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Stone Skipping Physics

Charles F. Babbs

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Charles F. Babbs, Purdue University, West Lafayette, IN

As a boy I spent many hours skipping stones across the surface of the water on the shore of Lake Michigan. The challenge was to get as many skips or as much distance as possible. Zero skips was a bust. One or two skips was disappointing, three or four skips was mediocre, and six or more skips was exhilarating. The angle of the stone with respect to the surface of the water seemed to be critical. A large angle approaching 45° produced one large jump and perhaps one or two more after that. A smaller angle produced more skips and a longer run. However, too small an angle would cause immediate sinking. Fine tuning the skill of stone skipping was a captivating pastime. Today stone skipping has become both a recreational and a competitive sport.

Motivated students can explore the underlying physics of stone skipping to explain why stones skip, derive equations for the trajectory of an idealized skipping stone, and predict the number of skips and the total distance of travel of the stone. This problem can provide an entertaining exercise to consolidate knowledge of first-year physics, and an organizing theme for problem-based learning, without requiring advanced mathematics or a detailed description of the fluid flow around the colliding stone in three dimensions.

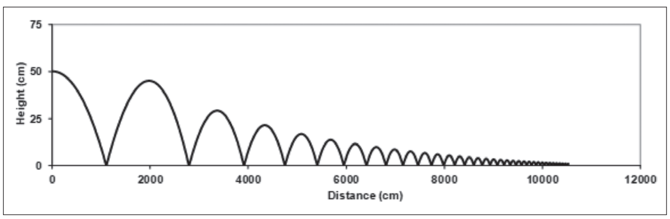

Fig. 1. Trajectory of a skipping stone with parameters of a championship throw. Launch speed 3600 cm/s. Note difference in horizontal and vertical distance scales, which distorts the apparent heights of the skips.

Indeed, one can do for sidearm stone skipping what Poljak[1] has done for overhand vs. underhand throwing to provide new insights: namely characterize physical variables needed to throw a stone as far horizontally as possible. Calculation of complete stone trajectories, including championship throws in competition, is possible. For example, Fig. 1 illustrates the trajectory of an idealized stone with model parameters tuned to enhance skipping behavior. The vertical distance scale is expanded to show detail. There are 67 skips over a distance of 114 meters. The pattern of a few early high skips, followed by a large number of low, quick skips is similar to that seen on recorded videos (search YouTube championship stone skipping).

An idealized skipping stone

Figure 2 shows sketches of an idealized flat stone, the angle of which with respect to the horizontal α is stabilized by rotational spin—the gyroscopic effect—and so is considered constant. The thickness of the stone is denoted h. The flat surface

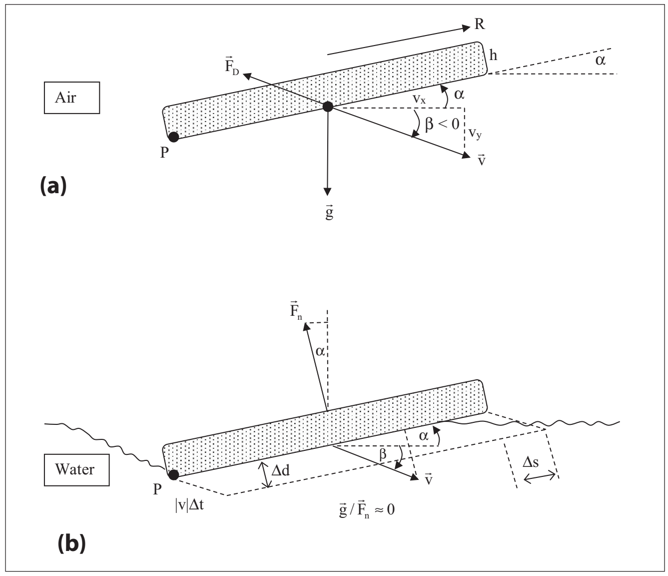

Fig. 2. (a) An idealized, spin-stabilized skipping stone in flight. The radius of the stone is R, and the thickness is h. The velocity of the stone in air is v. The tilt angle of the stone from the horizontal is α. The flight path angle is $\beta = \tan^{-1}(v_y/v_x)$, with $\beta < 0$ for a falling stone and $\beta > 0$ for a rising stone. Air resistance creates drag force, F_D, which opposes forward motion. (b) An idealized, spin-stabilized skipping stone in the water. Here the stone is moving into the water at velocity, v. The reactive force, F_n, acting on the bottom surface of the stone pushes upward with force $F_n \cos(\alpha)$ and backward with force $F_n \sin(\alpha)$. The stone displaces water when moving normal to its bottom surface through distance Δd. A free slip condition at the water-stone boundary means that there is no friction during movement over distance, Δs, perpendicular to F_n.

area of the stone is denoted $A = \pi R^2$ for a hockey puck-shaped stone of radius R. If ρ_s denotes mass density, then the mass of the stone is $m_s = \rho_s \pi R^2 h$. The stone moving through air is shown on top. The stone in contact with the surface of the water is shown on the bottom.

The stone moves with instantaneous horizontal and vertical velocity coordinates v_x and v_y in two dimensions. The flight path angle with respect to the horizon is $\beta = \tan^{-1}(v_y/v_x)$. A negative value of v_y or a negative value of β indicates that the stone is falling downward under the acceleration of gravity g. A positive value of v_y or a positive value of β indicates that the stone is rebounding upward. At time $t = 0$ the stone is launched over a flat surface of water from vertical height y_0 with initial horizontal velocity v_{x0} and initial vertical velocity v_{y0}. Subsequent motion of the stone requires successive passages through two domains: air and water. Motion in the air can be treated as ordinary projectile motion. Motion during brief collisions with the water can be treated approximately in terms of Newton's third and second laws with reasonable simplifying assumptions.

Consider a point P at the trailing bottom edge of the stone. For simplicity, let the trajectory of P as a function of x, y,

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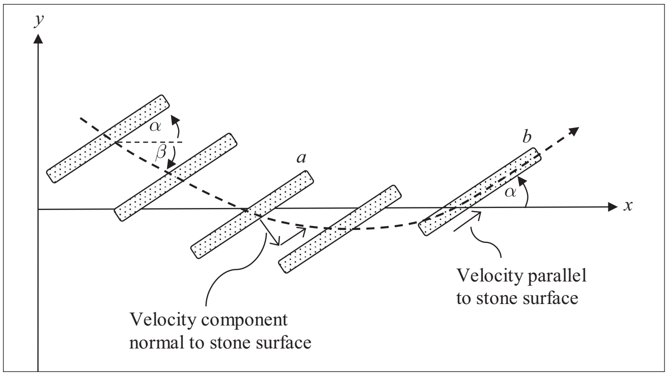

Fig. 3. An idealized collision model, in which a spin-stabilized, flat stone pushes water ahead of it during collision. In early positions (a) reactive force slows the stone in the direction normal to its surface, changing its trajectory until the flight path becomes parallel to the stone's tilt at angle α, after which no more force is exerted by the water on the stone (b). The stone exits the water at angle α. During such collisions the force of gravity is relatively small compared to other forces acting on the stone.

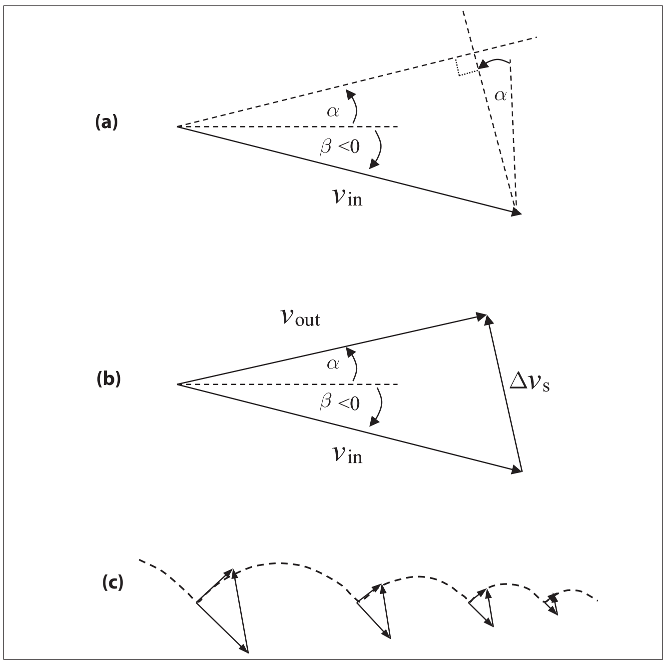

Fig. 4. (a) and (b) Vector addition $\boldsymbol{v}_{in} + \Delta \boldsymbol{v}_s = \boldsymbol{v}_{out}$ for computing outbound velocity of a skipping stone. (c) Serial application of the vector addition rule to reconstruct airborne segments of the stone's trajectory.

and time t represent the position of the stone in space and time. When $y > 0$ the stone is considered to be in the air and ordinary projectile kinetics apply. In flight, and ignoring air resistance, the acceleration of the stone in the x-direction, $a_x = 0$, and the acceleration in the y-direction, $a_y = -g$. If aerodynamic drag forces $\boldsymbol{F}_D$ are included, the stone experiences additional vector drag acceleration $\boldsymbol{a}_D = \boldsymbol{F}_D/m_s$ in a direction that opposes its forward motion.

The crux of the stone skipping problem, however, is to characterize the change in velocity of the stone after it hits the surface of the water. Reynolds numbers for this scenario of stone-water collision are $Re \sim 10^5$, so that viscous forces can be neglected, and reactive forces dominate.[2,3] The following treatment gives expressions for the vector change in velocity of the stone $\Delta \boldsymbol{v}_s$, with each skip for a subset of all possible collisions in which the motion of the stone normal to its flat bottom surface is stalled by reactive forces before water overtops the stone.

An idealized collision

To model the interaction of the stone with the water during successive skips, one can imagine the force of the stone pushing on the water, the equal and opposite force of the water pushing back on the stone, and, in turn, the change in velocity of the stone caused by the collision, which allows calculation of the trajectory of the stone through the air during the next skip. When the stone is in the air, typical projectile motion occurs, and gravity plays an important role. However, when the stone is in the water, the reactive forces from displacement of the water are much greater than gravity, which for simplicity can be neglected during initial analysis of the brief stone-water collisions.

Figure 3 illustrates the flight path of an idealized skipping stone colliding with the surface of the water at spin-stabilized angle α. The vertical scale is expanded to show detail. The water is regarded as an ideal fluid to allow frictionless slipping at the fluid-solid boundary. Hence, no work is done by the water on the stone as the stone moves parallel to its flat bottom surface (b). Substantial work is done only as the stone moves perpendicular to its bottom surface (a). (The small amount of work done against gravity to lift the mass of the stone from its low point in the water to its exit point from the water is considered subsequently.) By Newton's third law, the reactive force on the stone is equal in magnitude and opposite in direction from the force that the stone exerts on the water. By Newton's second law the product of the average reactive force and the brief time interval Δt of the collision equals the mass of the stone multiplied by the change in velocity of the stone: $\boldsymbol{F}\Delta t = m_s \Delta \boldsymbol{v}_s$. Both the reactive force and the change in velocity of the stone point in the direction perpendicular to the bottom surface of the stone. As long as water does not overtop the stone, this effect will change the stone's trajectory until the flight path becomes parallel to the stone's surface at angle α (Fig. 3). Thus, the water removes the perpendicular component of stone velocity and leaves the larger parallel component unaffected. Then the stone exits the water at angle α or very nearly α.

As shown in Fig. 4, the outbound velocity vector $\boldsymbol{v}_{out}$ must be approximately at angle α with respect to the horizontal. The reactive forces, and the consequent change in velocity vector $\Delta \boldsymbol{v}_s$, must be perpendicular to the surface of the stone, at angle α from the vertical. These two constraints define a right triangle for vector addition, $\boldsymbol{v}_{in} + \Delta \boldsymbol{v}_s = \boldsymbol{v}_{out}$, which determines the direction and magnitude of the outbound velocity $\boldsymbol{v}_{out}$ of the stone.

By deduction from Fig. 4, for total angle $-\beta + \alpha$ at water entry,

$$|\boldsymbol{v}_{out}| = |\boldsymbol{v}_{in}| \cos(-\beta + \alpha). \tag{1}$$

For realistic angles α, the dominant component of the normal force, $|\boldsymbol{F}_n|\cos\alpha$, is a vertical lifting force. It is this force that causes the stone to skip!

The next skip of the stone, beginning at zero height, will have initial velocity components

$$v_{x\mathrm{out}} = |\boldsymbol{v}_{\mathrm{in}}| \cos(-\beta + \alpha)\cos\alpha \tag{2}$$

and

$$v_{y\mathrm{out}} = |\boldsymbol{v}_{\mathrm{in}}| \cos(-\beta + \alpha)\sin\alpha\,. \tag{3}$$

Note that very small height skips at the end of the trajectory are difficult to distinguish from vibrations and also difficult to count. These short skips at end of run with little water showing between are known to stone skipping aficionados as "pitty-pat" (www.stoneskipping.com/glossary). Thus, to avoid unrealistic overestimation of the number of physically realistic skips, one might wish to establish a minimum legal height criterion, such as 0.5 cm above the smooth surface of the water.

Table I. Standard model.

Variable	Value	Units	Definition
h	1	cm	Stone thickness
R	4	cm	Stone radius
ρ_s	2.5	g/cm^3	Stone mass density
ρ_w	1.0	g/cm^3	Water mass density
ρ_{air}	0.00122	g/cm^3	Air mass density
y_0	50	cm	Launch height
y_{legal}	0 to 1	cm	Minimum legal skip height for counting
v_{x0}	1000	cm/s	Horizontal launch velocity
v_{y0}	0	cm/s	Vertical launch velocity (positive = up)
α	0.3	rad	Stone surface angle with horizon
	17	deg	
Δt	0.00001	s	Time step for numerical integration

Including gravity during collisions and the stopping criterion

There is a small elevation change as the stone slides parallel to its flat bottom surface from the low point in the water to the point of taking flight on the next skip. To obtain an approximate correction for the extra downward travel of the stone caused by the acceleration of gravity during the brief time of the collision, one can assume the typical maximal depth of point P at the trailing edge of the stone in the water is approximately $R\sin(\alpha)$. Then from conservation of energy the vertical velocity, corrected for the energy required to lift the stone a small, constant distance $R\sin(\alpha)$ out of the water, is

$$\hat{v}_{\mathrm{yout}} = \sqrt{v_{\mathrm{yout}}^2 - 2gR\sin(\alpha)}. \tag{4}$$

Further, if the stone is required to have a minimum legal bounce height, $0 \le y_{\mathrm{legal}}$, then the end-of-run stopping criteria are that if

$$v_{\mathrm{yout}}^2 - 2g[R\sin(\alpha) + y_{\mathrm{legal}}] < 0 \quad \text{or } v_{\mathrm{xout}} < 0, \tag{5}$$

then stop. If $y_{\mathrm{legal}} = 0$, then bounces of any height are allowed.

Numerical computations of stone trajectories in the air

The horizontal and vertical components of stone acceleration in air can be integrated numerically using the simple Euler method, implemented, for example, in Visual Basic code within an Excel spreadsheet. Typical initial conditions are described in Table I. Specifically, given initial height y_0 and initial velocity components v_{x0} and v_{y0}, integration is performed numerically for each successive time increment Δt,

$$v_x(t+\Delta t) \cong v_x(t) + a_x\Delta t, \text{ and } v_y(t+\Delta t) \cong v_y(t) + a_y\Delta t \tag{6}$$

and

$$x(t+\Delta t) \cong x(t) + v_x\Delta t, \text{ and } y(t+\Delta t) \cong y(t) + v_y\Delta t, \tag{7}$$

until the stone returns to height $y = 0$. A subroutine can be created to perform this calculation for both the initial throw ($y_0 > 0$, e.g., 50 cm) and subsequent skips ($y_0 = 0$). One may include the aerodynamic drag force on the stone, which has direction opposite the stone's velocity and magnitude

$$|F_{\mathrm{D}}| = \frac{1}{2}\rho_{\mathrm{air}} C_{\mathrm{D}} A |\boldsymbol{v}|^2, \tag{8}$$

where ρ_{air} is the mass density of air (0.00122 g/cm^3), constant C_{D} is a dimensionless drag coefficient or shape factor, typically ranging between 0 and 2, area A is the reference surface area, taken here as πR^2 for simplicity, and $\boldsymbol{v}$ is the forward velocity. From Hoerner,[4] $C_{\mathrm{D}} \approx 0.5$. Drag acceleration has magnitude

$$|a_{\mathrm{D}}| = \frac{|F_{\mathrm{D}}|}{m_s} = \frac{\frac{1}{2}\rho_{\mathrm{air}} C_{\mathrm{D}} A |\boldsymbol{v}|^2}{\rho_s A h} = \frac{1}{2}\frac{\rho_{\mathrm{air}}}{\rho_s}\frac{C_{\mathrm{D}}}{h}|\boldsymbol{v}|^2. \tag{9}$$

Horizontal and vertical components of drag acceleration are $-|a_{\mathrm{D}}|\cos\beta$ and $-|a_{\mathrm{D}}|\sin\beta$.

Numerical computations of changes in stone velocity in water

After either the initial throw or a skip, as soon as the computed height of the stone above the water, $y(t)$, becomes less than zero, the subsequent change in velocity of the stone caused by collision with the water is easily computed using Eqs. (1) through (4). The horizontal and vertical exit velocity components are then taken as initial conditions for the next flight, beginning at $y_0 = 0$. The short unknown horizontal distance that the stone travels in the water from its point of entry is estimated as $2R$. This process is repeated until a stopping criterion is met.

Table I shows standard model parameters for the idealized skipping stone. Figure 5(a) shows stone trajectories for the standard model. The horizontal axis ranges from 0 to 2000 cm. The vertical axis ranges from 0 to 75 cm to show detail of the skips. The apparent skip heights and water entry and exit angles in the figure are correspondingly exaggerated. For this standard model, including aerodynamic drag, there are nine skips and the flight distance is 13.71 m when the minimum legal skip height $y_{\mathrm{legal}} = 0$. When y_{legal} is increased to between 0.2 cm and 0.5 cm, there are only eight skips, and the

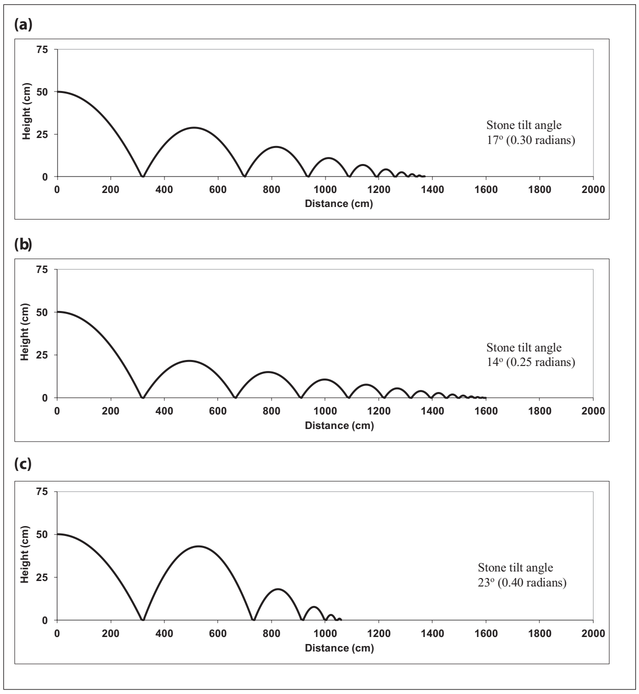

Fig. 5. Typical stone trajectories for the standard model. Note the difference in horizontal and vertical length scales, which exaggerates apparent skip height. Minimum legal skip height, y_{legal} = 0. Other model parameters are those listed in Table I. (a) Tilt angle = 17° (0.3 rad). (b) Tilt angle α = 14° (0.25 rad). (c) Tilt angle α = 23° (0.4 rad).

flight distance is only 13.57 m (data not shown).

Figure 5(b) shows the stone trajectory for the otherwise standard model in Fig. 5(a) with reduced stone tilt angle α = 14° (0.25 rad). There are 13 skips and a flight distance of 16.00 m. The last three skips are very low in height. If y_{legal} is increased to 0.5 cm, then only 10 skips are counted over 15.55 m. This feature of hard-to-judge terminal "pitty-pat" is characteristic of low tilt-angle throws.

Figure 5(c) shows the stone trajectory for the otherwise standard model with increased tilt angle α = 23° (0.4 rad). There are five skips and a flight distance of 10.6 m. The trajectory shows physically realistic skipping behavior.

Other such simulations can readily be done to explore effects of stone tilt angle, initial launch angle, initial launch velocity, and possible parameters of world record throws.

Discussion

Using elementary physics, based upon Newton's laws of motion, it is possible to estimate the trajectory of an idealized skipping stone using a simple computer program. Students can practice math and science skills, as well as coding skills, by creating a mathematical model of stone skipping and then testing its predictions experimentally with actual stones and throws through careful video analysis.

Students who wish to delve deeper into the fluid-structure interactions between water and a skipping stone can explore Refs. 1-13. For example, Richard Crane[5] describes the charming and entertaining experiments for measuring rotation frequency and angle using a mast placed on a man-made stone. This paper is a great source of inspiration for follow-on experiments in the real world, including studies of the effects of texture or dimples, similar to those on golf balls, on the under-surfaces of stones. High-speed video analysis has been conducted by Clanet, Hersen, and Bocquet.[6] A more detailed description of air resistance is provided by Mohazzabi.[7] For those who wish to spend less time on math and more time at the water's edge, helpful observations on technique can be found at www.stoneskipping.com and in a variety of articles and videos available online.[9-13]

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Charles F. Babbs, *MD (Baylor College of Medicine), PhD (pharmacology, Purdue University) is a founding member of the Biomedical Engineering Center and Continuing Lecturer at the Weldon School of Biomedical Engineering at Purdue University. His research has focused in the areas of biomedical engineering, cardiopulmonary resuscitation, tumor blood flow, heat therapy for cancer, digital mammography, the roles of free radicals and iron in tissue injury, biomechanics, and the development of new medical devices. His research reports and patents have been cited over 10,000 times according to Google Scholar. He has received multiple teaching awards for courses in medicine, physiology, and mathematical modeling. Dr. Babbs is married to Patricia Joan (Murphy) Babbs and has three grown children. His hobbies include soccer, chess, hiking, and mountaineering.*
Weldon School of Biomedical Engineering, Purdue University, West Lafayette, IN 47907; babbs@purdue.edu