omega = \Omega_{abc} dz^a \wedge dz^b \wedge dz^c \tag{2.2.9}$$

This holomorphic form is closed but not exact. The closeness follows from the fact that the bilinear with which it is constructed is covariantly constant and the non-exactness can be deduced from the fact that the combination $\Omega \wedge \bar{\Omega}$ is proportional to the volume form of the Calabi-Yau which clearly has a non-vanishing integral. In fact, a useful formula is:

$$\Omega \wedge \bar{\Omega} := -i ||\Omega||^2 vol \tag{2.2.10}$$

[8] As we saw in the previous section, this tensor defines the "obstruction" of an almost complex structure to be a complex structure. In local coordinates it takes the form:

$$N^p{}_{mn} = J_m{}^q \partial_{[q} J_{n]}{}^p - J_n{}^q \partial_{[q} J_{m]}{}^p$$

with:

$$||\Omega||^2 = 1/6\, g^{a_1\overline{b}_1} g^{a_2\overline{b}_2} g^{a_3\overline{b}_3} \Omega_{a_1a_2a_3} \overline{\Omega}_{\overline{b_1}\overline{b_2}\overline{b_3}} = 1/6\, \Omega_{abc} \overline{\Omega}^{abc} \tag{2.2.11}$$

In §2.1.4 we mentioned some properties that Betti and Hodge numbers possess on a given compact Kähler manifold. Since a Calabi-Yau is Kähler, these will hold even for Hodge and Betti numbers of Calabi-Yau manifolds. To these, however, we should add more constraints given by the extra structure that a Calabi-Yau possesses. For simply connected Calabi-Yau 3-folds (3-complex dimensional) $\mathscr{Y}$ the number of independent and non-trivial Hodge numbers drastically decreases to 2. The additional relations to that of §2.1.4, in fact, are:

(i) $b_1 = h^{(1,0)} + h^{(0,1)} = 0 \implies h^{(1,0)} = h^{(0,1)} = 0$ for simply connected $\mathscr{Y}$;

(ii) $h^{(p,0)} = h^{(n-p,0)}$ due to *Serre Duality* and the fact that $\mathscr{Y}$ admits a unique holomorphic $(3,0)$-form, making the canonical bundle $\bigwedge^3 T\mathscr{Y}^*$ trivial [Hub92].

Using all the relations between Hodge numbers, we can conclude that:

(i) $h^{(1,0)} = h^{(0,1)} = 0$;

(ii) $h^{(2,3)} = h^{(3,2)} = 0$ (from the above relation and Hodge star duality);

(iii) $h^{(2,0)} = h^{(0,2)} = 0$ (from the first relation and Serre duality);

(iv) $h^{(1,3)} = h^{(3,1)} = 0$ (from the above relation and Hodge star duality);

(v) $h^{(1,2)} = h^{(2,1)}$;

(vi) $h^{(1,1)} = h^{(2,2)}$;

(vii) $h^{(3,0)} = h^{(0,3)} = 1$;

(viii) $h^{(0,0)} = 1$ (from the above relation and Serre duality);

(ix) $h^{(3,3)} = 1$ (from the above relation and Hodge star duality).

In a more compact and useful way, these numbers are represented in the following *Hodge Diamond*:

$$\begin{array}{ccccccc} & & & h^{(3,3)} & & & \\ & & h^{(3,2)} & & h^{(2,3)} & & \\ & h^{(3,1)} & & h^{(2,2)} & & h^{(1,3)} & \\ h^{(3,0)} & & h^{(2,1)} & & h^{(1,2)} & & h^{(0,3)} \\ & h^{(2,0)} & & h^{(1,1)} & & h^{(0,2)} & \\ & & h^{(1,0)} & & h^{(0,1)} & & \\ & & & h^{(0,0)} & & & \end{array} \longrightarrow \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(2,2)} & & 0 & \\ 1 & & h^{(2,1)} & & h^{(1,2)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.2.12}$$

So we can see that the only non trivial Hodge numbers are $h^{(1,1)} = h^{(2,2)}$ and $h^{(1,2)} = h^{(2,1)}$. Later on, we are going to show that these numbers give the dimensions of the moduli space, with $h^{(1,1)}$ giving the number of Kähler moduli parametrising the shape of the Calabi-Yau volume and $h^{(1,2)}$ giving the number of complex structure moduli parametrising deformations of the complex structure. Since the Euler characteristic χ is defined as the alternating sum of betti numbers, we have that for a Calabi-Yau three-fold:

$$\begin{aligned}\chi &= b_0 - b_1 + b_2 - b_3 + b_4 - b_5 + b_6\\ &= 1 - 0 + h^{(1,1)} - (2 + 2h^{(2,1)}) + h^{(1,1)} - 0 + 1\\ &= 2\left(h^{(1,1)} - h^{(2,1)}\right)\end{aligned} \tag{2.2.13}$$

and clearly of the given Calabi-Yau three-fold has $h^{(1,1)} > h^{(2,1)}$, then its Euler characteristic will be positive, while, vice versa for $h^{(1,1)} < h^{(2,1)}$ the Euler characteristic will be negative.

2.2.2 Emergence of Elliptic Fibrations in Type IIB Compactifications with 7-branes

If we come back to the compactification $\mathscr{M} = \mathbb{R}^{1,9-2n} \times B_n$ with B_n a compact and n-dimensional complex manifold and with a (p,q) 7-brane wrapping a holomorphic cycle Σ_{n-1} in the compact manifold, as we saw in §1.6, due to the backreaction of the 7-brane, the axio-dilaton is not constant, but it is allowed to vary in the directions perpendicular to the 7-brane with a monodromy given by (1.6.6). This means that the dilaton contributes to the *Einstein Field Equations*, in such a way that B_n cannot be Ricci-flat and hence a Calabi-Yau. In order to see that, consider the trace-reversed Einstein's equations in 10 spacetime dimensions:

$$R_{MN} = 8\pi G_{10}\left(T_{MN} - \frac{1}{2}G_{MN}T\right) \tag{2.2.14}$$

Using the compactification ansatz:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{a\bar{b}}dx^a d\bar{x}^b \tag{2.2.15}$$

(where $\mu,\nu = 0,1,..,9-2n$ and $a,\bar{b} = 1,...,n$), the Einstein's equations for the compact space are the equations of motion for the metric $g_{a\bar{b}}$, and are found by varying the type IIB action with respect to the metric and imposing the vanishing condition:

$$\frac{\delta S_{IIB}}{\delta g^{a\bar{b}}} = 0 \iff R_{a\bar{b}} = T_{a\bar{b}} - \frac{1}{2}g_{a\bar{b}}T \tag{2.2.16}$$

And the energy-momentum tensor is just:

$$\begin{aligned}T_{a\bar{b}} &= -\frac{2k_{10}^2}{\sqrt{-g}}\frac{\delta S_{IIB}}{\delta g^{a\bar{b}}} =\\ &= -\frac{2k_{10}^2}{\sqrt{-g}}\frac{\delta}{\delta g^{a\bar{b}}}\left(\frac{1}{2k_{10}^2}\int d^{2n}x\sqrt{-g}\,4\,g^{c\bar{d}}\nabla_c\phi\nabla_{\bar{d}}\phi\right) =\\ &= -\frac{1}{\sqrt{-g}}\int d^{2n}x\left(\frac{\delta(\sqrt{-g})}{\delta g^{a\bar{b}}}4g^{c\bar{d}}\nabla_c\phi\nabla_{\bar{d}}\phi + \sqrt{-g}\,4\,\frac{\delta(g^{c\bar{d}})}{\delta g^{a\bar{b}}}\nabla_c\phi\nabla_{\bar{d}}\phi\right) =\\ &= -\frac{1}{\sqrt{-g}}\left(2\sqrt{-g}\,g_{a\bar{b}}g^{c\bar{d}}\nabla_c\phi\nabla_{\bar{d}}\phi + 4\sqrt{-g}\,\nabla_a\phi\nabla_{\bar{b}}\phi\right) =\\ &= -\nabla_a\phi\nabla_{\bar{b}}\phi - \frac{1}{2}g_{a\bar{b}}\nabla_c\phi\nabla^c\phi\end{aligned} \tag{2.2.17}$$

In this way, the Einstein's equations become:

$$R_{a\bar{b}} = -\nabla_a \phi \nabla_{\bar{b}} \phi \tag{2.2.18}$$

proving our claim that the compact space B_n in the presence of 7-branes cannot be a Calabi-Yau manifold.

Now, the variation of the axio-dilaton τ defines a *Holomorphic Line Bundle* $\mathscr{L}$ over B_n. The construction goes as follows [Wei18; BCM11]. Consider the 1-form:

$$A = \frac{i}{2}\frac{d(\tau+\bar{\tau})}{\tau-\bar{\tau}} = \frac{i}{2}(\bar{\partial}\phi - \partial\phi) \tag{2.2.19}$$

which can be seen to transform as a connection, so that we can identify it as the connection of a complex line bundle L over B_n. As we encircle the 7-brane in its normal direction, we saw that the axio-dilaton develops a monodromy depending on the type of the 7-brane. This means that we cannot define type IIB fields globally, but we can define them in different patches in which they are in a certain $SL(2,\mathbb{Z})$ "frame". The transition functions between these patches can be identified with $\exp(i \arg(c\tau + d))$. Also, to every complex line bundle with curvature of type $(1,1)$, one can associate a holomorphic line bundle $\mathscr{L}$ with, then, holomorphic transition functions. The Einstein equations (2.2.18) implies that the first Chern class of the compact space B_n should equals that of the holomorphic line bundle:

$$c_1(B_n) = c_1(\mathscr{L}) \tag{2.2.20}$$

in this way one can see that the non-triviality of the line bundle $\mathscr{L}$ is directly related to the failure of B_n to be a Calabi-Yau manifold.

Now, the line bundle $\mathscr{L}$ over the compact space B_n together with a choice of sections for $\mathscr{L}^4$ and $\mathscr{L}^6$ uniquely defines an elliptic fibration over B_n with varying elliptic parameter τ. In fact, the transition functions' transformations resemble the behaviour of the parameters in the *Weierstrass Model* [Wei18].

Aside

The Weierstrass model is a way to represent an elliptic curve $\mathbb{E}_\tau = \mathbb{C}/\Lambda$ as the vanishing locus of a homogeneous polynomial $P_W \in \mathbb{P}^2_{(231)}$ (see definition (B.0.1) of the weighted projective space and the following discussion of weighted homogeneous polynomials):

$$P_W = y^2 - (x^3 + f x z^4 + g z^6) \tag{2.2.21}$$

The map from the elliptic curve to the Weierstrass model is given by:

$$\mathbb{C} \ni w \mapsto \begin{cases} |4^{2/3}\wp(w;\tau) : 2\wp(w;\tau)' : 1| & w \notin \Lambda \\ |1:1:0| & w \in \Lambda \end{cases} \tag{2.2.22}$$

where $\wp$ is the Weierstrass's elliptic function and it is the unique meromorphic function doubly periodic (namely $\wp(w;\tau) = \wp(w+\tau;\tau)$) on $\mathbb{C}/\Lambda$ with double poles at the lattice points:

$$\wp(w;\tau) := \frac{1}{w^2} + \sum_{(m,n)\in\mathbb{Z}^2 \neq (0,0)} \left(\frac{1}{(w+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right) \tag{2.2.23}$$

It also satisfies a differential equation of the form:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \tag{2.2.24}$$

where g_2 and g_3 are functions of the elliptic parameter τ and are given by the Eisenstein's series:

$$\begin{aligned} g_2(\tau) &= 60 \sum_{(m,n)\in\mathbb{Z}^2 \neq (0,0)} (m+n\tau)^{-4} \\ g_3(\tau) &= 140 \sum_{(m,n)\in\mathbb{Z}^2 \neq (0,0)} (m+n\tau)^{-6} \end{aligned} \tag{2.2.25}$$

Now, if we define:

$$f(\tau) = -4^{1/3}\, g_2(\tau); \qquad g(\tau) := -4 g_3(\tau) \tag{2.2.26}$$

then by a straightforward substitution it can be seen that the differential equation satisfied by the Weierstrass's function $\wp$ becomes:

$$y^2 - (x^3 + f x z^4 + g z^6) = 0 \tag{2.2.27}$$

There is also a correspondence in the other way, namely given a Weierstrass model it is possible to deduce the elliptic parameter τ from the functions f and g (we leave the details to [Wei18]). What is important, though, is that under an $SL(2,\mathbb{Z})$ transformation $\tau \mapsto \frac{a\tau+b}{c\tau+d}$, the functions f and g of the Weierstrass model are mapped to:

$$\begin{aligned} f &\mapsto (c\tau+d)^4 f \\ g &\mapsto (c\tau+d)^6 g \end{aligned} \tag{2.2.28}$$

So, what we have found is that specifying the complex structure of an elliptic curve $\mathbb{E}_\tau$ is equivalent to specify the complex parameter f and g of the Weierstrass model, parameters that transforms according to (2.2.28), which are also the transformations of the transition functions of the bundles $\mathscr{L}^4$ and $\mathscr{L}^6$.

In order to construct the elliptic fibration associated with $\mathscr{L}$, let's promote the coordinates $|x : y : z|$ of the Weierstrass model and the functions f and g to sections of a suitable line bundle over B_n in such a way that the elliptic curve $\mathbb{E}_\tau$ varies over B_n to form an elliptic fibration. Given the transformations of f and g, it is then natural to identify them as:

$$f \in \Gamma(B_n, \mathscr{L}^4); \qquad g \in \Gamma(B_n, \mathscr{L}^6) \tag{2.2.29}$$

This means that, once the coordinates x, y and z of the Weierstrass's equation are taken as sections of suitable bundles in order to respect the homogeneity property of P_W, an elliptic fibration is obtained:

$$\pi : \mathbb{E}_\tau \to Y_{n+1} \to B_n \tag{2.2.30}$$

and it is readily find that the first Chern class of Y_{n+1} vanishes:

$$c_1(Y_{n+1}) = c_1(B_n) - c_1(\mathscr{L}) = 0 \tag{2.2.31}$$

making Y_{n+1} a Calabi-Yau manifold elliptically fibred over the base space B_n. What we have found is that even though the compact space B_n over which we compactify our type IIB theory is not Calabi-Yau, its elliptic fibration is.

2.2.3 Some Algebraic Geometry

The problem that we would like to address in this section is the following: when should we regard two Calabi-Yau manifolds as equivalent? This question is naturally embedded in the domain of mathematics dealing with the problem of *Classification*. If one has a class of mathematical objects what is desirable to obtain is a notion of when two of them should be treated as equivalent, i.e. some notion of equivalence relation should be defined. In this regard *Moduli Spaces* can be thought of as geometric solutions to geometric classification problems [Ben08]. These kind of spaces arise in various fields of mathematics such as *Algebraic/Differential Geometry* and *Algebraic/Differential Topology* (in topology moduli spaces are in general called classifying spaces). The idea is to consider all the objects that we want to classify and give them a geometric structure which will allow to a deeper study of the objects themselves by studying how these are related in the overall set of objects.

The origin of the study of Moduli Spaces can be traced back to Riemann in trying to classify *Riemann Surfaces*. These are complex manifolds of complex dimension one and are also *Algebraic Varieties*, which means that they can be represented as the loci of zeros of a collection of homogeneous polynomials. Algebraic varieties are connected to the *Complex Projective Space* $\mathbb{C}P^n$ since it is naturally endowed with homogeneous coordinates. A general criterion to determine when a compact complex manifold is an algebraic variety is given by the *Kodaira* embedding theorem:

Theorem 2.2.1 (*Kodaira*)

A compact complex manifold M is an algebraic variety if and only if it has a closed, positive (1,1)-form ω such that the cohomology class $[\omega] \in H^2(M, \mathbb{Q})$.

This is a useful tool since algebraic varieties can be studied in the context of algebraic geometry which allows to apply the powerful formalism and theorems of algebraic geometry.

How should we identify two Riemann surfaces? Answering this question will lead to the definition of the Moduli Space of a Riemann surface which will be crucial to understand what really goes on when one talks about the Moduli Space of a Calabi-Yau. Let's first give two equivalent definitions of what a Riemann surface is.

Definition 2.2.2 (Riemann Surface)**.**

(i) A Riemann surface is a one-dimensional complex manifold, which means that it is a 2-dimensional real manifold M such that its atlas $\mathscr{A}$ is defined by charts (U_α, ϕ_α) where the homeomorphisms $\phi_\alpha : U_\alpha \to D_1$ are glued together in a biholomorphic[9] way (D_1 is the unit open disk in $\mathbb{C}$). In other words, the transition functions of the manifold must be biholomorphic.

(ii) A Riemann surface is a two-dimensional, real and oriented manifold together with a conformal structure.

The last statement of the first definition is a very strong condition since holomorphic functions are more "rigid" than smooth functions and in fact complex manifolds are closer to algebraic varieties than differentiable manifolds. In the following we are going to define the conformal structure and give a heuristic argument explaining why these two definitions are equivalent.

Now that we have properly defined a Riemann surface we shall start to examine when we should regard two such Riemann surfaces as equivalent. There are various ways in which we can make this identification. With respect to the topology we could say that two Riemann surfaces are equivalent if they are homeomorphic. This will be however pretty useless since we would completely forget about the complex structures of such objects. There are even stronger conditions that one could employ, for example if we regard these surfaces as the loci of some algebraic

[9]A biholormorphic function is a bijective function whose inverse is also holomorphic.

equations in a suitable projective space, we could say that these Riemann surfaces are equivalent if they are equals as subset. Even this identification would be useless in practical applications since what we are interested in is the intrinsic geometry of Riemann surfaces and not such incidental features emerging due to the particular way in which we represent them.

As already hinted we should focus on the complex structure of the Riemann surfaces and define an equivalent relation among them by means of a suitable geometrisation. Our main reference for the next sections will be [Nas91].

2.2.4 Teichmüller Space

Consider a Riemannian metric on a Riemann surface Σ which locally would look like:

$$ds^2 = Adx^2 + 2Bdxdy + Cdy^2$$

It is always possible to recast this metric in terms of complex coordinates $(z, \bar{z})$ using appropriate functions $\lambda > 0, \mu$ as follows:

$$ds^2 = \lambda |dz + \mu d\bar{z}|^2$$

It is clear that a Weyl transformation $g \to e^{2\omega} g$ does not change the complex structure of the Riemann surface inasmuch as what it accomplish is a volume rescaling. In this regard it is suggested that there should be a one-to-one correspondence between conformal classes and complex structures (which is the equivalence between the two definitions of a Riemann surface). Let us rephrase this statement in a more geometrical way. Consider the set of all metrics on a given Riemann sphere Σ:

$$\mathcal{M}(\Sigma) := \left\{ g \in \mathfrak{X}^0_2(\Sigma) \mid g \text{ is a Riemannian metric on } \Sigma \right\}$$

where $\mathfrak{X}^0_2(\Sigma)$ is the set of all $\binom{0}{2}$-tensor fields on Σ (which are sections of the $\binom{0}{2}$-tensor bundle $\mathscr{T}_2{}^0\Sigma$ and in this way they can also be denoted as $\Gamma(\Sigma, \mathscr{T}_2{}^0\Sigma)$). Calling $C^\infty_+(\Sigma)$ the set of all smooth and positive functions on Σ, two metrics g_1, g_2 are conformally equivalent if $\exists f \in C^\infty_+(\Sigma)$ such that $g_1 = f\, g_2$. The action of such smooth functions can be defined as:

$$\begin{aligned} A : C^\infty_+(\Sigma) \times \mathcal{M}(\Sigma) &\to \mathcal{M}(\Sigma) \\ (f, g) &\mapsto [x \mapsto A(f(x), g(x)) := f(x) \cdot_{\mathbb{R}} g(x)] \end{aligned}$$

where $\cdot_{\mathbb{R}}$ denotes the multiplication on the *Field* of real numbers $\mathbb{R}$ and the conformal structure will be given by:

$$Conf(\Sigma) := \frac{\mathcal{M}(\Sigma)}{C^\infty_+(\Sigma)} \tag{2.2.32}$$

What we do not want to is consider two metrics as distinct if they are related by a diffeomorphism $\phi : \Sigma \to \Sigma$. These kind of diffeomorphisms are called *Isometries* and the infinitesimal generators of such isometries are named *Killing vector fields*. Diffeomorphisms act on tensor fields with their pull-back in the following way:

$$\begin{aligned} \phi^* : \mathfrak{X}^0_2(\Sigma) &\to \mathfrak{X}^0_2(\Sigma) \\ g &\mapsto \phi^* g \\ &\quad [p \mapsto (\phi^* g)_p] \\ &\qquad [(v_p, w_p) \mapsto (\phi^* g)_p(v_p, w_p) := g_{\phi(p)}(T_p\phi \cdot v_p, T_p\phi \cdot w_p)] \end{aligned}$$

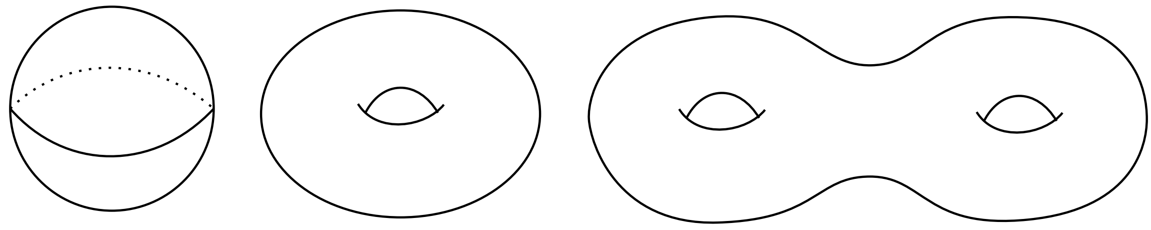


Figure 2.1: Representation of three different surfaces with genera $p = 0, 1, 2$. The sphere "has no holes", meaning that its genus is exactly zero. A torus has one hole, meaning that $p = 1$, while the third surface is easily seen to be the connected sum of two tori, meaning that its genus will be equals to 2.

where p is a point in Σ, v_p and w_p are tangent vectors at the point p, i.e. $v_p, w_p \in T_p\Sigma$ and $T_p\phi$ is the tangent map of ϕ, i.e. its "differential". We shall focus on *orientation* preserving diffeomorphisms, in this regard we will call this *Group*[10] $Diff^+(\Sigma)$. We are now ready to give our first "restricted" definition of *Teichmüller space*.

Definition 2.2.3 (Restricted Teichmüller Space)**.** *Given a closed Riemann surface Σ of genus $p \geq 2$, its Teichmüller space $\mathscr{T}_{p\geq 2}$ is defined as:*

$$\mathscr{T}_{p\geq 2} := \frac{Conf(\Sigma)}{Diff_0(\Sigma)}$$

where $Diff_0(\Sigma)$ is the subset of $Diff^+(\Sigma)$ which is continuously connected to the identity.

The *genus* p of an orientable surface, roughly speaking, counts the number of handles/holes. In a more precise way the genus for a connected and closed surface can be defined using the *Classification Theorem of Closed Surfaces*, which states that if a surface is connected, closed and orientable, then it is homeomorphic to either a sphere or a connected sum[11] of $p \geq 1$ tori; while a non-orientable, closed and connected surface is homeomorphic to the connected sum of $k \geq 1$ real projective spaces. Now, the number p (which for a sphere is $p = 0$) is called the genus of the surface, and in figure (2.1) there are shown some surfaces with different genera p. Using this theorem and the definition of the *Euler Characteristic* as the alternating sum of Betti numbers:

$$\chi = \sum_i (-1)^i b_i$$

[10]Since $(Diff(\Sigma), \circ)$ is a Group in order to prove that $Diff^+(\Sigma) \subset Diff(\Sigma)$ is a subgroup (and hence a group itself) it suffices to prove that it contains the identity and that it is closed under the restriction of the $\circ$-operation. The first requirement is fulfilled since the identity does not change the orientation. The second is also clearly verified since the orientation cannot change by a composition of two orientation-preserving functions. These statements can be made rigorous considering that for a n-dimensional differentiable manifold M an orientation is given by an equivalence class $[\omega]$ of top forms $\omega \in \Gamma(M, \bigwedge^n TM^*)$ where two of them ω_1, ω_2 are equivalent if there exists a positive function $f \in C^\infty_+(M)$ such that $\omega_1 = f\omega_2$.

[11]A connected sum between two manifolds, roughly speaking, is a manifold formed by deleting a ball inside each manifold and gluing together the resulting boundary spheres.

it can be proved that for an orientable and closed connected surface, the Euler characteristic can be recast as:

$$\chi = 2 - 2p \tag{2.2.33}$$

Coming back to the Teichmüller space we see that the definition is valid only for genera $p \geq 2$ and this is so because for $p = 0, 1$ the quotient space becomes singular due to the fact that $Diff_0(\Sigma)$ does not act freely anymore. This fact manifests itself with the presence of fixed points which is equivalent to say that there exists metrics in $\mathscr{M}(\Sigma)$ which are mapped to a conformally equivalent one by a non-trivial element of $Diff_0(\Sigma)$, i.e. there exists $\phi \in Diff_0(\Sigma)$ such that $\phi^*[g] = [g]$.

How should we deal with the genera $p = 0, 1$ then? A key insight which will enable us to give a more general definition of the Teichmüller space valid for all p is the *Riemann uniformisation theorem*.

Theorem 2.2.2 (*Riemann Uniformisation*)

Let M be a simply connected [12]Riemann surface of genus p. Then M is conformally equivalent to $\tilde{\Sigma}$, where $\tilde{\Sigma}$ is one of the following Riemann surfaces:

$$\tilde{\Sigma} = \begin{cases} S^2 & \text{if } p = 0; \\ \mathbb{C} & \text{if } p = 1; \\ H^2 & \text{if } p \geq 2. \end{cases}$$

(where H^2 is the upper-half $\mathbb{C}$-plane, $\mathfrak{Im}(z) > 0$, also called hyperbolic plane).

The conformal equivalence stated in the theorem means that given a simply connected Riemann surface Σ with metric g there is a $\tilde{\Sigma}$ surface given as in the theorem with metric $\tilde{g}$, in such a way that there exists a $C_+^\infty(\Sigma)$ function f such that $g = f\tilde{g}$. Another way to formulate the theorem is to say that every compact Riemann surface Σ admits a simply connected universal cover $\tilde{\Sigma}$ of the form given above. Since each of these $\tilde{\Sigma}$ has a unique complex structure determined by a metric of *constant curvature*, then the covering projection $\pi : \tilde{\Sigma} \to \Sigma$ induces a constant curvature metric on Σ too. In this way, we can characterise Riemann surfaces in terms of their constant curvature metrics. If we denote $R(g)$ the Gaussian curvature of a metric $g \in \mathscr{M}(\Sigma)$, then we can define the space:

$$\mathscr{M}_{const}(\Sigma) = \begin{cases} \{g \in \mathscr{M}(\Sigma) : R(g) = 1\} & \text{if } p = 0 \\ \{g \in \mathscr{M}(\Sigma) : R(g) = 0,\ vol(\Sigma) = 1\} & \text{if } p = 1 \\ \{g \in \mathscr{M}(\Sigma) : R(g) = -1\} & \text{if } p \geq 2 \end{cases}$$

where in the $p = 1$ case it must be specified the volume of Σ since it is the Torus T^2 and with a vanishing Euler characteristic (2.2.33) the *Gauss-Bonnet theorem* imposes no normalisation restriction on its volume [Nas91]. We are now able to give a complete definition for the Teichmüller space valid for all genera p.

Definition 2.2.4 (Teichmüller Space). *Given a closed Riemann surface Σ of genus $p \in \mathbb{N}$, its Teichmüller space $\mathscr{T}_p$ is defined as:*

$$\mathscr{T}_p := \frac{\mathscr{M}_{const}(\Sigma)}{Diff_0(\Sigma)}$$

[12]A simply connected manifold is a path-connected manifold such that each closed loop is homotopic to a point, where a path-connected manifold is a manifold in which every pair of points can be joined by a path.

2.2.5 Moduli Space

Having defined the Teichmüller space $\mathcal{T}_p$ for all genera $p \in \mathbb{N}$, we can now turn our attention to the more general case in which we do not restrict ourselves to the connected component of the identity $Diff_0(\Sigma)$ but consider the complete group of orientation-preserving diffeomorphisms $Diff^+(\Sigma)$. In this way we will take into consideration all the possible complex structures on the given Riemann surface Σ.

Definition 2.2.5 (Moduli Space). *The Moduli Space $\mathcal{M}_p$ of a closed Riemann surface Σ of genus $p \in \mathbb{N}$ is given by:*

$$\mathcal{M}_p = \frac{\mathcal{M}_{const}(\Sigma)}{Diff^+(\Sigma)}$$

If we define the *Mapping Class Group* Γ_Σ as the quotient of the $Diff^+(\Sigma)$ with respect to $Diff_0(\Sigma)$ then it can be readily seen that a relation between the Moduli space and the Teichmüller space arises:

$$\mathcal{M}_p = \frac{\mathcal{T}_p}{\Gamma_\Sigma} \tag{2.2.34}$$

with

$$\Gamma_\Sigma := \frac{Diff^+(\Sigma)}{Diff_0(\Sigma)} \tag{2.2.35}$$

Remark

We can say that the mapping class group Γ_Σ is a discrete group which counts the number of connected components of the orientation-preserving diffeomorphisms $Diff^+(\Sigma)$. What we have tacitly assumed when we introduced the connected to the identity component $Diff_0(\Sigma)$, is that $Diff^+(\Sigma)$ admits a topology in the first place (if we want to talk about connectedness). The construction of the mapping class group is completely general and can be defined for an arbitrary compact, closed and orientable surface S endowed with a topology given by a metric d on it. If we define the set of all orientation-preserving Homeomorphisms $Homeo^+(S)$, it can be made a topological space by defining the so called *compact-open topology* induced by the metric d on S in the following way:

$$\begin{aligned} \delta : Homeo^+(S) \times Homeo^+(S) &\to \mathbb{R} \\ (f,g) &\mapsto \delta(f,g) := \sup_{x\in S} d(f(x), g(x)) \end{aligned}$$

Once defined the topology, τ say, induced by this distance then it is fair to consider the normal subgroup[13] of it consisting of the connected component to the identity $Homeo_0(S)$ and then define the mapping class group of the surface S as:

$$\Gamma_S = \frac{Homeo^+(S)}{Homeo_0(S)}$$

If we replace the $Homeo$ with $Diff$ in the context of differential topology, then we obtain the previous (2.2.35).

[13] A subgroup H of a group $(G, \circ)$ is said to be a Normal subgroup if $\forall g \in G$ and $\forall h \in H$ it is true that $g \circ h \circ g^{-1}$ lies in H.

The moduli space is then the complete space of complex structures of a given Riemann surface Σ. Since the mapping class group Γ_Σ does not act freely on $\mathcal{M}_p$, the moduli space will have some fixed points, making it not a manifold but what is called an *Orbifold*.

The work need to obtain the dimension of the moduli space is quite involved and it can be found by working out the dimension of its tangent space (for details we re refer the reader to Nash [Nas91]). The result is the following:

$$\dim_{\mathbb{C}}(\mathcal{M}_p) = \begin{cases} 0 & p=0 \\ 1 & p=1 \\ 3p-3 & p \geq 2 \end{cases} \tag{2.2.36}$$

String Geometry

An important application of the construction of Moduli Spaces is in the quantisation of the Polyakov action S_p through functional integral. It will involve an integration over the space of worldsheet metrics with measure $\mathscr{D}g$ and an integral over the space of embeddings with measure $\mathscr{D}X$ for a worldsheet of genus p:

$$\mathscr{Z}_p = \int \mathscr{D}g\mathscr{D}Xe^{-S_P} \tag{2.2.37}$$

This partition function, in mathematical sense, is a trace and in quantum theory it represents the vacuum-to-vacuum amplitude where an object is emitted from the vacuum and re-absorbed. The worldsheet will then be a closed surface, namely a Riemann surface Σ. In this way, in the functional integral we must also sum over all possible genera p:

$$\mathscr{Z} = \sum_p \int \mathscr{D}g\mathscr{D}Xe^{-S_P} = \sum_{p=0}^{\infty} \mathscr{Z}_p \tag{2.2.38}$$

As we said in §1.2.1, the Polyakov action presents various symmetries, and in particular it is invariant under diffeomorphisms and Weyl transformations. This means that there is an action over the configuration space $\mathscr{C} := Emb(\Sigma, \mathbb{R}^D) \times \mathcal{M}(\Sigma)$ where $Emb(\Sigma, \mathbb{R}^D)$ is the space of embeddings from the worldsheet to the spacetime and $\mathcal{M}(\Sigma)$ is the usual space of metrics on the worldsheet:

$$\begin{aligned} A : (Diff^+(\Sigma) \times C^\infty_+(\Sigma)) \times \mathscr{C} &\to \mathscr{C} \\ ((\phi, f), (X, g)) &\mapsto [x \mapsto (X(\phi(x)), f(x)g(\phi(x)))] \end{aligned} \tag{2.2.39}$$

In the functional integral, the integration over the embeddings can be resolved, since the fields X^μ enters quadratically and it can be used a Gaussian integration to take out a suitable infinite dimensional determinant. The measure on $\mathcal{M}(\Sigma)$, instead, given the symmetries of the Polyakov action, should be thought of as:

$$\mathscr{D}g = \frac{\mathscr{D}\mathcal{M}(\Sigma)}{\mathrm{vol}(Diff^+(\Sigma))\mathrm{vol}(C^\infty_+(\Sigma))} \tag{2.2.40}$$

This is because if we do not quotient out the volumes of the orbits of the invariance groups, we will always have a divergent functional integral since in the integration we are over counting the effective elements by taking into consideration all their orbits. This measure $\mathscr{D}g$ will then be defined on a suitable quotient space, which is nothing but the moduli space (2.2.5) defined previously. On the space of metrics there is a natural metric:

$$\langle \cdot, \cdot \rangle_{met} : T_g \mathcal{M}(\Sigma) \times T_g \mathcal{M}(\Sigma) \to \mathbb{R} \tag{2.2.41}$$

defined by:

$$\langle X, Y \rangle_{met} := \int_{\Sigma} \langle X^I, Y^I \rangle_{g(x)} \sqrt{g}\, dx \tag{2.2.42}$$

where in a local patch $(U, g^1, ..)$ of $\mathcal{M}(\Sigma)$ the tangent vectors are expanded as $X = X^J \frac{\partial}{\partial g^I}$ and the inner product $\langle \cdot, \cdot \rangle_p$ on Σ is defined as: $\langle X^I, Y^I \rangle_{g(x)} := g^{mn}(x) g^{pq}(x) X^I_{mp}(x) Y^I_{nq}(x)$.

In this way $(\mathcal{M}(\Sigma), \langle \cdot, \cdot \rangle)$ can now be regarded as an infinite dimensional Riemannian manifold.

The moduli space (2.2.5) is defined using the $\mathcal{M}_{const}(\Sigma)$ space, namely considering the subspace of $\mathcal{M}(\Sigma)$ consisting of constant curvature metrics, which can be regarded as the conformal space of metrics valid for all genera p. If we restrict the metric $\langle \cdot, \cdot \rangle_{met}$ to the space $\mathcal{M}_{const}(\Sigma)$ and denote it by (we are indicating g_{const} the metrics in $\mathcal{M}_{const}(\Sigma)$ and the meaning of the subscript is that they are of constant curvature, not that they are themselves constant):

$$\langle \cdot, \cdot \rangle_{const} : T_{g_{const}} \mathcal{M}_{const}(\Sigma) \times T_{g_{const}} \mathcal{M}_{const}(\Sigma) \to \mathbb{R} \tag{2.2.43}$$

then the pair $(\mathcal{M}_{const}(\Sigma), \langle \cdot, \cdot \rangle_{const})$ will be a Riemannian submanifold of $(\mathcal{M}(\Sigma), \langle \cdot, \cdot \rangle)$. Connected to identity diffeomorphisms $(Diff_0(\Sigma))$ acts on $\mathcal{M}_{const}(\Sigma)$ as *Isometries*, in fact, given the definitions of $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{const}$, it can be readily verified that $\forall \alpha \in Diff_0(\Sigma)$ and $\forall X, Y \in \mathcal{M}_{const}(\Sigma)$ it is true that $\langle \alpha_* X, \alpha_* Y \rangle_{const} = \langle X, Y \rangle_{const}$.

On the Teichmüller space $\mathcal{T}_p = \mathcal{M}_{const}(\Sigma)/Diff_0(\Sigma)$ there will be an induced metric from $\langle \cdot, \cdot \rangle_{const}$ and it is the *Weil-Petersson metric* and it's written as $\langle \cdot, \cdot \rangle_{WP}$. Given the projection π from $\mathcal{M}_{const}(\Sigma)$ to its quotient space $\mathcal{T}_p$, the Weil-Petersson metric is defined as follows, $\forall U, V \in T_{[g_{const}]} \mathcal{T}_p$ and $X, Y \in \mathcal{M}_{const}(\Sigma)$ such that $\pi_* X = U$ and $\pi_* Y = V$:

$$\langle U, V, \rangle_{WP} = \langle X, Y \rangle_{const} \tag{2.2.44}$$

and because for $X', Y' \in T_{g'_{const}} \mathcal{T}_p$, satisfying $\pi_* X' = U$ and $\pi_* Y' = V$, with $g'_{const} = \alpha_* g_{const}$ for some $\alpha \in Diff_0(\Sigma)$ (which means $[g_{const}] = [g'_{const}]$), we have that $X' = \alpha_* X$ and $Y' = \alpha_* Y$ and because diffeomorphisms are isometries, then $\langle X', Y' \rangle_{const} = \langle X, Y \rangle_{const}$ which tells us that the Weil-Petersson metric is well-defined since it is independent of the representative g_{const} chosen in its class $[g_{const}]$.

The tangent space $T_g \mathcal{M}(\Sigma)$ can be regarded as the space of small perturbations δg of the metric $g \in \mathcal{M}(\Sigma)$. A combined action of diffeomorphisms $Diff(\Sigma)$ and Weyl transformations $C^\infty(\Sigma)$ leaves the Polyakov action invariant and the resultant variation of the metric under such a transformation is of the form:

$$\begin{aligned} \delta g_{mn} &= \nabla_m \epsilon_n + \nabla_n \epsilon_m + 2\rho g_{mn} \\ &= 2\tilde{\rho} g_{mn} + (P\epsilon)_{mn} \end{aligned} \tag{2.2.45}$$

where ϵ is the infinitesimal parameter associated to the action of the $Diff(\Sigma)$ group, ρ is the parameter associated to a Weyl transformation, $\tilde{\rho} = \rho + 1/2\, \nabla^p \epsilon_p$ and P is an operator associating to each vector a symmetric traceless 2-tensor of the form:

$$(P\epsilon)_{mn} = \nabla_m \epsilon_n + \nabla_n \epsilon_m - (\nabla^p \epsilon_p) g_{mn} \tag{2.2.46}$$

When we try to construct the Teichmüller space, the action of $Diff^{+}(\Sigma)$ and $C^{\infty}_{+}(\Sigma)$ must be projected out, and as we can see from the expression above, the metric transformations which cannot be reversed by $Diff(\Sigma)\times C^{\infty}(\Sigma)$ should be of the form $(\mathrm{Im}(P))^{\perp}$ (perpendicular with respect to the metric (2.2.41)). The set of small perturbations, namely the tangent space $T_g\mathscr{M}(\Sigma)$ can then be decomposed into:

$$T_g\mathscr{M}(\Sigma)=\{\rho\, g_{mn}\}\oplus \mathrm{Im}(P)\oplus(\mathrm{Im}(P))^{\perp} \tag{2.2.47}$$

but $(\mathrm{Im}(P))^{\perp}=\ker(P^{\dagger})$ so that:

$$T_g\mathscr{M}(\Sigma)=\{\rho\, g_{mn}\}\oplus \mathrm{Im}(P)\oplus\ker(P^{\dagger}) \tag{2.2.48}$$

Elements of $\ker(P^{\dagger})$ are indeed moduli deformations as it can be read from the variation of the curvature with respect to a variation of the metric [DP88]:

$$\delta R=-\frac{1}{2}g^{mn}\delta g_{mn}R-\frac{1}{2}\nabla^{p}\nabla_{p}(g^{mn}\delta g_{mn})+\frac{1}{2}\nabla^{m}\nabla^{n}(\delta g_{mn}) \tag{2.2.49}$$

Deformations in $\ker(P^{\dagger})$ does not change the curvature and hence are tangent to $\mathscr{M}_{const}(\Sigma)$ and in particular define the tangent space of the moduli space:

$$T_{[g]}\mathscr{M}_p=\ker(P^{\dagger}) \tag{2.2.50}$$

Using these facts, the measure on the space of metrics can be decomposed according to:

$$\mathscr{D}\mathscr{M}(\Sigma)=N\,\mathscr{D}\rho\mathscr{D}(P\epsilon)d\mathscr{T}_p \tag{2.2.51}$$

where N is a normalising constant depending on the frames used.

For $p\geq 2$ the measure $\mathscr{D}(P\epsilon)$ can be written as:

$$\sqrt{\det(P^{\dagger}P)}\mathscr{D}\epsilon \tag{2.2.52}$$

in this way, the string partition function (2.2.37) can be recast as an integral over the Teichmüller space and then over the moduli space [Nas91]:

$$\begin{aligned}\mathscr{Z}_p&=\frac{1}{|\Gamma_\Sigma|}\int d\mu(\mathscr{T}_p)\left(\frac{\det(\triangle_g/2)}{\langle 1,1\rangle_g}\right)^{-13}\sqrt{\det(P^{\dagger}P)}\\&=\int d\mu(\mathscr{M}_p)\left(\frac{\det(\triangle_g/2)}{\langle 1,1\rangle_g}\right)^{-13}\sqrt{\det(P^{\dagger}P)}\end{aligned} \tag{2.2.53}$$

where the determinant factor comes from the integration[14] of the Polyakov action with respect to the embeddings X^{μ}, the power -13 comes from the dimension $D=26$ in which the bosonic string lives and $|\Gamma_\Sigma|$ is the cardinality of the mapping class group (2.2.35).

[14]The Polyakov action on a Riemann surface worldsheet, can be written using complex coordinates on the worldsheet as:

$$S_P=-\frac{T}{2}\int_\Sigma dzd\bar{z}\,\sqrt{g}\,g^{\alpha\beta}\,\partial_\alpha X^m\partial_\beta X_m=-\frac{T}{2}\int_\Sigma\langle\partial X,\partial X\rangle_g=\frac{T}{2}\int_\Sigma\langle X,\triangle_g X\rangle_g \tag{2.2.54}$$

2.2.6 Calabi-Yau Moduli Space

We have seen how the classification of Riemann surfaces is not a trivial task and the concepts of Teichmüller and Moduli spaces have to be introduced. These spaces allows to study the complex structure of complex manifolds in a systematic way by analysing the space of conformal metrics, in particular by defining a suitable equivalence relation which enables to extrapolate this "complex behaviour" out of our given complex manifold.

String theory phenomenology requires compactifications of the 10-dimensional spacetime $\mathscr{M}$ into a Calabi-Yau manifold $(\mathscr{Y}, g)$, this vacuum allows for a reduction of supersymmetries in the 4-dimensional effective supergravity theory on the non-compact Minkowski space $\mathbb{R}^{1,3}$. However, parameters like the size and shape of the Calabi-Yau manifold appears as massless scalar fields in the 4-dimensional supergravity action. Their appearance is dictated by the fact that there is a whole continuous degeneracy of possible backgrounds $(\mathscr{Y}', g')$ which differs from $(\mathscr{Y}, g)$ in shape and complex structure but which leave the condition $\mathscr{R}[g] = \mathscr{R}[g'] = 0$ unchanged. These fields are called *Moduli Fields* and in order to avoid long-ranged unobserved forces in the effective field theory, it should be employed a mechanism to generate a potential and let them acquire a non-zero mass.

We have said that moduli fields arise as deformations of a Calabi-Yau manifold leaving unchanged its structure and in fact they can also be interpreted as transformations of background fields mapping vacuum configurations in our theory to other vacuum configurations.

If we consider a Calabi-Yau manifold $\mathscr{Y}$ - i.e. a Kähler manifold with Kähler metric given on a local patch by $\sum g_{i\bar{j}} dx^i \otimes d\bar{x}^j$ (and Kähler form $J = \sqrt{-1} g_{i\bar{j}} dx^i \wedge d\bar{x}^j$) and such that the Ricci (1,1)-form $\mathscr{R}[g] = \sqrt{-1}/2\pi R_{i\bar{j}}[g] dx^i \wedge d\bar{x}^j$ (which is equals to the first Chern class of $\mathscr{Y}$) vanishes - and consider a small variation of the metric such that it does not spoil the Ricci-flatness:

$$\mathscr{R}[g] = \mathscr{R}[g + \delta g] = 0 \tag{2.2.57}$$

then this condition leads to the *Lichnerowicz equation* for the variation of the metric once the gauge condition[15] $\nabla^m \delta g_{mn} = 1/2 \nabla_n \delta g^m_m$ is imposed on (2.2.57), obtaining:

$$\nabla^k \nabla_k \delta g_{mn} + 2 R_{m\ n}^{\ i\ j} \delta g_{ij} = 0 \tag{2.2.58}$$

Due to the properties of a Calabi-Yau manifold, the equations for the mixed type $\delta g_{i\bar{j}}$ and pure type $\delta g_{ij}, \delta g_{\bar{i}\bar{j}}$ decouple. The Lichnerowicz equation with metric variation of the mixed type is equivalent to say that $\sqrt{-1} \delta g_{i\bar{j}} dx^i \wedge d\bar{x}^j$ is a harmonic form (lies in $\mathscr{H}^{1,1}(\mathscr{Y})$) while with

with $\alpha, \beta = z, \bar{z}$ and $\triangle_g = \partial\partial^* + \partial^*\partial$. In the partition function (2.2.37) there will be a term like:

$$\int_{Emb(\Sigma, \mathbb{R}^D)} \mathscr{D}X \exp\left(-\frac{T}{2} \int \langle X, \triangle_g X \rangle_g\right) \tag{2.2.55}$$

Now, this term can be integrated to give an infinite dimensional determinant, using, for example, zeta function regularisation. The result will be:

$$\int_{Emb(\Sigma, \mathbb{R}^D)} \mathscr{D}X \exp\left(-\frac{T}{2} \int \langle X, \triangle_g X \rangle_g\right) = \left(\frac{\det(\triangle_g/2)}{\langle 1, 1 \rangle_g}\right)^{D/2} \tag{2.2.56}$$

where $\langle 1, 1 \rangle_g$ is the volume of the worldsheet Σ with respect to the metric g.

[15]It eliminates coordinate changes which are not of interest.

the metric variation of the pure type it is equivalent to say that $\Omega_{ij}{}^{\bar{k}}\delta g_{\bar{k}\bar{l}}dx^i \wedge dx^j \wedge d\bar{x}^l$ is a harmonic form (lies in $\mathcal{H}^{2,1}(\mathcal{Y})$), where $\Omega = \Omega_{ijk}dx^i \wedge dx^j \wedge dx^k$ is the holomorphic (3,0)-form of the Calabi-Yau (2.2.1). The first case can be seen as deformations of the Kähler structure since the Kähler form is nothing but $J = \sqrt{-1}g_{i\bar{j}}dx^i \wedge d\bar{x}^j$. The second case can be regarded instead as a modification of the complex structure and this can be seen by noticing that due to the fact that on a Kähler manifold components of the pure type g_{ij} are identically zero, in order to remove this variation, a non-holomorphic coordinate change has to be performed (since holomorphic coordinate transformations do not change the index structure). This means that we have to study separately *Kähler structure deformations* and *Complex structure deformations*.

As we have seen previously in §2.2.5, the moduli space is given by the space of metrics of constant curvature quotientized with the group of diffeomorphisms. On the space of metrics it can be defined a metric (2.2.41) which we called $\langle\cdot,\cdot\rangle_g$, in particular over the constant curvature metrics we had $\langle\cdot,\cdot\rangle_{const}$, i.e. (2.2.43). We saw that this latter metric allows to define a metric in the Teichmüller space (and hence in the moduli space) called the Weil-Petersson metric (2.2.44). Moreover, the small metric deformations which we would like to inspect belong to the tangent space of the Teichmüller/moduli space and a line element in these spaces will then be given by following Candelas, Green, and Hübsch [CGH89][CGH90] and Candelas and Ossa [CO91]:

$$d\mathfrak{s}^2 = \frac{1}{4\mathcal{V}}\int_{\mathcal{Y}} d^6x\,\sqrt{g}\left(\delta g_{ij}\delta g_{kl} + \delta B_{ij}\delta B_{kl}\right)g^{ik}g^{jl} \tag{2.2.59}$$

where $\mathcal{V}$ is the volume of the Calabi-Yau.
The first to introduce such a metric was DeWitt [DeW67] and analogous formulations can be recovered using our, previously defined, Weyl-Petersson metric or the Zamolodchikov metric (in [CHS90] the authors indeed proved that all these metrics are equivalent in this case). The appearance of the 2-form B is due to the fact that every SUGRA low-energy effective action of the five different string theories contains a NS-NS 2-form bosonic field B_2. This field get split in the compactification leaving a background 2-form $B = B_{i\bar{j}}dx^a \wedge d\bar{x}^b$ on the internal manifold $\mathcal{Y}$ with $i, j = 1, ..., h^{(1,1)}$ which is related to the metric by supersymmetry transformations. In turn this means that if we want the most general metric on the moduli space $\mathcal{M}$ we must take also the variation δB into consideration.
The line element (2.2.59) can be recast in a more suggestive form using the fact that $g_{ij} = g_{\bar{i}\bar{j}} = 0$ for a Calabi-Yau, obtaining:

$$d\mathfrak{s}^2 = \frac{1}{4\mathcal{V}}\int_{\mathcal{Y}} d^6x\,\sqrt{g}\left[\delta g_{ij}\delta g_{\bar{k}\bar{l}} + \left(\delta g_{j\bar{k}}\delta g_{i\bar{l}} + \delta B_{j\bar{k}}\delta B_{\bar{l}i}\right)\right]g^{i\bar{k}}g^{j\bar{l}} \tag{2.2.60}$$

What can be noted is that the metric is block diagonal allowing us to factorise the moduli space into two blocks (as it was hinted before):

$$\mathcal{M} = \mathcal{M}^{\mathrm{cs}} \times \mathcal{M}^{\mathrm{ks}}$$

allowing us to study separately the Riemannian manifold $(\mathcal{M}^{\mathrm{ks}}, G^{\mathrm{ks}})$ of the Kähler structure deformations and $(\mathcal{M}^{\mathrm{cs}}, G^{\mathrm{cs}})$ of the complex structure deformations. Actually the space $\mathcal{M}^{\mathrm{ks}}$ is not a moduli space since technically speaking the moduli space is constructed considering conformal classes of metrics and the only truly moduli space should be $\mathcal{M}^{\mathrm{cs}}$. However in the physics literature the term "moduli space" turns out to take an enlarged meaning, incorporating also the space of Kähler deformations.

Kähler Structure Deformations

By considering a Kähler structure deformation $g_{i\bar{j}} + \delta g_{i\bar{j}}$ the resulting metric should be positive definite and this can be seen as a set of constraints on the Kähler form $J = \sqrt{-1} g_{i\bar{j}} dx^i \wedge d\bar{x}^j$ as follows:

$$\int_{M_r} \underbrace{J \wedge \cdots \wedge J}_{r\text{-times}} > 0 \tag{2.2.61}$$

where M_r is any complex r-dimensional submanifold of the Calabi-Yau $\mathscr{Y}$. These set of constraints define the so-called *Kähler cone* which is the subset of metric transformations leading to a positive definite metric.
We want now to recast the metric (2.2.60) in terms of Cohomology classes (which are equivalent to harmonic forms thanks isomorphisms between Dolbeault cohomology and harmonic forms (2.1.2)) in order show that $(\mathscr{M}^{\text{ks}}, G^{\text{ks}})$ is a Kähler manifold. The inner product on $H^{1,1}_{\bar{\partial}}(\mathscr{Y})$ defined by (2.2.60) is then:

$$G^{\text{ks}}(\rho, \sigma) = \frac{1}{2\mathscr{V}} \int_{\mathscr{Y}} d^6 x \sqrt{g}\, \rho_{j\bar{k}} \sigma_{i\bar{l}} g^{i\bar{k}} g^{j\bar{l}} \equiv \frac{1}{2\mathscr{V}} \int_{\mathscr{Y}} \rho \wedge \star\sigma \tag{2.2.62}$$

where $\rho, \sigma \in H^{1,1}_{\bar{\partial}}(\mathscr{Y})$ are real (1,1)-forms. If we define the cubic form:

$$k(\rho, \sigma, \zeta) = \int \rho \wedge \sigma \wedge \zeta$$

as was shown by Strominger [Str85], we can recast the inner product as:

$$G^{\text{ks}}(\rho, \sigma) = -3 \left[\frac{k(\rho, \sigma, J)}{k(J, J, J)} - \frac{3}{2} \frac{k(\rho, J, J) k(\sigma, J, J)}{k^2(J, J, J)} \right]$$

where we recall that by its definition $k(J, J, J) = 3!\mathscr{V}$.
Let now $\{\hat{D}_i\}_{i=1,..,h^{(1,1)}}$ be a basis of $H^{1,1}_{\bar{\partial}}(\mathscr{Y})$ and consider the *Complexified Kähler cone* $\mathscr{J} = J + iB_2$ expanded in this basis:

$$\mathscr{J} = v^i \hat{D}_i \tag{2.2.63}$$

with $v^i = t^i + i b^i$ parametrising the complexified Kähler cone. It is easy to see that the metric is generated by a Kähler potential of the form:

$$G^{\text{ks}}_{i\bar{j}} = \frac{1}{2} G^{\text{ks}}\left(\hat{D}_i, \hat{D}_j\right) = -\frac{\partial}{\partial v^i} \frac{\partial}{\partial v^{\bar{j}}} \ln(k(J, J, J)) = -6 \frac{\partial}{\partial v^i} \frac{\partial}{\partial v^{\bar{j}}} \ln(\mathscr{V}) = \frac{\partial}{\partial v^i} \frac{\partial}{\partial v^{\bar{j}}} K^{\text{ks}} \tag{2.2.64}$$

meaning that the scalars v^i span the moduli space of Kähler deformations and $\mathscr{M}^{\text{ks}}$ is a Kähler manifold. Moreover, if we define the *Intersection numbers* as:

$$k_{ijk} := k\left(\hat{D}_i, \hat{D}_j, \hat{D}_k\right) = \int_{\mathscr{Y}} \hat{D}_i \wedge \hat{D}_j \wedge \hat{D}_k \tag{2.2.65}$$

then we can see that the Kähler potential is determined by a holomorphic prepotential F:

$$K^{\text{ks}} = -\ln(k(J, J, J)) = -\ln(k_{ijk} t^i t^j t^k) = -\ln(F(t))$$

which makes $\mathscr{M}^{\text{ks}}$ a *Special Kähler manifold*. The parameters v^i are indeed moduli fields of the Kähler structure (note that $v^i : \mathscr{Y} \to \mathbb{C}$).

Complex Structure Deformations

As it has already been said, the Lichnerowicz equation (2.2.58) corresponds to the fact that the (2,1)-form $\Omega_{ij}{}^{\bar{k}}\delta g_{\bar{k}\bar{l}} dx^i \wedge dx^j \wedge d\bar{x}^l$ is a harmonic form, i.e. it belongs to $\mathscr{H}^{2,1}(\mathscr{Y})$. We can set [CGH89; CGH90; CO91] (following Tian [Tia87]):

$$(\chi_\alpha)_{jk\bar{l}} := -\frac{1}{2}\Omega_{jk}{}^{\bar{m}}\frac{\delta g_{\bar{m}\bar{l}}}{U^\alpha} \qquad \chi_\alpha = \frac{1}{2}(\chi_\alpha)_{jk\bar{l}}\, dx^j \wedge dx^k \wedge d\bar{x}^l$$

with $\alpha = 1, ..., h^{(2,1)}$ and in this way we have that the complex structure deformations of the metric will take the following form:

$$\delta g_{ab} = \frac{i}{||\Omega||^2}(\overline{\chi}_\alpha)_{a\bar{b}\bar{c}}\Omega^{\bar{b}\bar{c}}{}_b \overline{U}^\alpha \tag{2.2.66}$$

where we recall (2.2.11) for $||\Omega||^2$ and $\{[\chi_\alpha]\}_{\alpha=1,...,h^{(2,1)}}$ is a basis for $H^{2,1}_{\bar{\partial}}(\mathscr{Y})$, while the U^α are local coordinates spanning the complex structure moduli space $\mathscr{M}^{\text{cs}}$. $\{U^\alpha\}_{\alpha=1,...,h^{(2,1)}}$ are the *Complex Structure Moduli Fields* and $\mathscr{M}^{\text{cs}}$ is a Kähler manifold as can be seen by the fact that its metric is given by [CO91; Gri05]:

$$\begin{aligned} G^{\text{cs}}_{\alpha\bar{\beta}} &= -\frac{\int_{\mathscr{Y}}\chi_\alpha \wedge \overline{\chi}_\beta}{\int_{\mathscr{Y}}\Omega\wedge\overline{\Omega}} \\ &= \frac{\partial^2 K^{\text{cs}}}{\partial U^\alpha \partial \overline{U}^\beta} \\ &= \frac{\partial^2}{\partial U^\alpha \partial \overline{U}^\beta}\left[-\ln\left(-i\int_{\mathscr{Y}}\Omega\wedge\overline{\Omega}\right)\right] \end{aligned} \tag{2.2.67}$$

with K^{cs} the Kähler potential (2.1.5). Recalling the decomposition (2.1.38) coming from the Hodge theorem (2.1.3), we have that $H^3(\mathscr{Y}) = \mathscr{H}^{3,0}(\mathscr{Y}) \oplus \mathscr{H}^{2,1}(\mathscr{Y}) \oplus \mathscr{H}^{1,2}(\mathscr{Y}) \oplus \mathscr{H}^{0,3}(\mathscr{Y})$. Moreover, from §2.2.1, the hodge numbers $h^{(3,0)} = h^{(0,3)}$ (and hence the dimensions of the harmonic/cohomology groups) are known to be exactly 1, which means that the dimension of $H^3(\mathscr{Y})$ is equals to $2h^{(2,1)} + 2$. On this space it is possible to define a real symplectic basis $\{(\alpha_I, \beta^J)\}_{I,J=1,...,h^{(2,1)}+1}$ and its Poincaré dual basis on the Homology $H_3(\mathscr{Y})$ as $\{(A^I, B_J)\}_{I,J=1,...,h^{(2,1)}+1}$ and these will indeed satisfy (using (2.1.101)):

$$\int_{A^J}\alpha_I = \int_{\mathscr{Y}}\alpha_I \wedge \beta^J = \delta^J_I \qquad \int_{B_I}\beta^J = \int_{\mathscr{Y}}\beta^J \wedge \alpha_I = -\delta^J_I \tag{2.2.68}$$

Using this basis, we can construct the *Periods* of the holomorphic $(3,0)$-form Ω as follows:

$$Z^I(U^\alpha) = \int_{A^I}\Omega(U^\alpha) = \int_{\mathscr{Y}}\Omega(U^\alpha)\wedge\beta^I \tag{2.2.69}$$

$$F^I(U^\alpha) = \int_{B_I}\Omega(U^\alpha) = \int_{\mathscr{Y}}\Omega(U^\alpha)\wedge\alpha_I \tag{2.2.70}$$

It was shown by Bryant and Griffiths [BG83] that the complete Hodge structure $\{H^{p,q}(\mathscr{Y})\}$ is fully determined by the $Z^I(U^\alpha)$ (locally), namely the F^I can be regarded as functions of Z^I.

Now, the holomorphic $(3,0)$-form can be expanded in the symplectic basis, and its expansion coefficients will be exactly the periods defined above:

$$\Omega = Z^I(U^\alpha)\alpha_I - F_I(Z^J(U^\alpha))\beta^I \tag{2.2.71}$$

Using this form for Ω, the Kähler potential defined in (2.2.67) can be recast as:

$$K^{\mathrm{cs}} = -\ln\left[i\left(\overline{Z}^I F_I - Z^I \overline{F}_I\right)\right] \tag{2.2.72}$$

Because the periods F^I can be written as the Z^I-derivative of a holomorphic prepotential $\mathscr{F} = Z^I F_I$, the Kähler potential is completely determined by $\mathscr{F}$ in terms of the new coordinates Z^I:

$$K^{\mathrm{cs}} = -\ln\left[i\left(\overline{Z}^I \frac{\partial \mathscr{F}}{\partial Z^I} - Z^I \frac{\partial \overline{\mathscr{F}}}{\partial \overline{Z}^I}\right)\right] \tag{2.2.73}$$

In turn this means that $\mathscr{M}^{\mathrm{cs}}$ is a *Special Kähler manifold* like $\mathscr{M}^{\mathrm{ks}}$.

As we already noted in §2.1.2, the Kähler potential is not uniquely defined, in particular, in this case if we rescale the holomorphic $(3,0)$-form with respect to a holomorphic function $e^{-h(U)}$ as:

$$\Omega \to \Omega e^{-h(U)} \tag{2.2.74}$$

then the Kähler potential will undergo the following Kähler transformation:

$$K^{\mathrm{cs}} \to K^{\mathrm{cs}} + h + \overline{h} \tag{2.2.75}$$

This sort of symmetry makes one of the periods unphysical, since we can always put $Z^0 = 1$. The result is that the complex structure deformations can be identified with the remaining $h^{(2,1)}$ periods by defining affine coordinates $u^\alpha = Z^\alpha / Z^0$, in terms of which the Kähler potential becomes [CO91]:

$$K^{\mathrm{cs}} = -\ln\left[i\left(2(\mathscr{F} - \overline{\mathscr{F}}) - (u^\alpha - \overline{u}^\alpha)\left(\frac{\partial \mathscr{F}}{\partial u^\alpha} + \frac{\partial \overline{\mathscr{F}}}{\partial \overline{u}^\alpha}\right)\right)\right] \tag{2.2.76}$$

In the 4-dimensional low-energy supergravity effective theory, the parameters U^α spanning the complex structure moduli space and the parameters t^i spanning the Kähler moduli space will become moduli fields when the graviton will be expanded in the Kaluza-Klein reduction process. These, together with other fields arising in the expansion of the 10-dimensional gauge-potentials, will give rise to $\mathscr{N} = 2$ multiplets for our type IIB supergravity theory.

2.2.7 Mirror Symmetry

Among the dualities pervading string theory, mirror symmetry is maybe one of the most interesting but also enigmatic duality that relates different Calabi-Yau manifolds. The mathematical results that have been proved thanks to mirror symmetry were long standing problems in mathematics, like the counting of rational curves on a generic quintic [Can+91; MT12]. The work made by Candelas, Lynker, and Schimmrigk [CLS90] in analysing the Euler characteristics of different Calabi-Yau manifolds, in particular taking into consideration manifolds arising as vanishing polynomials in Weighted Projective Spaces (see appendix B), led to the discovery of an approximate symmetry under $\chi_E \to -\chi_E$, which can be appreciated in the figure (2.2). More

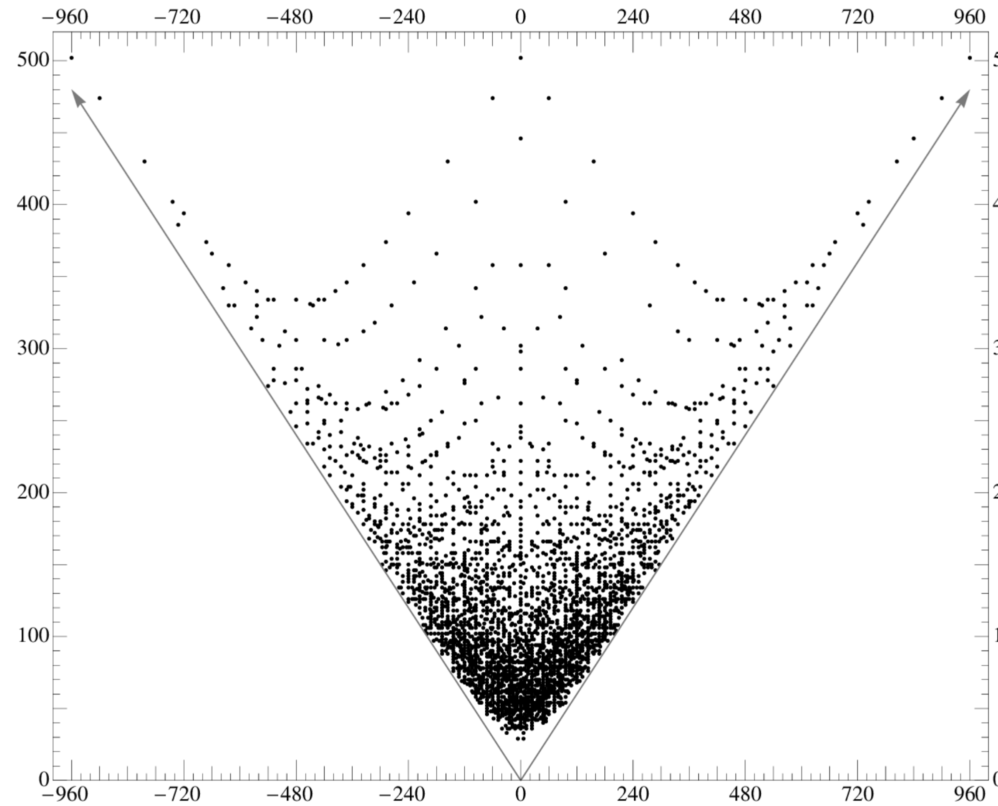


Figure 2.2: Plot of the sum of the hodge numbers $h^{(1,1)} + h^{(2,1)}$ versus the Euler Characteristics $\chi_E = 2(h^{(1,1)} - h^{(2,1)})$ for different Calabi-Yau manifolds, making manifest the approximate symmetry between hodge numbers.

precisely, the mirror map associates with almost any Calabi-Yau three-fold $\mathscr{Y}$ another Calabi-Yau three-fold $\hat{\mathscr{Y}}$ such that:

$$H^{p,q}_{\bar{\partial}}(\mathscr{Y}) = H^{3-p,q}_{\bar{\partial}}(\hat{\mathscr{Y}}) \tag{2.2.77}$$

and in particular this relates the hodge numbers in the following way:

$$h^{(1,1)}(\mathscr{Y}) = h^{(2,1)}(\hat{\mathscr{Y}}); \qquad h^{(2,1)}(\mathscr{Y}) = h^{(1,1)}(\hat{\mathscr{Y}}) \tag{2.2.78}$$

namely

$$(h^{(1,1)}, h^{(2,1)}) \xrightarrow{\text{Mirror Map}} (h^{(2,1)}, h^{(1,1)}) \tag{2.2.79}$$

These observations led to the conjecture that type IIA superstring theory compactified on a Calabi-Yau three-fold $\mathscr{Y}$ should be the same as type IIB string theory compactified on the mirror manifold $\hat{\mathscr{Y}}$. Since moduli spaces are generated by elements of the above cohomologies, mirror symmetry exchanges the complex structure deformations and Kähler structure deformations moduli spaces. How then can we make use of this symmetry? What has to be noted is that the Yukawa couplings of the low-energy effective field theory (once the string theory has been compactified on the Calabi-Yau) depends independently on the complex structure deformations and on the Kähler structure deformations. The former vary with the parameters, while the latter are topological numbers, and since both $\mathscr{Y}$ and its mirror $\hat{\mathscr{Y}}$ corresponds to the same superconformal field theory and the role of the two moduli spaces are exchanged, one can combine the calculations to obtain exact results. In their paper, Greene and Plesser [GP90] managed to construct and study the mirror of the quintic (see §3.4) by constructing a quotient of the quintic by a suitable group and then resolving the singularities to obtain a Calabi-Yau manifold (this can be appreciated in [Can+91; Can+94]), they also managed to establish that mirror pairs when used as a basis for Calabi-Yau σ-models give the same string theory.

2.3 Type IIB String Theory on Calabi-Yau

We are going to consider only compactifications of type IIB string theory, this is because our models will be in this context and in string phenomenology type IIB string theory is the most used string framework to construct string vacua reproducing the Standard Model and eventually Beyond Standard Model physics. In §1.4.2 we have seen the low-energy type IIB supergravity action in 10-spacetime dimensions retrieved by considering the point-particle limit of the superstring theory (namely taking into consideration only massless modes) and by GSO-projecting onto a chiral spectrum. The resulting theory was a $D = 10$ and $\mathcal{N} = 2$ supergravity with massless spectrum divided into 4 sectors: two bosonic (NS-NS and R-R) and two fermionic (NS-R and R-NS). What we would like to achieve now is a dimensional reduction by compactify 6 dimensions into a Calabi-Yau manifold. For type II theories, this kind of compactification, leaves a $D = 4$ and $\mathcal{N} = 2$ supergravity theory where the zero modes of the Calabi-Yau have to assemble into massless multiplets. Now, in order for our theory to reproduce a chiral Standard Model, the supersymmetries have to be reduced to $\mathcal{N} = 1$, since $\mathcal{N} > 1$ cannot reproduce a chiral spectrum. The way this reduction is done is by making an orientifold projection, in this manner half of the supersymmetries are projected out leaving a $\mathcal{N} = 1$ supergravity theory in 4 spacetime dimensions. Let's analyse first the Kaluza-Klein dimensional reduction when compactifying the theory onto a Calabi-Yau and then we are going to how to orientifold projects the theory.

2.3.1 4-Dimensional $\mathcal{N} = 2$ Type IIB Supergravity

Let's compactify the type IIB string theory on a Calabi-Yau manifold $\mathscr{Y}$, in this way the 10-dimensional spacetime will be factorised according to:

$$\mathscr{M} = \mathbb{R}^{1,3} \times \mathscr{Y} \tag{2.3.1}$$

where we are going to denote the coordinates by $\mathscr{M} \ni x^M = (x^\mu, (y^m, \bar{y}^m)) \in \mathbb{R}^{1,3} \times \mathscr{Y}$, namely $M = 0, ..., 9$, $\mu = 0, 1, 2, 3$ and $m = 1, 2, 3$.

The metric on $\mathscr{M}$ is supposed to be block-diagonal by making the ansatz (in local coordinates):

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{m\bar{n}} dy^m d\bar{y}^n \tag{2.3.2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric on the Minkowski space $\mathbb{R}^{1,3}$ and $g_{m\bar{n}}$ is the metric on the Calabi-Yau, namely a Kähler metric with a Kähler form given by $J = \sqrt{-1} g_{m\bar{n}} dy^m \wedge d\bar{y}^n$. The reduction of the 10-dimensional metric then gives a 4-dimensional graviton $g_{\mu\nu}$, a scalar V_0, $h^{(1,1)}$ Kähler structure moduli t^i and $h^{(2,1)}$ complex structure moduli U^α. The fields t^i were introduced in the expansion of the Kähler deformations of the metric $\delta g_{m\bar{n}} = -\sqrt{-1}\, t^i (\hat{D}_i)_{m\bar{n}} = -\sqrt{-1} J_{m\bar{n}}$ (since both $\sqrt{-1}\, \delta g_{m\bar{n}} dy^m \wedge d\bar{y}^n$ and J belong to the same cohomology group $H^{1,1}_{\bar{\partial}}(\mathscr{Y}) \simeq \mathscr{H}^{1,1}(\mathscr{Y})$), while the fields U^α were introduced to parametrise deformations δg_{mn} of the complex structure which can be expanded in a basis of the cohomology group $H^{2,1}_{\bar{\partial}}(\mathscr{Y}) \simeq \mathscr{H}^{2,1}(\mathscr{Y})$.

The other bosonic fields in type IIB are also expanded in terms of cohomology classes of the Calabi-Yau $\mathscr{Y}$ according to (the hats on the fields indicate the 10-dimensional ones):

- NS-NS sector:
$$\hat{\phi} = \phi(x) \qquad \hat{B} = B(x) + b^i(x)\hat{D}_i \tag{2.3.3}$$
with $i = 1, ..., h^{(1,1)}$.

- R-R sector:
$$\hat{C}_0 = C_0(x) \qquad \hat{C}_2 = C_2(x) + c^i(x)\hat{D}_i \tag{2.3.4}$$
$$\hat{C}_4 = Q_2^i(x) \wedge \hat{D}_i + V^I(x) \wedge \alpha_I - \tilde{V}_I(x) \wedge \beta^I + \rho_i(x)\tilde{D}^i \tag{2.3.5}$$

where we recall that $\{\hat{D}_i\}_{i=1,...,h^{(1,1)}}$ is a basis for the cohomology group $H^{1,1}_{\overline{\partial}}(\mathscr{Y})$ (or analogously a basis of harmonic $(1,1)$-form due to the isomorphism (2.1.2)), while $\{\tilde{D}^i\}_{i=1,...,h^{(1,1)}}$ is a basis of $H^{2,2}_{\overline{\partial}}(\mathscr{Y})$ (which is dual to $H^{1,1}_{\overline{\partial}}(\mathscr{Y})$ due to the Hodge star isomorphism (recall $h^{(p,q)} = h^{(n-p,n-q)}$)). Also $\{(\alpha_I, \beta^J)\}_{I,J=0,...,h^{(2,1)}}$ is the symplectic basis of $H^3(\mathscr{Y})$. The 4-dimensional fields $b^i(x)$, $C_0(x)$, $c^i(x)$ and $\rho_i(x)$ are scalars while $V^I(x)$ and $\tilde{V}_I(x)$ are 1-forms and $Q_2^i(x)$ are 2-forms. Now as we saw in §1.4.2, the $\tilde{F}_5$ is self-dual and this translates in the fact that half of the degrees of freedom of $\hat{C}_4$ can be eliminated. Conventionally we choose to keep ρ_i and V^I. All these fields can be reorganized in $\mathcal{N} = 2$ multiplets as follows (only the bosonic components):

- 1 gravity multiplet: $(g_{\mu\nu}, V_0)$;
- $h^{(2,1)}$ vector multiplets: (V^α, U^α);
- $h^{(1,1)}$ hypermultiplets: (t^i, b^i, c^i, ρ^i);
- 1 double-tensor multiplet: (B_2, C_2, ϕ, C_0).

The tree-level effective action can then be written in the standard $\mathcal{N} = 2$ supergravity form as [Cic10; GL04; Gri05]:

$$\begin{aligned} S^{4D}_{IIB} = -\int \Big(&\frac{1}{2}\mathscr{R} \star \mathbb{1} - \frac{1}{4}\mathfrak{Re}(M_{\alpha\beta})F^\alpha \wedge F^\beta - \frac{1}{4}\mathfrak{Im}(M_{\alpha\beta})F^\alpha \wedge \star F^\beta \\ &+ g_{\alpha\overline{\beta}}\, dU^\alpha \wedge \star d\overline{U}^\beta + h_{AB}\, dq^A \wedge \star dq^B \Big) \end{aligned} \tag{2.3.6}$$

The matrix $M_{\alpha\beta}$ is the gauge kinetic matrix and depends on the complex structure moduli U^α and can be found for example in [Gri05]. $F^\alpha = dV^\alpha$ and $g_{\alpha\overline{\beta}}$ is the metric on the moduli space of complex deformations which has been introduced in (2.2.67) with a slightly different notation. Finally we have defined the vector q^A as the collection of the scalar fields in the hypermultiplets and h_{AB} as the metric on the space of these $4(h^{(1,1)} + 1)$ moduli fields [Gri05].

2.3.2 Orientifolding and $\mathcal{N}=1$ Type IIB Supergravity

In order to break some supersymmetries and get a 4-dimensional $\mathcal{N}=1$ supergravity theory an orientifold projection is employed. This consists in gauging a discrete symmetry of the form:

$$\mathcal{O}=(-1)^{F_L}\Omega_p\sigma^* \tag{2.3.7}$$

where F_L is the left-moving fermion number, Ω_p is the worldsheet parity operator and $\sigma:\mathcal{Y}\to\mathcal{Y}$ is an isometric and holomorphic involution of the Calabi-Yau (with σ^* its pull-back acting on k-forms). Fixed points Σ of the involution σ will give rise to orientifold planes of topology $\mathbb{R}^{1,3}\times\Sigma$. Given our orientifold projection (2.3.7) these planes will only be $O3/O7$-planes, meaning that σ leaves invariant a zero- or two-complex dimensional submanifold of the Calabi-Yau $\mathcal{Y}$ (because the involution leaves the Minkowski space $\mathbb{R}^{1,3}$ invariant, the orientifold planes must be spacetime filling). There is another projection which can be employed, namely $\mathcal{O}'=\Omega_p\sigma^*$ and this will carry $O5/O9$-planes, however we will not get into it since we are going to use the first orientifold projection.

Since σ is a holomorphic isometry, it leaves unchanged the metric and the complex structure of the Calabi-Yau and this means that also the Kähler form J is left invariant due to the compatibility condition (see §2.1.2) with respect to metric:

$$\sigma^*J=J \tag{2.3.8}$$

Also, the fact that σ is holomorphic, implies that it respects the Hodge decomposition (2.1.3) which for Calabi-Yau manifolds is explicated by the Hodge diamond (2.2.12). In particular this means that $\sigma^*H^{3,0}_{\bar\partial}(\mathcal{Y})=H^{3,0}_{\bar\partial}(\mathcal{Y})$. With our definition of $\mathcal{O}$, the holomorphic $(3,0)$-form Ω of the Calabi-Yau (2.2.1) transforms according to:

$$\sigma^*\Omega=-\Omega \tag{2.3.9}$$

Since $\sigma^2=\mathbb{1}$, the other possibility would be $\sigma^*\Omega=\Omega$, which as we said, would bring into play $O5/O9$-planes and represents the other possible orientifold projection.

The orientifold planes naturally carry R-R charges since they couple to the R-R forms of the theory. This means that in order to cancel these charges one has to introduce also Dp-branes and in this case $D3/D7$-branes.

Since σ is a holomorphic involution, the Dolbeault cohomology groups $H^{p,q}_{\bar\partial}(\mathcal{Y})$ split into even and odd eigenspaces according to [Gri05; Cic10]:

$$H^{p,q}_{\bar\partial}(\mathcal{Y})=(H^{p,q}_{\bar\partial}(\mathcal{Y}))^+\oplus(H^{p,q}_{\bar\partial}(\mathcal{Y}))^- \tag{2.3.10}$$

and their dimensions are given by:

$$\dim(H^{p,q}_{\bar\partial}(\mathcal{Y})^+)=h_+^{(p,q)}\qquad\qquad \dim(H^{p,q}_{\bar\partial}(\mathcal{Y})^-)=h_-^{(p,q)} \tag{2.3.11}$$

The Hodge numbers are then constrained in the following way:

(i) Because σ preserves the orientation and the metric, the Hodge $\star$-operator commutes with σ^* so that $h_\pm^{(1,1)}=h_\pm^{(2,2)}$;

(ii) The holomorphicity of σ implies that $h_\pm^{(2,1)}=h_\pm^{(1,2)}$;

(iii) Due to (2.3.9) we have $h_+^{(3,0)} = h_+^{(0,3)} = 0$ while $h_-^{(3,0)} = h_-^{(0,3)} = 1$;

(iv) The volume form $vol = i\Omega \wedge \overline{\Omega}/||\Omega||^2$ (see (2.2.10)) is invariant under σ^*, thus $h_+^{(0,0)} = h_+^{(3,3)} = 1$ and $h_-^{(0,00)} = h_-^{(3,3)} = 0$.

Spectrum of type IIB Orientifolded with O3/O7-planes

Type IIB string theory in 10 spacetime dimensions has a bosonic spectrum which consists of NS-NS fields: $(\hat{\phi}, \hat{B}_2, \hat{G})$ and R-R fields $(\hat{C}_0, \hat{C}_2, \hat{C}_4)$. Under $(-1)^{F_L}$ the NS-NS states are even while R-R ones are odd. Conversely, the world-sheet parity operator Ω_p maps $\hat{\phi}, \hat{G}$ and $\hat{C}_2$ to themselves while $\hat{B}_2, \hat{C}_0$ and $\hat{C}_4$ to their opposite. These parity behaviours can be summarised as follows:

$10D$ Field	$(-1)^{F_L}$	Ω_p	$(-1)^{F_L}\Omega_p$
$\hat{\phi}$	+	+	+
$\hat{B}_2$	+	-	-
$\hat{G}$	+	+	+
$\hat{C}_0$	-	-	+
$\hat{C}_2$	-	+	-
$\hat{C}_4$	-	-	+

The orientifolded spectrum consists of states which are left invariant under the orientifold projection $\mathcal{O}$ and so it is constructed by throwing away those who are not invariant. From the above table it is clear that we should keep for the NS-NS and R-R fields only those which obey:

- NS-NS sector:
$$\sigma^*\hat{\phi} = \hat{\phi} \qquad \sigma^*\hat{B}_2 = -\hat{B}_2 \qquad \sigma^*\hat{G} = \hat{G} \tag{2.3.12}$$

- R-R sector:
$$\sigma^*\hat{C}_0 = \hat{C}_0 \qquad \sigma^*\hat{C}_2 = -\hat{C}_2 \qquad \sigma^*\hat{C}_4 = \hat{C}_4 \tag{2.3.13}$$

The expansions (2.3.3), (2.3.4) and (2.3.5) are then recast in the following form:

- NS-NS sector:
$$\hat{\phi} = \phi(x) \qquad \hat{B} = b^{i_-}(x)\hat{D}_{i_-} \tag{2.3.14}$$
  with $i_- = 1, ..., h_-^{(1,1)}$.

- R-R sector:
$$\hat{C}_0 = C_0(x) \qquad \hat{C}_2 = c^{i_-}(x)\hat{D}_{i_-} \tag{2.3.15}$$
$$\hat{C}_4 = Q_2^{i_+}(x) \wedge \hat{D}_{i_+} + V^{I_+}(x) \wedge \alpha_{I_+} - \tilde{V}_{I_+}(x) \wedge \beta^{I_+} + \rho_{i_+}(x)\tilde{D}^{i_+} \tag{2.3.16}$$
  where $i_+ = 1, ..., h^{(1,1)_+}$, $I_+ = 1, ..., h_+^{(2,1)}$.

Here $\{\tilde{D}^{i_+}\}_{i_+=1,\dots,h_+^{(1,1)}}$ is a basis of the cohomology $H^{2,2}_{\bar\partial}(\mathscr{Y})^+$ dual to $\{\hat{D}^{i_+}\}_{i_+=1,\dots,h_+^{(1,1)}}$ and $\{(\alpha_{I_+},\beta^{I_+})\}_{I_+=1,\dots,h_+^{(2,1)}}$ is a symplectic basis for $H^3_{dR}(\mathscr{Y},\mathbb{C})^+ \simeq H^{2,1}_{\bar\partial}(\mathscr{Y})^+ \oplus H^{(1,2)}_{\bar\partial}(\mathscr{Y})^+$ (there is no $H^{3,0}_{\bar\partial}(\mathscr{Y})^+$ and $H^{0,3}_{\bar\partial}(\mathscr{Y})^+$ because as we said earlier $h_+^{(3,0)} = h_+^{(0,3)} = 0$).

Finally, the Kähler form J, since it left invariant under σ^*, is expanded as:

$$J = t^{i_+}\hat{D}_{i_+} \tag{2.3.17}$$

while, due to the fact that the holomorphic $(3,0)$-form takes a negative sign upon acting on σ^*, the only surviving modes in complex structure deformations (2.2.66) will be:

$$\delta g_{ab} = \frac{i}{||\Omega||^2}(\overline{\chi}_{\alpha_-})_{a\bar b\bar c}\,\Omega^{\bar b\bar c}{}_b\,\overline{U}^{\alpha_-} \tag{2.3.18}$$

These give the following spectrum:

Multiplet	Number	Fields
Gravity Multiplet	1	$g_{\mu\nu}$
Vector multiplets	$h_+^{(2,1)}$	V^{I_+}
Chiral multiplets	$h_-^{(2,1)}$	U^{α_-}
	$h_-^{(1,1)}$	(b^{i_-},c^{i_-})
	1	(ϕ,C_0)
Chiral/linear multiplets	$h_+^{(1,1)}$	(t^{i_+},ρ_{i_+})

The low-energy effective action for orientifolded type IIB string theory can be obtained from the $\mathcal{N}=2$ action (2.3.6) by means of the truncation mentioned above and it can be written in the standard $\mathcal{N}=1$ supergravity form by first rearranging the fields in the spectrum and define:

- Axio-dilaton: $S = e^{-\phi} - iC_0$ (we have already encountered it when F-theory was discussed, at that time it took a slightly different form, namely $\tau = C_0 + ie^{\phi}$);
- 2-form moduli: $G^{i_-} = c^{i_-} - iSb^{i_-}$;
- Complex structure moduli: U^{α_-};
- Kähler moduli: $T_{i_+} = \tau_{i_+} - \frac{1}{2(S+\bar S)}k_{i_+j_-k_-}G^{j_-}(G-\overline{G})^{k_-} + i\rho_{i_+}$, where τ_{i_+} are implicit functions of t_{i_+} and k_{ijk} are the intersection numbers given by (2.2.65).

The orientifold projections which we will employ will be such that $h_-^{(1,1)} = 0$ meaning that $h^{(1,1)} = h_+^{(1,1)}$ so that the fields B_2 and C_2 will be projected out and the Kähler moduli will simply be given by:

$$T_i = \tau_i + ib_i \tag{2.3.19}$$

where we have simply redefined $b_i := \rho_i$.
If we consider the expansion:

$$J = t^i\hat{D}_i \tag{2.3.20}$$

then the volume of the Calabi-Yau $\mathscr{Y}$ will be given by (in Einstein frame and in units of ℓ_s):

$$\mathcal{V} = \frac{1}{6}\int J \wedge J \wedge J = \frac{1}{6}\int t^i \hat{D}_i \wedge t^j \hat{D}_j \wedge t^k \hat{D}_k = \frac{1}{6} t^i t^j t^k k_{ijk} \tag{2.3.21}$$

The τ_i turns out to be the volume of the cycle $D_i \in H_4(\mathscr{Y})$ Poincaré dual to $\hat{D}_i$. This means that:

$$\tau_i = \frac{1}{2}\int_{D_i} J \wedge J = \frac{1}{2}\int_{\mathscr{Y}} J \wedge J \wedge \hat{D}_i = \frac{1}{2} t^j t^k k_{ijk} = \frac{\partial \mathcal{V}}{\partial t^i} \tag{2.3.22}$$

and its axionic partner b_i will be the component of the R-R form C_4 along the cycle D_i, namely: $b_i = \int_{D_i} C_4$.

Tree-Level Effective Action

The $\mathcal{N} = 1$ supergravity action will be:

$$\begin{aligned} S_{IIB}^{4D} = -\int \Big(&\mathscr{R} \star \mathbb{1} + K_{I\bar{J}} D\Phi^I \wedge \star D\Phi^{\bar{J}} + \frac{1}{2}\mathfrak{Re}(f_{ab}) F^a \wedge \star F^b \\ &+ \frac{1}{2}\mathfrak{Im}(f_{ab}) F^a \wedge F^b + V \star \mathbb{1} \Big) \end{aligned} \tag{2.3.23}$$

where Φ^I denotes all the scalar fields in the theory, i.e. the *axio-dilaton* $S = e^{-\phi} - i C_0$, the *complex structure moduli* U^α (with $\alpha = 1, \ldots, h_-^{(2,1)}$) and the *Kähler structure moduli* $T^i = \tau^i + i b^i$ (with $i = 1, \ldots, h^{(1,1)}$):

$$\Phi^I = (S, U^\alpha, T^i)$$

F^a are the field strength of the vector fields in the vector multiplet $dV^a = F^a$ (with $a = 1, \ldots, h_+^{(2,1)}$), while $\mathscr{R}$ is the Ricci $(1,1)$-form given with respect to the metric g and f_{ab} are the gauge-kinetic functions. The Kähler metric is given by the Hessian of the Kähler potential K (recall (2.1.5)):

$$K_{I\bar{J}} = \partial_I \partial_{\bar{J}} K(S, U, T) \tag{2.3.24}$$

while the scalar potential V is given by two factors $V = V_F + V_D$, the first is called *F-term potential* while the second *D-term potential* and these are expressed in function of the superpotential W and Kähler potential K in the following form:

$$V_F = e^K \left(K^{I\bar{J}} D_I W D_{\bar{J}} \overline{W} - 3|W|^2 \right) \tag{2.3.25}$$

where the covariant derivatives are given by:

$$\begin{cases} D_I W = \partial_I W + W \partial_I K \\ D_{\bar{J}} \overline{W} = \partial_{\bar{J}} \overline{W} + \overline{W} \partial_{\bar{J}} K \end{cases} \tag{2.3.26}$$

while the D-term is:

$$V_D = \frac{1}{2} (\mathfrak{Re}(f))^{-1ab} D_a D_b \tag{2.3.27}$$

with:

$$D_a = \left[\partial_I K + \frac{\partial_I W}{W} \right] (T_a)_{IJ} \partial_J \Phi$$

At the lowest order in perturbation theory with respect to (α') and (g_s) expansions, the Kähler potential takes the following tree-level form (in M_P units):

$$K_{tree} = -2\ln(\mathcal{V}) - \ln(S + \bar{S}) - \ln\left(-i \int \Omega \wedge \bar{\Omega}\right) \tag{2.3.28}$$

where Ω depends implicitly on the complex structure moduli U^α, whereas the volume mode $\mathcal{V}$ depends on the real part τ_i of the Kähler moduli T_i. This Kähler potential gives rise to a block-diagonal metric, the moduli space has in fact the factorised form:

$$\mathcal{M} = \mathcal{M}^{\text{cs}}_{h_-^{(2,1)}} \times \mathcal{M}^{\text{ks}}_{h^{(1,1)}+1} \tag{2.3.29}$$

In absence of internal fluxes, the superpotential W is identically zero and thus no scalar potential V is generated, making (S, U^α, T^i) completely flat directions in the Moduli space and in order to provide a sensible framework in which try to make phenomenologically testable models, we must give them a non-zero potential term allowing to fix their VEVs and ruling out unobserved long-range forces from our theory. This procedure is called *Moduli stabilisation* and the process leading to the stabilisation of all the moduli is quite involved and we are going to explore it in the next chapter.

Chapter 3

Moduli Stabilisation in Type IIB

In the previous chapters we have seen how various consistent theory of superstrings can be defined on a 10-dimensional background manifold. In the point-particle limit, namely for energies lower that the string scale M_s, massive modes on the worldsheets can be discarded and a consistent low-energy effective supergravity theory with a number of supersymmetries depending on the string theory considered can be recovered. The fact that it seems like we are living in a 4-dimensional spacetime, suggests that we should find vacuum solutions of these theories in which the 10-dimensional spacetime is factorised into a non-compact 4-dimensional Minkowski space and a compact 6-(real)dimensional manifold. In the compactification, Kaluza-Klein reduction can be used to retrieve the 4-dimensional spectrum. For general compactification spaces, the number of supersymmetries in the 4-dimensional theory can be as big as $\mathcal{N} = 8$. In fact, $\mathcal{N} = 2$ supersymmetry in D-dimensions means that there are 2 supercharges Q^i_α (i.e. 2 Majorana spinors), with $i = 1,2$ and $\alpha = 1,...,2^{[D/2]-1}$ which implement the supersymmetry transformations. In particular, in $D = 10$ the spinor index takes the values $\alpha = 1,...,16$ for each supercharge. When reducing the theory on a 4-dimensional spacetime by means of the Kaluza-Klein reduction, if no supersymmetries are broken, then all these 32 components will rearrange into $\mathcal{N} = 8$ supersymmetry in $D = 4$. This very large number of supersymmetries is not very appealing for phenomenology. Calabi-Yau compactifications for type II superstring theories leave $\mathcal{N} = 2$ supersymmetries which is more acceptable but still cannot reproduce a chiral spectrum. The orientifold projection is what is needed to truncate further the spectrum leaving finally $\mathcal{N} = 1$ supersymmetry in 4-dimensions. As we have seen, the compactification on a Calabi-Yau manifold brings in the spectrum some scalar fields, called *Moduli Fields*, whose potential is completely flat. Phenomenologically trustable models should then implement a procedure under which these fields get fixed at their vacuum expectation values, giving them large enough masses to overcome their potential coupling to matter particles which would lead to fifth forces not yet observed. In order to stabilise these moduli it is necessary to generate a superpotential W_{tree} at the tree-level and this was firstly proposed by Gukov, Vafa, and Witten [GVW00] in the context of compactifications of M-theory. When the 4-form dA_3 acquire a non-zero flux due to the introduction of membranes, for example, a superpotential of the form:

$$W \sim \int dA_3 \wedge \Omega \tag{3.0.1}$$

is generated, and in type IIB the field strength dA_3 becomes the combined 3-form flux $G_3 = F_3 - iSH_3$. The superpotential can thus be generated by turning on gauge fluxes and in the next section we are going to inspect in more details this process. Also, because the superpotential generated in this way cannot fix the Kähler moduli due to the *No Scale Structure*, other mechanisms should be employed to deal with these moduli and the other sections will be dedicated to these processes.

3.1 Axio-Dilaton and Complex Structure Moduli Stabilisation

In order to fix the axio-dilaton S and the complex structure moduli U^α with $\alpha = 1, ..., h_-^{(2,1)}$ we turn on internal gauge fluxed for the NS-NS 3-form $dB_2 = H_3$ and for the R-R 3-form $dC_2 = F_3$. Let's then study first these generalised fluxes and then inspect how these exactly fix the axio-dilaton and the complex structure moduli.

3.1.1 Generalised Fluxes

As we have already seen in §1.3 one can generalise the definition of a flux for an arbitrary p-form field strength F_p. In this case, through a p-cycle γ_p in the Calabi-Yau manifold $\mathscr{Y}$, we will have the following flux:

$$\int_{\gamma_p} F_p = n \tag{3.1.1}$$

If we turn on non-zero electric and magnetic fluxes for the R-R 3-form $F_3 = dC_2$ and NS-NS 3-form $H_3 = dB_2$ on 3-cycles (γ_I, γ^J) which are Poincaré duals[1] to the symplectic basis $\{(\alpha_I, \beta^J)\}_{I,J=1,...,h^{(2,1)}+1}$ of $H^3_{dR}(\mathscr{Y})$, we obtain:

$$\int_{\gamma_I} F_3 = n_I^R \qquad \int_{\gamma^J} F_3 = m^{R,J}$$
$$\int_{\gamma_I} H_3 = n_I^{NS} \qquad \int_{\gamma^J} H_3 = m^{NS,J}$$

These numbers n_I^R, n_I^{NS} and $m^{R,J}, m^{NS,J}$ are constrained to be integers from the Dirac quantisation condition and can be thought of as the coefficients in the expansion of the 3-forms F_3 and H_3 in the symplectic basis $\{(\alpha_I, \beta^J)\}_{I,J=1,...,h^{(2,1)}+1}$:

$$F_3 = m^{R,I}\alpha_I + n_J^R \beta^J$$
$$H_3 = m^{NS,I}\alpha_I + n_J^{NS} \beta^J$$

Using these expansion, the 3-form flux G_3 becomes:

$$G_3 = F_3 - iSH_3 = (m^{R,I} - iSm^{NS,I})\alpha_I + (n_J^R - iSn_J^{NS})\beta^J \tag{3.1.3}$$

[1]Recall that by Poincaré duality (from §2.1) we can associate to each element $[\omega]$ in the cohomology $H^k_{dR}(\mathscr{Y})$ an element Ω in the homology $H_{n-k}(\mathscr{Y})$ such that $\forall [\alpha] \in H^k_{dR}(\mathscr{Y})$:

$$\int_\Omega [\alpha] = \int_{\mathscr{Y}} \alpha \wedge \omega \tag{3.1.2}$$

in particular, if we take a symplectic basis for the cohomology group $H^3_{dR}(\mathscr{Y})$ and its dual basis, we will have the relations (2.2.68).

The effect of these non-vanishing background fluxes is to generate a scalar potential of the form:

$$V(U,S) = -(\overline{n} - \overline{M} \cdot \overline{m})_I (\mathfrak{Im}(M))^{-1IJ}(n - M \cdot m)_J \tag{3.1.4}$$

where M is the gauge kinetic matrix appearing in (2.3.6). As can be noted, this potential depends only on the axio-dilaton S and the complex moduli U^α. The superpotential generated by it is of the Gukov-Vafa-Witten form:

$$W_{\text{tree}}(S,U) \sim \int_{\mathscr{Y}} G_3 \wedge \Omega \tag{3.1.5}$$

Now, when the orientifold projection is taken into account, the 3-form flux G_3 will be projected onto:

$$G_3 = (m^{R,I_-} - iSm^{NS,I_-})\alpha_{I_-} + (n^R_{J_-} - iSn^{NS}_{J_-})\beta^{J_-} \tag{3.1.6}$$

with $I_-, J_- = 0, \dots, h^{(2,1)}_-$ (there are then $2(h^{(2,1)}_- + 1)$ flux coefficients since $H^3_{dR}(\mathscr{Y})^- = H^{(3,0)}_{\overline{\partial}}(\mathscr{Y})^- \oplus H^{(2,1)}_{\overline{\partial}}(\mathscr{Y})^- \oplus H^{(1,2)}_{\overline{\partial}}(\mathscr{Y})^- \oplus H^{(0,3)}_{\overline{\partial}}(\mathscr{Y})^-$).

The backreaction of these fluxes on the internal geometry causes the metric to warp, so that locally the metric becomes:

$$ds^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + e^{-2A(y)} g_{m\overline{n}} dy^m d\overline{y}^n \tag{3.1.7}$$

where $A(y)$ is the warp factor which is allowed to vary in the compact space $\mathscr{Y}$ in order to not break the Poincaré invariance in $\mathbb{R}^{1,3}$.

Remark

Before the discovery of Dp-branes by Dai, Leigh, and Polchinski [DLP89], there was a no-go theorem due to Wit, Smit, and Hari Dass [WSH87] which didn't permit the existence of fluxes in supergravity theories since they would have led to a warped background metric and then to inconsistent equations of motion. This can be seen as follows: take the type IIB action (1.4.13) in the manifestly $SL(2,\mathbb{Z})$-invariant form and consider a warped metric of the form (3.1.7). For this scope we come back to the definition of the axio-dilaton as $\tau = C_0 + ie^{-\phi}$. Now, the 5-form field $\tilde{F}_5$ (defined in §1.4.2) was self-dual, and this self-duality condition has to be imposed on the equations of motion as a constraint. Bianchi identities and Poincaré invariance restrict the 3-form flux G_3 to have only components along the compact directions [GKP02] while the self-dual $\tilde{F}_5$ should take the form:

$$\tilde{F}_5 = (1 + \star) d\alpha \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{3.1.8}$$

where $\alpha(y)$ is a function of the compact coordinates and $\star$ is the Hodge $\star$-operator. Now, the Einstein's equations trace reversed are:

$$R_{MN} = k^2_{10}(T_{MN} - \frac{1}{8} G_{MN} T) \tag{3.1.9}$$

with T_{MN} the stress-energy tensor given by the variation of the type IIB action with respect to the 10D metric G_{MN}. By inspecting the explicit form of the Einstein's equations one gets [GKP02]:

$$\triangle A = e^{-2A} \frac{|G_3|^2}{8\,\mathfrak{Im}(\tau)} + \frac{e^{-6A}\partial_m \alpha \partial^m \alpha}{4} \tag{3.1.10}$$

with $|G_3|^2 = 1/3!\, G_{ABC}\overline{G}_{MNO}G^{AM}G^{BN}G^{CO}$ and $\triangle$ the *Laplace Beltrami* operator in the compact space, i.e. using the metric $g_{m\bar{n}}$.

Integrating both sides of (3.1.10), since the LHS (left-hand side) vanishes and the RHS (right-hand side) is the sum of two non-negative contributions, then the only possible solutions are those with A, α constant and vanishing G_3.

The interesting consideration which allows to bypass this no-go theorem was made by Giddings, Kachru, and Polchinski [GKP02] and consists in realising that localised sources such as D3-branes, wrapped D7-branes and eventually O3-planes, generate a contribution to the equations of motion (3.1.10) which can balance non-zero G_3 and non-constant warp factors. The above mentioned localised sources are in fact of negative tension and their contribution to the RHS of (3.1.10) will be of the form:

$$\frac{k_{10}^2 e^{-2A}}{8}(T_m^m - T_\mu^\mu) \overset{\text{For a Dp-brane}}{\widehat{=}} \frac{k_{10}^2 e^{-2A}}{8}(7-p)T_p\delta(\Sigma) \tag{3.1.11}$$

where $\delta(\Sigma)$ is the delta function on the cycle Σ which is wrapped by the Dp-brane. Then, for $p \leq 7$, negative tension sources give the negative required contribution. The consistency condition can be translated in the *Tadpole-Cancellation Condition*: the Bianchi identity (equivalent to the equations of motion for self-dual fields) for $\tilde{F}_5$ in presence of localised sources is modified according to:

$$d\tilde{F}_5 = H_3 \wedge F_3 + 2k_{10}^2 T_3 \rho_3 \tag{3.1.12}$$

where T_3 is the tension of a D3-brane and ρ_3 is the D3 charge density form from the localised sources. Integrating the above Bianchi identity, one gets to the tadpole-cancellation condition for type IIB strings:

$$\frac{1}{2k_{10}^2 T_3}\int_{\mathcal{Y}} H_3 \wedge F_3 + Q_3 = 0 \tag{3.1.13}$$

where Q_3 is the total charge associated with ρ_3. A more useful form exploits the lifting to F-theory of type IIB. In this context the condition is given by:

$$N_{D3} + \frac{1}{2k_{10}^2 T_3}\int_{\mathcal{Y}} H_3 \wedge F_3 = \frac{\chi(Y)}{24} \tag{3.1.14}$$

In the F-theory interpretation, $\chi(Y)$ is the Euler characteristic of the fibred Calabi-Yau 4-fold Y and N_{D3} are the number of $(D3 - \overline{D3})$ branes. In type IIB language, $\chi(Y)$ counts the D3 charge coming from O3-planes and D7-branes, while the term N_{D3} counts the net charge from transverse branes and fluxes.

When turning on NS-NS and R-R fluxes, the metric becomes warped and the 3-form flux G_3 can be seen to become *imaginary self-dual*:

$$\star G_3 = i G_3 \tag{3.1.15}$$

Also, the localised sources needed to cancel the tadpole are precisely D3/D7-branes and/or O3-planes.

3.1.2 Stabilisation of S and U^α

With this set-up, consider again the form of the F-term potential V_F in (2.3.23):

$$V_F = e^K \left(K^{I\bar{J}} D_I W D_{\bar{J}} \overline{W} - 3|W|^2 \right) \tag{3.1.16}$$

Expanding this expression considering that I indexes the moduli fields, namely the axio-dilaton S, the Kähler structure moduli T^i (with $i = 1, ..., h^{(1,1)}$ since we are considering orientifolds with $h_-^{(1,1)} = 0$ so that $h_+^{(1,1)} = h^{(1,1)}$) and the complex structure moduli U^α (with $\alpha = 1, ..., h_-^{2,1}$), and using also the factorisation of the moduli space (2.3.29) making the Kähler metric block-diagonal, we obtain:

$$V_F = e^K \left(K^{T^i \overline{T}^j} D_{T^i} W D_{\overline{T}^j} \overline{W} + K^{S\bar{S}} D_S W D_{\bar{S}} \overline{W} + K^{U^\alpha \overline{U}^\alpha} D_{U^\alpha} W D_{\overline{U}^\alpha} \overline{W} - 3|W|^2 \right) \tag{3.1.17}$$

Now, taking into consideration the Gukov-Vafa-Witten superpotential generated through non-zero fluxes:

$$W_{\text{tree}}(S, U^\alpha) \sim \int G_3 \wedge \Omega \tag{3.1.18}$$

we can see that:

$$D_{T^i} W_{\text{tree}} = \partial_{T^i} W_{\text{tree}} + W_{\text{tree}} \partial_{T^i} K = W_{\text{tree}} \partial_{T^i} K \tag{3.1.19}$$

By plugging this back into the F-term potential we get:

$$V_F = e^K \left(K^{S\bar{S}} D_S W D_{\bar{S}} \overline{W} + K^{U^\alpha \overline{U}^\alpha} D_{U^\alpha} W D_{\overline{U}^\alpha} \overline{W} + (K^{T^i \overline{T}^j} \partial_{T^i} K \, \partial_{\overline{T}^j} K - 3)|W|^2 \right) \tag{3.1.20}$$

At this point what can be noted is that at the classical level the Kähler potential K_{tree} given by (2.3.28) enjoys the *No-Scale Structure* for the Kähler moduli:

Proposition 3.1.1 (*No-Scale*)

The tree-level Kähler potential K_{tree} satisfies the No-Scale identity:

$$3 \left(\frac{\partial^2 K_{tree}}{\partial T^i \overline{T}^j} \right) = \frac{\partial K_{tree}}{\partial T^i} \frac{\partial K_{tree}}{\partial \overline{T}^j} \tag{3.1.21}$$

In order to prove that, we will use the Euler's theorem for homogeneous functions and the fact that the volume mode $\mathcal{V}$ of the Calabi-Yau is a homogeneous function of degree $3/2$ in the τ_i.

Theorem 3.1.1 (*Euler's theorem*)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function of degree n, namely $\forall \lambda \in \mathbb{R}$, $f(\lambda \vec{x}) = \lambda^n f(\vec{x})$. Then:

$$\vec{x} \cdot \vec{\nabla} f(\vec{x}) = n f(\vec{x}) \tag{3.1.22}$$

Proof 1. Consider the characterising property of homogeneous functions and derive with respect to the parameter λ both sides, obtaining (using chain rule):

$$\vec{x} \cdot \nabla_{\vec{x}'} f(\vec{x}') = n \lambda^{n-1} f(\vec{x}) \tag{3.1.23}$$

Now, by taking the particular case in which $\lambda = 1$, we get the desired result. □

Lemma 3.1.1

The volume mode $\mathcal{V}$ of the Calabi-Yau is a homogeneous function of degree 3/2 with respect to the 4-cycle volumes τ_i.

Proof 2. Recalling the formulae (2.3.21) and (2.3.22) furnishing the volume of the Calabi-Yau and the volumes of its 4-cycles:

$$\mathcal{V} = \frac{1}{6} t^i t^j t^j k_{ijk}; \qquad \tau_i = \frac{\partial \mathcal{V}}{\partial t^i} = \frac{1}{2} k_{ijk} t^j t^k \tag{3.1.24}$$

we can see that:

$$\mathcal{V} = \frac{1}{3} t^i \tau_i \tag{3.1.25}$$

and by defining $A_{ij} = t^k k_{ijk}$, we have the following relations:

$$t^i A_{ij} t^j = 6\mathcal{V}; \qquad A_{ij} t^j = t^j t^j k_{ijk} = 2\tau_i \iff \frac{1}{2} t^j = A^{ji} \tau_i \tag{3.1.26}$$

where A^{ij} is the inverse of A_{ij}.

Now, since the Euler's theorem (3.1.1) is actually an "if and only if" (we have only stated and proved it as an implication $\Longrightarrow$), it will suffices to prove that there exists a number $n \in \mathbb{R}$ such that $\tau_i \frac{\partial \mathcal{V}}{\partial \tau_i} = n\mathcal{V}$ in order to prove that $\mathcal{V}$ is a homogeneous function of degree n. Consider then:

$$\frac{\partial \mathcal{V}}{\partial \tau_i} = \frac{\partial \mathcal{V}}{\partial t^j} \frac{\partial t^j}{\partial \tau_i} = \tau_j A^{ji} \iff \tau_i \frac{\partial \mathcal{V}}{\partial \tau_i} = \tau_i A^{ji} \tau_j = \frac{1}{2} t^j \tau_j = \frac{3}{2} \mathcal{V} \tag{3.1.27}$$

□

We can now finally prove the no-scale property of K_{tree}:

Proof 3 (3.1.1). Recall that $\tau_i = 1/2(T_i + \overline{T}_i)$ and $\mathcal{V}$ depends only on τ_i (not on the axionic part of T_i), in this way:

$$\begin{aligned} K_{i\bar{j}} = \frac{\partial}{\partial T^i} \frac{\partial}{\partial \overline{T}^j} (K_{\text{tree}}) &= \frac{\partial}{\partial T^i} \frac{\partial}{\partial \overline{T}^j} (-2\ln(\mathcal{V})) \\ &= \frac{1}{4} \frac{\partial}{\partial \tau^i} \frac{\partial}{\partial \tau^j} (-2\ln \mathcal{V}) \\ &= -\frac{1}{2} \frac{\partial}{\partial \tau^i} \left(\frac{1}{\mathcal{V}} \frac{\partial \mathcal{V}}{\partial \tau^j} \right) \\ &= \frac{1}{2\mathcal{V}^2} \frac{\partial \mathcal{V}}{\partial \tau^i} \frac{\partial \mathcal{V}}{\partial \tau^j} - \frac{1}{2\mathcal{V}} \frac{\partial^2 \mathcal{V}}{\partial \tau^i \partial \tau^j} \end{aligned} \tag{3.1.28}$$

Contract with $\tau^i \tau^j$ to obtain:

$$\begin{aligned} \tau^i K_{ij} \tau^j &= \frac{1}{2\mathcal{V}^2} \left(\tau^i \frac{\partial \mathcal{V}}{\partial \tau^i} \right) \left(\tau^j \frac{\partial \mathcal{V}}{\partial \tau^j} \right) - \tau^i \tau^j \frac{1}{2\mathcal{V}} \frac{\partial^2 \mathcal{V}}{\partial \tau^i \partial \tau^j} \\ &= \frac{1}{2\mathcal{V}^2} \frac{3}{2} \mathcal{V} \frac{3}{2} \mathcal{V} - \tau^i \tau^j \frac{1}{2\mathcal{V}} \frac{\partial^2 \mathcal{V}}{\partial \tau^i \partial \tau^j} \\ &= \frac{9}{8} - \tau^i \tau^j \frac{1}{2\mathcal{V}} \frac{\partial^2 \mathcal{V}}{\partial \tau^i \partial \tau^j} \end{aligned} \tag{3.1.29}$$

Consider now:

$$\tau^i \frac{\partial \mathcal{V}}{\partial \tau^i} = \frac{3}{2} \mathcal{V} \xrightarrow{\frac{\partial}{\partial \tau^j}} \frac{\partial \tau^i}{\partial \tau^j} \frac{\partial \mathcal{V}}{\partial \tau^i} + \tau^i \frac{\partial^2 \mathcal{V}}{\partial \tau^j \partial \tau^i} = \frac{3}{2} \frac{\partial \mathcal{V}}{\partial \tau^j} \iff \frac{\partial \mathcal{V}}{\partial \tau^j} + \tau^i \frac{\partial^2 \mathcal{V}}{\partial \tau^j \partial \tau^i} = \frac{3}{2} \frac{\partial \mathcal{V}}{\partial \tau^j} \tag{3.1.30}$$

and contract the last term with τ^j so that we get to an expression for the second derivate:

$$\tau^i\tau^j\frac{\partial^2\mathcal{V}}{\partial\tau^j\partial\tau^i}=\frac{3}{2}\tau^j\frac{\partial\mathcal{V}}{\partial\tau^j}-\tau^j\frac{\partial\mathcal{V}}{\partial\tau^j}=\frac{3}{2}\cdot\frac{3}{2}\mathcal{V}-\frac{3}{2}\mathcal{V}=\frac{3}{4}\mathcal{V} \tag{3.1.31}$$

Plugging the above expression into (3.1.29) we get:

$$\tau^i K_{ij}\tau^j=\frac{9}{8}-\frac{3}{8}=\frac{3}{4} \tag{3.1.32}$$

If we compute now:

$$\tau^i\frac{\partial K_{\text{tree}}}{\partial T^i}=\frac{1}{2}\tau^i\frac{\partial K_{\text{tree}}}{\partial\tau^i}=-\frac{2}{\mathcal{V}}\tau^i\frac{\partial\mathcal{V}}{\partial\tau^i}=-3 \tag{3.1.33}$$

then we can see that:

$$\tau^i\tau^j\left(3K_{i\bar{j}}-\frac{\partial K_{\text{tree}}}{\partial T^i}\frac{\partial K_{\text{tree}}}{\partial\overline{T}^j}\right)=3\tau^i K_{ij}\tau^j-\frac{\tau^i}{2}\frac{\partial K_{\text{tree}}}{\partial\tau^i}\frac{\tau^j}{2}\frac{\partial K_{\text{tree}}}{\partial\tau^j}=\frac{9}{4}-\frac{9}{4}=0 \tag{3.1.34}$$

Since this must be true for all τ^i and τ^j, then:

$$3K_{i\bar{j}}-\frac{\partial K_{\text{tree}}}{\partial T^i}\frac{\partial K_{\text{tree}}}{\partial\overline{T}^j}=0 \tag{3.1.35}$$

which is exactly the no-scale property. □

Thanks to this cancellation at tree level, the F-term potential is reduced to:

$$V_F=e^K\left(K^{S\bar{S}}D_S W D_{\bar{S}}\overline{W}+K^{U^\alpha\overline{U}^\alpha}D_{U^\alpha}WD_{\overline{U}^\alpha}\overline{W}\right) \tag{3.1.36}$$

Since it is positive definite, we can locate the minima for S and U^α by imposing and solving:

$$\begin{cases} D_S W|_{\langle S\rangle,\langle U^\alpha\rangle}=0 \\ D_{U^\alpha}W|_{\langle S\rangle,\langle U^\alpha\rangle}=0 \quad \forall\alpha\in\{1,\ldots,h_-^{(2,1)}\}\end{cases} \tag{3.1.37}$$

Once their Vacuum Expectation Values have been found, these moduli can be integrated out at tree level by fixing them at their VEVs. This stabilisation, even if at tree level, can be trusted even when quantum corrections are taken into account since these will be only subleading corrections to their VEVs. In this way, we get:

$$W_0:=\left\langle\int_{\mathcal{Y}}G_3\wedge\Omega\right\rangle \tag{3.1.38}$$

and:

$$K_{\text{tree}}=-2\ln(\mathcal{V})-\ln\left(\frac{2}{g_s}\right)+K_{\text{cs}} \tag{3.1.39}$$

where:

$$\begin{cases} g_s:=\left\langle\dfrac{2}{S+\bar{S}}\right\rangle=e^{\langle\phi\rangle} \\ K_{\text{cs}}:=-\ln\left\langle-i\displaystyle\int_{\mathcal{Y}}\Omega\wedge\overline{\Omega}\right\rangle\end{cases} \tag{3.1.40}$$

The resulting potential V is completely flat in the Kähler moduli directions, making these moduli still non-stabilised due to the no-scale property of the Käahler potential. These solutions are however non-supersymmetric, in fact, the F-terms $F_{T^i} := D_{T^i} W$ do not vanish in general. Also the no-scale structure can be - and in fact is - broken by both string loop corrections and sigma model corrections ((α')-corrections). These corrections are what is needed to fix the other moduli in the theory together with non-perturbative corrections to the superpotential. In the next section we are going to inspect these mechanisms which will allow us to, at the end, fix all the moduli fields in play.

3.2 Kähler Structure Moduli Stabilisation

The tree-level Kähler potential K_{tree} and the tree-level superpotential W_{tree} do not allow to fix deformations of the Kähler structure, namely the shape deformations of the Calabi-Yau upon which we compactify our type IIB string theory. Our vacuum is still degenerate in those directions (in the T^i-directions on the moduli space of Kähler deformations $\mathscr{M}_{\text{ks}}$) and in order to fix them and uplift these directions we will make use of non-perturbative corrections of the superpotential (by wrapping D7-branes which undergo gaugino condensation or by employing Euclidean D3-brane instantons) and perturbative corrections in (α') and (g_s) to the Kähler potential. These processes, however, are not painless and consistency checks must be made in order to not run into problems. Before presenting how the aforementioned mechanisms are implemented we are going to list the issues that could arise in this process, mainly following [CMV12]:

(i) Tension between Kähler moduli stabilisation via non-perturbative effects and chirality:

Problem:
Non-perturbative effects for the superpotential can be generated by gaugino condensation on D7-branes wrapping a blow-up mode or by Euclidean D3-brane instantons. Now, chiral intersections between instanton and visible sector divisors induces a prefactor for the non-perturbative superpotential. Since this will depend on the vacuum expectation values of chiral matter, in order to not break any visible sector gauge symmetry, these VEVs have to vanish, killing the instanton contribution to the superpotential (see [BMP08]).

Solution:
The D7-branes supporting the standard model should not be wrapped around the same divisor wrapped by the stack of branes giving the non-perturbative effects.

(ii) Tension between Kähler moduli stabilisation via non-perturbative effects and the cancellation of Freed-Witten anomalies:

Problem:
The Freed-Witten anomaly is a global anomaly of the worldsheet path integral first pointed out by Freed and Witten [FW99]. Consider $\mathscr{M}$ to be the 10-dimensional spacetime and $Q \subset \mathscr{M}$ be an oriented subset given by a D-brane, so that strings can end on Q. Take $X : \Sigma \to \mathscr{M}$ as the embedding of the string worldsheet onto the spacetime

which maps its boundary to the D-brane, namely $X(\partial\Sigma) \subset Q$. The worldsheet measure will contain a term of the form (with NS-NS field $B_2 = 0$):

$$\mathrm{Pfaff}(D) \cdot \exp\left(i \oint_{\partial\Sigma} A \right) \tag{3.2.1}$$

where D is the Dirac operator on the world-sheet, $(\mathrm{Pfaff}(D))^2 = \det(D)$ is the Pfaffian of D and A is the gauge field on the D-brane (remember that in order to have spinors defined in the space-time manifold $\mathcal{M}$, the first and second *Stiefel-Whitney classes* have to vanish $w_1(\mathcal{M}) = w_2(\mathcal{M}) = 0$, i.e. $\mathcal{M}$ must admit a *Spin-structure*). The Pfaffian, however, is not well-defined inasmuch as it possesses a sign ambiguity. This ambiguity is compensated if $\exp\left(i \oint_{\partial\Sigma} A\right)$ also possesses the same "sign behaviour". This fact leads to the conclusion that spinors charged under A are defined as sections of the tensor bundle $S(Q) \otimes L$ where $S(Q)$ is the *Spinor bundle* of Q and L is the *Line bundle* on which A is a connection and this defines a $Spin^c$-structure of Q. The conclusion is that in order for a D-brane to give rise to a well-defined string theory, it must possess a $Spin^c$-structure which is captured by the vanishing of its third Stiefel-Whitney class $w_3(Q) = 0$. Moreover it is seen that in order to not have this anomaly, non-Spin D-branes must support a $U(1)$ gauge field with *half-integral* flux. Sadly these fluxes create problems in the stabilisation procedures which rely on more than one non-perturbative correction to the superpotential. Turning on Freed-Witten fluxes makes the instanton configuration (generating the non-perturbative effect) no more orientifold-invariant. This can be dealt with by adjusting the B_2-field in order to compensate these fluxes in such a way that the combination $\mathcal{F} = F_2 - B_2$ is still invariant (F_2 is the field on the D-brane). Since B_2 can be fixed only once, there cannot be more than one non-perturbative instantons contributions since these would be, indeed, not orientifold-invariant.

Solution:
The moduli stabilisation technique called "LARGE Volume Scenario" (LVS), which we are going to inspect later on, naturally solve the problem since it relies on just one non-perturbative effect which allows to fix the small blow-up mode (small Kähler mode which contributes negatively to the total volume mode).

(iii) D-term problem:

Problem:
In type IIB string theory, GUT or MSSM-like visible sectors are built through stacks of spacetime filling D7-branes wrapping holomorphic 4-cycles D_i in the compact internal space. These divisors (cycles which are of complex codimension one, i.e. 4-cycles for a Calabi-Yau manifold) are chosen to be rigid in order to avoid unwanted matter in the adjoint representation. Chiral matter is obtained at the intersection with a second stack of D7-branes via turning on an internal gauge flux $\mathcal{F}_i = \tilde{f}_i^k \hat{D}_k$ (where $\hat{D}_i$ are the Poincaré duals to the 4-cycles D_i). Gauge fluxes generate a Fayet-Iliopoulos (FI) term ξ_i which will depend on Kähler moduli as:

$$\xi_i = \frac{1}{4\pi\mathcal{V}} \int_{\mathcal{Y}} \hat{D}_i \wedge J \wedge \mathcal{F}_i = \frac{t^j \tilde{f}_i^k}{4\pi\mathcal{V}} \int_{\mathcal{Y}} \hat{D}_i \wedge \hat{D}_j \wedge \hat{D}_k = \frac{1}{4\pi\mathcal{V}} q_{ij} t^j \tag{3.2.2}$$

where $\mathscr{V}$ is the volume of the Calabi-Yau $\mathscr{Y}$ and we have defined the $U(1)$ charges $q_{ij} = k_{ijk}\tilde{f}_i^k$ (the i is not summed here) of the j-th Kähler modulus T_j induced by the flux on the i-th divisor D_i. If matter fields φ_j charged under the $U(1)$, with charges c_{ij}, are introduced, then there will be a non-zero D-term potential of the form:

$$V_D = \frac{g_i^2}{2}\left(\sum_j c_{ij}|\varphi_j|^2 - \xi_i\right)^2 \tag{3.2.3}$$

This contribution to the scalar potential V dominates over the F-term potential V_F and could destabilise the process of stabilisation of the volume mode τ_{vis} of the 4-cycle wrapped by the brane supporting the visible sector. If there are no visible sector singlets φ_j which can balance and partially cancelling the Fayet-Iliopuolos term, the supersymmetric locus V_D is realised by imposing the vanishing of $\xi_i = 0$. This, however, will result in the shrinking of some rigid 4-cycles to zero size and this singular regime is poorly understood in terms of α' and quantum corrections.

Solution:
If the visible sector is wrapped on a non-diagonal *del Pezzo*[2] rigid divisor, the FI-term will depend on a linear combination of various Kähler moduli and requiring the vanishing of the FI term will just fix the corresponding combination of Kähler moduli, making it possible to avoid the shrinking of any cycle.

(iv) Stabilisation within Kähler cone and phenomenological requirements:

Problem:
When in §2.2.6 we analysed the Kähler structure deformations, we said that in order for the resulting metric $g_{i\bar{j}} + \delta g_{i\bar{j}}$ to be positive definite, the Kähler form J has to satisfy, for a Calabi-Yau 3-fold, a set of 3 constraints defining a subset of $\mathbb{R}^{h^{(1,1)}}$ which goes under the name of Kähler cone:

$$\int_{\mathscr{C}} J > 0; \qquad \int_{\mathscr{S}} J \wedge J > 0; \qquad \int_{\mathscr{Y}} J \wedge J \wedge J > 0 \tag{3.2.4}$$

where these relations must be true for each complex curve $\mathscr{C}$ and surface $\mathscr{S}$ on the Calabi-Yau $\mathscr{Y}$. Any stabilisation process should then respect the Kähler cone, namely it should fix the moduli within it. However, when combined with the requirement to get phenomenologically viable scales, these constraints are not trivial to satisfy.

Solution:

[2] Del Pezzo surfaces are two-dimensional complex variety which admits a Kähler metric with strictly positive Ricci curvature. On a Calabi-Yau, del Pezzo submanifolds are arbitrarily contractible to a point without affecting the rest of the geometry. Diagonal del Pezzo divisors appear in the intersection form in a diagonal way. For such D_i, in fact, it can be found a basis in which only the self-intersection number k_{ijk} with $i = j = k$ is not zero. These can be thought of as local effects and they will appear in the volume form of the Calabi-Yau in a diagonal way. Also they arise as resolutions of point-like singularities. Non-diagonal del Pezzo, instead, can't be brought to a diagonal form by a suitable choice of basis. They are indeed global effects and will appear in a non-diagonal form in the Calabi-Yau volume mode.

There is no general solution to this problem, it relies, in fact, on the model considered, the brane set-up and the fluxes chosen.

In our explicit models of the next chapter, we are not going to construct the Standard Model (or GUT or MSSM sectors), but we are only going to focus on the brane and flux set-ups to get the right hidden gauge sector which will kinetically couple to the Standard Model's electroweak gauge boson. Our Calabi-Yau manifold will be chosen with a given volume mode $\mathcal{V}$ in function of the 4-cycle volumes: $\mathcal{V} = \mathcal{V}(\tau_i)$ in a multiple-hole Swiss cheese form (see [CCQ08a; Cic10] for details). The 4-cycle where the Standard Model D-branes should be wrapped will not be included in $\mathcal{V}$ and this in turn will allow us to forget these global-like problems and focus more on our local model building.
That begin said, we can now explore the mechanisms to stabilise the Kähler moduli. In $\mathcal{N} = 1$ supergravity in 4-dimensions, the Kähler potential receives corrections at every order in perturbation theory, while the superpotential receives only non-perturbative corrections due to the non-renormalisation theorem [DS86]. The Kähler potential and superpotential then enjoy an expansion of the form:

$$\begin{cases} K = K_{\text{tree}} + \delta K_{(\alpha')} + \delta K_{(g_s)} \\ W = W_{\text{tree}} + \delta W_{\text{np}} \end{cases} \tag{3.2.5}$$

where $\delta K_{(\alpha')}$ are the perturbative corrections of the Kähler potential in the (α')-expansion, while $\delta K_{(g_s)}$ are the quantum loop corrections in the string coupling g_s. The latter are, in general, the less understood corrections which can be employed in the stabilisation procedure. For general Calabi-Yau manifolds, these functions are not explicitly known, however, there are some considerations which can be made in order to, at least, come up with a functional dependence of $\delta K_{(g_s)}$ with respect to the Kähler moduli.

3.2.1 Non-perturbative Corrections to the Superpotential

Non-perturbative effects play a very important role in string theory and they have been studied since the '80s (see [Din+86; Din+87; Wit96; Wit00; BW06; DIN85; Din+85; Bur+96; Gor+04]), where these corrections can be seen to arise in two different ways:

- Via D7-branes wrapping 4-cycles in the Calabi-Yau undergoing gaugino condensation;
- Via Euclidean D3-brane instantons.

Either way, the form of the non-perturbative correction is given by (in M_P-units):

$$\delta W_{\text{np}}(T_i, U^\alpha) = \sum_i A_i(U^\alpha) e^{-a_i T_i} \tag{3.2.6}$$

where a_i is 2π for D3-brane instantons, and $a_i = 2\pi/N$ for a stack of N D7-branes undergoing gaugino condensation. The sum is over 4-cycles generating the non-perturbative contributions to the superpotential W and $A_i(U^\alpha)$ are threshold effects that can depend on the complex structure moduli U^α. There also may be higher instanton effects of the form: $e^{-2a_i T_i}, e^{-3a_i T_i}, \ldots$, which can be neglected in the regime $a_i \tau_i \gg 1$. From the form of the F-term potential (2.3.25) we can see that the contribution of δW_{np} generates a scalar potential for the Kähler structure moduli as (we write here $W_{\text{tree}} \equiv W_0$ and $\delta W_{\text{np}} \equiv \delta W$ for notational convenience, moreover we write $K_0 \equiv -2\ln(\mathcal{V})$ for the classical Kähler potential for the Kähler structure moduli space):

$$\begin{aligned}
\delta V_{\text{np}}(T_i) =& e^K \left(K_0^{i\bar{j}} (\partial_i W + W \partial_i K_0)(\partial_{\bar{j}} \overline{W} + \overline{W} \partial_{\bar{j}} K_0) - 3|W|^2 \right) \\
=& e^K \Big[K_0^{i\bar{j}} \left(\partial_i W \partial_{\bar{j}} \overline{W} + (\partial_i W) \overline{W} \partial_{\bar{j}} K_0 + (\partial_{\bar{j}} \overline{W}) W \partial_i K_0 \right) \\
&+ |W|^2 (K_0^{i\bar{j}} \partial_i K_0 \partial_{\bar{j}} K_0 - 3) \Big] \\
\boxed{1} \rightarrow =& e^K K_0^{i\bar{j}} \Big(\partial_i (W_0 + \delta W) \partial_{\bar{j}} (\overline{W}_0 + \delta \overline{W}) + (\partial_i (W_0 + \delta W))(\overline{W}_0 + \partial \overline{W}) \partial_{\bar{j}} K_0 \\
&+ (\partial_{\bar{j}} (\overline{W}_0 + \delta \overline{W}))(W_0 + \delta W) \partial_i K_0) \Big) \\
\boxed{2} \rightarrow =& e^K K_0^{i\bar{j}} \left(\partial_i \delta W \partial_{\bar{j}} \delta \overline{W} + (\partial_i \delta W) W \partial_{\bar{j}} K_0 + (\partial_{\bar{j}} \delta \overline{W}) W \partial_i K_0 \right) \\
\boxed{3} \rightarrow =& e^K K_0^{i\bar{j}} \left(A_i a_i \overline{A}_j a_j e^{-a_i T_i - a_j \overline{T}_j} - (A_i a_i \overline{W} e^{-a_i T_i} \partial_{\bar{j}} K_0 + \overline{A}_j a_j W e^{-a_j \overline{T}_j} \partial_i K_0) \right)
\end{aligned} \tag{3.2.7}$$

where in $\boxed{1}$ we have used the no-scale property of the tree-level Kähler potential and expanded the superpotential as $W = W_0 + \delta W$, in $\boxed{2}$ we have used the fact that the tree-level superpotential W_0 does not depend on the Kähler moduli so that $\partial_i W_0 = 0$ and finally in $\boxed{3}$ we have used the expansion of δW as in (3.2.6). Notice that the exponential is in function of the total Kähler potential K, while in the other instances it appears K_0. This is simply because the Kähler potential is block-diagonal, so that $\partial_i K = \partial_i K_0$.

In the LARGE volume scenario, the non-perturbative correction to the superpotential allows to fix the 4-cycle volume of a small blow-up resolving a pointlike singularity, which is required to exists by the LARGE volume claim [CCQ08a].

3.2.2 Perturbative (α')-Corrections to the Kähler Potential

The classical no-scale structure of the scalar potential does not allow to fix the Kähler moduli. However, this structure is violated by stringy corrections in (α') as was first conjectured in [GKP02] and proved by Becker et al. [Bec+02]. In [Bec+02] the authors managed to explicitly find these corrections to the scalar potential (sometimes called BBHL-stringy corrections) up to orders $O(\alpha'^3)$, in this way, the scalar potential will lose the no-scale property. Following their work, the relevant terms in the 10-dimensional type IIB supergravity action are:

$$S = -\frac{1}{2k_{10}^2} \int d^{10}x \sqrt{-G}\, e^{-2\phi} \left(R + 4\partial_M \phi \partial^M \phi + (\alpha')^3 c_1 J_0 \right) \tag{3.2.8}$$

where $c_1 = \zeta(3)/3 \cdot 2^{11}$, the $\zeta(z)$ is the Riemann ζ-function and the higher order interaction is [FKT02]:

$$J_0 = 3 \cdot 2^8 \left(R^{HMNK} R_{PMNQ} R_H{}^{RSP} R^Q{}_{RSK} + \frac{1}{2} R^{HKMN} R_{PQMN} R_H{}^{RSP} R^Q{}_{RSK} \right) \tag{3.2.9}$$

Along with this contribution coming from the structure of the Green-Schwarz 4-point massless string scattering amplitude in the NS-NS sector, there is a contribution due to the fact that the

β-function receives 4-loop contributions which do not vanish in Ricci-flat spaces (as opposed to 1-, 2- and 3-loop contributions)[GVZ86; GW73b]. The term that must be added is of the form:

$$\frac{1}{k_{10}^2}\int d^{10}x\sqrt{-G}\,e^{-2\phi}(\alpha')^3(\nabla^2\phi)Q \tag{3.2.10}$$

where Q is given by:

$$Q = R_{IJ}{}^{KL}R_{KL}{}^{MN}R_{MN}{}^{IJ} - 2R_{I}{}^{K}{}_{J}{}^{L}R_{M}{}^{I}{}_{N}{}^{J} \tag{3.2.11}$$

As pointed out in [Can+91], it can be recognized that for a Calabi-Yau 3-fold $\mathscr{Y}$, it is true that $Q\sqrt{-g}\,d^6x = 12(2\pi)^3c_3(T\mathscr{Y})$, where $c_3(T\mathscr{Y})$ is the third Chern class of the tangent bundle of $\mathscr{Y}$, also, by making use of the Gauss-Bonnet theorem (by noticing that for a complex n-dimensional manifold, the Euler class of the real tangent space $e(T\mathscr{Y}_{\mathbb{R}})$ is equals to the n-Chern class $c_n(T\mathscr{Y})$), it can be shown that:

$$\int_{\mathscr{Y}} d^6x\,\sqrt{-g}\,Q = 12(2\pi)^3\,\chi_E(\mathscr{Y}) \tag{3.2.12}$$

where $\chi_E(\mathscr{Y})$ is the Euler characteristic of $\mathscr{Y}$.

Using all these ingredients Becker et al. [Bec+02] came up with the quantum-corrected Kähler potential:

$$K = -2\ln\left(\mathscr{V} + \frac{\xi}{2g_s^{3/2}}\right) - \ln\left(\frac{2}{g_s}\right) + K_{\text{cs}} \tag{3.2.13}$$

where:

$$\xi = -\frac{\chi_E(\mathscr{Y})\zeta(3)}{2(2\pi)^3} \tag{3.2.14}$$

By plugging (3.2.13) into the F-term potential (2.3.25) we get a new term involving the volume mode $\mathscr{V}$ of the form (defining $\hat{\xi} \equiv \xi/g_s^{3/2}$):

$$\delta V_{(\alpha')} = 3e^K\hat{\xi}\,\frac{(\hat{\xi}^2 + 7\hat{\xi}\mathscr{V} + \mathscr{V}^2)}{(\mathscr{V}-\hat{\xi})(2\mathscr{V}+\hat{\xi})^2}|W|^2 \sim e^K\frac{3}{4}\frac{\hat{\xi}}{\mathscr{V}}|W|^2 \tag{3.2.15}$$

where in the last step we have shown the asymptotic behaviour of this term in the decompactification limit, namely when the volume of the Calabi-Yau is sent to infinity $\mathscr{V}\to\infty$. Since we will be using the LARGE volume scenario stabilisation scheme, in our scalar potential we will use the asymptotic form of the above expression.

The interplay between non-perturbative corrections to W and (α')-corrections to K will in general fix all the small blow-up modes and the total volume mode of the Calabi-Yau manifold. What are left unfixed are, however, those cycles which corresponds to fibrations and which are "big" with respect to the blow-ups (but clearly small with respect to the exponentially large volume). In order to fix these last cycle volumes, we need to take into account (g_s)-corrections, namely string loop corrections.

3.2.3 Perturbative (g_s)-Corrections to the Kähler Potential

The explicit form of these corrections for Calabi-Yau manifolds are still unknown, however, from their previous computation on $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ in [BHK05], Berg, Haack, and Pajer [BHP07] made an educated guess for the scaling of these loop corrections with the Kähler moduli and the dilaton S respecting their previous computation on the torus. Their conclusion is that there should be two kind of perturbative corrections to be taken into account:

$$\delta K_{(g_s)} = \delta K^{KK}_{(g_s)} + \delta K^{W}_{(g_s)} \tag{3.2.16}$$

The first one $\delta K^{KK}_{(g_s)}$ comes from the exchange of Kaluza-Klein modes between D7-branes (or O7-planes) and D3-branes (or O3-planes). The conjectured parametric form given by [BHP07] can be written as:

$$\delta K^{KK}_{(g_s)} \sim \sum_{i=1}^{h^{(1,1)}} \frac{C_i^{KK}(U^\alpha)\cdot m_{KK}^{-2}}{\mathfrak{Re}(S)\mathcal{V}} \sim \sum_{i=1}^{h^{(1,1)}} \frac{C_i^{KK}(U^\alpha)\cdot(a_{ij}t^j)}{\mathfrak{Re}(S)\mathcal{V}} \tag{3.2.17}$$

where $\mathcal{V}$ is the volume of the Calabi-Yau, m_{KK}^{-2} is the mass of the exchanged mode, $C_i^{KK}(U^\alpha)$ are unknown functions of the complex structure moduli which parametrise our ignorance regarding the true form of the correction, and $(a_{ij}t^j)$ is a linear combination of the 2-cycle basis t^j giving the path along which the mode between the D7-brane and the D3-brane propagates. The second passage in (3.2.17) is justified by noticing that for Kaluza-Klein modes $m_{KK}^{-2} \sim t$ (which is nothing but the correspondence between (energy)$^{-2}$ of the mode and the (space)^2 travelled).

The second term in (3.2.16), namely $\delta K^{W}_{(g_s)}$, comes from the exchanging of winding modes between intersecting stacks of D7-branes. From the fact that $m_{KK}^{-2} \sim t$, for winding modes, due to T-duality, it will be true that $m_W^2 \sim t^{-1}$. In this way, we can write an analogous expression for the loop correction given by exchanging of winding modes:

$$\delta K^{W}_{(g_s)} = \sum_{i=1}^{h^{(1,1)}} \frac{C_i^{W}(U^\alpha)\cdot m_{W}^{-2}}{\mathcal{V}} \sim \sum_{i=1}^{h^{(1,1)}} \frac{C_i^{W}(U^\alpha)}{\mathcal{V}\cdot(a_{ij}t^j)} \tag{3.2.18}$$

where $a_{ij}t^j$ gives the 2-cycle over which the D7-branes are intersecting and again, $C_i^W(U^\alpha)$ are unknown functions of the complex structure moduli and, as $C_i^{KK}(U^\alpha)$, should be regarded as free parameters since the complex structure moduli U^α are stabilised at tree-level thanks to the Gukov-Vafa-Witten superpotential generated by background fluxes.

Extended No-Scale Structure and Correction to the Scalar Potential

The (g_s)-corrections could, in principle, spoil the stabilisation procedure obtained by using non-perturbative and (α')-corrections. However, what can be shown is that their leading contribution to the scalar potential vanishes, extending further the no-scale property at tree level to one more order, and for that reason it is referred to as the *Extended No-Scale Structure*. This is true for perturbative loop corrections $\delta K_{(g_s)}$ which are homogeneous functions of degree $n = -2$ in the 2-cycle volumes [CCQ08b]. This leading order cancellation is then crucial to render $\delta V_{(g_s)}$ subdominant with respect to $\delta V_{(\alpha')}$, in fact, the first non-vanishing contribution to the scalar potential of the (g_s)-corrections above are found to be [CCQ08b] (using the fact that the axio-dilaton at this stage is fixed at $\langle (S+\bar{S})/2\rangle = g_s^{-1}$):

$$\delta V^{\text{1-loop}}_{(g_s)} = \frac{|W|^2}{\mathcal{V}^2}\sum_i \Big((g_s C_i^{KK})^2 (K_0)_{i\bar{i}} - 2\delta K^{W}_{(g_s)} \Big) \tag{3.2.19}$$

Also, there can be given a field theory interpretation of the above potential through the *Coleman-Weinberg Potential* in supergravity, for details see [CCQ08b].

3.3 KKLT Scenario

In order to give a first example of the moduli stabilisation process, we consider the first model that has been proposed with all the moduli supersymmetrically fixed and with a suitable uplift to get a de Sitter vacuum: the *KKLT Scenario* [Kac+03]. The set-up is a warped type IIB flux compactification on a Calabi-Yau orientifold $\mathscr{Y}$ (more generally F-theory) with the number of Kähler moduli given by:

$$h^{(1,1)} = h_{+}^{(1,1)} = 1 \tag{3.3.1}$$

Following §3.1 we turn on background fluxes with respect to the NS-NS 3-form H_3 and R-R 3-form F_3 in such a way to generate the tree-level superpotential of the Gukov-Vafa-Witten form:

$$W_{\text{tree}} \sim \int_{\mathscr{Y}} G_3 \wedge \Omega \tag{3.3.2}$$

with $G_3 = F_3 - iSH_3$ imaginary self-dual ($\star G_3 = iG_3$) due to the background fluxes which also serve as sources for a warp factor for the metric that in general will be given by (3.1.7).

The tree-level Kähler potential can be inferred by dimensional reducing the 10-dimensional action (for details one can consult the appendix of [GKP02]), which will give:

$$K_{\text{tree}} = -3\ln((\rho+\bar{\rho})) - 2\ln(S+\bar{S}) - \ln\left(-i\int_{\mathscr{Y}} \Omega \wedge \bar{\Omega}\right) \tag{3.3.3}$$

where ρ is the radial Kähler modulus which governs the volume of the Calabi-Yau. Also the scalar potential will be of the no-scale form:

$$V = V_F = e^K \left(K^{S\bar{S}} D_S W D_{\bar{S}} \overline{W} + K^{U^\alpha \bar{U}^\beta} D_{U^\alpha} W D_{\overline{U}^\beta} \overline{W} - 3|W|^2\right) \tag{3.3.4}$$

where U^α are the complex structure moduli with $\alpha = 1, \ldots, h_{-}^{(2,1)}$. This allows to supersymmetrically fix the axio-dilaton S and all the complex structure moduli by imposing:

$$\begin{cases} D_S W|_{\langle S \rangle, \langle U^\alpha \rangle} = 0 \\ D_{U^\alpha} W|_{\langle S \rangle, \langle U^\alpha \rangle} = 0 \end{cases} \tag{3.3.5}$$

Since in this model there is only one Kähler modulus ρ, it means that the volume $\mathcal{V}$ is regulated only by the real part of ρ, namely $\mathcal{V} = \mathcal{V}(\rho+\bar{\rho})$. After the fixing of S and U^α, it remains a flat direction in $\mathcal{V}$ which need to be uplifted to resolve the vacuum degeneracy. The stabilisation of $\mathcal{V}$ can be done via non-perturbative corrections to the superpotential, as presented in §3.2.1, thanks to D7-branes wrapping the 4-cycle with volume ρ. The non-perturbative correction will be given by:

$$\delta W_{\text{np}} = Ae^{-a\rho} \tag{3.3.6}$$

with A and a constant such that $a\rho \gg 1$ in order to avoid higher order instanton corrections. At a supersymmetric vacuum we have: $D_\rho W = 0$, namely:

$$D_\rho W|_{\langle\rho\rangle} = (\partial_\rho W + W\partial_\rho K)|_{\langle\rho\rangle} = -aAe^{-a\langle\rho\rangle} + (W_0 + Ae^{-a\langle\rho\rangle})\frac{-3}{\langle\rho+\bar{\rho}\rangle} = 0 \tag{3.3.7}$$

in such a way that we can write the tree-level superpotential as:

$$W_0 = Ae^{-a\langle\rho\rangle}\left[-1+\frac{\langle\rho+\bar{\rho}\rangle a}{3}\right] \tag{3.3.8}$$

And this allows to get an Anti-de Sitter vacuum:

$$\begin{aligned} V_{\mathrm{AdS}} =&(-3e^{K}|W|^2)|_{\langle\rho\rangle} \\ =&-3e^{-3\ln\langle\rho+\bar{\rho}\rangle}|W_0+Ae^{-a\langle\rho\rangle}|^2 \\ =&-3\left(\frac{1}{\langle\rho+\bar{\rho}\rangle^3}\right)\left|\frac{\langle\rho+\bar{\rho}\rangle}{3}aAe^{-a\langle\rho\rangle}\right|^2 \\ =&-\frac{a^2A^2e^{-2a\langle\rho\rangle}}{3\langle\rho+\bar{\rho}\rangle} \end{aligned} \tag{3.3.9}$$

We can see that all the moduli have been fixed while preserving supersymmetry. In order to trust the supergravity approximation the modulus ρ is required to be large $\rho \gg 1$, making also the (α')-corrections to the Kähler potential under better control. These conditions are met if background fluxes are turned on such that $W_0 \ll 1$. This, however, is not "natural" and in order to get to it, some fine tuning is needed. If too much flux is turned on, the tadpole cancellation condition (3.1.14) can be satisfied provided that one introduce one or more $\overline{D3}$-branes. These leave an extra energy density term in the potential slightly breaking supersymmetry and which allowed [Kac+03] to uplift their vacuum to a metastable de Sitter minimum.

3.4 Calabi-Yau Embedded in $\mathbb{P}^4_{(1,1,2,2,6)}$: 2-Parameter K3 Fibration

Calabi-Yau manifolds can be constructed in different ways and in general lots of algebraic geometry techniques are required. Thanks to the Yau's theorem [Yau77; Yau78] we are allowed to construct manifolds of $SU(3)$ holonomy (so that the supersymmetries are reduced to $\mathcal{N}=2$ for type II string theories and to $\mathcal{N}=1$ for heterotic strings) by looking for manifolds of vanishing first Chern class (which are much easier to construct than that of $SU(3)$ holonomy). As Candelas et al. [Can+85] presented in their original work when they first managed to connect Calabi-Yau manifolds to compactifications of string theories, the easiest Kähler manifold that one can think about is the complex projective space $\mathbb{P}^n$, which is nothing but $(n+1)$-dimensional complex plane $\mathbb{C}^{n+1}$ subject to the equivalence relation $(z^i \sim w^i) \iff z^i = \lambda w^i$ for a complex number $\lambda \in \mathbb{C}$. Also, every complex dimensional subspace of $\mathbb{P}^n$ defined as the vanishing locus of analytic functions $\phi_\alpha(z^i)$ is a Kähler manifold due to the fact the induced metric from $\mathbb{P}^n$ is Kähler. This means that to construct a three-dimensional Kähler manifold, one can consider the 4-dimensional projective space $\mathbb{P}^4$ and define a hypersurface embedded in this projective space as the zero locus of a homogeneous polynomial. The simplest case is the *Quintic* which is the manifolds defined as the subspace of $\mathbb{P}^4$ by:

$$z_1^5+z_2^5+z_3^5+z_4^5+z_5^5=0 \tag{3.4.1}$$

This space has complex dimension three and is in particular a Calabi-Yau manifold since its first Chern class vanishes. As Candelas et al. [Can+85] already noted, one can generalise this construction to k vanishing polynomials of degree $d_1, ..., d_k$ in $\mathbb{P}^{k+3}$, in such a way that the simultaneous vanishing locus of all these polynomials will be a three-complex-dimensional Kähler manifold. It is found that the first Chern class vanishes if and only if the sum of the degree of the polynomials is the dimension of the projective space plus one: $d_1 + ... + d_k = k + 4$. Besides, since a linear subspace of $\mathbb{P}^n$ is $\mathbb{P}^{n-1}$, there are only five cases one needs to study in order to be able to construct a Calabi-Yau manifold in this way.

Now, over the last decades the techniques to construct Calabi-Yau manifolds have grown and datasets of explicit constructions have been developed, ranging from *Complete-Intersection Calabi-Yaus* (CICYs) to *Elliptically Fibred manifolds over Toric Bases* and *Hypersurfaces in Toric Ambient Spaces*. The CICYs three-folds were for the first time classified by Candelas et al. [Can+85], while the CICYs four-folds by Gray, Haupt, and Lukas [GHL13]. Four-folds are clearly useful in the context of F-theory, namely for the construction of interesting string vacua based on elliptically fibred Calabi-Yau. All the smooth toric bases supporting elliptically fibred Calabi-Yau manifolds 3-folds were classified and constructed by Morrison and Taylor [MT12]. Finally, Hypersurfaces in Toric ambient spaces were constructed by Kreuzer and Skarke [KS02] and represents the biggest known database for Calabi-Yau three-folds. Moreover, [Alt+15] provided an online database for these toric based Calabi-Yau manifolds with a detailed inventory of all the relevant quantities needed by physicists, like topological data and geometric information. Lots of algebraic geometry notions are needed to construct and study Calabi-Yau manifolds in toric ambient spaces and these can be gathered in the appendix A of [Alt+15].

The Calabi-Yau manifold we are going to consider in this section is defined as the vanishing locus of a homogeneous polynomial in a weighted projective space (see appendix B). This example was analysed byCandelas et al. [Can+94], and the mathematical context to realise the mirror symmetry used to study this model are indeed ambient toric spaces. Calabi-Yau manifolds are described in terms of reflexive polyhedra [KS02] allowing the study of all the useful topological/geometrical characteristics needed in the applications such as the hodge numbers, Chern classes, divisors, intersections, Kähler cones, etc.

We consider then a Calabi-Yau three-fold $\mathscr{Y}$ which is obtained by resolving singularities of a degree 12 hypersurface $\hat{\mathscr{Y}}$ embedded in the weighted projective space $\mathbb{P}^4_{(1,1,2,2,6)}$.

Remark

We recall that a hypersurface defined as the zero locus of a homogeneous polynomial p in the weighted projective space $\mathbb{P}^n_{(a_0,...,a_n)}$ is a Calabi-Yau if the degree of the polynomial d is equals to the sum of the weights: $d = a_0 + \cdots + a_n$. Also we remind that weighted projective spaces in general are not smooth as opposed to projective spaces. This can be easily seen considering a simple example: in $\mathbb{P}^3_{(1,1,2,3)}$ we have the equivalence relation relating $(z_0, z_1, z_2, z_3) \simeq (\lambda z_0, \lambda z_1, \lambda^2 z_2, \lambda^3 z_3)$ for $\lambda \in \mathbb{C}$. Taken $\lambda = -1$, in a neighbourhood of the point $(0, 0, 1, 0)$, it will be true that $(z_0, z_1, 1, z_3) \simeq (-z_0, -z_1, 1, -z_3)$, meaning that there will be a cyclic quotient singularity, namely a $\mathbb{Z}_2$ identification. In general a weighted projective space can be locally viewed as a quotient of the projective space by a suitable group action, leaving singularities which may intersect an embedded surface.

A typical defining polynomial for the 12 degree hypersurface $\hat{\mathscr{Y}}$ is [Can+94]:

$$p = z_0^{12} + z_1^{12} + z_2^6 + z_3^6 + z_4^2 \tag{3.4.2}$$

There is a curve of singularities described by:

$$\begin{cases} z_0 = z_1 = 0 \\ z_2^6 + z_3^6 + z_4^2 = 0 \end{cases} \tag{3.4.3}$$

and to resolve them, the base locus $z_0 = z_1 = 0$ must be blown-up. The curve of singularities is replaced by a divisor E on the resolved hypersurface $\mathscr{Y}$, namely every singular point is replaced by a sphere $\mathbb{P}_1$. In order to study the moduli space of $\mathscr{Y}$, it is considered the mirror manifold defined as the family of Calabi-Yau of the form $\{p = 0\}/G$ where:

$$p = z_0^{12} + z_1^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi z_0 z_1 z_2 z_3 z_4 z_5 - 2\phi z_0^6 z_1^6 \tag{3.4.4}$$

with the group G as in [Can+94]. In terms of 2-cycles, the volume form can be expressed as:

$$\mathscr{V} = t_1 t_2^2 + \frac{2}{3} t_2^3 \tag{3.4.5}$$

where we recall that if $\mathscr{Y}$ is a Calabi-Yau manifold, we decompose its Kähler form J into a basis of harmonic $(1,1)$-form $\{\hat{D}_i\}_{i=1,\dots,h^{(1,1)}}$ with $h^{(1,1)} = \dim(\mathscr{H}^{(1,1)}(\mathscr{Y}))$ which thanks to the Hodge decomposition theorem (2.1.3) are the same as the elements of the Dolbeault Cohomology group $H^{(1,1)}_{\bar{\partial}}(\mathscr{Y})$. We then write $J = t^i \hat{D}_i$ and the moduli space $\mathscr{M}_{\text{ks}}$ defined by Kähler structure deformations for type IIB orientifolds with $h_-^{(1,1)} = 0$ and D3/D7-branes, is spanned by the moduli $T_i = \tau_i + i b_i$ with τ_i related to t_i by $\tau_i = \partial \mathscr{V} / \partial t^i$. The 4-cycle moduli can then be written as functions of t_i in the following way:

$$\begin{aligned} \tau_1 &= \frac{\partial \mathscr{V}}{\partial t^1} = t_2^2 \\ \tau_2 &= \frac{\partial \mathscr{V}}{\partial t^2} = 2t_1 t_2 + 2t_2^2 \end{aligned} \tag{3.4.6}$$

Inverting these relations, we can express $t_i(\tau_j)$ as:

$$\begin{aligned} t_1 &= \frac{\tau_2 - 2\tau_1}{2\sqrt{\tau_1}} \\ t_2 &= \sqrt{\tau_1} \end{aligned} \tag{3.4.7}$$

By substituting these expressions in the volume form, we see that the Calabi-Yau is a $K3$ fibration with base volume given by τ_1 and fibre modulus t_1. In fact in the large anisotropic limit $\tau_1 \ll \tau_2$, we get:

$$\mathscr{V} = \frac{1}{2}\sqrt{\tau_1}(\tau_2 - \frac{2}{3}\tau_1) \simeq \frac{1}{2}\sqrt{\tau_1}\tau_2 = t_1 \tau_1 \tag{3.4.8}$$

Also, the hodge numbers are given by $(h^{(1,1)}, h^{(2,1)}) = (2, 128)$, so that from (2.2.13) the Euler characteristic χ is:

$$\chi = 2(h^{(1,1)} - h^{(2,1)}) = -252 \tag{3.4.9}$$

The Kähler potential at tree level will be given by:

$$K_{\text{tree}} = -2\ln(\mathscr{V}) - \ln(S + \bar{S}) - \ln\left(-i \int_{\mathscr{Y}} \Omega \wedge \bar{\Omega}\right) \tag{3.4.10}$$

with S the axio-dilaton and Ω the holomorphic $(3,0)$-form of the Calabi-Yau defined in terms of complex structure moduli U^α with $\alpha = 1,...,h_-^{(2,1)}$. Turning on fluxes for the NS-NS 3-form $H_3 = dB_2$ and the R-R 3-form $F_3 = dC_2$, we can fix the axio-dilaton and complex structure moduli to their vacuum expectation values $\langle U^\alpha \rangle$ and $\langle S \rangle$, since a tree-level superpotential of the Gukov-Vafa-Witten form W_0 is generated. The tree-level Kähler potential becomes:

$$K_{\text{tree}} = -2\ln(\mathcal{V}) - \ln(2/g_s) + K_{\text{cs}} \tag{3.4.11}$$

with g_s and K_{cs} as in (3.1.40). The Kähler metric for the Kähler structure deformation moduli space in the anisotropic limit $\tau_1 \ll \tau_2$ will be:

$$(K_{\text{tree}})_{ij} := \frac{\partial^2 K_{\text{tree}}}{\partial T^i \partial T^j} = \frac{1}{4}\frac{\partial^2 K_{\text{tree}}}{\partial \tau^i_i \partial \tau^j} \simeq \frac{1}{4}\begin{pmatrix} \tau_1^{-2} & 0 \\ 0 & 2\tau_2^{-2} \end{pmatrix} \tag{3.4.12}$$

with its inverse given by:

$$(K_{\text{tree}})^{ij} = 4\begin{pmatrix} \tau_1^2 & 0 \\ 0 & \tau_2^2/2 \end{pmatrix} \tag{3.4.13}$$

As it is, this model does not allow the existence of an exponentially large volume, this is because it does not satisfy the requirements of the LARGE volume scenario stabilisation framework [CCQ08a]. In fact this Calabi-Yau does not admit a small blow-up resolving a pointlike singularity, which is needed to achieve the exponentially large volume. This is easily seen as follows. In order to stabilise the Kähler moduli we consider (α')-corrections and non-perturbative corrections due to gaugino condensation on stack of D7-branes wrapped along the divisors D_1 and D_2 (with $vol(D_1) = \tau_1$ and $vol(D_2) = \tau_2$). Since $\tau_1 \ll \tau_2$, we can neglect the contributions of the condensate on the stack wrapping τ_2, so that we will have the following corrections:

$$\begin{cases} W &= W_0 + \delta W_{(np)} = W_0 + A_1 e^{-a_1 T_1} \\ \\ K &= K_{\text{tree}} + \delta K_{(\alpha')} = -2\ln\left(\mathcal{V} + \dfrac{\xi}{2g_s}\right) - \ln\left(\dfrac{2}{g_s}\right) + K_{\text{cs}} \end{cases} \tag{3.4.14}$$

As [CCQ08a] explicitly showed, in the limit of large volume $\mathcal{V} \gg 1$ the F-term scalar potential will take the form:

$$V \simeq \frac{4}{\mathcal{V}^2}[(a_1\tau_1 e^{-a_1\tau_1} - W_0)a_1\tau_1 e^{-a_1\tau_1}] + \frac{3}{4}\frac{\xi}{\mathcal{V}^3}W_0^2 \tag{3.4.15}$$

which in every case $W_0 \sim O(1), W \ll 1$ and $W_0 = 0$, it does not allow exponentially large volume minima. However, with the addition of a small blow-up it can be obtained a K3-fibred Calabi-Yau with three Kähler moduli admitting a stabilisation at an exponentially large volume with all the moduli stabilised making also use of the string loop corrections. We will analyse this model in search for a good candidate to realise an explicit brane set-up implementing the sought parameter region.

Chapter 4

Hidden Gauge Bosons from Kinetic Mixing in Type IIB String Models

As we mentioned in the first chapter, there are various issues that the Standard Model fails to explain, in fact lots of models have been constructed in order to account for these problems under the framework of string theory. In string compactifications it is known that various $U(1)$ gauge sectors arise from D-branes wrapping holomorphic cycles in the compact space. These gauge bosons can couple to the Standard Model hypercharge $U(1)$ boson, giving rise to a so called *Kinetic Mixing* which could, in principle, give rise to measurable effects. The $U(1)$ gauge bosons are called *hidden* if the D-branes supporting them do not intersect the Standard Model's brane and if they are put on a suitable distance from it (more than the string length), in such a way that no matter will be charged under both the Standard Model $U(1)$ and the hidden $U(1)$s, making them indeed "hidden". The kinetic mixing parameter does not suffer from any kind of mass suppression, making this mechanism a very promising probe of high scale physics and its measurement could provide valuable clues for energy scales probably never accessible to colliders. In this regard, we would like to present a model in which kinetic mixing appears and explains the non-standard interaction proposed to describe the $\sim 2\sigma$ deviation from the Standard Model found by Dutta et al. [Dut+19] in the analysis of the *Coherent Elastic Neutrino-Nucleus Scattering* (CEνNS) energy and timing data extrapolated by the COHERENT collaboration [Aki+17]. The next section is devoted to present a summary of the COHERENT experiment and how the non-standard interaction could arise. After that, we will present a brief account of the Kinetic Mixing in the context of string phenomenology and the D-term potential (Fayet-Iliopoulos term) generated due to the Stückelberg mechanism employed to generate a mass for the hidden gauge boson. Finally, explicit models will be introduced and developed with a great care in following and respecting the moduli stabilisation processes outlined in the preceding chapter in order to reproduce the correct mass and coupling of the hidden gauge sector.

4.1 Non-Standard Interactions from CEνNS

The coherent elastic neutrino-nucleus scattering (CEνNS) was predicted for the first time by Freedman [Fre74] and finally experimentally discovered by the COHERENT collaboration [Aki+17] within a confidence level of $\sim 6.7\sigma$. The CEνNS is an important source for Beyond Standard Model (BSM) physics but also a potential disturbance background for dark matter detectors. There are in fact lots of potential disturbing neutrinos coming from different sources. Solar neutrinos, anti-neutrinos produced in radioactive decays in the Earth's mantle and core, atmospheric neutrinos coming from decays of cosmic rays, supernova and nuclear fission anti-neutrinos are all sources of a background flux of neutrinos which cannot be shielded against. The study of CEνNS is then of utmost importance in order to be able to handle in the right way dark matter experimental results.

At the Spallation Neutron Source (SNS) at Oak Ridge National Laboratory, a pulsed proton beam is produced, impinging on a dense target of Mercury. The proton-mercury interaction will produce pions π^- and π^+ which are quickly stopped in the target with a small probability of decay-in-flight. The former are captured by target nuclei while the latter experience a decay-at-rest with a production of mono-energetic 30 MeV muon-netruinos ν_μ called PROMPT neutrinos, via:

$$\pi^+ \to \mu^+ + \nu_\mu \tag{4.1.1}$$

The anti-muon μ^+ produced in this process travels for about a tenth of millimetre before decaying-at-rest with a production of $\bar{\nu}_\mu$ and ν_e referred to as DELAYED neutrinos, via:

$$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu \tag{4.1.2}$$

The neutrinos escaping the shielding monolith surrounding the mercury target, swamps a CEνNS detector composed of Na-doped Caesium Iodine (CsI[Na]) which will measure the number of recoil events due to the coherent interaction:

$$\nu + \text{Nucleus} \to \nu + \text{Nucleus} \tag{4.1.3}$$

driven by a long wavelength neutral boson Z_0. In a ~ 300 days of exposure of proton beams, the COHERENT collaboration measured a best fit of 134 ± 22 recoil events which favour the CEνNS process instead of the null hypothesis at the $\sim 6.7\sigma$.

Studying the full energy and timing distribution of these nuclear recoil events, Dutta et al. [Dut+19] tested deviations from pure Standard Model interactions by considering as example light mediators that couple to the Standard Model. The Standard Model cross section for the CEνNS process may be, indeed, modified by introducing a new mediating particle which couples to the neutrinos and either electrons or quarks. Taken this new boson to be Z'_μ, there will be a new interaction term in the Lagrangian of the form:

$$\mathscr{L} \supset Z'_\mu(g'_\nu \bar{\nu}_L \gamma^\mu \nu_L + g'_{f,v}\bar{f}\gamma^\mu f + g'_{f,a}\bar{f}\gamma^\mu\gamma^5 f) \tag{4.1.4}$$

where clearly g'_ν, $g'_{f,v}$ and $g'_{f,a}$ are the coupling constants associated to new physics and $g'_{f,v}$, $g'_{f,a}$ referred to the vector and axial couplings of the fermions f to the new Z' boson. It can also be considered the case where quarks/electrons couple to the new gauge boson Z' via a loop containing hidden sector particles χ, however, we will not be concerned about this case.

Considering the COHERENT set-up, [Dut+19] focus only on g'_e and g'_μ, associated to the couplings of the electron and muon to the Z'. Also, by imposing the couplings to be of the same order $g'_\nu = g'_u = g'_d = g'$ (with u for up-quark and d for down-quark), the Non-Standard Interaction (NSI) parameter will become:

$$\epsilon = \frac{g'^2}{2\sqrt{2}\,G_F(q^2 + M^2_{Z'})} \tag{4.1.5}$$

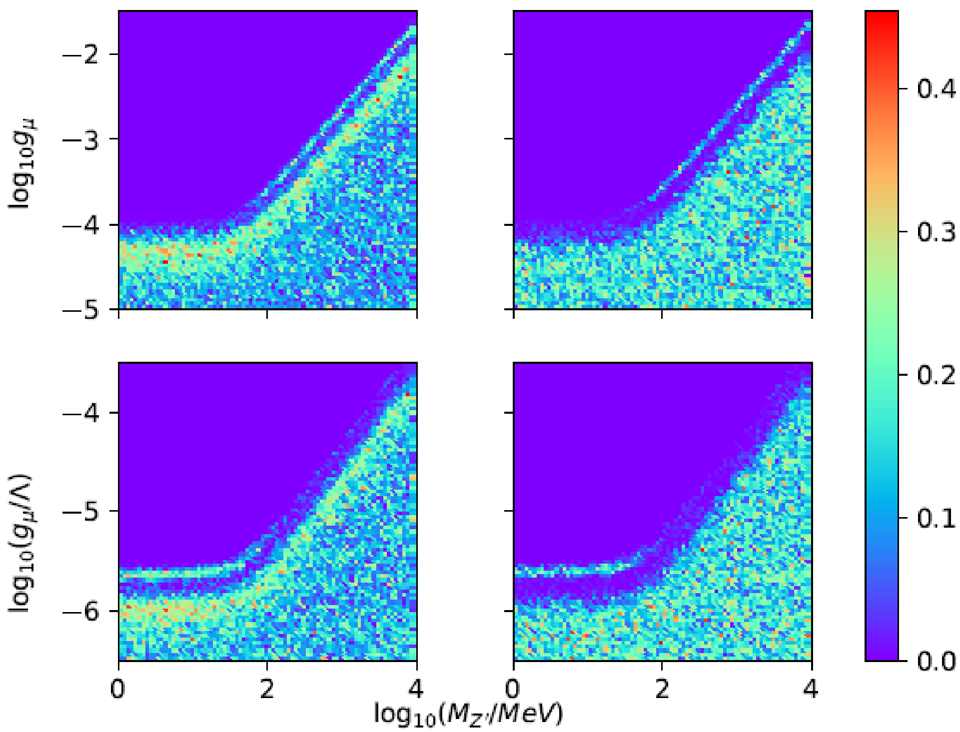


Figure 4.1: Probability densities found by Dutta et al. [Dut+19] in the coupling g' versus the mass of the hidden gauge boson $M_{Z'}$ space. The right probabilities were found considering only the energy data, while on the left we can appreciate the heat maps resulting from the combined energy and timing data. The second row of figures refers to a model in which the Standard Model interaction is modified due to a loop of hidden sector particles and we are not going to consider them in the present work.

where q^2 is the momentum transfer, $M_{Z'}$ is the mass of the new mediator and this (NSI) parameter ϵ enters the differential cross section of a neutrino scattering off of a target quark/electron as a shift in the vector-axial couplings:

$$(g_v, g_a) \mapsto (g_v, g_a) + (\epsilon, \epsilon) \tag{4.1.6}$$

Using the energy and timing distributions from the COHERENT data, Dutta et al. [Dut+19] performed a joint analysis in order to identify possible contributions from Beyond Standard Model physics. What they obtained is a probability in $\log_{10}(M_{Z'})$ vs $\log_{10}(g'_\mu)$ space (where g'_μ is the non-standard interaction parameter between the muon-neutrinos and the gauge boson Z') given by the following heat maps of figure (4.1). As we can see from the upper left heat map, for mediator masses $M_{Z'} \lesssim 10^{1.7}$ MeV, the coupling is constant $g'_\mu \sim 10^{-4.3}$, while for masses $M_{Z'} > 10^{1.7}$MeV, the coupling grows linearly in the mass as $g'_\mu \sim 10^{-6} M_{Z'}/$ MeV. This can be summarised as:

$$g'_\mu \sim \begin{cases} 10^{-4.3} & \text{for } M_{Z'} \lesssim 10^{1.7} \text{ MeV} \\ 10^{-6} \cdot \dfrac{M_{Z'}}{\text{MeV}} & \text{for } M_{Z'} > 10^{1.7} \text{ MeV} \end{cases} \tag{4.1.7}$$

We note that the above relations can be relaxed, since as we can see from figure (4.1) what we have is a probability density.

We are now going to see how in string theory a non-standard interaction via kinetic mixing can arise and how the above values of couplings and masses can be reproduced in an explicit compactification model.

4.2 Hidden Gauge Sectors in Type IIB

4.2.1 Kinetic Mixing in String Phenomenology

In the search for hidden sector particles with masses below the TeV scale and weak coupling to the Standard Model, superstring theories provide good extensions of the Standard Model accommodating hidden gauge sectors living on D-branes. Extra $U(1)$ gauge bosons, hidden from the Standard Model, are a prime candidate for Beyond Standard Model Physics. At low energies, the interaction between the hidden sector and the visible sector is primarily given through kinetic mixing (analysed by [Oku82; Hol86; BKM98] in QFT context). If we consider the hidden gauge group to be $U(1)_b$ and the visible one to be $U(1)_a$, then we will have [Abe+08; Goo+09]:

$$\mathscr{L} \supset -\frac{1}{4g_a^2}F^{(a)}_{\mu\nu}F^{\mu\nu}_{(a)} - \frac{1}{4g_b^2}F^{(b)}_{\mu\nu}F^{\mu\nu}_{(b)} + \frac{\chi_{ab}}{2g_a g_b}F^{(a)}_{\mu\nu}F^{\mu\nu}_{(b)} + m^2_{ab}A^{(a)}_{\mu}A^{\mu}_{(b)} \tag{4.2.1}$$

where χ_{ab} is the kinetic mixing parameter and m_{ab} is the mass mixing term, which, in string theory, arises via Stückelberg mechanism. There will be a stack of D-branes supporting the Standard Model, and in particular the $U(1)_a$ hypercharge gauge boson $A^{(a)}_{\mu}$, with field strength $F^{(a)}_{\mu\nu}$ and coupling g_a, and a D-brane supporting the hidden gauge sector $U(1)_b$ with gauge boson $A^{(b)}_{\mu}$, with field strength $F^{(b)}_{\mu\nu}$ and coupling g_b. The kinetic mixing appears in the Lagrangian where massive modes, which couples to different $U(1)$s, are integrated out [Abe+08]. These heavy modes correspond to open strings stretched between the visible and the hidden branes or, in the closed string channel, correspond to the exchange of light massless closed string modes (see figure (4.2)). In order to discuss the physical implications of the above kinetic mixing, we should rotate the fields $(A^{(a)}_{\mu}, A^{(b)}_{\mu})$ in such a way to get to the physical (or mass) eigenbasis for the system. At lowest order, we can take $g_a = g_b = 1$ and by shifting the gauge bosons by:

$$\begin{aligned} A^{(a)}_{\mu} &= \hat{A}^{(a)}_{\mu} + \chi_{ab}\hat{A}^{(b)}_{\mu} \\ A^{(b)}_{\mu} &= \hat{A}^{(b)}_{\mu} \end{aligned} \tag{4.2.2}$$

the gauge kinetic term in (4.2.1) becomes diagonal (with $\hat{A}^{(a)}_{\mu}$ the physical visible gauge boson and $\hat{A}^{(b)}_{\mu}$ the physical hidden gauge boson). The result is also a modification of the interactions, in fact, a new coupling between the visible matter particles and the hidden gauge boson arises:

$$\bar{f}A^{(a)}_{\mu}\gamma^{\mu}f = \bar{f}\hat{A}^{(a)}_{\mu}\gamma^{\mu}g + \chi_{ab}\bar{f}\hat{A}^{(b)}_{\mu}\gamma^{\mu}f \tag{4.2.3}$$

where f are Standard Model's particles. This in turn, corresponds to a possibly small coupling between the visible matter and the hidden sector of the form:

$$g' = g_a\chi_{ab} \tag{4.2.4}$$

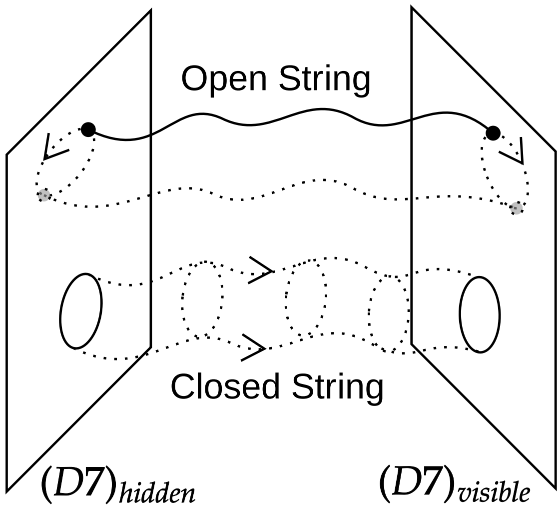


Figure 4.2: Representation of the two different string channels leading to Kinetic Mixing between the visible sector supported on $(D7)_{\text{visible}}$ and the hidden sector supported on $(D7)_{\text{hidden}}$. Above it can be appreciated a string stretched between the (stack of) D-branes which makes a loop and it is immediate to realise its equivalence with the below tree-level exchange of a closed string. This duality in general is referred to as the *Channel Duality* and it is at the heart of holography and AdS/CFT correspondence.

It is natural to ask whether it is possible to kinetically mix non-anomalous $U(1)$s (i.e. massless gauge boson) considered that the mixing parameter χ and the mass term, which mixes visible and hidden $U(1)$s, come from the same diagram. This is indeed possible, since the mass generated via Stückelberg mechanism does not depend upon the masses of the particles (namely the length of the string stretched between the D-branes) while the kinetic mixing does. In a set-up with a D-brane and an anti-D-brane separated by an orientifold plane together with a D-brane supporting the Standard Model, this could happen, because the strings stretched between these branes and the visible sector will give rise to massless $U(1)$s (anomaly free), but with a non-zero kinetic mixing [Abe+08].

The Stückelberg mechanism allows for the hidden $U(1)$ gauge boson to acquire a mass through a completely stringy process. If in (4.2.1) we consider that the photon is massless (after the spontaneous breaking of the symmetry $SU(2)_L \times U(1)_Y$ into $U(1)_{\text{EM}}$), then we have a mass term for the hidden gauge boson $m_b^2 A_\mu^{(b)} A_{(b)}^\mu$, and together with (4.2.4), these are nothing but the mass $M_{Z'}$ and coupling g'_μ of the preceding section, and which we would like to fix in such a way to satisfy (4.1.7). To compute g' we then need an explicit form of the kinetic mixing χ_{ab}. As showed in [Goo+09], before the breaking of supersymmetry, the kinetic mixing appears as a holomorphic function in the kinetic part of the supergravity Lagrangian of the 4D effective theory. The holomorphic kinetic mixing depends on the complex structure moduli, generically in polynomial or exponential form, which will typically be numbers of order one, making the physical kinetic mixing to be of the form:

$$\chi_{ab} \sim \frac{g_a g_b}{16\pi^2} \tag{4.2.5}$$

We are now left to explain the process for generating the mass of the hidden gauge boson, namely the Stückelberg mechanism, and the form of the coupling of a $U(1)$ sector living on a Dp-brane wrapping a holomorphic cycle in the compact space. This is however not the end of the story, since as we shall see, in order to generate a mass for the hidden gauge boson we need to turn on magnetic fluxes on internal cycles wrapped by the Dp-brane. In turn, these will generate a *Fayet-Iliopoulos* (FI) term entering the D-term potential, threatening the stabilisation processes we outlined before because of its dominating character over the F-term potential. Stabilise the moduli without taking into account the FI-term will be incorrect, and in order to partially cancel its contribution, the presence of charged scalar fields (open string modes) will come at rescue.

4.2.2 Stückelberg Mechanism from Internal Fluxes

The process which allows to generate a mass for anomalous $U(1)$ in string theory, is well described in the appendix of [Cic+11], here we will just give a brief review stating the important results that we will be using later.

In this derivation we will not be concerned about multiplicative constants, we are going to focus on why and how does this process takes place, the final result, however, will be furnished with the correct constants in order to reproduce later the correct mass of the hidden boson. Let's then consider the Chern-Simons (or Wess-Zumino) action term (1.3.14) for a D7-brane wrapping a holomorphic cycle D_i, taking only the part with the coupling to the 10-dimensional R-R 4-form C_4:

$$S \propto \int_{\mathbb{R}^{1,3}\times D_i} F_2 \wedge C_4 \wedge F_2 \tag{4.2.6}$$

The R-R 4-form can be expanded in terms of the basis $\{\hat{D}_i\}_{i=1,\dots,h^{(1,1)}}$ of $\mathscr{H}^{1,1}(\mathscr{Y})$ and $\{\tilde{D}_i\}_{i=1,\dots,h^{(2,2)}=h^{(1,1)}}$ of $\mathscr{H}^{2,2}(\mathscr{Y})$ (which we recall from §2.1.4 and §2.2.1 that due to the Hodge $\star$-isomorphism and Hodge decomposition theorem, we have $\mathscr{H}^{2,2}(\mathscr{Y}) \simeq \mathscr{H}^{1,1}(\mathscr{Y})$) as:

$$C_4 = Q_2^j(x) \wedge \hat{D}_j(y) + b_i(x)\tilde{D}^i(y) \tag{4.2.7}$$

Due to the self-duality of the 5-form (1.4.8), Q_2^i are duals to the axions b_i. If the gauge field strength F_2 has no components in the compact dimensions, reducing the action along the divisor leads to:

$$S \propto \left(\int_{D_i} C_4\right)\left(\int_{\mathbb{R}^{1,3}} F_2 \wedge F_2\right) \propto b_i(x) \int_{\mathbb{R}^{1,3}} F_2 \wedge F_2 \tag{4.2.8}$$

since we recall that the Kähler moduli T_i are defined as:

$$T_i = \tau_i + i\, b_i \propto \int_{D_i} \sqrt{-g}\, d^4y + i \int_{D_i} C_4 \tag{4.2.9}$$

Now, as we shall see in the next section, the $U(1)$ gauge theory living on a D7-brane is encoded in a kinetic action of the form:

$$S_{\text{kin}} = -\frac{1}{4g_i}\int_{\mathbb{R}^{1,3}} F_{\mu\nu}F^{\mu\nu} d^4x \qquad \text{with } g_i = \frac{2\pi}{\tau_i} \tag{4.2.10}$$

so that the combined expression for the gauge kinetic function reads:

$$f_{D7_i} = \frac{T_i}{2\pi} \tag{4.2.11}$$

On the other hand, when we consider one of the F_2 in (4.2.6) to have components in the compact dimensions, namely to be the compact flux F_2^c, then:

$$S \propto \left(\int_{D_i} F_2^c \wedge \hat{D}_j\right)\left(\int_{\mathbb{R}^{1,3}} Q_2^j \wedge F_2\right) \tag{4.2.12}$$

and expanding $F_2^c = f_k \hat{D}_k$, we obtain:

$$S \propto f_k \left(\int_{D_i} \hat{D}_k \wedge \hat{D}_j\right)\left(\int_{\mathbb{R}^{1,3}} Q_2^j \wedge F_2\right) = k_{ijk} f_k \int_{\mathbb{R}^{1,3}} Q_2^j \wedge F_2 =: q_{ij} \int_{\mathbb{R}^{1,3}} Q_2^j \wedge F_2 \tag{4.2.13}$$

where we have defined the charge $q_{ij} := k_{ijk} f_k$. This is the charge of the 2-form Q_2^j under the $U(1)$ living of the divisor D_i. From the definition of q_{ij} we see that the intersection numbers k_{ijk} and the flux components f_k determine which 2-form Q_2^j couples to the gauge boson living on D_i. The Kähler moduli which get charged under the $U(1)$ are those parametrising the volume of the 4-cycles that intersect the 2-cycle supporting the gauge flux. In other words, the charged Kähler moduli are a combination of 4-cycles corresponding to the 4-cycle Poincaré dual to the 2-cycle supporting the flux. Take now the kinetic term for the Q_2^j:

$$\begin{aligned} -\int_{\mathbb{R}^{1,3}\times\mathcal{Y}} dC_4 \wedge \star dC_4 &\propto -\left(\int_{\mathcal{Y}} \hat{D}_j \wedge \hat{D}_k\right)\left(\int_{\mathbb{R}^{1,3}} dQ_2^j \wedge \star dQ_2^k\right) \\ &\propto -\frac{(K_{\text{tree}})^{jk}}{\mathcal{V}}\left(\int_{\mathbb{R}^{1,3}} dQ_2^j \wedge \star dQ_2^k\right) \end{aligned} \tag{4.2.14}$$

where $(K_{\text{tree}})^{jk}$ is the inverse of $(K_{\text{tree}})_{jk} := \partial^2 K_{\text{tree}}/\partial\tau_j \partial\tau_k$ and we have also used the self-duality property of dC_4. The Lagrangian then will contains terms of the form:

$$\begin{aligned} \mathcal{L}_{\text{kin}}^{Q} &\propto -\frac{(K_{\text{tree}})^{jk}}{\mathcal{V}} H^j_{\mu\nu\rho} H^{k,\mu\nu\rho} \\ \mathcal{L}_{\text{kin}}^{A} &= -\frac{1}{4g_i^2} F_{\mu\nu} F^{\mu\nu} \\ \mathcal{L}_{\text{int}} &\propto q_{ij}\, Q_2^j \wedge F_2 \end{aligned} \tag{4.2.15}$$

By canonically normalise the fields and dualise the 2-form Q_2 to get the corresponding axion b, then a Lagrangian of the following form arises:

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{m_{Z'}}{2}(A_\mu + \partial_\mu b)(A^\mu + \partial^\mu b) \tag{4.2.16}$$

with the mass of the hidden gauge boson given by:

$$m_{Z'} = g_{D7}^2 \frac{M_P^2}{4\pi^2} q_{D7i} (K_{\text{tree}})_{ij} q_{D7j} \tag{4.2.17}$$

where g_{D7} is the gauge coupling of the $U(1)$ sector supported on the D7 and q_{D7i} are the charges of the 2-form Q_2^i with respect to the flux on the 4-cycle D_{D7} wrapped by the $D7$, namely:

$$q_{D7i} = \int_{D_{D7}} \hat{D}_i \wedge \mathcal{F}_2 \tag{4.2.18}$$

Note that the expression (4.2.17) for the mass of the gauge boson supported on the D7, is valid when there isn't any cycle odd under the orientifold involution, otherwise it should be modified by the addition of another contribution [Cic+11]. However, since in LARGE volume scenarios the orientifold involution is taken such that $h_-^{(1,1)} = 0 = h_-^{(2,2)}$, then there will be no more contributions to (4.2.17).

4.2.3 Effective $U(1)$ Gauge Theory Living on a D7-Brane

In §1.3 and in particular in §1.3.1, we saw that Dirichlet branes are not static objects living in the bulk but are dynamical objects with dynamics described by the Dirac-Born-Infeld action S_{DBI} and the Wess-Zumino (Chern-Simons) action S_{WZ}. For a Dp-brane with worldvolume $\mathscr{W}$ embedded in the 10-dimensional spacetime $\mathscr{M}$ via the map $\varphi : \mathscr{W} \to \mathscr{M}$, these actions take the following form:

$$\begin{aligned} S_{Dp} = & -T_p e^{-\phi} \int_{\mathscr{W}} d^{p+1}\xi \sqrt{-\det[\varphi^*(G+B_2)+(\ell_s^2/2\pi)F_2]} \\ & + T_p e^{-\phi} \int_{\mathscr{W}} \sum_i \varphi^* C_p \wedge e^{\varphi^* B_2 + (\ell_s^2/2\pi)F_2} \end{aligned} \tag{4.2.19}$$

where T_p is the tension of the Dp-brane, G is the ten-dimensional graviton, B_2 is the NS-NS 2-form, ϕ is the dilaton and F_2 is the field strength associated to the $U(1)$ gauge boson living on the Dp-brane. Stable Dp-branes in type IIB string theory have odd p, and given our orientifold projection (see §2.3.2), only D3/D7-branes are brought into play. D3-branes must be extended to all the Minkowski $\mathbb{R}^{1,3}$ in order to maintain Poincaré invariance, so that they are just single points in the compact space. D7-branes, on the other hand, will have a worldvolume of the form $\mathscr{W} = \mathbb{R}^{1,3} \times D$ where D will be a holomorphic 4-cycle in the compact directions. The Dirac-Born-Infeld action (where we will infer the form of the kinetic term for the gauge boson, and so the form of the gauge coupling) for D7-branes becomes:

$$S_{D7}^{DBI} = -T_7 e^{-\phi} \int_{\mathbb{R}^{1,3}\times D} d^8\xi \sqrt{-\det[\varphi^*(G+B_2)+(\ell_s^2/2\pi)F_2]} \tag{4.2.20}$$

At this point we can expand in powers of the field strength F_2 in order to find the corresponding Maxwell action allowing us to learn the precise form of the gauge coupling constant. In order to simplify our notation let's write for now $\mathscr{G} := \varphi^*(G)$ and $kF := \ell_s^2/2\pi F_2$, we will also neglect the NS-NS field B_2 since the orientifold projection we will be considering is such that $h_-^{(1,1)} = 0$. The determinant is then:

$$\det(\mathscr{G}+kF) = \det[(\mathscr{G}+kF)^T] = \det(\mathscr{G}-kF) \tag{4.2.21}$$

where in the first equality we have used the fact that the determinant is invariant upon taking the transpose and in the second equality we have used the fact that F is anti-symmetric. The above relation tells us that this determinant is even with respect to k. Let's then define $M := k\mathscr{G}^{-1}F$ and we will have:

$$\begin{aligned} \sqrt{-\det(\mathscr{G}+kF)} &= \sqrt{-\det\mathscr{G}}\sqrt{\det(\mathbb{1}+M)} \\ &= \sqrt{-\det\mathscr{G}}\left[\det(\mathbb{1}+M)^2\right]^{1/4} \\ &= \sqrt{-\det\mathscr{G}}\left[\det(\mathbb{1}+M)\det(\mathbb{1}+M)\right]^{1/4} \\ &= \sqrt{-\det\mathscr{G}}[\det(\mathbb{1}-M^2)]^{1/4} \end{aligned} \tag{4.2.22}$$

We can then use the identity relating the logarithm of the determinant to the logarithm of the trace:

$$\ln(\det(A)) = \mathrm{tr}(\ln(A)) \tag{4.2.23}$$

to obtain:

$$\ln(\det(\mathbb{1} - M^2)) = \mathrm{tr}(\ln(\mathbb{1} - M^2)) = -\mathrm{tr}\left(M^2 + \frac{1}{2}M^4 + ...\right) \tag{4.2.24}$$

so that:

$$[\det(1 - M^2)]^{1/4} = e^{-\frac{1}{4}\mathrm{tr}(M^2) - \frac{1}{8}\mathrm{tr}(M^4) + ...} = 1 - \frac{1}{4}\mathrm{tr}(M^2) - \frac{1}{8}\mathrm{tr}(M^4) + \frac{1}{32}\left(\mathrm{tr}(M^2)\right)^2 + ... \tag{4.2.25}$$

Putting all together and plugging back the constants and the pull-back, the term in the DBI-action will be:

$$\sqrt{-\det \varphi^* G}\left(1 - \frac{1}{4}\frac{\ell_s^4}{4\pi^2}F_{MN}F^{MN} - \frac{1}{8}\frac{\ell_s^8}{16\pi^4}F_{MN}F^{NP}F_{PQ}F^{QM} + \frac{1}{32}\frac{\ell_s^8}{16\pi^4}...\right) \tag{4.2.26}$$

where clearly $M, N, P, Q = 0, ..., 9$ are the indices in the 10-dimensional spacetime. The Maxwell-like term will just be the one proportional to $F_{MN}F^{MN}$, so that by dimensional reducing from 8D to 4D, we obtain:

$$S_{D7}^{DBI} = -\frac{T_7 \ell_s^4 e^{-\phi}}{16\pi^2}\int_{\mathbb{R}^{1,3}\times D} F_{MN}F^{MN} \implies S_{\mathrm{Maxwell}} = -\left(\frac{T_7 \ell_s^8}{16\pi^2}\right)\tau_D \int d^4x F_{\mu\nu}F^{\mu\nu} \tag{4.2.27}$$

where τ_D is the volume of the 4-cycle D wrapped by the D7-brane $\tau_D = (e^{-\phi}/\ell_s^4)\int_D \sqrt{g}\, d^4y$ and $\mu, \nu = 0, 1, 2, 3$ are indices in the Minkowski spacetime $\mathbb{R}^{1,3}$. Since the D7-brane tension is $T_7 = 2\pi/\ell_s^8$, we have:

$$S_{\mathrm{Maxwell}} = -\frac{1}{4g_D^2}\int_{\mathbb{R}^{1,3}} F_{\mu\nu}F^{\mu\nu} \tag{4.2.28}$$

where the gauge coupling is given by:

$$g_D^2 = \frac{2\pi}{\tau_D} \tag{4.2.29}$$

4.2.4 Fayet-Iliopoulos Term

Turning on magnetic fluxes for D7-branes wrapping holomorphic cycles, allows to generate a mass via Stückelberg mechanism as we saw in §4.2.2. However, this is not the only result we get when internal fluxes do not vanish, in fact, a *Fayet-Iliopoulos term* is also generated [JL05]. From the reduction we have already seen in the previous section of the DBI-action for a D7-brane wrapping a holomorphic 4-cycle $\mathscr{D}$ in the Calabi-Yau $\mathscr{Y}$, we have:

$$S_{D7}^{DBI} = -T_7\, e^{-\phi}\, \Gamma_{\mathscr{D}} \int_{\mathbb{R}^{1,3}} d^4x \sqrt{-\det g_{(4)}}\sqrt{\mathbb{1} + (\ell_s^2/2\pi)F_{(4)}} \tag{4.2.30}$$

with:

$$\Gamma_{\mathscr{D}} = \int_{\mathscr{D}} d^4y \sqrt{-\det(\varphi^* g_{\mathscr{Y}}) + F} \tag{4.2.31}$$

where we are denoting $g_{(4)}$ the metric in $\mathbb{R}^{1,3}$, $F_{(4)}$ the four-dimensional field strength and F the field strength restricted to the divisor $\mathscr{D}$ (namely the component of the $U(1)$ field strength living on the D7-brane in the compact directions). What it is found using the BPS-calibration condition, is that $\Gamma_{\mathscr{D}}$ is of the following form [Haa+07]:

$$\Gamma_{\mathscr{D}} = \tilde{\Gamma}_{\mathscr{D}} e^{-i\theta} = |\tilde{\Gamma}_{\mathscr{D}}| e^{-i(\theta-\tilde{\theta})} \tag{4.2.32}$$

where:

$$\tilde{\Gamma}_{\mathscr{D}} = \frac{1}{2}\int_{\mathscr{D}} (\varphi^* J \wedge \varphi^* J - F \wedge F) + i \int_{\mathscr{D}} \varphi^* J \wedge F \tag{4.2.33}$$

and:

$$\tilde{\theta} = \arctan\left(\frac{2\int_{\mathscr{D}} \varphi^* J \wedge F}{\int_{\mathscr{D}} (\varphi^* J \wedge \varphi^* J - F \wedge F)} \right) \tag{4.2.34}$$

Since the tension of the brane should be real and positive, we have that: $\theta = \tilde{\theta} + 2\pi n$, which leads to:

$$\Gamma_{\mathscr{D}} = |\tilde{\Gamma}_{\mathscr{D}}| = \sqrt{\left(\frac{1}{2}\int_{\mathscr{D}} (\varphi^* J \wedge \varphi^* J - F \wedge F) \right)^2 + \left(\int_{\mathscr{D}} \varphi^* J \wedge F \right)^2} \tag{4.2.35}$$

If the supersymmetry is preserved on the D7-brane, then the imaginary part of $\tilde{\Gamma}_{\mathscr{D}}$ must vanish, forcing to have no flux on the divisor:

$$\mathfrak{Im}(\tilde{\Gamma}_{\mathscr{D}}) = \int_{\mathscr{D}} \varphi^* J \wedge F = 0 \tag{4.2.36}$$

However, if one allows a small supersymmetry breaking, one can expand $\Gamma_{\mathscr{D}}$ (4.2.35) as[1]:

$$\Gamma_{\mathscr{D}} = \frac{1}{2}\int_{\mathscr{D}} (\varphi^* J \wedge \varphi^* J - F \wedge F) + \frac{\left(\int_{\mathscr{D}} \varphi^* J \wedge F \right)^2}{\int_{\mathscr{D}} (\varphi^* J \wedge \varphi^* J - F \wedge F)} \tag{4.2.37}$$

Thus, the deviation from the BPS condition is a measure of supersymmetry breaking and leads to the appearance of a D-term potential because a new moduli-dependent Fayet-Iliopoulos term $\xi_{\mathscr{D}}$ appears (with the correct constants and in M_P units):

$$V_D = \frac{g_{\mathscr{D}}^2}{2}\xi_{\mathscr{D}}^2 = \frac{g_{\mathscr{D}}^2}{2}\frac{1}{(4\pi\mathscr{V})^2}\left(\int_{\mathscr{D}} \varphi^* J \wedge \frac{\ell_s^2}{2\pi} F \right)^2 \Longrightarrow \xi_{\mathscr{D}} = \frac{1}{4\pi\mathscr{V}} \int_{\mathscr{D}} \varphi^* J \wedge \frac{\ell_s}{2\pi} F \tag{4.2.38}$$

Also, due to the non-vanishing fluxes on the worldvolume of the brane, the gauge coupling gets modified to:

$$\frac{2\pi}{g_{\mathscr{D}}^2} = \mathfrak{Re}(T_{\mathscr{D}}) - e^{-\phi}\frac{f^j q_{\mathscr{D}j}}{2} \tag{4.2.39}$$

[1]For $z = \mathfrak{Re}(z) + i\mathfrak{Im}(z)$ with $\mathfrak{Im}(z) \ll \mathfrak{Re}(z)$, we can expand:

$$|z| = \sqrt{\mathfrak{Re}(z)^2 + \mathfrak{Im}(z^2)} = \mathfrak{Re}(z)\sqrt{1 + \frac{\mathfrak{Im}(z)^2}{\mathfrak{Re}(z)^2}} = \mathfrak{Re}(z)\left(1 + \frac{1}{2}\frac{\mathfrak{Im}(z)^2}{\mathfrak{Re}^2} + O\left(\frac{\mathfrak{Im}(z)^3}{\mathfrak{Re}(z)^3}\right)\right) \simeq \mathfrak{Re}(z) + \frac{1}{2}\frac{\mathfrak{Im}(z)^2}{\mathfrak{Re}(z)}$$

where $q_{\mathscr{D}j}$ are the charges defined as in §4.2.2 and $T_{\mathscr{D}}$ is the Kähler modulus with $\mathfrak{Re}(T_{\mathscr{D}}) = vol(\mathscr{D}) = \tau_{\mathscr{D}}$.

In the presence of charged matter fields ψ_j (open string states) with $U(1)$ charges given by the $O(1)$ numbers $c_{\mathscr{D}j}$, the D-term potential will look like:

$$V_D = \frac{\pi}{(\tau_{\mathscr{D}} - f^j q_{\mathscr{D}j}/2g_s)}\left(\sum_j c_{\mathscr{D}j}\psi^j \frac{\partial \tilde{K}}{\partial \psi^j} - \xi_{\mathscr{D}}\right) \tag{4.2.40}$$

where $\tilde{K}$ is the Kähler matter metric and depends on which open string states are allowed due to the brane configuration.

4.3 K3 Fibration With a Blow-Up Mode

We finally start here to present some explicit models and see whether they can provide interesting set-ups of D-branes and fluxes allowing us to reproduce the interesting parameter region given by (4.1.7). We consider a generalisation of the 2-parameter K3-fibred Calabi-Yau embedded in $\mathbb{P}^4_{(1,1,2,2,6)}$ (considered in §3.4) by the addition of a small blow-up mode τ_3 in such a way that the volume form can be written as:

$$\mathscr{V} = \alpha\left[\sqrt{\tau_1}(\tau_2 - \beta\tau_1) - \gamma\tau_3^{3/2}\right] \tag{4.3.1}$$

where α, β and γ are real numbers determined by the intersection numbers k_{ijk}. The LARGE Volume Scenario limit is given by considering:

$$\begin{cases} \tau_2 \to \infty \\ \tau_3 \ll \tau_1 < \tau_2 \end{cases} \tag{4.3.2}$$

As we have said before, non-perturbative corrections to the superpotential are necessary to find a LVS minimum and are given by gaugino condensation on a D7-brane wrapping the small blow-up moduli. We also do not want the same contributions coming from the D7-brane supporting the hidden gauge sector, indeed since we have the relation $\tau_3 \ll \tau_{hid}$, then $e^{-a_3\tau_3} \gg e^{-a_{\text{hid}}\tau_{\text{hid}}}$ and this allows us to discard possible non-perturbative effects generated on the hidden D7-brane wrapping τ_{hid} (whether it will be τ_1 or τ_2). Also, we will require a very high degree of anisotropy for the volume in the form of demanding $\tau_1 \ll \tau_2$. In this anisotropic limit, the volume reduces to:

$$\mathscr{V} \simeq \alpha\sqrt{\tau_1}\tau_2 \tag{4.3.3}$$

and a representation of the volume can be appreciated in figure (4.3).
The tree-level Kähler potential K_{tree} given by (2.3.28) will give rise to the following Kähler metric:

$$(K_{\text{tree}})_{ij} = \frac{\partial^2 K_{\text{tree}}}{\partial\tau_i \partial\tau_j} \simeq \begin{pmatrix} \tau_1^{-2} & 0 \\ 0 & 2\tau_2^{-2} \end{pmatrix} \tag{4.3.4}$$

In order for our model to support a hidden gauge sector, we need to wrap a D7-brane on a 4-cycle. We have then two choices, namely we can support the hidden gauge sector either on τ_1 or on τ_2, let's then inspect these two possibilities. In the following we are going to take:

$$\tau_{\text{vis}} = 12.5 \tag{4.3.5}$$

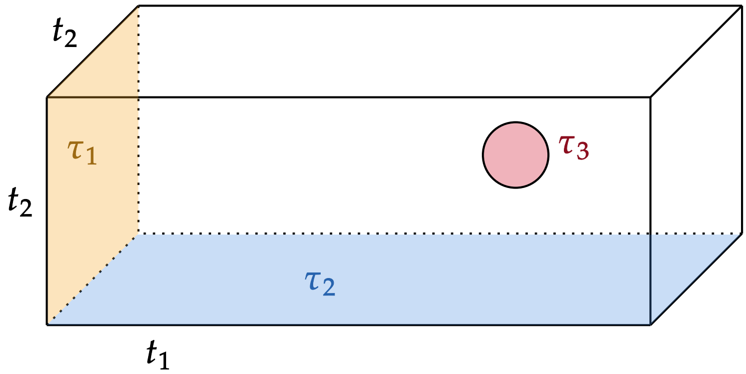


Figure 4.3: Pictorial representation of the Calabi-Yau volume, where the 2-cycles are represented as 1-dimensional segments and 4-cycles as 2-dimensional surfaces. We see that the volume is given by the product $\sqrt{\tau_1}\tau_2$ or equivalently $t_1\tau_1$ with the subtraction of the small blow-up τ_3, making the configuration of a Swiss cheese type.

leading to a visible gauge coupling of:

$$g_{\text{vis}} \simeq 0.709 \tag{4.3.6}$$

Which means that we are considering the visible coupling to be determined by the GUT-fine structure constant $\alpha_{\text{GUT}}^{-1} = 25$ in such a way that:

$$g_{\text{vis}} = \sqrt{4\pi\alpha_{\text{GUT}}} = \sqrt{\frac{2\pi}{\tau_{GUT}}} \tag{4.3.7}$$

4.3.1 Hidden Sector on D_2

Wrapping a D7-brane on D_2 (where $vol(D_2) = \tau_2$) will generate the hidden $U(1)$ gauge sector with the coupling given by:

$$g_{\text{hid}}^2 = \frac{2\pi}{\tau_2} \tag{4.3.8}$$

The kinetic mixing for a hidden sector living on τ_2 with the electromagnetic gauge boson on the visible sector is of the order:

$$\chi \sim \frac{g_{\text{vis}}g_{\text{hid}}}{16\pi^2} \sim \frac{1.1 \cdot 10^{-2}}{\sqrt{\tau_2}} \tag{4.3.9}$$

giving rise to a coupling g':

$$g' \sim \frac{8 \cdot 10^{-3}}{\sqrt{\tau_2}} \tag{4.3.10}$$

To generate a mass for this $U(1)$ gauge boson, we turn on magnetic fluxes:

$$F_2 = f_i \hat{D}_i \tag{4.3.11}$$

and the mass will be:

$$m_{Z'}^2 = g_{\text{hid}}^2 \frac{M_P^2}{4\pi^2} q_{2j}(K_{\text{tree}})_{jk} q_{2k} \tag{4.3.12}$$

Since the charge q_{ij} of the 2-form Q_2^j (recall (4.2.7)) under the $U(1)$ living on the divisor D_i is:

$$q_{ij} = \int_{D_i} \hat{D}_j \wedge \mathscr{F} = f_k k_{ijk} \implies q_{2i} = f^k k_{2ik} \tag{4.3.13}$$

and that the only non-zero k_{2jk} intersection number is k_{122} (k_{ijk} is symmetric in all indices), we see that the mass becomes the sum of two contributions:

$$m_{Z'}^2 = g_{\text{hid}}^2 \frac{M_P^2}{4\pi^2}\left(q_{21}^2 (K_{\text{tree}})_{11} + q_{22}^2 (K_{\text{tree}})_{22}\right) = g_{\text{hid}}^2 \frac{M_P^2}{4\pi^2}\left(\frac{(f_2 k_{122})^2}{\tau_1^2} + \frac{2(f_1 k_{122})^2}{\tau_2^2}\right) \tag{4.3.14}$$

This means that depending on which fluxes we turn on, we will have a different mass for the hidden gauge boson. However, we can immediately note that if both $(f_2, f_1) \neq (0,0)$, then the second term is suppressed with the respect to the first due to the anisotropic limit $\tau_2 \gg \tau_1$, this means that the case $f_1 \neq 0, f_2 \neq 0$ is equivalent to the case $f_1 = 0, f_2 \neq 0$. With these considerations in mind let's analyse both cases.

Gauge flux on t_2: $f_1 = 0$

Imposing $f_1 = 0$, we are left with:

$$m_{Z'}^2 = \frac{2\pi}{\tau_2}\frac{M_P^2}{4\pi^2}\frac{(f_2 k_{122})^2}{\tau_1^2} = \frac{(M_P f_2 k_{122})^2}{2\pi \tau_2 \tau_1^2} \implies m_{Z'} = \frac{M_P f_2 k_{122}}{\sqrt{2\pi}\sqrt{\tau_2}\,\tau_1} \tag{4.3.15}$$

We can now isolate the $\sqrt{\tau_2}$ from the above equation and put it into the coupling g', obtaining:

$$g' \sim 8 \cdot 10^{-3} \frac{\tau_1}{f_2 k_{122}} \frac{m_{Z'}}{M_P} \sim 8 \cdot 10^{-21} \frac{\tau_1}{f_2 k_{122}} \frac{m_{Z'}}{\text{GeV}} \tag{4.3.16}$$

It is immediate to see that, since τ_1 is small and both f_2 and k_{122} are of order $O(1)$, the coupling g' cannot reach the desired values (4.1.7). In fact, employing the anisotropic limit, we can come up with a lower bound for the mass as follows:

$$\frac{\tau_1}{\tau_2} \ll 1 \iff \tau_1 \ll \frac{6.4 \cdot 10^{-5}}{(g')^2} \iff m_{Z'} \gg 1.1 \cdot 10^{24} (g')^3\, \text{GeV} \tag{4.3.17}$$

which brings to a phenomenologically uninteresting parameter region.

Gauge flux on t_1: $f_2 = 0$

Requiring $f_2 = 0$, or equivalently just taking $f_1 \neq 0$ without worrying about f_2, we have:

$$m_{Z'}^2 = \frac{(M_P f_1 k_{122})^2}{\pi \tau_2^3} \implies m_{Z'} = \frac{M_P f_1 k_{122}}{\sqrt{\pi}\,\tau_2^{3/2}} \tag{4.3.18}$$

by substituting again $\sqrt{\tau_2}$ into the coupling g' we get:

$$g' \sim 8 \cdot 10^{-3} \left(\frac{1}{f_1 k_{122}} \frac{m_{Z'}}{M_P}\right)^{1/3} \sim 8 \cdot 10^{-9} \left(\frac{1}{f_1 k_{122}}\right)^{1/3} \left(\frac{m_{Z'}}{\text{GeV}}\right)^{1/3} \tag{4.3.19}$$

By direct comparison with the relation (4.1.7), we see that they are not compatible, since in (4.3.19) the coupling g' scales as the cubic root of the mass of the hidden gauge boson, while in (4.1.7) it scales linearly with the respect to $m_{Z'}$.

4.3.2 Hidden Sector on D_1

Wrapping a D7-brane on τ_1 will generate the hidden $U(1)$ gauge sector with the coupling given by:

$$g^2_{\rm hid} = \frac{2\pi}{\tau_1} \tag{4.3.20}$$

To generate a mass for this $U(1)$ gauge boson, we turn on magnetic fluxes:

$$F_2 = f_i \hat{D}_i \tag{4.3.21}$$

and the mass will be:

$$m^2_{Z'} = g^2_{\rm hid} \frac{M_P^2}{4\pi^2} q_{1j} (K_{\rm tree})_{jk} q_{1k} \tag{4.3.22}$$

Since the charge q_{ij} of the 2-form Q_2^j (recall (4.2.7)) under the $U(1)$ living on the divisor D_i is:

$$q_{ij} = \int_{D_i} \hat{D}_j \wedge \mathcal{F} = f_k k_{ijk} \tag{4.3.23}$$

given that the only non-zero k_{1jk} intersection number is k_{122}, we thus need to turn on gauge fluxes just on t_2, namely $F_2 = f_2 \hat{D}_2$, so that the mass will be of the form:

$$m^2_{Z'} = \frac{2\pi}{\tau_1} \frac{M_P^2}{4\pi^2} (q_{12})^2 (K_{\rm tree})_{22} = \frac{(M_P f_2 k_{122})^2}{\pi \tau_1 \tau_2^2} \simeq \frac{(f_2 k_{122})^2}{\pi \mathcal{V}^2} M_P^2 \tag{4.3.24}$$

The kinetic mixing for a hidden sector living on τ_1 with the electromagnetic gauge boson on the visible sector is of the order:

$$\chi \sim \frac{g_{\rm vis} g_{\rm hid}}{16\pi^2} \sim \frac{1.1 \cdot 10^{-2}}{\sqrt{\tau_1}} \tag{4.3.25}$$

giving rise to a coupling g':

$$g' \sim \frac{8 \cdot 10^{-3}}{\sqrt{\tau_1}} \tag{4.3.26}$$

For a hidden gauge boson of mass of the GeV order, we know from the figure (4.1) that the coupling g' in $\log_{10}$ space can be of order ~ -3. In turn, from the above expression we can infer that the modulus τ_1 should be fixed at values of order:

$$\tau_1 = O(10) \div O(100) \tag{4.3.27}$$

Note that we cannot require the mass of the hidden boson to be bigger than the GeV order, namely something like 10 GeV, this is because the resulting kinetic mixing would be too big, falling in a parameter region ruled out by experiments. This can be appreciated from the figure at page 3 of [Cic+11]. So, for a hidden gauge boson mass of $m_{Z'} = 1$ GeV we can use the formula giving $m_{Z'}$ via Stückelberg mechanism (4.3.24) in order to get an estimate of the volume mode:

$$1 \text{ GeV} = m_{Z'} = \frac{(f_2 k_{122})}{\sqrt{\pi \mathcal{V}}} M_P \iff \mathcal{V} \simeq 10^{18} \tag{4.3.28}$$

(we have used that natural values for the flux f_2 and intersection numbers are of order unity). We also note that once the modulus τ_1 and the volume mode $\mathcal{V}$ are known, then the big 4-cycle volume τ_2 will be fixed.

Fayet-Iliopoulos Term

This reasoning leads us to believe that the above stet-up could indeed reproduce the suitable parameter region we are searching for, provided we manage to correctly stabilise all the moduli at the above values. However, as we saw previously, the fluxes for the Stückelberg mechanism give rise to also a Fayet-Iliopoulos (FI) term, contributing to the D-term scalar potential. In our case the FI-term will be of the form:

$$\xi_1 = \frac{q_{1j}t^j}{4\pi\mathcal{V}} = \frac{f_2 k_{122} t_2}{\mathcal{V}} \tag{4.3.29}$$

Considering matter fields, namely open string modes, charged with respect to the $U(1)$, these can contribute to a partial cancellation of the FI-term, since its behaviour will dominate the F-term potential. The D-term potential considering one charged field will become (with charge c_1):

$$V_D = \frac{\pi}{\tau_1}(c_1|\phi|^2 - \xi_1)^2 + m_\phi^2|\phi|^2 \tag{4.3.30}$$

The VEV of the matter field is readily obtained:

$$\left.\frac{\partial V_D}{\partial \phi}\right|_{\langle|\phi|\rangle} = 0 \iff \langle|\phi|\rangle^2 = \frac{\xi_1}{c_1} - \frac{\tau_1 m_\phi^2}{2\pi c_1^2} \simeq \frac{\xi}{c_1} \tag{4.3.31}$$

where the last equality comes from the fact that the mass of the matter field ϕ is of the order the gravitino mass $m_\phi^2 \sim m_{3/2}^2$ which is of the order $O(1/\mathcal{V}^2)$, so that its contribution is highly suppressed with respect to the term ξ_1. As can be noted, this results in a leading order cancellation of the FI-term with the charged matter fields. The presence of these fields, which in general are written as $\phi = |\phi|e^{i\theta}$ with θ their axionic part, is, however, problematic for our case since the hidden gauge boson via the Stückelberg mechanism eats a combination of the open string axion and the closed string axion (recall $T_i = \tau_i + i\,b_i$) (as pointed out in [Cho+11; CQV16]):

$$m_{Z'}^2 \simeq \frac{M_P^2}{\tau_1}(f_\theta^2 + f_{b_2}^2) \tag{4.3.32}$$

where these two terms are proportional to the open and closed string axion decay constants f_θ and f_{b_2} given by:

$$\begin{aligned} f_\theta^2 &= |\phi|^2 \simeq \xi_1 \\ f_{b_i}^2 &= \frac{\partial^2 K_{\text{tree}}}{\partial \tau_2 \partial \tau_2} \simeq \xi_1^2 \end{aligned} \tag{4.3.33}$$

Because $\xi_1 \gg \xi_1^2$ (since $\xi \sim \sqrt{\tau_1}/\mathcal{V}$), then $f_\theta^2 \gg f_{b_2}^2$, meaning that the axionic part of the charged fields dominates over the axionic part of the Kähler moduli, resulting in a mass for the hidden gauge boson of the order Kaluza-Klein modes:

$$m_{Z'} \simeq M_P \frac{f_\theta}{\sqrt{\tau_1}} \simeq \frac{M_P}{\tau_1^{1/4}\sqrt{\mathcal{V}}} \tag{4.3.34}$$

which is clearly too big to reproduce a mass of the order 1 GeV. Now, the presence of charged matter fields is crucial to get D-term potential comparable to the F-term potential. As it is, the

FI-term scales as $\sim 1/\mathcal{V}$ while the F-term potential in LARGE volume scenario goes like $\sim 1/\mathcal{V}^3$. The FI-term will then induce a run away for the volume mode making impossible its stabilisation to a fixed minimum. When charged string modes are introduced to balance the FI-term, however, their axionic parts contribute too much to the mass of the hidden gauge boson, nullifying whatever process we make to stabilise the moduli since we would still get a mass $m_{Z'}$ out of our interesting parameter region (4.1.7). Imposing the vanishing of the FI-term, because of the simplicity of the Calabi-Yau intersection numbers k_{ijk}, would force the 4-cycle volume τ_1 to go to zero size, making the model inconsistent. A way out could be to consider the D-brane supporting the hidden sector to wrap a cycle of the form $D_{D7} = D_1 + D_3$, this would result in a more complicated FI-term and forcing it to be zero will just fix a particular combination of 2-cycles. Let's see if this could work.

4.3.3 Hidden Sector on $D_{D7} = D_1 + D_3$

The Calabi-Yau volume can be written in terms of 2-cycles as $\mathcal{V} = 1/6\, k_{ijk} t^i t^j t^k$, this means that (4.3.1)is also equivalent to:

$$\mathcal{V} = k_{122} t_1 t_2^2 + k_{222} t_2^3 + k_{333} t_3^3 \tag{4.3.35}$$

In the anisotropic limit $\tau_2 \gg \tau_1$ the volume can be written as $\mathcal{V} \simeq \sqrt{\tau_1}\tau_2 - \tau_3^{3/2}$ (without worrying about intersection numbers and taking into account that the 2-cycle t_3 will be fixed negative by the Kähler cone constraints) and the diagonal part of the Kähler metric will be:

$$(K_{\text{tree}})_{ij} = \frac{\partial^2 K_{\text{tree}}}{\partial \tau_i \partial \tau_j} \simeq \begin{pmatrix} \tau_1^{-2} & & \\ & 2\tau_2^{-2} & \\ & & 3(2\mathcal{V}\sqrt{\tau_3})^{-1} \end{pmatrix} \tag{4.3.36}$$

Take now the hidden gauge sector D7-brane to wrap $D_{D7} = D_1 + D_3$, this will have a volume given by:

$$vol(D_{D7}) =: \tau_{D7} = \int_{D_1+D_3} J \wedge J = \int_{\mathscr{Y}} (\hat{D}_1 + \hat{D}_3) \wedge t^i \hat{D}_i \wedge t^j \hat{D}_j = \tau_1 + \tau_3 \tag{4.3.37}$$

Also, the coupling will just be:

$$g_{D7}^2 = \frac{2\pi}{\tau_{D7}} = \frac{2\pi}{\tau_1 + \tau_3} \tag{4.3.38}$$

Suppose to turn on magnetic gauge fluxes $\mathscr{F} = f^k \hat{D}_k$ which will give rise to FI-term of the form:

$$\xi_{D7} = \frac{1}{4\pi\mathcal{V}} \int_{D_{D7}} J \wedge \mathscr{F} = \frac{1}{4\pi\mathcal{V}} \int_{\mathscr{Y}} (\hat{D}_1 + \hat{D}_2) \wedge J \wedge \mathscr{F} = \frac{1}{4\pi\mathcal{V}} (t_2 f_2 k_{122} + t_3 f_3 k_{333}) \tag{4.3.39}$$

Imposing:

$$\xi_{D7} = 0 \tag{4.3.40}$$

will then fix the combination $t_2 f_2 k_{122} + t_3 f_3 k_{333}$ to zero, namely the fact that the 2-cycle t_2 should be of the same order of t_3 (given that the fluxes and intersection numbers are of order $O(1)$):

$$t_2 \sim t_3 \tag{4.3.41}$$

In turn, from the relation $\tau_i = \partial \mathcal{V}/\partial t^i$, we see that $\tau_1 \simeq t_2^2$ while $\tau_3 \simeq t_3^2$. This will be translated in the relation:

$$\tau_1 \sim \tau_3 \Longrightarrow g_{D7}^2 \simeq \frac{\pi}{\tau_1} \simeq \frac{\pi}{\tau_3} \tag{4.3.42}$$

The situation seems pretty promising at this stage, since, effectively, this set-up reduces to the case analysed before of the D7-brane wrapping τ_1 without introducing charged fields to cancel the FI-term. What needs to be inspected is the form of the mass generated via Stückelberg mechanism for the hidden gauge boson, given that the 4-cycle wrapped by the brane is now a combination of D_1 and D_3. Using the general formula:

$$m_{Z'}^2 = g_{D7}^2 \frac{M_P^2}{4\pi^2} q_{D7\,j} (K_{\text{tree}})_{ik}\, q_{D7\,k} \tag{4.3.43}$$

we see that there are two contributions given by:

$$\begin{aligned} q_{D7\,2} (K_{\text{tree}})_{22}\, q_{D7\,2} &= \frac{2(f_2 k_{122})^2}{\tau_2^2} \\ q_{D7\,3} (K_{\text{tree}})_{33}\, q_{D7\,3} &= \frac{3(f_3 k_{333})^2}{2\mathcal{V}\sqrt{\tau_3}} \end{aligned} \tag{4.3.44}$$

where we have used the fact that $q_{D7\,i}$ is not zero only for $i = 2, 3$, since:

$$q_{D7\,i} = \int_{D_{D7}=D_1+D_3} \hat{D}_i \wedge \mathcal{F} = \int_{\mathcal{Y}} (\hat{D}_1 + \hat{D}_3) \wedge \hat{D}_i \wedge f_k \hat{D}_k = q_{1i} + q_{3i} \tag{4.3.45}$$

From the expressions in (4.3.44) and the fact that $\mathcal{V} \sim \sqrt{\tau_1}\tau_2$ and $\tau_2 \gg \tau_1$, it is clear that the first term is subdominant with respect to the second one. The mass of the hidden boson will then just be (using $\tau_3 \sim \tau_1$):

$$m_{Z'}^2 = g_{D7}^2 \frac{M_P^2}{4\pi^2} \frac{3(f_3 k_{333})^2}{2\mathcal{V}\sqrt{\tau_3}} \simeq \frac{3(M_P f_3 k_{333})^2}{4\pi \tau_3^{3/2} \mathcal{V}} \Longrightarrow m_{Z'} \simeq \frac{\sqrt{3} M_P f_3 k_{333}}{2\sqrt{\pi}\, \tau_3^{3/4} \sqrt{\mathcal{V}}} \tag{4.3.46}$$

Sadly the above relation resembles a lot the mass (4.3.34) we found when we took into account the axionic part of the charged matter fields and we can see even here that since τ_3 is a small blow-up, the above relation cannot give phenomenologically interesting masses for the hidden gauge boson. Even if at first wrapping the hidden D7-brane on a combination of 4-cycles seemed to be promising since it naturally led to a cancellation of the FI-term without the introduction of "dangerous" charged matter fields, at the end, the mass generated is almost of the same order of that generated by the charged open string modes. This can be attributed to the fact that new charges $q_{D7\,i}$ pop-up, coupling the R-R 4-form also to fluxes on the 2-cycle t_3. Note that by looking at (4.3.44) one can think to turn on gauge fluxes only on t_2, while keeping $f_3 = 0$. However, in this case the FI-term will just depend upon t_2, making it inconsistent to impose $\xi_{D7} = 0$ and falling back to the model considered before where charged matter fields were needed, giving rise to a wrong hidden gauge boson mass.

4.4 A Triple K3 Fibration With a Blow-Up Mode

As we said in §3.4 there are various ways to construct Calabi-Yau manifolds and the biggest database is constructed by means of toric geometries. Also, all the relevant characteristics of these manifolds can be explicitly calculated by means of the methods developed in [Alt+15]. Every Calabi-Yau manifold is constructed starting from a hypersurface in a toric ambient space. What has been shown is that if the hypersurface in the ambient space is constructed from lattice polytopes obeying the condition of reflexivity, then it will be a Calabi-Yau manifold. For these construction one can refer to [Alt+15; Lou+12; KS02]. Here, we are going to consider a Calabi-Yau three-fold $\mathcal{Y}$ with 4 Kähler moduli embedded in toric ambient space with a triple K3 fibration, with volume mode of the form:

$$\mathcal{V} = k_{123}\, t_1 t_2 t_3 + k_{444}\, t_4^3 \tag{4.4.1}$$

where t_1, t_2, t_3 are bulk moduli and t_4 is the 2-cycle associated to a small blow-up τ_4 where non-perturbative effects will be turned on. An example of such a Calabi-Yau is the polytope ID #1206 of the database constructed by [Alt+15]. This Calabi-Yau was also considered by Cicoli et al. [Cic+17] as a model for fibre inflation and it has $k_{123} = 2$ and $k_{444} = 1/3$. We can then consider this Calabi-Yau to have the same Hodge numbers as the polytope mentioned above:

$$(h^{(1,1)}, h^{(2,1)}) = (4, 98) \implies \chi_E = -188 \tag{4.4.2}$$

The 4-cycle D_4 is a del Pezzo divisor dP_7, while the other 4-cycles are non-del Pezzo rigid divisors, resulting in three $K3$ fibrations over different $\mathbb{P}^1$ bases. We see that this Calabi-Yau satisfies the two topological conditions for the existence of a LVS minimum [CCQ08a], namely the fact that the Euler characteristic is negative (more precisely $h^{2,1} > h^{(1,1)} > 1$) and the presence of a diagonal blow-up mode.
The 4-cycle volumes will be given by:

$$\begin{aligned} \tau_1 &= \frac{\partial \mathcal{V}}{\partial t^1} = k_{123}\, t_2 t_3; & \tau_2 &= \frac{\partial \mathcal{V}}{\partial t^2} = k_{123}\, t_1 t_3; \\ \tau_3 &= \frac{\partial \mathcal{V}}{\partial t^3} = k_{123}\, t_1 t_2; & \tau_4 &= \frac{\partial \mathcal{V}}{\partial t^4} = 3k_{444}\, t_4^2. \end{aligned} \tag{4.4.3}$$

and the constraints given by the Kähler cone (2.2.61) will impose $t_4 < 0$. This allows to rewrite the volume in function of these 4-cycles:

$$\mathcal{V} = \frac{1}{\sqrt{k_{123}}}\sqrt{\tau_1 \tau_2 \tau_3} - \frac{1}{\sqrt{27 k_{444}}}\tau_4^{3/2} \tag{4.4.4}$$

The diagonal part of the Kähler metric in the limit $\tau_4 \ll \tau_1, \tau_2 \tau_3$ (since τ_4 is a small blow-up mode which will be used to fix the volume exponentially large):

$$(K_{\text{tree}})_{ij} = \frac{\partial^2 K_{\text{tree}}}{\partial \tau_i \partial \tau_j} \sim \begin{pmatrix} \tau_1^{-2} & & & \\ & \tau_2^{-2} & & \\ & & \tau_3^{-2} & \\ & & & (2\sqrt{3k_{444}}\mathcal{V}\sqrt{\tau_4})^{-1} \end{pmatrix} \tag{4.4.5}$$

We know that when we consider the Stückelberg mechanism, we need to turn on fluxes and these in turn will generate a Fayet-Iliopoulos term of the form (4.2.38). Since we would like to bring us back to the promising case of §4.3.2 but without worrying about the FI-term, we are going to wrap a D7-brane on τ_1 supporting the hidden gauge sector. The Stückelberg mechanism will also generate the following FI-term:

$$\xi_1 = \frac{q_{1j}t_j}{4\pi\mathcal{V}} = \frac{1}{4\pi\mathcal{V}}(q_{12}t_2 + q_{13}t_3) = \frac{k_{123}}{4\pi\mathcal{V}}(f_3t_2 + f_2t_3) \tag{4.4.6}$$

where the charges are given by $q_{1j} = k_{1jk}f_k$. We can the impose $\xi_1 = 0$ which fixes the combination $f_3t_2 + f_2t_3 = 0$. This means that since f_2, f_3 should be of order unity, the 2-cycles t_2 and t_3 should be of the same order: $t_2 \sim t_3$, leading to the 4-cycle volumes:

$$\begin{aligned} \tau_1 &\simeq k_{123}\, t_2^2 \simeq k_{123}\, t_3^2 \\ \tau_2 &= k_{123}\, t_1 t_3 \simeq k_{123}\, t_1 t_2 \\ \tau_3 &= k_{123}\, t_1 t_2 \simeq k_{123}\, t_1 t_3 \end{aligned} \tag{4.4.7}$$

making:

$$\tau_2 \simeq \tau_3 \tag{4.4.8}$$

in such a way that the volume mode will be of the form:

$$\mathcal{V} \simeq \frac{1}{\sqrt{k_{123}}}\sqrt{\tau_1}\tau_2 - \frac{1}{\sqrt{27k_{444}}}\tau_4^{3/2} \tag{4.4.9}$$

which is the same as the one considered in §4.3.2, meaning that we can transpose here all the considerations that we have made there. Namely, we could get a hidden gauge sector supported on a D7-brane wrapping the small cycle D_1 (with $vol(D_1) = \tau_1$) kinetically coupled to the visible sector supported on a small blow-up mode τ_{vis} (which is not included in the above model, but it is supposed to be there and its presence is in fact irrelevant in the stabilisation process). As we saw in §4.3.2 this set-up is very promising in reproducing the suitable mass and coupling of the hidden gauge boson. We are then going to search for values of the moduli reproducing (4.3.27) and (4.3.28). What we have to do now is to check whether the mass generated via Stückelberg mechanism is of the right order of magnitude (without worrying about the presence of charged fields since we are imposing the vanishing of the FI-term). Considering the general formula:

$$m_{Z'}^2 = g_{\text{hid}}^2 \frac{M_P^2}{4\pi^2} q_{1j}\,(K_{\text{tree}})_{jk}\, q_{1k} \tag{4.4.10}$$

the contributions to the mass will be:

$$\begin{aligned} q_{12}\,(K_{\text{tree}})_{22}\, q_{12} &= \left(\frac{k_{123}\, f_3}{\tau_2}\right)^2 \\ q_{13}\,(K_{\text{tree}})_{33}\, q_{13} &= \left(\frac{k_{123}\, f_2}{\tau_3}\right)^2 \end{aligned} \tag{4.4.11}$$

and now, contrary to the case considered in §4.3.3, these two contributions will be of the same order of magnitude since $\tau_2 \simeq \tau_3$. The resulting mass for the hidden gauge boson will be:

$$m_{Z'}^2 = \frac{M_P^2}{2\pi\tau_1}\left(\left(\frac{k_{123}\, f_3}{\tau_2}\right)^2 + \left(\frac{k_{123}\, f_2}{\tau_3}\right)^2\right) \simeq \frac{M_P^2}{\tau_1\tau_2^2} \implies m_{Z'} \simeq \frac{M_P}{\sqrt{\tau_1}\tau_2} \simeq \frac{M_P}{\mathcal{V}} \tag{4.4.12}$$

which is of the same form as (4.3.24). Thus, provided we manage to fix the modulus τ_1 small, while $\mathcal{V}$ of order (4.3.28), we will be able to reach the interesting parameter region given by (4.1.7). In order to do that, we are going to wrap a D7-brane undergoing gaugino condensation on τ_4 so that a non-perturbative correction to the superpotential will be generated, thanks to which, by also considering the (α')-correction to the Kähler potential, a LVS potential $V_{\text{LVS}}(\mathcal{V}, \tau_4)$ will appear. By the interplay between a higher order (α')-correction which will be a function of the modulus τ_1 (found in [CLW15] and applied to a string inflationary model in [Cic+16]) and string loop corrections, we will be able to fix τ_1 to small values and the volume mode exponentially large in function of the small blow-up mode τ_4 as the LVS claims. Let's now put all of these together.

4.4.1 (α') and Non-Perturbative Corrections

Since τ_4 is a small diagonal blow-up, we can wrap a D7-brane around it undergoing gaugino condensation without incurring in any tension between this effect and chirality since the visible sector is supported on a D7-brane wrapped around another blow-up with volume τ_{vis} (see §3.2). This will then generate a correction to the superpotential of the form:

$$\delta W_{\text{np}} = A_4 e^{-a_4 T_4} \tag{4.4.13}$$

where we are discarding higher order contributions of the form $e^{-2a_4 T_4}, e^{-3a_4 T_4}, \ldots$ by demanding $a_4 \tau_4 \gg 1$. Using (3.2.7) we can infer that the correction to the scalar potential will be of the form (where here $(K_{\text{tree}})_{44} = \partial^2 K_{\text{tree}} / \partial T_4^2 = (1/4) \partial^2 K_{\text{tree}} / \partial \tau_4^2$):

$$\begin{aligned}
\delta V_{\text{np}} &= e^K ((K_{\text{tree}})_{44})^{-1} \left(A_4^2 a_4^2 e^{-2a_4\tau_4} - \frac{1}{2} W_0 A_4 a_4 \frac{\partial K_{\text{tree}}}{\partial \tau_4} (e^{-a_4 T_4} + e^{-a_4 \overline{T}_4}) \right) \\
&= \frac{e^{K_{\text{cs}}} g_s}{2\mathcal{V}^2} (8\sqrt{3k_{444}}\, \mathcal{V} \sqrt{\tau_4}) \left(A_4^2 a_4^2 e^{-2a_4\tau_4} - \frac{1}{2} W_0 A_4 a_4 \frac{\sqrt{\tau_4}}{\sqrt{3k_{444}}\, \mathcal{V}} e^{-a_4\tau_4} (e^{ia_4 b_4} + e^{-ia_4 b_4}) \right) \\
&= \frac{e^{K_{\text{cs}}} g_s}{2} \left(\frac{8\sqrt{3k_{444}} A_4^2 a_4^2 e^{-a_4\tau_4} \sqrt{\tau_4}}{\mathcal{V}} - \frac{4 W_0 A_4 a_4 \tau_4 e^{-a_4\tau_4}}{\mathcal{V}^2} \cos(a_4 b_4) \right)
\end{aligned} \tag{4.4.14}$$

In the above expression we see that it appears the axionic part b_4 of the Kähler modulus T_4. We can fix it by:

$$\left. \frac{\partial V_{\text{np}}}{\partial b_4} \right|_{\langle b_4 \rangle} = 0 \iff \sin(a_4 \langle b_4 \rangle) = 0 \iff \langle b_4 \rangle = \frac{n\pi}{a_4}; \quad n \in \mathbb{Z} \tag{4.4.15}$$

The non-perturbative potential, with the correct pre-factor (see the appendix of [Bur+10]) will then be (in M_P-units):

$$\delta V_{\text{np}} = \frac{e^{K_{\text{cs}}} g_s}{8\pi} \left(\frac{c\sqrt{x} e^{-2x}}{\mathcal{V}} - \frac{b x e^{-x}}{\mathcal{V}^2} \right) \tag{4.4.16}$$

with the following definitions:

$$\begin{aligned}
x &:= a_4 \tau_4 \\
b &:= 4 W_0 A_4 \\
c &:= 8\sqrt{3k_{444}} A_4^2 a_4^{3/2}
\end{aligned} \tag{4.4.17}$$

Taking into account (α')-corrections to the Kähler potential given by (3.2.15), namely of the form (again with the correct pre-factor):

$$\delta V_{(\alpha')} \simeq \frac{e^{K_{cs}} g_s}{8\pi} \left(\frac{3\xi |W_0|^2}{4 g_s^{3/2} \mathcal{V}^3} \right) \tag{4.4.18}$$

together with the non-perturbative corrections, we get to the LVS potential:

$$V_{\text{LVS}}(\mathcal{V}, \tau_4) = \delta V_{\text{np}} + \delta V_{(\alpha')} = \frac{e^{K_{cs}} g_s}{8\pi} \left(\frac{c\sqrt{x} e^{-2x}}{\mathcal{V}} - \frac{b x e^{-x}}{\mathcal{V}^2} + \frac{3\xi W_0^2}{4 g_s^{3/2} \mathcal{V}^3} \right) \tag{4.4.19}$$

As we can see, we have found a potential term for the volume mode $\mathcal{V}$ and the blow-up modulus τ_4, without any contribution for the modulus τ_1. We could then stabilise at this order the above mentioned moduli, and then employ string loop corrections and higher order (α')-corrections to stabilise at the next order τ_1. However, with a bit of prescience, the contribution of string loop and higher order (α') corrections cannot be completely ignored in the stabilisation of τ_4 and $\mathcal{V}$, since, even if its contribution to τ_4 is small, it will have a significant contribution to the volume mode (because it will be fixed at an exponentially large value in function of the blow-up τ_4 as the LVS prescription predicts). We will then stabilise all the moduli at the order of the string loop corrections.

4.4.2 String Loop Corrections and Higher Order (α')-Corrections

In order to fix the VEV of τ_1, we consider the string loop corrections presented in §3.2.3. As we saw, there are two kind of contributions that can arise: corrections due to the exchange of Kaluza-Klein modes between D7-branes and D3-branes and/or corrections due to the exchange of winding modes between intersecting (stacks of) D7-branes. The former are given by:

$$\delta V^{\text{KK}}_{(g_s)} = g_s^2 \left(\frac{g_s e^{K_{cs}}}{8\pi} \right) \frac{W_0^2}{\mathcal{V}^2} \sum_i (C_i^{KK})^2 (K_{\text{tree}})_{i\bar{i}} \tag{4.4.20}$$

with $(K_{\text{tree}})_{i\bar{i}} = \partial^2 K_{\text{tree}} / \partial T^i \partial \overline{T}^i = (1/4) \partial^2 K_{\text{tree}} / \partial \tau^i \partial \tau^i$. While the latter will be:

$$\delta V^{\text{W}}_{(g_s)} = -\left(\frac{g_s e^{K_{cs}}}{8\pi} \right) \frac{W_0^2}{\mathcal{V}^2} \sum_i \frac{2 C_i^W}{\mathcal{V}(a_{ij} t^j)} \tag{4.4.21}$$

where $a_{ij} t^j$ is the 2-cycle where the D-branes intersect.

If we wrap D7-branes on every 4-cycle τ_i and consider the existence of a D3-brane thanks to which the exchange of Kaluza-Klein modes can take place, we will get the following contributions (for $i, j, k = 1, 2, 3$):

$$\begin{aligned} \delta V^{\text{KK}}_{(g_s),\tau_i} &= \frac{g_s e^{K_{cs}}}{8\pi} \frac{\Delta_1}{\mathcal{V}^2 \tau_i^2}; \qquad \Delta_i = \frac{(g_s C_i^{KK})^2 W_0^2}{4} \\ \delta V^{W}_{(g_s),\tau_i \cap \tau_j} &= -\frac{g_s e^{K_{cs}}}{8\pi} \frac{\Xi_k}{\mathcal{V}^3 t_k}; \qquad i \neq j \neq k; \qquad \Xi_k = 2 W_0^2 C_k^W \end{aligned} \tag{4.4.22}$$

and:

$$\delta V^{\mathrm{KK}}_{(g_s),\tau_4} = \frac{g_s e^{K_{cs}}}{8\pi} \frac{\Delta_4}{2\sqrt{3k_{444}}\mathcal{V}^3\sqrt{\tau_4}} \tag{4.4.23}$$

In the stabilisation of τ_1, this last term won't contribute, so clearly we are not considering it in the following discussion. The other terms can be manipulated considering the fact that $\tau_2 \simeq \tau_3$ and $\tau_4 \ll \tau_1 \ll \tau_2$ making the volume of the form $\mathcal{V} \sim \sqrt{\tau_1}\tau_2 \sim \sqrt{\tau_1}\tau_3$. Using these facts, the Kaluza-Klein corrections can be re-written as:

$$\delta V^{\mathrm{KK}}_{(g_s),\tau_1} = \frac{g_s e^{K_{cs}}}{8\pi}\frac{\Delta_1}{\mathcal{V}^2\tau_1^2}; \quad \delta V^{\mathrm{KK}}_{(g_s),\tau_2} \simeq \frac{g_s e^{K_{cs}}}{8\pi}\frac{\Delta_2\tau_1}{\mathcal{V}^4}; \quad \delta V^{\mathrm{KK}}_{(g_s),\tau_3} \simeq \frac{g_s e^{K_{cs}}}{8\pi}\frac{\Delta_3\tau_1}{\mathcal{V}^4} \tag{4.4.24}$$

while the winding corrections become:

$$\delta V^{W}_{(g_s),\tau_1\cap\tau_2} \simeq -\frac{g_s e^{K_{cs}}}{8\pi}\frac{\Xi_3}{\mathcal{V}^3\sqrt{\tau_1}}; \quad \delta V^{W}_{(g_s),\tau_1\cap\tau_3} \simeq -\frac{g_s e^{K_{cs}}}{8\pi}\frac{\Xi_2}{\mathcal{V}^3\sqrt{\tau_1}}; \quad \delta V^{W}_{(g_s),\tau_2\cap\tau_3} \simeq -\frac{g_s e^{K_{cs}}}{8\pi}\frac{\Xi_3\tau_1}{\mathcal{V}^4}; \tag{4.4.25}$$

The resulting scalar potential for τ_1 is given by the sum of all the above contributions, leading to the following:

$$\delta V_{(g_s)}(\tau_1,\mathcal{V}) = \frac{g_s e^{K_{cs}}}{8\pi}\left(\frac{\Delta_1}{\mathcal{V}^2\tau_1^2} + \frac{\Delta_{233}\tau_1}{\mathcal{V}^4} - \frac{\Xi_{23}}{\mathcal{V}^3\sqrt{\tau_1}}\right) \tag{4.4.26}$$

where clearly:

$$\Delta_{233} = \Delta_2 + \Delta_3 - \Xi_3; \qquad \Xi_{23} = \Xi_2 + \Xi_3 \tag{4.4.27}$$

If we minimise this potential with respect to τ_1, we get:

$$\left.\frac{\partial\,\delta V_{(g_s)}}{\partial\tau_1}\right|_{\langle\tau_1\rangle,\langle\mathcal{V}\rangle} = 0 \iff \langle\tau_1\rangle^{-3/2} = \frac{\Xi_{23}}{8\Delta_1\langle\mathcal{V}\rangle}\left(1+\mathrm{sgn}(B)\sqrt{\frac{32\Delta_1\Delta_{233}}{\Xi_{23}}+1}\right) \tag{4.4.28}$$

where $\mathrm{sgn}(B)$ is the sign-function.

Natural values for the loop coefficients C_i^{KK} and C_k^W are of order $O(1)$, and from the above expression, we see that $\langle\tau_1\rangle \sim \langle\mathcal{V}\rangle^{2/3}$. This relation is problematic, since we are seeking a stabilisation which will make τ_1 small, certainly not of the same order of the volume. This brings us to try a different set-up of string loop corrections and eventually taking into consideration higher order (α')-corrections thanks to which we will be able to fix τ_1 small.

D7-brane Wrapping τ_1 and τ_2, Without D3-branes

If there are no D3-branes, corrections induced by the exchange of Kaluza-Klein modes do not appear. We are then left with the correction induced by winding modes exchanged in the intersection $\tau_1 \cap \tau_2$:

$$\delta V^{W}_{(g_s),\tau_1\cap\tau_2} \simeq -\frac{g_s e^{K_{cs}}}{8\pi}\frac{\Xi_3}{\mathcal{V}^3\sqrt{\tau_1}} \tag{4.4.29}$$

This set-up is represented in the figure (4.4). Needless to say, the above contribution alone cannot be used to fix the modulus τ_1 to a finite value. In addition to the loop correction we are then going to consider higher order (α')-corrections coming from terms of the form $F^4 O(\alpha'^3)$ in the

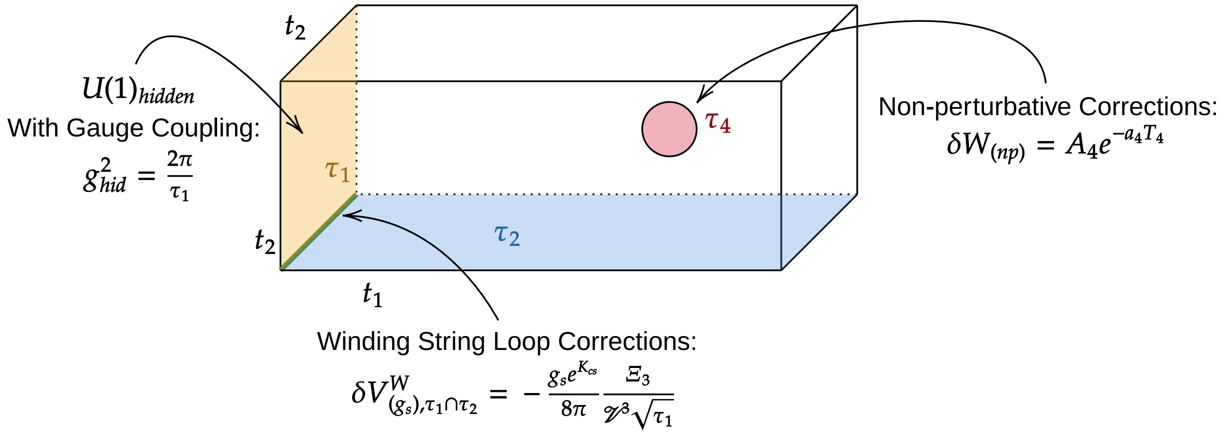


Figure 4.4: Pictorial representation of the Calabi-Yau volume in terms of 2-cycles (represented as segments) and 4-cycles (represented as surfaces) with the blow-up modulus τ_4 which has the effect of decreasing the effective volume, making it of Swiss cheese type. It is also shown the origin of the non-perturbative and g_s-string loop corrections, as well as the place where the hidden gauge sector is supported.

10D supergravity action. These type of subleading corrections were computed by Ciupke, Louis, and Westphal [CLW15] and take the following form:

$$\delta V_{F^4} = -\left(\frac{g_s e^{K_{cs}}}{8\pi}\right)^2 \frac{\lambda W_0^4}{g_s^{3/2}\mathcal{V}^4}\Pi_i t_i \tag{4.4.30}$$

where the Π_is are topological integers defined as:

$$\Pi_i = \int_{\mathcal{Y}} c_2(T\mathcal{Y})\wedge \hat{D}_i \tag{4.4.31}$$

with $c_2(T\mathcal{Y})$ the second Chern class of the Calabi-Yau manifold $\mathcal{Y}$ with Kähler form $J = t_i\hat{D}_i$ and from appendix A we see that $c_2(T\mathcal{Y}) \propto \mathrm{tr}(\mathcal{R}^2) - (\mathrm{tr}(\mathcal{R}))^2$ with $\mathcal{R} = R_{i\bar{j}} dz^i \wedge d\bar{z}^j$ be the curvature 2-form. The natural range in which these topological numbers are defined is something like $O(1)\div O(10)$, as can be seen for the Calabi-Yau considered in [Cic+17] (which has the same volume form of our Calabi-Yau). The parameter λ is an overall unknown numerical factor which is believed to take values in the range $10^{-2}\div 10^{-3}$ (as can be seen by an explicit computation for one Kähler modulus performed by Grimm, Mayer, and Weissenbacher [GMW18]).
When applied to our Calabi-Yau, the V_{F^4} term will become:

$$\begin{aligned}\delta V_{F^4} &= -\left(\frac{g_s e^{K_{cs}}}{8\pi}\right)^2 \frac{\lambda W_0^4}{g_s^{3/2}\mathcal{V}^4}(\Pi_1 t_1 + \Pi_2 t_2 + \Pi_3 t_3)\\ &\simeq -\left(\frac{g_s e^{K_{cs}}}{8\pi}\right)^2 \frac{\lambda W_0^4}{g_s^{3/2}\mathcal{V}^4}\left(\Pi_1\frac{\mathcal{V}}{\tau_1} + \Pi_{23}\sqrt{\tau_1}\right)\end{aligned} \tag{4.4.32}$$

with:

$$\Pi_{23} = \Pi_2 + \Pi_3 \tag{4.4.33}$$

The scalar potential for the modulus τ_1 will be:

$$\delta V(\tau_1, \mathcal{V}) = \delta V_{(g_s)} + \delta V_{F^4} = -\frac{g_s e^{K_{cs}}}{8\pi}\frac{1}{\mathcal{V}^3}\left(\frac{\Xi_3}{\sqrt{\tau_1}} - \frac{|\Theta_1|}{\tau_1} + \frac{\Pi_{23}\sqrt{\tau_1}}{\mathcal{V}}\right) \tag{4.4.34}$$

with:

$$|\Theta_1| := \frac{e^{K_{cs}}|\lambda| W_0^4 \Pi_1}{8\pi\sqrt{g_s}} \tag{4.4.35}$$

The minus sign before the term $|\Theta_1|$ appears because the parameter $\lambda = -|\lambda|$ is negative as can be inferred from the explicit computation provided in [GMW18]. Minimising the above potential considering that the volume will be fixed exponentially large, leads to:

$$\left.\frac{\partial\, \delta V}{\partial \tau_1}\right|_{\langle\tau_1\rangle,\langle\mathcal{V}\rangle} = 0 \iff \langle\tau_1\rangle = \frac{4\Theta_1^2}{\Xi_3^2} + O\left(\frac{1}{\mathcal{V}^{1/2}}\right) = \left(\frac{e^{K_{cs}}\lambda W_0^2 \Pi_1}{8\pi\sqrt{g_s}C_3^W}\right)^2 \tag{4.4.36}$$

We can also verify that it is indeed a minimum by computing the second derivative:

$$\left.\frac{\partial^2\, \delta V}{\partial \tau_1^2}\right|_{\langle\tau_1\rangle,\langle\mathcal{V}\rangle} \propto -\frac{3\,\Xi_3}{4\langle\tau_1\rangle^{5/2}} - \frac{2\Theta_1}{\langle\tau_1\rangle^3} = \frac{-3\sqrt{\langle\tau_1\rangle}\,\Xi_3 + 8|\Theta_1|}{4\langle\tau_1\rangle^3} = \frac{-6|\Theta_1| + 8|\Theta_1|}{4\langle\tau_1\rangle^3} > 0 \tag{4.4.37}$$

Integrating out the modulus τ_1, namely substituting its VEV into its contribution δV to the scalar potential V, we get:

$$\delta V = -\frac{g_s e^{K_{cs}}}{8\pi}\frac{\Xi_3^2}{4|\Theta_1|\mathcal{V}^3} \tag{4.4.38}$$

What we have to do now, is stabilise the other directions, namely τ_4 and the volume mode $\mathcal{V}$, also we can see that the effect of fixing τ_1 is a shift in the term $3\xi W_0^2/4g_s^{3/2}\mathcal{V}^3$ coming from (α') corrections.

4.4.3 Stabilisation of $\mathcal{V}$ and τ_4

Now that we have stabilised τ_1, we can consider the resulting scalar potential (where we recall that $x = a_4\tau_4$):

$$V(x, \mathcal{V}) = V_{LVS} + \delta V = \frac{g_s e^{K_{cs}}}{8\pi}\left(\frac{c\sqrt{x}e^{-2x}}{\mathcal{V}} - \frac{b x e^{-x}}{\mathcal{V}^2} + \frac{3\xi W_0^2}{4g_s^{3/2}\mathcal{V}^3} - \frac{\Xi_3^2}{4|\Theta_1|\mathcal{V}^3}\right) \tag{4.4.39}$$

and by defining:

$$\Lambda := \frac{3\xi W_0^2}{4g_s^{3/2}} - \frac{\Xi_3^2}{4|\Theta_1|} \tag{4.4.40}$$

we get to:

$$V(x, \mathcal{V}) = \frac{g_s e^{K_{cs}}}{8\pi}\left(\frac{c\sqrt{x}e^{-2x}}{\mathcal{V}} - \frac{b x e^{-x}}{\mathcal{V}^2} + \frac{\Lambda}{\mathcal{V}^3}\right) \tag{4.4.41}$$

Let's now minimise the above potential in order to see if we can find suitable minima for τ_4 and $\mathcal{V}$. The critical points will be given by the solutions of:

$$\nabla V|_{\langle x\rangle,\langle\mathcal{V}\rangle} = \vec{0} \tag{4.4.42}$$

namely:

$$\left.\frac{\partial V}{\partial x}\right|_{\langle x\rangle,\langle\mathcal{V}\rangle}=0 \iff \mathcal{V}=\frac{2be^{\langle x\rangle}(-1+\langle x\rangle)\sqrt{\langle x\rangle}}{c(-1+4\langle x\rangle)}\simeq\frac{be^{\langle x\rangle}\sqrt{\langle x\rangle}}{2c} \tag{4.4.43}$$

$$\left.\frac{\partial V}{\partial \mathcal{V}}\right|_{\langle x\rangle,\langle\mathcal{V}\rangle}=0 \iff \frac{ce^{-2\langle x\rangle}\langle\mathcal{V}\rangle^2\sqrt{\langle x\rangle}-2be^{-\langle x\rangle}\langle\mathcal{V}\rangle\langle x\rangle+3\Lambda}{\langle\mathcal{V}\rangle^4}=0 \tag{4.4.44}$$

now, using the approximate form of the volume mode (4.4.43) we obtain:

$$\frac{12c^3e^{-4\langle x\rangle}(b^2\langle x\rangle^{3/2}-4c\Lambda)}{b^4\langle x\rangle^2}=0 \iff \langle x\rangle\simeq\left(\frac{4c\Lambda}{b^2}\right)^{2/3} \tag{4.4.45}$$

And the volume mode becomes:

$$\langle\mathcal{V}\rangle\simeq\frac{b}{2c}e^{(4c\Lambda/b^2)^{2/3}}\left(\frac{4c\Lambda}{b^2}\right)^{1/3} \tag{4.4.46}$$

realising the exponentially large volume required by the LARGE volume scenario stabilisation mechanism. If we try to evaluate the potential (4.4.41) in the approximate minima $\langle x\rangle$ and $\langle\mathcal{V}\rangle$, we obtain (by plugging (4.4.43) back into (4.4.41) and using 4.4.45):

$$\begin{aligned} V(\langle x\rangle,\langle\mathcal{V}\rangle) &= \frac{1}{\langle\mathcal{V}\rangle^3}\left(\frac{b\sqrt{\langle x\rangle}}{2c}\left(\frac{b\sqrt{\langle x\rangle}}{2c}\right)^2-b\langle x\rangle\frac{b\sqrt{\langle x\rangle}}{2c}+\Lambda\right) \\ &= \frac{1}{\langle\mathcal{V}\rangle^3}\left(-\frac{b^2\langle x\rangle^{3/2}}{4c}+\frac{b^2\langle x\rangle^{3/2}}{4c}\right)=0 \end{aligned} \tag{4.4.47}$$

This means that we can't say whether our expectation values provide a *de Sitter* or *Anti-de Sitter* minimum. In order to extrapolate this information, we should take into account the exact form of (4.4.43), namely:

$$\langle\mathcal{V}\rangle=\frac{2be^{\langle x\rangle}(\langle x\rangle-1)\sqrt{\langle x\rangle}}{c(4\langle x\rangle-1)} \tag{4.4.48}$$

in this way, from (4.4.44) we will obtain an expression for Λ:

$$\Lambda=\frac{4b^2\langle x\rangle^{5/2}(\langle x\rangle-1)}{c(4\langle x\rangle-1)^2} \tag{4.4.49}$$

From the above relation, since $x\gg 1$ and $c>0$, we see that $\Lambda>0$, which means that the following condition must be satisfied:

$$\Lambda=\frac{3\xi W_0^2}{4g_s^{3/2}}-\frac{\Xi_3^2}{4|\Theta_1|}=\frac{3\xi W_0^2}{4g_s^{3/2}}\left[1-\frac{32\pi g_s^2(C_3^W)^2}{3\,e^{K_{cs}}|\lambda|\xi W_0^4\Pi_1}\right]>0 \iff 1>\frac{32\pi g_s^2(C_3^W)^2}{3\,e^{K_{cs}}|\lambda|\xi W_0^4\Pi_1} \tag{4.4.50}$$

Now, by substituting the (4.4.49) and (4.4.48) back into the potential, we finally obtain:

$$V(\langle\tau_1\rangle,\langle x\rangle,\langle\mathcal{V}\rangle)=-\frac{2b^2\langle x\rangle^{3/2}(\langle x\rangle-1)}{c\langle\mathcal{V}\rangle^3(4\langle x\rangle-1)^2}\cdot\frac{g_se^{K_{cs}}}{8\pi} \tag{4.4.51}$$

As we can see, since $\langle x\rangle\gg 1$, the minimum that we found is Anti-de Sitter, $V<0$, this means that we will have to employ an uplift mechanism in order to get a de Sitter vacuum.

4.4.4 Considerations on Gravitino Mass and Free Parameters of the Theory

The LARGE Volume Scenario vacua in general lead naturally to an intermediate string scale, namely:

$$M_s \simeq 10^{11} \text{ GeV} \tag{4.4.52}$$

since via the relation (see appendix B of [CQV16]):

$$\frac{1}{\ell_s} = M_s = \frac{g_s^{1/4} M_P}{\sqrt{4\pi \mathcal{V}}} \tag{4.4.53}$$

and for values of $\mathcal{V} \simeq 10^{12}$ and $g_s \simeq 10^{-1}$, we indeed get (4.4.52). The gravitino mass in these type IIB compactifications is generally given by the following leading order expression:

$$m_{3/2} \simeq e^{K/2}|W| = \sqrt{\frac{g_s}{8\pi}} \frac{M_P W_0}{\mathcal{V}} \tag{4.4.54}$$

And in our case, we see that if we want to reproduce a volume of the magnitude $\mathcal{V} \simeq 10^{18}$, with the string coupling $g_s = 0.1$, we have:

$$m_{3/2} \simeq 0.151 \cdot W_0 \text{ GeV} \tag{4.4.55}$$

In order to reproduce TeV scale supersymmetry, we are then fixing the value of W_0 to the order:

$$W_0 = O(10^4) \tag{4.4.56}$$

leading to a gravitino mass of the TeV scale: $m_{3/2} = O(1\,\text{TeV})$.

Consistency Conditions - Neglect of Stringy Kaluza-Klein Modes

What we have to check is if these parameters are consistent with the supergravity approximation. Since we are discarding Kaluza-Klein (KK) massive modes when considering the 10D supergravity action (1.4.5), we should check whether the resulting vacuum energy V_0, once all the moduli have been stabilised at their VEVs, is much smaller than the KK modes [CQS05]:

$$|V_0| \ll m_{KK}^4 \tag{4.4.57}$$

Now, the precise value of m_{KK} is unknown, but by making use of toroidal compactifications, one can at least infer its scaling. In fact when one compactify the 10-dimensional manifold in which strings propagate on a circle, one can see that the tower of Kaluza-Klein massive modes is given by:

$$m_{KK}^2 = \left[\left(\frac{n}{R}\right)^2 + \left(\frac{wR}{\alpha'}\right)^2 \right] \tag{4.4.58}$$

where n is the KK momentum and w is the winding number. In order to get the mass scale for general compactifications, we can consider $R = R_s \ell_s$ with $R \gg 1$, so that:

$$m_{KK} \sim \frac{M_s}{R_s}; \qquad m_W \sim (2\pi)^2 R_s M_s \tag{4.4.59}$$

For a Calabi-Yau $\mathcal{Y}$ with anisotropic compactification as in our case, the lightest Kaluza-Klein mode will scale as [CQS05]:

$$m_{KK}^4 \sim \frac{M_s^4}{\tau_{\text{big}}} \sim \frac{M_P^4}{\mathcal{V}^2 \tau_{\text{big}}} \tag{4.4.60}$$

This means that we can already see whether the condition (4.4.57) is satisfied or not. In fact, we are searching for values of the volume mode of order $\mathcal{V} \simeq 10^{18}$ and of the small modulus $\tau_1 = O(100)$. Since we have that effectively $\mathcal{V} \simeq \sqrt{\tau_1}\tau_2$, it means that the biggest 4-cycle τ_2 will be of order:

$$\tau_2 = O(10^{17}) \tag{4.4.61}$$

Using then the values of:

$$\mathcal{V} = 10^{18}; \qquad \tau_2 = 10^{17}; \tag{4.4.62}$$

we see that by combining (4.4.51) with (4.4.57) and (4.4.60) we get to an upper bound for W_0 roughly of (we recall that the potential was in units of M_P):

$$V_0 \sim \frac{g_s}{8\pi} \frac{M_P^4 W_0^2}{\mathcal{V}^3} \ll \frac{M_P^4}{\mathcal{V}^2 \tau_2} \iff W_0 \ll O(10) \tag{4.4.63}$$

Which seems to be incompatible with our choice of $W = O(10^4)$. However, we should note that after the uplift, the resulting vacuum energy density will be of order the cosmological constant, namely almost zero. In turn, the condition (4.4.57) will always be satisfied even for large value of W_0.

Consistency Conditions - Decoupling of Particle Masses

In order to get a trustable supergravity approximation, we also need to require/check a separation between the masses. Starting from the complex structure moduli, from [CQS05] we see that these masses will scale as:

$$m_{\text{cs}} \sim \frac{M_s}{\sqrt{\mathcal{V}}} \tag{4.4.64}$$

which means that the ratio m_{KK}/m_{cs} will be indeed much greater than 1:

$$\frac{m_{KK}}{m_{\text{cs}}} \sim \frac{1}{\mathcal{V}^{1/3}} \gg 1 \tag{4.4.65}$$

thanks to the fact the volume will be fixed at an exponentially large value and of order $O(10^{18})$. What can be seen is that a hierarchy of the following form appear:

$$m_{3/2} \ll m_{KK} \ll M_s \ll M_P \tag{4.4.66}$$

with the soft masses of order the gravitino mass.

Fixing the Parameters

We can already see whether this stabilisation leads to the searched VEVs for the moduli fields using natural values for the constants in play before employing the uplift mechanism. We then fix $W_0 = 10^4$ and $g_s = 0.1$ due to the above considerations on TeV scale supersymmetry. If we fix the other parameters to the seemingly most natural values[2]:

$$\begin{aligned} &A_4 = 1; \quad a_4 = 2\pi; \quad C_3^W = 1; \quad f_2 = 1; \quad f_3 = -1 \\ &e^{K_{cs}} = 1; \quad \xi = \frac{188 \cdot 1.2}{2(2\pi)^3}; \quad |\lambda| = 10^{-3}; \quad k_{444} = \tfrac{1}{3} \quad \Pi_1 = 1 \end{aligned} \tag{4.4.67}$$

from (4.4.36) and by numerically solving (4.4.48) and (4.4.49) we get:

$$\langle \tau_1 \rangle \simeq 1.58 \cdot 10^8; \qquad \langle x \rangle \simeq 49.0; \qquad \langle \mathcal{V} \rangle \simeq 2.09 \cdot 10^{24} \tag{4.4.68}$$

We can see that these values are not compatible with our requirements (4.3.27) and (4.3.28). If we play around with the values of the winding loop coefficient C_3^W and the λ, we see that by taking them as:

$$C_3^W = 50; \qquad |\lambda| = 10^{-4} \tag{4.4.69}$$

while leaving all the other parameters fixed, we get to the following VEVs for the moduli:

$$\langle \tau_1 \rangle \simeq 633; \qquad \langle x \rangle \simeq 42.8; \qquad \langle \mathcal{V} \rangle \simeq 4.14 \cdot 10^{21} \tag{4.4.70}$$

These are indeed closer to our searched values, but not closed enough. We would like then to study a bit the parameter space given by the pairs $(C_3^W, |\lambda|)$. First of all we note that little changes in these parameters can result in a big variation in the moduli's VEVs as can be appreciated in the figure (4.5). We see, there, that many values of C_3^W and $|\lambda|$ can reproduce a volume of the order $\mathcal{V} \simeq 10^{18}$ and a small modulus τ_1 of order $O(100)$. In fact, the points in the intersection of $\langle x(C_3^W, |\lambda|) \rangle$ with the horizontal plane $z = 34.6$ (i.e. the pairs $(C_3^W, |\lambda|)$ making $\langle x \rangle \simeq 34.6$) represents the values which give rise to a volume of 10^{18}, and we can see that for for such pairs the modulus τ_1 is of the right order of magnitude. If we want to find the best values for C_3^W and $|\lambda|$, we can impose by hand:

$$\begin{cases} \Lambda = \dfrac{4b^2 \langle x \rangle^{5/2} (\langle x \rangle - 1)}{c(4\langle x \rangle - 1)^2} \\ \langle \tau_1 (C_3^W, |\lambda|) \rangle = 300 \\ 10^{18} = \langle \mathcal{V} \rangle = \dfrac{2b e^{\langle x \rangle} (\langle x \rangle - 1) \sqrt{\langle x \rangle}}{c(4\langle x \rangle - 1)} \end{cases} \tag{4.4.71}$$

and by numerically solving this system (which are 3 equations in three unknowns: $\langle x \rangle, C_3^W$ and $|\lambda|$) with the other parameters fixed as in (4.4.67), we get the following values:

$$C_3^W \simeq 76.4; \qquad |\lambda| \simeq 1.05 \cdot 10^{-4}; \qquad \langle x \rangle = 34.63 \tag{4.4.72}$$

[2]We note here that the combinations of all possible values of the free parameters are clearly infinite, but these values are, however, fixed for each Calabi-Yau manifold, brane set-up and fluxes. In phenomenological models they are considered as free parameters since their exact values are either unknown or left as free in order to not restrict the study to any particular case, leaving the considerations as general as possible since phenomenological requirements will indeed guide the best configuration. Some of these parameters like Π_i, λ, $e^{K_{cs}}$, etc.. have been calculated, allowing us to infer what kind of ranges should be more natural to consider than others.

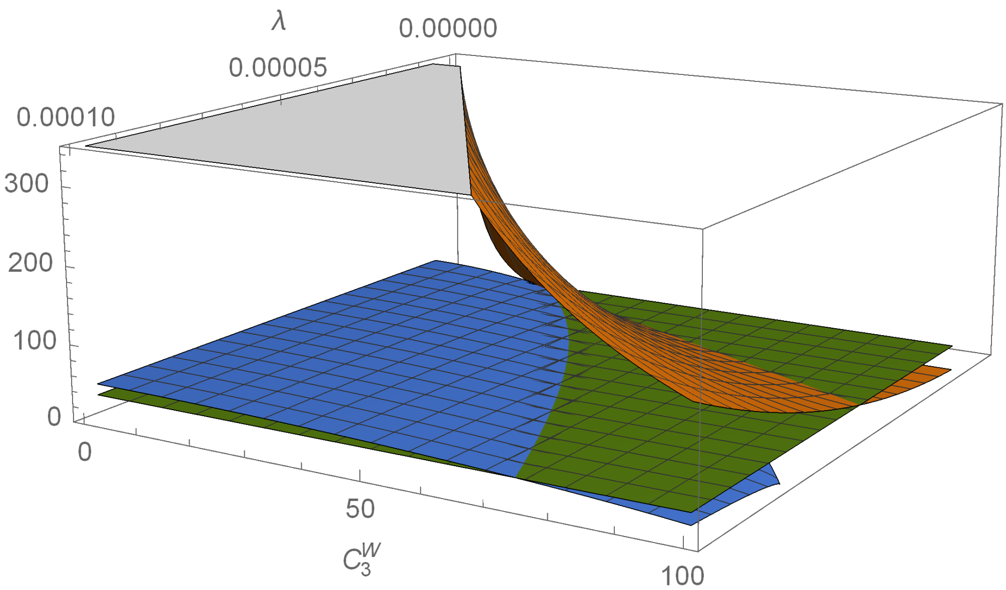


Figure 4.5: 3D plot of the stabilised moduli $\langle\tau_1\rangle$ (orange) and $\langle x\rangle$ (blue) in function of the winding loop coefficient $C_3^W \in [1, 100]$ and $|\lambda| \in [10^{-5}, 10^{-4}]$. It is also displayed the plane $z = 34.6$ (green) which enables to visually see the parameters' pairs making $\langle x\rangle = 34.6$ which in turn leads to a volume of the order $O(10^{18})$.

A posteriori the value of $\langle x\rangle = 34.63$ justifies our previous choice of the plane $z = 34.6$ in figure (4.5), claiming that it would have led to a volume of the order $O(10^{18})$.
Choosing $\tau_1 = 300$ leads, through (4.3.26), to the following coupling:

$$g' \sim 4.62 \cdot 10^{-4} \tag{4.4.73}$$

In $\log_{10}$ space, we have $\log_{10} g' \simeq -3.3$ which, by looking at the heat map (4.1), is compatible with a mass of the hidden gauge boson of 1 GeV. Also the values of C_3^W and λ (4.4.72) do not seem to actually deviate much from their expected natural values. We can also compute the vacuum energy through (4.4.51), obtaining:

$$V(\langle\tau_1\rangle, \langle x\rangle, \langle\mathcal{V}\rangle) \simeq -3.61 \cdot 10^{-49} \tag{4.4.74}$$

as the figure (4.6) shows.

4.4.5 Uplifting to a De Sitter

The vacuum state of our universe seems to be of de Sitter type since our universe is expanding with a non zero acceleration attributed to the existence of a positive vacuum energy density, namely a positive cosmological constant. In LVS models with the stabilisation techniques we have mentioned, one inevitably lands on an Anti-de Sitter space once all the moduli have been stabilised at their vacuum expectation values. In order to reproduce a de Sitter universe, additional terms contributing positive energy density should be considered. In the KKLT scenario [Kac+03], the authors brought into the scene anti-D3-branes which in their models allowed to uplift the vacuum to a de Sitter. However, there are other ways to do the exact same job, for example one can exploit T-branes [CQV16], Dilaton-dependent non-perturbative effects [Cic+12] or non-vanishing F-terms due to the fluxes introduced to fix the complex structure moduli and

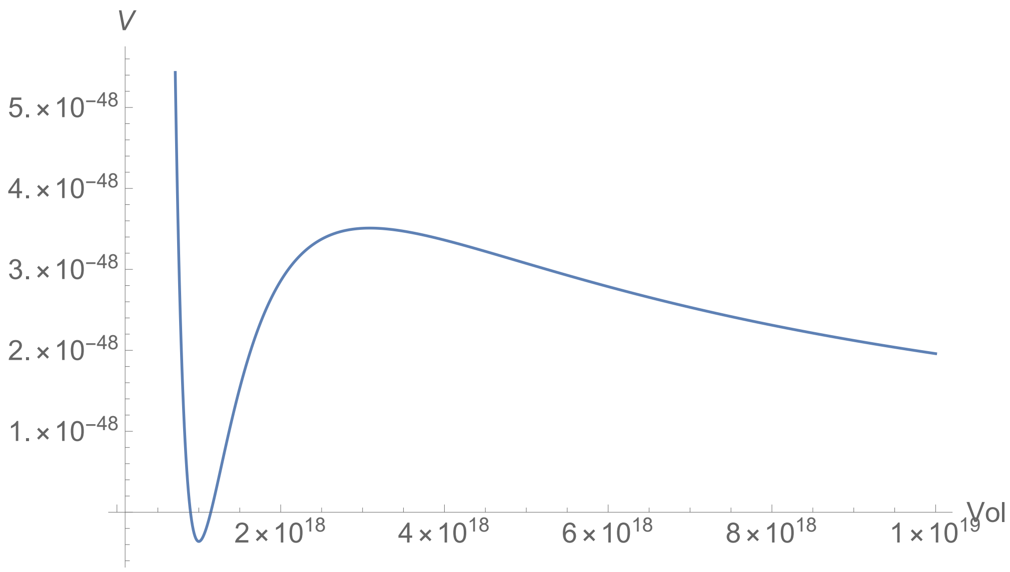


Figure 4.6: Plot of the potential $V(\mathcal{V})$ in function of the volume $\mathcal{V}$ once the other moduli τ_1 and $x = a_4\tau_4$ have been fixed. It is evident its metastable minimum with a negative value of the energy density.

the axio-dilaton [Gal+17]. In this work, we are going to employ the latter uplifting technique, namely we will follow the road smoothed out by Gallego et al. [Gal+17]. In §3.1 the stabilisation of the axio-dilaton S and the complex structure moduli U^α led us to impose at tree-level the supersymmetric conditions $D_S W = D_U W = 0$, which allowed us to fix S and U at their vacuum expectation values $\langle S\rangle$ and $\langle U\rangle$. However we can introduce a small amount of flux-induced supersymmetry breaking for these moduli by imposing:

$$F_I := D_I W = \epsilon W f_I \tag{4.4.75}$$

with $\epsilon \ll 1$, f_I a unit vector and $I = S, U^\alpha$. By introducing this new supersymmetry-breaking term for the axio-dilaton and complex structure moduli, it could be possible that the compactification is spoiled since the following term will appear in the scalar potential:

$$\delta V_{\mathrm{dS}} = \frac{g_s e^{K_{\mathrm{cs}}}}{8\pi} \frac{F_I \overline{F}^{\bar{I}}}{\mathcal{V}^2} \tag{4.4.76}$$

Since $F_I \overline{F}^{\bar{I}} \sim |W|^2$ the new term could cause a dangerous run-away for the volume mode. However if it is suitably small, $F_I \overline{F}^{\bar{I}}$ can provide an uplifting for the Large Volume Scenario potential from Anti-de Sitter to a de Sitter vacuum. In order to develop this behaviour the ϵ coefficient must be taken of the order:

$$\epsilon = O\left(\frac{1}{\sqrt{\mathcal{V}}}\right) \tag{4.4.77}$$

The total scalar potential considering all the contributions (non-perturbative effects via gaugino condensation on the brane wrapped around τ_4, (α')-corrections, string loop corrections, higher order (α')-corrections and this new term) will result in:

$$\begin{aligned} V(\tau_1,x,\mathcal{V}) &= \delta V_{np} + \delta V_{(\alpha')} + \delta V_{(g_s)} + \delta V_{F^4} + \delta V_{\mathrm{dS}} \\ &= \frac{g_s e^{K_{\mathrm{cs}}}}{8\pi}\left(\frac{c\sqrt{x}e^{-2x}}{\mathcal{V}} - \frac{bxe^{-x}}{\mathcal{V}^2} + \frac{3\xi W_0^2}{4g_s^{3/2}\mathcal{V}^3} - \frac{\Xi_3}{\mathcal{V}^3\sqrt{\tau_1}} + \frac{|\Theta_1|}{\mathcal{V}^3\tau_1} - \frac{\Pi_{23}\sqrt{\tau_1}}{\mathcal{V}^4} + \frac{\epsilon^2 W_0^2}{\mathcal{V}^2}\right) \end{aligned} \tag{4.4.78}$$

where x, b and c are defined as (4.4.17) The critical points for the potential will be given by the vanishing of the gradient:

$$\nabla V|_{\langle\tau_1\rangle,\langle x\rangle,\langle\mathcal{V}\rangle} = \vec{0} \iff \begin{cases} \left.\dfrac{\partial V}{\partial \tau_1}\right|_{\langle\tau_1\rangle,\langle x\rangle,\langle\mathcal{V}\rangle} = 0 \\ \left.\dfrac{\partial V}{\partial x}\right|_{\langle\tau_1\rangle,\langle x\rangle,\langle\mathcal{V}\rangle} = 0 \\ \left.\dfrac{\partial V}{\partial \mathcal{V}}\right|_{\langle\tau_1\rangle,\langle x\rangle,\langle\mathcal{V}\rangle} = 0 \end{cases} \tag{4.4.79}$$

The stabilisation of the τ_1 modulus goes precisely as before since the newly added term does not carry any factor which is function of τ_1:

$$\left.\frac{\partial V}{\partial \tau_1}\right|_{\langle\tau_1\rangle,\langle x\rangle,\langle\mathcal{V}\rangle} = 0 \iff \langle\tau_1\rangle = \frac{4\Theta_1^2}{\Xi_3^2} + O\left(\frac{1}{\mathcal{V}^{1/2}}\right) = \left(\frac{e^{K_{\mathrm{cs}}}\lambda W_0^2 \Pi_1}{8\pi\sqrt{g_s}C_3^W}\right)^2 \tag{4.4.80}$$

This means that the potential we will be using to minimise x and $\mathcal{V}$ is just:

$$\tilde{V}(\mathcal{V},x) = \frac{g_s e^{K_{\mathrm{cs}}}}{8\pi}\left(\frac{c\sqrt{x}e^{-2x}}{\mathcal{V}} - \frac{bxe^{-x}}{\mathcal{V}^2} + \frac{\Lambda}{\mathcal{V}^3} + \frac{\epsilon^2 W_0^2}{\mathcal{V}^2}\right) \tag{4.4.81}$$

where we have defined Λ as before in (4.4.40).
So, as far as the stabilisation of x and $\mathcal{V}$ is concerned, we have:

$$\begin{cases} \left.\dfrac{\partial \tilde{V}}{\partial x}\right|_{\langle x\rangle,\langle\mathcal{V}\rangle} = 0 \iff e^{-\langle x\rangle} = \dfrac{2b(-1+\langle x\rangle)\sqrt{\langle x\rangle}}{c(-1+4\langle x\rangle)\langle\mathcal{V}\rangle} \\ \left.\dfrac{\partial \tilde{V}}{\partial \mathcal{V}}\right|_{\langle x\rangle,\langle\mathcal{V}\rangle} = 0 \iff \Lambda + \frac{2}{3}\epsilon^2 W_0^2\langle\mathcal{V}\rangle = \dfrac{4b^2(-1+\langle x\rangle)\langle x\rangle^{5/2}}{c(1-4\langle x\rangle)^2} \end{cases} \tag{4.4.82}$$

Plugging these expressions back in (4.4.81) we obtain:

$$\tilde{V} = \frac{g_s e^{K_{\mathrm{cs}}}}{8\pi}\frac{1}{3\langle\mathcal{V}\rangle^3}\left[-\frac{6b^2(-1+\langle x\rangle)\langle x\rangle^{3/2}}{c(1-4\langle x\rangle)^2} + \epsilon^2 W_0^2\langle\mathcal{V}\rangle\right] \tag{4.4.83}$$

and since we are seeking a non-negative vacuum, the condition $\tilde{V}(\langle x\rangle,\langle\mathcal{V}\rangle) \geq 0$ gives:

$$\frac{6b^2(-1+\langle x\rangle)\langle x\rangle^{3/2}}{c(1-4\langle x\rangle)^2} \leq \epsilon^2 W_0^2\langle\mathcal{V}\rangle \tag{4.4.84}$$

To this, we can add another constraint, namely the fact that the squared mass of the volume mode should be strictly positive. To this end, we can calculate the Hessian matrix $\mathcal{H}(\tilde{V})$ of the potential (4.4.81) and evaluate it in the critical point $(\langle\mathcal{V}\rangle, \langle x\rangle)$:

$$\begin{pmatrix} \frac{2}{\mathcal{V}^5}\left(-\frac{2b^2\langle x\rangle^{3/2}(1+\langle x\rangle-2\langle x\rangle^2)}{c(1-4\langle x\rangle)^2}-\epsilon^2 W_0^2\langle\mathcal{V}\rangle\right) & -\frac{2b^2(-1+\langle x\rangle)^2\sqrt{\langle x\rangle}}{c\langle\mathcal{V}\rangle^4(-1+4\langle x\rangle)} \\ -\frac{2b^2(-1+\langle x\rangle)^2\sqrt{\langle x\rangle}}{c\langle\mathcal{V}\rangle^4(-1+4\langle x\rangle)} & \frac{b^2(-1-2\langle x\rangle+9\langle x\rangle^2-14\langle x\rangle^3+8\langle x\rangle^4)}{c\langle\mathcal{V}\rangle^3(1-4\langle x\rangle)^2\sqrt{\langle x\rangle}} \end{pmatrix}$$

Its eigenvalues will be the squared masses of the volume and τ_4 ($x = a_4\tau_4$) modes. In order to get an approximate but still reliable form for the mass squared eigenvalues, we can consider that, once diagonalized, it will be of the form:

$$\mathcal{D} = \begin{pmatrix} m_{\mathcal{V}}^2 & 0 \\ 0 & m_x^2 \end{pmatrix} \tag{4.4.85}$$

The cross-terms in the matrix for the mass of the blow-up $x = a_4\tau_4$ do not contribute much since they lead to very small corrections [Gal+17]. The squared mass for x will then be given by:

$$m_x^2 = \left.\frac{\partial^2\tilde{V}}{\partial x^2}\right|_{\langle\mathcal{V}\rangle,\langle x\rangle} = \frac{b^2(-1-2\langle x\rangle+9\langle x\rangle^2-14\langle x\rangle^3+8\langle x\rangle^4)}{c\langle\mathcal{V}\rangle^3(1-4\langle x\rangle)^2\sqrt{\langle x\rangle}} \tag{4.4.86}$$

In this way, since the determinant is invariant upon diagonalization, we can see that:

$$\det(\mathcal{H}(\tilde{V})) = \det(\mathcal{D}) = m_{\mathcal{V}}^2 \cdot m_x^2 \iff m_{\mathcal{V}}^2 = \frac{\det(\mathcal{H}(\tilde{V}))}{m_x^2} \tag{4.4.87}$$

resulting in:

$$m_{\mathcal{V}}^2 = \frac{2}{\langle\mathcal{V}\rangle^5}\left(\frac{6b^2\langle x\rangle^{5/2}(-5+16\langle x\rangle-23\langle x\rangle^2+12\langle x\rangle^3)}{c(1-\langle x\rangle)^2(1+3\langle x\rangle-6\langle x\rangle^2+8\langle x\rangle^3)}-\epsilon^2 W_0^2\langle V\rangle\right) \tag{4.4.88}$$

Because this mass squared must be non-negative, together with (4.4.84), we get two conditions bounding the value of ϵ:

$$\begin{cases} \frac{6b^2x^{5/2}(-5+16\langle x\rangle-23\langle x\rangle^2+12\langle x\rangle^3)}{c(1-\langle x\rangle)^2(1+3\langle x\rangle-6\langle x\rangle^2+8\langle x\rangle^3)} > \epsilon^2 W_0^2\langle V\rangle \\ \frac{6b^2(-1+\langle x\rangle)\langle x\rangle^{3/2}}{c(1-4\langle x\rangle)^2} \leq \epsilon^2 W_0^2\langle\mathcal{V}\rangle \end{cases} \tag{4.4.89}$$

Now, by a simple manipulation, the final condition can be written as:

$$1 \leq \frac{c(1-4\langle x\rangle)^2}{6b^2(-1+\langle x\rangle)\langle x\rangle^{3/2}} \cdot \epsilon^2 W_0^2\langle\mathcal{V}\rangle < \frac{\langle x\rangle(5-11\langle x\rangle+12\langle x\rangle^2)}{1+3\langle x\rangle-6\langle x\rangle^2+8\langle x\rangle^3} \tag{4.4.90}$$

Now, the far right term, asymptotically (for $x \to \infty$) tends to $3/2$ and this means that there

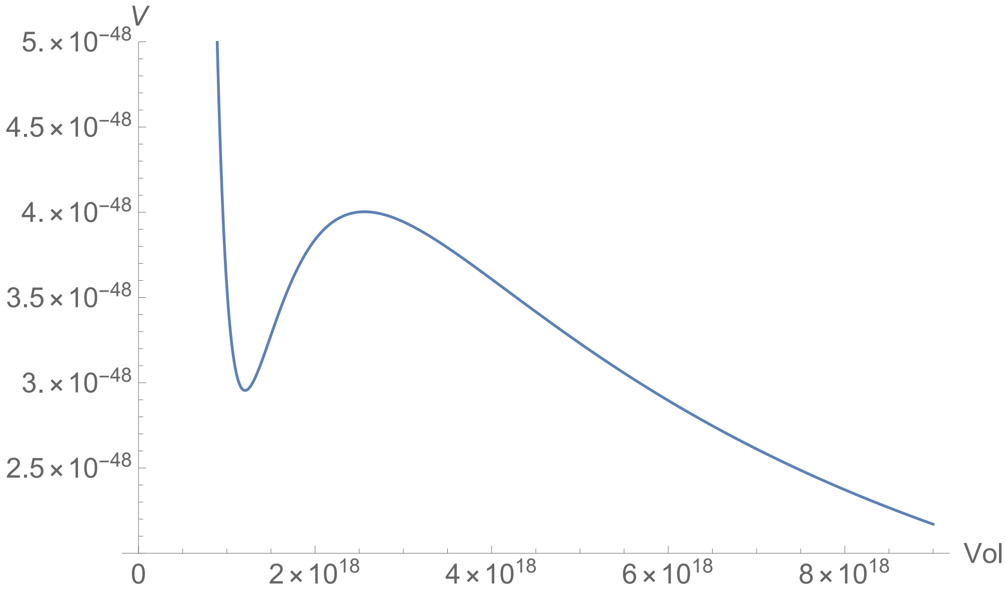


Figure 4.7: Plot of the potential $V(\mathcal{V})$ once the moduli τ_1 and τ_4 have been fixed at their VEVs using the uplifting term.

will always be a region for ϵ to be chosen in order to make the uplift happen. From the second equation of (4.4.82) we see that we cannot determine, even numerically, the value of $\langle x \rangle$ since the uplifting parameter ϵ appears. What we can do is, however, stabilise x as before by discarding the uplifting term $2/3\,\epsilon^2 W_0^2 \mathcal{V}$. Is seems reasonable to do that since from the considerations we made before on the order of ϵ, namely $\epsilon = O(1/\sqrt{\mathcal{V}})$, we see that the term in Λ should dominate over it. In a first approximation we can then stabilise x using again (4.4.49), and then numerically solve the second equation of (4.4.82) for the volume mode taking, then, into account the uplifting term. Using the values (4.4.67) with C_3^W and $|\lambda|$ as in (4.4.72) and (with some prescience in choosing ϵ since we know that the volume mode will be of order 10^{18}):

$$\epsilon^2 = 10^{-18} \tag{4.4.91}$$

we will again obtain:

$$\langle \tau_1 \rangle = 300; \qquad \langle x \rangle \simeq 34.63; \tag{4.4.92}$$

while for the volume mode we get:

$$\langle \mathcal{V} \rangle = 1.21 \cdot 10^{18} \tag{4.4.93}$$

which is in perfect agreement with our requests. Also, as we can see from the graph (4.7), the vacuum will be of de Sitter type, in fact:

$$V(\langle \tau_1 \rangle, \langle x \rangle, \langle \mathcal{V} \rangle) \simeq 2.95 \cdot 10^{-48} \tag{4.4.94}$$

We then managed to obtain a de Sitter vacuum for our model with the moduli stabilised at the correct values in order to reproduce the correct mass and coupling of the hidden gauge sector matching the values obtained in the analysis made by [Dut+19] of the energy and timing data extrapolated by the COHERENT collaboration. We should point out that in our model there isn't an explicit blow-up mode supporting the visible sector. We have, in fact, tacitly considered

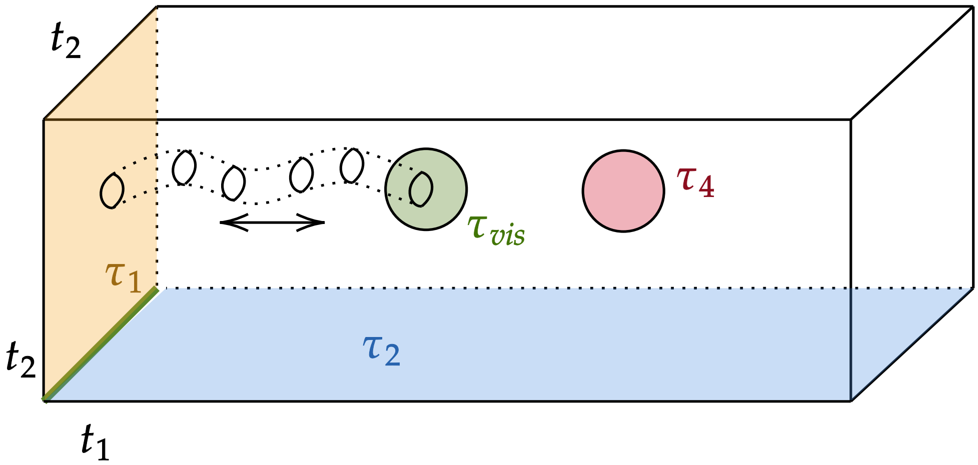


Figure 4.8: Pictorial representation of our Calabi-Yau volume with the brane set-up and the presence of an additional blow-up mode τ_{vis} supporting the visible sector, with exchange of heavy closed string modes giving rise to the kinetic mixing. In green it is also represented the intersection between the branes wrapping τ_1 and τ_2 where the string loop corrections arise.

that another blow-up mode τ_{vis} is present, with D7-branes wrapping it and supporting the visible sector. Kinetic mixing as we have seen comes from integrating out heavy modes coming from either open strings stretched between the visible and the hidden sector (in the open string channel) or from the exchange of closed strings between these two (stack of) branes (in the closed string channel). The emerging picture can be appreciated in figure (4.8).

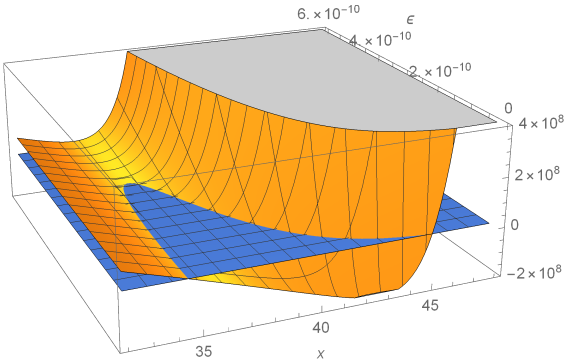
Additional Considerations on the Uplift

[*Here, we are always going to consider the free parameters fixed at the values (4.4.67) with* C_3^W *and* $|\lambda|$ *as in (4.4.72)*]

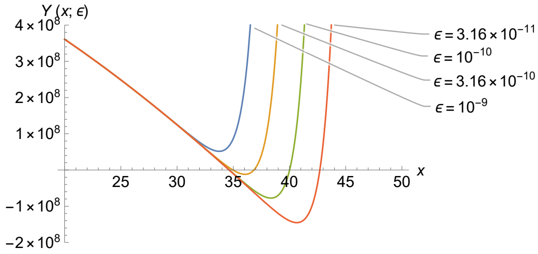
In the evaluation of $\langle x\rangle$ in (4.4.92), we said that the uplifting term could be discarded since it would not contribute significantly in the computation. Here, we would like to, instead, not make any approximation of this kind and see how far we can go. Starting from (4.4.82), we see that the value of the blow-up mode $\langle x\rangle$ is given by the vanishing locus of:

$$Y(\langle x\rangle;\epsilon)=\Lambda+\frac{2}{3}W_0^2\epsilon^2\frac{2be^{\langle x\rangle}(-1+\langle x\rangle)\sqrt{\langle x\rangle}}{c(-1+4\langle x\rangle)}-\frac{4b^2(-1+\langle x\rangle)\langle x\rangle^{5/2}}{c(1-4\langle x\rangle)^2}=0 \tag{4.4.95}$$

Fixed the value of ϵ, the function Y does not always admits a zero, in fact, from the figure (4.9b), we see that for ϵ sufficiently big, $Y=0$ is never attained, while starting from a value of $\epsilon_{\max}$ and taking $\epsilon<\epsilon_{\max}$, $Y=0$ always admits two distinct solutions. We also present the 3D plot of Y in function of both variables $\langle x\rangle$ and ϵ in figure (4.9a). If we want to find the maximum value of ϵ which allows to get a real $\langle x\rangle$, from the figure (4.9b) we see that this will be given by the



(a) 3D Plot of the function $Y(\langle x \rangle, \epsilon)$ and the plane $z = 0$. The intersection is clearly the locus of the points $(\langle x \rangle, \epsilon)$ satisfying $Y = 0$.



(b) Plot of the function $Y(\langle x \rangle; \epsilon)$ when ϵ has been fixed at some different values. It can be noted that as ϵ grows, from a certain value on, there will be no solutions to $Y = 0$.

Figure 4.9

conditions:

$$\begin{cases} \left.\dfrac{\partial Y}{\partial x}\right|_{\epsilon_{\max}, \langle \tilde{x} \rangle} = 0 \\ \\ Y|_{\epsilon_{\max}, \langle \tilde{x} \rangle} = 0 \end{cases} \tag{4.4.96}$$

where $\langle \tilde{x} \rangle$ is the value of $\langle x \rangle$ in which the ϵ is the maximum value $\epsilon_{\max}$ such that for $\epsilon < \epsilon_{\max}$ we will have two distinct real solutions of $\langle x \rangle$. By numerically solving the system of equations (4.4.96), it is found:

$$\epsilon_{\max} \simeq 3.91 \cdot 10^{-10}; \qquad \langle \tilde{x} \rangle \simeq 35.6 \tag{4.4.97}$$

We stress that the value $\langle \tilde{x} \rangle$ is not the maximum value of $\langle x \rangle$ consistent with the de Sitter requirement, this can be seen by noticing (looking at figure (4.9b) that for $\epsilon < \epsilon_{\max}$ one of the solutions is indeed bigger than $\langle \tilde{x} \rangle$. We also note that our guess (4.4.91) does not satisfy this condition $\epsilon < \epsilon_{\max}$ even though we managed to find a de Sitter minimum and this is because if one tries to solve the equation $Y = 0$ using the ϵ in (4.4.91), they will find solutions for $\langle x \rangle$ only in $\mathbb{C}$. This means that we should be more careful in the approximations we make, even though the correct result will not deviate much from the approximate one. We can also get a lower bound $\langle x \rangle_{\min}$ such that values above it will give a de Sitter vacuum, while values below will give an Anti-de Sitter vacuum. In fact, we can take the condition (4.4.84) and eliminate the ϵ by making use of (4.4.82) to obtain:

$$\frac{4b^2(\langle x \rangle - 1)^2 \langle x \rangle^{3/2} - c\Lambda(1 - 4\langle x \rangle)^2}{2c\langle \mathcal{V} \rangle^3 (1 - 4\langle x \rangle)^2} \geq 0 \tag{4.4.98}$$

Which can be numerically solved to yield:

$$0.231886 < \langle x \rangle < 0.269277 \wedge \langle x \rangle > 35.3$$

Where $\wedge$ here stands for the "or" logic. Since we are requiring $\langle x \rangle \gg 1$, the only condition surviving will be:

$$\langle x \rangle_{\min} > 35.3 \tag{4.4.99}$$

which corresponds to a value of ϵ of:

$$\tilde{\epsilon} \simeq 3.77 \cdot 10^{-10} \tag{4.4.100}$$

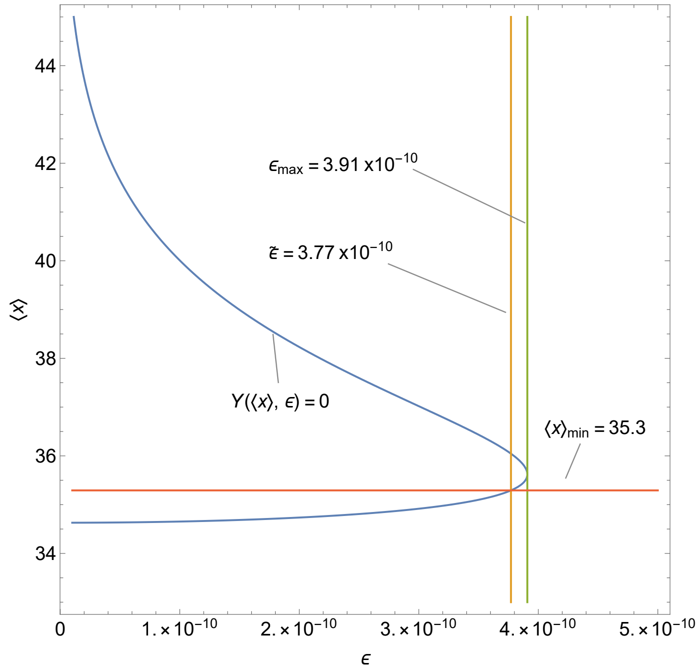
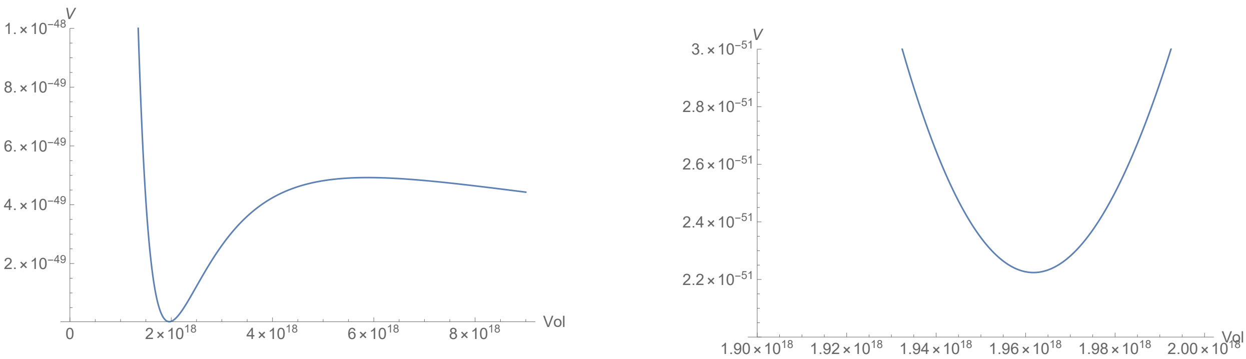


Figure 4.10: Vanishing locus of $Y(\langle x\rangle;\epsilon)$, where the allowed region to obtain a de Sitter vacuum is for pairs above the horizontal line $\langle x\rangle = 35.3$.

We can then plot the vanishing locus of $Y(\langle x\rangle;\epsilon)$ in $\mathbb{R}^2$ which represents the compatible pairs $(\langle x\rangle,\epsilon)$ which make Y vanish, obtaining the figure (4.10). Also, we can notice that in the interval $[\tilde{\epsilon},\epsilon_{\text{max}}]$, for each ϵ^* it will correspond two $\langle x\rangle^*_{1,2}$ making $Y(\langle x\rangle^*_{1,2},\epsilon^*) = 0$. While, for values of ϵ smaller than $\tilde{\epsilon}$, only one value of $\langle x\rangle$ will be compatible with the requirement of a de Sitter vacuum.

We see that the closest value of $\langle x\rangle$ to the wanted $\langle x\rangle = 34.63$ of (4.4.92) and compatible with the de Sitter requirement is indeed $\langle x\rangle_{\text{min}} = 35.3$. This corresponds to the value $\tilde{\epsilon} \simeq 3.77\cdot 10^{-10}$ and a volume mode of $\mathcal{V} \simeq 1.96\cdot 10^{18}$ which is still ok with our requirement (4.3.28). From figures (4.11a) and (4.11b) it can be appreciated that the minimum we have found is indeed a de Sitter one. In order to obtain an actual uplift and phenomenologically interesting values for the moduli's VEVs, the parameter ϵ should then be of order 10^{-10}. The contribution in which the ϵ appears is given by a supersymmetry breaking term caused by the fluxes introduced to stabilise the axio-dilaton and complex structure moduli. We remind that the fluxes are constrained by the tadpole cancellation condition (3.1.14) without the presence of D3-branes for this model, since they would induce unwanted string loop corrections. As [Gal+17] pointed out, a small ϵ of order $\epsilon = O(1/\sqrt{\mathcal{V}})$ is always possible to be achieved by a suitable tuning of continuous fluxes, but for quantised fluxes (which are the physical ones due to the Dirac quantisation condition) this is expected to still be true for many cycles and a large flux tadpole.



(a) Plot of the potential $V(\mathcal{V})$ in function of the volume mode $\mathcal{V}$ once the other moduli have been stabilised and the correct uplift has been employed, i.e. using $\epsilon = 3.77 \cdot 10^{-10}$ corresponding to $\langle x \rangle \simeq 35.3$

(b) Zoom in of the neighbourhood of the minimum for the potential plotted on the left. We can appreciate here the fact that $V(\langle \mathcal{V} \rangle) > 0$, obtaining a de Sitter vacuum.

Figure 4.11

Conclusions

In this thesis we have explored the realm of string theory, starting from its very foundations to the more recent developments in string phenomenology. Our goal was to, at the end, construct a model supporting a hidden gauge sector with hidden gauge boson kinetically coupled to the visible one. In our final model, for a Calabi-Yau with four Kähler moduli (three K3 fibrations and one small blow-up) in a toric ambient space, we managed to find a suitable D7-brane configuration and fluxes set-up to generate the right mass for the hidden gauge boson as well as the right coupling to the visible sector compatible with experimental bounds (see page 3 of [Cic+11]) and compatible with the values found by [Dut+19]. Moreover, all the moduli have been stabilised with the free parameters all being chosen within (more or less) expected natural ranges also reproducing TeV scale supersymmetry.
More precisely for a Calabi-Yau with volume given by four 4-cycles of the form $\mathcal{V} \simeq \sqrt{\tau_1\tau_2\tau_3} - \tau_4^{3/2}$, we found for the mass of the hidden gauge boson supported on a D7-brane wrapped on τ_1, the value:

$$m_{Z'} = 1\ \text{GeV} \tag{4.4.101}$$

coming from the Stückelberg mechanism as a result of the switching on of internal gauge fluxes. The fluxes also generated a Fayet-Iliopoulos (FI) term of the form $\xi = (1/4\pi\mathcal{V})(f_2t_3+f_3t_2)$ (where t_i are the 2-cycles) which in principle could destabilise the stabilisation process since it dominates the F-term scalar potential, because in general, in LVS models, the F-term scalar potential is given by (after considering non-perturbative corrections to the superpotential and (α')-corrections to the Kähler potential):

$$V_{\text{LVS}}(x,\mathcal{V}) \sim \frac{c\sqrt{x}e^{-2x}}{\mathcal{V}} - \frac{bxe^{-x}}{\mathcal{V}^2} + \frac{\Lambda}{\mathcal{V}^3} \sim O\left(\frac{\ln(\mathcal{V})}{\mathcal{V}^3}\right) \ll \xi \sim O\left(\frac{1}{\mathcal{V}}\right) \tag{4.4.102}$$

with c, b, Λ suitable constants and x the small blow-up mode supporting the non-perturbative correction. Our previous model (the K3 fibration with a blow-up mode of §4.3) could not reproduce the suitable parameter region due to the following facts:

- If the D7-brane supporting the hidden sector was wrapped around a single divisor D_1, then we had to require the existence of charged matter fields (open string modes) which partially would have balanced the FI-term. However, it turned out that the massive axionic parts of these charged fields were eaten up by the hidden gauge boson, and these contribution dominated the one furnished by the axion of the Kähler modulus, making, in turn, the mass of the hidden boson too large.

- If the D7-brane supporting the hidden sector was wrapped around a divisor of the form $D_1 + D_3$, then we could require the vanishing of the FI-term (given that it was a combination of 2-cycles), fixing the size of the 4-cycle volumes τ_3 and τ_1 of the same order. This promising set-up quickly turned out to fail too, since the mass from the Stückelberg mechanism received a contribution due to the new charges q_{ij} that came out with respect to the previous case which were too big to reproduce the interesting parameter region.

Learning from the above attempts, we knew that the new model was free of these problems since, there, we could have imposed the vanishing of the FI-term resulting in the fixing of a specific combination of 2-cycles without generating a too large contribution in the Stückelberg mass of the hidden boson. Once the model was found adequate to reproduce the right mass and coupling of the hidden gauge boson, we employed all the stabilisation machinery presented in §3 to, indeed, stabilise the Kähler moduli in such a way to reproduce the values we were seeking and also reproduce TeV scale supersymmetry. With the volume's value of the Calabi-Yau we needed in order to reproduce $m_{Z'} = 1$ GeV, namely $\mathcal{V} = 10^{18}$, we had to consider the flux term W_0 (which is the VEV of the tree-level Gukov-Vafa-Witten superpotential) to be of order 10^4, and with a string coupling of $g_s = 0.1$ (which is the VEV of the axio-dilaton $2/g_s = \langle \bar{S} + S \rangle$) we got to a gravitino's mass in the TeV range:

$$m_{3/2} \simeq 1.51 \text{ TeV} \tag{4.4.103}$$

As is customary in LVS models, the stabilised moduli led us to an AdS vacuum, which we were able to uplift to a de Sitter one by employing small supersymmetry breaking contributions coming from non-vanishing F-terms, following [Gal+17]. This process leads always to a metastable de Sitter vacuum provided one has enough freedom to choose these contributions suitably small. Constraints may come from the tadpole cancellation condition and the complete brane set-up, it is however believed that for a suitably large flux tadpole these values can always be found.
In conclusion we found that anisotropic compactifications of Calabi-Yau with volume of the form $\mathcal{V} \simeq \sqrt{\tau_1 \tau_2 \tau_3} - \tau_4^{3/2}$ are particularly interesting for studying hidden gauge sectors in type IIB string models, since they naturally lead to parameter regions interesting for phenomenological applications. It would be interesting to explicitly construct the visible sector (whether it would be the MSSM, or other extensions of the Standard Model) on another blow-up mode (enlarging the Calabi-Yau volume form with the addition of another τ_{vis}), and see whether the construction presented here can be realised when all the constraints are carefully checked.

Acknowledgements

I'm very grateful to my supervisor Michele Cicoli for introducing me to the intricate realm of string phenomenology, for teaching me how to do research in this field and for all the helpful discussions we had in the preparation of this thesis, to not mention his final careful review of the entirety of the script before the submission.
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I would also like to address a special thanks to my better half Giorgia, for her sweet, delightful and pleasant presence in my life which has always been a great shelter against all the external problems.

Appendix A

Characteristic Classes

Chern classes are particular *Characteristic classes* which are tools defined to see how far from trivial a given principal bundle is, and in particular they are topological invariants. In this regard it is first studied the so called *Universal Bundle* denoted by $\xi(n-k-1, O(k))$, which is the $O(k)$-principal bundle which has as its base space the *Grassmann manifold of k-dimensional planes*[1]: $Gr(n,k,\mathbb{R}) = O(n)/(O(k)\times O(n-k))$ and as total space $O(n)/O(n-k)$. As shown in [Ste60] every $O(k)$-principal bundle P with base space M can be recovered as the pull-back bundle of $\xi(n-k-1, O(k))$ by a suitable map $f : M \to Gr(n,k,\mathbb{R})$. Moreover this construction can be extended to any connected Lie group G instead of $O(k)$, defining the universal bundle by $\xi(n,k,G)$ with base space given by $O(n)/(G\times O(n-k))$ and total space as before. In order to study how far a principal G-bundle P is from being trivial, a function c - which will tell how non-trivial is the universal bundle $\xi(n,k,G)$ - must be developed and subsequently pulled it back in some way to P [NS83]. What is required of this c are the following two conditions:

(i) $c(P) = c(P')$ if P and P' are equivalent (as principal bundles)

(ii) $f^*(c(\xi)) = c(f^*\xi)$

where P, ξ and f are defined as above.

If the object $c(P)$ is taken to be an element of $H^p(M,\mathbb{R})$ (where M is the base space of P), then these two properties are satisfied. A key fact to prove these statements is that equivalent bundles arise from homotopic maps and the pull-back bundles of homotopic maps are equivalent.

This means that if we are able to compute the *characteristic class* $c(\xi)\in H^p(B,\mathbb{R})$, where B is the base space of ξ, then we can compute the characteristic class of the bundle P by computing the pull-back $f^*(c(\xi))\in H^p(M,\mathbb{R})$ if $f^*\xi = P$.

[1] The Grassmann manifold $Gr(n,k,\mathbb{R})$ is defined as the set of all k-dimensional subspaces of $\mathbb{R}^n$ ("set of k-dimensional planes" is also used). In order to give a topology to this set and making it a manifold, we can consider a map $\sigma : O(n) \to Gr(n,k,\mathbb{R})$ which maps an orthogonal matrix $A\in O(n)$ to $\sigma(A) := Aw_0$ where $w_0 \subset \mathbb{R}^n$ is the k-dimensional subspace spanned by $\{e_1,...,e_k\}$. Considering also the *Stiefel manifold*

$$V(n,k,\mathbb{R}) := \{(v_1,...,v_k)\in \underbrace{\mathbb{R}^n\times\cdots\times\mathbb{R}^n}_{k-times} \mid \sum \lambda_i v_i = 0 \iff \lambda_i = 0, \forall i = 1,..,k\}$$

which is the set of $(v_1,...,v_k)$ which are linearly independent, then we can define a map $\rho : V(n,k,\mathbb{R}) \to Gr(n,k,\mathbb{R})$ as: $(v_1,...,v_k) \mapsto \rho((v_1,...,v_k)) := \text{span}\{v_1,...,v_k\} \in Gr(n,k,\mathbb{R})$. In this way, considering the continuous map $\alpha : O(n) \to V(n,k,\mathbb{R})$ defined by $A \mapsto \alpha(A) := (Ae_1,...,Ae_k)$, then we have that $\rho\circ\alpha = \sigma$. Using this fact, it can be seen that $Gr(n,k,\mathbb{R})$ can be represented as the left coset space $O(n)/(O(k)\times O(n-k))$. For a more complete account of this fact see [Spi99].

A.1 Pontrjagin, Euler and Chern Classes

Having considered a general Lie group G, we can now specialise to the most frequently encountered groups, namely $O(k), SO(k)$ and $U(k)$. The characteristic classes for principal G-bundles with these Lie groups are called respectively *Pontrjagin, Euler and Chern classes*. The reasoning made before can be extended in a trivial way to these groups by a suitable modification of the universal bundle. What we do want to compute is the Cohomology of the Grassmann manifold modified in order to take into account these different Lie groups G. Indeed we have the following Grassmann manifolds:

(i) Pontrjagin: $Gr(n,k,\mathbb{R}) = O(n)/(O(k) \times O(n-k)$

(ii) Euler: $\tilde{Gr}(n,k,\mathbb{R}) := SO(n)/(SO(k) \times SO(n-k))$

(iii) Chern: $Gr(n,k,\mathbb{C}) = U(n)/(U(k) \times U(n-k))$

and we want to compute the corresponding Cohomology groups:

(i) Pontrjagin: $H^i(Gr(n,k,\mathbb{R});\mathbb{R})$;

(ii) Euler: $H^i(\tilde{Gr}(n,k,\mathbb{R});\mathbb{R})$;

(iii) Chern: $H^i(Gr(n,k,\mathbb{C});\mathbb{R})$

By a complete computation (see [Spi99]) it is seen that:

(i) $H^i(Gr(n,k,\mathbb{R});\mathbb{R})$ is non-zero only if i is a multiple of 4. We will indicate the Pontrjagin class of a bundle P with base space M as $p_i(P) \in H^{4i}(M;\mathbb{R})$ (this is an abuse of notation that we will always employ, since the cohomology groups are defined by cohomology classes, and more precisely we should write $[p_i(P)] \in H^{4i}(M;\mathbb{R})$, but we it will be understood that $p_i(P)$ stands for the cohomology class in order to make the notion less cumbersome) and it is calculated from the fact that $f^*H^{4i}(Gr(n,k,\mathbb{R});\mathbb{R}) = H^{4i}(M;\mathbb{R})$ if $f^*\xi = P$ (where ξ is the universal bundle over $Gr(n,k,\mathbb{R})$).

(ii) $H^i(\tilde{Gr}(n,k,\mathbb{R});\mathbb{R})$ is non-zero only if i is a multiple of 2. The characteristic classes are still the Pontrjagin classes with the addition of a new one (provided that k is even): the Euler class. We will indicate the Euler class of a $SO(2m)$-bundle P with base space M as $e(P) \in H^{2m}(M;\mathbb{R})$. From the fact that, taken an integer $2m$, we have that the Euler class $e(P)$ is a $2m$-form, then $e(P) \wedge e(P)$ is a $4m$-form and thus belongs to $H^{4m}(M;\mathbb{R})$. In this way, the Euler class is defined in terms of the Pontrjagin class: take the $4m$-form $p_m(P)$, then the Euler class is defined to be the $2m$-form such that $e(P) \wedge e(P) = p_m(P)$.

(iii) $H^i(Gr(n,k,\mathbb{C});\mathbb{R})$ is non-zero only if i is a multiple of 2. We will indicate the Chern class of a bundle P with base space M as $c_i(P) \in H^{2i}(M;\mathbb{R})$.

Now that we have classified the most important characteristic classes we can try to compute them. Leaving the details to [NS83], what can be said is that $c_i(P)$, $p_i(P)$ and $e(P)$ are all given by polynomials in the curvature 2-form F. In the following we are going to firstly recall how a connection and its curvature on a principal bundle arise before continuing to analyse and compute characteristic classes.

A.1.1 Connection on a Principal Bundle

Recall that for a principal G-bundle (P,π,M,G) (with P total space, M base space, π the projection and G its structure group) we can choose a *connection* which is nothing but a smooth assignment of *horizontal* spaces in the tangent bundle of P, in the sense that for all $p \in P$, T_pP is split into a vertical component $\mathscr{V}_pP$ - whose vectors are pushed-forward by the projection π to zero - and a horizontal component $\mathscr{H}_pP$ in such a way that:

(i) $T_pP = \mathscr{V}_pP \oplus \mathscr{H}_pP$;

(ii) $(\triangleleft g)_* \mathscr{H}_pP = \mathscr{H}_{p\triangleleft g}P$;

(iii) each smooth vector field $X : P \to TP$ can be split into two smooth vector fields: $X = X_{ver} + X_{hor}$ where $X_{ver} : P \to \mathscr{V}_pP$ and $X_{hor} : P \to \mathscr{H}_pP$.

where we have denoted by $\triangleleft$ the action of the group G onto P.

Connection 1-form

A connection in this form is not that useful however, in fact in general it is used what is called a *connection 1-form* ω, which is a differential form of degree 1 in P with values in the *Lie Algebra* $\mathfrak{g}$ of G, i.e. $\omega \in \Omega^1(P) \otimes \mathfrak{g}$. In other words taken a point $p \in P$ we have that:

$$\omega_p : T_pP \to \mathfrak{g}$$

This connection 1-form is defined by some characterising properties which match the definition of a connection:

(i) $p \mapsto \omega_p$ is smooth

(ii) $\forall g \in G, \forall p \in, P \forall v_p \in T_pP$ it is true that:

$$((\triangleleft g)_* \omega)_p(v_p) = (Ad_{g^{-1}})_*(\omega_p(v_p)) = g^{-1} \cdot \omega_p(v_p) \cdot g \tag{A.1.1}$$

with $Ad_g : G \to G$ the adjoint map $Ad_g(h) := g \circ h \circ g^{-1}$ and in the second equality it has been taken the Lie group G to be a matrix group, namely a subgroup of the general linear group $GL(n,\mathbb{R})$ (if the dimension of P is n).

(iii) $\forall A \in \mathfrak{g}, p \in P$ we have that

$$\omega_p(X_p^A) = A \tag{A.1.2}$$

where X^A is the vector field (vertical) generated by the Lie algebra element A by the flux $\phi(t,p) := p \triangleleft \exp(tA)$.

Once chosen a section σ of the principal bundle, then ω can be pulled-back to the base manifold, defining the so called *gauge potential*: $A_\sigma = \sigma^*\omega \in \Omega^1(M) \otimes \mathfrak{g}$. What is remarkable is that if we have a gauge potential $A_U \in \Omega^1(U) \otimes \mathfrak{g}$ defined on a chart (U,ϕ) and a section $\sigma : U \to P$, then there exists a unique lifted connection $\omega \in \Omega^1(\pi^{-1}(U)) \otimes \mathfrak{g}$ defined by:

$$\omega_{\pi^{-1}(U)} = g^{-1}(\pi^* A_U) g + g^{-1} dg \tag{A.1.3}$$

where those $g : \pi^{-1}(U) \to G$ allows to reach every point in the fibre $\pi^{-1}(\pi(p))$ starting from the local section by means of $\sigma(m) \triangleleft g(p)$ with $m = \pi(p)$.
Moreover, if we have two gauge potentials A_U and $A'_{U'}$ defined on two overlapping charts (U, ϕ) and (U', ϕ') with sections $\sigma_U : U \to P$ and $\sigma'_{U'} : U' \to P$, then these are related by:

$$A'_{U'} = g_{UU'}^{-1} \cdot A_U \cdot g_{UU'} + g_{UU'}^{-1} d\, g_{UU'} \tag{A.1.4}$$

where $g_{UU'} : U \cap U' \to G$ are the functions relating the sections $\sigma'_{U'} = \sigma_U \triangleleft g_{UU'}$ and $\sigma_U = \sigma'_{U'} \triangleleft g_{U'U}$.

Covariant Derivative and Curvature 2-form

Taken a principal bundle (P, π, M, G) with a connection 1-form ω, we can define a covariant derivative acting on $\mathfrak{g}$-valued k-forms:

$$D : \Omega^k(P) \otimes \mathfrak{g} \to \Omega^{k+1}(P) \otimes \mathfrak{g} \tag{A.1.5}$$

by:

$$(DA)_p(v_1, ..., v_{k+1}) = dA(v_1^{hor}, ..., v_{k+1}^{hor}) \tag{A.1.6}$$

where v_i^{hor} are the horizontal components of $v_i \in T_pP$ with respect to the connection ω. Using this covariant derivative we can define the *Curvature 2-form* $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$ as:

$$\Omega = D\omega \tag{A.1.7}$$

which will satisfy $\forall g \in G$:

$$(\triangleleft g)_* \Omega = g^{-1} \cdot \Omega \cdot g \tag{A.1.8}$$

and an important characterisation of the curvature form is the following *Cartan's Structure Equation*, $\forall p \in P, \forall v_p, w_p \in T_pP$:

$$\Omega_p(v_p, w_p) = d\omega_p(v_p, w_p) + [\omega_p(v_p), \omega_p(w_p)] \tag{A.1.9}$$

where $[\cdot, \cdot]$ is the commutator in the Lie algebra $\mathfrak{g}$.
Now, same as before, if we take a section $\sigma : M \to P$, then we can pull-back the curvature 2-form to the base manifold M, obtaining a 2-form $F \in \Omega^2(M) \otimes \mathfrak{g}$:

$$F = \sigma^* \Omega \tag{A.1.10}$$

and using Cartan's structure equation, if $A = \sigma^* \omega$, then $\forall m \in M$:

$$F_m(v_m, w_m) = (dA)_m(v_m, w_m) + [A_m(v_m), A_m(w_m)] \tag{A.1.11}$$

However, it can be proved that the compatibility on overlapping charts is translated in the covariance of the curvature form (in contrast with the local connection 1-form):

$$F'_{U'} = g_{UU'}^{-1} \cdot F_U \cdot g_{UU'} \tag{A.1.12}$$

(in the same conditions as the above (A.1.4)).

A.2 Computation of Characteristic Classes

Coming back to our characteristic classes, namely the Pontrjagin, Chern and Euler classes, we anticipated that given a principal bundle (P, π, M, G) these will be given by some polynomials in the curvature 2-form $F \in \Omega^2(M) \otimes \mathfrak{g}$. Now, at first this could seems to be unlikely since changing the connection on P, the curvature will also change and F will be different, but what we are seeking is some intrinsic topological character possessed by our bundle. However the polynomials used to define the characteristic classes are independent of the connection because these are chosen to be invariants of the Lie algebra $\mathfrak{g}$. If we consider $\{T_i\}$ to be the set of generators of the Lie algebra $\mathfrak{g}$, then it is known that the *Casimir operator* defined in terms of these generators for a compact and semi-simple Lie algebra is $T_1^2 + ... + T_n^2$ with $\dim(G) = n$ and invariants of $\mathfrak{g}$ are formed by taking polynomials in the generators. All these polynomials are obtained by expanding in power series of t the following $m \times m$ determinant (if the group is $O(k)$, the elements of the lie algebra $\mathfrak{o}(k)$ will be $k \times k$ real matrices):

$$\det\left(\mathbb{1} + t\sum_k a_k T_k\right) = \sum_i^m P_i(a_k)t^i \tag{A.2.1}$$

This equation defines the polynomials $P_i(a_k)$ and the invariants of $\mathfrak{g}$ are obtained by substituting a_k with T_k, making $P_i(T_k)$.

Chern Classes

For the case $G = U(k)$, if we now substitute the curvature 2-form F into the polynomials in (A.2.1) we obtain:

$$\det\left(\mathbb{1} + \frac{it}{2\pi}F\right) = \sum_i P_i(F)t^i \tag{A.2.2}$$

where $i = \sqrt{-1}$ has been introduced to make the P_i real and the factor of 2π to have that the $c_i(P)$ actually determine integral cohomology classes. Now, since P_i's are homogeneous of degree i, then since F is a 2-form, $P_i(F)$ is a $2i$-form. It turns out that the Chern class $c_i(P)$ is given by $P_i(F)$:

$$c_i(P) = P_i(F) \tag{A.2.3}$$

What can be proved is that indeed $P_i(F)$ is closed and independent of the connection 1-form A used to compute F so that $c_i(P) \in H^{2i}(M;\mathbb{R})$. In this way we have that the Chern Classes can be computed by expanding the determinant in power series:

$$\det\left(\mathbb{1} + \frac{it}{2\pi}F\right) = \sum_i c_i(P)t^i \tag{A.2.4}$$

Pontrjagin Classes

For $G = O(k)$, we can substitute $a_k T_k$ with $-F/2\pi$ in (A.2.1), obtaining:

$$\det\left(\mathbb{1} - \frac{t}{2\pi}F\right) = \sum_i P_i(F)t^i \tag{A.2.5}$$

However, in this case we see that since the determinant is invariant with respect to taking the transpose, the only surviving polynomials $P_i(F)$ are those with even i, in fact (using that F is a $\mathfrak{o}(k)$-valued 2-form, meaning that $F^T = -F$):

$$\det\left(\mathbb{1} - \frac{t}{2\pi}F\right) = \det\left(\mathbb{1} - \frac{t}{2\pi}F\right)^T = \det\left(\mathbb{1} + \frac{t}{2\pi}F\right) \tag{A.2.6}$$

leading to:

$$(-1)^m \sum_i P_i(F)t^i = \sum_i P_i(F)t^i \implies P_i(F) = 0 \quad \forall i \text{ odd} \tag{A.2.7}$$

This means that we can define the Pontrjagin classes as:

$$p_i(P) = P_{2i}(F) \tag{A.2.8}$$

and since $P_i(F)$ are $(2i)$-forms, this means that the Pontrjagin classes are $4i$-forms, in particular they belongs to the cohomology groups $p_i(P) \in H^{4i}(M;\mathbb{R})$.

Euler Classes

For $G = SU(2k)$, another characteristic classes adds to the Pontrjagin's ones and is the Euler class. This is the $(2k)$-form $e(P)$ defined in terms of the Pontrjagin class $p_k(P) \in H^{4k}(M;\mathbb{R})$ as follows:

$$e(P) \wedge e(P) = p_k(P) \tag{A.2.9}$$

also, since $p_k(P) = P_{2k}(F)$, then equivalently:

$$e(P) \wedge e(P) = P_{2k}(F) \tag{A.2.10}$$

Explicitly, since the 2-form field F is $SO(2k)$ valued, we can write F^a_b for its matrix content with $a, b = 1, \ldots, 2k$, then the Euler class is given by the following expression:

$$\begin{aligned} e(P) &= \frac{1}{(2\pi)^k}\mathrm{Pfaff}(F) \\ &= \frac{(-1)^k}{2^k(2\pi)^k k!}\epsilon_{a_1,\ldots,a_k} F^{a_1}_{a_2} \wedge \cdots \wedge F^{a_{2k-1}}_{a_{2k}} \end{aligned} \tag{A.2.11}$$

where $\mathrm{Pfaff}(F)$ is the Pfaffian of the matrix F^a_b.

A.2.1 Chern Classes for SU(2) Principal Bundles

Let's take $(P, \pi, M, SU(2))$ to be a Principal bundle with total space P, base space M, projection π and structure group $SU(2)$. Using the formula (A.2.4) above, we can easily compute the Chern classes for this kind of principal bundles. The first ingredient that we can employ is the fact that for a Hermitian operator A it is true that:

$$\log(\det(A)) = \mathrm{tr}(\log(A)) \tag{A.2.12}$$

which implies that (A.2.4) can be recast as:

$$\det\left(\mathbb{1} + \frac{it}{2\pi}F\right) = \exp\left(\mathrm{tr}\left(\log\left(\mathbb{1} + \frac{it}{2\pi}F\right)\right)\right) \tag{A.2.13}$$

Using the expansion of $\log(\mathbb{1}+M)\simeq M-M^2/2$ we get:

$$\mathrm{tr}\left(\log\left(\mathbb{1}+\frac{it}{2\pi}F\right)\right)\simeq \mathrm{tr}\left(\frac{it}{2\pi}F+\frac{t^2F^2}{8\pi^2}\right)=\mathrm{tr}\left(\frac{it}{2\pi}F\right)+\mathrm{tr}\left(\frac{t^2F^2}{8\pi^2}\right) \tag{A.2.14}$$

and by expanding the exponential too as $\exp(A)\simeq \mathbb{1}+M+M^2/2$ the result is:

$$\det\left(\mathbb{1}+\frac{it}{2\pi}F\right)\simeq \mathbb{1}+\frac{it}{2\pi}\mathrm{tr}(F)+\frac{\mathrm{tr}(F^2)-\mathrm{tr}(F)^2}{8\pi^2}t^2 \tag{A.2.15}$$

where we have discarded terms of order $O(t^3)$ and since the curvature F is a 2-form with values in the Lie algebra $\mathfrak{su}(2)$, the square is intended to be a wedge product $F^2 := F\wedge F$. Comparing the definition of Chern classes (A.2.4) with this expansion, we can infer that:

$$\begin{aligned} c_0(P)&=\mathbb{1}\\ c_1(P)&=\frac{i}{2\pi}\mathrm{tr}(F)\\ c_2(P)&=\frac{\mathrm{tr}(F^2)-\mathrm{tr}(F)^2}{8\pi^2} \end{aligned} \tag{A.2.16}$$

Since $F\in\Omega^2(M)\otimes\mathfrak{su}(2)$ we can write it as:

$$F=\frac{F^a\otimes\sigma_a}{2i}=\frac{F^a_{\mu\nu}}{4i}dx^\mu\wedge dx^\nu\otimes\sigma_a \tag{A.2.17}$$

where σ^a are the *Pauli* matrices (generators of the Lie Algebra), $a=1,2,3$ and we have chosen a chart (U,x) of M with $\mu=1,..,\dim(M)$. In this we can see that the first Chern class vanishes since $\mathrm{tr}(\sigma_a)=0$, while the second Chern class can be rewritten as:

$$\begin{aligned} c_2(P)&=-\frac{\mathrm{tr}(F^a\otimes\sigma_a\wedge F^b\otimes\sigma_b)-\mathrm{tr}(F^a\otimes\sigma_a)\wedge\mathrm{tr}(F^b\otimes\sigma_b)}{32\pi^2}\\ &=-\frac{F^a\wedge F^b\mathrm{tr}(\sigma_a\sigma_b)-F^a\wedge F^b\mathrm{tr}(\sigma_a)\mathrm{tr}(\sigma_b)}{32\pi^2}\\ &=-\frac{1}{32\pi^2}F^a\wedge F^b(2\delta_{ab})\\ &=-\frac{1}{16\pi^2}F^a\wedge F^a=-\frac{1}{8\pi^2}\mathrm{tr}(F\wedge F) \end{aligned} \tag{A.2.18}$$

where it has been used the fact that the sigma matrices satisfy: $\mathrm{tr}(\sigma_a\sigma_b)=2\delta_{ab}$ obtained from the characterising property: $\sigma_a\sigma_b=\delta_{ab}\mathbb{1}+i\epsilon_{ab}{}^c\sigma_c$.

A.2.2 Pontrjagin and Euler Classes for the Tangent Bundle

Every smooth n-dimensional manifold M admits a tangent bundle TM which is defined as the disjoint union of all the tangent spaces at each point $p\in M$. The tangent spaces are of the same dimension as the base manifold, namely n-dimensional. If M is a Riemannian manifold, Once one has the tangent spaces T_pM, one can define the so-called *Reference Frame Bundle* which is a principal $GL(n,\mathbb{R})$ bundle constructed as follows. Consider the set of all the frame references for the tangent space at the point $p \in M$:

$$L_pM := \left\{ (e_1, ..., e_n) \in \underbrace{T_pM \times \cdots \times T_pM}_{n-\text{times}} \middle| \{e_1, ..., e_n\} \text{ is a basis of } T_pM \right\} \tag{A.2.19}$$

By considering the disjoint union LM of all the above spaces and defining a right action:

$$\triangleleft : LM \times GL(n;\mathbb{R}) \to LM \tag{A.2.20}$$

with:

$$((e_1, ..., e_n), R) \mapsto (e_1, ..., e_n) \triangleleft R := (e_1, ..., e_n) \cdot R = (e_i R_{i1}, ..., e_i R_{in}) \tag{A.2.21}$$

which is nothing but the matrix multiplication, the 4-tuple $(LM, \pi, M, GL(n;\mathbb{R}))$ is a principal bundle. Reducing the group $GL(n,\mathbb{R})$ to $O(n)$ (see §7 of [NS83]) amounts to a continuous assignment of an orthogonal frame at each point $p \in M$, namely it provides an inner product on the tangent space T_pM. We thus have provided a Riemannian metric g to the base space M, which allows to construct a unique affine connection (Levi-Civita connection) and a curvature form $\mathscr{R}$.

As an example we consider the tangent bundle TM of a 4-dimensional manifold M, which is chosen to be Riemannian with metric g and curvature form $\mathscr{R}$. The group of the bundle is the orthogonal group $O(4)$, which means that the Euler class $e(TM)$ should be a 4-form $e(TM) \in H^4(M;\mathbb{R})$. The Pontrjagin classes $p_i(TM)$ are: $p_0(TM) = 1$, $p_1(TM) \in H^4(M;\mathbb{R})$ and $p_i(TM) = 0$ for $i > 1$. Since $p_2(TM) = e(TM) \wedge e(TM)$, it could be tempting to conclude that $e(TM) = 0$. However this is not the case, in fact using the expression for the Euler class, we see that:

$$e(TM) = \frac{(-1)^2}{2^4 \pi^2 2!} \text{Pfaff}(\mathscr{R}) = \frac{1}{32\pi^2} \epsilon_{abcd} \mathscr{R}^a_b \wedge \mathscr{R}^c_d \neq 0 \tag{A.2.22}$$

A.2.3 Stiefel-Whitney Class

Stiefel-Whitney classes are characteristic classes of vector bundles with group $O(k)$ or principal vector bundles with group $O(k)$, which are not given by polynomials in the curvature form F but nonetheless encodes important topological properties of the underlying base manifold M. The calculation of these classes requires a calculation of the cohomologies $H^i(Gr(n, k\mathbb{R}); \mathbb{Z}/2)$ and it can be seen that:

$$w_1(TM) = 0 \iff M \text{ is orientable} \tag{A.2.23}$$

and

$$w_2(TM) \neq 0 \implies M \text{ has no Spin Structure} \tag{A.2.24}$$

If the first class is not zero, then the second cannot be non-vanishing, namely one cannot define a spin structure if the manifold is not orientable. The first two Stiefel-Whitney classes are then used to see if one can define spinors on a manifold M, since spinors are sections of the Spinor Bundle which is the associated bundle of the lift of the orthonormal frame bundle. In fact, one can consider the frame bundle LM defined above but instead of defining its elements as general frames, we use the Riemannian metric on M to define them as orthonormal frames, obtaining a principal

$SO(n)$-bundle $(F_{SO}(M), \pi, M, SO(n))$. Using the double cover $\rho : Spin(n) \to SO(n)$ one can lift this bundle to obtain a $Spin(n)$-principal bundle $(P, \pi_P, M, Spin(n))$ with an equivariant 2-fold covering map $F_p : P \to F_{SO}(M)$ such that $\pi \circ F_P = \pi_P$ and $F_P(p \triangleleft A) = F_P(p) \triangleleft \rho(A)$ (where the $\triangleleft$ are the action of the group on $F_{SO}(M)$ and P and $A \in SO(n)$, $p \in P$). The pair (P, F_P) is called a spin structure and the associated bundle to the $Spin(n)$-bundle is the Spinor Bundle, whose sections are spinor fields. In particular we note that a spinor cannot be defined on a manifold without priorly define a Riemannian structure, since its existence is related to the construction of the Spinor bundle which itself is related to the orthonormal frame bundle, which requires the presence of a metric g.

A.2.4 Physical Significance of Chern Classes

Consider a gauge theory, namely the $SU(2)$ gauge theory describing the proton-neutron system subject to strong force. The fact that protons and neutrons are almost indistinguishable from the point of view of strong interactions (since their masses are almost the same) suggests that we can arrange them in a *strong isospin doublet*:

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \tag{A.2.25}$$

and take their masses to be exactly the same $m_p = m_n = m$. The kinetic term for this matter field will be given by (defining $M = \mathrm{diag}(m, m)$):

$$\mathscr{L}^{\psi}_{\text{kin}} = \overline{\psi}(i\partial\!\!\!/ - M)\psi \tag{A.2.26}$$

and using the *gauge principle* we can introduce the interaction with a gauge boson $A_\mu = A^a_\mu \tau_a$ (where τ_a are the generators of $\mathfrak{su}(2)$) by substituting the ordinary derivative with the covariant derivative $D_\mu = \partial_\mu - igA_\mu$, obtaining:

$$\mathscr{L}^{\psi}_{\text{kin}} + \mathscr{L}^{\psi - A}_{\text{int}} = \overline{\psi}(iD\!\!\!\!/ - M)\psi \tag{A.2.27}$$

If we add the gauge field strength term for the gauge boson, we obtain the complete Lagrangian density for the system (with a massless gauge boson):

$$\mathscr{L} = -\frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \overline{\psi}(iD\!\!\!\!/ - M)\psi \tag{A.2.28}$$

where the trace is taken over the Lie algebra $\mathfrak{su}(2)$. The gauge field strength is given by the commutator: $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ and it comprises the kinetic term $\partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ and the interaction between gauge bosons $-ig[A_\mu, A_\nu]$. The action is given by:

$$\mathscr{S}[\psi, A_\mu] = \int \mathscr{L}\, d^4x \tag{A.2.29}$$

and the equations of motion for the gauge field are given by the usual *Euler-Lagrange* equations:

$$\partial_\mu \frac{\delta \mathscr{L}}{\delta \partial_\mu A_\nu} - \frac{\delta \mathscr{L}}{\delta A_\nu} = 0 \tag{A.2.30}$$

When in Quantum Field Theory one considers the most generic Lagrangian for a gauge field, one realises that an additional term can be added: $F_{\mu\nu}F_*^{\mu\nu}$ where $F_*^{\mu\nu} = 1/2\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. This is, however, always discarded since it is a total derivative and in QFT boundary terms are thrown away by demanding the asymptotic flatness of the gauge field strength and considering the integration over the hole spacetime manifold.

Instantons

From a differential geometry point of view, a gauge theory is described by a principal bundle over the spacetime manifold, with structure group equals to the gauge group and a gauge field is nothing but the pull-back of the connection 1-form with respect to a given section. The gauge field strength is the curvature 2-form pulled-back to the spacetime manifold.

Let $(P, \pi, M, SU(2))$ be a principal $SU(2)$-bundle, with M a Riemannian manifold of dimension 4 endowed with a connection 1-form ω. Choosing a local section $\sigma : U \to P$ in a given chart (U, x), provides a gauge choice and allows to define the gauge potential on the base manifold as $A = \sigma^*\omega$. The curvature 2-form Ω given by (A.1.9), will be also pulled-back to the base manifold giving rise to the gauge field strength $F = \sigma^*\Omega$. The action on M for the gauge field will be written as:

$$\mathscr{S}[A_\mu] = -\int_M \mathrm{tr}(F \wedge \star F) =: ||F||^2 =: \langle F, F\rangle \tag{A.2.31}$$

The equations of motion are obtained by considering the family of gauge potentials:

$$A_t = A + ta \tag{A.2.32}$$

The field strength will be:

$$\begin{aligned} F(A_t) &= dA_t + A_t \wedge A_t \\ &= dA + tda + (A + ta) \wedge (A + ta) \\ &= dA + tda + A \wedge A + t(A \wedge a + a \wedge A) + t^2 a \wedge a \\ &= F(A) + t(da + a \wedge A + A \wedge a) + t^2 a \wedge a \\ &= F(A) + t d_A a + t^2 a \wedge a \end{aligned} \tag{A.2.33}$$

where $F(A)$ is the field strength arising from the 1-form A and d_A is defined as $d_A A = dA + A \wedge A$.

The action for the family of gauge potentials can be expanded near $t = 0$:

$$\begin{aligned} \mathscr{S}[A_t] &= \langle F(A_t), F(A_t)\rangle \\ &= \left\langle F(A) + t \frac{d}{dt}\bigg|_0 \langle F(A_t), F(A_t)\rangle + O(t^2), F(A) + t\frac{d}{dt}\bigg|_0 \langle F(A_t), F(A_t)\rangle + O(t^2)\right\rangle \\ &= \langle F(A), F(A)\rangle + t\left[\langle d_A a, F(A)\rangle + \langle F(A), d_A a\rangle\right] + O(t^2) \\ &= \langle F(A), F(A)\rangle + 2\langle d_A a, F(A)\rangle \end{aligned} \tag{A.2.34}$$

and A is a critical point for the action if:

$$\frac{d\mathscr{S}}{dt}\bigg|_0 = 0 \iff \langle d_A a, F(A)\rangle = 0 \iff \langle a, d_A^* F(A)\rangle = 0 \tag{A.2.35}$$

where d_A^* is the formal adjoint of d_A with respect to the inner product $\langle\cdot,\cdot\rangle$. Since $a \neq 0$, the equations of motion will be:

$$d_A^* F(A) = 0 \tag{A.2.36}$$

If we take into consideration the fact that the field strength F naturally satisfies the *Bianchi* identity:

$$d_A F = 0 \tag{A.2.37}$$

then, since $d_A^* = -\star d_A \star$, we see that (A.2.36) is satisfied provided that $F = \pm \star F$ and if F satisfies one of these conditions, then it is called self-dual (+ case) or anti-self-dual (— case). In principle these solutions could be maxima or minima, but we are going to show that they are indeed always minima for the action and these minima are called *Instantons*. In order to see that $F = \pm \star F$ are always minima for the action, we can decompose F into its self-dual and anti-self-dual components:

$$F = \frac{1}{2}[F + (\star F)] + \frac{1}{2}[F - (\star F)] := F^+ + F^- \tag{A.2.38}$$

in this way, it can be readily seen that the action becomes:

$$\mathscr{S} = ||F^+ + F^-||^2 = \langle F^+ + F^-, F^+ + F^-\rangle = \langle F^+, F^+\rangle + 2\langle F^+, F^-\rangle + \langle F^-, F^-\rangle = ||F^+||^2 + ||F^-||^2 \tag{A.2.39}$$

where in the last step it has been used the fact $\langle F^+, F^-\rangle$ is identically equals to zero. If we consider now a representative of the second Chern class calculated before (A.2.18) and integrate it, we have:

$$\begin{aligned}
-\int_M c_2(P) &= \int_M \frac{\operatorname{tr}(F \wedge F)}{8\pi^2} \\
&= \frac{1}{8\pi^2} \int_M \operatorname{tr}[(F^+ + F^-) \wedge (F^+ + F^-)] \\
&= \frac{1}{8\pi^2} \int_M [\operatorname{tr}(F^+ \wedge F^+) + \operatorname{tr}(F^- \wedge F^-)] \\
&= \frac{1}{8\pi^2} \int_M [\operatorname{tr}(F^+ \wedge \star F^+) - \operatorname{tr}(F^- \wedge \star F^-)] \\
&= \frac{1}{8\pi^2} [||F^+||^2 - ||F^-||^2]
\end{aligned} \tag{A.2.40}$$

Now, it is clear that we have the following inequality:

$$\frac{1}{8\pi^2}\left(||F^+||^2 + ||F^-||^2\right) \geq \frac{1}{8\pi^2}\left|||F^+||^2 - ||F^-||^2\right| \tag{A.2.41}$$

which is nothing but the fact that the action is always greater than or equals to the expression (A.2.40) and the minimum is realised when we have the equality. By calling k the expression (A.2.40), the absolute minima are obtained when:

$$\mathscr{S} = 8\pi^2 |k| \tag{A.2.42}$$

which corresponds to $F^{\pm} = 0$, namely the self-duality and anti-self-duality conditions on F. We have thus shown that Instantons are indeed always minima of the Yang-Mills action.

Let's now analyse further (A.2.40). Since $c_2(P)$ is a representative of the cohomology $H^4(M,\mathbb{R})$, it is a closed 4-form and if the bundle is trivial, it is also exact. This means that we can find a 3-form ω such that $d\omega = c_2(P)$. This 3-form is found to be:

$$\omega = \mathrm{tr}\left(A \wedge A + \frac{2}{3} A \wedge A \wedge A\right) \tag{A.2.43}$$

and is called *Chern-Simons form*. So, if the bundle is trivial, the integral (A.2.40) is zero. We see that the instantons are topological entities which, in fact, depends on the topological characteristics of the bundle. If one considers the $U(1)$-principal bundle $\pi : S^3 \to S^2$ (Hopf bundle), then since it is not trivial, (A.2.40) will not be zero, and it is found that k is equals to an integer characterising the homotopy type of the transition functions which are maps from the equator of S^2 (namely an S^1) to the structure group $U(1)$ (which is topologically equivalent to S^1). This means that we are characterising the maps from S^1 to S^1, which are defined by their winding number. For the case of a Euclideanized Yang-Mills theory, the principal bundle is $(P, \pi, \mathbb{R}^4, SU(2))$, and after the compactification of $\mathbb{R}^4$ to S^4 by adding the point at infinity $|x| = +\infty$, we get the bundle $(P, \pi, S^4, SU(2))$. A cover of the 4-sphere S^4 is given by two open sets $\{U_N, U_S\}$, one covering the north pole, U_N, and one covering the south pole, U_S, intersecting on the equator giving rise to an infinitesimal strip homeomorphic to a 3-sphere, namely $U_N \cap U_S \simeq S^3$. The transition functions are then maps from S^3 to $SU(2) \simeq S^3$, and these too are classified by their homotopy type, so the integral $-\int_{S^4} \mathrm{tr}(F \wedge F)/8\pi^2$ will be given by a number $n \in \mathbb{Z} = \pi_3(SU(2))$ (where $\pi_3(SU(2))$ is the 3rd homotopy group of $SU(2)$).

Appendix B

Varieties in Weighted Projective Spaces

There are various ways to find and realise Calabi-Yau manifolds: as intersection manifolds embedded in products of ordinary projective spaces, as hypersurfaces embedded in weighted projective spaces, as orbifolds and as fibrations. It is clear that the domain of mathematics this kind of construction belongs to is *Algebraic Geometry*. The study of *Mirror Symmetry* (see §2.2.7) for Calabi-Yau manifolds led to very profound results in this branch of mathematics. One example is the result in enumerative geometry concerning the counting of rational curves on a generic quintic which was done on a combination of physical based assumption and mathematics by Candelas et al. [Can+91] and later on interpreted in a more mathematical fashion by Morrison [Mor92]. It is seems compelling, then, to give a brief introduction to weighted projective spaces, homogeneous polynomials and weighted projective varieties in order to understand the realisation of Calabi-Yau manifolds as embedded hypersurfaces. We will mainly follow [Hos16].

Definition B.0.1 (Weighted Projective Space). *Let $a=(a_0,\dots,a_n)\in\mathbb{N}^{n+1}$ be a collection of integers called weights and let ρ_a be the action of $\mathbb{C}$ on $\mathbb{A}^{n+1}\setminus\{0\}$ defined as:*

$$\begin{aligned}\rho_a : \mathbb{C}\times\mathbb{A}^{n+1}\setminus\{0\} &\to \mathbb{A}^{n+1}\setminus\{0\}\\ (\lambda,(x_0,\dots,x_n)) &\mapsto \rho_a(\lambda,(x_0,\dots,x_n)) := (\lambda^{a_0}x_0,\dots,\lambda^{a_n}x_n)\end{aligned} \tag{B.0.1}$$

The Weighted Projective Space $\mathbb{P}^n_{(a_0,\dots,a_n)}$ is then defined as the quotient of $\mathbb{A}^{n+1}\setminus\{0\}$ with respect to the equivalence relation induced by the action:

$$\mathbb{P}^n_{(a_0,\dots,a_n)} = \frac{\mathbb{A}^{n+1}\setminus\{0\}}{\sim} \tag{B.0.2}$$

where $\forall(x_0,\dots,x_n),(x'_0,\dots,x'_n)\in\mathbb{A}^{n+1}\setminus\{0\}$:

$$(x_0,\dots,x_n)\sim(x'_0,\dots,x'_n) \iff \forall\lambda\in\mathbb{C},\exists a\in\mathbb{N}^{n+1} \text{ such that } (x_0,\dots,x_n)=(\lambda^{a_0}x'_0,\dots,\lambda^{a_n}x'_n) \tag{B.0.3}$$

In the definition we have used $\mathbb{A}^{n+1}$ which is the affine n-space of $\mathbb{C}$ and it is defined as the spectrum[1] of the ring of polynomials:

$$\mathbb{A}^{n+1} := \mathrm{Spec}(\mathbb{C}[x_0,\dots,x_n]) \tag{B.0.4}$$

[1]The spectrum of a commutative ring is the set given by the collection of all prime ideals, where a prime ideal P of a commutative ring R is an ideal such that P is not the whole ring R and for each $a,b\in R$ such that $a\cdot b\in P$, then it must be true that either $a\in P$ or $b\in P$.

Remark

> The affine n-space $\mathbb{A}^{n+1}$ can be thought to just be the vector space $\mathbb{C}^{n+1}$ without the origin, in the sense that we add translations to linear maps and forget about the origin. From a categorical point of view, there two categories, one given by affine spaces over a field k (in the sense of vector spaces without the origin) and one of varieties over the field k (zero loci of polynomials). There is a functor from these two categories allowing us to identify these two kind of (at first sight) different definitions.

Points in $\mathbb{P}_{(a_0,...,a_n)}$ are written as $|x_0 : \cdots : x_n|_a$ for $a = (a_0,...,a_n) \in \mathbb{N}^{n+1}$, also $\mathbb{P}^n_{(a_0,...,a_n)}$ inherits naturally the quotient topology given the Zariski topology on $\mathbb{A}^{n+1}$.

Definition B.0.2 (Weighted Polynomial Ring)**.** *For $a \in \mathbb{N}^{n+1}$ we define the weighted polynomial ring of $n+1$ variables as $k_a[x_0,...,x_n]$ (k is a ring) with the weight of x_i to be a_i, in particular:*

$$deg\left(\prod_{i=0}^{n} x_i^{c_i}\right) = \sum_{i=0}^{n} a_i c_i \tag{B.0.5}$$

For example, we can consider $\mathbb{R}_{(1,5,4)}[x,y,z]$ and take the polynomial $p(x,y,z) = 2x^2 + \sqrt{3}yz^3x - \pi y^3$. The degree of each monomial will then be given by the formula (B.0.5) as:

$$\begin{aligned}
&\deg(2x^2) = \overbrace{1}^{a_0} \cdot \overbrace{2}^{c_0} \\
&\deg(\sqrt{3}yz^3x) = 5 \cdot 1 + 4 \cdot 3 + 1 \cdot 1 = 18 \\
&\deg(-\pi y^3) = 3 \cdot 5 = 15
\end{aligned} \tag{B.0.6}$$

It seems natural now to consider those polynomials made up of monomials with the same degree. These are called weighted homogeneous polynomials and we collect this in the following definition:

Definition B.0.3 (Weighted Homogeneous Polynomials)**.** *Let $k_a[x_0,...,x_n]$ be a weighted polynomial ring with $a \in \mathbb{N}^{n+1}$. We say that $f \in k_a[x_0,...,x_n]$ is a-weighted-homogeneous of degree d if each monomial in f is of weighted degree d.*

This means that for a polynomial $f \in k_a[x_0,...,x_n]$ to be a-weighted-homogeneous of degree d there must exists $b_i \in k$ and $m \in \mathbb{N}$ such that:

$$f = \sum_{i=0}^{m} b_i \left(\prod_{j=0}^{n} x_j^{c_j^{(i)}}\right) \tag{B.0.7}$$

in such a way that:

$$\sum_{j=0}^{n} a_j c_j^{(i)} = d \qquad \forall i = 0,...,m \tag{B.0.8}$$

As an example consider again as before the weighted polynomial ring $\mathbb{R}_{(1,5,4)}[x,y,z]$, but take the following polynomial $f(x,y,z) = x^{10} - y^2 + z^2x^2 + xyz$. The degree of each monomial will be:

$$\begin{aligned}
&\deg(x^{10}) = 10 \\
&\deg(-y^2) = 5 \cdot 2 = 10 \\
&\deg(z^2x^2) = 4 \cdot 2 + 2 = 10 \\
&\deg(xyz) = 1 + 5 + 4 = 10
\end{aligned} \tag{B.0.9}$$

and this means that f is $(1,5,4)$-weighted-homogeneous of degree $d=10$.

The fact that a polynomial $f \in k_a[x_0,...,x_n]$ is a-weighted-homogeneous of degree d means also that $\forall \lambda \in k$ it is true that:

$$f(\lambda^{a_0}x_0,...,\lambda^{a_n}x_n)=\lambda^d f(x_0,...,x_n) \tag{B.0.10}$$

which can be seen to arise from (B.0.7) in the following straightforward way:

$$\begin{aligned} f(\lambda^{a_0}x_0,...,\lambda^{a_n}x_n) &= \sum_{i=0}^{m} b_i \left(\prod_{j=0}^{n}(\lambda^{a_j}x_j)^{c_j^{(i)}}\right) \\ &= \sum_{i=0}^{m} b_i \left(\prod_{j=0}^{n}\lambda^{a_j c_j^{(i)}} x_j^{c_j^{(i)}}\right) \\ &= \sum_{i=0}^{m} b_i \lambda^{\sum_{j=0}^{n} a_j c_j^{(i)}} \left(\prod_{j=0}^{n} x_j^{c_j^{(i)}}\right) \\ &= \lambda^d f(x_0,...,x_n) \end{aligned} \tag{B.0.11}$$

If we now take an element of the weighted projective space, namely $p \in \mathbb{P}^n_{(a_0,...,a_n)}$, and write it as $|p_0 : \cdots : p_n|_a$, we could try to evaluate a weighted homogeneous polynomial $f \in k_a[x_0,...,x_n]$ at this point p. But recalling that for each $\lambda \in k$ it is true by definition that $|p_0 : \cdots : p_n|_a = |\lambda^{a_0}p_0 : \cdots : \lambda^{a_n}p_n|_a$, then we have the following situation:

$$\begin{cases} f(\lambda^{a_0}p_0,...,\lambda^{a_n}p_n)=f(p_0,...,p_n) \\ f(\lambda^{a_0}p_0,...,\lambda^{a_n}p_n)=\lambda^d f(p_0,...,p_n) \end{cases} \tag{B.0.12}$$

where in the first we have used the property of the elements in the weighted projective space and in the second we have used the property of weighted homogeneous polynomials. Combining these two, it can be seen that for generally $\lambda^d - 1 \neq 0$, in order to make sense of these expressions it must be true that:

$$f(p_0,...,p_n)=0 \tag{B.0.13}$$

The evaluation of a homogeneous polynomial on a point in the weighted projective space makes sense only if one considers the zeroes of such a function at that point. This understanding is of fundamental importance in order to get to the definition of weighted projective varieties. The last ingredient will be the concept of weighted homogeneous ideals.

Definition B.0.4. *Let $k_a[x_0,...,x_n]$ be a weighted polynomial ring with $a \in \mathbb{N}^{n+1}$. We will say that an ideal $I \triangleleft k_a[x_0,...,x_n]$ is a-weighted-homogeneous if it is generated by a-weighted-homogeneous elements.*

Aside

We recall that for a ring of polynomials $k[x_0,...x_n]$, there always can be constructed ideals I generated by polynomials of $k[x_0,...,x_n]$. Namely, for $\{p_i\}_{i=1,...,m}$ sequence of polynomials in $k[x_0,...,x_n]$, the space:

$$I := \langle f_1,...,f_m\rangle := \left\{ p \in k[x_0,...,x_n] \;\middle|\; p=\sum_{i=0}^{n} g_i f_i \text{ with } g_i \in k[x_0,...,x_n] \right\} \tag{B.0.14}$$

is an ideal of $k[x_0, ..., x_n]$.

For example take again $\mathbb{R}_{(1,5,4)}[x, y, z]$ and consider the ideals:

$$I = \langle x^4 + y, z - yx, x^9 + yz \rangle \qquad J = \langle x^3 y - 3yz, zy + x^4 z \rangle \tag{B.0.15}$$

By direct inspection, every polynomial generating I is weighted-homogeneous, while those generating J are not. We remark that it is not required for the generating polynomials to be of the same degree. This means that the ideal I is $(1,5,4)$-weighted-homogeneous, while J is not.

We are now in a position to define the idea of weighted projective variety, given the previous discussion on the vanishing of weighted homogeneous polynomials on points of the weighted projective space.

Definition B.0.5 (Weighted Projective Variety). *Let $I \lhd k_a[x_0, ..., x_n]$ be a weighted homogeneous ideal of $k_a[x_0, ..., x_n]$. The weighted projective variety (associated to the ideal I) is the subspace of the weighted projective space $\mathbb{P}^n_{(a_0,...,a_n)}$ given by:*

$$\mathbb{V}(I) := \{p \in \mathbb{P}^n_{(a_0,...,a_n)} \mid \forall f \in I : f(p) = 0\} \tag{B.0.16}$$

This definition makes sense since polynomials in a weighted homogeneous ideal can be written as a sum of a-weighted-homogeneous polynomials of different degrees. It then makes sense to consider $I \ni f(p) = \sum_i g_i(p) f_i(p) = 0$ where each f_i is a weighted homogeneous polynomial of degree i while $g_i \in k_a[x_0, ..., x_n]$.

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