Robustness and Games with Linear Best Replies: Theory and Applications

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Keler Marku

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Dedication

To my parents.

Abstract

This thesis consists of three chapters on games with linear best replies. In the first chapter[1] we show how in the context of a common agency game, when principals seek robustness, then linearity in total output emerges as an equilibrium outcome. More specifically we consider a game between several principals and a common agent, where principals design contracts that are robust to misspecification of the agent's technology. The principals know a subset of the actions available to the agent, but other unknown actions could exist. Principals demand robustness and evaluate contracts on the worst-case performance over all possible actions of the agent. Despite the complexity of the game, we show that a pure strategy equilibrium always exists, by constructing a pseudo-potential for the game. Equilibrium contracts are linear in total output and imply that all players (the principals and the agent) receive a share of total output. The higher the share of total output accruing to the agent, the more efficient the outcome of the game. The crisp characterization of the equilibrium allows us to revisit the classical question of the efficiency of competitive outcomes relative to collusion among principals. We also consider a game where principals collude and offer a joint contract. The efficiency of the competitive outcome depends crucially on the ability of principals to offer side-payments to one another through the agent.

In the second chapter we consider several applications of the framework introduced in the first chapter. Linearity allows for sharp predictions of the model in several contexts. The main application of the model in the first chapter is in analyzing the taxation of multinational firms where we study the effects of tax competition among countries. We show that a flat tax on domestic and foreign profits with a full deduction of foreign taxes provides the best worst-case guarantee for each country's revenues. Furthermore we consider a procurement auction setup, as well as an application of the model to private provision of public goods.

In the third chapter we depart from the robustness framework and focus on the network structure of games with linear best replies. Games played on fixed networks capture a

[1]Chapters one and two are products of a collaboration with Sergio Ocampo Diaz.

variety of economic settings including public goods, peer effects, and technology adaption. Bramoullé et al. (2014) analyze a large class of one dimensional linear best reply games and provides general results on how the network affect social and economic outcomes. In this paper we first provide an isomorphism between games with linear best replies and the threshold-linear recurrent networks used in neuroscience to study the encoding of memory patterns in the brain, connecting two seemingly unrelated literatures. Inspired by the isomorphism we extend games of linear best replies in understanding Lindahl equilibria and to games with multidimensional actions. In particular we show how Cournot competition among several firms leads to specialization in production. We show how the network structure of competition in demand for consumers shape the decision of firms in which goods to specialize.

Contents

List of Tables

List of Figures

Chapter 1

Robust Contracts in Common Agency

Strategic settings where several actors try to influence a common party have attracted abundant attention in many areas of economics. Examples include political economy, industrial organization, mechanism design, public finance, international trade, and auctions.[1] Starting with the work of Bernheim and Whinston (1986a,b) these settings have been modeled as a common agency game where several principals simultaneously and non-cooperatively contract with a single agent.

In this chapter we present a general moral hazard common agency game, where the principals seek to design contracts that are *robust* to misspecification of the environment. More specifically contracts that perform well if the principals have an incomplete knowledge of the agent's technology, or if principals cannot renegotiate contracts when technology

[1] For example, in political economy, lobbying is modeled as a game between lobbyists (the principals) influencing a politician (the agent), see Grossman and Helpman (1994), Dixit et al. (1997), Le Breton and Salanie (2003), and Martimort and Semenov (2008). In public finance, a firm (the agent) is taxed and regulated by the local, state and federal government (the principals), or a multinational company (the agent) has to pay taxes in several countries (the principals), see Martimort (1996), and Bond and Gresik (1996). In combinatorial auctions an auctioneer (the agent) wants to sell several items, and multiple bidders (the principals) bid on all or a subset of the items is studied as a common agency game (Milgrom, 2007). In the voluntary provision of public goods, the public good provider (the agent) elicits payment from consumers (the principals), see Laussel and Le Breton (1998).

changes.

The game has two stages. First, risk neutral principals simultaneously and non - cooperatively offer contracts to a risk neutral agent that has limited liability.[2] Given the contracts, the agent chooses an action that induces a joint distribution over the output of each principal at an unobservable cost. Each principal observes her output realization along with that of the other principals, and can condition her contract on all these observations, as in Bernheim and Whinston (1986a). The action taken by the agent is not observed, moreover, the principals do not know the full set of actions available to the agent, as in Carroll (2015). This captures instances where the principals are contracting with a new agent, or instances where the agent's technology can change after the contracts have been set in place.[3] Principals evaluate contracts on a worst-case basis, considering their performance under all potential actions that the agent might take.

We show that, in equilibrium, each principal offers a linear contract that is increasing in her output and decreasing in the other principals' output. Furthermore, contracts are such that the payoffs of all players are linearly tied, and payments depend only on total output. Specifically, all players (the principals and the agent) receive a share of total output. This is also the case when principals collude and offer the agent a joint contract. Equilibrium contracts deal with the distributional concerns that arise from competition and the potential misspecification of the environment. While previous papers on moral hazard common agency games, such as Dixit (1996) and Maier and Ottaviani (2009), restrict attention to linear contracts for tractability, we place no such restrictions. Linear contracts arise as equilibrium outcomes.

Linearity in total output allows us to further characterize the equilibrium. First, we show that a pure strategy equilibrium always exist. This is done by constructing a pseudo-potential for the game, similar to that of the standard Cournot competition model. The application of the pseudo-potential to a common agency game is new to the literature. This

[2]Limited liability implies that the aggregate payment to the agent has to be non-negative for any output realization, as in Martimort and Stole (2012).

[3]In a lobbying game, this would imply that lobbyists (the principals) are unsure about the preferences and goals of new politicians. In the problem of taxing multinational firms, countries want to design tax policy taking into account that the production technology of multinational corporations might change, but they will unable to change their tax policy due to political constraints.

approach allows us to establish existence of equilibrium without imposing any assumptions over the action set of the agent, while previous papers can only obtain existence under restrictive assumptions (e.g. Bernheim and Whinston (1986a), Fraysse (1993), Carmona and Fajardo (2009)).

Second, we show that the higher the share of total output accruing to the agent, the higher the surplus of the action chosen by the agent, as well as the sum of payoffs of all players. The monotonicity of surplus and welfare in the share of output accruing to the agent allows us to compare the outcomes under competition and collusion. The share is weakly lower when the principals compete, thus leading to lower expected surplus and welfare than if the principals collude.

This result is similar to the one obtained in the moral hazard models of Bernheim and Whinston (1986a), Holmstrom and Milgrom (1988), and Martimort and Stole (2012), as well as the adverse selection models of Martimort and Stole (2012) and Bond and Gresik (1996), where the agent's effort is lower under competition than collusion, due to free-riding among principals.

We also consider a stronger form of limited liability, where each principal's contract should specify non-negative payments to the agent for any output realization. In this case the equilibrium contract of each principal rewards the agent with a share of her output, and with a share of output that the agent did not produce (but could have) for the other principals. Contracts are such that each principal gets a share of total output for a fee. This fee is proportional to the share of total output that the principal appropriates for herself.

Since the agent still gets a share of total output we can compare the outcome of the game under this stronger version of limited liability to the outcome under collusion and the weaker version of limited liability. In this case the share of output accruing to the agent is higher than under collusion. If the agent receives a lower share than what he gets under collusion, we show that one of the principals is better off lowering her own share to increase the agent's share to its collusion level. This lowers the fee she pays and increases expected total output, enough to make up for the decrease in her share of output. This highlights the role of limited liability in the provision of incentives.

As noted above, we depart from the common agency literature by dropping usual assumptions on the information set of the principals. In particular, we deal with an extreme version of moral hazard of the type introduced in the principal-agent framework by Hurwicz (1977) and Hurwicz and Shapiro (1978), and recently explored by Chassang (2013), Frankel (2014), Garrett (2014), Antic (2014), Carroll (2015), and Carroll and Meng (2016).[4] This degree of informational asymmetry over the possible set of actions makes the design of incentive compatible contracts challenging. Instead, we look for robust contracts that maximize the minimum guaranteed payoff for the principals, as in Carroll (2015). Our work adds to this literature in a crucial way by allowing strategic interaction between several principals that simultaneously contract with a single agent. To our knowledge we are the first to study robust contracts in common agency. Dai and Toikka (2017) study an analogous problem of moral hazard in teams (one principal and multiple agents), where they find that the optimal contract for the principal is to give each agent a share of total output.

Robust contracts ensure performance over a wide range of possible settings that the principals may face (Chassang, 2013). In the absence of a complete characterization of the agent's technology (action set), or a well-formed system of priors over possible technologies, robust contracts will guarantee the highest lower bound for the principals' payoffs.[5] For instance, in the context of taxing multinationals, the need for robustness to profit shifting strategies is evident. Particularly since tax reforms are often slow, complicated and expensive processes (we expand on this in Section 2.2). Furthermore, the sensitivity of some of the results in common agency to the details of the information structure (Martimort, 2006) justifies our focus on robust contracts. We find that, despite its increased complexity, our setting allows for a tractable yet general solution.

A central feature of the agent in the classical papers of common agency, like Bernheim and

[4] The information setup considered here most closely resembles the work of Carroll (2015). Hurwicz (1977) and Hurwicz and Shapiro (1978) study cases where the principal knows the technology of the agent belongs to a certain class, but does not know the actual technology. We build on Carroll (2015), because his framework is more amenable to generalizations.

[5] When the space of possible technologies of the agent is large the problem becomes intractable, even when the principals have a well-formed system of priors over this space, see Frankel (2014). Robust contracts avoid specifying such a system.

Whinston (1985, 1986a), is that of facilitating collusion among the principals. Collusive behavior in those models comes from the fact that each principal sells the firm to the agent, making him the residual claimant of all output. However, this is not possible when the agent has limited liability.[6] In our setup, under the stronger version of limited liability, incentives align more intuitively since, instead of selling her firm, each principal buys a share of all the firms for a fee that she pays to the agent (the fee disappears under the weaker version of limited liability). In equilibrium all players own a share of the 'conglomerate' of firms, caring only about the aggregate output. The alignment of incentives in equilibrium takes the form of mergers and acquisitions facilitated by having a common agent.[7]

To summarize, in this chapter we make the following contributions: First, we provide a model of common agency where principals seek robustness. Second, we characterize equilibrium contracts. Third, we establish how the efficiency properties of the equilibrium depend on the form of the contracts offered, and compare the non-cooperative and cooperative solutions of the game.

The remainder of this chapter is organized as follows: Section 1.1 lays out the model. Section 1.2 compares our common agency results to the collusive outcome of the game. Finally, Section 1.3 presents extensions.

[6] Without limited liability our solution converges to the Bernheim and Whinston (1986a) case where the agent becomes the residual claimant, see Section 1.3.3. The only other moral hazard common agency paper that considers some form of limited liability on the agent is Martimort and Stole (2012). They extend the model of Innes (1990) to a common agency setting. Their model is a special case of ours, where output is perfectly correlated between all principals.

[7] In the 1990s the uncertainty over government policy after the Reagan presidency and the fall of the Soviet Union pushed defense contractors to merge in order to guarantee their profitability, most famously the merger of Lockheed with Martin Marietta, see Censer (2014). Also, many mergers and acquisitions happen between firms that share a financial adviser. Agrawal et al. (2013) find that the smaller firms (targets) are usually hurt by the M&A process relatively more when they share a common advisor with the acquirer. Our model captures this fact by showing how smaller principals get a smaller share of total output in competition.

1.1 Model

Consider a game played between two principals, indexed by $i \in \{1, 2\}$, and one agent A, all risk neutral.[8] The payoff space for the principals is $Y = Y_1 \times Y_2 \subset \mathbb{R}^2$. Y_i is compact with $\min\{Y_i\} = 0$ and $\max\{Y_i\} = \overline{y}_i$.[9] The agent has access to a compact technology set $\mathcal{A} \subset \Delta(Y) \times \mathbb{R}_+$. An action is a pair $(F, c) \in \mathcal{A}$, where F is a probability distribution over payoffs $y = (y_1, y_2)$ and $c \geq 0$ is the cost of the action. We endow the space of Borel distributions, $\Delta(Y)$ with the weak-$\star$ topology and $\Delta(Y) \times \mathbb{R}$ with the natural product topology.

The game has two stages. First both principals offer a contract to the agent; this is done simultaneously and in a non-cooperative fashion. Second, the agent chooses an action in its technology set $\mathcal{A}$, and payments realize. Each principal observes her output realization along with that of the other principals, and can condition her contract on all these observations, as in Bernheim and Whinston (1986a). The action taken by the agent is not observed, moreover, the principals do not know the full set of actions available to the agent ($\mathcal{A}$). As in Carroll (2015) the principals both know a subset $\mathcal{A}_0$ of $\mathcal{A}$. We assume that both principals know the same $\mathcal{A}_0$ for notational convenience and to facilitate comparison across principals. Only three other assumptions are placed on the set $\mathcal{A}_0$:

Assumption 1. ***(Inaction)*** *The agent can always choose not to produce:* $(\delta_0, 0) \in \mathcal{A}_0$, *where* δ_0 *is the degenerate distribution on* $y = (0, 0)$.

Assumption 2. ***(Positive Cost)*** *For all* $\mathcal{A} \supseteq \mathcal{A}_0$*, If* $(F, c) \in \mathcal{A}$ *and* $c = 0$*, then* $F = \delta_0$.

Assumption 3. ***(Non-triviality)*** $\exists_{(F,c)\in\mathcal{A}_0} E_F[y_1 + y_2] - c > 0$.

Assumption 1 says that choosing the minimum output is costless for the agent, so that the agent can always choose not to produce. Assumption 2 is a technical assumption requiring the agent to pay a cost in order to produce. This cost can be arbitrarily small. Assumption 3 ensures that the principals and the agent will, potentially, find it beneficial to participate in the game.

[8] All the results are extended to the general case with n principals in Section 1.3.2.

[9] This assumption can be relaxed by letting $Y \subseteq \mathbb{R}^2$ be an arbitrary compact set with $\min_{y \in Y} y_i = 0$ for $i \in \{1, 2\}$, allowing any degree of complementarity or substitutability.

Although no other assumptions are needed for our results, we can strengthen them when the following holds:

Assumption 4. ***(Full Support)*** *For all* $(F,c) \in \mathcal{A}_0$ *if* $(F,c) \neq (\delta_0, 0)$ *then* $supp\,(F) = Y$.

A contract is a continuous function $w_i : Y_1 \times Y_2 \to \mathbb{R}$, and a contract scheme is a vector of functions $w = (w_1, w_2)$. The agent has **limited liability** so that the aggregate payment to the agent has to be non-negative for any output realization, i.e. $w_1(y) + w_2(y) \geq 0$ for all $y \in Y$. A principal can charge the agent up to the amount that the other principal is paying. This is the same restriction imposed in Martimort and Stole (2012). We consider different limited liability assumptions in Sections 1.2.1, 2.2 and 1.3.3. Furthermore, in Section 1.3.1 we consider the *private* common agency case where principals are restricted to contract only on their own output, as opposed to the *public* common agency considered in this section.

Given a contract scheme and a technology $\mathcal{A}$, the agent will choose an action to maximize his expected payoff. The set of optimal actions and the value they give to the agent are:

$$A^{\star}(w|\mathcal{A}) = \underset{(F,c)\in\mathcal{A}}{\operatorname{argmax}}\; E_F\left[w_1(y) + w_2(y)\right] - c V_A(w|\mathcal{A}) = \max_{(F,c)\in\mathcal{A}} \; E_F\left[w_1(y) + w_2(y)\right] - c. \tag{1.1}$$

We define the value of a principal, given a contract scheme w, as the minimum payoff guarantee offered by the contracts, as in Carroll (2015).[10] The payoff to the principal is:

$$V_i(w) = \inf_{\mathcal{A}\supseteq\mathcal{A}_0} V_i(w|\mathcal{A}) \tag{1.2}$$

where $V_i(w|\mathcal{A})$ is the value for a given technology $\mathcal{A}$, given by:

$$V_i(w|\mathcal{A}) = \min_{(F,c)\in A^{\star}(w|\mathcal{A})} E_F\left[y_i - w_i(y)\right]. \tag{1.3}$$

The principal doesn't know which action in $A^{\star}$ the agent will choose, so she assigns the value of the minimum payoff across those actions. In this we depart from what is usually

[10]Behind the definition of the principal's guaranteed payoff there is a strong assumption on the type of technologies that the principals consider. All technologies $\mathcal{A}$ that contain $\mathcal{A}_0$ are allowed. This includes technologies for which the agent has almost zero cost of inducing distributions that are detrimental for the principal. It is possible to relax this assumption by allowing for a lower bound on the cost that the agent faces. Doing so does not change our main results. In particular a version of Theorem 1 can be proven and Proposition 1 goes unchanged. See Appendix.

assumed in the robustness literature, where the principal believes that the agent will take the best action for her among those in $A^\star(w|\mathcal{A})$ (see Frankel (2014) and Carroll (2015)). This change is motivated by the the principal's goal to maximize her guaranteed payoff. Any other tie-braking rule can potentially lead to cases where the expected payoff the principal would actually get is lower than $V_i(w|\mathcal{A})$.

The best response of principal i to a contract w_j is:

$$\mathrm{BR}_i(w_j) = \underset{w_i \geq 0}{\mathrm{argmax}} \quad V_i(w_1, w_2). \tag{1.4}$$

We call the contracts in the best response of the principal **robust**, since they maximize the guaranteed payoff of the principal across all possible technologies. We can now define an equilibrium:

Definition. A Nash equilibrium is a contract scheme $w^\star = (w_1^\star, w_2^\star)$ such that $w_i^\star \in \mathrm{BR}_i\left(w_j^\star\right)$, along with a best response of the agent $A^\star(w^\star|\mathcal{A})$ given the true technology set.

An implicit assumption in the definition of equilibrium is that principals correctly predict the behavior of the other principals and the only uncertainty is about the technology of the agent. In a lobbying game, this would imply that lobbyists know each other but they are unsure about the preferences and goals of new politicians. In reality new politicians come every election cycle however lobbying firms are there longer term. In the problem of taxing multinational firms, countries know the tax policy of other countries, however they are unsure about future changes in the production technology of the multinational firm. Thus our assumption that worst case only considers the agent's technology (i.e. the multinational) and not the other principal's strategy (the other country's tax policy) is not very demanding. Dai and Toikka (2017) employ a similar informational structure, where the agents have complete knowledge about the true technology and the only misinformed party is the principal.

Note that we are restricting attention to pure strategies, and the equilibrium actions and payoffs of the principals are independent of the agent's true technology set, $\mathcal{A}$. They instead depend on $\mathcal{A}_0$.

1.1.1 Principal's best response

In this section we characterize the behavior of a principal who maximizes her guaranteed payoff, taking as given the contract of the other principal. We propose a set of contracts that imply linear revenue sharing between a principal and the agent and show that they are robust to misspecification of the agent's technology. That is, they maximize the principal's guaranteed payoff. The strategy to determine optimality is similar to that of Carroll (2015) and the detailed proofs can be found in Appendix A.1.1.

We proceed by defining the class of linear revenue sharing contracts:

Linear Revenue Sharing (LRS) Contracts: Given a contract $w_j(y)$, a contract $w_i(y)$ is a LRS contract for principal i if it ties the principal's ex-post payoff linearly to the total revenue of the agent. That is, for some $\alpha \in (0,1]$ and $k \in \mathbb{R}$:

$$w_i(y) = \alpha y_i - (1-\alpha) w_j(y) - \alpha k \qquad \forall y \in Y \tag{1.5}$$

It is easy to see that the ex-post payoff of the principal and the revenue of the agent are linearly tied because A.43 implies that

$$y_i - w_i(y) = \frac{(1-\alpha)}{\alpha} (w_1(y) + w_2(y)) + k \tag{1.6}$$

LRS contracts deal with the dual objective of the principal: providing incentives to the agent to increase her output and competing against the offers made by other principals. The contract rewards the agent when he produces for the principal, and partially undoes the payments the agent receives from the other principal. The first part of the contract is reminiscent of the results in the literature on the max-min optimality of linear contracts in principal-agent settings[11]. The second part resembles the *principle of aggregate congruence* in Bernheim and Whinston (1986a), where the principals first offset the payments of the other principals and then design their preferred incentive scheme. Under LRS contracts however the payments of the other principal are only partially offset. The principal claims a fraction of his output from the agent and the same fraction of the payments of the other principal. This results in the sharing of the agent's revenue.[12]

[11] See Carroll (2015), Chassang (2013), Hurwicz (1977) and Hurwicz and Shapiro (1978).

[12] Formally, the problem of a principal can then be thought of in two steps: first undoing the payments of other principals, and then offering the agent an aggregate contract satisfying limited liability. We call this

Moreover, the defining property of LRS contracts, i.e. the affine relationship between the ex-post payments of the agent and the principal, allows for linking the principal's guaranteed payoff to the agent's payoff under $\mathcal{A}_0$ in an affine way. Lemmas 35, 36 and 37 in Appendix A.1.1 show this explicitly. Formally, given a contract scheme (w_1, w_2) where w_i is a LRS contract as in (A.43), the principal's guaranteed payoff is given by:

$$V_i(w) = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) - k \tag{1.7}$$

Equation (1.7) provides a way to compute principals' guaranteed payoffs, since the agent's payoff under $\mathcal{A}_0$ is known given a contract scheme w. The affine link between the agent's payoff and the principal's payoff is a crucial element in providing incentives. Given the incomplete knowledge of the agent's set of actions the principals' optimal strategy is to tie their payoff to that of the agent, thus aligning the agent's objectives with their own. LRS contracts implement this strategy. This is the same mechanism at the heart of the optimal contracts in Carroll (2015) and Hurwicz and Shapiro (1978).

The main result of this section is summarized in the following theorem. It states that offering a LRS contract is always a best response for the principal, that is: *a LRS contracts is always robust.* Furthermore, under Assumption 4 only LRS contracts are robust or the principal can only guarantee herself the payoff given by offering the agent zero aggregate incentives and induce the agent to pick inaction. In the later case, principal i sets $w_i(y) = -w_j(y)$ and the agent picks $(F, c) = \left(\delta_{(0,0)}, 0\right)$. The proof can be found in Appendix A.1.1[13].

Theorem 1. *For any contract w_j there exists LRS contract $\overline{w}_i$ such that $\overline{w}_i \in BR_i(w_j)$, where $\min_{y \in Y} \{\bar{w}_i(y) + w_j(y)\} = 0$. That is, there is always a LRS contract that is **robust** for principal i.*

aggregate contract $\tilde{w}_i$. Then the ex-post payoff of principal i is: $y_i + w_j(y) - \tilde{w}_i(y)$. Principal i's actual contract is of course: $w_i(y) = \tilde{w}_i(y) - w_j(y)$. When w_i is an LRS contract the implied aggregate contract is: $\tilde{w}_i(y) = \alpha(y_i + w_j(y)) - \alpha k$. Thus the ex-post payoffs of the principal are linearly tied to those of the agent, with principal i receiving $1-\alpha$ of the payoff $y_i + w_j(y)$ and the agent receiving α of it. k acts like a lump-sum transfer between the principal and the agent and is determined by limited liability.

[13]As mentioned above when we impose a lower bound on the cost of the agent a similar version of Theorem 1 can be obtained. LRS contracts are always a best response to LRS contracts. This will allow us to still construct an equilibrium in LRS contracts. The proof relies on similar arguments as the one of Theorem 1 and is presented in the online appendix.

If $\mathcal{A}_0$ satisfies the full support property, then any robust contract for principal i is a LRS contract or principal i cannot guarantee a positive guaranteed higher than $w_j(0,0)$.

1.1.2 Equilibrium

We turn now to study the Nash equilibrium of the model. Theorem 1 allows us to focus on equilibria where both principals offer LRS contracts. In this section we establish that an equilibrium in LRS contracts always exists and we characterize equilibrium payoffs in this case. In common agency with incomplete information, as in many sequential games, establishing existence under general conditions has proven difficult, mostly because of the failure of convexity of the principals' best responses (see Bernheim and Whinston (1986a), Fraysse (1993) and Carmona and Fajardo (2009)). However our robust approach allows us to prove existence of a pure strategy Nash equilibrium in a novel way, by showing that the common agency game has a pseudo-potential function as in Dubey et al. (2006). The proof for existence is presented at the end of the section.

Recall from (A.43) that an LRS contract depends on the contract offered by the other principal, by partially undoing the payments she makes to the agent. In equilibrium, when both principals play LRS contracts, that satisfy limited liability with equality, we obtain a sharper characterization of the form of the contract and of the principal's payoffs. Each principal gets a share of total output by rewarding the agent for not working for the other principal.

The following proposition characterizes contract schemes in LRS contracts precisely. If assumption 4 holds then all equilibria are in LRS contracts.

Proposition 1. *Let w be a LRS contract scheme satisfying limited liability. Then there exist $\{\theta_i, k_i\}_{i\in\{1,2\}}$ such that:*

$$w_i(y) = (1-\theta_i)\,y_i - \theta_i y_j - k_i \qquad \text{and} \qquad k_1 = -k_2 \qquad \theta_i \in [0, 1-\theta_j] \tag{1.8}$$

The guaranteed payoff of principal i is:

$$V_i(w) = \theta_i \max_{(F,c)\in\mathcal{A}_0} \left\{ E_F[y_1+y_2] - \frac{c}{1-\theta_1-\theta_2} \right\} - k_i \tag{1.9}$$

Proof. Since both $w_1(y)$ and $w_2(y)$ are LRS contracts, for some shares $\alpha_1, \alpha_2 \in [0,1]$ and constants k_1 and k_2 such that $w_i(y)$ is as in (A.43) for $i \in \{1,2\}$. Then the aggregate contract offered to the agent is:

$$w_1(y) + w_2(y) = \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2}(y_1 + y_2 - k_1 - k_2)$$

In order to satisfy limited liability for all $y \in Y$ it must be that $k_1 = -k_2$.
We can further define $\theta_i = \frac{(1-\alpha_i)\alpha_j}{\alpha_1+\alpha_2-\alpha_1\alpha_2}$ to characterize the contracts:

$$w_i(y) = (1-\theta_i)\, y_i - \theta_i y_j - k_i$$

note that $\theta_i \in [0, 1-\theta_j]$ and that the aggregate contract faced by the agent is: $w_1(y) + w_2(y) = (1-\theta_1-\theta_2)(y_1+y_2)$. The principals' guaranteed payoffs are obtained from equation (1.7) (see Lemma 37 in Appendix A.1.1). □

It is worthwhile noting that the transfers k_1 and k_2 do not affect the action chosen by the agent. In this sense the constants k_1 and k_2 act as transfers between the principals, channeled through the agent. This is similar to the way in which the agent is used to make side payments between principals in Bernheim and Whinston (1985, 1986a). Moreover, since the effect of k_i on the payoff of principal i is independent of θ_i, and the agent's payoff (and thus her optimal action) does not depend on the value of k_1 or k_2, the value of θ_1 and θ_2 can be computed separately from the value of the transfers (k_1 and k_2). Finally, the transfers are not pinned down in equilibrium.[14]
When both principals use LRS contracts the payoffs of all players depend only on aggregate output. In fact principal i receives a share θ_i of aggregate output. This is a more explicit form of the principle of aggregate congruence in Bernheim and Whinston (1986a):

> "[W]e underscore the need to make principals' objectives congruent in equilibrium: since all principals can effect the same changes in the aggregate incentive scheme, none must find any such change worthwhile. One can think of this congruence as being accomplished through implicit side payments among principals."

Under LRS contract one can equivalently cast the problem of each principal as choosing the share of output going to the agent. In an equilibrium all principals have to agree on

[14] In Bernheim and Whinston (1986a) the value of transfers between principals is also indeterminate in equilibrium. A participation constraint is assumed for the agent and each principal can induce the agent not to participate. If that happens principals get some outside payoff. The value of transfers in an equilibrium with participation ensures that each principal receives at least her outside payoff.

the agent's share. Moreover, side payments are made explicit through the transfers k_1 and k_2, as mentioned above these transfers have to sum to zero, using the agent as conduit for payments between principals.

LRS contracts maximize the guaranteed payoffs of the principals by balancing the dual objective of incentivizing the agent and competing with the other principal. The contract described in (1.8) gives the agent a fraction $(1-\theta_i)$ of principal i's output, and takes a share θ_i of principal j's output. Furthermore, the contract deals in an effective way with the distributional concerns that lie behind the competition between principals. Under LRS contracts it is irrelevant who the agent chooses to work for when determining realized payoffs. Each player receives a share of total output and as a consequence the guaranteed payoffs of all principals are linearly tied with those of the agent (see (1.9)).

We now present the main result of this section using the results in Proposition 1 to show that an equilibrium in LRS contracts exists. We do this by showing that our game allows for a pseudo-potential as in Dubey et al. (2006). The use of a potential function to show equilibrium existence is new to the common agency literature and can be useful in showing equilibrium existence in common agency games with incomplete information that do not take a robust contracting approach.

Theorem 2. *Under assumption 3 a pure strategy Nash Equilibrium in LRS contracts, with $\theta_i > 0$ for $i \in \{1,2\}$, exists.*

Proof. First note from Proposition 1 that a pair of LRS contracts is characterized by two shares (θ_1, θ_2) and two transfers (k_1, k_2) satisfying $k_1 = -k_2$. The shares are chosen to maximize the principal's guaranteed payoff, as in equation (1.9). This is equivalent to maximizing:

$$\tilde{V}_i(\theta, \theta_j) = \max_{\theta \in [0, \theta_j]} \theta G(\theta + \theta_j) \tag{1.10}$$

where we define $G : \mathbb{R}_+ \to \mathbb{R}$ as follows:

$$G(x) = \frac{1}{1-x} \max_{(F,c) \in \mathcal{A}_0} \{(1-x) E_F[y_1 + y_2] - c\} \tag{1.11}$$

Note that if $x \geq 1$ then $G(x) = 0$, G is continuous.

We prove existence of an equilibrium in which $\theta_i > 0$ for $i \in \{1,2\}$, so we will only consider best responses to strictly positive actions. As in Monderer and Shapley (1996) we consider an ordinal potential function for the game:

$$P(\theta_1, \theta_2) = \theta_1 \theta_2 G(\theta_1 + \theta_2) \tag{1.12}$$

The function P is an ordinal potential for the game if the shares θ_1, θ_2 are positive since the function P induces the same order over θ_i as the function V_i, that is for all $\theta_j > 0$ and $\theta, \theta' \in [0,1]$:

$$\tilde{V}_i(\theta, \theta_j) - \tilde{V}_i\left(\theta', \theta_j\right) > 0 \quad \Longleftrightarrow \quad P(\theta, \theta_j) - P\left(\theta', \theta_j\right) > 0 \tag{1.13}$$

However the strategy space that we are considering allows for θ_1 or θ_2 to be zero, so the function P is not an ordinal potential, however it is a pseudo-potential since its maxima in $[0,1]^2$ are interior. It is immediate from (1.13) that any maximum of P such that $\theta_1, \theta_2 > 0$ is a pure strategy equilibrium of the common agency game. Under assumption 3 such a maximum exist. First note that P attains a maximum in $[0,1]^2$ by Weierstrass' theorem.Assumption 3 allows for an action that generates enough (expected) output to cover the cost of production, formally there exists $\theta_1, \theta_2 > 0$ such that $G(\theta_1 + \theta_2) > 0$, then $P(\theta_1, \theta_2) > 0$. Then for all $(\theta_1^\star, \theta_2^\star) \in \underset{(\theta_1,\theta_2)\in[0,1]^2}{\text{argmax}} P(\theta_1, \theta_2)$, it holds that $\theta_1^\star, \theta_2^\star > 0$. All these pairs are Nash equilibria of the common agency game. If assumption 3 is violated then it is not possible to induce the agent to produce and the game has a trivial solution.[15] □

The potential function structure provides an interesting connection of the common agency game with the standard Cournot competition. Once we show that equilibrium contracts are LRS, the problem of each principal can be interpreted as maximizing profits $\left(\tilde{V}_i\right)$ by choosing a quantity of production (θ_i) and facing an inverse demand function given by G, defined in (1.11), and a constant marginal cost of zero.

Finally, it is possible to characterize an equilibrium in LRS contracts more tightly by analyzing the guaranteed payoff of the principals. An equilibrium is completely characterized by a pair of shares (θ_1, θ_2) and a pair of transfers (k_1, k_2). Interestingly the equilibrium has an anonymity property, pinning down the actions taken in equilibrium, but not the identity of the principal taking them. The conditions they satisfy are summarized in the following proposition:

[15] An equilibrium still exists if assumption 3 is violated, for instance it is a best response for both principals to set $\theta_i = 0$. That makes $\tilde{V}_i = 0$.

Proposition 2. *A Nash equilibrium in LRS contracts is a pair of shares* $\left(\theta^1, \theta^2\right)$ *and transfers* (k_1, k_2) *such that* $k_1 = -k_2$ *and there are actions* $\left(F^1, c^1\right), \left(F^2, c^2\right) \in \mathcal{A}_0$ *such that:*

$$\left(1-\theta^1-\theta^2\right)^2 = \frac{\left(1-\theta^j\right)c^i}{E_{F^i}\left[y_1+y_2\right]} \qquad \left(F^i, c^i\right) \in \underset{(F,c)\in\mathcal{A}_0}{argmax}\left\{\left(\sqrt{\left(1-\theta^j\right)E_F\left[y_1+y_2\right]}-\sqrt{c}\right)^2\right\}$$

Proof. From (1.9) in Proposition 1 we can find the shares and the transfers independently. The transfer don't have any constraint other than summing to zero, so $k_1 = -k_2$. The share θ_i of principal i is chosen to solve:

$$\max_{(F,c)\in\mathcal{A}_0}\max_{\theta\in[0,1-\theta_j]}\left\{\theta_i E_F\left[y_1+y_2\right]-\frac{\theta_i}{1-\theta_1-\theta_2}c\right\} \tag{1.14}$$

for a fixed $(F_i, c_i) \in \mathcal{A}_0$ the solution to this problem is characterized by:

$$\left(1-\theta_i-\theta_j\right)^2 = \frac{\left(1-\theta_j\right)c_i}{E_{F_i}\left[y_1+y_2\right]} \tag{1.15}$$

Since both principals satisfy this equation in equilibrium we have:

$$1-\theta_1-\theta_2 = \sqrt{\frac{\left(1-\theta_j\right)c_i}{E_{F_i}\left[y_1+y_2\right]}} = \sqrt{\frac{\left(1-\theta_i\right)c_j}{E_{F_j}\left[y_1+y_2\right]}}$$

We obtain (F_i, c_i) by replacing (1.15) in (1.14):

$$\left(F_i, c_i\right) \in \underset{(F,c)\in\mathcal{A}_0}{\text{argmax}}\left\{\left(\sqrt{\left(1-\theta_j\right)E_F\left[y_2+y_2\right]}-\sqrt{c}\right)^2\right\}$$

The problem of each principal does not depend on her identity, because of that the solution will be anonymous.

An equilibrium is then a pair of shares $\left(\theta^1, \theta^2\right)$ and actions $\left(\left(F^1, c^1\right), \left(F^2, c^2\right)\right)$ such that:

$$\left(1-\theta^1-\theta^2\right)^2 = \frac{\left(1-\theta^j\right)c^i}{E_{F^i}\left[y_1+y_2\right]} \qquad \left(F^i, c^i\right) \in \underset{(F,c)\in\mathcal{A}_0}{\text{argmax}}\left\{\left(\sqrt{\left(1-\theta^j\right)E_F\left[y_1+y_2\right]}-\sqrt{c}\right)^2\right\}$$

We use superscripts to reinforce anonymity, so that θ_i is not necessarily given by θ^i. □

1.1.3 Collusion

When colluding, principals seek to maximize guaranteed joint payoff. They offer a single contract that satisfies limited liability of the form $w : Y_1 \times Y_2 \to \mathbb{R}_+$. The principals' problem is a generalization of the principal-agent problem studied in Carroll (2015) to a multi-task principal-agent model. In this case the agent controls two components or accounts (y_1, y_2).

Given technology $\mathcal{A}$, the agent's optimal actions and payoff are now given by:

$$A^\star (w|\mathcal{A}) = \underset{(F,c)\in\mathcal{A}}{\text{argmax}} \quad E_F [w(y)] - c \qquad\qquad V_A (w|\mathcal{A}) = \max_{(F,c)\in\mathcal{A}} E_F [w(y)] - c.$$

The guaranteed joint payoff for the principals is:

$$V_P (w) = \inf_{\mathcal{A}\supseteq\mathcal{A}_0} V_P (w|\mathcal{A}) \quad \text{where} \quad V_P (w|\mathcal{A}) = \min_{(F,c)\in A^\star(w|\mathcal{A})} E_F [y_1 + y_2 - w(y_1, y_2)].$$

The optimal contract under collusion is linear in total output $(y_1 + y_2)$. By making the contract dependent on total output the principals leave to the agent the decision of which output (y_1 or y_2) to favor when producing. The decision depends on the agent's true technology, which is unknown to the principals when contracting. Even if, under the known technology ($\mathcal{A}_0$) the principals want to incentivize differently across outputs, say because the agent is more productive in generating y_1 than y_2, the same incentives do not generalize across all possible technologies, and thus do not provide the best guarantee for the principals.

The linear contract offered by the principals also ties linearly the value of the agent and the principal, and it makes ex-post payoffs depend only on total output. We summarize these results in the following theorem. The proof can be found in the Online Appendix.

Theorem 3. *Let* $(F^\star, c^\star) \in \underset{(F,c)\in\mathcal{A}_0}{\mathit{argmax}} \left\{ \left(\sqrt{E_F [y_1 + y_2]} - \sqrt{c} \right)^2 \right\}$. *When principals collude, the contract:*

$$w_c (y) = (1 - \theta_c) (y_1 + y_2) \qquad \textit{where} \quad 1 - \theta_c = \sqrt{\frac{c^\star}{E_{F^\star} [y_1 + y_2]}} \tag{1.16}$$

maximizes V_P. *Moreover, for any contract of the form* $w(y) = (1 - \theta)(y_1 + y_2)$ *that guarantees a positive payoff,* V_P *can be expressed as:*

$$V_P (w|\mathcal{A}_0) = \frac{\theta}{1-\theta} \max_{(F,c)\in\mathcal{A}_0} \{(1-\theta) E_F [y_1 + y_2] - c\} \tag{1.17}$$

Corollary 1. *If $\mathcal{A}_0$ satisfies assumption 4, then all optimal contracts are of this form.*

As mentioned above, when the principals collude, the model boils down to a multi-task principal-agent model. This type of problem has received extensive attention by the literature, most notably by Holmstrom and Milgrom (1987). A key question is how the incentives should depend on the different tasks (or outputs) controlled by the agent. In the model developed in Holmstrom and Milgrom (1987) an agent controls the drift of a multi-dimensional Brownian motion, the principal chooses how to reward the agent given the terminal value of the Brownian motion. Importantly they find that the optimal scheme is not generally linear in total output (principal's profits), instead it rewards the agent differently for different tasks. They specifically note (Holmstrom and Milgrom, 1987, p.306):

> "The optimal scheme for the multidimensional Brownian model is a linear function of the end-of-period levels of the different dimensions of the process. ...If... the compensation paid must be a function of profits alone (perhaps because reliable detailed accounts are unavailable), or if the manager has sufficient discretion in how to account for revenues and expenses then the optimal compensation scheme will be a linear function of profits. *This is a central result, because it explains the use of schemes which are linear in profits even when the agent controls a complex multi-dimensional process."* [Emphasis added]

However, in our model, robustness leads to linearity in profits no matter how complex the multi-dimensional process the agent controls is. The alignment of incentives between the principal and the agent requires linearity in profits.

1.2 Efficiency

In this section we examine the efficiency properties of the equilibrium, and compare them to those of the collusive outcome. In games of complete information the issue of efficiency was tackled by considering *truthful* equilibria (Bernheim and Whinston, 1986b), which are always efficient. However, in environments with asymmetric information competition among principals can lead to inefficiencies.[16] In this section we verify this result by showing

[16]See Martimort and Stole (2015), Martimort and Moreira (2010) and Bond and Gresik (1996) under adverse selection and Bernheim and Whinston (1986a) and Holmstrom and Milgrom (1988)under moral

that *competition between principals leads to an outcome as much as efficient as that under collusion.*

This efficiency result parallels finding in the literature, see for instance Bernheim and Whinston (1986a), Holmstrom and Milgrom (1988), and Martimort and Stole (2012), as well as the adverse selection models of Martimort and Stole (2012) and Bond and Gresik (1996). As noted in Section 1.1.2 the objectives of all the principals are made congruent in the equilibrium (they all receive a share of total output), this gives rise to a "free-rider" problem since principals are made aware of the incentives provided by their competitors to the agent. The free riding problem appears here because each principal does not internalize the effect of an increase in her share of total output (which lowers the share of the agent) on the payoffs of the competing principal. This leads to the equilibrium share of output accruing to the agent to be lower relative to what he gets under collusion. As we show in Lemmas 1 and 2 this implies a less efficient equilibrium outcome.

We consider two notions of efficiency:

Total expected surplus (TES): Given a contract scheme w and a known technology set $\mathcal{A}$, total expected surplus measures the sum of the expected payoffs of all players. This is given by the difference between total expected output and the cost for the actions preferred action of the agent given the contract scheme w and the action set $\mathcal{A}$.[17]

Let $(F, c) \in \underset{(F,c)\in\mathcal{A}}{\operatorname{argmax}} \{w_1(y) + w_2(y) - c\}$, Total expected surplus is given by:

$$\text{TES} = E_F[y_1 + y_2] - c$$

Total guaranteed surplus (TGS): Given a contract scheme w and a known technology set $\mathcal{A}_0$, total guaranteed surplus measures the surplus to be guaranteed to all players.

$$TGS(w) = V_1(w|\mathcal{A}_0) + V_2(w|\mathcal{A}_0) + V_A(w|\mathcal{A}_0)$$

and under collusion:

$$TGS(w) = V_P(w|\mathcal{A}_0) + V_A(w|\mathcal{A}_0)$$

The first notion of efficiency (TES) is standard in the literature, while the second one (TGS) is considered here because of the game's information structure. Given a technology

hazard.

[17] A contract scheme can then induce more than one total expected surplus.

set $\mathcal{A}$ it is possible to compute TES for any action on that set. Yet, the principals only know a minimal technology set $\mathcal{A}_0$. Given that knowledge and a contract scheme it is still possible to compute the guaranteed surplus of a principal, V_i.[18] This allows for a notion of ex-ante efficiency from the point of view of the principals. Theorem 4 establishes the main result of this section, namely that collusion leads to a more efficient outcome than competition between principals. Its proof will be developed in lemmas 1, 2 and 3.

Theorem 4. *Total expected surplus and total guaranteed surplus are higher under a collusion between the principals than under a Nash equilibrium in LRS.*

Note that under both competition and collusion the agent gets paid a share of total output. This will allow for a clean comparison of the efficiency properties of the two scenarios. Lemmas 1 and 2 show that total expected and total guaranteed surplus are increasing in the share of output that the agent gets. Then Lemma 3 shows that the agent will always get a higher share of output when principals collude than when they compete.
Recall that, given the equilibrium contracts (or contract in the case of collusion), the agent will choose an action (F, c) in order to maximize her payoff $V_A(w|\mathcal{A})$ given a technology set $\mathcal{A}$. Under both competition and collusion the agent's problem reduces to:

$$\tilde{V}_A(\theta|\mathcal{A}) = \max_{(F,c)\in\mathcal{A}} \{\theta E_F[y_1 + y_2] - c\} \tag{1.18}$$

for some share θ. Thus, the action taken by the agent is, in general, not efficient (in the sense that it does not maximize TES). Yet, we can establish how total (expected) surplus varies with the equilibrium contracts. Contracts for which the agent captures a larger share of realized output are more efficient. This is intuitive, since as θ goes to one, the agent's problem converges to that of maximizing total surplus. We formalize this argument in Lemmas 1 and 2 below.

Lemma 1. *Total expected surplus (TES) is weakly increasing in the share of total output going to the agent.*

[18] We can think of V_i as the utility function of principal i, which is quasilinear in lump sum transfers. Because of the quasilinear environment we can consider the sum of utilities as a measure of welfare or efficiency.

Proof. Let θ and $\theta^{'}$ be shares of total output going to the agent with $\theta < \theta^{'}$. Let $(F_\theta, c_\theta) \in A^\star(\theta y|\mathcal{A})$ and $\left(F_{\theta'}, c_{\theta'}\right) \in A^\star\left(\theta^{'} y|\mathcal{A}\right)$, then:

$$\tilde{V}_A(\theta|\mathcal{A}) = \theta E_{F_\theta}[y] - c_\theta < \theta^{'} E_{F_\theta}[y] - c_\theta \leq \theta^{'} E_{F_{\theta'}}[y] - c_{\theta'} = \tilde{V}_A\left(\theta^{'}|\mathcal{A}\right) \tag{1.19}$$

where $y = y_1 + y_2$. The first inequality follows from $\theta < \theta^{'}$ and the second one from (F_θ, c_θ) being feasible at $\theta^{'}$. Furthermore, its easy to check that $E_{F_{\theta'}}[y] \geq E_{F_\theta}[y]$, otherwise $(F_\theta, c_\theta) \notin A^\star(\theta y|\mathcal{A})$. Finally, using the second inequality in (1.19) we have:

$$\begin{aligned} c_{\theta'} - c_\theta &\leq \theta^{'}\left[E_{F_{\theta'}}[y] - E_{F_\theta}[y]\right] \\ c_{\theta'} - c_\theta &\leq E_{F_{\theta'}}[y] - E_{F_\theta}[y] \\ E_{F_\theta}[y] - c_\theta &\leq E_{F_{\theta'}}[y] - c_{\theta'} \end{aligned}$$

This proves the monotonicity of expected total surplus surplus on θ. □

Lemma 2. *Total guaranteed surplus (TGS) is weakly increasing in the share of total output going to the agent.*

Proof. First note that total guaranteed surplus depends exclusively on the share of output going to the agent, regardless of whether or not the principals cooperate. Using (1.9) and (1.17) we have:[19]

$$\text{TGS} = \frac{1}{\theta} \max_{(F,c)\in\mathcal{A}_0} \{\theta E_F[y] - c\}$$

under both competition and collusion, where θ is the share of total output going to the agent, and $y = y_1 + y_2$.
Let $\theta < \theta^{'}$ and $(F_\theta, c_\theta) \in A^\star(\theta y|\mathcal{A})$ and $\left(F_{\theta'}, c_{\theta'}\right) \in A^\star\left(\theta^{'} y|\mathcal{A}\right)$, then:

$$\theta^{'} E_{F_\theta}[y] - c_\theta \leq \theta^{'} E_{F_{\theta'}}[y] - c_{\theta'} \tag{1.20}$$

by optimality of the agent. Since $\theta < \theta^{'}$ we have:

$$\text{TGS}(\theta) = E_{F_\theta}[y] - \frac{1}{\theta} c_\theta < E_{F_\theta}[y] - \frac{1}{\theta'} c_\theta \leq E_{F_{\theta'}}[y] - \frac{1}{\theta'} c_{\theta'} = \text{TGS}\left(\theta^{'}\right)$$

The inequalities follow from $\theta < \theta^{'}$ and (1.20), respectively. □

[19] Note that this implies that under competition θ_i is not only the share of output going to the principal, but also the share of total guaranteed surplus.

Finally we compare the share that the agent gets under collusion and in a Nash equilibrium. To do this we first note that the condition of the agent's share under collusion given in (1.16) of Theorem 3 is equivalent to the problem of a principal under competition facing $\theta_j = 0$. This is immediate from Proposition 2. Then, in order to show that the share of the agent under competition is lower than the share under collusion, it is sufficient to show that the share that principal i wants to induce for the agent is decreasing in the share of principal j. Since in equilibrium $\theta_j \geq 0$ it follows that the share of the agent will be lower than under collusion.

Intuitively, as in Bernheim and Whinston (1986a), where principals only internalize "1/Jth" of the gain when making the principals' objectives congruent, here the principal only internalizes only $(1-\theta_j)$ of the increases in output, note that the objective function is:

$$\begin{aligned}
V_i(\theta_i,\theta_j) &= \theta_i \left(E_F[y_1+y_2] - \frac{c}{1-\theta_j-\theta_i} \right) + k_i \\
&= \frac{\theta_i}{1-\theta_j} \left((1-\theta_j) E_F[y_1+y_2] - \frac{(1-\theta_j)c}{1-\theta_j-\theta_i} \right) + k_i \\
&= \frac{\theta_i}{1-\theta_j} \left(E_F[(1-\theta_j)y_1 + (1-\theta_j)y_2] - \frac{c}{1-\frac{\theta_i}{1-\theta_j}} \right) + k_i \\
&= \tilde{\theta}_i \left(E_F[\tilde{y}_1+\tilde{y}_2] - \frac{c}{1-\tilde{\theta}_i} \right) + k_i
\end{aligned}$$

Then the problem of principal i of choosing $\theta_i \in [0, 1-\theta_j]$ given θ_j is equivalent to the problem of a single principal facing a multitasking agent over a "reduced" output space $\tilde{Y} = (1-\theta_j)Y$, choosing $\tilde{\theta}_i \in [0,1]$. The constant k_i plays no role in choosing θ_i. Since the principal does not internalize all of the output it also does not want to give as much incentives to the agent. This is the same force at the heart of the "free-rider" problem described in Bernheim and Whinston (1986a), Holmstrom and Milgrom (1988), Maier and Ottaviani (2009) andMartimort and Stole (2012).

The formal result is presented in the following Lemma:

Lemma 3. *Let $\theta_j^L < \theta_j^H$ and denote by θ_i^L and θ_i^H any elements of the best response of principal i to θ_j^L and θ_j^H respectively. It holds that:*

$$1-\theta_i^L-\theta_j^L \geq 1-\theta_i^H-\theta_j^H$$

Proof. Suppose for a contradiction that the best response of principal i implies a higher share for the agent when responding to θ_j^H than when responding to θ_j^L:

$$1-\theta_i^L-\theta_j^L<1-\theta_i^H-\theta_j^H$$

Since θ_i^L is in the best response to θ_j^L it must give at least as much payoff to principal i as any other share, given a fixed level of transfers (k_1, k_2). In particular consider an alternative share for principal i given by: $\tilde{\theta}_i=\theta_i^H-\left(\theta_j^H-\theta_j^L\right)$. This alternative share implies that the share of the agent is the same as under the high share: $1-\theta_i^H-\theta_j^H$. It must be that:

$$\theta_i^L\left(E_{F^L}\left[y_1+y_2\right]-\frac{c^L}{1-\theta_i^L-\theta_j^L}\right)\geq\left(\theta_i^H-\left(\theta_j^H-\theta_j^L\right)\right)\left(E_{F^H}\left[y_1+y_2\right]-\frac{c^H}{1-\theta_i^H-\theta_j^H}\right)$$

Where $\left(F^l,c^l\right)\in\underset{(F,c)\in\mathcal{A}_0}{\operatorname{argmax}}\left\{\left(1-\theta_i^l-\theta_j^l\right)E_F\left[y_1+y_2\right]-c\right\}$ for $l\in\{L,H\}$.The pair (F,c) is determined by the agent's problem and thus depends only on the share of the agent. Similarly, since θ_i^H is in the best response to θ_j^H we can consider an alternative share for principal i given by: $\tilde{\theta}_i^H=\theta_i^L-\left(\theta_j^H-\theta_j^L\right)$. As before, this alternative share implies that the share of the agent is $1-\theta_i^L-\theta_j^L$. It must be that:

$$\theta_i^H\left(E_{F^H}\left[y_1+y_2\right]-\frac{c^H}{1-\theta_i^H-\theta_j^H}\right)\geq\left(\theta_i^L-\left(\theta_j^H-\theta_j^L\right)\right)\left(E_{F^L}\left[y_1+y_2\right]-\frac{c^L}{1-\theta_i^L-\theta_j^L}\right)$$

By subtracting these inequalities we get:

$$\left(\theta_j^H-\theta_j^L\right)\left(E_{F^L}\left[y_1+y_2\right]-\frac{c^L}{1-\theta_i^L-\theta_j^L}\right)\geq\left(\theta_j^H-\theta_j^L\right)\left(E_{F^L}\left[y_1+y_2\right]-\frac{c^L}{1-\theta_i^H-\theta_j^H}\right)$$
$$\max_{(F,c)\in\mathcal{A}_0}\left\{E_F\left[y_1+y_2\right]-\frac{c}{1-\theta_i^L-\theta_j^L}\right\}\geq\max_{(F,c)\in\mathcal{A}_0}\left\{E_F\left[y_1+y_2\right]-\frac{c}{1-\theta_i^H-\theta_j^H}\right\}$$
$$\text{TGS}\left(1-\theta_i^L-\theta_j^L\right)\geq\text{TGS}\left(1-\theta_i^H-\theta_j^H\right)$$

This contradicts Lemma (2) since $1-\theta_i^L-\theta_j^L<1-\theta_i^H-\theta_j^H$. □

1.2.1 Limited liability and efficiency

In this section we provide a better understanding of the efficiency result derived in the previous section and in most of the common agency literature. Even though principals are competing among them trying to influence the agent, the agent gets a larger share of total output when the principals collude. This in turn leads to a more efficient action being taken by the agent. We show now that the ability of principals to implicitly make side-payments through the agent is crucial for this result. To see this we impose individual limited liability constraints on the contracts offered by the principals.

Up until now we have assumed that the agent has limited liability, implying that the aggregate payments to the agent must be non-negative for any realization of output. Critically, this allows the principals to extract payments from the agent. This form of punishment is at the heart of two of the mechanisms described in Section 1.1.2: competition among principals and the congruence of objectives of principals in equilibrium. As mentioned above, the contract described in (1.8) allows principal i to take a share θ_i of principal j's output, and then receive a share θ_i of total output. By doing this one of the principals can end out receiving net payments from the agent, $w_i(y) < 0$ for some $y \in Y$. Moreover, the contract (1.8) stipulates transfers between the agent and the principals given by k_1 and k_2. In equilibrium these transfers are such that $k_1 = -k_2$, so that they have no effect over the agent's problem, they are in effect transfers between the principals.

Individual limited liability implies that the contract of each principal has to be non-negative, $w_i(y) \geq 0$ for all $y \in Y$ and $i \in \{1, 2\}$. This constraints the ability to transfer resources between principals, and to demand payments from the output produced for competitors. As we show below this will not change the optimality of LRS contracts for the principals, but it does affect the ability of a principal to free-ride on the incentives provided by her competitors as now limited liability forces each principal to internalize the externality imposed on the other principal. At the end of this section we show that under individual limited liability an equilibrium in LRS contracts is more efficient than the outcome under collusion.

Regardless of the limited liability constraints it is still optimal for the principals to tie their payoffs to the payoffs of the agent. The desire for robustness requires that link to ensure that the incentives of the agent and the principals are aligned. A version of Theorem 1 establishing the optimality of LRS contracts applies and is presented below, its proof is almost identical to the proof of Theorem 1 and is presented in the Online Appendix.

Theorem 5. *For any contract w_j there exists a LRS contract $\overline{w}_i$ such that $\overline{w}_i \in BR_i(w_j)$,* *where* $\min_y \bar{w}_i(y) = 0$. *That is, there is always a LRS contract that is* ***robust*** *for principal* i.

If $\mathcal{A}_0$ *satisfies the full support property, then any robust contract for principal* i *is a LRS contract or principal* i *cannot guarantee a positive profit.*

Theorem 5 suggests that since LRS contracts are always optimal for the principals it is worthwhile focusing in what happens when we restrict attention to this class of strategy, just as we did in Section 1.1.2. Unfortunately the added restrictions of individual limited liability do not allow for a general proof of existence as that in Theorem 2. We are nonetheless able to provide two different sufficient conditions for equilibrium existence and characterize an equilibrium in which both principals use LRS contracts. The exercise sheds light on the role of limited liability in the provision of incentives to the agent.

When both principals use LRS contracts we can characterize the equilibrium more fully, obtaining a similar result as that of Proposition 1. The main difference is that individual limited liability pins down the transfer of the principal to the agent, this in turns allows for a nice interpretation of the contract: each principal gets a share of total output by rewarding the agent for not working for the other principal. Transfers are such that each principal pays the agent for the output that did not, but could have, produced for her competitors. The following proposition makes this precise:

Proposition 3. *Let* w *be a LRS contract scheme satisfying individual limited liability. Then there exist* $(\theta_1, \theta_2) \in [0,1]$ *such that:*

$$w_i(y) = (1-\theta_i)\, y_i + \theta_i \left(\overline{y}_j - y_j\right) \qquad \text{and} \qquad \theta_i \in [0, 1-\theta_j] \tag{1.21}$$

The guaranteed payoff of principal i *is:*

$$V_i(w) = \theta_i \max_{(F,c)\in\mathcal{A}_0} \left\{ E_F\left[y_1 + y_2\right] - \frac{c}{1-\theta_1-\theta_2} \right\} - \theta_i \overline{y}_j \tag{1.22}$$

Proof. Since both w_1 and w_2 are LRS contracts there are shares α_1 and α_2 and constants k_1 and k_2 such that w_i is as in (A.43) for $i \in \{1,2\}$. Then the aggregate contract offered to the agent is:

$$w_1(y) + w_2(y) = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2-\alpha_1\alpha_2}\left(y_1 + y_2 - k_1 - k_2\right)$$

Defining $\theta_i = \frac{(1-\alpha_i)\alpha_j}{\alpha_1+\alpha_2-\alpha_1\alpha_2}$ this implies:

$$w_i(y) = (1-\theta_i) y_i - \theta_i (y_j - k_i - k_j) - k_i$$

where $\theta_i \in [0, 1-\theta_j]$. Note that the contract is increasing in y_i and decreasing in y_j, so $\min_y w_i(y)$ is attained at $y_i = 0$ and $y_j = \overline{y}_j$.[20] In order to satisfy limited liability for all $y \in Y$ it must be that $\min w(y) = w(0, \overline{y}_j) = 0$. This implies $k_i = -\frac{\theta_i}{1-\theta_1-\theta_2}\left((1-\theta_j)\overline{y}_j + \theta_j \overline{y}_i\right)$. Replacing we get:

$$w_i(y) = (1-\theta_i) y_i + \theta_i (\overline{y}_j - y_j)$$

The payoffs are obtained directly from equations (1.6) and (1.7). The aggregate contract faced by the agent is: $w_1(y) + w_2(y) = (1-\theta_1-\theta_2)(y_1+y_2) + \theta_1\overline{y}_2 + \theta_2\overline{y}_1$. □

The first change induced by the strengthening of limited liability is that the upper bound of the output space plays now an explicit role in how incentives are provided to the agent. In this setting the dual objective of the principal (offering incentives to the agent and competing against the other principal) is served by giving the agent a fraction $(1-\theta_i)$ of her output, and also a share θ_i of the output, that he did not (but could have) produced for the other principals. Doing this still ties the payoffs of all players, they all get a share of realized total output, but it also implies that transfers are now given entirely to the agent. The equilibrium payment to the agent is then a combination of high and low powered incentives, in the form of a share of total output and a constant fee. Notice that the fee that principal i pays $(\theta_i \bar{y}_j)$ depends only on the maximum output of the other principal. Hence we can interpret $\bar{y}_j$ as the price per unit share of the conglomerate that principal i has to pay. A consequence of this is that increasing the share of output going to the principal comes at a cost, not only in terms of the share of output left to the agent (which potentially lowers production), but in terms of a fee payed with certainty.[21]

[20] The points $(0, \overline{y}_2)$ and $(\overline{y}_1, 0)$ are in Y by the assumption that Y is a cross product. This assumption is not a necessary one, and is just convenient for determining the values of (y_1, y_2) for which $\min_y w_i(y)$ is attained. If the assumption is lifted (as we do in the example in Section 2.2 and Example 2.4 of Appendix A.3.1), only the constants k_1 and k_2 are directly affected. For instance if output is perfectly and positively correlated $\min_y w_i(y)$ is attained when $y_1 = y_2 = 0$ and then $k_1 = k_2 = 0$.

[21] The dependency of the contract in the maximum output (size) of the competing principal also breaks

It is useful to understand the characteristics of the game that induce a principal to offer high powered versus low powered incentives. A lower θ gives the principal a lower share of total output, and, all else equal increases the share of the agent. It also reduces the fee that the principal pays. Hence incentives are 'high powered'. Conversely a higher θ gives the agent a smaller share of output, and it increases the fee the principal pays to the agent. Hence incentives are 'low powered'.[22] This allows for understanding the effect of competition and productivity on the use of 'high powered' incentives by simply analyzing how they affect the share of output θ.

Existence of an equilibrium as the one just described can be guaranteed under the following sufficient conditions:

Assumption 5. ***(Symmetry)*** *The output space is such that* $\max\{Y_1\} = \max\{Y_j\} = \overline{y}$.

Assumption 6. ***(Convexity of*** $\mathcal{A}_0$***)*** *Consider the known technology set* $\mathcal{A}_0$ *and define a function* $f : \mathbb{R} \to \mathbb{R}$ *as:*

$$f(x) = \min\{c|\, (F, c) \in \mathcal{A}_0 \text{ and } E_F[y_1 + y_2] = x\}. \tag{1.23}$$

The set $\mathcal{F}_{\mathcal{A}_0} = \{F \in \Delta(Y) \,|\, (F, c) \in \mathcal{A}_0\}$ *is convex, and the function* f *is continuous, and its square root is a convex function.*

Theorem 6. *If assumption 5 or assumption 6 hold then a Nash equilibrium in LRS contracts that satisfy individual limited liability exists.*

Assumption 5 imposes a type of symmetry across principals. It allows to prove existence of equilibrium using the potential approach of Theorem 2. The symmetry imposed on the principals is actually very mild, since only the maximum output that can be produced for them is required to be the same across principals. This leaves unconstrained the rest of the output space, and the known actions of the agent. In particular the agent can be known to

the anonymity that characterized the equilibrium described in Section 1.1.2 (see Proposition 2). It also implies stronger conditions on what a principal needs to guarantee herself a positive payoff. In particular Assumption 3 (non-triviality) is not enough, a necessary condition for principal i to guarantee herself a positive payoff is that there exists an action $(F, c) \in \mathcal{A}_0$ such that $E_F[y_1 + y_2] - c > \overline{y}_j$.

[22]Note that the payment of fees to the agent implies that the ex post payoffs of the principals can be negative. Yet, our results do not rely on the ability of principals to make unbounded payments to the agent. In the Online Appendix we augment the model by adding limited liability on the principal's side.

favor production for one of the principals, or one of the principals can have just extreme realizations of output (only high and low values of y_i in Y_i), while the other one can have intermediate values of production.

Assumption 6 imposes more structure over the known set of actions. This structure adds enough convexity to ensure that the principals' best responses are single valued. As mentioned before failure of convexity in the principals' best responses is a common issue faced when establishing existence in common agency games.

The conditions stated in assumption 6 use the results in Proposition 3. In particular, since only expected total output is relevant in determining payoffs, it is without loss to have the agent choose expected total output, x, and an associated cost, c. Moreover, if two actions have the same expected total output the agent will always pick the one with lower cost. This gives rise to a cost function for the agent, defined in (1.23). Only the expected output that an agent can induce, and its associated cost, matter for determining the behavior of the agent and the principals.

It is worthwhile mentioning that these conditions are only sufficient, we can also show existence of equilibrium in special cases. For instance when the cost function is linear, or when the agent is indifferent between actions. This latter case has been used in the literature, for example, Bernheim and Whinston (1986a) establish that an equilibrium of the common agency game exists and implements the efficient outcome when the agent is indifferent between actions (their condition (ii)).[23] Although the assumption is restrictive, it is well suited to describe situations such as auctions or lobbying, were the agent is expected not to have preferences over the actions.

We can now turn to the final result of this section. We show that under individual limited liability the efficiency result established in Theorem 4 is overturned. We do this by showing that the agent receives a higher share of total output when principals compete than when they collude. Or else, there exist a profitable deviation for one of the principals. The deviation consists in reducing her own share of output so that the agent receives the same share as under collusion. The critical step is that doing this increases expected output and reduces the fees that the principal has to pay, increasing the principal's own payoff.

[23] We reproduce Bernheim and Whinston (1986a) results under this condition in Examples 4 and 2.3 of Appendix A.3.1.

The formal result is:

Theorem 7. *Total expected surplus and total guaranteed surplus are higher under a Nash equilibrium in LRS contracts satisfying individual limited liability than under collusion.*

Proof. First note that Lemmas 1 and 2 still apply since fees do not play a role in the agent's decisions and are cancelled out across players. Then it is only left to show that the share of output accruing to the agent in a Nash equilibrium in LRS contracts is higher than in collusion.

Let w^N be a contract scheme in LRS contracts as the one in equation (1.21) characterized by shares $\left(\theta_1^N, \theta_2^N\right)$. The share of output going to the agent is: $\theta_A^N = 1 - \theta_1^N - \theta_2^N$. Under collusion the principals get a share θ^C of output and the agent a share $\theta_A^C = 1 - \theta^C$, see Theorem 3. The problem of the agent is then equivalent to that in (1.18). To simplify notation we define the following objects:

$$\tilde{V}_A^N = \tilde{V}_A\left(\theta_A^N | \mathcal{A}_0\right) \qquad \tilde{V}_A^C = \tilde{V}_A\left(\theta_A^C | \mathcal{A}_0\right)$$

It will be also useful to define the value of principal i under LRS contracts as in (1.22), given his share (θ_i) and that of the agent (θ_A):

$$V_i\left(\theta_i, \theta_A\right) = \frac{\theta_i}{\theta_A}\tilde{V}_A\left(\theta_A\right) - \theta_i \overline{y}_j$$

Suppose that w^N is such that $\theta_A^N < \theta_A^C$, where θ_A^C is the highest share that the agent gets if principals were to collude. We will show that this leads to a contradiction. There are five cases to consider.

Case 1. Both principals can reduce their share of output so as to give the agent the same share of output as under collusion. This is:

$$\theta_1^N \geq \theta_A^C - \theta_A^N \qquad \text{and} \qquad \theta_2^N \geq \theta_A^C - \theta_A^N$$

Thus any principal can unilaterally deviate and induce the collusive outcome. Now, let $V_i\left(\theta_i^C, \theta_A^C\right) - V_i\left(\theta_i^N, \theta_A^N\right) \leq 0$ be the gain that principal i gets by unilaterally deviating to the collusion outcome, i.e. by reducing his share to

$\theta_i^C = \theta_i^N - \left(\theta_A^C - \theta_A^N\right)$. Since w^N is a Nash equilibrium the gain must be non-positive. Letting $V_i\left(\theta_i^C, \theta_A^C\right) = V_i^C$ and $V_i\left(\theta_i^N, \theta_A^N\right) = V_i^N$ we can write

$$V_i^C - V_i^N = \left(\theta_A^C - \theta_A^N\right)\bar{y}_j + \frac{\theta_i^C}{\theta_A^C}\tilde{V}_A^C - \frac{\theta_i^N}{\theta_A^N}\tilde{V}_A^N$$

Summing across the two principals we get:

$$\begin{aligned}
0 &\geq \left(V_1^C - V_1^N\right) + \left(V_2^C - V_2^N\right) \\
&= \left(\theta_A^C - \theta_A^N\right)(\bar{y}_1 + \bar{y}_2) + \frac{\theta_1^C + \theta_2^C}{\theta_A^C}\tilde{V}_A^C - \frac{\theta_1^N + \theta_2^N}{\theta_A^N}\tilde{V}_A^N \\
&= \left(\theta_A^C - \theta_A^N\right)(\bar{y}_1 + \bar{y}_2) + \left(\frac{1 - \theta_A^C}{\theta_A^C} - \frac{\theta_A^C - \theta_A^N}{\theta_A^C}\right)\tilde{V}_A^C - \frac{1 - \theta_A^N}{\theta_A^N}\tilde{V}_A^N \\
&= \left(\theta_A^C - \theta_A^N\right)\left(\bar{y}_1 + \bar{y}_2 - \frac{\tilde{V}_A^C}{\theta_A^C}\right) + \frac{1 - \theta_A^C}{\theta_A^C}\tilde{V}_A^C - \frac{1 - \theta_A^N}{\theta_A^N}\tilde{V}_A^N
\end{aligned}$$

Now note that

$$\frac{1 - \theta_A^C}{\theta_A^C}\tilde{V}_A^C - \frac{1 - \theta_A^N}{\theta_A^N}\tilde{V}_A^N \geq 0$$

by the fact that when agents act collusively they maximize $\frac{1-\theta}{\theta}\tilde{V}_A(\theta|\mathcal{A}_0)$, see (1.17) in Theorem 3. Also note that

$$\bar{y}_1 + \bar{y}_2 - \frac{\tilde{V}_A^C}{\theta_A^C} = \min_{(F,c)\in\mathcal{A}_0}\left\{(\bar{y}_1 + \bar{y}_2 - E_F[y_1 + y_2]) + \frac{c}{\theta_A^C}\right\} > 0$$

since the highest action possible that the agent can choose is to put full support on $(\bar{y}_1, \bar{y}_2)$, and by assumption 2 $c > 0$ in $\mathcal{A}_0$. This leads to a contradiction since it violates the assumption that w^N is a Nash equilibrium contract scheme. At least one principal has a profitable deviation.

Case 2. Only one principal can unilaterally deviate to collusion, say principal i, and $\theta_j > 0$.

$$\theta_i^N \geq \theta_A^C - \theta_A^N \qquad \text{and} \qquad 0 < \theta_j^N < \theta_A^C - \theta_A^N$$

As in Case 1, it is still the case that

$$\left(V_i^C - V_i^N\right) + \left(V_j^C - V_j^N\right) > 0$$

However to get a contradiction we must show that $\left(V_j^C - V_j^N\right) \le 0$:

$$\begin{aligned} V_j^C - V_j^N &= \left(\theta_A^C - \theta_A^N\right) \bar{y}_i + \frac{\theta_j^C}{\theta_A^C} \tilde{V}_A^C - \frac{\theta_j^N}{\theta_A^N} \tilde{V}_A^N \\ &= \left(\theta_A^C - \theta_A^N\right) \left(\bar{y}_i - \frac{\tilde{V}_A^C}{\theta_A^C}\right) + \theta_j^N \left(\frac{\tilde{V}_A^C}{\theta_A^C} - \frac{\tilde{V}_A^N}{\theta_A^N}\right) \\ &= \left(\theta_A^C - \theta_A^N\right) \left(\bar{y}_i - \frac{\tilde{V}_A^N}{\theta_A^N}\right) + \left(\theta_A^C - \theta_A^N - \theta_j^N\right) \left(\frac{\tilde{V}_A^N}{\theta_A^N} - \frac{\tilde{V}_A^C}{\theta_A^C}\right) \le 0 \end{aligned}$$

The inequality follows since both terms are non-positive. $\left(\theta_A^C - \theta_A^N - \theta_j^N\right) > 0$ by assumption and $\left(\bar{y}_i - \frac{\tilde{V}_A^N}{\theta_A^N}\right) \le 0$ since it must be the case that $V_j^N \ge 0$ if w^N is a Nash equilibrium. Finally, $\left(\frac{\tilde{V}_A^N}{\theta_A^N} - \frac{\tilde{V}_A^C}{\theta_A^C}\right) \le 0$, this follows from the proof of lemma 2 since $\theta_A^N \le \theta_A^C$. Hence it must be that $V_i^C - V_i^N \ge 0$, proving that principal i has a profitable deviation. Then w^N is not an equilibrium.

Case 3. None of the principals can unilaterally deviate to induce the collusive outcome, and $\theta_1, \theta_2 > 0$. Then $\theta_1^N < \theta_A^C - \theta_A^N$ and $\theta_2^N < \theta_A^C - \theta_A^N$. As shown in Case 2 this implies that: $V_1^C - V_1^N \le 0$ and $V_2^C - V_2^N \le 0$, but this leads to a contradiction since, as in Case 1, it still holds that: $\left(V_i^C - V_i^N\right) + \left(V_j^C - V_j^N\right) > 0$.

Case 4. Finally we consider the case where one principal, say j, has no share of the surplus, so that: $\theta_j^N = 0$. This case can be shown directly. Recall the problem of principal i given $\theta_j = 0$:

$$\theta_A^N = \underset{\theta_A \in [0,1]}{\operatorname{argmax}} \left\{ \frac{1 - \theta_A}{\theta_A} \tilde{V}_A(\theta_A) - (1 - \theta_A) \overline{y}_j \right\}$$

Note that for all $\theta_A < \theta_A^C$:

$$\frac{1 - \theta_A}{\theta_A} \tilde{V}_A(\theta_A) - (1 - \theta_A) \overline{y}_j \le \frac{1 - \theta_A^C}{\theta_A^C} \tilde{V}_A\left(\theta_A^C\right) - \left(1 - \theta_A^C\right) \overline{y}_j$$

Since

$$\frac{1 - \theta_A}{\theta_A} \tilde{V}_A(\theta_A) \le \frac{1 - \theta_A^C}{\theta_A^C} \tilde{V}_A\left(\theta_A^C\right)$$

from the collusion problem (1.17), and $-(1 - \theta_A) \overline{y}_j \le -\left(1 - \theta_A^C\right) \overline{y}_j$ by assumption. This implies that $\theta_A^N \ge \theta_A^C$.

Case 5. Finally we consider the case where both principals have no share of the surplus in equilibrium, so that $\theta_1^N = \theta_2^N = 0$. This cannot be since it would imply that $\theta_A^N = 1$, contradicting $\theta_A^C > \theta_A^N$.

□

The crucial element that overturns the efficiency result is that individual limited liability forces the principals to internalize the externality that they impose on the other principals. The fees $\left(\theta_i \overline{y}_j\right)$ force each principal to lower her share of total output that she claims in equilibrium implying a higher share for the agent and a more efficient outcome under competition than under collusion.

1.3 Extensions

1.3.1 Private common agency

In this section we consider the case where principals are restricted to contract only on their own output.[24] The study of private common agency is appealing when considering certain applications of common agency. For instance, when home buyers and sellers hire a realtor, they do not explicitly reward him for not working for other home buyers and sellers. Celebrities or professional athletes usually simply give their agents a share of their earnings, regardless of the earnings of others that are also represented by the same agent. The equilibrium contract that arises in this setting provides a rationale for this behavior.
The essential feature of the LRS contracts in Section 1.1 was that they allowed each principal to tie their payoff to the payoff of the agent in an affine way. The principals did so by partially offsetting the contract given to the agent by their competitors. In the private common agency framework such contracts are not allowed. A contract now is a continuous function $w_i^r : Y_i \to \mathbb{R}_+$. The best response for a principal is to give the agent a share of her output and offset the payment of the other principal by charging the agent the maximum value of the other principal's contract. However, limited liability does not allow the principal to charge the agent, so the best response is a **linear** contract; she gives the agent a

[24]This can be due to their inability to observe the other principal's output, or because of regulation that prohibits contracting on output other than your own.

share of her output.
We formalize this in the following theorem. All the proofs of this section can be found in the online appendix.

Theorem 8. *For any contract* w_j^r*, the best response of principal* i *contains a linear contract. i.e. there exists* $\theta_i \in [0,1]$ *such that:*

$$w_i^r(y_i) = (1-\theta_i)\, y_i \qquad \textit{and} \qquad w_i^r \in BR_i\left(w_j^r\right)$$

Corollary 2. *If* $\mathcal{A}_0$ *has the full support property then, for any* w_j^r*, if* $w_i^r \in BR_i\left(w_j^r\right)$ *then* $w_i^r = (1-\theta_i)\, y_i$ *for some* $\theta_i \in [0,1]$*. All best responses are linear contracts, or principal* i *cannot guarantee a positive payoff.*

This does not imply that the problem is reduced to a simple principal agent problem like the one studied in Carroll (2015). Because of the interaction between principals, the share of output that each principal offers in equilibrium is affected by the need to compete for the agent's services.
We can now further characterize the equilibrium. When both principals play linear contracts as in 8 the best response of principal i is:

$$\mathrm{BR}_i(\theta_j) = \operatorname*{argmax}_{\theta_i \in [0,1]} \left[\max_{(F,c)\in\mathcal{A}_0} \left\{ \frac{\theta_i}{1-\theta_i} E_F\left[(1-\theta_i)\, y_i - (1-\theta_j)\left(\overline{y}_j - y_j\right) - c\right] \right\} \right] \tag{1.24}$$

Given the equilibrium action of the agent $(F,c) \in \mathcal{A}_0$, in an interior equilibrium (i.e. $\theta_i \in (0,1)$) we have that:

$$1-\theta_i = \frac{c + (1-\theta_j)\left(\overline{y}_j - E_F[y_j]\right)}{(1-\theta_i)\, E_F[y_i]}$$

First note that the numerator in the ratio above is the opportunity cost of the agent (as perceived by the principal) of taking action (F,c). This cost is formed by the accounting cost of the action (c), plus the expected forgone earnings from the other principal. The share of output that a principal gives to the agent is then equal to the ratio between this cost and the expected payment that the agent receives from the principal. From the agent's stand point, in equilibrium, the marginal revenue she gets from principal i's output is equal to the average cost of taking the action.

The principal increases the share of output given to the agent as the forgone earnings from the other principal increase. This resembles the second term in the equilibrium contract (1.21) found in Proposition 3. When contracts were not restricted, each principal was able to compensate the agent for the forgone earnings from the other principal. Under the restricted contracting domain this explicit form of competition is not possible. Instead, principals implicitly compete with each other by offering higher shares of their own output to the agent. In the absence of this competition motive (e.g. if $\theta_j = 1$ or $E_F[y_j] = \overline{y}_j$) the share obtained in equilibrium is equal to that found in Carroll (2015) for the single principal setup.

It is possible that in equilibrium one principal behaves as though she is the only principal in the game, while the other one offers the zero contract. If $c+(1-\theta_j)\left(\overline{y}_j - E_F[y_j]\right) \geq E_F[y_i]$ then the principal cannot generate sufficient incentives given the output that she expects to receive. In this case it is optimal for the principal to offer $\theta_i = 1$ and thus get zero payoff. See Appendix A.1.3.

1.3.2 Multiple Principals

The model considered in Section 1.1 can be extended to multiple principals. Our main results are preserved in this case. Below we summarize them and we leave all details in the Online Appendix. We denote the number of principals by n, and we define the vector of competing contracts as $w_{-i}(y) = (w_1(y), \ldots, w_{i-1}(y), w_{i+1}(y), \ldots, w_N(y))$. The definition of a LRS contract is the natural extension to that in Section 1.1:

Linear Revenue Sharing (LRS) Contracts: A contract w_i is a LRS contract for principal i if, given a vector of competing contracts w_{-i}, it ties the principal's ex-post payoff linearly to the total revenue of the agent. That is, for some $\alpha \in (0,1]$ and $k \in \mathbb{R}$:

$$y_i - w_i(y) = \frac{(1-\alpha)}{\alpha}\left(\sum_{n=1}^{N} w_n(y)\right) - k \tag{1.25}$$

Just as before we can show that that there is always a LRS contract in each principal's best response. And if all principals play LRS contracts then we can characterize them as in Theorem 2:

$$w_i(y) = (1-\theta_i) y_i + \theta_i \sum_{j \neq i} \left(\overline{y}_j - y_j\right) \tag{1.26}$$

Although this contract has the same form, the principal's best response functions are more sensitive to competition than before. In fact, we show in the Appendix that, in order to have $\theta_i > 0$ in equilibrium, the principal must expect the agent to choose an action (F, c) such that:

$$E_F [y_i] > \sum_{j \neq i} E_F [\overline{y}_j - y_j]$$

This condition is stronger than non-triviality and increasingly difficult to satisfy as the number of principals increases. Intuitively the LRS contract is compensating the agent for her forgone earnings from other principals, this compensation requires principal i's payoff to be large enough in order to guarantee a positive payoff.

1.3.3 No limited liability on the agent

If we dispense with the limited liability on the agent and instead impose a participation constraint on the agent, guaranteeing the agent a given expected payoff (normalized here to 0), then we get the Bernheim and Whinston (1986b) solution where each principal sells "her firm" to the agent.

Suppose that $\mathcal{A}_0$ is common knowledge among the principals. Also let s_0 equal the total surplus under $\mathcal{A}_0$, that is:

$$s_0 = \max_{(F,c) \in \mathcal{A}_0} \{E_F [y_1 + y_2] - c\}$$

Now consider the best response of principal i to the strategy of principal j that sells the firm to the agent for a price $s_j \leq s_0$ (i.e. $w_j (y) = y_j - s_j$). Principal i cannot be guaranteed a payoff higher than $s_0 - s_j$. Otherwise the participation constraint of the agent would be violated. This payoff is achieved if principal i offers $w_i (y) = y_i - (s_0 - s_j)$. Thus selling the firm is a best response of principal i. We note that there is an indeterminacy in how the total surplus is divided between the principals as is the case in Bernheim and Whinston (1986b).

Furthermore it is obvious the same equilibria are also valid if we restrict attention to private common agency, since the optimal contracts do not depend on the other principal's output.

1.4 Conclusion

Taking a robust contracting approach provides a crisp characterization of equilibrium strategies and payoffs in the complicated problem of common agency. The central issue in the literature of how competition among principals affects the efficient provision of incentives can be easily pinned down to one component, namely the share of total output that the agent receives in equilibrium. We show that when principals can make side payments (through the agent) to each other, in equilibrium a free-riding problem appears. Free riding leads to lower incentives given to the agent, compared to the collusive outcome. When such side payments are not possible because of limited liability, then principals are forced to internalize their externality, which leads to the competitive outcome being more efficient than the collusive outcome.

Our results can provide interesting insights in a dynamic common agency game, because of their implication on renegotiation possibilities. We leave this work for future research.

Chapter 2

Applications of Robustness and Linearity in Common Agency

2.1 Introduction

In this chapter we apply the theoretical framework established in Chapter 1 to study a variety of economic questions, such as taxation of multinational corporations, government procurement auctions, and provision of public goods. The first application of our theoretical results is to study the problem of tax competition among countries in the presence of a multinational firm, using a common agency framework. Our robust contracting approach is especially relevant to the problem of taxing multinational companies, where the two primary concerns for policymakers are the eroding corporate tax base, due to aggressive profit shifting by multinational companies, and the complexity of tax law.

With the increase of globalization and advances in technology, the number of multinational corporations and their ability to shift profits to low-tax countries have increased tremendously. This issue has received considerable attention in the news and in political and economic debates in the United States and other developed countries. An estimated $2 trillion dollars of U.S. multinational corporations' profits are "parked" overseas, mostly in tax havens like the Bahamas, Bermuda, and the Cayman Islands, that implies a loss in tax revenue of about $50 billion dollars every year (Hungerford, 2014).[1] This issue has implications beyond tax revenue, for instance, Guvenen et al. (2017) show that profit shifting understates measured U.S. GDP in official statistics, helping to explain the slowdown in U.S. productivity seen since the mid 2000s.

The debate among tax policy experts and lawmakers in the United States has centered on whether to adopt a territorial approach–taxing only profits generated in the U.S.–or a worldwide approach–taxing all profits, foreign and domestic, the same.[2] Our model provides a rationale for why a worldwide approach along the lines of the system proposed in the Bipartisan Tax Fairness and Simplification Act of 2011 by Sen. Wyden and Sen. Coats (Senate Bill 727, 2011) addresses the two primary concerns regarding corporate income tax. We show that taxing domestic and foreign profits at a flat rate, with a full deduction for taxes paid in the foreign country, provides the best guarantee for corporate tax revenues as well as a simplified tax code.

The second application is understanding how robustness affects procurement auctions, specifically when the bidders (suppliers) face uncertainty about the preferences of the buyer.

The third application considers the implications of the private common agency and public common agency in the efficient provision of public goods.

As mentioned in Section 1.3.1 of Chapter 1, when principals are restricted to contract only on their own output, that significantly restricts the ability of each principal to achieve a large guarantee. In the third application we show such restriction affects the provision of public good.

2.2 Taxing Multinationals

We now show how the setup developed in Chapter 1 can be applied to study the taxation of multinationals. As mentioned earlier, there is a big debate among tax policy experts and lawmakers on how to reform the corporate income tax with a particular focus on foreign profits. The debate in the United States has centered on whether to adopt a territorial

[1] Using a different data source Zucman (2014) estimates that profit shifting activities have reduced the tax burden of corporations by about 20%.

[2] The current U.S. system is between the two approaches. Foreign profits are taxed (almost) the same as domestic profits, but not until they are paid to a U.S. parent company. This is known as deferral.

approach–taxing only the profits generated in the U.S.–or a worldwide approach–taxing all profits, foreign and domestic, the same.

The need for tax systems to be robust to profit shifting strategies is evident. Tax reforms are slow, complicated and expensive processes, hard to adapt to changes in the strategies used by firms. Having a robust tax system also implies having a more simple tax system. This is beneficial for firms, since it gives a stable legal framework to work with, and for governments, since it avoids the costs of changing tax laws. Our common agency approach characterizes the main features of a robust tax system for multinationals. We show that a worldwide tax with a deduction paid for taxes in the foreign countries is indeed robust. This is the tax system proposed in the Bipartisan Tax Fairness and Simplification Act of 2011 by Senators Wyden and Coats (Senate Bill 727, 2011).

The theoretical literature on taxation of multinational corporations has primarily focused on the issue of how taxes affect multinational corporations who allocate their business operations and capital abroad-see Feldstein and Hartman (1979). The main concern of that literature is to achieve neutrality in investment allocation, and they abstract from informational asymmetries. However, today's biggest multinational corporations (like Google and Apple) rely heavily on intangible capital (patents, brands, etc.), which can easily shift ownership to a subsidiary in a tax haven without affecting the firm's productivity. These transfer pricing practices have allowed firms like Microsoft and Google to pay an overall tax rate of less than 3% on non-U.S. profits (Zucman, 2014).

The literature has addressed profit shifting by multinationals in the context of an adverse selection problem–see Bond and Gresik (1996) and Olsen and Osmundsen (2001). They use the revelation principle to deal with profit shifting strategies. As noted in Martimort (2006) their solution is highly sensitive to the information structure of the problem, while our focus is on solutions that are robust to potential misspecification of the environment.

We define an isomorphic problem to the common agency problem in Chapter 1. Denote the multinational firm by A and let π_i be the firm's profit in country i. The firm's actions are then distributions (F) over the profits in Π, and a cost (c) associated with each distribution.[3] The firm's action set ($\mathcal{A}$) is then composed by pairs $(F, c) \in \Delta(\Pi) \times \mathbb{R}_+$.

[3]The cost can be interpreted as an economic cost (after accounting costs are deducted) of engaging in transfer pricing between the firm's subsidiaries in each country. Alternatively, the cost can be interpreted as unobservable effort from the firm's manager as in Laffont and Tirole (1986).

Each country's government chooses a tax function to maximize their guaranteed corporate tax revenue when they only know a subset $\mathcal{A}_0 \subset \mathcal{A}$, all assumptions on $\mathcal{A}$ and $\mathcal{A}_0$ are as in Chapter 1. The tax function for country i is a continuous function $t_i : \Pi \to \mathbb{R}$. The country's objective can be interpreted directly as maximizing its guaranteed revenue, or as seeking robustness in the tax system, which is obtained by solving the problem just described.

We consider two different restrictions over the range of the taxes. We refer to them as weak and strong enforceability:

Weak Enforceability: A country has weak enforceability if it can only tax up to the amount of profits declared in its territory. This implies: $t_i(\pi_1, \pi_2) \leq \pi_i$.

For small countries that have a subsidiary of a big multinational firm this is a reasonable restriction[4].

Strong Enforceability: A country has strong enforceability if it can collect taxes on all profits generated by the firm. This implies: $t_i(\pi_1, \pi_2) \leq \pi_1 + \pi_2$.

For large countries like the United States where the multinational corporation has most of its activity this restriction is more reasonable.

The firm's problem is to maximize after tax profits:

$$A^\star(t|\mathcal{A}) = \operatorname*{argmax}_{(F,c)\in\mathcal{A}} E_F\left[(\pi_1 - t_1(\pi_1, \pi_2)) + (\pi_2 - t_2(\pi_1, \pi_2))\right] - c \tag{2.1}$$

where $t = (t_1, t_2)$ is a tax scheme. The guaranteed expected revenues of government i are given by:

$$R_i(t_1, t_2) = \inf_{\mathcal{A}\supseteq\mathcal{A}_0} \left\{ \min_{(F,c)\in A^\star(t|\mathcal{A})} E_F\left[t_i(\pi_1, \pi_2)\right] \right\} \tag{2.2}$$

It is easy to show from Theorem 1 that given the tax system of country j, country i's best response contains the following tax:

Worldwide Tax: A tax function t_i is a worldwide (flat) tax rate if the firm's global profits are taxed at a constant rate α, allowing for the full deduction of taxes payed to country j, and a potential tax incentive (in the form of a lump sum subsidy). That is, for some

[4]Weak enforceability is equivalent to the limited liability assumption imposed in the main section. It does not amount to a territorial approach to taxation. A territorial approach would amount to restricting the domain of the taxes, so that $t_i(\pi_1, \pi_2) = t_i(\pi_i)$.

$\alpha \in (0,1]$ and $k \in \mathbb{R}$:

$$t_i(\pi_1, \pi_2) = (1-\alpha)(\pi_1 + \pi_2 - t_j(\pi_1, \pi_2)) + \alpha k_i \tag{2.3}$$

The tax proposed by Senators Wyden and Coats has this form. It proposes a flat tax rate for all profits independently of origin. While the model does not provide a literal description of reality, it provides a robustness property for the Wyden-Coats tax reform, that a territorial tax system does not have. This property has been informally articulated among tax policy experts (Hungerford, 2014), and thus the model provides a rigorous treatment of the policy debate. Note that a Worldwide Tax is not just an equilibrium outcome of the game. A Worldwide tax is a best response for country i to any arbitrary tax system of country j.

Interestingly, the worldwide tax has the same form as the taxes found by Feldstein and Hartman (1979) without a lump sum subsidy. Unlike us they have a complete information setup and restrict attention to linear tax functions, and their "full taxation after deduction" result rests on concerns on the optimal allocation of capital between countries. In our setup the lump sum subsidy k depends on the enforceability assumption that we make. Under strong enforceability there is no lump sum transfer, and under weak enforceability we get the same lump sum transfer as in the main section of the paper.

Proposition 4. *If both countries use a worldwide tax functions, then taxes in country i are:*

$$\begin{aligned} &\textit{Weak Enforceability:} && t_i^\star(\pi_i, \pi_j) = \theta_i(\pi_i + \pi_j) - \theta_i \overline{\pi}_j \\ &\textit{Strong Enforceability:} && t_i^\star(\pi_i, \pi_j) = \theta_i(\pi_i + \pi_j) \end{aligned}$$

where $\theta_i \in [0,1]$, π_i is profits declared in country i and and $\overline{\pi}_j$ is the maximum profit the firm can declared in country j.

Moreover the ex-post payoffs of the company (π_A) and the revenue of each country (π_i) are:

$$\begin{aligned} &\textit{Weak Enforceability:} && \pi_A = (1-\theta_1-\theta_2)(\pi_1+\pi_2) + \theta_1\overline{\pi}_2 + \theta_2\overline{\pi}_1 - c \\ & && r_i = \theta_i(\pi_1+\pi_2) - \theta_i\overline{\pi}_j \\ &\textit{Strong Enforceability:} && \pi_A = (1-\theta_1-\theta_2)(\pi_1+\pi_2) - c \\ & && r_i = \theta_i(\pi_1+\pi_2) \end{aligned}$$

It is easy to see that weak enforceability leads to significantly lower tax rates than strong enforceability. The following illustrating example can be useful in understanding this. Suppose that the total profits of the firm $\pi_1 + \pi_2 = \bar{\pi}$ (if it produces) are fixed and the firm is simply deciding how much of the total profits to declare in each country. So the output space is $[(\bar{\pi}, 0), (0, \bar{\pi})] \cup \{0, 0\}$.[5] Then, only a country that has strong enforceability would be able to guarantee positive revenue. The country with weak enforceability would choose not to tax (set $\theta = 0$). Intuitively a country with weak enforceability is always threatened by the possibility that the firm does not produce in its territory, this induces the country to offer subsidies. [6]

Another important issue is that of efficiency, as shown in Section 1.2, competition between countries would lead to a lower overall tax rate on the multinational, relative to cooperation between countries. This would also imply that tax competition leads to higher efficiency due to lower distortionary taxes. This contrasts with the results in Bond and Gresik (1996) where competition between countries lead to higher taxes and lower efficiency compared to the cooperative outcome.

Beyond guaranteeing revenue

An important assumption made above is that each country only cares about the guaranteed tax revenue that it collects from the multinational. Yet, it is natural to consider cases where the government cares also about the profits of the firm (because of spillovers on the economy, or lobbying). If we were to consider, like in Bond and Gresik (1996), that country i cares about the after (total) tax profits of the multinational as well as its tax revenue, then qualitatively nothing would change in the tax structure, except that the tax rate that country i would charge would be lower. Hence if country i's guaranteed payoff are given by

$$R_i(t_1, t_2) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} \left\{ \min_{(F,c) \in A^{\star}(t|\mathcal{A})} E_F \left[\rho ATP + t_i(\pi_1, \pi_2) \right] \right\}$$

[5] Note that here we have dispensed with the cross product assumption on the output space

[6] This also relates to inter-state competition for firms or sport teams, as discussed in Burstein and Rolnick (1995).

where $ATP = \pi_1 - t_1(\pi_1, \pi_2) + \pi_2 - t_2(\pi_1, \pi_2)$ and $\rho \in [0, 1]$. Taxes are given by.

$$t_1(\pi_1, \pi_2) = \left(1 - \frac{\alpha}{1 - \alpha\rho}\right)(\pi_1 + \pi_2 - t_2(\pi_1, \pi_2)) + \frac{\alpha}{1 - \alpha\rho}k$$

Alternatively, the government can be interested in increasing the profits that the firm makes locally. The policy and political debate has considered the benefits of attracting foreign profits by allowing a 'tax holiday' (like the one enacted in 2005), or as in the proposal of Senator Max Baucus to tax foreign profits at a lower rate than domestic profits (Chairman Staff Discussion Draft - MCG13834, 2013). However, if we were to model such considerations in our set up we would get that domestic profits should be taxed at a lower rate than foreign profits. If country i's guaranteed payoffs are given by

$$R_i(t_1, t_2) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} \left\{ \min_{(F,c) \in A^\star(t|\mathcal{A})} E_F[\rho\pi_i + t_i(\pi_1, \pi_2)] \right\}$$

then we would have that taxes are:

$$t_1(\pi_1, \pi_2) = ((1 - \alpha) - \alpha\rho)\pi_1 + (1 - \alpha)(\pi_2 - t_2(\pi_1, \pi_2)) + \alpha k$$

Details for both of these cases are provided in the Online Appendix.

As was the case in LRS contracts, a worldwide tax system implies a linear relation between country i's payoff and the company's after tax profits in all the cases above. For instance if taxes are as in (2.3):

$$t_1(\pi_1, \pi_2) = \frac{1 - \alpha}{\alpha}(\pi_1 - t_1(\pi_1, \pi_2) + \pi_2 - t_2(\pi_1, \pi_2)) + k \tag{2.4}$$

This link is at the heart of the robustness properties of the tax system. Robust tax systems align the incentives of the multinational with those of taxing authority, thus making profit shifting strategies futile (or as beneficial to the taxing authority as they are to the multinational).

2.2.1 Two Period Model of Taxation

We can extend the model in the previous section to a two period model of taxation

Consider a government and a firm that operates for two periods. The firm has access to some unspecified technology of which the government only knows some subset. An action of the firm is a pair of a distribution over output and a cost. Taxes depend on realized output. In each period the firm takes an action (a_t) that induces a distribution over output (F_t) at some cost (c_t). The government only observes realized output and taxes it to collect revenue. Taxes depend on current and past output. The period profits of the firm are:

$$y_1 - t_1(y_1) - c_1 \qquad \text{and} \qquad y_2 - t_2(y_1, y_2) - c_2$$

Government revenue is:

$$t_1(y_1) + t_2(y_2)$$

There is limited liability on the firm, so that $t_1(y_1) \leq y_1$ and $t_2(y_1, y_2) \leq y_2$. Both the government and the firm are assumed risk neutral. The government has commitment.
The timing of the game is as follows:

1. The government sets a tax scheme (t_1, t_2).

2. The firm chooses a_1, output y_1 realizes, and taxes are payed.

3. The firm chooses a_2, output y_2 realizes, and taxes are payed.

4. The game ends.

The action set of the firm ($\mathcal{A}$) is unknown to the government, save from a minimal set of actions $\mathcal{A}_0 \subseteq \mathcal{A}$.
Given a tax scheme (t_1, t_2) and an action set ($\mathcal{A}$) the problem of the firm is:

$$V_A(t_1, t_2, \mathcal{A}) = \max_{a_1 \in \mathcal{A}} \left\{ E_{F_1}\left[(y_1 - t_1(y_1) - c_1) + \max_{a_2 \in \mathcal{A}} \{ (E_{F_2}[y_2 - t_2(y_1, y_2)] - c_2) \} \right] \right\}$$

Notice that $a_1 = (F_1, c_1) \in \Delta Y \times \mathbb{R}_+$ is only a function of the contract scheme, while $a_2(y_1) = (F_2(y_1), c_2(y_1)) \in \Delta Y \times \mathbb{R}_+$ is a function of y_1. Let $A^\star(t_1, t_2 | \mathcal{A})$ be the set of maximizers for the firm.
The government's guaranteed revenue is given by:

$$V(t_1, t_2) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} \left\{ \min_{a_1, a_2 \in A^\star(t_1, t_2 | \mathcal{A})} \left\{ E_{F_1}\left[t_1(y_1) + E_{F_2(y_1)}[t_2(y_1, y_2)] \right] \right\} \right\}$$

We can show know that the robust taxes have the form:

$$t_1(y_1) = \alpha y_1 \qquad t_2(y_1, y_2) = \alpha y_2$$

The proof of these results are in the Appendix.

2.3 Government Contracting (First Price Auction)

Consider a setup where two competing firms bid for a government contract (such as a contract for the provision of services to the government, the construction of a public good, or the privatization of a government asset). The government announces that the contracting process has a fixed cost $c > 0$, and that the contract will be awarded with the objective of maximizing the government's profits. The cost of the contract can be interpreted as the social benefit of carrying out the project that the contract stipulates, or the valuation of a government asset that is being privatized. Both firms have their own valuation of the contract, we denote them by $\overline{y}_1$ and $\overline{y}_2$. We assume without loss that $\overline{y}_1 > \overline{y}_2 > c$.

If the government is known to be corrupt the firms would have reasons to doubt the announcement. For instance, the government can potentially (and secretly) favor one of the firms, or 'under the table' payments can make the cost vary depending on who is awarded the contract; it is also possible that the government is willing to randomize between the firms and lower the cost, this might be the case if bids are hard to assess and the government can lower costs at the expense of adding error to the contracting process, or if technicalities can arise that create the chance of a lower bid to be awarded the contract.[7]

The possible outcomes of the contracting process are that firm 1 is awarded the contract, firm 2 is awarded the contract, or the process is declared null and neither firm gets it. Note that, in a perfect information setting, this setup is that of a first price auction. The game would then have no solution since firm 1 would try to marginally outbid firm 2, leaving the bids undefined. Instead we show that there is a unique equilibrium in robust contracts for this game.

To formalize the model consider a payoff space $Y = \{(0,0), (\bar{y}_1, 0), (0, \bar{y}_2)\}$ and a known set of actions for the agent given by:

$$\mathcal{A}_0 = \left\{(\delta_0, 0), \left(\delta_{\overline{y}_1}, c\right), \left(\delta_{\overline{y}_2}, c\right)\right\}$$

We now solve for the equilibrium when firms are allow to offer bids that depend who the

[7] Randomness in who is assigned the contract can also arise from last minute changes in the rules (not uncommon in developing countries), or from challenges made in courts to the rules or the decision of the government. It is worth pointing out that randomization is not itself necessary for our results. The firms could simply be worried that the government can allocate the good with certainty to the other contractor for a zero cost. This is in fact the worst case scenario they face.

contract is awarded to. LRS contracts will have the following form:

$$w_i = \begin{cases} \theta_i \overline{y}_j & \text{if } y = (0,0) \\ \overline{y}_i - \theta_i \left(\overline{y}_i - \overline{y}_j\right) & \text{if } y = (\overline{y}_i, 0) \\ 0 & \text{if } y = \left(0, \overline{y}_j\right) \end{cases}$$

Note that since $\theta_i \geq 0$ and $\overline{y}_j > 0$ it holds that $w_i\left(0, \overline{y}_j\right) \leq w_i(0,0) \leq w_i(\overline{y}_i, 0)$. That is, the principals always pay more if they win the auction, followed by no one winning and lastly if the auction is won by the other principal.

It is left to characterize the optimal actions of the government and the two firms, given the form of the contracts. The government's problem is:

$$V_A(w|\mathcal{A}_0) = \max\left\{ \theta_i \overline{y}_j + \theta_j \overline{y}_i,\ \ (1-\theta_i)\overline{y}_i + \theta_i \overline{y}_j - c,\ \ (1-\theta_j)\overline{y}_j + \theta_j \overline{y}_i - c \right\}$$

Note that for any strategy of the firms (θ_1, θ_2) the government will either award the contract to the firm with the highest valuation (firm 1) or not award it at all.

This implies that the best response of firm 2 is to set $\theta_2 = 0$ or to offer the zero contract. In turn, this gives rise to two equilibria of the game. One in which firm 2 sets $\theta_2 = 0$, thus bidding $w_2(y) = y_2$, where firm 1 optimally sets

$$\theta_1 = \begin{cases} 1 - \sqrt{\frac{c}{\overline{y}_1 - \overline{y}_2}} & \text{if } c\left(\frac{\overline{y}_1 - \overline{y}_2}{\overline{y}_1}\right) < \overline{y}_1 \wedge \overline{y}_2 + c < \overline{y}_1 \\ 0 & \text{otw} \end{cases}$$

The condition above is just guaranteeing that the government will prefer awarding the contract to firm 1 over declaring null the process, and that the valuation of firm 1 is enough to pay the cost to the government and compensate for not awarding the contract to firm 2 (this ensures that $\theta_1 \geq 0$).

And another equilibrium in which firm 2 walks away from the bid, setting $w_2(y) = 0$, where firm 1 optimally sets $w_1(y) = (1-\theta_1) y_1$ with $\theta_1 = 1 - \sqrt{\frac{c}{\overline{y}_1}}$. For this to be an equilibrium it must be a best response by principal 2 to offer the zero contract when principal 1 offers this contract. That is the case when:

$$\overline{y}_2 < \sqrt{c\overline{y}_1}$$

If $\overline{y}_1 = \overline{y}_2$ then there are no eligible contracts for the firms, since the agent will be indifferent between them and neither firm can guarantee to be awarded the contract. Because of this the only equilibrium is for both of them to set $w_i(y) = y_i$. In all cases the government ends up awarding the contract to the firm with the highest valuation.

2.4 Provision of public goods (Public vs Private Contracting Domain)

In this section we consider a classical example on the provision of a public good, taken from Martimort and Stole (2012), under the assumption of constant marginal costs.
Consider an agent that produces one unit of a public good with variable quality $q \in [0, 1]$. The cost function of the agent depends on q and is given by $f(q) = \gamma q$. Each principals values the public good with $y_i = \nu_i q$. The output space is then:

$$Y = \left\{(y_1, y_2) \in \mathbb{R}_+^2 | \exists_{q \in [0,1]} \quad y_1 = v_1 q \quad \wedge \quad y_2 = v_2 q\right\}$$

This abandons the assumption on the set Y being a cross product. Output is now assumed to be perfectly correlated across principals. This will only change the intercept of the LRS contract. The efficient outcome is of course to provide the good at highest quality if $\nu_1 + \nu_2 \geq \gamma$.
We first characterize the equilibrium in public common agency. In this case there is no competition factor since output is perfectly correlated across principals. Each principal "partially" free rides on the other by lowering compensation by a fraction of the other principal's payoff. Moreover the agent optimally chooses to set $q = 1$. An interesting feature of this equilibrium is that no matter how different the valuations are, *all principals get the same share of expected output and the same guaranteed payoff. Moreover the agent picks the efficient action.*

Proposition 5. ***(Public good provision - Public common agency)*** *In the public common agency equilibrium both principals offer contracts of the form:*

$$w_i(y) = (1-\theta) y_i - \theta y_j \qquad \text{where } \theta \text{ is such that} \qquad \frac{1-\theta}{(1-2\theta)^2} = \frac{\nu_i + \nu_j}{\gamma}$$

if $\frac{(\nu_j - \nu_i)^2}{\max\{\nu_i, \nu_j\}} \leq \gamma \leq \nu_1 + \nu_2$.

Proof. We first note that the LRS contracts in equilibrium change because of our assumption on the output space. If principal j offers a contract $w_j = (1-\theta_j)\,y_j - \theta_j y_i$, then the LRS contract of principal i (given by (A.43)) is increasing in both y_i and y_j as long as:

$$(\alpha + (1-\alpha)\,\theta_j)\,\nu_i - (1-\alpha)\,(1-\theta_j)\,\nu_j \geq 0 \tag{2.5}$$

In this case the minimum is achieved when $y_i = y_j = 0$. This implies $k = 0$, and thus: $w_i = (1-\theta_i)\,y_i - \theta_i y_j$, with $\theta_i = (1-\alpha)\,(1-\theta_j)$ and no fees payed to the agent. Condition (2.5) is verified later.

The value of the agent is given by:

$$V_A\,(w|\mathcal{A}_0) = \max\left\{0 \quad , \quad (1-\theta_1-\theta_2)\,(\nu_1+\nu_2) - \gamma\right\}$$

The agent will choose either to induce the highest quality of not to produce at all.

The best response of principal i is then:

$$\text{BR}_i\,(w_j) = \underset{\theta_i \in [0,1-\theta_j]}{\text{argmax}} \left\{ \begin{cases} \theta_i\,(\nu_1+\nu_2) - \frac{\theta_i}{1-\theta_1-\theta_2}\gamma & \text{if } (1-\theta_1-\theta_2)\,(\nu_1+\nu_2) > \gamma \\ -\theta_i \overline{y}_j & \text{if } (1-\theta_1-\theta_2)\,(\nu_1+\nu_2) \leq \gamma \end{cases} \right\}$$

The interior solution assuming that the agent produces is given by:

$$\theta_i^\star = (1-\theta_j) - \sqrt{\frac{(1-\theta_j)\,\gamma}{\nu_1+\nu_2}}$$

Moreover, in equilibrium it must be that:

$$\frac{1-\theta_j}{(1-\theta_i-\theta_j)^2} = \frac{\nu_i+\nu_j}{\gamma} \qquad \wedge \qquad \frac{1-\theta_i}{(1-\theta_i-\theta_j)^2} = \frac{\nu_i+\nu_j}{\gamma}$$

which implies that $\theta_i = \theta_j = \theta$, where θ is such that: $\frac{1-\theta}{(1-2\theta)^2} = \frac{\nu_i+\nu_j}{\gamma}$. This characterizes the equilibrium contract. It is left to verify the assumptions, namely condition (2.5) which is satisfied if $\frac{(\nu_j-\nu_i)^2}{\max\{\nu_i,\nu_j\}} \leq \gamma$, and profitability of the agent $((1-\theta_1-\theta_2)\,(\nu_1+\nu_2) > \gamma)$, feasibility of the share θ $\left(0 \leq \theta \leq \frac{1}{2}\right)$ and profitability of the principals, which are always satisfied. □

Restricting the principals to a private contracting domain changes the equilibrium outcomes significantly. In this case principals cannot "free ride" on each other. Even though there is no competition each principal will act as if the public good had to be financed by herself alone. In fact, in equilibrium, the value of the principal does not directly depend on θ_j, and so neither does the optimal value of θ_i. Because of this, the equilibrium share $(1-\theta_i)$ is the same share that would have been optimal in the single principal framework (see Carroll (2015)). When restricted to contract only on their own output principals optimally behave *as if they were financing the public good on their own.*

When $\nu_i < \gamma$ (principal i cannot unilaterally pay for the costs of the public good) principal i cannot guarantee herself a positive payoff, and sets $\theta_i = 0$, or offers the zero contract. This implies that when $\nu_i < \gamma$, $\nu_j < \gamma$ and $\nu_1 + \nu_2 > \gamma$, in equilibrium both principals can offer the zero contract and the public good does not get built (the agent chooses $q = 0$), despite it being socially valuable. Only when either ν_i and ν_j are greater than γ can we guarantee the provision of the public good, however each principal has to foot the bill by herself. This implies that the principals overpay the agent for the good, since $(1-\theta_i)\,\nu_i > \gamma$.

Proposition 6. ***(Public good provision - Private common agency)*** *In a private common agency equilibrium principals offer contracts of the form:*

$$w_i^r(y) = (1-\theta_i)\,y_i \qquad \textit{where} \qquad 1-\theta_i = \min\left\{\sqrt{\frac{\gamma}{\nu_i}}, 1\right\}$$

Proof. LRS contracts are of the form $w_i(y) = (1-\theta_i)\,y_i$. Then the agent's problem is:

$$V_A(w) = \max_{q\in[0,1]}\left\{((1-\theta_i)\,\nu_i + (1-\theta_j)\,\nu_j - \gamma)\,q\right\}$$

We guess that $(\nu_i + \nu_j - \gamma) \geq \theta_i\nu_i + \theta_j\nu_j$ so that the optimal choice is $q = 1$ in equilibrium. The principal's value is then:

$$V_i(w) = \begin{cases} \theta_i\nu_i - \frac{\theta_i}{1-\theta_i}\gamma & \text{if } (1-\theta_i)\,\nu_i + (1-\theta_j)\,\nu_j \geq \gamma \\ -\frac{\theta_i}{1-\theta_i}(1-\theta_j)\,\nu_j & \text{otw} \end{cases}$$

The optimal share if the agent produces is given by:

$$1-\theta_i = \sqrt{\frac{\gamma}{\nu_i}}$$

which is the solution to the single principal problem of **Carroll (2015).** Note that this is interior only if $\gamma < \nu_i$. Now we can check the guesses. Namely that the principal gets a positive payoff, and that the agent will produce. Both conditions are satisfied as long as $\nu_i \geq \gamma$. □

Chapter 3

Network Games with Linear Best Replies

3.1 Introduction

The structure of connections among economic agents affects their actions in many areas of economics. The production of firms and the choice of their markets depends on the action of other firms and the substitutability or complementarity of their goods with those of other firms. People's information, opinions and actions are influenced by the information of their social circles. Technology adaptation and the spread of innovation depends on the underlying channels of information.

The structure of these connections can be formally represented by a network, or graph and the interaction of two agents does not depend only on the intensity of their link, but on the links that they have with other agents as well, i.e. the entire network structure. A major issue is understanding how the the network structure shapes equilibrium outcomes.

Bramoullé et al. (2014) study a large class of games where agents have linear best replies. As they point out many games fall in this category- including games of investment, belief formation, public good provision, social interaction and oligopoly. For a review of papers that fall in this category see Bramoullé et al. (2014). The first part of this chapter focuses on extending results for such games when agent's choice set is uni-dimensional, with a particular focus on games of private provision of public goods. More specifically exploiting an equivalence of stable equilibria in games with linear best replies to the steady states of the Threshold-Linear neural network Hahnloser et al. (2003); Yi et al. (2009), used in recent developments in theoretical neuroscience to explain the formation of memory[1], we can provide further characterization of such the Nash equilibra. Furthermore in particular in the context of public goods we show how results from the theoretical neuroscience literature allow for characterizing Lindahl equilibria [2] and determination of conditions under which Lindahl equilibra have the same stability and properties as the Nash equilibria[3].

Furthermore, as Bramoullé and Kranton (2015) point out, there is little research done in network games where the agent's choice is multidimensional. However in many contexts, players' actions are multidimensional. Firms have multiple products that they sell, and they choose both quality and quantity of the goods that they produce. Individuals act differently toward some people than toward others. For example BourlÃšs et al. (2017) advance a model of altruism in networks where players care about their neighbor's utility and choose a profile of transfers.

In the later part of this chapter we propose a framework to extend network games with linear best replies to a multidimensional setting inspired by an isomorphism between network games with linear best replies as in Bramoullé et al. (2014), Bramoullé and Kranton (2007),Ballester et al. (2006) to the threshold-linear recurrent neural network Hahnloser et al. (2003) used in theoretical neuroscience as a model of memory encoding and retrieval. Then we provide an extension to multidimensional games extending results from the Competitive Layer Model (CLM) used for feature binding and sensory segmentation Wersing et al. (2001).

In particular we focus on a Cournot competition game between firms that can produce multiple products. While models of oligopolistic competition usually abstract from the multi-dimensional nature of competition, in reality oligopolistic firms compete with one another across several different markets. For example in electricity and airline markets,

[1] The stable steady states of the network correspond to patters of firing neurons which represent memory

[2] Lindahl equilibria are equilbria where each agent faces an individual price for the public good, and that price is set to the individual's marginal benefit from the public good and thus they implement a socially efficient outcome.

[3] Elliott and Golub (January 17 2017) also characterize Lindahl equilibria in a similar setting, however their approach and characterizations are different and they do not focus on stability and structure of Lindahl equilibria.

firms compete with one another in different geographical markets, and are constrained by the network like interactions that these firms have in each market. Multi-market contact is prevalent in many oligopolistic industries[4]. Recently, Bimpikis et al. (2019) models the multi-dimensional nature of Cournot competition, with a similar structure the the one we study. However they do not focus on stable Nash equilbria, and our results focus on cases when this multi-market competition leads to the specialization of each firm producing only one good. Which good is produced depends on the centrality that each firm has on each market.

3.2 One Dimensional Network Games

Before proceeding to the multidimensional case, for exposition it is worth revisiting the one dimensional version of the game. The discussion below follows Bramoullé et al. (2014). There are n agents. Each agent i simultaneously chooses an action $x_i \geq 0$ and let $\boldsymbol{x} = (x_1, ..., x_n)$. Let $\boldsymbol{x}_{-i}$ denote the actions of all agents other than i. $G = [g_{ij}]$ denotes the adjacency matrix where $g_{ij} = 1$ if agents i and j are linked and $g_{ij} = 0$ otherwise. However there is no loss in considering arbitrary numbers for g_{ij} as long as $g_{ij} = g_{ji}$ which implies that we have a weighted undirected network. $\delta \geq 0$ measures how much i and j affect each others payoffs. Player i's payoff is $U(x_i, \boldsymbol{x}_{-i}; \delta, G)$. Also let $f_i(\boldsymbol{x}_{-i}; \delta, G)$ denote agent i's best response. A Nash equilibrium is $\boldsymbol{x}^*$ where $x_i^* = f_i\left(\boldsymbol{x}_{-i}^*; \delta, G\right)$. We will focus on games where the best response is linear.

$$f_i(\boldsymbol{x}_{-i}; \delta, G) = max\left(0, \bar{x}_i - \delta \sum_j g_{ij} x_j\right) \tag{3.1}$$

There are plenty of games where the best response has the above form. For example in a game of private provision of public good as in Bramoullé and Kranton (2007) we have that the agents have the following utility function.

$$\hat{U}(x_i, \boldsymbol{x}_{-i}; \delta, G) = b_i\left(x_i + \delta \sum_j g_{ij} x_j\right) - \kappa_i x_i$$

[4]See Bimpikis et al. (2019) for a review of several industries.

where $b_i\left(\right)$ is twice differentiable strictly increasing and concave. In this game the agents have the best response given by equation (3.1).
Another game is that of peer effects considered in Ballester et al. (2006) where the utility function is given by

$$\tilde{U}\left(x_i, \boldsymbol{x}_{-i}; \delta, G\right) = \bar{x}_i x_i - \frac{1}{2} x_i^2 - \delta \sum_j g_{ij} x_i x_j$$

and as they show equilibrium outcomes are related to players Bonacich centralities. This game has the same best response as the one in equation (3.1).
Another game that is important and that we will carry as an example in the multi dimensional game is a game of Cournot competition.
Firm i faces price

$$p_i = a - s\left(x_i + 2\delta \sum_j g_{ij} x_j\right)$$

which imply a linear inverse demand function from the consumers. Vives (2001) shows that when consumers have strictly concave quadratic utility functions over substitute products, then firms face linear inverse demand functions.
The profits of the firm are given by

$$\Pi_i\left(x_i, \boldsymbol{x}_{-i}; \delta, G\right) = x_i p_i - d x_i$$

, where d is the constant marginal cost.
This game also has the same best reply given by equation (3.1).
More generally all games that have the following generalized payoff function:

$$\Pi_i\left(x_i, \boldsymbol{x}_{-i}; \delta, G\right) = v_i\left(x_i - x_i^0 + \delta \sum_j g_{ij} x_j\right) + w_i\left(\mathbf{x}_{-i}\right)$$

where v_i is increasing on $(-\infty, 0]$, decreasing on $[0, +\infty)$ and symmetric around 0, so that 0 is the unique maximum of v_i and since it does not affect incentives w_i can be an arbitrary function. x_i^0 represents the action that the agent would pick in the absence of network effects.
Bramoullé et al. (2014) show that the game in Ballester et al. (2006) has a potential function of the form

$$\phi\left(\boldsymbol{x}; \delta, G\right) = \boldsymbol{x}^T \mathbf{1} - \frac{1}{2} \boldsymbol{x}^T\left(\boldsymbol{I} + \delta \boldsymbol{G}\right) \boldsymbol{x}$$

which then implies that $\boldsymbol{x}^*$ is a Nash equilibrium if and only if $\boldsymbol{x}^*$ satisfies the Kuhn-Tucker conditions of the problem below

$$\max_{\boldsymbol{x}} \phi\left(\boldsymbol{x}; \delta, G\right) \quad \text{s.t. } \forall_i x_i \geq 0$$

3.3 Linear Threshold Recurrent Neural Network

Consider the following system of differential equations

$$\begin{aligned}
\dot{x}_1 &= f_1\left(\boldsymbol{x}; \delta, G\right) - x_1 \\
&\vdots \\
\dot{x}_n &= f_n\left(\boldsymbol{x}; \delta, G\right) - x_n
\end{aligned}$$

$\boldsymbol{x}$ is a Nash Equilibrium of the game if and only if it is a steady state of the system above, by construction.

It turns out that this non-linear system of equations is the Threshold-Linear Recurrent Neural Network model used in neuroscience Hahnloser et al. (2003).

$$\frac{dx_i}{dt} + x_i = \left[b_i + \sum_j W_{ij} x_j\right]^+$$

Here there are n neurons and $x_i = x_i\left(t\right)$ is the firing rate of neuron i. W_{ij} denotes the effective strength of the recurrent connection from jth to ith neuron. b_i is the external input to neuron i and it is constant over time. It is easy to see that both systems are the same with $b_i = \bar{x}_i$ and $W_{ij} = -\delta g_{ij}$.

Hahnloser et al. (2003) show essentially the same results as Bramoullé et al. (2014) using similar techniques. Hahnloser et al. (2003) use an energy function formulation of the threshold-linear network while Bramoullé et al. (2014) use a potential for their game. Stable Nash equilibria correspond to stable steady states of the threshold-linear network. When $W_{ij} = -\delta g_{ij}$ and $g_{ij} \in \{0, 1\}$ Xie et al. (2002) show results about winner-take-all networks, which are similar to results regarding the relationship between maximal independent sets and Nash equilibria in Bramoullé and Kranton (2007). In the next section we show how the results of Hahnloser et al. (2003) can be used to provide characterizations of stable Lindahl equilibria in these games with linear best replies.

Both these literatures have advanced several different properties of the effect of networks on the Nash equilibrium outcomes and the steady states of the dynamical system. While the motivations are different the results are meaningful in both of these literatures.
For example Bramoullé et al. (2014) prove the following results that also are easily shown using similar techniques using Hahnloser et al. (2003).
Some of those results are the following:

1. If $|\lambda_{\min}(G)| < \frac{1}{\delta}$, there is a unique Nash equilibrium.

2. For any δ and G, if $|\lambda_{\min}(G)| > \frac{1}{\delta}$ there exists at least one Nash equilibrium with inactive agents (their action is $x_i = 0$).

3. Consider a graph G and a Nash equilibrium $\boldsymbol{x}$ with active agents A and strictly inactive agents. $\boldsymbol{x}$ is stable if and only if $|\lambda_{\min}(G_A)| < \frac{1}{\delta}$.

4. For $|\lambda_{\min}(G)| > \frac{1}{\delta}$, all stable equilibria involve at least one inactive agent.

5. Consider a stable equilibrium $\boldsymbol{x}$ with active agents A. There is no other equilibrium $\boldsymbol{x}'$with active agents $A' \subset A$

3.4 Lindahl Equilibria and public goods

While the questions of public good provision are as old as economics itself, there is a recent and growing literature that takes a social network approach to public goods and focuses on understanding the impact of the heterogeneity of externalities across agents. Many public goods, like innovation or technological spillover (as well as other actions with positive and negative externalities, like crime, smoking, quitting smoking etc) spread locally so the social and geographical structure can have substantial effects.
This local nature of public goods and of the spread of influence have raised new questions in the public goods literature: How does the social or geographic structure affect the level of public good provision? Do people exert effort themselves or rely on others? Who is the most influential agent in the economy? Whose effort generates the highest positive or negative externality? How do new links—links between communities or firms, for example—affect contributions and welfare? What are the efficient allocation of a particular

network structure and how do they compare to non-cooperative outcomes? What policies can remedy inefficiencies that can arise due to the underlying network structure of the economy?

Consider the game of private provision of public goods of Bramoullé and Kranton (2007) where the agents have the following utility function.

$$\hat{U}\left(x_i, \boldsymbol{x}_{-i}; \delta, G\right) = b_i\left(x_i + \delta \sum_j g_{ij} x_j\right) - \kappa_i x_i$$

but now we allow for transfers where

$$\hat{U}\left(x_i, \boldsymbol{x}_{-i}; \delta, G\right) = b_i\left(x_i + \delta \sum_j g_{ij} x_j\right) - \kappa_i x_i + \tau_i x_i$$

is quasilinear in money transfers ($\tau_i x_i$).

The best response of agent i is of the form

$$f_i\left(\boldsymbol{x}_{-i}; \delta, G\right) = max\left(0, \bar{\bar{x}}_i - \delta \sum_j g_{ij} x_j\right)$$

where $b_i^{'}\left(\bar{\bar{x}}_i\right) = \kappa_i - t_i$.

Then we ask the following question. Given the network structure for what allocations $\boldsymbol{x} \geq 0$ do there exist transfers or Lindahl prices $\boldsymbol{t} = (t_1, ..., t_n)$ such that $\boldsymbol{x}$ is a stable Nash equilibrium? This leads us to the following results.

Theorem 9. *Let $\boldsymbol{x} \geq 0$ be an allocation. Then there exist some linear transfer rates $\boldsymbol{t} = (t_1, ..., t_n)$ for which $\boldsymbol{x}$ is a Nash equilibrium.*

Moreover if $\left|\lambda_{min}\left(G\right)\right| < \frac{1}{\delta}$, all these Nash equilibra are stable equilibria.

Proof. Let $\bar{\bar{x}}_i = x_i + \delta \sum_j g_{ij} x_j$ if $x_i > 0$ and $\bar{\bar{x}}_i = \delta \sum_j g_{ij} x_j$ otherwise.

Then $\boldsymbol{x}$ is a Nash equilibrium since it is a steady state of the system

$$\frac{dx_i}{dt} + x_i = \left[\bar{\bar{x}}_i - \delta \sum_j g_{ij} x_j\right]^+$$

If $\left|\lambda_{\min}\left(G\right)\right| < \frac{1}{\delta}$ then the matrix $I + \delta G$ is positive definite. That implies the function $L = \frac{1}{2} x^T \left(I + \delta G\right) x - \bar{\bar{x}}_i^T x$ is lower bounded and radially unbounded.

In $\mathbb{R}^n_+$ the function L is non-increasing under the network dynamics and it is constant only at the steady state. By Lyapunov's Stability theorem, the stable steady states are globally asymptotically stable, hence a stable Nash equilibrium. □

Using this framework we can further characterize Lindahl equilibria of this game, where we use the definition of permitted and forbidden sets.

Definition. A set of agents A is called **permitted** if for some tax rates $\boldsymbol{t} = (t_1, ..., t_n)$ there is a stable NE where all agents in A are active.

also

Definition. A set of agents A is called **forbidden** if there are no tax rates $\boldsymbol{t} = (t_1, ..., t_n)$ for which there exists a stable NE where all agents in A are active .
Then using the Threshold linear equilibrium framework we can have the following results.

Theorem 10. *Any subset of a permitted set is permitted. Also any superset of a forbidden set is forbidden.*

Proof. Follows by Cauchy's Interlacing Theorem. □

Theorem 11. *If $|\lambda_{min}(G)| > \frac{1}{\delta}$, then there exits a forbidden set. Also there exits $\mathbf{t} = (t_1, ..., t_n)$, for which there are multiple stable Nash equilibria.*

Proof. Follows from Hahnloser et al. (2003) □

Theorem 12. *Consider a stable equilibrium $\boldsymbol{x}$ with active agents A. There is no other equilibrium $\boldsymbol{x}'$ with active agents $A' \supset A$.*

3.5 Multidimensional network games

Now let us consider the Cournot game mentioned earlier, modified such that each firm can produce multiple goods and face different markets. The framework below allows for understanding Cournot competition between multiproduct firms where firms face some resource constraint or supply chain difficulties where holding everything else fixed, increasing output in one of the markets leads to lower output in the other markets of that firm.

Formally there are n firms and L goods that each firm can choose to produce from. Indices i, j will be reserved for agents, and indices k, l will be reserved for goods. Firm i chooses $\left(x_i^1, x_i^2, ...x_i^L\right) \geq 0$ in order to maximize its profits.
Assume that each firm i has the following best response for good k.

$$x_i^k = max\left(0, \bar{x}^k - A_i^{kl}\sum_l x_i^l - \delta^k \sum_j g_{ij}^k x_j^k + x_i^k\right)$$

where we need $A_i^{kl} = A_i^{lk} > 0$ for all $l, k \in \{1, 2, ..., L\}$ and $i \in \{1, 2, ..., N\}$. The Nash equilibria of this game are the same as the fixed points of the following system of equations:

$$\frac{dx_i^k}{dt} + x_i^k = max\left(0, \bar{x}_i - A_i^{kl}\sum_{l\neq k} x_i^l - \delta^k \sum_j g_{ij}^k x_j^k\right) \tag{3.2}$$

Now this system of equations is called the Competitive Layer Model of Wersing et al. (2001) and has an energy function given by

$$E = -\sum_{ik}\bar{x}_i x_i^k + \frac{1}{2}\sum_i\sum_{kl} A_i^{kl} x_i^k x_i^l - \frac{1}{2}\sum_k\sum_{ij} g_{ij}^k x_i^k x_j^k$$

Wersing et al. (2001) provides conditions for the existence of a stable Nash equilibrium each agent chooses at most one dimension. Essentially those conditions boil down to strong substitutability, given by A_i^{kl} being large enough for all k and l and $k \neq l$.
Now for simplicity assume that there are only two good $\{k, l\}$.
Consider a simplified version of the model in Bimpikis et al. (2019). The price of good k that firm i faces is given by

$$p_i^k = a - s\left(x_i^k + 2\delta^k \sum_j g_{ij}^k x_j^k\right)$$

Implicitly assumed here is that the overall demand for these two goods is the same in the sense that $a^k = a^l = a$ and $s^k = s^l = s$. We will relax this assumption later but maintain it for now for ease of exposition. $G^k = \{g_{ij}^k\}$ denotes the market structure for good k and δ^k denotes the degree of substitutability among the two goods. For now $g_{ij}^k \in \{0, 1\}$. Again this assumption will be reduced later. Note that this also implies that there are no complementarities between goods since $g^k_{ij} \geq 0$ and $\delta > 0$. The firm tries to maximize profits which are given by

$$\Pi_i = x^k_i p^k_i + x^l_i p^l_i - d\left(x^k_i + x^l_i\right)^2$$

The quadratic cost function here implies that there are interaction terms that mean that marginal cost of production in one good is increasing in the level of the other good.

$$\Pi_i = p^k_i x^k + p^l_i x^l_i - d\left(x^k_i + x^l_i\right)^2$$
$$= \left(a - s\left(x^k_i + 2\delta^k \sum_{j \neq i} g^k_{ij} x^k_j\right)\right) x^k_i + \left(a - s\left(x^l_i + 2\delta^l \sum_{j \neq i} g^l_{ij} x^l_j\right)\right) x^l_i - d\left(x^k_i + x^l_i\right)^2$$

Taking first order conditions we get

$$\frac{\partial \Pi}{\partial x^k_i} = \left(a - s\left(2\delta^k \sum_{j \neq i} g^k_{ij} x^k_j\right) - 2dx^l_i\right) - 2\left(d + s\right) x^k_i$$

Which implies that

$$x^k_i = max\left(0, \frac{\left(a - s\left(2\delta^k \sum_{j \neq i} g^k_{ij} x^k_j\right) - 2dx^l_i\right)}{2\left(s + d\right)}\right) \quad (3.3)$$

It will be useful to write the best response in equation (3.3) in the following way

$$x^k_i = max\left(0, \left(\frac{a}{2} - dx^l_i - \left(d + 2s\right) x^k_i + s\left(x^k_i + \delta^k \sum_{j \neq i} g^k_{ij} x^k_j\right) + x^k_i\right)\right)$$

Now the Nash equilibria of this game are the fixed points of the following system of equations

$$\frac{dx^k_i}{dt} + x^k_i = max\left(0, \left(\frac{a}{2} - dx^l_i - \left(d + 2s\right) x^k_i + s\left(x^k_i - \delta^k \sum_{j \neq i} g^k_{ij} x^k_j\right) + x^k_i\right)\right) \quad (3.4)$$

However in this particular setup with linear demand and quadratic costs, the sufficient conditions of Wersing et al. (2001) for the existence of stable Nash equilibrium where each firm produces only in one market are not satisfied, as it essentially requires that $s < 0$, which would mean upward sloping demand curves.

It would be useful to have sufficient conditions for the existence of stable Nash equilibria where firms produce in all markets. We leave that for future work.

Bibliography

Anup Agrawal, Tommy Cooper, Qin Lian, and Qiming Wang. Common advisers in mergers and acquisitions: Determinants and consequences. *Journal of Law and Economics*, 56(3): 691 – 740, 2013. URL `https://ideas.repec.org/a/ucp/jlawec/doi10.1086-673322.html`.

Nemanja Antic. Contracting with unknown technologies. Northwestern Working Paper, 2014.

Coralio Ballester, Antoni Calvó-Armengol, and Yves Zenou. Who's who in networks. wanted: The key player. *Econometrica*, 74(5):1403–1417, 2006.

B. Douglas Bernheim and Michael Whinston. Common marketing agency as a device for facilitating collusion. *RAND Journal of Economics*, 16(2):269–281, 1985. URL `http://EconPapers.repec.org/RePEc:rje:randje:v:16:y:1985:i:summer:p:269-281`.

B. Douglas Bernheim and Michael D. Whinston. Common agency. *Econometrica*, 54(4): 923–942, 1986a. ISSN 00129682. URL `http://www.jstor.org/stable/1912844`.

B. Douglas Bernheim and Michael D. Whinston. Menu auctions, resource allocation, and economic influence. *The Quarterly Journal of Economics*, 101(1):1–32, 1986b. ISSN 00335533. URL `http://www.jstor.org/stable/1884639`.

Kostas Bimpikis, Shayan Ehsani, and Rahmi İlkılıç. Cournot competition in networked markets. *Management Science*, 65(6):2467–2481, 2019.

Eric W. Bond and Thomas A. Gresik. Regulation of multinational firms with two active governments: A common agency approach. *Journal of Public Economics*, 59(1):33–53, January 1996. URL https://ideas.repec.org/a/eee/pubeco/v59y1996i1p33-53.html.

Renaud BourlÃšs, Yann BramoullÃ©, and Eduardo Perez-Richet. Altruism in networks. *Econometrica*, 85(2):675–689, 2017. ISSN 1468-0262. doi: 10.3982/ECTA13533. URL http://dx.doi.org/10.3982/ECTA13533.

Yann Bramoullé and Rachel Kranton. Public goods in networks. *Journal of Economic Theory*, 135(1):478–494, 2007.

Yann Bramoullé and Rachel Kranton. Games played on networks. 2015.

Yann Bramoullé, Rachel Kranton, and Martin D'amours. Strategic interaction and networks. *The American Economic Review*, 104(3):898–930, 2014.

Melvin Burstein and Arthur J. Rolnick. Congress should end the economic war between the states. *Federal Reserve Bank of Mineapolis Annual Report*, (Mar):2–20, 1995. URL https://ideas.repec.org/a/fip/fedmar/y1995imarp2-20nv.9no.1.html.

Guilherme Carmona and JosÃ© Fajardo. Existence of equilibrium in common agency games with adverse selection. *Games and Economic Behavior*, 66(2):749 – 760, 2009. ISSN 0899-8256. doi: http://dx.doi.org/10.1016/j.geb.2008.10.003. URL http://www.sciencedirect.com/science/article/pii/S089982560800184X. Special Section In Honor of David Gale.

Gabriel Carroll. Robustness and linear contracts. *American Economic Review*, 105(2): 536–63, 2015. doi: 10.1257/aer.20131159. URL http://www.aeaweb.org/articles.php?doi=10.1257/aer.20131159.

Gabriel Carroll and Delong Meng. Robust contracting with additive noise. *Journal of Economic Theory*, 166:586–604, 2016. URL http://EconPapers.repec.org/RePEc:eee:jetheo:v:166:y:2016:i:c:p:586-604.

Marjorie Censer. Defense companies brace for a different kind of consolidation this time around. *The Washington Post*, 2014.

Chairman Staff Discussion Draft - MCG13834. *International Business Tax Reform Discussion Draft - Senator Max Baucus, Chairman U.S. Senate Committee on Finance*. 113th Congress (2012-2013), 2013. URL https://www.finance.senate.gov/chairmans-news/baucus-unveils-proposals-for-international-tax-reform.

Sylvain Chassang. Calibrated incentive contracts. *Econometrica*, 81(5):1935–1971, 09 2013. URL http://ideas.repec.org/a/ecm/emetrp/v81y2013i5p1935-1971.html.

Tianjiao Dai and Juuso Toikka. Robust incentives for teams. MIT Working Paper, 2017.

Avinash Dixit, Gene M Grossman, and Elhanan Helpman. Common agency and coordination: General theory and application to government policy making. *Journal of Political Economy*, 105(4):752–69, August 1997. URL https://ideas.repec.org/a/ucp/jpolec/v105y1997i4p752-69.html.

Avinash K. Dixit. *The making of economic policy : a transaction-cost politics perspective*. MIT Press Cambridge, Mass., 1996. ISBN 0262041553.

Pradeep Dubey, Ori Haimanko, and Andriy Zapechelnyuk. Strategic complements and substitutes, and potential games. *Games and Economic Behavior*, 54(1):77–94, 2006.

Matthew Elliott and Benjamin Golub. A network approach to public goods. January 17 2017.

Martin Feldstein and David Hartman. The optimal taxation of foreign source investment income. *The Quarterly Journal of Economics*, 93(4):613–629, 1979. doi: 10.2307/1884472. URL +http://dx.doi.org/10.2307/1884472.

Alexander Frankel. Aligned delegation. *American Economic Review*, 104(1):66–83, January 2014. URL http://ideas.repec.org/a/aea/aecrev/v104y2014i1p66-83.html.

Jean Fraysse. Common agency: Existence of an equilibrium in the case of two outcomes. *Econometrica*, 61(5):1225–1229, 1993. ISSN 00129682, 14680262. URL http://www.jstor.org/stable/2951499.

Daniel Garrett. Robustness of simple menus of contracts in cost-based procurement. *Games and Economic Behavior*, 87:631–641, 2014. URL http://EconPapers.repec.org/RePEc:eee:gamebe:v:87:y:2014:i:c:p:631-641.

Gene M Grossman and Elhanan Helpman. Protection for sale. *American Economic Review*, 84(4):833–50, September 1994. URL https://ideas.repec.org/a/aea/aecrev/v84y1994i4p833-50.html.

Fatih Guvenen, Jr. Raymond J. Mataloni, Dylan G. Rassier, and Kim J. Ruhl. Offshore profit shifting and domestic productivity measurement. Working Paper 23324, National Bureau of Economic Research, April 2017. URL http://www.nber.org/papers/w23324.

Richard HR Hahnloser, H Sebastian Seung, and Jean-Jacques Slotine. Permitted and forbidden sets in symmetric threshold-linear networks. *Neural computation*, 15(3):621–638, 2003.

Bengt Holmstrom and Paul Milgrom. Aggregation and linearity in the provision of intertemporal incentives. *Econometrica*, 55(2):303–328, 1987.

Bengt Holmstrom and Paul Milgrom. Common agency and exclusive dealing. April 1988. URL http://economics.mit.edu/files/3779.

Thomas L. Hungerford. The simple fix to the problem of how to tax multinational corporations - ending deferral. Issue Brief 378, Economic Policy Institute, March 2014. URL http://www.epi.org/publication/how-to-tax-multinational-corporations/.

Leonid Hurwicz. On the interaction between information and incentives in organizations. In Klaus Krippendorf, editor, *Communication and Control in Society*, pages 123–147. Gordon and Breach, Science Publishers, 1977.

Leonid Hurwicz and Leonard Shapiro. Incentive structures maximizing residual gain under incomplete information. *Bell Journal of Economics*, 9(1):180–191, Spring 1978. URL http://ideas.repec.org/a/rje/bellje/v9y1978ispringp180-191.html.

Robert D. Innes. Limited liability and incentive contracting with ex-ante action choices. *Journal of Economic Theory*, 52(1):45–67, October 1990. URL https://ideas.repec.org/a/eee/jetheo/v52y1990i1p45-67.html.

Jean-Jacques Laffont and Jean Tirole. Using cost observation to regulate firms. *Journal of Political Economy*, 94(3, Part 1):614–641, 1986. doi: 10.1086/261392. URL `https://doi.org/10.1086/261392`.

Didier Laussel and Michel Le Breton. Efficient private production of public goods under common agency. *Games and Economic Behavior*, 25(2):194–218, November 1998. URL `https://ideas.repec.org/a/eee/gamebe/v25y1998i2p194-218.html`.

Michel Le Breton and Francois Salanie. Lobbying under political uncertainty. *Journal of Public Economics*, 87(12):2589–2610, December 2003. URL `https://ideas.repec.org/a/eee/pubeco/v87y2003i12p2589-2610.html`.

Norbert Maier and Marco Ottaviani. Information sharing in common agency: When is transparency good? *Journal of the European Economic Association*, 7(1):162–187, 03 2009. URL `https://ideas.repec.org/a/tpr/jeurec/v7y2009i1p162-187.html`.

David Martimort. The multiprincipal nature of government. *European Economic Review*, 40(3-5):673–685, April 1996. URL `https://ideas.repec.org/a/eee/eecrev/v40y1996i3-5p673-685.html`.

David Martimort. Multi-contracting mechanism design. In Richard Blundell, Whitney K. Newey, and Torsten Persson, editors, *Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress*, volume 1, pages 57–101. Cambridge University Press, 008 2006. doi: 10.1017/CBO9781139052269.004. URL `https://www.cambridge.org/core/books/advances-in-economics-and-econometrics/multi-contracting-mechanism-design/7B24A552767F424950973110A4CD4447`.

David Martimort and Humberto Moreira. Common agency and public good provision under asymmetric information. *Theoretical Economics*, 5(2):159–213, May 2010. URL `https://ideas.repec.org/a/the/publsh/507.html`.

David Martimort and Aggey Semenov. Ideological uncertainty and lobbying competition. *Journal of Public Economics*, 92(3-4):456–481, April 2008. URL `https://ideas.repec.org/a/eee/pubeco/v92y2008i3-4p456-481.html`.

David Martimort and Lars Stole. Representing equilibrium aggregates in aggregate games with applications to common agency. *Games and Economic Behavior*, 76(2):753–772, 2012. URL https://ideas.repec.org/a/eee/gamebe/v76y2012i2p753-772.html.

David Martimort and Lars Stole. Menu auctions and influence games with private information. MPRA Paper 62388, University Library of Munich, Germany, February 2015. URL https://ideas.repec.org/p/pra/mprapa/62388.html.

Paul Milgrom. Package auctions and exchanges. *Econometrica*, 75(4):935–965, 07 2007. URL https://ideas.repec.org/a/ecm/emetrp/v75y2007i4p935-965.html.

Dov Monderer and Lloyd S. Shapley. Potential games. *Games and Economic Behavior*, 14(1):124 – 143, 1996. ISSN 0899-8256. doi: https://doi.org/10.1006/game.1996.0044. URL http://www.sciencedirect.com/science/article/pii/S0899825696900445.

Trond E. Olsen and Petter Osmundsen. Strategic tax competition; implications of national ownership. *Journal of Public Economics*, 81(2):253 – 277, 2001. ISSN 0047-2727. doi: http://dx.doi.org/10.1016/S0047-2727(00)00114-6. URL http://www.sciencedirect.com/science/article/pii/S0047272700001146.

Senate Bill 727. *Bipartisan Tax Fairness and Simplification Act of 2011*. 112th Congress (2011-2012), 2011. URL https://www.congress.gov/bill/112th-congress/senate-bill/727.

Xavier Vives. Oligopoly pricing: old ideas and new tools. 2001.

Heiko Wersing, Jochen J Steil, and Helge Ritter. A competitive-layer model for feature binding and sensory segmentation. *Neural Computation*, 13(2):357–387, 2001.

Xiaohui Xie, Richard HR Hahnloser, and H Sebastian Seung. Selectively grouping neurons in recurrent networks of lateral inhibition. *Neural computation*, 14(11):2627–2646, 2002.

Zhang Yi, Lei Zhang, Jiali Yu, and Kok Kiong Tan. Permitted and forbidden sets in discrete-time linear threshold recurrent neural networks. *IEEE transactions on neural networks*, 20(6):952–963, 2009.

Gabriel Zucman. Taxing across borders: Tracking personal wealth and corporate profits. *Journal of Economic Perspectives*, 28(4):121–48, November 2014. doi: 10.1257/jep.28.4.121. URL `http://www.aeaweb.org/articles?id=10.1257/jep.28.4.121`.

Appendix A

Robust Contracts in Common Agency

A.1 Proofs

A.1.1 Best Response - 2 Principals

First consider the implications of the common limited liability assumption. Under this assumption contracts have to guarantee that $w_1(y)+w_2(y) \geq 0$. From the point of view of an individual principal this allows to charge the agent up to the amount that the opposing party is paying. From the point of view of the equilibrium this allows for transfers between principals (through the agent), as in Bernheim and Whinston (1986a,b). The problem of a principal can then be thought of in two steps: first undoing the payments of other principals, and then offering the agent an aggregate contract satisfying limited liability. We call this aggregate contract $\tilde{w}_i$. Then the ex-post payoff of principal i is: $y_i+w_j(y)-\tilde{w}_i(y)$. Principal i's actual contract is of course: $w_i(y)=\tilde{w}_i(y)-w_j(y)$.
One option that is always available to a principal when facing a competing contract is to undo all payments and offer the agent the "zero contract", i.e. $\tilde{w}_i(y)=0$. Under assumption (2) the agent's unique optimal action, under any technology set, given the zero contract is to choose inaction. This allows us to define a lower bound on the payoff of the principal. We call a contract **eligible** if it guarantees the principal a payoff higher than

the zero contract[1]:

$$V_i(w) > w_j(0,0) \tag{A.1}$$

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition 7. *Let $(F,c) \in A^\star(w|\mathcal{A})$. For $\mathcal{A} \supseteq \mathcal{A}_0$, it holds that:*

$$E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0)$$

Moreover, if $(F,c) \in A^\star(w|\mathcal{A})$ then $F \in \mathcal{F}$ where:

$$\mathcal{F} = \{F \in \Delta(Y) \,|E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0)\}$$

Proof. To see the first inequality let $(F,c) \in A^\star(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F[w_1(y) + w_2(y)] \geq E_F[w_1(y) + w_2(y)] - c \geq V_A(w|\mathcal{A}) \geq V_A(w|\mathcal{A}_0)$$

Then $F \in \mathcal{F}$. □

Lemma 45 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above). Note that $\mathcal{F}$ only depends on the contract and the known set of actions $\mathcal{A}_0$. In this way we replace the complexity of the definition of $V_i(w)$ in (1.2) with an object that depends only on known elements.

Lemma 4. *Let w be an eligible contract for principal i. Then*

$$V_i(w) = \min_{F \in \mathcal{F}} \quad E_F[y_i - w_i(y)]$$

Moreover if $F \in \underset{F \in \mathcal{F}}{argmin} E_F[y_i - w_i(y)]$ then $E_F[w_1(y) + w_2(y)] = V_A(w|\mathcal{A}_0)$.

Proof. We first establish the first claim: Let w be an eligible contract scheme then: $V_i(w) = \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$.
It must be that: $V_i(w) \geq \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$. Using the definition of $V_i(w)$:

$$V_i(w) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} \min_{(F,c) \in A^\star(w|\mathcal{A})} E_F[y_i - w_i(y)] \geq \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$$

[1] In the case of a single principal dealing with a single agent this is similar to allowing the output of the principal $(\tilde{y})$ to go from $\underline{\tilde{y}} = \min_{y \in Y}\{y_i + w_j(y)\}$ to $\bar{\tilde{y}} = \max_{y \in Y}\{y_i + w_j(y)\}$. In that case a contract is eligible if it gives a guaranteed payoff above the minimum possible output $(\underline{\tilde{y}})$.

Where the inequality follows because if $(F, c) \in A^{\star}(w|\mathcal{A})$ then $F \in \mathcal{F}$.

To prove equality suppose that $V_i(w) > \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$, and let

$$F^{'} \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i(y)]$$

Note that We have that $E_{F'}[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0)$ from Proposition 7. There are two options:

1. $F^{'}$ does not place full support on the values of y that maximize $w_1 + w_2$.

   Let $\hat{y} \in \operatorname{argmax}\{w_1(y) + w_2(y)\}$, and $\hat{F} = \delta_{\hat{y}}$ be a distribution with full mass on $\hat{y}$. Let $\epsilon \in [0, 1]$ and $F_\epsilon = (1 - \epsilon) F^{'} + \epsilon \hat{F}$.

   Note then that for all ϵ there exists a $\xi_\epsilon > 0$ such that: $E_{F_\epsilon}[w_1(y) + w_2(y)] - \xi_\epsilon > V_A(w|\mathcal{A}_0)$.

   Define and $\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \{(F_\epsilon, \xi_\epsilon)\}$. It follows that the unique optimal action of the agent in $\mathcal{A}_\xi$ is $(F_\epsilon, \xi_\epsilon)$. Then:

   $$V_i(w) \leq V_i(w|\mathcal{A}_\epsilon) = E_{F_\epsilon}[y_i - w_i(y)] = (1 - \epsilon) E_{F'}[y_i - w_i] + \epsilon E_{\hat{F}}[y_i - w_i]$$

   This condition holds for all $\epsilon > 0$. Letting $\epsilon \to 0$ we arrive at a contradiction:

   $$V_i(w) \leq E_{F'}[y_i - w_i] = \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i]$$

2. $F^{'}$ places full support on the values of y that maximize $w_1 + w_2$. There are still two possible cases:

   (a) $E_{F'}[w_1 + w_2] > V_A(w|\mathcal{A}_0)$. Then there exists $\xi > 0$ and a technology $\mathcal{A}^{'} = \mathcal{A}_0 \cup \left\{\left(F^{'}, \xi\right)\right\}$ such that $\left(F^{'}, \xi\right)$ is the unique optimal action for the agent in $\mathcal{A}^{'}$. Then we arrive at a contradiction:

   $$V_i(w) \leq V_i\left(w|\mathcal{A}^{'}\right) = E_{F'}[y_i - w_i] = \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i]$$

   (b) $E_F[w_1 + w_2] = V_A(w|\mathcal{A}_0)$. This implies $V_A(w|\mathcal{A}_0) = \max_{y \in Y}\{w_1 + w_2\}$ which can only be satisfied if $F^{'}$ is available in $\mathcal{A}_0$ at zero cost. By assumption 2 this implies that $F = \delta_{(0,0)}$ and that $w_1(0, 0) + w_2(0, 0) = \max_{y \in Y}\{w_1(y) + w_2(y)\}$.

In this case the unique optimal action for the agent under any technology is $(\delta_0, 0)$, so the value of the principal is $V_i(w) = -w_i(0,0) \leq w_j(0,0)$, where the inequality follows from limited liability. This contradicts eligibility.

Now we establish the second claim: Let w be an eligible contract scheme for principal i. If $F \in \underset{F\in\mathcal{F}}{\text{argmin}} E_F[y_i - w_i]$ then $E_F[w_1 + w_2] = V_A(w|\mathcal{A}_0)$.
To prove this, let $F^{'} \in \underset{F\in\mathcal{F}}{\text{argmin}} E_F[y_i - w_i(y)]$ and suppose for a contradiction that

$$E_F[w_1(y) + w_2(y)] > V_A(w|\mathcal{A}_0).$$

Let $\epsilon \in [0,1]$ and define $F_\epsilon = (1-\epsilon)F^{'} + \epsilon\delta_0$.
For low enough ϵ it holds that: $E_{F_\epsilon}[w_1(y) + w_2(y)] > V_A(w|\mathcal{A}_0)$. Then there exists $\xi_\epsilon > 0$ such that $\{(F_\epsilon, \xi_\epsilon)\} = A^\star(w|\mathcal{A}_\epsilon)$ where $\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \{(F_\epsilon, \xi_\epsilon)\}$. The payoff to the principal is then:

$$\begin{aligned}
V_i(w|\mathcal{A}_\epsilon) &= (1-\epsilon)E_F[y_i - w_i(y)] + \epsilon(-w_i(0,0)) \\
&= (1-\epsilon)V_i(w) + \epsilon(w_j(0,0) - (w_1(0,0) + w_2(0,0))) \\
&= V_i(w) - \epsilon(V_i(w) - w_j(0,0) + (w_1(0,0) + w_2(0,0))) \\
&\leq V_i(w) - \epsilon(V_i(w) - w_j(0,0)) \\
&< V_i(w)
\end{aligned}$$

This gives a contradiction. □

Given the known action set $\mathcal{A}_0$, the next lemma links the principal's guaranteed payoff to the agent's payoff in an affine way. This link allows the principal to increase her own guaranteed payoff by controlling the payoff given to the agent. Lemma 46 also offers a relation between any contract w_i, the outcome y_i and the contract w_j offered by the other principal.
The affine link between the agent's payoff and the principal's payoff is a crucial element in providing incentives for the agent. Given the partial knowledge over the agent's set of actions the principals' optimal strategy is to tie their payoff to that of the agent, thus aligning the agent's objectives with their own. This is the same mechanism at the heart of the optimal contracts in Hurwicz and Shapiro (1978) and Carroll (2015), and will be crucial in establishing the optimality of affine (LRS) contracts in the setting we develop.

Lemma 5. *Let w be an eligible contract scheme. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$w_i(y) \leq \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k \tag{A.2}$$

$$V_i(w) = k + \lambda V_A(w|\mathcal{A}_0) \tag{A.3}$$

Proof. For the proof define the following two sets:

1. Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points $(w_1(y) + w_2(y), y_i - w_i(y))$ for $y \in Y$.
2. Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u > V_A(w|\mathcal{A}_0)$ and $v < V_i(w)$.

We first establish that $S \cap T = \emptyset$. Let $(u, v) \in T$ then let $F \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i(y)]$, by definition of T and Lemma (45):

$$u > V_A(w|\mathcal{A}_0) = E_F[w_1(y) + w_2(y)]$$

$$v < V_i(w) = E_F[y_i - w_i(y)]$$

now, suppose for a contradiction that $(u, v) \in S$, then there exists $F^{'} \in \Delta(Y)$ such that:

$$u = E_{F'}[w_1(y) + w_2(y)] \qquad \text{and} \qquad v = E_{F'}[y_i - w_i(y)]$$

Note that $F^{'}$ guarantees a payoff to the agent larger than $V_A(w|\mathcal{A}_0)$ so $F^{'} \in \mathcal{F}$ but:

$$E_F[y_i - w_i(y)] > E_{F'}[y_i - w_i(y)]$$

which contradicts minimality of F. Then $S \cap T = \emptyset$

Second, since $S \cap T = \emptyset$ we can apply the separating hyperplane theorem which implies that there exist constants (k, λ, μ) such that $(\lambda, \mu) \neq (0, 0)$ and:

$$k + \lambda u - \mu v \leq 0 \qquad (u, v) \in S \tag{A.4}$$

$$k + \lambda u - \mu v \geq 0 \qquad (u, v) \in T \tag{A.5}$$

Let $F^{\star} \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i(y)]$.

Note that the pair $(E_{F^\star}[w_1(y) + w_2(y)], E_{F^\star}[y_i - w_i(y)])$ lies in the closures of both S and T. Then:

$$k + \lambda E_{F^\star}[w_1(y) + w_2(y)] - \mu E_{F^\star}[y_i - w_i(y)] = 0 \tag{A.6}$$

It is left to show that $\lambda, \mu > 0$.

Note that $(u, v) \in T$ admits u arbitrarily high and v arbitrarily low. So for (B.8) to hold it must be that $\lambda \geq 0$ and $\mu \geq 0$. There are then two cases to rule out:

1. Suppose $\mu = 0$, then it must be that $\lambda > 0$ (since $(\lambda, \mu) \neq (0, 0)$). From (B.7) and (B.8)
$$u \leq -\frac{k}{\lambda} \quad (u, v) \in S \qquad \text{and} \qquad u \geq -\frac{k}{\lambda} \quad (u, v) \in T$$
So $\max_{y \in Y} [w_1(y) + w_2(y)] = \max_{u \in S} u \leq -\frac{k}{\lambda} \leq \inf_{u \in T} u = V_A(w|\mathcal{A}_0)$. Which implies:
$$\max_{y \in Y} [w_1(y) + w_2(y)] = V_A(w|\mathcal{A}_0)$$
This can only be satisfied if the agent takes an action with zero cost. By assumption 2 this implies that $F = \delta_{(0,0)}$ and that $w_1(0,0) + w_2(0,0) = \max_{y \in Y} \{w_1(y) + w_2(y)\}$. In this case the unique optimal action for the agent under any technology is $(\delta_0, 0)$, so the value of the principal is $V_i(w) = -w_i(0,0) \leq w_j(0,0)$, where the inequality follows from limited liability. This contradicts eligibility. Then $\mu > 0$.

2. Suppose $\lambda = 0$, then it must be that $\mu > 0$ (since $(\lambda, \mu) \neq (0, 0)$). From (B.7) and (B.8)
$$v \geq \frac{k}{\mu} \quad (u, v) \in S \qquad \text{and} \qquad v \leq \frac{k}{\mu} \quad (u, v) \in T$$
So $\min_{y \in Y} [y_i - w_i(y)] = \min_{v \in S} v \geq \frac{k}{\mu} \geq \sup_{v \in T} v = V_i(w)$, then:
$$V_i(w) \leq \min_{y \in Y} [y_i - w_i(y)] \leq \min_{y \in Y} [y_i + w_j(y)] \leq w_j(0,0)$$
which violates eligibility (the second inequality follows from limited liability). So $\lambda > 0$.

Note that since λ and μ are greater than zero μ can be normalized to 1, giving from (B.7):

$$k + \lambda(w_i(y) + w_j(y)) - (y_i - w_i(y)) \leq 0$$

And from (B.9):

$$V_i(w) = k + \lambda V_A(w|\mathcal{A}_0)$$

□

The following two lemmas (35 and 36) use the relation between the principals' contracts derived in Lemma 46 to construct an alternative contract that dominates the original one, in the sense that it weakly increases principal i's guaranteed payoff. Since the relation obtained in (B.5) is affine in output and the other principal's contract, the alternative contract constructed below will inherit that form. These contracts form the LRS contracts defined in (A.43).

Lemma 6. *Let $w = (w_i, w_j)$ be an eligible contract scheme that satisfies limited liability and $w_1(0,0) + w_2(0,0) < \max_{y \in Y} \{w_1(y) + w_2(y)\}$. Then there exists $\lambda > 0$ and k such that the contract*

$$w_i^{'}(y) = \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k \tag{A.7}$$

satisfies $V_i\left(w_i^{'}, w_j\right) \geq V_i(w)$, moreover it is also eligible and satisfies limited liability.

Proof. From Lemma 46 w_i satisfies equations (B.5) and (B.6). Clearly $w_i^{'}$ satisfies (B.5) as an equality, rearrange it as:

$$\left(y_i - w_i^{'}(y)\right) = k + \lambda\left(w_i^{'}(y) + w_j(y)\right)$$

then let $(F, c) \in A^{\star}(w|\mathcal{A})$ for any $\mathcal{A} \supseteq \mathcal{A}_0$ and taking expectations one gets:

$$E_F\left[y_i - w_i^{'}(y)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \tag{A.8}$$

This applies to any (F, c) under any technology, so this guarantees a payoff for principal i. Note that $w_i^{'}(y) \geq w_i(y)$ for all $y \in Y$ so the agent is always at least as well off under $w_i^{'}$. Moreover, $w_i^{'}$ satisfies limited liability. Then from equations (B.6) and (A.8):

$$E_F\left[y_i - w_i^{'}(y)\right] \geq k + \lambda V_A(w|\mathcal{A}_0) = V_i(w)$$

Since this holds for all $(F, c) \in A^{\star}(w|\mathcal{A})$, by Lemma 45:

$$V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right) = \min_{F \in A^{\star}(w|\mathcal{A})} E_F\left[y_i - w_i^{'}(y)\right] \geq V_i(w)$$

Then $V_i(w)$ is a lower bound for $V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right)$ under arbitrary $\mathcal{A} \supseteq \mathcal{A}_0$.
Thus $V_i\left(w_i^{'}, w_j\right) \geq V_i(w)$ by definition. Finally since w is an eligible contract scheme, so is $\left(w_i^{'}, w_j\right)$. $\square$

Lemma 7. *Let* (w_i, w_j) *be a contract scheme satisfying limited liability strictly:*

$$\min_{y \in Y} \{w_i(y) + w_j(y)\} = \beta > 0.$$

The alternative contract $w_i^{'}(y) = w_i(y) - \beta$ *outperforms* w_i *for principal* i*:* $V_i\left(w_i^{'}, w_j\right) > V_i(w_i, w_j)$.

Proof. Note that by limited liability $\min_{y \in Y} \{w_i(y) + w_j(y)\} = \beta > 0$. Let $w_i^{'}(y) = w_i(y) - \beta$, this contract satisfies limited liability with equality: $w_i^{'}(y) + w_j(y) = 0$. Note that $A^\star\left(\left(w_i^{'}, w_j\right) | \mathcal{A}\right) = A^\star\left((w_i, w_j) | \mathcal{A}\right)$ for all $\mathcal{A} \supseteq \mathcal{A}_0$. This implies $V_i\left(w_i^{'}, w_j\right) = V_i(w_i, w_j) + \beta \geq V_i(w_i, w_j)$. □

From the previous two lemmas we see that an eligible contract that satisfies limited liability is weakly dominated by an LRS contract of the form:

$$w_i(y) = \alpha y_i - (1 - \alpha) w_j(y) - \alpha k_i \qquad \forall y \in Y$$

satisfying limited liability with equality. For an LRS contract to satisfy limited liability with equality it must be that:

$$k = \min_{y \in Y} \{y_i + w_j(y)\}$$

The last two lemmas (37 and 38) establish the form of the principal's payoffs under LRS contracts and the existence of an optimal contract in that class.

Lemma 8. *Let* w *an eligible contract scheme, such that* w_i *is an LRS contract given* w_j *characterized by* $\alpha \in (0, 1]$*. Then:*

$$V_i(w) = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) + k = \max_{(F,c) \in \mathcal{A}_0} \left((1-\alpha) E_F[y_i + w_j(y)] - \frac{1-\alpha}{\alpha} c\right) + \alpha k \quad \text{(A.9)}$$

This also holds for $\alpha = 0$ *if we interpret the term* $\frac{1-\alpha}{\alpha} c$ *as 0 when* $c = 0$ *and* ∞ *for* $c > 0$.

Proof. Let $F \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i(y)]$ by Lemma 45 one has:

$$V_i(w) = E_F[y_i - w_i(y)] = \frac{1-\alpha}{\alpha} E_F[w_1(y) + w_2(y)] + k = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) + k$$

The second equality follows by replacing $V_A(w|\mathcal{A}_0)$. □

Its worthwhile to highlight that when $\alpha = 0$ the LRS contract offsets the other principal's payments to the agent and implies the zero aggregate contract, i.e. $w_i(y) = -w_j(y)$. In this case Lemma (37) gives $V_i(w) = w_j(0,0)$, corresponding to $(F, c) = (\delta_0, 0)$. Note that this is the only optimal action for the agent under any technology under assumption (2) and a zero aggregate contract.

Lemma 9. *In the class of LRS contracts that satisfy limited liability with equality there exists an optimal one for principal i.*

Proof. From Lemma 37 we can express $V_i(w)$ directly as a function of α as in (A.22). Recall that $k = \min_{y \in Y} \{y_i + w_j(y)\}$ is independent of α.
Moreover, The function $(1-\alpha) E_F[y_i + w_j(y)] - \frac{1-\alpha}{\alpha} c$ is continuous in α, thus its maximum over $\mathcal{A}_0$ is continuous as well. Since the RHS in equation (A.22) is continuous in α it achieves a maximum in $[0, 1]$. This α gives the optimal guarantee over all contracts of this class. □

Theorem 1. *For any contract w_j there exists LRS contract $\overline{w}_i$ such that $\overline{w}_i \in BR_i(w_j)$, where $\min_{y \in Y} \{\bar{w}_i(y) + w_j(y)\} = 0$. That is, there is always a LRS contract that is* ***robust*** *for principal i.*

Proof. Consider a contract w_j by the competing principal. By Lemma 38 among the class of LRS contracts satisfying limited liability with equality there is an optimal one, call it $w_i^\star$. There are two cases to consider:

1. The contract $w_i^\star$ is eligible.

   Suppose for a contradiction that there is an arbitrary contract w_i that satisfies limited liability and that does strictly better than $w_i^\star$: $V_i(w_i, w_j) > V_i(w_i^\star, w_j)$. This contract is itself eligible. Then by Lemmas 46, 35 and 36 there exists an LRS contract $w_i^{'}$ that satisfies limited liability with equality such that $V_i\left(w_i^{'}, w_j\right) \geq V_i(w_i, w_j)$. This contradicts $w_i^\star$ being optimal among LRS contracts that satisfy limited liability with equality.

2. The contract $w_i^\star$ is not eligible, i.e. $V_i(w_i^\star, w_j) \leq w_j(0,0)$.

Note that since $w_i(y) = -w_j(y)$ is a LRS contract that satisfies limited liability we know that:

$$V_i(w_i^\star, w_j) \leq w_j(0,0) = V_i((-w_j, w_j)) \leq V_i(w_i^\star, w_j)$$

Then $w^\star$ attains the bound: $V_i(w_i^\star, w_j) = w_j(0,0)$. We now claim that $w_i^\star \in \mathrm{BR}_i(w_j)$, if it were not then there exists a contract w_i that satisfies limited liability and $V_i(w_i, w_j) > V_i(w_i^\star, w_j)$, then this contract is eligible. Just as in the first case this leads to a contradiction of $w_i^\star$ being optimal among LRS contracts that satisfy limited liability with equality.

□

Corollary 1. *If $\mathcal{A}_0$ has the full support property (Assumption 4) then any robust contract for principal i is a LRS contract, or there are no eligible contracts.*

Proof. Consider a contract w_j by the competing principal. Suppose that there exists an eligible contract, then any contract in the best response is eligible. Suppose w_i is an optimal contract for principal i. Define $w_i^{'}$ as in Lemma 35. Note that $w_i^{'}$ satisfies:

$$E_F\left[y_i - w_i^{'}(y)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right)$$

Since w_i satisfies Equation (B.6) from Lemma 46 we can replace for k to obtain:

$$E_F\left[y_i - w_i^{'}(y)\right] \geq V_i(w) + \lambda\left(V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) - V_A((w_i, w_j)|\mathcal{A}_0)\right) \tag{A.10}$$

Because of full support, since $w_i^{'}(y) \geq w_i(y)$ pointwise and any action under $\mathcal{A}_0$ gives a (weakly) higher payoff to the agent under $w_i^{'}$ than under w_i, it follows that

$$V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \geq V_A((w_i, w_j)|\mathcal{A}_0)$$

with strict inequality unless $w_i^{'}$ is identical to w_i.
Since the equation (A.23) holds for all F we get: $V_i\left(w_i^{'}, w_j\right) \geq V_i(w)$, with strict inequality when w_i is not identical to $w_i^{'}$. Then $w_i = w_i^{'}$, or else optimality would be contradicted. Moreover, w_i has to be a LRS contract, or else by Lemma 36 there is a LRS contract that strictly improves on w_i. □

A.1.2 Private Common Agency

In the case of private output we employ the same procedure as in the case of the public common agency with the appropriate changes. To avoid repetition we provide only the statements of the crucial lemma in this new environment.

Lemma 10. *Let w be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$w_i(y_i) \leq \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} \overline{w}_j - \frac{1}{1+\lambda} k \tag{A.11}$$

$$V_i(w) = k + \lambda V_A(w|\mathcal{A}_0) \tag{A.12}$$

where $\overline{w}_j = \max_{y_j \in Y_j} w_j(y_j)$.

As noted earlier in the private common agency framework the principal gives the agent a share of his output and punishes the agent based on the maximum value of the other principal's contract. (i.e. the only difference is the use of $\bar{w}_j$ instead of $w_j(y)$).

Lemmas similar to (35), (36), (37), and (38) provide the optimality of linear contracts. Details are available in the Online Appendix.

A.1.3 Participation and competition in equilibrium

In the discussion above a form of non-participation by the principals is discussed. By conceding rights to the agent over all of her output a principal "opts out" of the game. This contracts gives the principal zero guaranteed payoff, but this form of opting out of the game is not without effects. In this case the remaining principal has to provide incentives for the agent not to work for herself. Thus the form of the LRS contracts is not modified. There is an alternative way of opting out of the game: instead of giving the agent rights over all of her output the principal can play the zero contract, thus not giving the agent any compensation for any action taken.(note that this contract also implies zero guaranteed payoffs). If a principal offers the zero contract then the other principal faces no competition, in the sense that hers are the only incentives the agent gets. The problem reduces to a principal-agent game studied by Carroll (2015). The best response to a zero contract is then:

$$w_j(y_i, y_j) = \theta y_j \qquad \theta = \sqrt{\frac{c}{E_F[y_j]}} \tag{A.13}$$

where (F, c) are such that:

$$(F, c) \in \underset{(F,c)\in\mathcal{A}_0}{\operatorname{argmax}} \left(\sqrt{E_F[y_j]} - \sqrt{c}\right)^2$$

For this to be an equilibrium it must be that the zero contract is a best response to the contract in (A.13); this happens if and only if there are no admissible contracts available to principal i. A sufficient and necessary condition for this, is for the best LRS contract (A.43) to have $\theta_i = 0$, when w_j is as in (A.13). This happens if and only if:

$$E_{F^\star}[y_i + \theta y_j] - c^\star \leq \theta \overline{y}_j$$

for $(F^\star, c^\star)$ such that:

$$(F^\star, c^\star) \in \underset{(F,c)\in\mathcal{A}_0}{\operatorname{argmax}} \left[E_F[y_i + \theta y_j] - c\right]$$

This condition has a simple interpretation: $\theta_i = 0$ is optimal if the agent is better off by inducing $\left(0, \overline{y}_j\right)$ with full probability and zero cost, than under any action available in $\mathcal{A}_0$, when principal i is already giving away the rights to her output. If this is the case then principal i's maximum guaranteed payoff is zero and the zero contract is a best response.

A.2 Existence of Equilibrium - Examples

As shown previously only expected total output is relevant to determine payoffs. This allows us to consider the agent's action as choosing an expected total output x and a cost c. Moreover, as noted earlier, if two actions have the same expected total output the agent will always pick the one with lower cost. These actions form the lower envelope of the action set in the (x, c) space, and define the cost function o the agent:

$$f(x) = \min_{\{(F,c)\in\mathcal{A}_0 | E_F[y_1+y_2]=x\}} c$$

The function f has domain on the set $X = \left\{x \in [0, \overline{y}_1 + \overline{y}_2] \,|\, \exists_{(F,c)\in\mathcal{A}_0} x = E_F[y_1 + y_2]\right\}$. By assumption this is a compact set, let $\bar{x} = \max_{x\in X} x$ and note that the minimum x is always zero.

In the examples below we consider different specifications for f. In all cases we consider the problem of principal i when $w_j = (1-\theta_j) y_j + \theta_j (\overline{y}_i - y_i)$. As shown previously the corresponding LRS contract for principal i is: $w_i = (1-\theta_i) y_i + \theta_i \left(\overline{y}_j - y_j\right)$, where $\theta_i = (1-\alpha)(1-\theta_j)$.

Example 1. ***Constant marginal cost (linear cost)***

Under Assumption 7 the cost function has the form: $f(x) = \gamma x$ for some constant $\gamma > 0$. The value of the agent and his optimal action are:

$$V_A(w|\mathcal{A}_0) = \max_{x\in X} \{ \ ((1-\theta_1-\theta_2) - \gamma) x\} + \theta_1 \overline{y}_2 + \theta_2 \overline{y}_1 \qquad x^\star = \begin{cases} \overline{x} & \text{if } 1-\theta_1-\theta_2 > \gamma \\ 0 & \text{if } 1-\theta_1-\theta_2 < \gamma \\ X & \text{if } 1-\theta_1-\theta_2 = \gamma \end{cases}$$

Then the best response of principal i is characterized by:

$$\text{BR}_i(w_j) = \underset{\theta_i\in[0,1-\theta_j]}{\text{argmax}} \left\{ \begin{cases} \theta_i\left(\overline{x} - \overline{y}_j\right) - \frac{\theta_i}{1-\theta_1-\theta_2}\gamma\overline{x} & \text{if } 1-\theta_1-\theta_2 > \gamma \\ -\theta_i \overline{y}_j & \text{if } 1-\theta_1-\theta_2 \leq \gamma \end{cases} \right\}$$

The function in the first case is strictly concave, its critical value if $\overline{x} > \overline{y}$ is given by:

$$\theta_i^\star = (1-\theta_j) - \sqrt{\frac{(1-\theta_j)\gamma\overline{x}}{\overline{x} - \overline{y}_j}}$$

This is an interior solution if:

$$1-\theta_j-\theta_i^\star>\gamma \qquad \text{and} \qquad 0\le\theta_i^\star\le(1-\theta_j)$$

these conditions are satisfied if and only if: $\frac{\overline{x}-\overline{y}_j}{\overline{x}}>\frac{\gamma}{1-\theta_j}$. This condition amounts to there being enough expected output to pay for the cost of the agent and the fees. The best response of principal i is:

$$\mathrm{BR}_i(\theta_j)=\begin{cases}1-\theta_j-\sqrt{\frac{(1-\theta_j)\gamma\overline{x}}{\overline{x}-\overline{y}_j}} & \text{if} \quad (1-\theta_j)\left(\overline{x}-\overline{y}_j\right)>\gamma\overline{x}\\ 0 & \text{otw}\end{cases}$$

The best response of each principal is then single valued. As before this implies the existence of an equilibrium.

Example 2. ***Constant cost***

Assume now that the agent is indifferent between actions, so that $f(x)=\gamma$, with $\gamma>0$, if $x>0$, and $f(0)=0$. This function is not convex. Since the agent's payoff under LRS contracts is increasing in expected total output, hence the agent will choose to induce the maximum expected total output, as long as it covers the cost γ.

$$x^\star(\theta_1,\theta_2)=\begin{cases}\overline{x} & \text{if} \quad (1-\theta_1-\theta_2)\,\overline{x}>\gamma\\ 0 & \text{otw}\end{cases}$$

Then the best response of principal i is characterized by:

$$\mathrm{BR}_i(w_j)=\underset{\theta_i\in[0,1-\theta_j]}{\operatorname{argmax}}\left\{\begin{cases}\theta_i\left(\overline{x}-\overline{y}_j\right)-\frac{\theta_i}{1-\theta_1-\theta_2}\gamma & \text{if } (1-\theta_1-\theta_2)\,\overline{x}>\gamma\\ -\theta_i\overline{y}_j & \text{if } (1-\theta_1-\theta_2)\,\overline{x}\le\gamma\end{cases}\right\}$$

The function in the first case is strictly concave, its critical value if $\overline{x}>\overline{y}$ is given by:

$$\theta_i^\star=(1-\theta_j)-\sqrt{\frac{(1-\theta_j)\,\gamma}{\overline{x}-\overline{y}_j}}$$

This is an interior solution if:

$$(1-\theta_j-\theta_i^\star)\,\overline{x}>\gamma \qquad \text{and} \qquad 0\le\theta_i^\star\le(1-\theta_j)$$

these conditions are satisfied if and only if: $\overline{x} - \overline{y}_j > \frac{\gamma}{1-\theta_j}$. This condition amounts to there being enough expected output to pay for the cost of the agent and the fees. The best response of principal i is:

$$\mathrm{BR}_i\left(\theta_j\right) = \begin{cases} 1-\theta_j-\sqrt{\frac{(1-\theta_j)\gamma}{\overline{x}-\overline{y}_j}} & \text{if} \quad \left(1-\theta_j\right)\left(\overline{x}-\overline{y}_j\right) > \gamma \\ 0 & \text{otw} \end{cases}$$

The best response of each principal is then single valued. As before this implies the existence of an equilibrium.

A.3 Individual Limited Liability

The model presented in the main text assumes that contracts are subject to common limited liability, so that the contracts satisfy $w_1(y) + w_2(y) \geq 0$ for all $y \in Y$. We can instead think of a stronger requirement and ask that the contract offered by each principal have to satisfy $w_i(y) \geq 0$ for all $y \in Y$ and $i \in \{1, 2\}$. In this case a principal cannot charge the agent, regardless of what the other principal is paying . Unlike previously, there are no equilibrium transfers between principals (through the agent).

Importantly, changing limited liability does not change our analysis on the principal's best response. We show thatLRS contracts are still best responses:

$$w_i(y) = \alpha_i y_i - (1-\alpha_i)\, w_j(y) - \alpha_i k_i \qquad \forall y \in Y \tag{A.14}$$

Theorem 1 of the main text then goes through with just one modification, namely the limited liability requirement. The formal statement and proof are provided at the end of this section. We now state and prove a series of lemmas to establish the result. They follow closely the arguments in the proof of Theorem 1 in the text, with the appropriate modifications.

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition 8. *Let $(F, c) \in A^\star(w|\mathcal{A})$. For $\mathcal{A} \supseteq \mathcal{A}_0$, it holds that:*

$$E_F\left[w_1(y) + w_2(y)\right] \geq V_A\left(w|\mathcal{A}_0\right)$$

Moreover, if $(F,c) \in A^{\star}(w|\mathcal{A})$ *then* $F \in \mathcal{F}$ *where:*

$$\mathcal{F} = \{F \in \Delta(Y) \,|E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0)\}$$

Proof. To see the first inequality let $(F,c) \in A^{\star}(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F[w_1(y) + w_2(y)] \geq E_F[w_1(y) + w_2(y)] - c \geq V_A(w|\mathcal{A}) \geq V_A(w|\mathcal{A}_0)$$

Then $F \in \mathcal{F}$. □

Lemma 11 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above). Note that $\mathcal{F}$ only depends on the contract and the known set of actions $\mathcal{A}_0$. The following results are valid for any scheme w that provides positive guarantees for principal i

We formally define them as follows:

Eligibility: A contract w is *eligible* for principal i if: $V_i(w) > 0$.

Lemma 11. *Let* w *be an eligible contract for principal* i*, then*

$$V_i(w) = \min_{F \in \mathcal{F}} \quad E_F[y_i - w_i(y)]$$

Moreover if $F \in \underset{F \in \mathcal{F}}{argmin} E_F[y_i - w_i(y)]$ *then* $E_F[w_1(y) + w_2(y)] = V_A(w|\mathcal{A}_0)$.

Proof. We first establish the first claim: Let w be an eligible contract scheme then: $V_i(w) = \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$.

It must be that: $V_i(w) \geq \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$. Using the definition of $V_i(w)$:

$$V_i(w) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} \min_{(F,c) \in A^{\star}(w|\mathcal{A})} E_F[y_i - w_i(y)] \geq \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$$

Where the inequality follows because if $(F,c) \in A^{\star}(w|\mathcal{A})$ then $F \in \mathcal{F}$.

To prove equality suppose that

$$V_i(w) > \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$$

and let $F' \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i(y)]$. We have that $E_{F'}[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0)$. There are two options:

1. $F^{'}$ does not place full support on the values of y that maximize $w_1 + w_2$.

   Let $\hat{y} \in \text{argmax}\left\{w_1(y) + w_2(y)\right\}$, and $\hat{F} = \delta_{\hat{y}}$ be a distribution with full mass on $\hat{y}$. Let $\epsilon \in [0,1]$ and $F_\epsilon = (1-\epsilon) F^{'} + \epsilon \hat{F}$.

   Note then that for all ϵ there exists a $\xi_\epsilon > 0$ such that: $E_{F_\epsilon}[w_1(y) + w_2(y)] - \xi_\epsilon > V_A(w|\mathcal{A}_0)$.

   Define and $\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \{(F_\epsilon, \xi_\epsilon)\}$. It follows that the unique optimal action of the agent in $\mathcal{A}_\xi$ is $(F_\epsilon, \xi_\epsilon)$. Then:

   $$V_i(w) \leq V_i(w|\mathcal{A}_\epsilon) = E_{F_\epsilon}[y_i - w_i(y)] = (1-\epsilon) E_{F'}[y_i - w_i] + \epsilon E_{\hat{F}}[y_i - w_i]$$

   This condition holds for all $\epsilon > 0$. Letting $\epsilon \to 0$ we arrive at a contradiction:

   $$V_i(w) \leq E_{F'}[y_i - w_i] = \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i]$$

2. $F^{'}$ places full support on the values of y that maximize $w_1 + w_2$. There are still two possible cases:

   (a) $E_{F'}[w_1 + w_2] > V_A(w|\mathcal{A}_0)$. Then there exists $\xi > 0$ and a technology $\mathcal{A}^{'} = \mathcal{A}_0 \cup \left\{\left(F^{'}, \xi\right)\right\}$ such that $\left(F^{'}, \xi\right)$ is the unique optimal action for the agent in $\mathcal{A}^{'}$. Then we arrive at a contradiction:

   $$V_i(w) \leq V_i\left(w|\mathcal{A}^{'}\right) = E_{F'}[y_i - w_i] = \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i]$$

   (b) $E_{F'}[w_1 + w_2] = V_A(w|\mathcal{A}_0)$. This implies $V_A(w|\mathcal{A}_0) = \max_{y \in Y}\{w_1 + w_2\}$ which can only be satisfied if $F^{'}$ is available in $\mathcal{A}_0$ at zero cost. By the positive cost assumption this implies that $F = \delta_{(0,0)}$ and that $w_1(0,0) + w_2(0,0) = \max_{y \in Y}\{w_1(y) + w_2(y)\}$. In this case the unique optimal action for the agent under any technology is $(\delta_0, 0)$, so the value of the principal is $V_i(w) = -w_i(0,0) \leq 0$, where the inequality follows from limited liability. This contradicts eligibility.

Now we establish the second claim: Let w be an eligible contract scheme for principal i. If $F \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i]$ then $E_F[w_1 + w_2] = V_A(w|\mathcal{A}_0)$.

To prove this, let $F^{'} \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F \left[y_i - w_i \left(y \right) \right]$ and suppose for a contradiction that

$$E_F \left[w_1 \left(y \right) + w_2 \left(y \right) \right] > V_A \left(w | \mathcal{A}_0 \right)$$

.

Let $\epsilon \in [0, 1]$ and define $F_\epsilon = (1 - \epsilon) F^{'} + \epsilon \delta_0$.
For low enough ϵ it holds that:

$$E_{F_\epsilon} \left[w_1 \left(y \right) + w_2 \left(y \right) \right] > V_A \left(w | \mathcal{A}_0 \right)$$

Then there exists $\xi_\epsilon > 0$ such that

$$\left\{ \left(F_\epsilon, \xi_\epsilon \right) \right\} = A^\star \left(w | \mathcal{A}_\epsilon \right)$$

where

$$\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \left\{ \left(F_\epsilon, \xi_\epsilon \right) \right\}$$

The payoff to the principal is then:

$$V_i \left(w | \mathcal{A}_\epsilon \right) = (1 - \epsilon) E_F \left[y_i - w_i \left(y \right) \right] + \epsilon \left(-w_i \left(0, 0 \right) \right) \leq (1 - \epsilon) E_F \left[y_i - w_i \left(y \right) \right]$$
$$= (1 - \epsilon) V_i \left(w \right) < V_i \left(w \right)$$

This gives a contradiction. □

Given the known action set $\mathcal{A}_0$, the next lemma links the principal's guaranteed payoff to the agent's payoff in an affine way. This link allows the principal to increase her own guaranteed payoff by controlling the payoff given to the agent. Lemma 12 also offers a relation between any contract w_i, the outcome y_i and the contract w_j offered by the other principal.
The affine link between the agent's payoff and the principal's payoff is a crucial element in providing incentives for the agent. Given the partial knowledge over the agent's set of actions the principals' optimal strategy is to tie their payoff to that of the agent, thus aligning the agent's objectives with their own. This is the same mechanism at the heart of the optimal contracts in Hurwicz and Shapiro (1978) and Carroll (2015), and will be crucial in establishing the optimality of affine (LRS) contracts in the setting we develop.

Lemma 12. *Let w be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$
\begin{aligned}
w_i(y) &\leq \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k && \text{(A.15)} \\
V_i(w) &= k + \lambda V_A(w|\mathcal{A}_0) && \text{(A.16)}
\end{aligned}
$$

Proof. This lemma is proven with the following two propositions. In both propositions define:

Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points $(w_1(y) + w_2(y), y_i - w_i(y))$ for $y \in Y$.

Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u > V_A(w|\mathcal{A}_0)$ and $v < V_i(w)$.

Proposition 9. $S \cap T = \emptyset$.

Proof. Let $(u, v) \in T$ then let $F \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i(y)]$, by definition of T and Lemma (11):

$$
\begin{aligned}
u &> V_A(w|\mathcal{A}_0) = E_F[w_1(y) + w_2(y)] \\
v &< V_i(w) = E_F[y_i - w_i(y)]
\end{aligned}
$$

now, suppose for a contradiction that $(u, v) \in S$, then there exists $F' \in \Delta(Y)$ such that:

$$
u = E_{F'}[w_1(y) + w_2(y)] \qquad \text{and} \qquad v = E_{F'}[y_i - w_i(y)]
$$

Note that F' guarantees a payoff to the agent larger than $V_A(w|\mathcal{A}_0)$ so $F' \in \mathcal{F}$ but:

$$
E_F[y_i - w_i(y)] > E_{F'}[y_i - w_i(y)]
$$

which contradicts minimality of F. Then $S \cap T = \emptyset$ □

Since $S \cap T = \emptyset$ we can apply the separating hyperplane theorem which implies that there exist constants (k, λ, μ) such that $(\lambda, \mu) \neq (0, 0)$ and:

$$
k + \lambda u - \mu v \leq 0 \qquad (u, v) \in S \tag{A.17}
$$

$$
k + \lambda u - \mu v \geq 0 \qquad (u, v) \in T \tag{A.18}
$$

Let $F^\star \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i(y)]$.

Note that the pair $(E_{F^\star}[w_1(y)+w_2(y)], E_{F^\star}[y_i - w_i(y)])$ lies in the closures of both S and T. Then:

$$k + \lambda E_{F^\star}[w_1(y)+w_2(y)] - \mu E_{F^\star}[y_i - w_i(y)] = 0 \tag{A.19}$$

It is left to show that $\lambda, \mu > 0$.

Note that $(u,v) \in T$ admits u arbitrarily high and v arbitrarily low. So for (A.18) to hold it must be that $\lambda \geq 0$ and $\mu \geq 0$. There are then two cases to rule out:

1. Suppose $\mu = 0$, then it must be that $\lambda > 0$ (since $(\lambda,\mu) \neq (0,0)$). From (A.17) and (A.18)

$$u \leq -\frac{k}{\lambda} \quad (u,v) \in S \qquad \text{and} \qquad u \geq -\frac{k}{\lambda} \quad (u,v) \in T$$

So $\max_{y\in Y}[w_1(y)+w_2(y)] = \max_{u\in S} u \leq -\frac{k}{\lambda} \leq \inf_{u\in T} u = V_A(w|\mathcal{A}_0)$. Which implies:

$$\max_{y\in Y}[w_1(y)+w_2(y)] = V_A(w|\mathcal{A}_0)$$

This can only happen if the agent has an action $(F,0) \in \mathcal{A}_0$ such that

$$E_F[w_1(y)+w_2(y)] = \max[w_1(y)+w_2(y)]$$

the only action in $\mathcal{A}_0$ with zero cost is $(\delta_0, 0)$, so

$$\max(w_1(y)+w_2(y)) = w_1(0,0)+w_2(0,0)$$

. This is also the unique action in $A^\star(w|\mathcal{A}_0)$ so:

$$V_i(w) \leq V_i(w|\mathcal{A}_0) = -w_i(0,0) \leq 0$$

This violates eligibility $(V_i(w) > 0)$.

2. Suppose $\lambda = 0$, then it must be that $\mu > 0$ (since $(\lambda,\mu) \neq (0,0)$). From (A.17) and (A.18)

$$v \geq \frac{k}{\mu} \quad (u,v) \in S \qquad \text{and} \qquad v \leq \frac{k}{\mu} \quad (u,v) \in T$$

So $\min_{y\in Y}[y_i - w_i(y)] = \min_{v\in S} v \geq \frac{k}{\mu} \geq \sup_{v\in T} v = V_i(w)$.

But we know that $\min_{y\in Y}[y_i - w_i(y)] \leq 0 - w(0,0) \leq 0$ this implies $V_i(w) \leq 0$ which contradicts eligibility. So $\lambda > 0$.

Note that since λ and μ are greater than zero μ can be normalized to 1, giving from (A.17):

$$k + \lambda \left(w_i\left(y\right) + w_j\left(y\right)\right) - \left(y_i - w_i\left(y\right)\right) \leq 0$$

And from (A.19):

$$V_i\left(w\right) = k + \lambda V_A\left(w|\mathcal{A}_0\right)$$

□

The following two lemmas (13 and 14) use the relation between the principals' contracts derived in Lemma 12 to construct an alternative contract that dominates the original one, in the sense that it weakly increases principal i's guaranteed payoff. Since the relation obtained in (A.15) is affine in output and the other principal's contract, the alternative contract constructed below will inherit that form. These contracts form the LRS contracts defined in (A.43).

Lemma 13. *Let $w = (w_i, w_j)$ with w_i satisfying (A.15) and (A.16). Then the contract*

$$w_i^{'}\left(y\right) = \frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j\left(y\right) - \frac{1}{1+\lambda}k \tag{A.20}$$

satisfies $V_i\left(w_i^{'}, w_j\right) \geq V_i\left(w\right)$.

Proof. Clearly $w_i^{'}$ satisfies (A.15) as an equality, rearrange it as:

$$\left(y_i - w_i^{'}\left(y\right)\right) = k + \lambda\left(w_i^{'}\left(y\right) + w_j\left(y\right)\right)$$

then let $(F, c) \in A^{\star}\left(w|\mathcal{A}\right)$ for any $\mathcal{A} \supseteq \mathcal{A}_0$ and taking expectations one gets:

$$E_F\left[y_i - w_i^{'}\left(y\right)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \tag{A.21}$$

This applies to any (F, c) under any technology, so this guarantees a payoff for principal i. Note that $w_i^{'}\left(y\right) \geq w_i\left(y\right)$ for all $y \in Y$ so the agent is always at least as well off under $w_i^{'}$. Then from equations (A.16) and (A.21):

$$E_F\left[y_i - w_i^{'}\left(y\right)\right] \geq k + \lambda V_A\left(w|\mathcal{A}_0\right) = V_i\left(w\right)$$

Since this holds for all $(F, c) \in A^{\star}\left(w|\mathcal{A}\right)$, by Lemma 11:

$$V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right) = \min_{F \in A^{\star}(w|\mathcal{A})} E_F\left[y_i - w_i^{'}\left(y\right)\right] \geq V_i\left(w\right)$$

Then $V_i(w)$ is a lower bound for $V_i\left(\left(w_i', w_j\right)|\mathcal{A}\right)$ under arbitrary $\mathcal{A} \supseteq \mathcal{A}_0$.
Thus $V_i\left(w_i', w_j\right) \geq V_i(w)$ by definition. □

Lemma 14. *Let $\left(w_i', w_j\right)$ with w_i' an affine contract on y_i and w_j, there is an affine contract w_i'' that does at least as well as w_i' for principal i: $V_i\left(w_i'', w_j\right) \geq V_i\left(w_i', w_j\right)$, with strict inequality unless $\min_y w_i'(y) = 0$.*

Proof. Note that by limited liability $\min_y w_i'(y) \geq 0$ let $\beta = \min_y w_i'(y)$ and $w_i''(y) = w_i'(y) - \beta$ which is a valid contract $\left(w_i''(y) \geq 0\right)$ and is affine on y_i and w_j. Note that $A^\star\left(\left(w_i'', w_j\right)|\mathcal{A}\right) = A^\star\left(\left(w_i', w_j\right)|\mathcal{A}\right)$ for all $\mathcal{A} \supseteq \mathcal{A}_0$. This implies $V_i\left(w_i'', w_j\right) \geq V_i\left(w_i', w_j\right)$, with strict inequality if $\beta > 0$. □

The last two lemmas (15 and 16) establish the form of the principal's payoffs under LRS contracts and the existence of an optimal contract in that class.

Lemma 15. *For w an eligible contract scheme such that w_i is an LRS contract given w_j satisfying limited liability with equality. w_i is characterized by $\alpha \in (0,1]$. Then:*

$$V_i(w) = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) + k = \max_{(F,c)\in\mathcal{A}_0} \left((1-\alpha) E_F[y_i + w_j(y)] - \frac{1-\alpha}{\alpha} c\right) + \alpha k \quad \text{(A.22)}$$

This also holds for $\alpha = 0$ if we interpret the term $\frac{1-\alpha}{\alpha} c$ as 0 when $c = 0$ and ∞ for $c > 0$.

Proof. Let $F \in \underset{F\in\mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i(y)]$ by Lemma 11 one has:

$$V_i(w) = E_F[y_i - w_i(y)] = \frac{1-\alpha}{\alpha} E_F[w_1(y) + w_2(y)] + k = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) + k$$

The second equality follows by replacing $V_A(w|\mathcal{A}_0)$. □

Lemma 16. *In the class of LRS contracts that satisfy limited liability with equality there exists an optimal one for principal i.*

Proof. From Lemma 15 we can express $V_i(w)$ directly as a function of α as in (A.22). Recall that $k(\alpha) = \min_y \left[y_i - \frac{1-\alpha}{\alpha} w_j(y)\right]$ is function is continuous in α for a given w_j. Moreover, The function $(1-\alpha) E_F[y_i + w_j(y)] - \frac{1-\alpha}{\alpha} c$ is continuous in α, thus its maximum over $\mathcal{A}_0$ is continuous as well. Since the RHS in equation (A.22) is continuous in α it achieves a maximum in $[0,1]$. This α gives the optimal guarantee over all contracts of this class. □

Theorem 2. *For any contract w_j there exists LRS contract $\overline{w}_i$ such that $\overline{w}_i \in BR_i(w_j)$, where $\min_y \overline{w}_i(y) = 0$. That is, there is always a LRS contract that is* ***robust*** *for principal i.*

Proof. By Lemma 16 among the class of LRS contracts there is an optimal one, call it $w_i^\star$. Suppose there is an arbitrary contract w_i that does strictly better than $w_i^\star$: $V_i(w_i, w_j) > V_i(w_i^\star, w_j)$. Note that $V_i(w_i^\star, w_j) \geq V_i(y_i, w_j) \geq 0$. Hence it must be the case that w_i is eligible, i.e. $V_i(w_i, w_j) > 0$. Then by Lemmas 12, 13 and 14 there exists an LRS contract $w_i^{'}$ such that $V_i\left(w_i^{'}, w_j\right) \geq V_i(w_i, w_j)$. This contradicts $w_i^\star$ being optimal among LRS contracts. □

Corollary 2. *If $\mathcal{A}_0$ has the full support property then any robust contract for principal i is a LRS contract, or she cannot guarantee a positive payoff.*

Proof. Suppose w_i is an optimal contract for principal i and define $w_i^{'}$ as in Lemma 13. Note that $w_i^{'}$ satisfies:

$$E_F\left[y_i - w_i^{'}(y)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right) | \mathcal{A}_0\right)$$

Since w_i satisfies Equation (A.16) from Lemma 12 we can replace for k to obtain:

$$E_F\left[y_i - w_i^{'}(y)\right] \geq V_p(w) + \lambda\left(V_A\left(\left(w_i^{'}, w_j\right) | \mathcal{A}_0\right) - V_A\left((w_i, w_j) | \mathcal{A}_0\right)\right) \tag{A.23}$$

Because of full support, since $w_i^{'}(y) \geq w_i(y)$ pointwise and any action under $\mathcal{A}_0$ gives a (weakly) higher payoff to the agent under $w_i^{'}$ than under w_i, it follows that

$$V_A\left(\left(w_i^{'}, w_j\right) | \mathcal{A}_0\right) \geq V_A\left((w_i, w_j) | \mathcal{A}_0\right)$$

with strict inequality unless $w_i^{'}$ is identical to w_i.

Since the equation (A.23) holds for all F we get: $V_i\left(w_i^{'}, w_j\right) \geq V_i(w)$, with strict inequality when w_i is not identical to $w_i^{'}$. Then $w_i = w_i^{'}$, or else optimality would be contradicted. Moreover, w_i has to be a LRS contract, or else by Lemma 14 there is a LRS contract that strictly improves on w_i. □

A.3.1 Existence of Equilibrium - Examples

In the examples below we consider different specifications for f. In all cases we consider the problem of principal i when $w_j = (1-\theta_j)\, y_j + \theta_j\,(\overline{y}_i - y_i)$. As shown in Lemma 11 of Appendix A.2 in the main text, the corresponding LRS contract for principal i is: $w_i = (1-\theta_i)\, y_i + \theta_i\left(\overline{y}_j - y_j\right)$, where $\theta_i = (1-\alpha)(1-\theta_j)$.

Example 3. ***Constant marginal cost (linear cost)***

To better understand the determinants of the share θ we consider the case where the agent's production technology exhibits constant marginal cost of production in total output. Note that if two actions have the same expected total output the agent will always pick the one with lower cost. Below we assume that this lowest cost is a constant fraction of total expected surplus. We formalize this notion in the following assumption

Assumption 7. *For any $x \in [0, \bar{y}_1 + \bar{y_2}]$ there exists $(F,c) \in \mathcal{A}_0$ such that $E_F[y_1+y_2] = x$ and*

$$\gamma x = \min\left\{c \mid (F,c) \in \mathcal{A}_0 \text{ and } E_F[y_1+y_2] = x\right\}$$

where $\gamma < 1$ is the marginal cost.

Note that this allows to replace the maximization of the agent over $(F,c) \in \mathcal{A}_0$ with one over the expected value of total output $x \in [0, \bar{y}_1 + \bar{y}_2]$. Under Assumption 7 we can characterize the equilibrium strategies of the principals and the agent.

Proposition 10. *Under Assumption 7 if principal j plays the contract $w_j(y) = (1-\theta_j)\, y_j + \theta_j\,(\overline{y}_i - y_i)$ for some $\theta_j \in [0,1]$, then principal i best responds with a contract of the form $w_i(y) = (1-\theta_i)\, y_i + \theta_i\left(\overline{y}_j - y_j\right)$ with:*

$$\theta_i = \begin{cases} (1-\theta_j) - \sqrt{(1-\theta_j)\,\gamma\frac{\overline{y}_1+\overline{y}_2}{\overline{y}_i}} & \text{if } \theta_j < 1-\gamma\frac{\overline{y}_1+\overline{y}_2}{\overline{y}_i} \\ 0 & otw \end{cases} \tag{A.24}$$

Moreover, an equilibrium exists and in equilibrium , if the true technology is $\mathcal{A}_0$, the agent chooses (F,c) such that $E_F[y_1+y_2] = \overline{y}_1 + \overline{y}_2$ and $c = \gamma\,(\overline{y}_1+\overline{y}_2)$.

Proof. Under Assumption 7 the cost function has the form: $f(x) = \gamma x$ for some constant $\gamma > 0$. The value of the agent and his optimal action are:

$$V_A(w|\mathcal{A}_0) = \max_{x \in X} \{ ((1-\theta_1-\theta_2)-\gamma)x\} + \theta_1\overline{y}_2 + \theta_2\overline{y}_1 \qquad x^\star = \begin{cases} \overline{x} & \text{if } 1-\theta_1-\theta_2 > \gamma \\ 0 & \text{if } 1-\theta_1-\theta_2 < \gamma \\ X & \text{if } 1-\theta_1-\theta_2 = \gamma \end{cases}$$

□

Then the best response of principal i is characterized by:

$$\text{BR}_i(w_j) = \underset{\theta_i \in [0,1-\theta_j]}{\text{argmax}} \left\{ \begin{cases} \theta_i\left(\overline{x}-\overline{y}_j\right) - \frac{\theta_i}{1-\theta_1-\theta_2}\gamma\overline{x} & \text{if } 1-\theta_1-\theta_2 > \gamma \\ -\theta_i\overline{y}_j & \text{if } 1-\theta_1-\theta_2 \leq \gamma \end{cases} \right\}$$

The function in the first case is strictly concave, its critical value if $\overline{x} > \overline{y}$ is given by:

$$\theta_i^\star = (1-\theta_j) - \sqrt{\frac{(1-\theta_j)\gamma\overline{x}}{\overline{x}-\overline{y}_j}}$$

This is an interior solution if:

$$1-\theta_j-\theta_i^\star > \gamma \qquad \text{and} \qquad 0 \leq \theta_i^\star \leq (1-\theta_j)$$

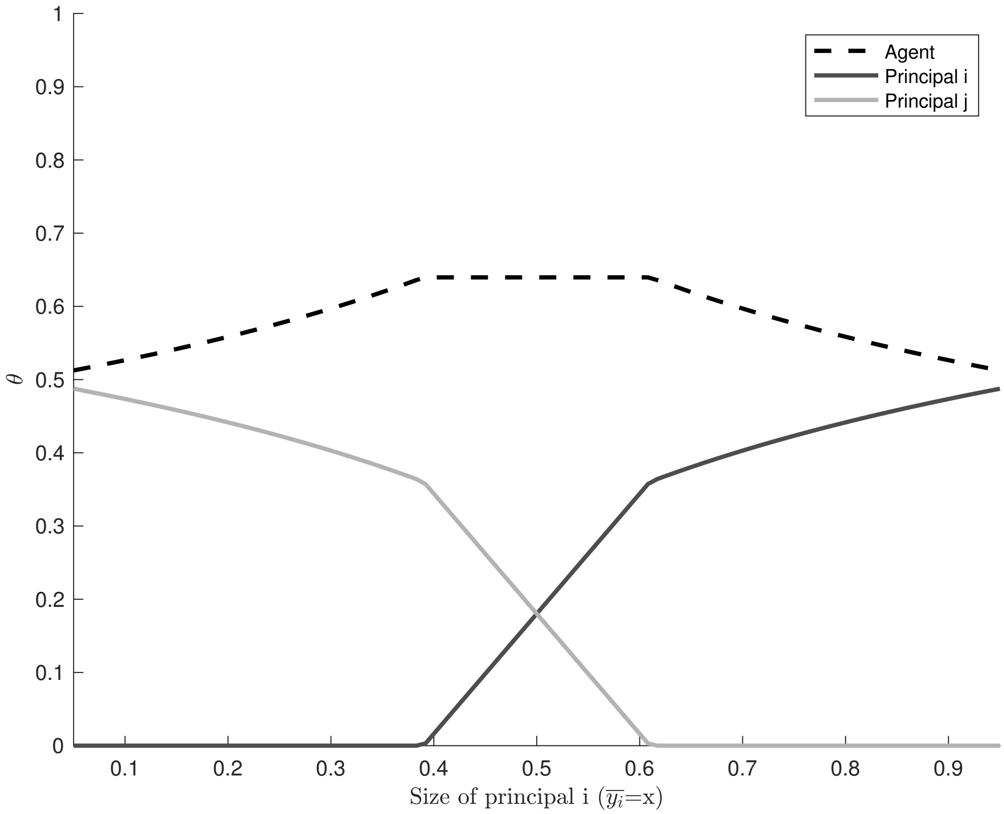
these conditions are satisfied if and only if: $\frac{\overline{x}-\overline{y}_j}{\overline{x}} > \frac{\gamma}{1-\theta_j}$. This condition amounts to there being enough expected output to pay for the cost of the agent and the fees. The best response of principal i is:

$$\text{BR}_i(\theta_j) = \begin{cases} 1-\theta_j-\sqrt{\frac{(1-\theta_j)\gamma\overline{x}}{\overline{x}-\overline{y}_j}} & \text{if} \quad (1-\theta_j)\left(\overline{x}-\overline{y}_j\right) > \gamma\overline{x} \\ 0 & \text{otw} \end{cases}$$

The best response of each principal is then single valued. As in Lemma 11 of Appendix A.2 in the main text, this implies the existence of an equilibrium.

Recall that when $\theta_i = 0$ principal i's guaranteed payoff, V_i, is zero as well. If this is the case in equilibrium we say that the principal has been driven out of the game. Effectively the principal renounces her output by setting $w_i(y) = y_i$. In particular, we see from equation (A.24) that if $\overline{y}_i < \gamma(\overline{y}_1 + \overline{y}_2)$ the principal cannot guarantee herself a positive payoff,

Figure A.1: Constant Marginal Cost of Production - Share of total output - by principal size



regardless of θ_j. For a principal to be able to profit in the game, she must be able to cover the (total) production cost of the agent. Clearly, when $w_i(y) = y_i$, the principal can always opt for the zero contract ($w_i(y) = 0$). This is another way to opt out of the game since the principal cannot guarantee herself a positive payoff without incentivizing the agent.

The figures show the equilibrium of the game in LRS contracts under Assumption 7. We let $\bar{y}_1 = \overline{x}$ and $\bar{y}_2 = 1 - \overline{x}$. Figures A.1 and A.3 vary $\overline{x}$ and fix $\gamma = 1/4$. Figures A.2 and A.4 vary γ and fix $\overline{x} = 1/2$.

We can now analyze the equilibrium contracts and payoffs for different values $(\overline{y}_1, \overline{y}_2)$ and γ. This allows for determining the effect of changes in competitor's size (Figures A.1, A.3) and productivity (Figures A.2, A.4) on the equilibrium outcomes.

The share of output that a principal can appropriate for herself decreases as her competitor

Figure A.2: Constant Marginal Cost of Production - Share of total output - marginal cost

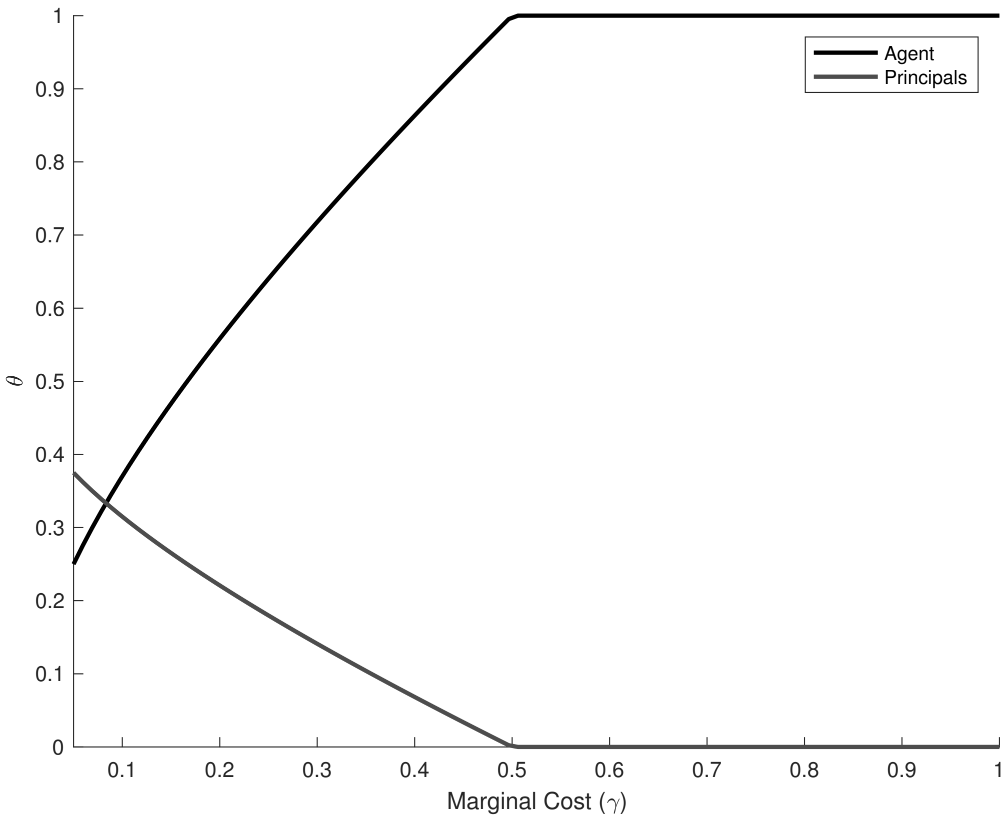


Figure A.3: Constant Marginal Cost of Production - Guaranteed Surplus - by principal size

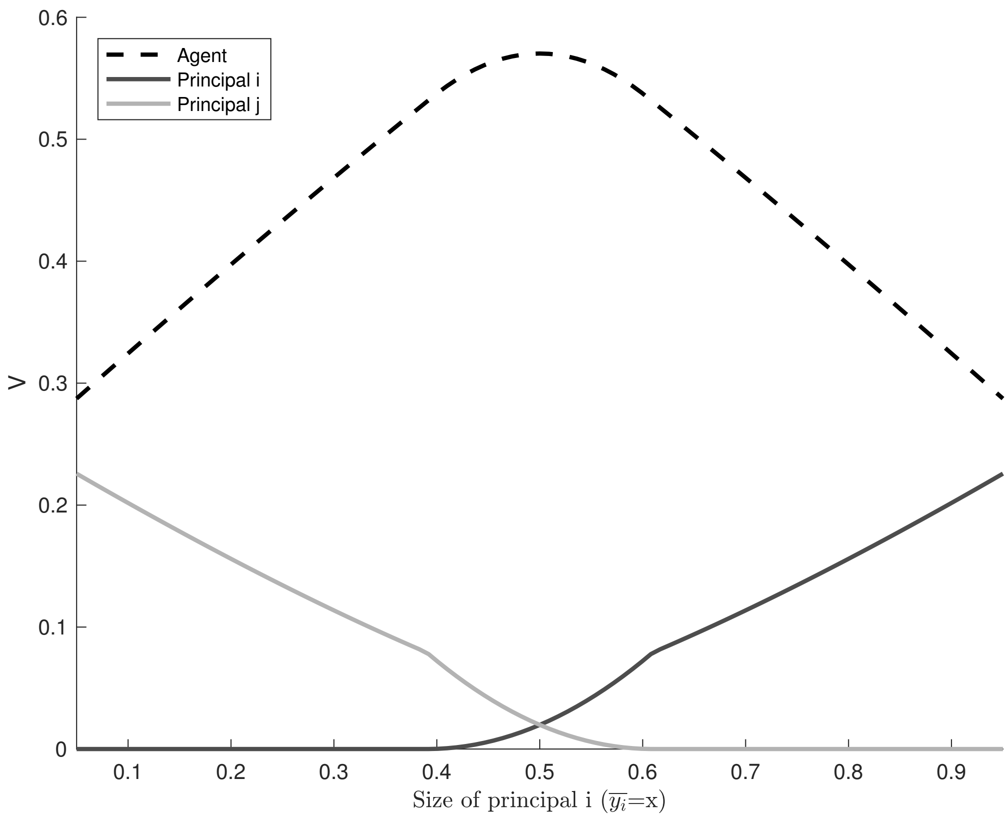
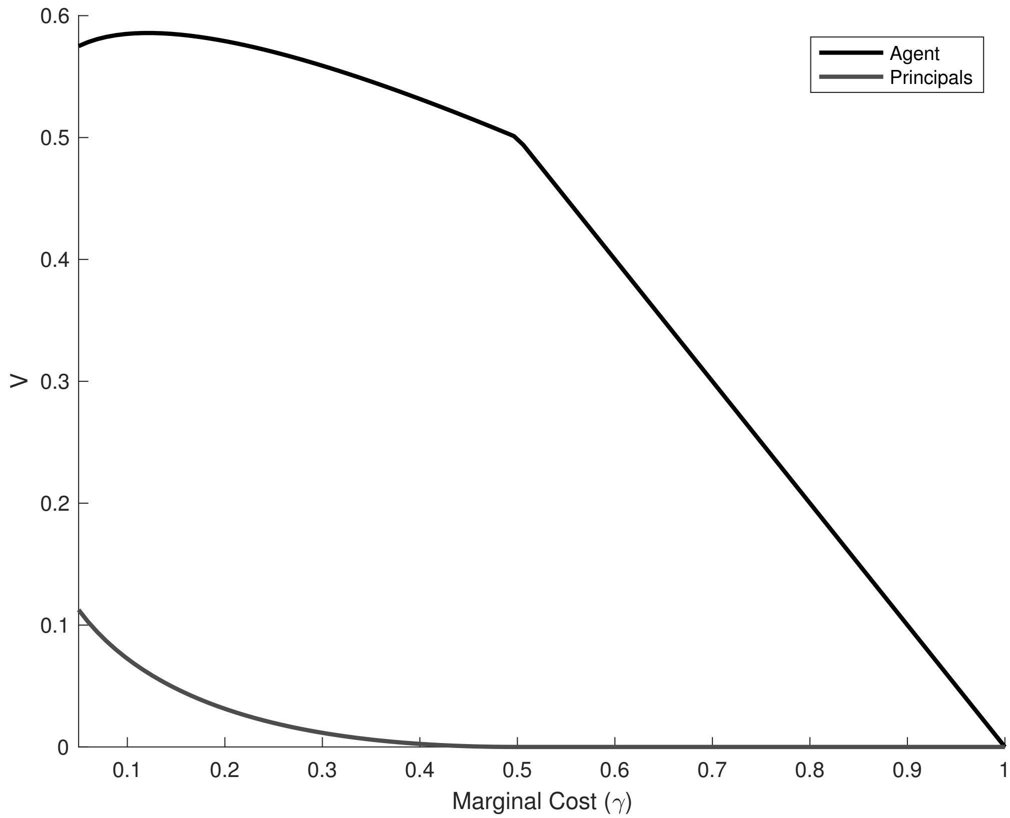


Figure A.4: Constant Marginal Cost of Production - Guaranteed Surplus - Marginal cost



becomes larger. Eventually if a principal is too small relative to her competition she is driven driven out of the game and cannot guarantee any positive payoffs. When a principal is relatively large she opts for increasing her share of total output, this lowers the share of the agent but increases his fees. As the productivity of the agent goes down (γ increases), higher incentives are needed to induce him to produce. This is achieved by reducing the share of output going to the principals. Eventually both principals end up giving up their output in equilibrium. In this case, competition drives their guaranteed payoffs to zero.

Example 4. ***Constant cost***

Assume now that the agent is indifferent between actions, so that $f(x) = \gamma$, with $\gamma > 0$, if $x > 0$, and $f(0) = 0$. This function is not convex. Since the agent's payoff under LRS contracts is increasing in expected total output, hence the agent will choose to induce the maximum expected total output, as long as it covers the cost γ.

$$x^{\star}(\theta_1, \theta_2) = \begin{cases} \overline{x} & \text{if} \quad (1-\theta_1-\theta_2)\,\overline{x} > \gamma \\ 0 & \text{otw} \end{cases}$$

Then the best response of principal i is characterized by:

$$\mathrm{BR}_i(w_j) = \underset{\theta_i \in [0, 1-\theta_j]}{\operatorname{argmax}} \left\{ \begin{cases} \theta_i\left(\overline{x} - \overline{y}_j\right) - \frac{\theta_i}{1-\theta_1-\theta_2}\gamma & \text{if } (1-\theta_1-\theta_2)\,\overline{x} > \gamma \\ -\theta_i \overline{y}_j & \text{if } (1-\theta_1-\theta_2)\,\overline{x} \leq \gamma \end{cases} \right\}$$

The function in the first case is strictly concave, its critical value if $\overline{x} > \overline{y}$ is given by:

$$\theta_i^{\star} = (1-\theta_j) - \sqrt{\frac{(1-\theta_j)\,\gamma}{\overline{x} - \overline{y}_j}}$$

This is an interior solution if:

$$(1-\theta_j-\theta_i^{\star})\,\overline{x} > \gamma \qquad \text{and} \qquad 0 \leq \theta_i^{\star} \leq (1-\theta_j)$$

these conditions are satisfied if and only if: $\overline{x} - \overline{y}_j > \frac{\gamma}{1-\theta_j}$. This condition amounts to there being enough expected output to pay for the cost of the agent and the fees. The best response of principal i is:

$$\mathrm{BR}_i(\theta_j) = \begin{cases} 1 - \theta_j - \sqrt{\frac{(1-\theta_j)\gamma}{\overline{x}-\overline{y}_j}} & \text{if} \quad (1-\theta_j)\left(\overline{x} - \overline{y}_j\right) > \gamma \\ 0 & \text{otw} \end{cases}$$

The best response of each principal is then single valued. As in Lemma 11 of Appendix A.2 in the main text, this implies the existence of an equilibrium.

A.4 Multiple Principals

The model considered in the main text can be extended to multiple principals. Our main result is preserved in this case. Letting N be the number of principals we have:

$$\mathrm{BR}_i(w_{-i}) = \underset{w_i}{\mathrm{argmax}} V_i(w_i, w_{-i}) \tag{A.25}$$

where $w_{-i}(y) = (w_1(y), \ldots, w_{i-1}(y), w_{i+1}(y), \ldots, w_N(y))$.

Theorem 3. *For any set of contracts w_{-i}, there exists an LRS contract $\overline{w}_i$ such that $\overline{w}_i \in BR_i(w_j)$, where $\min_{y\in Y}\left\{\bar{w}_i(y) + \sum_{j\neq i} w_j(y)\right\} = 0$ or $\min_{y\in Y}\{\bar{w}_i(y)\} = 0$ according to limited liability. That is, there is always a LRS contract that is **robust** for principal i.*
If $\mathcal{A}_0$ satisfies the full support property, then any robust contract for principal i is a LRS contract or principal i cannot guarantee a payoff higher than $\sum_{j\neq i} w_j(0,0)$ or 0, according to limited liability.

Proof. The proof is virtually identical to that of Theorem 1 in the main text. Lemmas 45 to 38 follow by defining the aggregate competing contract $w^c(y) = \sum_{j\neq i} w_j(y)$. □

We can characterize them as in Propositions 1 and 3 of the main text, depending on the limited liability restrictions:

Proposition 11. *Let w be a LRS contract scheme satisfying limited liability with equality. There exist $(\theta_1, ..., \theta_N)$ and $(k_1, \ldots, k_N)$ such that the for all $i \in \{1, 2, \ldots, N\}$ contracts are:*

$$w_i(y) = (1-\theta_i)\, y_i - \theta_i \sum_{j\neq i} y_j - k_i \qquad \textit{Liminted Liability} \tag{A.26}$$

$$w_i(y) = (1-\theta_i)\, y_i + \theta_i \sum_{j\neq i} \left(\overline{y}_j - y_j\right) \qquad \textit{Individual Limited Liability} \tag{A.27}$$

where $\sum_{i=1}^{N} k_i = 0$ *when limited liability is placed over the aggregate payment to the agent.* θ_i *is the share of total output and total guaranteed surplus going to principal* i *in equilibrium. Guaranteed surplus is computed relative to the payoffs under inaction.*

Proof. By Theorem 3 there is a LRS contract in the best response of each principal. Then consider contracts of the following form for all i:

$$w_i(y) = y_i - \frac{1-\alpha_i}{\alpha_i} \sum_{j=1}^{n} w_j(y) - k_i$$

Letting $\beta_i = \frac{1-\alpha_i}{\alpha_i}$ we obtain the following expression for the sum of contracts:

$$\sum_{i=1}^{n} w_i(y) = \frac{\sum (y_i - k_i)}{1 + \sum \beta_i}$$

When limited liability is placed over the aggregate payment to the agent this implies $\sum_{i=1}^{N} k_i = 0$. Replacing into the contract we get:

$$w_i(y) = y_i - \frac{\beta_i}{1 + \sum \beta_i} \sum (y_i - k_i) - k_i$$

When limited liability applies to each individual contract it must be that $\min w_i(y) = 0$, the minimum is achieved when $y_i = 0$ and $y_j = \overline{y}_j$ for $j \neq i$, then one can solve for k_i:

$$k_i = -\frac{\beta_i}{1 + \sum \beta_i} \sum_{j \neq i} \overline{y}_j + \frac{\beta_i}{1 + \sum \beta_i} \left(\sum k_i \right)$$

Replacing one last time we get the equilibrium wage and defining $\theta_i = \frac{\beta_i}{1+\sum \beta_i}$:

$$w_i(y) = (1 - \theta_i) y_i + \theta_i \sum_{j \neq i} \left(\overline{y}_j - y_j \right) \tag{A.28}$$

From Lemma 37 we can establish that the share of total guaranteed surplus going to principal i in equilibrium is equal to θ_i. To see this note from equilibrium contract, equation (A.28), that principal i's payoff given inaction is $-\frac{\beta_i}{1+\sum \beta_i} \sum_{j \neq i} \overline{y}_j$ and that total surplus given inaction is by construction zero. Then we have:

$$\theta_i = \frac{V_i(w) + \frac{\beta_i}{1+\sum \beta_i} \sum_{j \neq i} \overline{y}_j}{\sum_i V_i(w) + V_A(w|\mathcal{A}_0)} = \frac{\beta_1 V_A(w|\mathcal{A}_0) + k_1 + \frac{\beta_i}{1+\sum \beta_i} \sum_{j \neq i} \overline{y}_j}{(1 + \sum \beta_i) V_A(w|\mathcal{A}_0) + \sum k_i} = \frac{\beta_i}{1 + \sum \beta_i}$$

□

In order to further characterize the equilibrium we first present the best response in LRS contracts of principal i, given LRS contracts played by the other principals, characterized by θ_{-i}:

$$\mathrm{BR}_i\left(\theta_{-i}\right) = \underset{\theta_i \in [0,1]}{\operatorname{argmax}} \left[\max_{(F,c)\in\mathcal{A}_0} \left\{ E_F \left[\theta_i \left(\sum y_i - \sum_{j\neq i} \overline{y}_j \right) \right] - \frac{c}{1-\sum \theta_i} \right\} \right] \tag{A.29}$$

From the FOC of the principal's problem we can obtain an expression for θ_i given θ_{-i} and a pair (F, c):

$$\left(1+\theta_i\right)\Gamma_i = \frac{1}{1-\sum \theta_i}$$

where $\Gamma_i = \frac{E_F[\sum y_i] - \sum_{j\neq i} \overline{y}_j}{c}$ and (F, c) are maximizers of $V_A(w)$. As before $\Gamma_i > 0$ it is necessary for the principal to have an interior solution. This implies that $\mathcal{A}_0$ must be such that there exists a pair (F, c) that satisfies:

$$E_F\left[y_i\right] > \sum_{j\neq i} E_F\left[\overline{y}_j - y_j\right]$$

This condition is stronger than non-triviality and increasingly difficult to satisfy as the number of principals increases.

To compute an interior equilibrium where a subset I of n_I principals have $\theta_i \in (0,1)$, and for all principal $k \notin I$ $\theta_i = 0$, note that the equilibrium condition above induces a linear system of $n_I - 1$ equations, holding $i \in I$ fixed these equations are of the form:

$$\left(1-\theta_j\right) = \frac{\Gamma_i}{\Gamma_j}\left(1-\theta_i\right)$$

for $j \in I$. Then we get:θ. For this to be an interior equilibrium $\theta_i \in (0,1)$ it is needed that:

$$\frac{1}{\Gamma_i^2} + \frac{1}{\Gamma_i}\left(n_I - 1\right) \leq \left(\sum_{i\in I} \frac{1}{\Gamma_i}\right)$$

To get a sense of this expression it is instructive to consider the case of a symmetric solution, then $\Gamma_i = \Gamma_j$ and the expression is reduced to $\Gamma_i \geq 1$.

A.5 Double Limited Liability

As mentioned before our equilibrium contracts require principals to pay a fee to the agent. This fee depends on the maximum potential payment that other principals can make, thus, in equilibrium, principals offer potentially large payments to the agent. These payments are motivated by competition among the principals. This form of competition can lead to solutions where both principals force each other to up their payments and reduce their final payoff. This has two practical implications: first, principals can have negative ex post payoffs; second, one principal can try to drive out the other by increasing her own payments to the agent. Both these implications can be dealt with by introducing limited liability on the principals. We show now that the core of our results does not rely on the principals to offer unbounded rewards to the agent.

Imposing limited liability on the principals amounts to restricting contracts so that $y_i - w_i(y) \geq 0$ for all $y \in Y$. Under this extra assumption only the definition of LRS contracts changes, adding a cap to the amount that the principal can pay to the agent.

Linear Revenue Sharing contracts (Principal's limited liability): A contract w_i is a LRS contract for principal i if, given a competing contract w_j, it ties the principal's ex-post payoff linearly to the total revenue of the agent. That is, for some $\alpha \in (0,1]$ and $k \in \mathbb{R}$:

$$y_i - w_i(y) = \min\left((1-\alpha)(y_i + w_j(y)) - \alpha k, 0\right) \tag{A.30}$$

The relation of the value of the principal and the agent (equation (7) in the paper) and Theorem 1 in the paper remain true as shown in detail below.

Consider a model with two principals $i \in \{1,2\}$ and one agent A, all risk neutral. The payoff space for the principals is $Y_1 \times Y_2 \subset \mathbb{R} \times \mathbb{R}$, it is assumed that Y_i is compact and that $\min\{Y_i\} = 0$. The agent has access to a technology $\mathcal{A} \subset \Delta(Y) \times \mathbb{R}_+$.An action is therefore a pair (F, c), where F is a probability distribution over payoffs $y = (y_1, y_2)$ and $c \geq 0$ is the cost of the action. $\Delta(Y)$ is endowed with the weak-$*$ topology and $\Delta(Y) \times \mathbb{R}$ with the natural product topology.

The game has two stages. First both principals offer a contract to the agent; this is done simultaneously and in a non-cooperative fashion. Second, the agent chooses an action in its technology set $\mathcal{A}$. Finally payments realize. The principals do not know $\mathcal{A}$, but they both know a subset $\mathcal{A}_0$ of $\mathcal{A}$. For now we assume that both principals know the same

$\mathcal{A}_0$, but this assumption is not necessary for any of the results below. Only three other assumptions are placed on the set $\mathcal{A}_0$:

Non-triviality: $\exists_{(F,c)\in\mathcal{A}_0} E_F[y_1+y_2]-c>0$. This guarantees that the principals can benefit from hiring the agent.

Positive Cost: If $(F,c)\in\mathcal{A}\supseteq\mathcal{A}_0$ and $c=0$ then $F=\delta_0$, where δ_0 is the degenerate distribution on $y=(0,0)$. This implies that generating output requires some cost for any action in $\mathcal{A}_0$.

Full support: A technology $\mathcal{A}$ has the full support property if for all $(F,c)\in\mathcal{A}$ such that $(F,c)\neq(\delta_0,0)$, $\mathrm{supp}\,(F)=Y_1\times Y_2$.

Contracts: A contract by principal i is a continuous function $w_i : Y_1\times Y_2\to[0,\infty)$. In what follows let $w_i=w_i(y_1,y_2)$. **Contracts must also satisfy limited liability on the principals' side, i.e.** $y_i-w_i\geq 0$**.** A contract scheme is a vector of functions $w=(w_1,w_2)$.

Given a contract scheme and a technology $\mathcal{A}$, the agent will choose from the set of actions that maximize its expected payoff. The set of optimal actions and the value they give are:

$$A^\star(w|\mathcal{A})=\underset{(F,c)\in\mathcal{A}}{\mathrm{argmax}}\;E_F[w_1+w_2]-c \qquad V_A(w|\mathcal{A})=\max_{(F,c)\in\mathcal{A}}\;E_F[w_1+w_2]-c \tag{A.31}$$

We define the value of a principal given a contract scheme w is given by the minimum payoff guarantee offered by the contract:

$$V_i(w)=\inf_{\mathcal{A}\supseteq\mathcal{A}_0}V_i(w|\mathcal{A}) \tag{A.32}$$

where $V_i(w|\mathcal{A})$ is the value for a given technology $\mathcal{A}$ that is is given by:

$$V_i(w|\mathcal{A})=\min_{(F,c)\in A^\star(w|\mathcal{A})}E_F[y_i-w_i] \tag{A.33}$$

We restrict our attention to contracts that are eligible to a principal in the sense that they guarantee more that the trivial payoff 0. Formally:

Eligibility: A contract w is eligible for principal i if: $V_i(w)>0$.

Finally we can define the best response of principal i to a contract w_j offered by principal j as:

$$\mathrm{BR}_i(w_j)=\underset{w_i}{\mathrm{argmax}}\quad V_i(w_i,w_j) \tag{A.34}$$

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition 12. *Let* $(F,c) \in A^\star(w|\mathcal{A})$*.For* $\mathcal{A} \supseteq \mathcal{A}_0$*, it holds that:*

$$E_F\left[w_1(y) + w_2(y)\right] \geq V_A(w|\mathcal{A}_0)$$

Moreover, if $(F,c) \in A^\star(w|\mathcal{A})$ *then* $F \in \mathcal{F}$ *where:*

$$\mathcal{F} = \left\{F \in \Delta(Y) \,|E_F\left[w_1 + w_2\right] \geq V_A(w|\mathcal{A}_0)\right\}$$

Proof. To see the first inequality let $(F,c) \in A^\star(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F\left[w_1(y) + w_2(y)\right] \geq E_F\left[w_1(y) + w_2(y)\right] - c \geq V_A(w|\mathcal{A}) \geq V_A(w|\mathcal{A}_0)$$

Then $F \in \mathcal{F}$. □

Lemma 17 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above). Note that $\mathcal{F}$ only depends on the contract and the known set of actions $\mathcal{A}_0$. In this way we replace the complexity of the definition of $V_i(w)$ in (A.32) with an object that depends only on known elements. The following results are valid for any scheme w that is eligible for principal i.

Lemma 17. *Let* w *be an eligible contract for principal* i*, then* $V_i(w) = \min_{F\in\mathcal{F}} \quad E_F\left[y_i - w_i\right]$*. Moreover if* $F \in \underset{F\in\mathcal{F}}{argmin} E_F\left[y_i - w_i\right]$ *then* $E_F\left[w_1 + w_2\right] = V_A(w|\mathcal{A}_0)$*.*

Proof. We first establish the first claim: Let w be an eligible contract scheme then: $V_i(w) = \min_{F\in\mathcal{F}} E_F\left[y_i - w_i(y)\right]$.
It must be that: $V_i(w) \geq \min_{F\in\mathcal{F}} E_F\left[y_i - w_i(y)\right]$. Using the definition of $V_i(w)$:

$$V_i(w) = \inf_{\mathcal{A}\supseteq\mathcal{A}_0} \min_{(F,c)\in A^\star(w|\mathcal{A})} E_F\left[y_i - w_i(y)\right] \geq \min_{F\in\mathcal{F}} E_F\left[y_i - w_i(y)\right]$$

Where the inequality follows because if $(F,c) \in A^\star(w|\mathcal{A})$ then $F \in \mathcal{F}$.

To prove equality suppose that

$$V_i(w) > \min_{F\in\mathcal{F}} E_F[y_i - w_i(y)]$$

and let $F^{'} \in \underset{F\in\mathcal{F}}{\mathrm{argmin}} E_F[y_i - w_i(y)]$. We have that $E_{F'}[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0)$. There are two options:

1. $F^{'}$ does not place full support on the values of y that maximize $w_1 + w_2$.

   Let $\hat{y} \in \mathrm{argmax}\{w_1(y) + w_2(y)\}$, and $\hat{F} = \delta_{\hat{y}}$ be a distribution with full mass on $\hat{y}$. Let $\epsilon \in [0,1]$ and $F_\epsilon = (1-\epsilon)F^{'} + \epsilon\hat{F}$.

   Note then that for all ϵ there exists a $\xi_\epsilon > 0$ such that: $E_{F_\epsilon}[w_1(y) + w_2(y)] - \xi_\epsilon > V_A(w|\mathcal{A}_0)$.

   Define and $\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \{(F_\epsilon, \xi_\epsilon)\}$. It follows that the unique optimal action of the agent in $\mathcal{A}_\xi$ is $(F_\epsilon, \xi_\epsilon)$. Then:

   $$V_i(w) \leq V_i(w|\mathcal{A}_\epsilon) = E_{F_\epsilon}[y_i - w_i(y)] = (1-\epsilon)E_{F'}[y_i - w_i] + \epsilon E_{\hat{F}}[y_i - w_i]$$

   This condition holds for all $\epsilon > 0$. Letting $\epsilon \to 0$ we arrive at a contradiction:

   $$V_i(w) \leq E_{F'}[y_i - w_i] = \underset{F\in\mathcal{F}}{\mathrm{argmin}} E_F[y_i - w_i]$$

2. $F^{'}$ places full support on the values of y that maximize $w_1 + w_2$. There are still two possible cases:

   (a) $E_{F'}[w_1 + w_2] > V_A(w|\mathcal{A}_0)$. Then there exists $\xi > 0$ and a technology $\mathcal{A}^{'} = \mathcal{A}_0 \cup \left\{\left(F^{'}, \xi\right)\right\}$ such that $\left(F^{'}, \xi\right)$ is the unique optimal action for the agent in $\mathcal{A}^{'}$. Then we arrive at a contradiction:

   $$V_i(w) \leq V_i\left(w|\mathcal{A}^{'}\right) = E_{F'}[y_i - w_i] = \underset{F\in\mathcal{F}}{\mathrm{argmin}} E_F[y_i - w_i]$$

   (b) $E_{F'}[w_1 + w_2] = V_A(w|\mathcal{A}_0)$. This implies $V_A(w|\mathcal{A}_0) = \max_{y\in Y}\{w_1 + w_2\}$ which can only be satisfied if $F^{'}$ is available in $\mathcal{A}_0$ at zero cost. By the positive cost assumption this implies that $F = \delta_{(0,0)}$ and that $w_1(0,0) + w_2(0,0) = \max_{y\in Y}\{w_1(y) + w_2(y)\}$. In this case the unique optimal action for the agent under any technology is $(\delta_0, 0)$, so the value of the principal is $V_i(w) = -w_i(0,0) \leq 0$, where the inequality follows from limited liability. This contradicts eligibility.

Now we establish the second claim: Let w be an eligible contract scheme for principal i. If $F \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F [y_i - w_i]$ then $E_F [w_1 + w_2] = V_A (w|\mathcal{A}_0)$.

To prove this, let $F^{'} \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F [y_i - w_i (y)]$ and suppose for a contradiction that

$$E_F [w_1 (y) + w_2 (y)] > V_A (w|\mathcal{A}_0)$$

.

Let $\epsilon \in [0, 1]$ and define $F_\epsilon = (1 - \epsilon) F^{'} + \epsilon\delta_0$. For low enough ϵ it holds that:

$$E_{F_\epsilon} [w_1 (y) + w_2 (y)] > V_A (w|\mathcal{A}_0)$$

Then there exists $\xi_\epsilon > 0$ such that $\{(F_\epsilon, \xi_\epsilon)\} = A^\star (w|\mathcal{A}_\epsilon)$ where $\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \{(F_\epsilon, \xi_\epsilon)\}$. The payoff to the principal is then:

$$V_i (w|\mathcal{A}_\epsilon) = (1 - \epsilon) E_F [y_i - w_i (y)] + \epsilon (-w_i (0, 0)) \leq (1 - \epsilon) E_F [y_i - w_i (y)] =$$
$$(1 - \epsilon) V_i (w) < V_i (w)$$

This gives a contradiction. □

Given the known action set $\mathcal{A}_0$, the next lemma links the principal's guaranteed payoff to the agent's payoff in an affine way. This link allows the principal to increase her own guaranteed payoff by controlling the payoff given to the agent. Lemma 18 also offers a relation between any contract w_i, the outcome y_i and the contract w_j offered by the other principal.

Lemma 18. *Let w be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$w_i (y) \leq \frac{1}{1 + \lambda} y_i - \frac{\lambda}{1 + \lambda} w_j (y) - \frac{1}{1 + \lambda} k \quad \text{(A.35)}$$

$$V_i (w) = k + \lambda V_A (w|\mathcal{A}_0) \quad \text{(A.36)}$$

Proof. This lemma is proven with the following two propositions. In both propositions define:

Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points $(w_1 (y) + w_2 (y), y_i - w_i (y))$ for $y \in Y$.

Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u > V_A(w|\mathcal{A}_0)$ and $v < V_i(w)$. Note T is convex.

□

Proposition 13. $S \cap T = \emptyset$.

Proof. Let $(u, v) \in T$ then let $F \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i]$, by definition of T and Lemma (17):

$$\begin{aligned} u &> V_A(w|\mathcal{A}_0) = E_F[w_i + w_j] \\ v &< V_i(w) = E_F[y_i - w_i] \end{aligned}$$

now, suppose for a contradiction that $(u, v) \in S$, then there exists $F^{'} \in \Delta(Y)$ such that:

$$\begin{aligned} u &= E_{F'}[w_i + w_j] \\ v &= E_{F'}[y_i - w_i] \end{aligned}$$

Note that $F^{'}$ guarantees a payoff to the agent larger than $V_A(w|\mathcal{A}_0)$ so $F^{'} \in \mathcal{F}$ but:

$$E_F[y_i - w_i] > E_{F'}[y_i - w_i]$$

which contradicts minimality of F. Then $S \cap T = \emptyset$

Since $S \cap T = \emptyset$ we can apply the separating hyperplane theorem which implies that there exist constants (k, λ, μ) such that $(\lambda, \mu) \neq (0, 0)$ and:

$$k + \lambda u - \mu v \leq 0 \qquad (u, v) \in S \tag{A.37}$$

$$k + \lambda u - \mu v \geq 0 \qquad (u, v) \in T \tag{A.38}$$

Let $F^{\star} \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i]$. Note that the pair $(E_{F^{\star}}[w_1 + w_2], E_{F^{\star}}[y_i - w_i])$ lies in the closures of both S and T. Then:

$$k + \lambda E_{F^{\star}}[w_1 + w_2] - \mu E_{F^{\star}}[y_i - w_i] = 0 \tag{A.39}$$

It is left to show that $\lambda, \mu > 0$.

Note that $(u, v) \in T$ admits u arbitrarily high and v arbitrarily low. So for (A.38) to hold it must be that $\lambda \geq 0$ and $\mu \geq 0$. There are then two cases to rule out:

□

1. Suppose $\mu = 0$, then it must be that $\lambda > 0$ (since $(\lambda, \mu) \neq (0,0)$). Also from (A.37) and (A.38)

$$u \leq -\frac{k}{\lambda} \qquad (u,v) \in S$$
$$u \geq -\frac{k}{\lambda} \qquad (u,v) \in T$$

So $\max_{y \in Y} [w_1(y) + w_2(y)] = \max_{u \in S_1} u \leq -\frac{k}{\lambda} \leq \inf_{u \in T_1} u = V_A(w|\mathcal{A}_0)$. Which implies:

$$\max_{y \in Y} [w_1(y) + w_2(y)] = V_A(w|\mathcal{A}_0)$$

This can only happen if the agent has an action $(F, 0) \in \mathcal{A}_0$ such that

$$E_F [w_1(y) + w_2(y)] = \max [w_1(y) + w_2(y)]$$

the only action in $\mathcal{A}_0$ with zero cost is $(\delta_0, 0)$, so

$$\max (w_1(y) + w_2(y)) = w_1(0,0) + w_2(0,0)$$

This is also the unique action in $A^\star(w|\mathcal{A}_0)$ so:

$$V_i(w) \leq V_i(w|\mathcal{A}_0) = -w_i(0,0) \leq 0$$

This violates eligibility ($V_i(w) > 0$).

   (a) Suppose $\lambda = 0$, then it must be that $\mu > 0$ (since $(\lambda, \mu) \neq (0,0)$). Also from (A.37) and (A.38)

$$v \geq \frac{k}{\mu} \qquad (u,v) \in S$$
$$v \leq \frac{k}{\mu} \qquad (u,v) \in T$$

   So $\min_{y \in Y} [y_i - w_i(y)] = \min_{v \in S_1} v \geq \frac{k}{\mu} \geq \sup_{v \in T_1} v = V_i(w)$. But we know that

$$\min_{y \in Y} [y_i - w_i(y)] \leq 0 - w(0,0) \leq 0$$

   this implies $V_i(w) \leq 0$ which contradicts eligibility. So $\lambda > 0$.

Proof. Note that since λ and μ are greater than zero μ can be normalized to 1, giving from (A.37):

$$k+\lambda\left(w_i\left(y\right)+w_j\left(y\right)\right)-\left(y_i-w_i\left(y\right)\right)\leq 0$$

And from (A.39):

$$V_i\left(w\right)=k+\lambda V_A\left(w|\mathcal{A}_0\right)$$

□

Corollary 3. *Let w be an eligible contract of where*

$$w_i\left(y\right)=\min\left(\frac{1}{1+\lambda}y_i-\frac{\lambda}{1+\lambda}w_j\left(y\right)-\frac{1}{1+\lambda}k,y_i\right)$$

with $\lambda>0$. Then

$$V_i\left(w\right) \quad = \quad k+\lambda V_A\left(w|\mathcal{A}_0\right) \tag{A.40}$$

Proof. Let $F^*\in\underset{F\in\mathcal{F}}{\operatorname{argmin}}E_F\left[y_i-w_i\right]$. By Lemma 17we have that

$$k+\lambda V_A\left(w|\mathcal{A}_0\right)-V_i\left(w\right)=k+\lambda E_{F^\star}\left[w_1+w_2\right]-E_{F^\star}\left[y_i-w_i\right]$$
$$=k+\left(1+\lambda\right)E_{F^*}\left(w_i\right)+\lambda E_{F^*}\left(w_j\right)-E_{F^*}\left(y_i\right)$$
$$=k+\left(1+\lambda\right)E_{F^*}\left(\min\left(\frac{1}{1+\lambda}y_i-\frac{\lambda}{1+\lambda}w_j\left(y\right)-\frac{1}{1+\lambda}k,y_i\right)\right)+\lambda E_{F^*}\left(w_j\right)-E_{F^*}\left(y_i\right)$$

Suppose for a contradiction that F^* places some positive probability $\delta>0$ on a set $\bar{Y}\subset Y$ such that such that $\frac{1}{1+\lambda}y_i-\frac{\lambda}{1+\lambda}w_j\left(y\right)-\frac{1}{1+\lambda}k>y_i$ for $y\in\overline{Y}$.
This implies that $0>\frac{\lambda y_i+\lambda w_j(y)+k}{1+\lambda}\implies\frac{-k}{\lambda}>y_i+w_j\left(y\right)$, where the RHS is the agents payment if output is y.
Now consider $\hat{y}\in Y$ for which $\frac{1}{1+\lambda}\hat{y}_i-\frac{\lambda}{1+\lambda}w_j\left(\hat{y}\right)-\frac{1}{1+\lambda}k=\hat{y}_i$. Then the agents payment at $\hat{y}$ is (computed in two ways)

$$\hat{y}_i+w_j\left(\hat{y}\right)=\frac{1}{1+\lambda}\hat{y}_i-\frac{\lambda}{1+\lambda}w_j\left(\hat{y}\right)-\frac{1}{1+\lambda}k+w_j\left(\hat{y}\right)$$

Rearranging we get that

$$\frac{\lambda}{1+\lambda}\left(\hat{y}_i + w_j\left(\hat{y}\right)\right) = -\frac{1}{1+\lambda}k$$
$$\hat{y}_i + w_j\left(\hat{y}\right) = -\frac{k}{\lambda} > y_i + w_j\left(y\right)$$

Also it must be the case that F^* puts positive probability on a $\tilde{y} \in Y$ for which the payoff to principal i is positive (by eligibility).

Now consider F' that is the same as F^* but shifts all the weight δ in $\overline{Y}$ to $\hat{y}$. Then $E_{F'}\left(w_i + w_j\right) > V_A\left(w|\mathcal{A}_0\right)$.

Now consider F'' that is the same as F' but shifts a small but positive weight from $\tilde{y}$ to $\hat{y}$ such that we still have $E_{F''}\left(w_i + w_j\right) \geq V_A\left(w|\mathcal{A}_0\right)$. Note that $F'' \in \mathcal{F}$. But also the payoff to principal i under F'' is worse than that under F' and F^* which violates the minimality of F^*.

Hence F^* places full support on $y \in Y$ for which $\frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j\left(y\right) - \frac{1}{1+\lambda}k \leq y_i$. Then we have

$$\begin{aligned}
k + \lambda V_A\left(w|\mathcal{A}_0\right) - V_i\left(w\right) &= k + \lambda E_{F^\star}\left[w_1 + w_2\right] - E_{F^\star}\left[y_i - w_i\right] \\
&= k + \left(1+\lambda\right)E_{F^\star}\left[w_i\right] + \lambda E_{F^\star}\left[w_2\right] - E_{F^\star}\left[y_i\right] \\
= k + \left(1+\lambda\right)E_{F^\star}&\left[\frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j\left(y\right) - \frac{1}{1+\lambda}k\right] + \lambda E_{F^\star}\left[w_2\right] - E_{F^\star}\left[y_i\right] \\
&= 0
\end{aligned}$$

Rearranging

$$V_i\left(w\right) = k + \lambda V_A\left(w|\mathcal{A}_0\right)$$

□

The following two lemmas (19 and 20) use the relation between the principals' contracts derived in Lemma 18 to construct an alternative contract that dominates the original one, in the sense that it weakly increases principal i's guaranteed payoff. Since the relation obtained in (A.35) is affine in output and the other principal's contract, the alternative contract constructed below will inherit that form.

Lemma 19. *Let $w = (w_i, w_j)$ with w_i satisfying (A.35) and (A.36). Then the contract*

$$w_i^{'}(y) = \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k$$

satisfies $V_i\left(w_i^{'}, w_j\right) \geq V_i(w)$.

Proof. Clearly $w_i^{'} \leq \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k$, rearrange it as:

$$\left(y_i - w_i^{'}(y)\right) = k + \lambda\left(w_i^{'}(y) + w_j(y)\right)$$

then let $(F, c) \in A^{\star}(w|\mathcal{A})$ for any $\mathcal{A} \supseteq \mathcal{A}_0$ and taking expectations one gets:

$$E_F\left[y_i - w_i^{'}(y)\right] = k + \lambda E_F\left[w_i^{'}(y) + w_j(y)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right)$$

$$E_F\left[y_i - w_i^{'}(y)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \tag{A.41}$$

This applies to any optimal (F, c) under any technology, so this guarantees a payoff for principal i.

Note that $w_i^{'}(y) \geq w_i(y)$ for all $y \in Y$ so the agent is always at least as well off under $w_i^{'}$ and it doesn't violate the agent's limited liability . Then:

$$V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \geq V_A(w|\mathcal{A}_0)$$

Joining with (A.41):

$$E_F\left[y_i - w_i^{'}(y)\right] \geq k + \lambda V_A(w|\mathcal{A}_0) = V_i(w)$$

Since this holds for all $(F, c) \in A^{\star}(w|\mathcal{A})$:

$$V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right) = \min_{F \in A^{\star}(w|\mathcal{A})} E_F\left[y_i - w_i^{'}(y)\right] \geq V_i(w)$$

Finally:

$$V_i\left(w_i^{'}, w_j\right) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right) \geq V_i(w)$$

□

Lemma 20. *Let $\left(w_i^{'}, w_j\right)$ with $w_i^{'}$ be the affine contract on y_i and w_j satisfying 19. There is an affine contract $w_i^{''}$ that does at least as well as $w_i^{'}$ for principal i: $V_i\left(w_i^{''}, w_j\right) \geq V_i\left(w_i^{'}, w_j\right)$, with strict inequality unless $\min_y w_i^{'}(y) = 0$.*

Proof. Note that by limited liability $\min_y w_i^{'}(y) \geq 0$ let $\beta = \min_y w_i^{'}$ and $w_i^{''}(y) = w_i^{'}(y) - \beta$ which is a valid contract $\left(w_i^{''}(y) \geq 0\right)$ and is affine on y_i and w_j. Note that

$$A^{\star}\left(\left(w_i^{''}, w_j\right)|\mathcal{A}\right) = A^{\star}\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right)$$

for all $\mathcal{A} \supseteq \mathcal{A}_0$ since subtracting a constant doesn't change the agent's incentives. This implies $V_i\left(w_i^{''}, w_j\right) \geq V_i\left(w_i^{'}, w_j\right)$, with strict inequality if $\beta > 0$. □

Lemma 21. *Let* $w^{''} = \left(w_i^{''}, w_j\right)$ *be the contract in 20and* w_i *satisfying (A.35) and (A.36). Then the contract*

$$\begin{aligned} w_i^{PLL}(y) &= \min\left(w_i^{''}, y_i\right) \\ &= \min\left(\frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j(y) - \frac{1}{1+\lambda}k^{''}, y_i\right) \end{aligned}$$

where $k^{''}$ *is such that* $\min_y w_i^{''}(y) = 0$ *satisfies* $V_i\left(w_i^{PLL}, w_j\right) \geq V_i(w)$.

Proof. First note that $k^{''} \geq k$. Also note that

$$w_i^{PLL}(y) \leq \frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j(y) - \frac{1}{1+\lambda}k^{''}$$

rearrange it as:

$$\left(y_i - w_i^{PLL}(y)\right) \geq k^{''} + \lambda\left(w_i^{PLL}(y) + w_j(y)\right)$$

then let $(F, c) \in A^{\star}(w|\mathcal{A})$ for any $\mathcal{A} \supseteq \mathcal{A}_0$ and taking expectations one gets:

$$E_F\left[y_i - w_i^{PLL}(y)\right] \geq k^{''} + \lambda E_F\left[w_i^{PLL}(y) + w_j(y)\right] \geq k^{''} + \lambda V_A\left(\left(w_i^{PLL}, w_j\right)|\mathcal{A}_0\right)$$

$$\begin{aligned} E_F\left[y_i - w_i^{PLL}(y)\right] &\geq k + \lambda V_A\left(\left(w_i^{PLL}, w_j\right)|\mathcal{A}_0\right) + (k^{''} - k) \\ &= k + \lambda V_A\left(\left(w_i^{PLL} + \frac{(k^{''} - k)}{\lambda}, w_j\right)|\mathcal{A}_0\right) \end{aligned} \quad \text{(A.42)}$$

This applies to any optimal (F, c) under any technology, so this guarantees a payoff for principal i.

Note that

$$\begin{aligned}
w_i^{PLL} + \frac{(k'' - k)}{\lambda} &\geq \min\left(w_i'', y_i\right) + \frac{(k'' - k)}{\lambda} \\
&= \min\left(\frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k'', y_i\right) + \frac{(k'' - k)}{\lambda} \\
&> \min\left(\frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k'', y_i\right) + \frac{(k'' - k)}{1+\lambda} \\
&= \min\left(\frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k'' + \frac{(k'' - k)}{1+\lambda}, y_i + \frac{(k'' - k)}{1+\lambda}\right) \\
&= \min\left(\frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k, y_i + \frac{(k'' - k)}{1+\lambda}\right) \\
&= min\left(w_i', y_i + \frac{(k'' - k)}{1+\lambda}\right) \\
&\geq w_i
\end{aligned}$$

for all $y \in Y$ because $w_i' \geq w_i$ and by since w_i satisfies principals limited liability then $w_i \leq y_i \leq y_i + \frac{(k''-k)}{1+\lambda}$.

So the agent is always at least as well off under $w_i^{PLL} + \frac{(k''-k)}{\lambda}$ as he was under w_i. Then:

$$V_A\left(\left(\left(w_i^{PLL} + \frac{(k'' - k)}{\lambda}, w_j\right), w_j\right) | \mathcal{A}_0\right) \geq V_A\left(w | \mathcal{A}_0\right)$$

Joining with (A.42):

$$E_F\left[y_i - w_i^{PLL}(y)\right] \geq k + \lambda V_A\left(w | \mathcal{A}_0\right) = V_i(w)$$

Since this holds for all $(F, c) \in A^\star(w|\mathcal{A})$:

$$V_i\left(\left(w_i^{PLL}, w_j\right) | \mathcal{A}\right) = \min_{F \in A^\star(w|\mathcal{A})} E_F\left[y_i - w_i^{PLL}(y)\right] \geq V_i(w)$$

Finally:

$$V_i\left(w_i^{PLL}, w_j\right) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} V_i\left(\left(w_i^{PLL}, w_j\right) | \mathcal{A}\right) \geq V_i(w)$$

□

Definition. Given a contract w_j, a contract w_i is an LRS contract if there exists $\alpha \in [0,1]$ and k such that:

$$w_i(y) = \min(\alpha y_i - (1-\alpha) w_j(y) - \alpha k, y_i) \tag{A.43}$$

and $\min_y (\alpha y_i - (1-\alpha) w_j(y) - \alpha k) = 0$.
For a given w_j let $\mathcal{W}_i(w_j)$ be the set of LRS contracts for principal i.

The last two lemmas (22 and 23) establish the form of the principal's payoffs under LRS contracts and the existence of an optimal contract in that class.

Lemma 22. *Let w be an eligible LRS contract scheme characterized by $\alpha \in (0,1]$, then:*

$$V_i(w) = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) + k$$

This also holds for $\alpha = 0$ if we interpret the term $\frac{1-\alpha}{\alpha}c$ as 0 when $c = 0$ and ∞ for $c > 0$.

Proof. This follows immediately by 3 by setting $\alpha = \frac{1}{1+\lambda}$. □

Lemma 23. *In the class of LRS contracts there exists an optimal one for principal i.*

Proof. First note that

$$\begin{aligned} V_A(w|\mathcal{A}_0) &= \max_{(F,c)\in\mathcal{A}_0} E_F[w_i(y) + w_j(y) - c] \\ &= \max_{(F,c)\in\mathcal{A}_0} E_F[\min(\alpha y_i - (1-\alpha) w_j(y) - \alpha k, y_i) + w_j(y) - c] \end{aligned}$$

is continuous in α.
The function $\min(\alpha y_i - (1-\alpha) w_j(y) - \alpha k, y_i) + w_j(y) - c$ is continuous in α and so by some functional analysis result it should be that

$$E_F[\min(\alpha y_i - (1-\alpha) w_j(y) - \alpha k, y_i) + w_j(y) - c]$$

is also continuous in α , thus its maximum over $\mathcal{A}_0$ is continuous as well.
Recall that:

$$k(\alpha) = \min_y \left[y_i - \frac{1-\alpha}{\alpha} w_j(y) \right]$$

This function is continuous in α for a given continuous w_j.

This implies that

$$\frac{1-\alpha}{\alpha} V_A\left(w|\mathcal{A}_0\right) + k$$

is continuous in α hence it achieves a maximum in $[0,1]$. This α gives the optimal guarantee over all contracts of this class.

Now let $\alpha^* \in \arg\max_{\alpha\in[0,1]} \frac{1-\alpha}{\alpha} V_A\left(w|\mathcal{A}_0\right) + k$. If the LRS contract characterized by α^* is eligible then this contract is optimal in the class of LRS contracts. If not, then all LRS contracts provide a non-positive (and non-negative by PLL) guarantee for principal i. Hence any LRS contract provides zero guarantee and thus is optimal . □

Theorem 4. *For any contract w_j there exists $\alpha \in [0,1]$ such that:*

$$w_i\left(y\right) = \min\left(\alpha y_i - \left(1-\alpha\right) w_j\left(y\right) - \alpha k\left(\alpha\right), y_i\right) \qquad w_i\left(w_j\right) \in BR_i\left(w_j\right)$$

where $k\left(\alpha\right)$ is such that $\min_y \left(\alpha y_i - \left(1-\alpha\right) w_j\left(y\right) - \alpha k\left(\alpha\right)\right) = 0$. That is, there is a LRS contract in the best response of principal i.

Proof. By Lemma 23 among the class of LRS contracts there is an optimal one, call it $w_i^\star$. Suppose there is an arbitrary contract w_i that does strictly better than $w_i^\star$: $V_i\left(w_i, w_j\right) > V_i\left(w_i^\star, w_j\right)$. Note that $V_i\left(w_i^\star, w_j\right) \geq V_i\left(y_i, w_j\right) \geq 0$. Hence it must be the case that w_i is eligible, i.e. $V_i\left(w_{i,} w_j\right) > 0$. Then by Lemmas 18, 19 , 20, and (21) there exists a LRS contract $w_i^{'}$ such that $V_i\left(w_i^{'}, w_j\right) \geq V_i\left(w_i, w_j\right)$. This contradicts $w_i^\star$ being optimal among the LRS contracts. □

Corollary 4. *Suppose$\mathcal{A}_0$ has the full support property. For any given w_j for which there exists an eligible contract for principal i then, $BR_i\left(w_j\right) \subseteq \mathcal{W}_i\left(w_j\right)$, that is, any optimal contract for principal i is LRS.*

Proof. Suppose w_i is an optimal contract for principal i.

□

Define w_i^{PLL} as in Lemma 21. Note that from equation A.42 for any $\left(F,c\right) \in A^\star\left(w|\mathcal{A}\right)$ for any $\mathcal{A} \supseteq \mathcal{A}_0$ it satisfies:

$$E_F\left[y_i - w_i^{PLL}\left(y\right)\right] \geq k + \lambda V_A\left(\left(w_i^{PLL} + \frac{\left(k^{''} - k\right)}{\lambda}, w_j\right)|\mathcal{A}_0\right)$$

Also note that $k'' - k \geq 0$ as in the proof of Lemma21 Note that w_i satisfies Equation (A.36) from Lemma 18:

$$V_i(w) = k + \lambda V_A((w_i, w_j) | \mathcal{A}_0)$$

Replacing for k:

$$E_F\left[y_i - w_i^{PLL}(y)\right] \geq V_i(w) + \lambda\left(V_A\left(\left(w_i^{PLL} + \frac{(k'' - k)}{\lambda}, w_j\right) | \mathcal{A}_0\right) - V_A((w_i, w_j) | \mathcal{A}_0)\right)$$

Because of full support, since $w_i^{PLL} + \frac{(k''-k)}{\lambda} \geq w_i(y)$ point wise and any action under $\mathcal{A}_0$ gives a (weakly) higher payoff to the agent under $w_i^{PLL} + \frac{(k''-k)}{\lambda}$ than under w_i, it follows that $V_A\left(\left(w_i^{PLL} + \frac{(k''-k)}{\lambda}, w_j\right) | \mathcal{A}_0\right) \geq V_A((w_i, w_j) | \mathcal{A}_0)$, with strict inequality unless $w_i^{PLL} + \frac{(k''-k)}{\lambda}$ is identical to w_i.

Since the equation above holds for all optimal F under any technology it must be true that:

$$V_i\left(w_i^{PLL}, w_j\right) \geq V_i(w) + \lambda\left(V_A\left(\left(w_i^{PLL} + \frac{(k'' - k)}{\lambda}, w_j\right) | \mathcal{A}_0\right) - V_A((w_i, w_j) | \mathcal{A}_0)\right)$$

$$> V_i(w)$$

where the strict inequality follows when w_i is not identical to $w_i^{PLL} + \frac{(k''-k)}{\lambda}$.

Then $w_i = w_i^{PLL} + \frac{(k''-k)}{\lambda}$ (or else optimality would be contradicted).

It must be that w_i is LRS (i.e $\frac{(k''-k)}{\lambda} = 0$) otherwise, w_i^{PLL} gives the principal a strictly greater guarantee compared to $w_i^{PLL} + \frac{(k''-k)}{\lambda}$ as the incentives of the agent are not changed and in w_i^{PLL} the limited liability for the agent binds.

Any optimal contract is LRS.

A.6 Private common agency

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition 14. *Let* $(F, c) \in A^{\star}(w|\mathcal{A})$ *for* $\mathcal{A} \supseteq \mathcal{A}_0$*, it holds that:*

$$E_F\left[w_1(y_1) + w_2(y_2)\right] \geq V_A(w|\mathcal{A}_0)$$

Moreover, if $(F, c) \in A^{\star}(w|\mathcal{A})$ *then* $F \in \mathcal{F}$ *where:*

$$\mathcal{F} = \left\{F \in \Delta(Y) \,|E_F\left[w_1 + w_2\right] \geq V_A(w|\mathcal{A}_0)\right\}$$

Proof. To see the first inequality let $(F, c) \in A^{\star}(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F\left[w_1(y_1) + w_2(y_2)\right] \geq E_F\left[w_1(y_1) + w_2(y_2)\right] - c = V_A(w|\mathcal{A}) \geq V_A(w|\mathcal{A}_0)$$

Then $F \in \mathcal{F}$. □

In this section we characterize the behavior of a principal facing extreme uncertainty on the set of actions of the agent. The principal acts taken as given the other principal's action. We proceed in a similar fashion as Carroll (2015) to establish the following lemmas.
Lemma 24 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above). Note that $\mathcal{F}$ only depends only on the contract and the known set of actions $\mathcal{A}_0$. In this way we replace the complexity of the definition of $V_i(w)$ in Equation (2) in the paper with an object that depends only on known elements. The following results are valid for any scheme w eligible for principal i.

Lemma 24. *Let* w *be an eligible contract for principal* i*, then* $V_i(w) = \min_{F \in \mathcal{F}} \quad E_F\left[y_i - w_i\right]$*. Moreover if* $F \in \underset{F \in \mathcal{F}}{argmin} E_F\left[y_i - w_i\right]$ *then* $E_F\left[w_1 + w_2\right] = V_A(w|\mathcal{A}_0)$*.*

Proof. The proof is identical to that of Lemma (45). □

Lemma 25 links the principal's payoff guarantee to the agent's payoff given the known action set $\mathcal{A}_0$ in an affine way. This link allows the principal to increase its own guaranteed payoff by controlling the payoff given to the agent. The lemma also offers a relation between any contract w_i, the outcome y_i and the contract w_j offered by the other principal.

The affine link between the agent's payoff and the principal's payoff is a crucial element in providing incentives for the agent. Given the lack of knowledge over the agent's set of actions the principals' optimal strategy is to tie their payoff to that of the agent, thus aligning the agent's objectives with their own. This is the same mechanism that lies at the center of Hurwicz and Shapiro (1978) and Carroll (2015) optimal contracts, and will be crucial to establish the optimality of affine contracts in the setting we develop.

Lemma 25. *Let w be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$w_i(y_i) \leq \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} \overline{w}_j - \frac{1}{1+\lambda} k \tag{A.44}$$

$$V_i(w) = k + \lambda V_A(w|\mathcal{A}_0) \tag{A.45}$$

where $\overline{w}_j = \max_{y_j \in Y_j} w_j(y_j)$.

Proof. This lemma is proven with the following two propositions. In both propositions define:

Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points $(w_i(y_i) + \overline{w}_j, y_i - w_i(y_i))$ for $y_i \in Y_i$ and $\overline{w}_j = \max_{y_j \in Y_j} w_j(y_j)$.

Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u > V_A(w|\mathcal{A}_0)$ and $v < V_i(w)$. Note T is convex.

Proposition 15. $S \cap T = \emptyset$.

Proof. Let $(u, v) \in T$ then let $F \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i]$, by definition of T and Lemma (24):

$$\begin{aligned} u &> V_A(w|\mathcal{A}_0) = E_F[w_i(y_i) + w_j(y_j)] \\ v &< V_i(w) = E_F[y_i - w_i(y_i)] \end{aligned}$$

now, suppose for a contradiction that $(u, v) \in S$, then there exists $F' \in \Delta(Y)$ such that:

$$\begin{aligned} u &= E_{F'}[w_i(y_i)] + \overline{w}_j \\ v &= E_{F'}[y_i - w_i(y_i)] \end{aligned}$$

Without loss F' is such that $E_{F'}[w_j(y_j)] = \overline{w}_j$.[2] Then:

$$u = E_{F'}[w_i(y_i) + w_j(y_j)] > V_A(w|\mathcal{A}_0)$$

[2] This uses our assumption on the output space being of the form $Y = Y_1 \times Y_2$.

That is, $F^{'}$ guarantees a payoff to the agent larger than $V_A\left(w|\mathcal{A}_0\right)$ so $F^{'}\in\mathcal{F}$ but:

$$E_F\left[y_i-w_i\right]>E_{F'}\left[y_i-w_i\right]$$

which contradicts minimality of F. Then $S\cap T=\emptyset$ □

Since $S\cap T=\emptyset$ we can apply the separating hyperplane theorem which implies that there exist constants (k,λ,μ) such that $(\lambda,\mu)\neq(0,0)$ and:

$$k+\lambda u-\mu v\leq 0\qquad (u,v)\in S \tag{A.46}$$

$$k+\lambda u-\mu v\geq 0\qquad (u,v)\in T \tag{A.47}$$

Let $F^{\star}\in\underset{F\in\mathcal{F}}{\operatorname{argmin}}E_F\left[y_i-w_i\left(y_i\right)\right]$ such that $E_{F^{\star}}\left[w_j\left(y_j\right)\right]=\overline{w}_j$. This $F^{\star}$ always exists since the objective function $E_F\left[y_i-w_i\left(y_i\right)\right]$ only depends on y_i, moreover, recall that

$$\mathcal{F}=\left\{F\in\Delta\left(Y\right)|E_F\left[w_1\left(y_1\right)+w_2\left(y_2\right)\right]\geq V_A\left(w|\mathcal{A}_0\right)\right\}$$

Then if $F\in\mathcal{F}$ the distribution $F^{\star}$ with the same marginal over y_i and full probability over $\overline{w}_j$ also belongs to $\mathcal{F}$.
Note that the pair $\left(E_{F^{\star}}\left[w_i\left(y_i\right)+w_j\left(y_j\right)\right],E_{F^{\star}}\left[y_i-w_i\left(y_i\right)\right]\right)$ lies in the closures of both S and T. Then:

$$k+\lambda E_{F^{\star}}\left[w_1+w_2\right]-\mu E_{F^{\star}}\left[y_i-w_i\right]=0 \tag{A.48}$$

It is left to show that $\lambda,\mu>0$.
Note that $(u,v)\in T$ admits u arbitrarily high and v arbitrarily low. So for (A.47) to hold it must be that $\lambda\geq 0$ and $\mu\geq 0$. There are then two cases to rule out:

1. Suppose $\mu=0$, then it must be that $\lambda>0$ (since $(\lambda,\mu)\neq(0,0)$). Also from (A.46) and (A.47)

$$u\leq-\frac{k}{\lambda}\qquad (u,v)\in S$$
$$u\geq-\frac{k}{\lambda}\qquad (u,v)\in T$$

So $\max\limits_{y_i\in Y_i}\left[w_i\left(y_i\right)+\overline{w}_j\right]=\max\limits_{u\in S}u\leq-\frac{k}{\lambda}\leq\inf\limits_{u\in T}u=V_A\left(w|\mathcal{A}_0\right)$. Which implies:

$$\max_{y_i\in Y_i}\left[w_i\left(y_i\right)+\overline{w}_j\right]=V_A\left(w|\mathcal{A}_0\right)$$

This can only happen if the agent has an action $(F, 0) \in \mathcal{A}_0$ such that

$$E_F\left[w_1(y_1) + w_2(y_2)\right] = \overline{w}_1 + \overline{w}_2$$

By the positive cost assumption, the only action in $\mathcal{A}_0$ with zero cost is $(\delta_0, 0)$, so $\overline{w}_1 + \overline{w}_2 = w_1(0) + w_2(0)$. This is also the unique action in $A^\star(w|\mathcal{A}_0)$ so:

$$V_i(w) \leq V_i(w|\mathcal{A}_0) = -w_i(0) \leq 0$$

This violates eligibility ($V_i(w) > 0$).

2. Suppose $\lambda = 0$, then it must be that $\mu > 0$ (since $(\lambda, \mu) \neq (0, 0)$). Also from (A.46) and (A.47)

$$\begin{aligned} v &\geq \frac{k}{\mu} \qquad (u, v) \in S \\ v &\leq \frac{k}{\mu} \qquad (u, v) \in T \end{aligned}$$

So $\min_{y_i \in Y_i}\left[y_i - w_i(y_i)\right] = \min_{v \in S} v \geq \frac{k}{\mu} \geq \sup_{v \in T} v = V_i(w)$.
But we know that $\min_{y_i \in Y_i}\left[y_i - w_i(y_i)\right] \leq 0 - w(0) \leq 0$ this implies $V_i(w) \leq 0$ which contradicts eligibility. So $\lambda > 0$.

Note that since λ and μ are greater than zero μ can be normalized to 1, giving from (A.46):

$$k + \lambda\left(w_i(y_i) + \overline{w}_j\right) - \left(y_i - w_i(y_i)\right) \leq 0$$

And from (A.48) and Lemma (24):

$$V_i(w) = k + \lambda V_A(w|\mathcal{A}_0)$$

□

The following two lemmas (26 and 27) use the relation between the principals' contracts derived in the previous lemma to construct an alternative contract that dominates the original one in the sense that it guarantees a higher or equal payoff to principal i. Since the relation obtained in (A.44) is affine on output and the other principal's contract the alternative contract constructed below will inherit that form.

Lemma 26. *Let $w = (w_i, w_j)$ with w_i satisfying (A.44) and (A.45). Then the contract*

$$w_i^{'}(y_i) = \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda}\overline{w}_j - \frac{1}{1+\lambda}k$$

satisfies $V_i\left(w_i^{'}, w_j\right) \geq V_i(w)$.

Proof. Clearly $w_i^{'}$ satisfies (A.44) as an equality, rearrange it as:

$$\left(y_i - w_i^{'}(y_i)\right) = k + \lambda\left(w_i^{'}(y_i) + \overline{w}_j\right)$$

then let $(F, c) \in A^{\star}(w|\mathcal{A})$ for any $\mathcal{A} \supseteq \mathcal{A}_0$ and taking expectations one gets:

$$E_F\left[y_i - w_i^{'}(y_i)\right] = k + \lambda E_F\left[w_i^{'}(y_i) + \overline{w}_j\right] \geq k + \lambda E_F\left[w_i^{'}(y_i) + w_j(y_i)\right]$$
$$\geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right)$$

$$E_F\left[y_i - w_i^{'}(y_i)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \tag{A.49}$$

This applies to any (F, c) under any technology, so this guarantees a payoff for principal i. Note that $w_i^{'}(y_i) \geq w_i(y_i)$ for all $y_i \in Y_i$ so the agent is always at least as well off under $w_i^{'}$ and it doesn't violate the agent's limited liability. Then:

$$V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \geq V_A(w|\mathcal{A}_0)$$

Joining with (A.49):

$$E_F\left[y_i - w_i^{'}(y_i)\right] \geq k + \lambda V_A(w|\mathcal{A}_0) = V_i(w)$$

Since this holds for all $(F, c) \in A^{\star}(w|\mathcal{A})$:

$$V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right) = \min_{F \in A^{\star}(w|\mathcal{A})} E_F\left[y_i - w_i^{'}(y_i)\right] \geq V_i(w)$$

Finally:

$$V_i\left(w_i^{'}, w_j\right) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right) \geq V_i(w)$$

□

Lemma 27. *Let* $\left(w_i^{'}, w_j\right)$ *with* $w_i^{'}$ *an affine contract on* y_i*, there is an affine contract* $w_i^{''}$ *that does at lest as well as* $w_i^{'}$ *for principal* i*:* $V_i\left(w_i^{''}, w_j\right) \geq V_i\left(w_i^{'}, w_j\right)$*, with strict inequality unless* $\min_{y_i} w_i^{'}(y_i) = 0$*.*

Proof. Note that by limited liability $\beta = \min_{y_i} w_i^{'}(y_i) \geq 0$. Let $w_i^{''}(y) = w_i^{'}(y) - \beta$ which is a valid contract $\left(w_i^{''}(y) \geq 0\right)$ and is affine on y_i. Note that $A^{\star}\left(\left(w_i^{''}, w_j\right)|\mathcal{A}\right) = A^{\star}\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right)$ for all $\mathcal{A} \supseteq \mathcal{A}_0$ since subtracting a constant doesn't change the agent's incentives. This implies $V_i\left(w_i^{''}, w_j\right) \geq V_i\left(w_i^{'}, w_j\right)$, with strict inequality if $\beta > 0$. □

The previous two lemmas show affine contracts weakly dominate any eligible contract. We will show that contracts that are linear in the principal's output improve on them.

Linear contracts: A contract w_i is linear, given a contract w_j, if:

$$w_i(y_i) = \alpha y_i$$

where $\alpha \in [0, 1]$. Note that $\min_{y} \; w_i(y) = 0$ and that w_i does not depend on w_j.

Let $\mathcal{W}_i$ be the set of all linear contracts of principal i. Note that any eligible contract (w_i, w_j) can be (weakly) improved for principal i by a contract of the form $\left(w_i^{'}, w_j\right)$ where $w_i^{'} \in \mathcal{W}_i$.

The last two lemmas (28 and 29) establish the form of the principal's payoffs under linear contracts and the existence of an optimal contract in that class.

Lemma 28. *Let* $F \in \underset{F\in\mathcal{F}}{argmin} E_F[y_i - w_i]$ *for* w *an eligible contract scheme such that* $w_i \in \mathcal{W}_i$ *is characterized by* $\alpha \in (0, 1]$*, then:*

$$V_i(w) = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) - \frac{1-\alpha}{\alpha}\overline{w}_j = \max_{(F,c)\in\mathcal{A}_0}\left(\frac{1-\alpha}{\alpha}\left(E_F\left[\alpha_i y_i - \left(\overline{w}_j - w_j(y)\right)\right] - c\right)\right)$$

This also holds for $\alpha = 0$ *if we interpret the term* $\frac{1-\alpha}{\alpha}c$ *as 0 when* $c = 0$ *and* ∞ *for* $c > 0$*.*

Proof. Let $F \in \underset{F\in\mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i(y_i)]$ such that $E_F[w_j(y_j)] = \overline{w}_j$, where

$$\overline{w}_j = \max_{y_j \in Y_j} w_j(y_j)$$

By Lemma 24 one has:

$$V_i(w) = E_F[y_i - w_i] \qquad E_F[w_1 + w_2] = V_A(w|\mathcal{A}_0)$$

Then

$$V_i(w) = \frac{1-\alpha}{\alpha} E_F\left[w_i(y_i) + w_j(y_j)\right] - \frac{1-\alpha}{\alpha} E_F\left[w_j(y_j)\right] = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) - \frac{1-\alpha}{\alpha}\overline{w}_j$$

Moreover:

$$\begin{aligned}
\frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) - \frac{1-\alpha}{\alpha}\overline{w}_j &= \frac{1-\alpha}{\alpha}\left(\max_{(F,c)\in\mathcal{A}_0} E_F\left[w_i(y_i) + w_j(y_j) - c\right]\right) - \frac{1-\alpha}{\alpha}\overline{w}_j \\
&= \frac{1-\alpha}{\alpha}\left(\max_{(F,c)\in\mathcal{A}_0} E_F\left[\alpha y_i + w_j(y_j) - c\right]\right) - \frac{1-\alpha}{\alpha}\overline{w}_j \\
&= \max_{(F,c)\in\mathcal{A}_0}\left(\frac{1-\alpha}{\alpha}\left(E_F\left[\alpha_i y_i - \left(\overline{w}_j - w_j(y)\right)\right] - c\right)\right)
\end{aligned}$$

□

Lemma 29. *In the class of linear contracts $w_i \in \mathcal{W}_i$ there exists an optimal one for principal i given contract w_j.*

Proof. From lemma (28) one gets that principal i's payoff is given by:

$$V_i(w) = \max_{(F,c)\in\mathcal{A}_0}\left(\frac{1-\alpha}{\alpha}\left(E_F\left[\alpha_i y_i - \left(\overline{w}_j - w_j(y)\right)\right] - c\right)\right)$$

The function $\frac{1-\alpha}{\alpha}\left(E_F\left[\alpha_i y_i - \left(\overline{w}_j - w_j(y)\right)\right] - c\right)$ is continuous in α, moreover it is also continuous in (F, c) (since w_j is a continuous function) and $\mathcal{A}_0$ is a compact set (constant with respect to α). Then V_i is continuous in α as well (by the Theorem of maximum). Since the RHS is continuous in α it achieves a maximum in $[0, 1]$. This α gives the optimal guarantee over all contracts of this class. □

The lemmas above allow us to characterize the behavior of a principal, in particular they imply that it is always a best response to offer a linear contract, as shown in Theorem 5. The result can be strengthened under the full support property.

Theorem 5. *For any contract w_j there exists $\alpha \in [0, 1]$ such that:*

$$w_i(y_i) = \alpha y_i \qquad w_i \in BR_i(w_j)$$

That is, there is a linear contract in the best response of principal i to any contract w_j.

Proof. By Lemma 29 among the class of linear contracts there is an optimal one, call it $w_i^\star$. Suppose there is an arbitrary contract w_i that does strictly better than $w_i^\star$:

$$V_i\left(w_i, w_j\right) > V_i\left(w_i^\star, w_j\right)$$

By Lemmas 25, 26 and 27 there exists a linear contract $w_i^{'}$ such that

$$V_i\left(w_i^{'}, w_j\right) \geq V_i\left(w_i, w_j\right)$$

This contradicts $w_i^\star$ being optimal among the linear contracts. □

Corollary 5. *If $\mathcal{A}_0$ has the full support property then, for any w_j, $BR_i\left(w_j\right) \subseteq \mathcal{W}_i$, that is, any optimal contract for principal i is linear.*

Proof. Suppose w_i is an optimal contract for principal i.

- Define $w_i^{'}$ as in Lemma 26. $w_i^{'}$ satisfies:

  $$E_F\left[y_i - w_i^{'}\left(y_i\right)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right)$$

  Note that w_i satisfies Equation (A.45) from Lemma 25:

  $$V_i\left(w\right) = k + \lambda V_A\left(\left(w_i, w_j\right)|\mathcal{A}_0\right)$$

  Replacing for k:

  $$E_F\left[y_i - w_i^{'}\left(y_i\right)\right] \geq V_i\left(w\right) + \lambda\left(V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) - V_A\left(\left(w_i, w_j\right)|\mathcal{A}_0\right)\right)$$

- Because of full support, since $w_i^{'}\left(y_i\right) \geq w_i\left(y_i\right)$ pointwise and any action under $\mathcal{A}_0$ gives a (weakly) higher payoff to the agent under $w_i^{'}$ than under w_i, it follows that $V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \geq V_A\left(\left(w_i, w_j\right)|\mathcal{A}_0\right)$, with strict inequality unless $w_i^{'}$ is identical to w_i.

- Since the equation above holds for all F it must be true that:

  $$V_i\left(w_i^{'}, w_j\right) \geq V_i\left(w\right) + \lambda\left(V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) - V_A\left(\left(w_i, w_j\right)|\mathcal{A}_0\right)\right) > V_i\left(w\right)$$

  where the strict inequality follows when w_i is not identical to $w_i^{'}$.

- Then $w_i = w_i'$ (or else optimality would be contradicted). Then w_i is linear in y_i.
- It must be that w_i is linear, or else by Lemma 27 there is a linear contract that strictly improves w_i.
- Any optimal contract is linear.

□

A.7 Collusion

To prove this we first obtain versions of Lemmas 45 and 46 for the case of collusion. The results we obtain allow us to apply Carroll (2015)'s Theorem 1 to our collusion environment. We begin by proving the following proposition relating the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition 16. *Let* $(F, c) \in A^{\star}(w|\mathcal{A})$ *for* $\mathcal{A} \supseteq \mathcal{A}_0$*, it holds that:*

$$E_F [w(y)] \geq V_A (w|\mathcal{A}_0)$$

Moreover, $A^{\star}(w|\mathcal{A}) \subseteq \mathcal{F}$ *where:*

$$\mathcal{F} = \{F \in \Delta(Y) \,| E_F [w(y)] \geq V_A (w|\mathcal{A}_0)\}$$

Proof. To see the first inequality let $(F, c) \in A^{\star}(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F [w(y)] \geq E_F [w(y)] - c \geq V_A (w|\mathcal{A}) \geq V_A (w|\mathcal{A}_0)$$

Then $A^{\star}(w|\mathcal{A}) \subseteq \mathcal{F}$ follows from the definition of $\mathcal{F}$. □

Now we prove Lemmas 30 and 31 that will allow us to characterize the optimal contracts under collusion.

Lemma 30. *Let* w *be an eligible contract then:*

$$V_P(w) = \min_F E_F [y_1 + y_2 - w(y)] \qquad \text{where } F \text{ is s.t. } E_F [w(y)] \geq V_A (w|\mathcal{A}_0)$$

moreover if $F \in \underset{F \in \mathcal{F}}{argmin} E_F [y_1 + y_2 - w(y)]$ *then* $E_F [w(y)] = V_A (w|\mathcal{A}_0)$.

Proof. The proof of this is virtually identical to that of Lemma 45. □

Lemma 31. *Let* w *be an eligible contract. There exits* k, λ *with* $\lambda > 0$ *such that for all* $y \in Y$*:*

$$\begin{aligned} w(y) &\leq \frac{1}{1+\lambda}(y_1 + y_2) - \frac{1}{1+\lambda}k && \text{(A.50)} \\ V_P(w) &= k + \lambda V_A (w|\mathcal{A}_0) && \text{(A.51)} \end{aligned}$$

Proof. The proof of this is virtually identical to that of Lemma 46. □

With this we can use the framework developed in Carroll (2015) to obtain:

Theorem 6. *Under collusion there exists a contract that is linear on the sum of payoffs that maximizes V_P.*

$$w(y) = \alpha_c (y_1 + y_2)$$

Proof. This follows from Lemmas (30) and (31) along with Lemmas 2,4,5 and 6 in Carroll (2015), using the same argument as in his main theorem and replacing his y for $y_1 + y_2$. □

Corollary 6. *If $\mathcal{A}_0$ has the full support property then all optimal contracts are of the form:*

$$w(y) = \alpha_c (y_1 + y_2) \qquad \textit{where:} \quad \alpha_c = \sqrt{\frac{c^\star}{E_{F^\star}[y_1 + y_2]}}$$

for $(F^\star, c^\star) \in \underset{(F,c)\in\mathcal{A}_0}{\textit{argmax}} \left[\sqrt{E_F[y_1 + y_2]} - \sqrt{c}\right]^2$. *The payoff of the principals is:*

$$V_P(w) = \left[\sqrt{E_{F^\star}[y_1 + y_2]} - \sqrt{c^\star}\right]^2$$

Proof. Just as in Carroll (2015). □

A.8 Lower bound on costs

The model allows for large amounts of output produced for free. The distribution that provides the worst case guarantee is one with zero cost. To rule this out we suppose that the principal knows a lower bound on the cost of producing any given level of expected output. In this section we prove that LRS contracts are a best response to LRS contracts when allowing for a lower bound on costs.

Let $b : \mathbb{R} \to \mathbb{R}_+$ be a convex function satisfying $b(0) = 0$. A technology is a compact set $\mathcal{A} \subset \Delta(Y) \times \mathbb{R}_+$ such that for any $(F, c) \in \mathcal{A}$ we have that $c \geq b(\mathbb{E}_F[y_1 + y_2])$. This holds also for any $(F, c) \in \mathcal{A}_0$ with a strict inequality (i.e. $c > b(\mathbb{E}_F(y))$ if $(F, c) \in \mathcal{A}_0$) This is similar to the positive cost assumption when there was no lower bound on costs. Now suppose that for all $V_i(w)$ is still the infimum of $V_i(w|\mathcal{A})$ over all technologies $\mathcal{A} \supset \mathcal{A}_0$. We furthermore assume that $\mathcal{A}_0$ satisfies the full support property.

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition 17. *Let $(F, c) \in A^\star(w|\mathcal{A})$. For $\mathcal{A} \supseteq \mathcal{A}_0$, it holds that:*

$$E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])$$

Moreover, if $(F, c) \in A^\star(w|\mathcal{A})$ then $F \in \mathcal{F}$ where:

$$\mathcal{F} = \{F \in \Delta(Y) \,|E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])\}$$

Proof. To see the first inequality let $(F, c) \in A^\star(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F[w_1(y) + w_2(y)] - b(E_F[y_1 + y_2]) \geq E_F[w_1(y) + w_2(y)] - c \geq V_A(w|\mathcal{A}) \geq V_A(w|\mathcal{A}_0)$$

where the first inequality holds since $c \geq b(E_F[y_1 + y_2])$. Then $F \in \mathcal{F}$. $\square$

The following results are valid for any scheme w that provides positive guarantees for principal i.

We formally define them as follows:

Eligibility: A contract w is *eligible* for principal i if: $V_i(w) > 0$.

Lemma 32. *Let w be an eligible contract for principal i, then*

$$V_i(w) = \min_{F \in \mathcal{F}} \quad E_F[y_i - w_i(y)]$$

.

Moreover if $F \in \underset{F \in \mathcal{F}}{argmin} E_F[y_i - w_i(y)]$ then

$$E_F[w_1(y) + w_2(y)] = V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])$$

.

Proof. The proof is broken into the following two propositions.

Proposition 18. *Let w be an eligible contract then: $V_i(w) = \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$*

Proof. First note that it must be that: $V_i(w) \geq \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$. Using the definition of $V_i(w)$:

$$V_i(w) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} \min_{(F,c) \in A^\star(w|\mathcal{A})} E_F[y_i - w_i(y)] \geq \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$$

Where the inequality follows because if $(F, c) \in A^\star(w|\mathcal{A})$ then $F \in \mathcal{F}$.

Now we can establish equality. Suppose not, then it must be that:

$$V_i(w) > \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$$

Let $F \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i(y)]$.

We have that $E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])$. Then consider the technology $\mathcal{A}' = \mathcal{A}_0 \cup \{(F, b(E_F[y_1 + y_2]))\}$. Then we have that $(F, b(E_F[y_1 + y_2])) \in A^\star\left(w|\mathcal{A}'\right)$, which implies that

$$V_i(w) \leq V_i\left(w|\mathcal{A}'\right) = \min_{(F,c) \in A^\star\left(w|\mathcal{A}'\right)} E_F[y_i - w_i(y)] \leq \min_{F \in \mathcal{F}} E_F[y_i - w_i].$$

□

Proposition 19. *Let w be an eligible contract for principal i. If $F \in \underset{F \in \mathcal{F}}{argmin} E_F[y_i - w_i(y)]$ then $E_F[w_1(y) + w_2(y)] = V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])$.*

Proof. To prove this, let $F \in \underset{F\in\mathcal{F}}{\text{argmin}} E_F [y_i - w_i]$ and suppose for a contradiction that $E_F [w_1 (y) + w_2 (y)] > V_A (w|\mathcal{A}_0) + b (E_F [y_1 + y_2])$.
Let $\epsilon \in [0, 1]$ and consider $F_\epsilon = (1 - \epsilon) F + \epsilon\delta_{(0,0)}$ and $\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \left\{ \left(F_\epsilon, b \left(E_{F'} (y_1 + y_2)\right)\right) \right\}$. It follows that $\left\{ \left(F_\epsilon, b \left(E_{F'} (y_1 + y_2)\right)\right) \right\} = A^\star (w|\mathcal{A}_\epsilon)$ for low enough ϵ since the payoff to the agent is strictly greater choosing F_ϵ at a cost of $b \left(E_{F'} (y_1 + y_2)\right)$, than choosing any $(F, c) \in \mathcal{A}_0$. Note that by convexity $b (E_{F_\epsilon} (y_1 + y_2)) \leq (1 - \epsilon) b (E_F [y_1 + y_2]) + \epsilon b (0)$.
The payoff to the principal is then:

$$V_i \left(w|\mathcal{A}_\epsilon^{'}\right) = (1 - \epsilon) E_F [y_i - w_i (y)] - \epsilon w_i (0, 0) < E_F [y_i - w_i (y)] = V_i (w) \leq V_i \left(w|\mathcal{A}_\epsilon^{'}\right)$$

which is a contradiction with the definition of $V_i (w)$. The strict inequality follows from $E_F [y_i - w_i (y)] > 0$ by eligibility and $w_i (0, 0) \geq 0$ by the agent's limited liability. □

Joining the two propositions the proof of the lemma is completed. □

Now suppose that principal j offers a contract of the form: $w_j (y) = (1 - \theta_j) y_j + \theta_j (\bar{y}_i - y_i)$. And consider $w_i : Y \to \mathbb{R}_+$ so that (w_1, w_2) is an eligible contract scheme for principal i. Furthermore suppose that there does not exist $\theta_i \in [0, 1 - \theta_j]$ and k such that $w_i (y_1, y_2) = (1 - \theta_i) y_i + \theta_i (\bar{y}_j - y_j) + k$. Our objective is to show that in this case there exist an alternative contract $w_i^{'}$ that dominates w_i, where $w_i^{'} (y) = \left(1 - \theta_i^{'}\right) y_i + \theta_i^{'} (\bar{y}_j - y_j)$ for some $\theta_i^{'} \in [0, 1 - \theta_j]$.
The same separation argument as in the main theorem will follow. However the separation is done in outcome space and not in payoff space.
Define

$$t (x) = \max \{b (x) + V_A (w|\mathcal{A}_0) , (1 - \theta_j) x + \theta_j \bar{y}_i - V_i (w)\}$$

Clearly $t (x)$ is convex.
Now let $S \in \mathbb{R}^2$ be the convex hull of pairs $(y_1 + y_2, w_i (y_1, y_2) + w_j (y_1, y_2))$ for all $(y_1, y_2) \in Y$, and let $T \in \mathbb{R}^2$ be the set of all pairs (x, z) such that x lies in the convex hull of points $y_1 + y_2$, and $z > t (x)$[3]. Both of these sets are convex[4].

[3] Formally $T = \left\{(x, z) \in \mathbb{R}^2 | x \in [\min_Y \{y_1 + y_2\} , \max_Y \{y_1 + y_2\}] \quad \wedge \quad z > t (x)\right\}$.

[4] The first one is a convex hull, so it is convex, the second one is the intersection of the upper contour set of a convex function (a convex set) with two half spaces (convex sets), so it is convex as well.

We claim that S and T are disjoint. If not then there exists $F \in \Delta Y$ such that

$$E_F\left[w_i\left(y_1, y_2\right) + w_j\left(y_1, y_2\right)\right] > t\left(E_F\left[y_1 + y_2\right]\right)$$

In particular we have that

$$E_F\left[w_i\left(y_1, y_2\right) + w_j\left(y_1, y_2\right)\right] > b\left(E_F\left[y_1 + y_2\right]\right) + V_A\left(w|\mathcal{A}_0\right)$$

Also we have that

$$E_F\left[w_i\left(y_1, y_2\right) + w_j\left(y_1, y_2\right)\right] > \left(1 - \theta_j\right) E_F\left[y_1 + y_2\right] + \theta_j \bar{y}_i - V_i\left(w\right)$$

Replacing by $w_j\left(y\right) = \left(1 - \theta_j\right) y_j + \theta_j\left(\bar{y}_i - y_i\right)$ the second inequality becomes:

$$V_i\left(w\right) > E_F\left[y_i - w_i\left(y_1, y_2\right)\right]$$

From Lemma 32 we know that $V_i\left(w\right) = \min_{F \in \mathcal{F}} E_F\left[y_i - w_i\left(y_1, y_2\right)\right]$, but from the first inequality we know that $F \in \mathcal{F}$, this is a contradiction.
Then by the separating hyperplane theorem. There exists λ and μ and k with $\left(\lambda, \mu\right) \neq \left(0, 0\right)$ such that

$$\begin{aligned} \lambda\left(y_1 + y_2\right) + \mu z \leq k \qquad & \forall\left(\left(y_1 + y_2\right), z\right) \in S \\ \lambda\left(y_1 + y_2\right) + \mu z \geq k \qquad & \forall\left(\left(y_1 + y_2\right), z\right) \in T \end{aligned} \tag{A.52}$$

The second inequality implies that $\mu \geq 0$. Now suppose $\mu = 0$ then it must be that $\lambda = 0$, which is a contradiction. This implies that $\mu > 0$.
Now

$$\lambda\left(y_1 + y_2\right) + \mu z \leq k \qquad \forall\left(\left(y_1 + y_2\right), z\right) \in S$$

implies that

$$w_i\left(y_1, y_2\right) + w_j\left(y_1, y_2\right) \leq \frac{k - \lambda\left(y_1 + y_2\right)}{\mu}$$

Now consider the following wage

$$\begin{aligned} w_i^{'}\left(y_1, y_2\right) &= \frac{k - \lambda\left(y_1 + y_2\right)}{\mu} - w_j\left(y_1, y_2\right) \\ &= \theta_i^{'} y_i + \left(1 - \theta_i^{'}\right)\left(\overline{y}_j - y_j\right) + k^{'} \end{aligned}$$

where $\theta_i^{'} = \theta_j - \frac{\lambda}{\mu}$ and $k^{'} = \frac{k}{\mu} - \theta_j \bar{y}_i - \left(1 - \theta_i^{'}\right) \overline{y}_j$. Note that $w_i^{'} \geq w_i$ pointwise, and recall that $w_i \neq w_i^{'}$ by assumption. Now we need to check that $V_i\left(w_i^{'}\right) \geq V_i\left(w_i\right)$.

Consider any technology $\mathcal{A} \supset \mathcal{A}_0$. Then we have that $V_A\left(w^{'}|\mathcal{A}\right) \geq V_A\left(w^{'}|\mathcal{A}_0\right) > V_A\left(w|\mathcal{A}_0\right)$. The last inequality follows because $\mathcal{A}_0$ has full support and $w_i^{'}\left(y\right) > w_i\left(y\right)$ for some $y \in Y$.

Now let $(F, c) \in \mathcal{A}$ such that:

$$(F, c) = \underset{(F,c)\in A^{\star}\left(w^{'}|\mathcal{A}\right)}{\arg\min} E_F\left[y_i - w_i^{'}\left(y\right)\right]$$

Then $V_i\left(w^{'}|A\right) = E_F\left[y_i - w_i^{'}\left(y\right)\right]$. Now we have that from equation A.52:

$$\begin{aligned}
t\left(E_F\left[y_1 + y_2\right]\right) &\geq E_F\left(\frac{k - \lambda\left(y_1 + y_2\right)}{\mu}\right) \\
&= E_F\left[w_1^{'}\left(y\right) + w_2\left(y\right)\right] \\
&= V_A\left(w^{'}|\mathcal{A}\right) + c \\
&> V_A\left(w|\mathcal{A}_0\right) + c \\
&\geq V_A\left(w|\mathcal{A}_0\right) + b\left(E_F\left[y_1 + y_2\right]\right)
\end{aligned}$$

Since the inequality is strict then we have that $t\left(E_F\left[y_1 + y_2\right]\right) = \left(1 - \theta_j\right) E_F\left[y_1 + y_2\right] + \theta_j \bar{y}_i - V_i\left(w\right)$

Then we have that

$$\begin{aligned}
V_i\left(w^{'}|A\right) &= E_F\left[y_i - w_i^{'}\left(y\right)\right] \\
&= E_F\left[y_i + w_j\left(y\right)\right] - E_F\left[w_i^{'}\left(y\right) + w_j\left(y\right)\right] \\
&= \left(1 - \theta_j\right) E_F\left[y_1 + y_2\right] + \theta_j \bar{y}_i - E_F\left[w_i^{'}\left(y\right) + w_j\left(y\right)\right] \\
&= t\left(E_F\left[y_1 + y_2\right]\right) + V_i\left(w\right) - E_F\left[w_i^{'}\left(y\right) + w_j\left(y\right)\right] \\
&\geq V_i\left(w\right)
\end{aligned}$$

Since this holds for all $\mathcal{A} \supset \mathcal{A}_0$. We get that $V_i\left(w^{'}\right) \geq V_i\left(w\right)$. So any contract w_i (as described above) can be dominated by a contract of the form:

$$w_i^{'}\left(y_1, y_2\right) = \theta_i^{'} y_i + \left(1 - \theta_i^{'}\right)\left(\overline{y}_j - y_j\right) + k^{'}$$

This contract can be improved upon by dropping the constant $k^{'}$. Doing so makes it satisfy limited liability with equality (when $y_i = 0$ and $y_j = \overline{y}_j$), it also does not affect the problem of the agent, and it weakly increase the value of the principal (strictly if $k' > 0$).

Appendix B

Applications of Robustness and Linearity in Common Agency

B.1 Taxing Multinational Companies

B.1.1 Caring only about domestic profits

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition 20. *Let $(F, c) \in A^\star(t_1, t_2|\mathcal{A})$. For $\mathcal{A} \supseteq \mathcal{A}_0$, it holds that:*

$$E_F[y_1 - t_1(y) + y_2 - t_2(y)] \geq V_A(t_1, t_2|\mathcal{A}_0)$$

Moreover, if $(F, c) \in A^\star(t_1, t_2|\mathcal{A})$ then $F \in \mathcal{F}$ where:

$$\mathcal{F} = \{F \in \Delta(Y) \,|E_F[y_1 - t_1(y) + y_2 - t_2(y)] \geq V_A(t_1, t_2|\mathcal{A}_0)\}$$

Proof. To see the first inequality let $(F, c) \in A^\star(t_1, t_2|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F[y_1 - t_1(y) + y_2 - t_2(y)] \geq E_F[y_1 - t_1(y) + y_2 - t_2(y)] - c \geq V_A(t|\mathcal{A}) \geq V_A(t|\mathcal{A}_0)$$

Then $F \in \mathcal{F}$. □

Lemma 45 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above). Note that $\mathcal{F}$ only depends on the contract and the known set of actions $\mathcal{A}_0$. In this way we replace the complexity of the definition of $V_i(t)$ with an object that depends only on known elements. The following results are valid for any scheme t that provides positive guarantees for principal i

We formally define them as follows:

Eligibility: A contract t is *eligible* for principal i if: $V_i(t) > 0$.

Lemma 33. *Let t be an eligible contract for principal i, then $V_i(t) = \min_{F\in\mathcal{F}} E_F[\rho y_i + t_i]$. Moreover if $F \in \underset{F\in\mathcal{F}}{argmin} E_F[\rho y_i + t_i(y)]$ then $E_F[y_1 - t_1(y) + y_2 - t_2(y)] = V_A(w|\mathcal{A}_0)$.*

Proof. The proof is almost identical to that of 11 with the appropriate modification of the payoff function. □

Given the known action set $\mathcal{A}_0$, the next lemma links the principal's guaranteed payoff to the agent's payoff in an affine way. This link allows the principal to increase her own guaranteed payoff by controlling the payoff given to the agent. Lemma 46 also offers a relation between any contract t_i, the outcome (y_i, y_j) and the contract t_j offered by the other principal.

Lemma 34. *Let t be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$t_i(y) \geq \frac{\lambda-\rho}{1+\lambda}y_i + \frac{\lambda}{1+\lambda}y_j - \frac{\lambda}{1+\lambda}t_j(y) + \frac{1}{1+\lambda}k \tag{B.1}$$

$$V_i(t) = k + \lambda V_A(t|\mathcal{A}_0) \tag{B.2}$$

Proof. Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points $(y_1 - t_1(y) + y_2 - t_2(y), \rho y_i + t_i)$ for $y \in Y$.

Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u > V_A(t|\mathcal{A}_0)$ and $v < V_i(t)$. The proof follows the steps of Lemma 12 □

The following two lemmas (35 and 36) use the relation between the principals' contracts derived in Lemma 46 to construct an alternative contract that dominates the original one, in the sense that it weakly increases principal i's guaranteed payoff. Since the relation obtained in (B.5) is affine in output and the other principal's contract, the alternative contract constructed below will inherit that form.

Lemma 35. *Let $t = (t_i, t_j)$ with t_i satisfying (B.5) and (B.6). Then the contract*

$$t_i^{'}(y) = \frac{\lambda - \rho}{1+\lambda} y_i + \frac{\lambda}{1+\lambda} y_j - \frac{\lambda}{1+\lambda} t_j(y) + \frac{1}{1+\lambda} k$$

satisfies $V_i\left(t_i^{'}, t_j\right) \geq V_i(t)$.

Proof. The proof is identical to that of Lemma 13. □

Lemma 36. *Let $\left(t_i^{'}, t_j\right)$ with $t_i^{'}$ an affine contract on y_i , y_j and t_j, there is an affine contract $t_i^{''}$ that does at least as well as $t_i^{'}$ for principal i: $V_i\left(t_i^{''}, t_j\right) \geq V_i\left(t_i^{'}, t_j\right)$, with strict inequality unless $\max_y t_i^{'}(y) = y_i$.*

Proof. The proof is identical to that of Lemma 14. □

The last two lemmas (37 and 38) establish the form of the principal's payoffs under the worldwide taxes and the existence of an optimal contract in that class.

Lemma 37. *For t an eligible contract scheme such that $t_i \in \mathcal{W}_i(t_j)$ is characterized by $\alpha \in (0,1]$, then:*

$$V_i(t) = \frac{1-\alpha}{\alpha} V_A(t|\mathcal{A}_0) + k = \max_{(F,c)\in\mathcal{A}_0} \left((1-\alpha) E_F\left[y_i - t_i(y) + y_j - t_j(y)\right] - \frac{1-\alpha}{\alpha} c \right) + \alpha k$$

This also holds for $\alpha = 0$ if we interpret the term $\frac{1-\alpha}{\alpha} c$ as 0 when $c = 0$ and ∞ for $c > 0$.

Proof. The proof is identical to that of Lemma 15. □

Lemma 38. *In the class of WT contracts $w_i \in \mathcal{W}_i(w_j)$ there exists an optimal one for principal i.*

Proof. The proof is identical to that of Lemma 16. □

Theorem 7. *For any contract t_j there exists $\alpha \in [0,1]$ such that:*

$$t_i^{'}(y) = (1-\alpha-\alpha\rho) y_i + (1-\alpha) y_j - (1-\alpha) t_j(y) + \alpha k \qquad \text{and} \qquad t_i^{'}(t_j) \in BR_i(t_j)$$

where $k(\alpha)$ is such that $\min_y \left(y_i - t_i^{'}(y)\right) = 0$.

Proof. The proof is identical to that of Theorem 2 □

B.1.2 Welfare as a weighted sum of taxes and profits of the multinational

Lemma 39 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above).

Eligibility: A contract t is *eligible* for principal i if: $V_i(t) > 0$.

Lemma 39. *Let t be an eligible contract for principal i, then*

$$V_i(t) = \min_{F \in \mathcal{F}} \quad E_F\left[\rho\left(y_1 - t_1 + y_2 - t_2\right) + t_i\right]$$

Moreover if $F \in \underset{F \in \mathcal{F}}{argmin} E_F\left[\rho\left(y_1 - t_1 + y_2 - t_2\right) + t_i(y)\right]$ then

$$E_F\left[y_1 - t_1(y) + y_2 - t_2(y)\right] = V_A\left(w|\mathcal{A}_0\right)$$

Proof. The proof is identical to that of Lemma 11. □

Lemma 40. *Let t be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$t_i(y) \geq \frac{\lambda - \rho}{1 + \lambda - \rho} y_i + \frac{\lambda - \rho}{1 + \lambda - \rho} y_j - \frac{\lambda - \rho}{1 + \lambda - \rho} t_j(y) + \frac{1}{1 + \lambda - \rho} k \qquad \text{(B.3)}$$

$$V_i(t) = k + \lambda V_A\left(t|\mathcal{A}_0\right) \qquad \text{(B.4)}$$

Proof. Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points

$$\left(y_1 - t_1(y) + y_2 - t_2(y), \rho\left(y_1 - t_1 + y_2 - t_2\right) + t_i\right)$$

for $y \in Y$.

Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u > V_A\left(t|\mathcal{A}_0\right)$ and $v < V_i(t)$. Note T is convex. The rest follows as in the proof of Lemma 12. □

Lemma 41. *Let $t = (t_i, t_j)$ with t_i satisfying (B.3) and (B.4). Then the contract*

$$t_i^{'}(y) = \frac{\lambda - \rho}{1 + \lambda - \rho} y_i + \frac{\lambda - \rho}{1 + \lambda - \rho} y_j - \frac{\lambda - \rho}{1 + \lambda - \rho} t_j(y) + \frac{1}{1 + \lambda - \rho} k$$

satisfies $V_i\left(t_i^{'}, t_j\right) \geq V_i(t)$.

Proof. The proof is identical to that of Lemma 13 □

Lemma 42. *Let* $\left(t_i^{'}, t_j\right)$ *with* $t_i^{'}$ *an affine contract on* y_i *,* y_j *and* t_j*, there is an affine contract* $t_i^{''}$ *that does at least as well as* $t_i^{'}$ *for principal* i*:* $V_i\left(t_i^{''}, t_j\right) \geq V_i\left(t_i^{'}, t_j\right)$*, with strict inequality unless* $\max_y t_i^{'}(y) = y_i$.

Proof. The proof is identical to that of Lemma 14 □

The last two lemmas (43 and 44) establish the form of the principal's payoffs under the worldwide taxes and the existence of an optimal contract in that class.

Lemma 43. *For* t *an eligible contract scheme such that* $t_i \in \mathcal{W}_i(t_j)$ *is characterized by* $\alpha \in (0, 1]$*, then:*

$$V_i(t) = \frac{1-\alpha}{\alpha} V_A(t|\mathcal{A}_0) + k = \max_{(F,c)\in\mathcal{A}_0}\left((1-\alpha) E_F\left[y_i - t_i(y) + y_j - t_j(y)\right] - \frac{1-\alpha}{\alpha}c\right) + \alpha k$$

This also holds for $\alpha = 0$ *if we interpret the term* $\frac{1-\alpha}{\alpha}c$ *as 0 when* $c = 0$ *and* ∞ *for* $c > 0$.

Proof. The proof is identical to that of Lemma 15. □

Lemma 44. *In the class of WT contracts* $w_i \in \mathcal{W}_i(w_j)$ *there exists an optimal one for principal* i*.*

Proof. The proof is identical to that of Lemma 16. □

Theorem 8. *For any contract* t_j *there exists* $\alpha \in [0, 1]$ *such that:*

$$t_i^{'}(y) = \frac{1-\alpha-\alpha\rho}{1-\alpha\rho}\left(y_i + y_j - t_j(y)\right) + \frac{\alpha}{1-\alpha\rho}k \qquad \text{and} \qquad t_i^{'}(t_j) \in BR_i(t_j)$$

where $k(\alpha)$ *is such that* $\min_y \left(y_i - t_i^{'}(y)\right) = 0$.

Proof. The proof is identical to that of Theorem 2. □

B.1.3 Two Period Model of Taxation Proof

By relabeling variables:

$$w_1(y_1) = y_1 - t_1(y_1) \qquad \text{and} \qquad w_2(y_1, y_2) = y_2 - t_2(y_1, y_2)$$

the taxation problem is isomorphic to the problem below.
Consider a principal and an agent that contract for two periods. In each period the agent takes an action (a_t) that induces a distribution over output (F_t) at some cost (c_t). The principal only observes realized output and is able to condition contracts on it subject to limited liability, so that $w_1 : Y \to \mathbb{R}_+$ and $w_2 = Y \times Y \to \mathbb{R}_+$. Both the principal and the agent are assumed risk neutral. The principal has commitment.
The timing of the game is as follows:

1. The principal offers a contract scheme (w_1, w_2) to the agent.

2. The agent chooses a_1, output y_1 realizes, and payments are delivered.

3. The agent chooses a_2, output y_2 realizes, and payments are delivered.

4. The game ends.

The action set of the agent ($\mathcal{A}$) is unknown to the principal, save from a minimal set of actions $\mathcal{A}_0 \subseteq \mathcal{A}$.
Given a contract scheme (w_1, w_2) and an action set ($\mathcal{A}$) the problem of the agent is:

$$V_A(w_1, w_2, \mathcal{A}) = \max_{a_1 \in \mathcal{A}} \left\{ E_{F_1} \left[(w_1(y_1) - c_1) + \max_{a_2 \in \mathcal{A}} \left\{ (E_{F_2}[w_2(y_1, y_2)] - c_2) \right\} \right] \right\}$$

Notice that $a_1 = (F_1, c_1) \in \Delta Y \times \mathbb{R}_+$ is only a function of the contract scheme, while $a_2(y_1) = (F_2(y_1), c_2(y_1)) \in \Delta Y \times \mathbb{R}_+$ is a function of y_1. Let $A^\star(w_1, w_2|\mathcal{A})$ be the set of maximizers for the agent.
The principal's guaranteed payoff is given by:

$$V(w_1, w_2) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} \left\{ \min_{a_1, a_2 \in A^\star(w_1, w_2|\mathcal{A})} \left\{ E_{F_1} \left[(y_1 - w_1(y_1)) + E_{F_2(y_1)} [y_2 - w_2(y_1, y_2)] \right] \right\} \right\}$$

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition 21. *Let* $(a_1, a_2) \in A^{\star}(w|\mathcal{A})$*. For* $\mathcal{A} \supseteq \mathcal{A}_0$*, it holds that:*

$$E_{F_1}\left[w_1(y_1) + E_{F_2(y_1)}\left[w_2(y_1, y_2)\right]\right] \geq V_A(w|\mathcal{A}_0)$$

Moreover, if $(a_1, a_2) \in A^{\star}(w|\mathcal{A})$ *then* $F_1, F_2(y_1) \in \mathcal{F}$ *where:*

$$\mathcal{F} = \left\{F_1 \in \Delta Y\,,\, F_2(y_1) \in \Delta Y \mid E_{F_1}\left[w_1(y_1) + E_{F_2(y_1)}\left[w_2(y_1, y_2)\right]\right] \geq V_A(w|\mathcal{A}_0)\right\}$$

Proof. To see the first inequality let $(a_1, a_2) \in A^{\star}(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_{F_1}\left[w_1(y_1) + E_{F_2(y_1)}\left[w_2(y_1, y_2)\right]\right] \geq E_{F_1}\left[w_1(y_1) - c_1 + E_{F_2(y_1)}\left[w_2(y_1, y_2) - c_2\right]\right]$$

$$\geq V_A(w|\mathcal{A}) \geq V_A(w|\mathcal{A}_0)$$

Then $F_1, F_2(y_1) \in \mathcal{F}$. □

Lemma 45 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above). Note that $\mathcal{F}$ only depends on the contract and the known set of actions $\mathcal{A}_0$. In this way we replace the complexity of the definition of V with an object that depends only on known elements. The following results are valid for any scheme (w_1, w_2) that provides positive guarantees for principal i

We formally define them as follows:

Eligibility: A contract scheme (w_1, w_2) is *eligible* for principal i if: $V(w_1, w_2) > 0$.

Lemma 45. *Let* (w_1, w_2) *be an eligible contract scheme for principal, then*

$$V(w_1, w_2) = \min_{(F_1, F_2) \in \mathcal{F}} E_{F_1}\left[y_1 - w_1(y_1) + E_{F_2(y_1)}\left[y_2 - w_2(y_1, y_2)\right]\right]$$

Moreover if

$$(F_1, F_2) \in \underset{(F_1, F_2) \in \mathcal{F}}{\mathit{argmin}}\, E_{F_1}\left[y_1 - w_1(y_1) + E_{F_2(y_1)}\left[y_2 - w_2(y_1, y_2)\right]\right]$$

then $E_{F_1}\left[w_1(y_1) + E_{F_2(y_1)}\left[w_2(y_1, y_2)\right]\right] = V_A(w|\mathcal{A}_0)$.

Proof. The proof is broken into the following two propositions.

Proposition 22. *Let* (w_1, w_2) *be an eligible contract scheme, then:*

$$V(w_1, w_2) = \min_{(F_1, F_2) \in \mathcal{F}} E_{F_1}\left[y_1 - w_1(y_1) + E_{F_2(y_1)}\left[y_2 - w_2(y_1, y_2)\right]\right]$$

Proof. First note that it must be that: $V(w_1, w_2) \geq \min\limits_{(F_1,F_2)\in\mathcal{F}} E_{F_1}[y_1 - w_1 + E_{F_2}[y_2 - w_2]]$.
Using the definition of $V(w_1, w_2)$:

$$V(w_1, w_2) = \inf_{\mathcal{A}\supseteq\mathcal{A}_0}\left\{\min_{a_1,a_2\in A^\star(w_1,w_2|\mathcal{A})}\left\{E_{F_1}\left[(y_1 - w_1) + E_{F_2(y_1)}[y_2 - w_2]\right]\right\}\right\}$$

$$\geq \min_{(F_1,F_2)\in\mathcal{F}} E_{F_1}\left[y_1 - w_1 + E_{F_2(y_1)}[y_2 - w_2]\right]$$

Where the inequality follows because if $(a_1, a_2) \in A^\star(w|\mathcal{A})$ then $F_1, F_2(y_1) \in \mathcal{F}$.
Now we can establish equality. Suppose not, then it must be that:

$$V(w) > \min_{(F_1,F_2)\in\mathcal{F}} \quad E_{F_1}\left[y_1 - w_1 + E_{F_2(y_1)}[y_2 - w_2]\right]$$

Let $(F_1, F_2) \in \operatorname*{argmin}\limits_{(F_1,F_2)\in\mathcal{F}} E_{F_1}\left[y_1 - w_1 + E_{F_2(y_1)}[y_2 - w_2]\right]$.
Define $F(y_1, y_2) = F_1(y_1) F_2(y_2|y_1)$ as the joint distribution implied by F_1 and F_2. There are two options:

1. F does not place full support on the values of y_1 and y_2 that maximize $w_1(y_1) + w_2(y_1, y_2)$.

    - Let $(\hat{y}_1, \hat{y}_2) \in \operatorname{argmax}[w_1 + w_2]$. Let $\hat{F} = \delta_{(\hat{y}_1,\hat{y}_2)}$ be a (joint) distribution with full probability on $(\hat{y}_1, \hat{y}_2)$.
    - Let $\epsilon \in [0, 1]$ and $F' = (1 - \epsilon) F + \epsilon\hat{F}$ and F_1', F_2' such that $F_1'(y_1) F_2'(y_2|y_1) = F'(y_1, y_2)$. Define an action set $\mathcal{A}' = \mathcal{A}_0 \cup \left\{\left(F_1', 0\right), \left(F_2', 0\right)\right\}$.
    - It follows that the unique optimal action of the agent in $\mathcal{A}'$ is $\left\{\left(F_1', 0\right), \left(F_2', 0\right)\right\}$. Note:

    $$E_{F_1'}\left[w_1 + E_{F_2'}[w_2]\right] = E_{F'}[w_1 + w_2]$$

    $$= (1 - \epsilon) E_F[w_1 + w_2] + \epsilon \max[w_1 + w_2] > E_F[w_1 + w_2]$$

    $$= E_{F_1}[w_1 + E_{F_2}[w_2]] \geq V_A(w|\mathcal{A}_0)$$

    so $\left\{\left(F_1', 0\right), \left(F_2', 0\right)\right\}$ dominates the agent's payoff for any action in $\mathcal{A}_0$.
    - Principal i's payoff under $\left\{\left(F_1', 0\right), \left(F_2', 0\right)\right\}$ is given by:

    $$V\left(w|\mathcal{A}'\right) = E_{F'}[y_1 - w_1 + y_2 - w_2]$$

$$= (1-\epsilon) E_F [y_1 - w_1 + y_2 - w_2] + \epsilon E_{\hat{F}} [y_1 - w_1 + y_2 - w_2]$$

Recall that $\left\{ \left(F_1^{\prime}, 0\right), \left(F_2^{\prime}, 0\right) \right\}$ is the unique solution for the agent. This has to be an upper bound for $V_i(w)$ by definition.

- Joining:

$$V_i(w) \leq (1-\epsilon) E_F [y_1 - w_1 + y_2 - w_2] + \epsilon E_{\hat{F}} [y_1 - w_1 + y_2 - w_2]$$

which holds for all $\epsilon > 0$. Letting $\epsilon \to 0$:

$$V_i(w) \leq E_F [y_1 - w_1 + y_2 - w_2] = E_{F_1} [y_1 - w_1 + E_{F_2} [y_2 - w_2]]$$

$$= \underset{(F_1, F_2) \in \mathcal{F}}{\operatorname{argmin}} E_{F_1} [y_1 - w_1 + E_{F_2} [y_2 - w_2]]$$

- This proves equality.

2. F places full support on the values of (y_1, y_2) that maximize $w_1 + w_2$. There are still two possible cases:

   (a) $E_F [w_1 + w_2] = E_{F_1} [w_1 + E_{F_2} [w_2]] > V_A (w|\mathcal{A}_0)$.
   Consider $\mathcal{A}^{\prime} = \mathcal{A}_0 \cup \{(F_1, 0), (F_2, 0)\}$. $\{(F_1, 0), (F_2, 0)\}$ is the unique optimal action for the agent in $\mathcal{A}^{\prime}$ since it gives higher payoff and has zero cost.
   Principal i's payoff under $\{(F_1, 0), (F_2, 0)\}$ is given by:

$$V_i \left(w|\mathcal{A}^{\prime}\right) = E_{F_1} [y_1 - w_1 + E_{F_2} [y_2 - w_2]]$$

   This has to be an upper bound for $V_i(w)$ by definition:

$$V_i(w) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} V_i(w|\mathcal{A}) \leq V_i \left(w|\mathcal{A}^{\prime}\right)$$

   Joining:

$$V_i(w) \leq E_{F_1} [y_1 - w_1 + E_{F_2} [y_2 - w_2]]$$

$$= \min_{(F_1, F_2) \in \mathcal{F}} \quad E_{F_1} \left[y_1 - w_1 + E_{F_2(y_1)} [y_2 - w_2]\right]$$

   This proves equality.

(b) $E_F[w_1 + w_2] = E_{F_1}[w_1 + E_{F_2}[w_2]] = V_A(w|\mathcal{A}_0)$.

This implies $V_A(w|\mathcal{A}_0) = \max[w_1 + w_2]$ which can only be satisfied when producing at zero cost. This violates the Positive-cost assumption hence this case cannot happen.

□

Proposition 23. *Let w be an eligible contract for principal i. If*

$$(F_1, F_2) \in \underset{(F_1,F_2)\in\mathcal{F}}{\textit{argmin}}\, E_{F_1}\left[y_1 - w_1(y_1) + E_{F_2(y_1)}[y_2 - w_2(y_1, y_2)]\right]$$

then

$$E_{F_1}[w_1 + E_{F_2}[w_2]] = V_A(w|\mathcal{A}_0)$$

Proof. To prove this Let

$$(F_1, F_2) \in \underset{(F_1,F_2)\in\mathcal{F}}{\text{argmin}}\, E_{F_1}\left[y_1 - w_1(y_1) + E_{F_2(y_1)}[y_2 - w_2(y_1, y_2)]\right]$$

and suppose for a contradiction that $E_{F_1}[w_1 + E_{F_2}[w_2]] > V_A(w|\mathcal{A}_0)$.
Let $F(y_1, y_2) = F_1(y_1) F_2(y_2|y_1)$ be the joint distribution implied by F_1 and F_2.
Consider $F^{'} = (1-\epsilon) F + \epsilon\delta_{(0,0)}$ and $F_1^{'}$ and $F_2^{'}$ such that $F^{'}(y_1, y_2) = F_1^{'}(y_1) F_2^{'}(y_2|y_1)$ and $\mathcal{A}_\epsilon^{'} = \mathcal{A}_0 \cup \left\{\left(F_1^{'}, 0\right), \left(F_2^{'}, 0\right)\right\}$. It follows that $\left\{\left(F_1^{'}, 0\right), \left(F_2^{'}, 0\right)\right\} = A^\star\left(w|\mathcal{A}_\epsilon^{'}\right)$ for low enough ϵ since the payoff to the agent is strictly greater choosing $F^{'}$ at zero cost than choosing any $(F, c) \in \mathcal{A}_0$.
The payoff to the principal is then:

$$V_i\left(w|\mathcal{A}_\epsilon^{'}\right) = (1-\epsilon) E_F[y_1 - w_1(y_1) + y_2 - w_2(y_1, y_2)] - \epsilon(w_1(0) + w_2(0,0))$$
$$< E_{F_1}[y_1 - w_1(y_1) + y_2 - w_2(y_1, y_2)] = V_i(w) \le V_i\left(w|\mathcal{A}_\epsilon^{'}\right)$$

which is a contradiction with the definition of $V_i(w)$.
The strict inequality follows from $E_F[y_1 - w_1(y_1) + y_2 - w_2(y_1, y_2)] > 0$ by eligibility and $w_i(0,0) \ge 0$ by the agent's limited liability. □

Joining the two propositions the proof of the lemma is completed. □

Given the known action set $\mathcal{A}_0$, the next lemma links the principal's guaranteed payoff to the agent's payoff in an affine way. This link allows the principal to increase her own guaranteed payoff by controlling the payoff given to the agent. Lemma 46 also offers a relation between any contract (w_1, w_2) and the outcomes (y_1, y_2).

The affine link between the agent's payoff and the principal's payoff is a crucial element in providing incentives for the agent. Given the partial knowledge over the agent's set of actions the principal's optimal strategy is to tie her payoff to that of the agent, thus aligning the agent's objectives with her owns. This is the same mechanism at the heart of the optimal contracts in Hurwicz and Shapiro (1978) and Carroll (2015), and will be crucial in establishing the optimality of affine contracts in the setting we develop.

Lemma 46. *Let (w_2, w_2) be an eligible contract scheme. There exits k, λ with $\lambda > 0$ such that for all $(y_1, y_2) \in Y \times Y$:*

$$w_1(y_1) + w_2(y_1, y_2) \leq \frac{1}{1+\lambda}(y_1 + y_2) - \frac{1}{1+\lambda}k \tag{B.5}$$

$$V(w) = k + \lambda V_A(w|\mathcal{A}_0) \tag{B.6}$$

Proof. This lemma is proven with the following two propositions. In both propositions define:

Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points

$$(w_1(y_1) + w_2(y_1, y_2), y_1 - w_1(y_1) + y_2 - w_2(y_1, y_2))$$

for $(y_1, y_2) \in Y \times Y$.

Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u > V_A(w|\mathcal{A}_0)$ and $v < V(w)$. Note T is convex.

Proposition 24. $S \cap T = \emptyset$.

Proof. Let $(u, v) \in T$ then let

$$(F_1, F_2) \in \underset{(F_1,F_2)\in\mathcal{F}}{\operatorname{argmin}} E_{F_1}\left[y_1 - w_1(y_1) + E_{F_2(y_1)}[y_2 - w_2(y_1, y_2)]\right]$$

by definition of T and Lemma (45):

$$u > V_A(w|\mathcal{A}_0) = E_{F_1}[w_1 + E_{F_2}[w_2]]$$

$$v < V(w) = E_{F_1}[y_1 - w_1 + E_{F_2}[y_2 - w_2]]$$

now, suppose for a contradiction that $(u,v)\in S$, then there exists a joint distribution F' on (y_1,y_2) such that:

$$\begin{aligned} u &= E_{F'}\left[w_1(y_1)+w_2(y_1,y_2)\right] = E_{F_1'}\left[w_1(y_1)+E_{F_2'(y_1)}\left[w_2(y_1,y_2)\right]\right] \\ v &= E_{F'}\left[y_1-w_1(y_1)+y_2-w_2(y_1,y_2)\right] = E_{F_1'}\left[y_1-w_1(y_1)+E_{F_2'(y_1)}\left[y_2-w_2(y_1,y_2)\right]\right] \end{aligned}$$

where $F_1(y_1)F_2(y_2|y_1)=F(y_1,y_2)$. Note that $\left(F_1',F_2'\right)$ guarantee a payoff to the agent larger than $V_A(w|\mathcal{A}_0)$ so $\left(F_1',F_2'\right)\in\mathcal{F}$ but:

$$\begin{aligned} & E_{F_1}\left[y_1-w_1(y_1)+E_{F_2(y_1)}\left[y_2-w_2(y_1,y_2)\right]\right] \\ & > E_{F_1'}\left[y_1-w_1(y_1)+E_{F_2'(y_1)}\left[y_2-w_2(y_1,y_2)\right]\right] \end{aligned}$$

which contradicts minimality of F. Then $S\cap T=\emptyset$ □

Since $S\cap T=\emptyset$ we can apply the separating hyperplane theorem which implies that there exist constants (k,λ,μ) such that $(\lambda,\mu)\neq(0,0)$ and:

$$k+\lambda u-\mu v\leq 0 \qquad (u,v)\in S \tag{B.7}$$

$$k+\lambda u-\mu v\geq 0 \qquad (u,v)\in T \tag{B.8}$$

Let $(F_1^\star,F_2^\star)\in\underset{(F_1,F_2)\in\mathcal{F}}{\operatorname{argmin}} E_{F_1}\left[y_1-w_1(y_1)+E_{F_2(y_1)}\left[y_2-w_2(y_1,y_2)\right]\right]$. Note that the pair $(E_{F^\star}[w_1+w_2],E_{F^\star}[y_i-w_i])$ lies in the closures of both S and T. Then:

$$k+\lambda E_{F^\star}[w_1+w_2]-\mu E_{F^\star}[y_i-w_i]=0 \tag{B.9}$$

It is left to show that $\lambda,\mu>0$.
Note that $(u,v)\in T$ admits u arbitrarily high and v arbitrarily low. So for (B.8) to hold it must be that $\lambda\geq 0$ and $\mu\geq 0$. There are then two cases to rule out:

1. Suppose $\mu=0$, then it must be that $\lambda>0$ (since $(\lambda,\mu)\neq(0,0)$). Also from (B.7) and (B.8)

$$\begin{aligned} u &\leq -\frac{k}{\lambda} \qquad (u,v)\in S \\ u &\geq -\frac{k}{\lambda} \qquad (u,v)\in T \end{aligned}$$

So $\max_{y\in Y}\left[w_1(y)+w_2(y)\right]=\max_{u\in S}u\leq-\frac{k}{\lambda}\leq\inf_{u\in T}u=V_A(w|\mathcal{A}_0)$. Which implies:

$$\max_{y\in Y}\left[w_1(y)+w_2(y)\right]=V_A(w|\mathcal{A}_0)$$

This can only happen if the agent has an action $(F,0)\in\mathcal{A}_0$ such that

$$E_F\left[w_1(y)+w_2(y)\right]=\max\left[w_1(y)+w_2(y)\right]$$

the only action in $\mathcal{A}_0$ with zero cost is $(\delta_0,0)$, so

$$\max\left(w_1(y)+w_2(y)\right)=w_1(0,0)+w_2(0,0)$$

This is also the unique action in $A^\star(w|\mathcal{A}_0)$ so:

$$V_i(w)\leq V_i(w|\mathcal{A}_0)=-w_i(0,0)\leq 0$$

This violates eligibility ($V_i(w)>0$).

2. Suppose $\lambda=0$, then it must be that $\mu>0$ (since $(\lambda,\mu)\neq(0,0)$). Also from (B.7) and (B.8)

$$v\geq\frac{k}{\mu}\qquad(u,v)\in S$$
$$v\leq\frac{k}{\mu}\qquad(u,v)\in T$$

So $\min_{y\in Y}\left[y_i-w_i(y)\right]=\min_{v\in S}v\geq\frac{k}{\mu}\geq\sup_{v\in T}v=V_i(w)$. But we know that

$$\min_{y\in Y}\left[y_i-w_i(y)\right]\leq 0-w(0,0)\leq 0$$

this implies $V_i(w)\leq 0$ which contradicts eligibility. So $\lambda>0$.

Note that since λ and μ are greater than zero μ can be normalized to 1, giving from (B.7):

$$k+\lambda\left(w_i(y)+w_j(y)\right)-\left(y_i-w_i(y)\right)\leq 0$$

And from (B.9):

$$V_i(w)=k+\lambda V_A(w|\mathcal{A}_0)$$

□

If we go through with the hyperplane argument we will get that:

$$w_1(y_1) + w_2(y_1, y_2) = \alpha(y_1 + y_2)$$

If there is double limited liability then it must be the case that $w_1(y_1) = \alpha y_1$ and $w_2(y_1, y_2) = \alpha y_2$. If there exists a $\overline{y}_1$ for which $w_1(\overline{y}_1) > \alpha\overline{y}_1$ Then $w_2(\overline{y}_1, y_2) < \alpha y_2$ for all y_2. This cannot hold if $y_2 = 0$ because of the agent's limited liability. Similarly, if there exists a $\overline{y}_1$ for which $w_1(\overline{y}_1) < \alpha\overline{y}_1$ then $w_2(\overline{y}_1, y_2) > \alpha y_2$ for all y_2. This cannot hold if $y_2 = 0$ because of the principal's limited liability. Then $w_1(y_1) = \alpha y_1$ which implies $w_2(y_1, y_2) = \alpha y_2$ by the hyperplane equation.