MONOTONIC FUNCTIONS OF FINITE POSETS

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TO

VAL and DAVE and BABS

CONTENTS

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DECLARATION

The material contained in this dissertation which has previously appeared in the literature is as follows (which also indicates any joint research):

"A simple lower bound technique for comparison problems using lattice inequalities" by J.W. Daykin and M.S. Paterson [DP, 1983];

"Applications of the FKG inequality and its relatives" by R.L. Graham [G2, 1983, cf. Theorem 3.7];

"On log concavity for order-preserving maps of partial orders" by D.E. Daykin, J.W. Daykin and M.S. Paterson [DDP, 1984];

"Inequalities for the number of monotonic functions of partial orders" by J.W. Daykin [JD, 1984];

"Order preserving maps and linear extensions of a finite poset" by D.E. Daykin and J.W. Daykin [DD, 1985].

SUMMARY

Classes of monotone functions from finite posets to chains are studied. These include order-preserving and strict order-preserving maps. When the maps are required to be bijective they are called linear extensions. Techniques for handling the first two types are closely related; whereas for linear extensions quite distinct methods are often necessary, which may yield results for order-preserving injections.

First, many new fundamental properties and inequalities of a combinatorial nature are established for these maps. Quantities considered here include the range, height, depth and cardinalities of subposets. In particular we study convexity in posets and similarly pre-images of intervals in chains. The problem of extending a map defined on a subposet to the whole poset is discussed.

We investigate positive correlation inequalities, having implications for the complexity of sorting algorithms. These express monotonicity properties for probabilities concerning sets of relations in posets. New proofs are given for existing inequalities and we obtain corresponding negative correlations, along with a generalization of the xyz inequality. The proofs involve inequalities in distributive lattices, some of which arose in physics. A characterization is given for posets satisfying necessary conditions for correlation properties under linear extensions.

A log concavity type inequality is proved for the number of strict or non-strict order-preserving maps of an element. We define an explicit injection whereas the bijective case is proved in the literature using inequalities from the theory of mixed volumes.

These results motivate a further group of such inequalities. But now we count numbers of strict or non-strict order-preserving maps of subposets to varying heights in the chain.

Lastly we consider the computational cost of producing certain posets which can be associated with classical sorting and selection problems. A lower bound technique is derived for this complexity, incorporating either a new distributive lattice inequality, or the log concavity inequalities.

CHAPTER 1 : PRELIMINARIES AND INTRODUCTION

1.1 PRELIMINARIES: NOTATION AND DEFINITIONS

We let $\mathbb{Z}$, $\mathbb{R}$ denote the integers and reals respectively, while their non-negative parts are denoted $\mathbb{Z}^+$ and $\mathbb{R}^+$.

1.1.1 Structures

A *partially ordered set* or *poset* is a set in which a binary relation $x \prec y$ is defined, which satisfies for all x,y,z the following conditions

For all x, $x \prec x$.	(Reflexive)
If $x \prec y$ and $y \prec x$, then $x = y$.	(Antisymmetry)
If $x \prec y$ and $y \prec z$, then $x \prec z$.	(Transitivity)

Now $\prec$ is called a partial ordering relation. If $x \prec y$ and $x \neq y$ we write $x < y$. Also P will denote a finite poset and we will assume $|P| \neq \emptyset$.

A *chain* or *linearly ordered set* or *totally ordered set* is a partially ordered set satisfying the additional condition

Given x and y, either $x \prec y$ or $y \prec x$.

In other words any two distinct elements in a chain are *comparable.* We let C denote a finite non-empty chain.

Conversely an *antichain* is a partially ordered set such that any two distinct elements are *incomparable* (i.e., not comparable).

Theorem 1.1: (see [Bi]).

Any subset S *of a poset* P *is itself a poset under the same inclusion relation (restricted to* S*). Any subset of a chain is a chain.*

We may refer to such a subset S as a subposet.

For x,y incomparable in P, let $P \cup \{x < y\}$ denote the smallest extension of P having $x < y$. That is $P \cup \{x < y\}$ stands for the transitive closure of P with the additional comparability $x < y$. Hence we have $p < q$ in $P \cup \{x < y\}$ if $p < q$ in P, or if $p < x$ in P and $y < q$ in P.

Let $\langle P, < \rangle$ be a partially ordered set. Clearly $>$ (the inverse of $<$) is also a partial ordering and so we call $\langle P, > \rangle$ the *dual* of $\langle P, < \rangle$. The *duality principle* says that if a statement on partially ordered sets is dualized, by replacing all $<$ by $>$, then if the statement is true, so is its dual.

For $x,y \in P$ we write $x \sim y$ if either $x < y$ or $x > y$. The opposite of $x \sim y$ is written $x|y$ and means that x and y are incomparable. To make a binary comparison between x and y is denoted $x?y$. If Q,R are posets we let $Q < R$ mean $q < r$ for all $q \in Q$, $r \in R$. Also $Q|R$ denotes $q|r$ for all $q \in Q$, $r \in R$.

The partition $P = Q \cup R$ where Q,R are posets means the following. Let $Q = \langle Q', < \rangle$ and $R = \langle R', < \rangle$. Then $Q' \cap R' = \emptyset$, but $Q \cup R$ does not define P because it is necessary to know what relations if any exist between the elements of Q and R.

An *upper bound* of a subset S of a poset P is an element $x \in P$ greater than or containing every $s \in S$. The *least upper bound* is an upper bound contained in every other upper bound; it is denoted l.u.b. S or sup S. By antisymmetry, sup S is unique if it exists. The notions of *lower bound* of S and *greatest lower bound* (g.l.b. S or inf S) of S are defined dually. Again by antisymmetry, inf S is unique if it exists.

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A *lattice* is a poset any two of whose elements have a g.l.b. or "meet" denoted by $x \wedge y$, and a l.u.b. or "join" denoted by $x \vee y$.

Any non-empty finite lattice contains a least element 0 and a greatest element I.

A lattice is *distributive* iff the following equivalent identities hold in it

(1.1) $\quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x,y,z,

(1.2) $\quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all x,y,z.

Note that for fixed individual elements x,y,z of a lattice the equations in (1.1) and (1.2) are not equivalent.

A fundamental triviality is:

Lemma 1.1: (see [Bi])

Any chain is a distributive lattice.

The *direct product* $J \times K$ of lattices J, K is given by the Cartesian product

$$J \times K = \{(j,k) : j \in J,\ k \in K\}$$

with the product ordering

$$\left.\begin{array}{l}(j_1,k_1) \wedge (j_2,k_2) = (j_1 \wedge j_2,\ k_1 \wedge k_2) \\ (j_1,k_1) \vee (j_2,k_2) = (j_1 \vee j_2,\ k_1 \vee k_2)\end{array}\right\} \text{ in } J \times K.$$

This is again a lattice because every identity on the operations $\wedge$, $\vee$, which holds in J × K, will necessarily hold (since it holds termwise) in their direct product. It follows that any direct product of distributive lattices is distributive.

A *complemented lattice* is a lattice with universal bounds 0 and I in which every element p has at least one complement x, with

$$p \wedge x = 0 \quad \text{and} \quad p \vee x = I.$$

A *Boolean lattice* is one which is both complemented and distributive.

An important Boolean lattice is the lattice of subsets of a set ordered by inclusion. For this lattice we define $\wedge = \cap$ (intersection) and $\vee = \cup$ (union).

A well-known result is:

Theorem 1.2: (see [Bi]).

Every distributive lattice can be embedded in the lattice of subsets of a (finite or infinite) set.

If A,B are subsets of a lattice and α is a real-valued function defined on the lattice, then we will let $\alpha(A) = \Sigma(a \in A)\alpha(a)$, $A \wedge B = \{a \wedge b: a \in A, b \in B\}$ and $A \vee B = \{a \vee b : a \in A, b \in B\}$.

1.1.2 Monotonic Functions

Let P be a finite poset and C a finite chain. Four classes of monotonic functions from the elements of P into C are defined as follows.

For (P,C), a map $\omega^0: P \to C$ is *order-preserving* if, for all $x,y \in P$, $x < y$ implies $\omega^0(x) \leq \omega^0(y)$. Let $\Omega^0 = \Omega^0(P,C)$ be the set of all such ω^0. (Some authors require $|P| = |C|$, but we do not need this restriction.)

For (P,C), a map $\omega: P \to C$ is *strict order-preserving* if, for all $x,y \in P$, $x < y$ implies $\omega(x) < \omega(y)$. Let $\Omega = \Omega(P,C)$ be the set of all such ω. (Note that ω need not be 1-1, and again we do not require $|P| = |C|$.)

For (P,C), a map $\lambda^I:P \to C$ is an *order-preserving injection* if λ^I is 1-1 and strict order-preserving. Let $\Lambda^I = \Lambda^I(P,C)$ be the set of all such λ^I.

A map $\lambda:P \to C_{|P|} \equiv \{1 < 2 < \dots < |P|\}$ is an *order-preserving bijection*, or a *linear extension* of P, if λ is an order-preserving injection. Let $\Lambda = \Lambda(P)$ be the set of all such λ.

Notice that the later conditions imply the earlier ones, namely

linear extension $\Rightarrow$ order-preserving injection $\Rightarrow$ strict order-preserving $\Rightarrow$ order-preserving.

The *range* of $S \subset P$ over Ω is the subset $\Omega S = \{\omega s:\omega \in \Omega,\ s \in S\}$ of C. That is ΩS is the union of all *images* ωS of S where $\omega \in \Omega$. For $x \in P$ we often call ωx the *rank* of x.

The *pre-image* of $K \subset C$ *under* ω is the subset $\omega^{-1}K = \{x \in P:\omega x \in K\}$, and *under* Ω it is $\Omega^{-1}K = \cup\ \{\omega^{-1}K:\omega \in \Omega\}$.

There are similar ranges, images, ranks and pre-images for each of Ω^0, Λ^I, Λ.

1.1.3 Subsets of Posets

Let S be a subset of P. We define the following

above $S = \{x \in P:\ \exists\ s \in S \text{ with } s < x\}$,

below $S = \{x \in P:\ \exists\ s \in S \text{ with } s > x\}$,

comp S = above S $\cup$ below S,

incomp $S = P \setminus \text{comp } S$.

We call $U \subset P$ an *up-set* or an *upper ideal* if U = above U. Similarly $D \subset P$ is a *down-set* or a *lower ideal* if D = below D. Alternatively the intersection of all up-sets containing S is above S; below S is the corresponding intersection of down-sets.

Now S is *convex* if $x, z \in S$ and $x < y < z$ in P imply $y \in S$. The *convex hull* $\bar{S}$ of S is the minimal set containing S such that $\bar{S}$ is convex. In other words $\bar{S}$ is (above S) $\cap$ (below S), and S is convex if $\bar{S} = S$. Clearly up-sets and down-sets are each convex, and the empty set $\emptyset$ is both an up-set and a down-set.

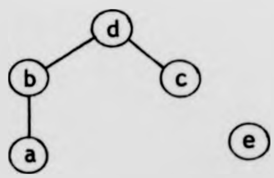
An *interval* in a chain denotes a convex subset of the chain.

If $z > x$ in P but $z > y > x$ for no y belonging to P we say z *covers* x; and a *path* of elements is a sequence of covering pairs. The elements and paths of P can be depicted by a *Hasse diagram*. For example, if

$$P = \{a < b \text{ and } a < d \text{ and } b < d \text{ and } c < d \text{ and } e\}$$

then an associated Hasse diagram is given by:

<u>Example 1.1</u>:



P

We call $x \in P$ a *minimal* (*maximal*) element in P if there is no $y \in P$ with $y < x$ ($y > x$). Hence x is minimal in P iff x is maximal in the dual of P. An element is *isolated* if it is both minimal and maximal in P.

1.1.4 Metrics for Posets

The *height* $ht(S)$ of $S \subset P$ is the maximum m for which there is a chain $s_1 < s_2 < \ldots < s_m$ in S. For $x \in P$ the *height* $ht_S(x)$ of x with respect to S is ht (below $\{x\}$) in S, and similarly the *depth* $dp_S(x)$ of x is ht (above $\{x\}$) in S. We clarify these definitions with:

Example 1.2:

P

Then

$$ht(P) = 3 > ht(S) = 2,$$

$$ht_P(x) = ht_S(x) = 1,$$

$$ht_P(y) = 2, \quad ht_S(y) \text{ - undefined},$$

$$ht_P(z) = 3 > ht_S(z) = 2.$$

We will denote $ht_P(x)$ by $ht(x)$ and $dp_P(x)$ by $dp(x)$.

For $x < y$ in P we let $ct(x,y)$ be the maximum integer m for which there is a chain $x = p_1 < p_2 < \ldots < p_m = y$ in P. So $ct(x,y) = ht(\overline{\{x,y\}})$.

The height function $x \to ht(x)$ (similarly depth function) partitions P into antichains $A_1, A_2, \ldots, A_{ht(P)}$ by taking A_i to be the set of elements of height i in P. This is known as the *canonical partition* of P.

1.2 INTRODUCTION

The basic concern of this dissertation is order. This is motivated by the binary relation "is contained in", "is less than or equal to". An ordered set consists of a set of elements together with an ordering relation. The theory of ordered sets bears on topics throughout mathematics. Important links have been forged with algebra, combinatorics, geometry, model theory, set theory, and topology, and this theory has applications throughout computer science, operations research, and the physical and social sciences. The ubiquity of ordered sets in mathematics has called for its identification as a mathematical discipline. This year (1984) has seen the appearance of a new journal, entitled "Order", to help serve this purpose.

Our bias is towards the combinatorial nature of ordered sets and the important branch of lattice theory, along with their applications to theoretical computer science. A typical problem encountered by a computer scientist in this area is topological sorting. That is to sort a given partial ordering topologically, which is equivalent to our notion of determining a linear extension of the partial order.

This kind of application can be naturally viewed as mapping the elements of the poset to the integers in such a way as to preserve any ordering relation among them. In this way we have the idea of monotonicity in relation to partial orders, and it followed that we became interested in studying various classes of monotonic functions of posets.

We will restrict our study to finite posets (and thus finite lattices), an area which is rich in both problems and applications. We usually map onto an initial segment of the integers, a chain.

An important combinatorial result about posets was given by Dilworth in 1950. This deep theorem states a basic relationship between chains and antichains.

Theorem 1.3: (Dilworth [Di]).

A poset whose largest antichain has c *elements is the union of* c *chains.*

This theorem remains true if the words "chain" and "antichain" are interchanged.

Theorem 1.4: (Mirsky [M]).

A poset whose longest chain has c *elements is the union of* c *antichains.*

More surprising than the fact that this is true is its triviality by comparison with Dilworth's theorem. For the result follows immediately from the canonical partition of P. We give a generalization of this result in Chapter 2.2, namely: the simultaneous decomposition of disjoint convex subposets into antichains.

Many fundamental combinatorial properties for monotonic functions are explored in the second chapter. We commence with the set of strict order-preserving maps, followed by inequalities for these maps. The features of interest are typically convex subposets, ranges and pre-images of elements, and height and depth functions. Corresponding results are established for linear extensions. For both kinds of maps we study the problem of completing partial mappings, and give necessary and sufficient conditions for a map defined on a subset to extend to the whole poset.

Combinatorics is a discipline which frequently draws upon methods and results from other topics in mathematics. For example, a problem in graph theory may get reformulated as an algebraic question.

Proof techniques in this dissertation are often based upon the application of various inequalities. These inequalities may be quite elementary like the arithmetic-geometric mean inequality, or more exotic as in the case of the Alexandrov-Fenchel inequalities from the theory of mixed volumes.

The latter were employed by Stanley [St] in 1981 to establish log concavity for the sequence $N_1^L, N_2^L, \ldots, N_{|P|}^L$, where N_i^L counts the number of linear extensions such that an element x in P has rank i. This fundamental theorem motivated the parallel results by D.E. Daykin, J.W. Daykin and M.S. Paterson [DDP] in 1984 for both strict and non-strict order-preserving maps. In this case we were able to construct an explicit injection to prove the inequality. These results are presented in Chapter 4.2.

In Chapter 5 the proof technique introduced in [DDP] is extended to derive a group of log concavity type inequalities. This constructive method has proved to be powerful enough to cover many generalizations.

An application of some of these inequalities from Chapters 4.2, 5.2 and 5.3 arises in Chapter 6, in which our main idea is a new lower bound technique in complexity theory for classes of problems based on binary comparisons between elements. For the lower bound we can use concepts from lattice theory, and an inequality that we give for distributive lattices. This states an upper bound of a quarter of the lattice elements for the minimum of two pairwise incomparable subsets.

Lattice inequalities also occur in the proofs in Chapter 3, where we look at positive and negative correlation properties of posets. The correlations refer to the probabilities of pairs of subposets being satisfied within the set of maps of a given poset. Usually the FKG inequality is used in the literature for these kinds of problems. This is a 1971 result of Fortuin, Kasteleyn and Ginibre, discovered in their research in physics. Special cases of their inequality are published in many different fields, and can be traced back to classical works of Chebyshev. In a different direction, extensions and related lattice results were given by R. Ahlswede and D.E. Daykin [AD] and D.E. Daykin and others (see [D2]). It is the latter which we find more applicable here.

Recently Graham [G2] remarked, "It is only natural to expect that many of the results which hold for linear extensions also hold for [strict] order-preserving maps. While this in fact may well be true, there are still relatively few theorems available for this class of maps (no doubt, due in part to the fact that they have not been studied as much)".

We verify this expectation for specific problems, and help to remedy the lack of such theorems by providing a collection of results for both strict and non-strict order-preserving maps. It will also be seen that proof techniques for these non-bijective maps are usually interchangeable with minor modifications. The contrast lies between these sets and the set of linear extensions, for which quite different methods are at times needed. Theorems for linear extensions may yield analogies for order-preserving injections, sometimes with the help of the binomial coefficients to choose image points in the chain.

The theory of linear extensions of ordered sets is rich and varied: in the infinite case the main results and methods are typically set-theoretic and model-theoretic in nature; whereas in the finite case they are characteristically motivated by algorithmic or constructive considerations.

Currently the most well known results about linear extensions are encountered in the infinite case. In the finite case the corresponding statements are often a triviality. An example is this connection between partial orders and linear orders, established in 1930:

Theorem 1.5: (Szpilrajn [Sz]).

Any partial order on a set can be extended to a linear order on the same set.

From this, it follows that any partial order is the intersection of its linear extensions; equivalently, every ordered set can be represented as some subset of a Cartesian product of chains.

However there is a substantial body of results recently emerging about finite ordered sets, and in particular linear extensions. These have often been prompted by questions arising in theoretical computer science.

An area where the infinite case is studied in computer science is domain theory. The domains in the semantics of programming languages, describing sets of information, are posets. Moreover the finite cases are usually trivial here.

In Sections 2.5 and 2.6 where we deal with the completion of monotonic functions, we cover mappings to both finite and infinite chains.

A reasonable starting point in the theory of monotonic functions is construction algorithms for generating the set of maps within a class, or algorithms for enumerating the class. The significance of enumerating linear extensions is that it provides a measure of the degree of ordering within the poset. At one extreme an n-element antichain has n! extensions, whilst a total order has but one; and similarly for the other monotonic maps. Wells [We] has given an enumeration algorithm for linear extensions. However it is felt that there is little hope for a unifying formula expressing this sum in terms of structural invariants of the set. Counting linear extensions in such a way has been found to be tractable only for some special and simple classes of ordered sets.

A possibility for tackling this problem is to see how likely is it that a typical extension of P contains $x < y$ where x,y are elements in P. One way is to define a "simple majority" criterion, i.e., the relation $x \beta y$ if

$$|\Lambda(P \cup \{x < y\})| < |\Lambda(P \cup \{x > y\})|.$$

Notice that β need not impose a total ordering on P: if P is an antichain then $|\Lambda(P \cup \{x < y\})| = |\Lambda(P \cup \{x > y\})|$ for each $x \neq y$, and so β leaves P totally unordered. Further, β need not even be a transitive relation on P. An example illustrating this was first put forward by Fishburn [Fi1], exhibiting a "β-cycle" in a 31-element set. An instance of the binary relation β is as follows. Let the elements of P represent candidates in an election. The order $p < q$ in P stands for q is unanimously preferred to p. Now suppose each of $\Lambda(P)$ electors ranks the candidates, i.e., gives a distinct linear extension of P. Then β summarizes the outcome according to simple majority rule.

In Chapter 3 we study a more encouraging direction along these lines, proposed in the series of articles D.E. Daykin [D3], Graham [G1, G2], R.L. Graham, A.C. Yao and F.F. Yao [GYY], Kleitman and Shearer [Kl,Sh], Shepp [Sh1, Sh2] and Winkler [Wi1,Wi2]. These have all appeared since 1980, and contain results expressing the correlation of certain types of comparabilities among the elements of P. To date perhaps the most important result is the xyz inequality of Shepp [Sh2]. This states that for elements x,y,z in P,

$$\mathrm{prob}(P \cup \{x < y\})\ \ \mathrm{prob}(P \cup \{x < z\}) \le \mathrm{prob}(P \cup \{x < y \text{ and } x < z\}),$$

where $\mathrm{prob}(\alpha)$ is the proportion of monotonic maps of P in a class which satisfy α. Assuming that all linear extensions of P are equally likely, then the quantities in this inequality have implications in the expected efficiency of comparison algorithms like sorting.

We look further into the theory of sorting in Chapter 6. In particular, classical comparison based problems may be understood as the construction of certain posets. Hence the complexity of the problem is given in terms of the cost of producing a specified poset.

The three principal themes encountered in the study of linear extensions of finite ordered sets are:

(i) what is the total number of linear extensions?

(ii) how is an optimal linear extension constructed?

(iii) what is the dimension of a poset?

We do not address these topics directly here but continue with briefly discussing the latter two.

Many situations arise where rather than determining the unique linear order underlying a poset, that is sorting, all that we require is some linear extension. Imagine sequencing the chapters for a book or a set of lecture courses. Constraints originate from the need to precede certain chapters or courses by others which contain the prerequisite skills. Any linear extension defines a sequential procedure through the contents. Moreover distinct extensions yield different benefits. A "breadth-first" linear extension, which traverses the canonical partition, gives a working knowledge of a variety of topics; whereas a "depth-first" approach, which traverses paths in the Hasse diagram, allows early specialization at the expense of diversity.

Questions concerning the construction of optimal linear extensions occur in the theory of deterministic sequencing and scheduling with precedence constraints. An important introduction with many applications is given by Lawler and Lenstra [LL]. Machine scheduling covers the allocation over time of scarce resources, in the form of machines or processors, to activities known as jobs or tasks. Jobs are considered independent if they can be performed in any order. Precedence constraints, for example technological considerations, impose a partial ordering on the job sequencing.

A linear extension defines a schedule but its optimality depends on the objectives of the schedule : earliest completion time, minimum maximum lateness, and so on.

Algorithmic results for machine scheduling problems are often complemented by results stating that satisfactory solution techniques for related problems will probably never be found, e.g., the problems are NP-hard.

The three classes of well-solved precedence constrained scheduling problems, i.e., those with polynomial time algorithms are:

(i) single machine problems, in the case of min-max optimization;

(ii) certain problems with series-parallel precedence constraints;

(iii) certain parallel machine problems.

As an example, suppose a single machine is to perform a set of jobs sequentially. A set of precedence constraints prohibits the start of particular jobs until others are completed. Any job x performed immediately after a job y, where $x \nsucc y$ requires a "setup" or "jump" entailing some fixed additional cost. The object is to schedule the jobs so as to minimize the number of jumps. The "jump number problem" has a further appeal due to known links with Dilworth's "chain decomposition" theorem.

Suppose x is incomparable to y in P. Then there are linear extensions λ,μ of P such that $\lambda x < \lambda y$ and $\mu x > \mu y$. We can deduce from $\lambda \cap \mu$ that $x|y$ in P; whereas if $x < y$ in $\lambda \cap \mu$ for every pair of extensions λ,μ then we infer $x < y$ in P. The (order) dimension is the least number of linear extensions required to determine P. For instance if P is an antichain then the dimension is two, as is seen by taking any extension and its dual.

To establish the dimension and to construct such a minimal set of linear extensions are quite old problems. An extensive literature with many deep results already exists about them; a definitive account is given by Kelly and Trotter [KT].

Another application of ordered sets to computer sciences is linear programming (see [Ho]). Ideas from ordered sets can be used to prove theorems in linear programming. Conversely, linear programming ideas help in proving theorems about ordered sets.

We see that a linear extension extends the partial order relation to a total order relation, without enlarging the underlying set. Therefore applications of these maps may resemble the nature of sequential processing. However strict order-preserving maps need only respect chains within the poset, so any incomparable collection can be mapped to the same point. Hence these non-bijective maps have connections with parallelism. Consider a set of computations to be performed on a parallel machine. Precedence constraints over the computations define a poset. The minimal time for the machine to compute the set is given by the height of a maximal chain in the corresponding poset. Clearly many instances of both strict and non-strict order-preserving maps along with order-preserving injections exist naturally, and hence we are motivated to study them. Like any mathematical structure or function, the more we understand it, the more far reaching the applications can be.

CHAPTER 2 : COMBINATORIAL PROPERTIES OF MONOTONIC FUNCTIONS

2.1. INTRODUCTION

We start by considering the combinatorial nature of partial orders in this chapter, and obtain many elementary yet fundamental properties for monotonic functions of posets. Applications of some of these results are found in later chapters, and especially Chapter 3.

For strict order-preserving maps we commence with the rake up or down of a map over a subposet. We then consider the image of a singleton and a convex subset under ω and under Ω. Conversely, pre-images of intervals in a chain under both ω and Ω are studied. Motivated by Mirsky's Theorem 1.4 we show the existence of a map linearly ordering a set of disjoint subsets of a poset.

Various inequalities for strict order-preserving maps are established. The quantities involved here are typically the height and depth of an element, and the cardinality of a subposet and its range or pre-image. From these results are derived the Graham and Harper normalised matching conditions [GH].

By defining the push up or down of a map over a subposet the parallel results are obtained for linear extensions. But here the analogous inequalities do not arise.

For both these kinds of monotonic functions we discuss the problem of extending a map defined on a subset of the poset to the whole poset, and give necessary and sufficient conditions for the completion of mappings. If a map on a subset cannot be completed we show how to partition the subset so that each part may be completed independently. These ideas arise in the concept of intricacy [B].

It is natural to start our study with the most general kind of map: order-preserving. We would then proceed to the class of strict such maps, and finally to the more restricted classes of order-preserving injections and linear extensions.

However, for $\Omega^0(P,C)$ and an element $p \in P$ then obviously the range of p is the chain, that is $\Omega^0 p = C$. It easily follows that $\Omega^0(P,C)$ is non-empty iff the chain C is non-empty, and so we may always assume that $\Omega^0 \neq \emptyset$. Since with order-preserving maps the whole poset can map onto a single point, any results about convex subsets, intervals and so on which are established for other kinds of maps, will carry over to these maps in a trivial way. So we will commence with strict order-preserving maps, followed by linear extensions, while results for order-preserving injections may also be omitted.

Most of the particular questions studied here originally occur in the literature in [DD].

2.2 THE SET Ω OF STRICT ORDER-PRESERVING MAPS

The map $p \to ht(p)$ is in Ω iff $ht(P) < |C|$. Dually the map $p \to |C| + 1 - dp(p)$ is in Ω iff $dp(P) < |C|$. Also

$$(2.1) \qquad ht(p) < \omega p < |C| + 1 - dp(p) \text{ if } p \in P, \omega \in \Omega.$$

Hence we immediately get:

Theorem 2.1:

$\Omega \neq \emptyset$ *iff* $ht(P) < |C|$.

From now on we think of P and C as fixed with $\Omega \neq \emptyset$. We remark that the set of strict order-preserving maps from a poset P to a poset Q is non-empty iff $ht(P) < ht(Q)$. For, using the canonical partition of Q, we can show without loss of generality that Q contains a chain and apply the above theorem.

Throughout this section the proofs rely on the concept of the rake down of a map over a subset S. Informally, given a mapping of S we want to move the image of S down the chain C. We start by moving down those elements having the highest rank, which may in turn push others down. Likewise we can define the rake up of a map over S.

So for any $\omega \in \Omega$ and $S \subset P$ we define a map $\pi : P \to \mathbf{Z}^+$ as follows. If $S = \emptyset$ then $\pi = \omega$. If $S \neq \emptyset$ we let $m = \max \{\omega s : s \in S\}$ and $T = \{s \in S : \omega s = m\}$. If there is a $t \in T$ with $ht(t) = m$ then $\pi = \omega$. Otherwise we construct $R \subset P$ by starting with $R = T$ and iterating the rule that, if $p \in P$, $r \in R$, $p < r$ and $1 + \omega p = \omega r$ then p must be adjoined to R. Finally π is defined by $\pi p = (\omega p) - 1$ if $p \in R$ but $\pi p = \omega p$ otherwise.

Clearly $\pi \in \Omega$ and we call π the *rake down of* ω *over* S. The *rake up of* ω *over* S is defined similarly.

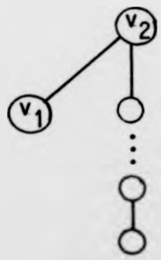
An obvious result is:

<u>Lemma 2.1</u>:

In the above notation, $|\omega S| - 1 < |\pi S| < |\omega S|$ *and if* ωS *is an interval then* πS *is an interval.*

The height function ht(p) may not map a convex set onto an interval as shown by:

<u>Example 2.1</u>:



But using this function it is easy to see:

<u>Lemma 2.2</u>:

If $D \subset P$ *is a down-set then* $\{ht(d) : d \in D\} = [1, ht(D)]$.

The following theorem generalises Mirsky's Theorem 1.4 by showing that pairwise disjoint convex subsets can be simultaneously decomposed into antichains. Mirsky gave the canonical partition of the whole poset, which is of course convex.

<u>Theorem 2.2</u>:

Let $S_1, S_2, \ldots, S_n$ *be pairwise disjoint subsets of* P *satisfying*

$$s_i \in S_i,\ s_j \in S_j,\ s_i < s_j,\ i \neq j \Rightarrow i < j.$$

Suppose that $(n+1)ht(P) \leq |C|$. *Then there is an* $\omega \in \Omega$ *satisfying both*

(2.2) $\quad 1 \leq i < j \leq n \Rightarrow \omega S_i < \omega S_j,$

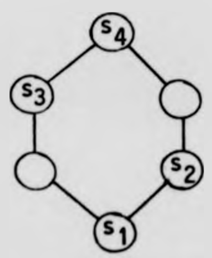
(2.3) $\quad$ *for* $1 \leq i \leq n$ *if* S_i *is convex then* ωS_i *is an interval of length* $ht(S_i)$.

Proof:

Put $U_0 = P$, $U_{n+1} = \emptyset$ and $U_i = \text{above}\,(S_i \cup S_{i+1} \cup \ldots \cup S_n)$ for $1 \leq i \leq n$ so $U_{n+1} \subset U_n \subset \ldots \subset U_0$. For $0 \leq i \leq n$ put $T_i = U_i \setminus U_{i+1}$. Then T_i considered as a possibly empty poset has its own height function ht_i. For $1 \leq i \leq n$ we have $S_i \subset T_i$ and if S_i is convex in P then S_i is a down-set in T_i and Lemma 2.2 applies. Finally define ω by $\omega p = i.ht(P) + ht_i(p)$ if $p \in T_i$ for $0 \leq i \leq n$. $\square$

Choosing S_i to be the minimal and maximal elements in a chain shows that, although $|\omega S_i| = ht(S_i)$, we need convexity in (2.3) for the image of S_i to be convex. Also, if S_i is not convex, we may get $|\omega S_i| \geq ht(S_i)$ even though the image of S_i is an interval. The latter point is illustrated in:

Example 2.2:



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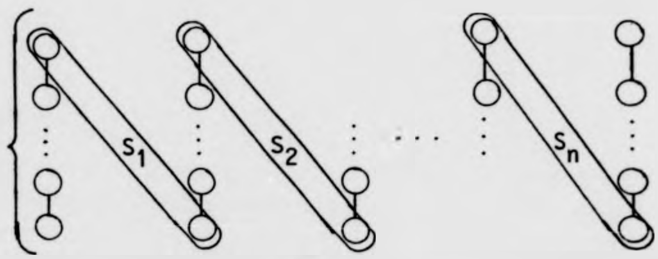
In this poset $S_i = \{s_1, s_2, s_3, s_4\}$, $ht(P) = 4$ and $4 = |\omega S_i| \geq ht(S_i) = 3$.

Two complementary examples help to justify the choice of the bound on $|C|$ in Theorem 2.2. In both cases, we take P to be a disjoint union of chains. If the S_i's are as in Example 2.3, then $|C|$ must be at least $(n+1)ht(P)-n$.

<u>Example 2.3</u>:



Alternatively, if P is defined by $P = S_1 \cup S_2 \cup \ldots \cup S_n$ where each S_i is a chain of ht(P) elements, then $|C|$ must be chosen $> n.ht(P)$.

We now have that the range of a point is convex.

<u>Theorem 2.3</u>:

If $p \in P$ *then* $\Omega p = [ht(p), |C| + 1 - dp(p)]$.

<u>Proof</u>:

Starting with the map $p \to ht(p)$ rake up repeatedly over $\{p\}$. □

In relation to the range of a convex subposet, and also the pre-images of intervals in the chain we give:

<u>Theorem 2.4</u>:

If $V \subset P$ *is convex,* $I \subset C$ *is an interval,* $c \in C$ *and* $\omega \in \Omega$ *then*

(2.4) ΩV *is an interval provided* $2ht(P) < |C|$,

(2.5) $\Omega^{-1} I$ *is convex,*

(2.6) $\omega^{-1} I$ *is convex and in particular* $\omega^{-1} c$ *is an antichain.*

Proof Part (2.4):

Provided that $2ht(P) < |C|$, then, by Theorem 2.2 with $n = 1$ and $S_1 = V$, there is an $\omega' \in \Omega$ with $\omega'V = [i,j]$. Suppose $v_1, v_2 \in V$, $\omega_1, \omega_2 \in \Omega$ are such that

$$\omega_1 v_1 = \min \{\omega v : v \in V,\ \omega \in \Omega\} = h$$

and

$$\omega_2 v_2 = \max \{\omega v : v \in V,\ \omega \in \Omega\} = k.$$

By Theorem 2.3 we have $\Omega v_1 \supset [h, \omega' v_1] \supset [h,i]$ and $\Omega v_2 \supset [\omega' v_2, k] \supset [j,k]$. Hence $[h,k] = [h,i] \cup [i,j] \cup [j,k] = \Omega V$.

Part (2.5):

Suppose $p,q,r \in P$ are such that $p < q < r$ and $p,r \in \Omega^{-1}I$. Let $I = [i,j]$. Now there exists $\omega_1, \omega_2 \in \Omega$ with $\omega_1 p$, $\omega_2 r \in I$. Suppose $\omega_1 q \notin I$. Since ω_1 is order-preserving we have $i < \omega_1 p < j < \omega_1 q$. Similarly if $\omega_2 q \notin I$ then $\omega_2 q < i < \omega_2 r < j$. However, $\omega_1 q, \omega_2 q \in \Omega q$ and thus by Theorem 2.3, $I \subset \Omega q$.

Part (2.6):

Suppose $p,q,r \in P$ satisfy $p < q < r$ and $p,r \in \omega^{-1}I$. Now ω must have $\omega p < \omega q < \omega r$ and so $\omega q \in I$. $\square$

Putting $n = 1$ in Example 2.3 shows that the condition on $|C|$ in (2.4) cannot be reduced below $2ht(P)-2 < |C|$, but we do not know the optimal bound.

The interval in C for which the image of a convex subposet is again convex is now specified.

Theorem 2.5:

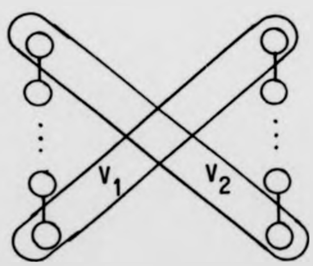
Suppose that $V \subset P$ *is convex, and put* $k = ht(V)$, $m = ht\ (\text{below } V)$, $n = ht\ (\text{above } V)$. *Also suppose that* $m + n - k < |C|$. *If the interval* $I \subset [1 + m-k,\ |C| - n + k]$ *has length* $|I| = k$ *then there is an* $\omega \in \Omega$ *with* $\omega V = I$.

Proof:

Starting with the map $p \to |C| + 1 - dp(p)$ for $p \in P$, rake down repeatedly over V. □

Suppose that $V_1, V_2 \subset P$ are convex. There does not always exist an $\omega \in \Omega$ such that both ωV_1 and ωV_2 are intervals for any $|C|$, as shown by:

Example 2.4:



This example can be seen to also hold for linear extensions.

2.3 INEQUALITIES FOR STRICT ORDER-PRESERVING MAPS

Theorem 2.6:

If $\emptyset \neq S \subset P$ *and* q *covers* p *in* P, *then,*

(2.7) $ht(S) + |C| - ht(P) \leq |\Omega S|$,

(2.8) $4 \leq dp(p) + ht(q) \leq 2 + |P|$,

(2.9) $ht(p) + dp(q) \leq ht(P)$,

(2.10) $ht(p) + dp(p) \leq 1 + ht(P)$.

Proof Part (2.7):

We use a modification of the height function. For $0 \leq i \leq |C| - ht(P)$ put $\omega_i p = ht(p) + i$ for all $p \in P$, and note that $|\omega_0 S| \geq ht(S)$. We achieve $|C| - ht(P)$ increments of the image of maximal elements in S.

Part (2.8):

For the R.H.S. consider a maximal chain in below {q} and a maximal chain in above {p}. □

Consider now the cardinality of the pre-image under Ω of a subset of the chain.

Theorem 2.7:

If $K \subset C$ *then* $|K| \leq |\Omega^{-1}K| \leq |P|$ *if* $|K| < ht(P)$ *but* $\Omega^{-1}K = P$ *otherwise.*

Proof:

We may assume that $|K| \le n = ht(P)$ and that $K \subset \{k_1, k_2, \ldots, k_n\} \subset C$. The result then follows because the map $p \to k_{ht(p)}$ for $p \in P$ is in Ω. □

Next we prove what Graham and Harper [GH] called normalized matching conditions (2.11), (2.12). It is well known that each implies the other.

Theorem 2.8:

If $|P| \le |C|$ *and* $S \subset P$ *and* $K \subset C$ *then both*

(2.11) $\quad |S|\ |C| \le |\Omega S|\ |P|,$

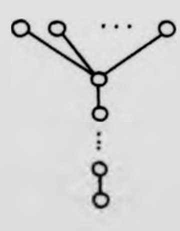
(2.12) $\quad |K|\ |P| \le |\Omega^{-1}K|\,|C|.$

Proof:

We prove (2.12) using Theorem 2.7 as follows. If $|K| < ht(P)$ then $|K| \le |\Omega^{-1}K|$ and we multiply this by $|P| \le |C|$. If $|K| \ge ht(P)$ then $|P| = |\Omega^{-1}K|$ and we multiply this by $|K| \le |C|$. □

The example where $K = C_m$ and $|C| = ht(P) > m$ and P is of the form in:

Example 2.5:



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shows that we need $|P| \le |C|$ in the last theorem, because $|K| = |\Omega^{-1}K|$ here.

In the following chapter we deal with results for the partition $P = Q \cup R$, and we go on to consider this partition.

Some obvious inequalities are:

Lemma 2.3:

Let $P = Q \cup R$ *and* $q < r$ *with* $q \in Q$, $r \in R$. *Then*

$$ht_P(r) \geq \begin{cases} ht_R(r) \\ 1 + ht_Q(q), \end{cases}$$

$$dp_P(q) \geq \begin{cases} dp_Q(q) \\ 1 + dp_R(r), \end{cases}$$

$$ht(P) \geq \quad ht_Q(q) + dp_R(r).$$

For $P = Q \cup R$, the condition on range:

$q \sim r$ implies $(\Omega q) \cap (\Omega r) = \emptyset$ where $q \in Q$, $r \in R$,

is shown to be an interesting property in Chapter 3. We start here with some basic results for this range condition.

Theorem 2.9:

If $p < q$ *in* P *then* $(\Omega p) \cap (\Omega q) = \emptyset$ *iff* $|C| \leq dp(p) + ht(q) - 2$.

Proof:

(i) Suppose $(\Omega p) \cap (\Omega q) = \emptyset$. By Theorem 2.3 we have $\Omega p=[ht(p),|C|+1-dp(p)]$ and $\Omega q=[ht(q),|C|+1-dp(q)]$. For any $\omega \in \Omega$, $\omega p < \omega q$. Thus $|C|+1-dp(p) < ht(q)$.

(ii) Suppose $|C| \leq dp(p) + ht(q) - 2$. By Theorem 2.3 $\Omega p \subset \{1,\ldots,|C| + 1 - dp(p)\}$ and $\Omega q \subset \{ht(q),\ldots,|C|\}$. Since $|C| + 1 - dp(p) < ht(q)$ these intervals are disjoint. □

Notice that in the above theorem it is not necessary for P to be partitioned. In Chapter 3.4 we give a complete characterization for posets satisfying an analogous range condition for Λ. No such nice characterization was forthcoming for Ω, although Theorem 2.9 suggests a fast algorithm for checking if P satisfies this condition for Ω.

As expected it is not always possible to satisfy $ht(P) \leq |C| \leq dp(p) + ht(q) - 2$. However we will show that when P is the union of two chains then there always exists a $|C|$ such that $\Omega(P,C)$ satisfies this condition on the range of q,r.

<u>Theorem 2.10</u>:

Suppose that $P = Q \cup R$ *where* Q,R *are disjoint chains. If* $q \in Q$, $r \in R$ *and* $q < r$ *then* $ht(P) \leq dp(q) + ht(r) - 2$.

<u>Proof</u>:

Let L be a maximal chain in P, so $|L| = ht(P)$. Let $A = \{q' \in Q \cap L: q \leq q'\} \cup \{r' \in R \cap L: r \leq r'\}$. Then there exists a chain up from q containing A, and so $dp(q) \geq |A|$. Now let $B = \{q' \in Q \cap L: q \geq q'\} \cup \{r' \in R \cap L: r \geq r'\}$. Similarly $ht(r) \geq |B|$. □

<u>Theorem 2.11</u>:

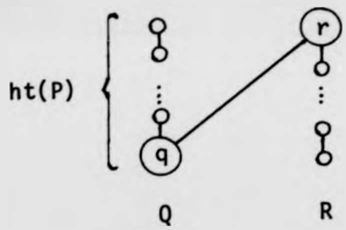
Suppose $P = Q \cup R$ *satisfies* $q \in Q$, $r \in R$, $q \sim r$ *implies* $(\Omega q) \cap (\Omega r) = \emptyset$. *If there exists a maximal length chain in* P *with an element* q *in* Q *and an element* r *in* R *then* $ht(P) = |C|$.

Proof:

First we have $q \sim r$, and we may assume that r covers q in P. Theorem 2.9 shows that $|C| \leq dp(q) + ht(r)-2$, and since q and r are in a maximal chain, $dp(q) + ht(r)-2 = ht(P)$. Equality follows from $ht(P) \leq |C|$. □

This theorem is not true if no maximal chain has an element in Q and an element in R, as shown by:

Example 2.6:



In this case $(\Omega q) \cap (\Omega r) = \emptyset$ provided that $|C| \leq 2ht(P) - 2$ by Theorem 2.9, and so $|C|$ does not necessarily equal $ht(P)$.

2.4 THE SET Λ OF LINEAR EXTENSIONS

In this case $|C| = |P|$, and these results contrast and correlate with those for Ω in Section 2.2. First we have

(2.13) $\quad |\text{below }\{p\}| \prec \lambda p \prec |P| + 1 - |\text{above }\{p\}|$ if $p \in P$, $\lambda \in \Lambda$.

Theorem 2.12:(Szpilrajn [Sz]).

$\Lambda \neq \emptyset$.

This section depends on the definition of the push up or down of a map over a subset S. This is an analogous concept to the rake up or down for Ω.

For any $\lambda \in \Lambda$ and $S \subset P$ we define a map $\mu : P \to \mathbb{Z}^+$ as follows. Put $\beta = \max\ \{\lambda s : s \in S\}$ with the value 0 by convention if $S = \emptyset$. If $|\text{below } S| = \beta$ then $\mu = \lambda$. Otherwise we put

$$Q = \{p \in P \setminus \text{below } S : \lambda p < \beta\} .$$

Since $|\text{below } S| < \beta$ there is a unique $q \in Q$ with λq maximal. We put

$$R = \{r \in \text{below } S : \lambda q < \lambda r \prec \beta\}$$

and observe that $q \| r$ for all $r \in R$. Finally μ is defined for this case by

$$\mu p = \begin{cases} \beta & \text{if } p = q, \\ (\lambda p) - 1 & \text{if } p \in R, \\ \lambda p & \text{otherwise.} \end{cases}$$

Clearly $\mu \in \Lambda$ and we call μ the *push down of* λ *over* S. The *push up of* λ *over* S is defined similarly.

An obvious result is:

Lemma 2.4:

In the above notation, if λS *is an interval then* μS *is an interval.*

Lemma 2.5:

If $D_1 \subset D_2 \subset \ldots \subset D_n$ *are down-sets of* P *there is a* $\lambda \in \Lambda$ *such that for all* $p \in P$, $1 \leq i \leq n$ *we have* $\lambda p \leq |D_i|$ *iff* $p \in D_i$.

Proof:

Push down repeatedly over $D_1, D_2, \ldots, D_n$ in any order. □

Theorem 2.13:

Let $S_1, S_2, \ldots, S_n$ *be pairwise disjoint subsets of* P *satisfying*

$$s_i \in S_i,\ s_j \in S_j,\ s_i < s_j,\ i \neq j \Rightarrow i < j.$$

Then there is a $\lambda \in \Lambda$ *satisfying both*

(2.14) $\quad s_i \in S_i,\ s_j \in S_j,\ i < j \Rightarrow \lambda s_i < \lambda s_j,$

(2.15) $\quad$ *for* $1 \leq i \leq n$ *if* S_i *is convex then* λS_i *is an interval.*

Proof:

For $1 \leq i \leq n$ let $D_i = \text{below}\,(S_1 \cup S_2 \cup \ldots \cup S_i)$. Then $D_1 \subset D_2 \subset \ldots \subset D_n$ are down-sets with $D_i \cap S_j = \emptyset$ for $1 \leq i < j \leq n$. Let λ_0 be the map of Lemma 2.5 which satisfies (2.14). For $1 \leq i \leq n$ consider the subposet $D_i \setminus D_{i-1}$ where $D_0 = \emptyset$. Let λ_i be λ_0 restricted to D_i. Let λ_i' be the result of repeatedly pushing up λ_i over S_i in D_i. Notice that if S_i is convex, then S_i is an up-set in D_i, and hence $\lambda_i'S_i$ is an interval by the dual version of Lemma 2.5. Finally define λ by $\lambda_1' \cup \ldots \cup \lambda_n' \cup \lambda_{n+1}'$ where λ_{n+1}' is λ_o restricted to $P \setminus D_n$. $\square$

As an application of the last theorem, observe that if $p_1, p_2, \ldots, p_m$ is an antichain in P then there is a $\lambda \in \Lambda$ with $\lambda p_{i+1} = 1 + \lambda p_i$ for $1 \leq i < m$. Start by putting $S_1 = \{p_1, p_2, \ldots, p_m\}$ and then clearly we can permute the order of $p_1, \ldots, p_m$ in the interval λS_1.

<u>Theorem 2.14</u>:

If $p \in P$ *then* $\Lambda p = [|\text{below } \{p\}|, |P| + 1 - |\text{above } \{p\}|]$.

<u>Proof</u>:

Push up and down repeatedly over $\{p\}$. $\square$

<u>Theorem 2.15</u>:

If $V \subset P$ *is convex,* $I \subset C$ *is an interval and* $\lambda \in \Lambda$ *then*

(2.16) ΛV *is an interval,*

(2.17) $\Lambda^{-1} I$ *is convex,*

(2.18) $\lambda^{-1} I$ *is convex.*

<u>Proof Part (2.16)</u>:

By Theorem 2.13 with n=1 and $S_1 = V$, there is a $\lambda' \in \Lambda$ with $\lambda' V = [i,j]$. Suppose $v_1, v_2 \in V$, $\lambda_1, \lambda_2 \in \Lambda$ are such that

$$\lambda_1 v_1 = \min \{\lambda v : v \in V, \lambda \in \Lambda\} = h \quad \text{and}$$

$$\lambda_2 v_2 = \max \{\lambda v : v \in V, \lambda \in \Lambda\} = k.$$

By Theorem 2.14 we have $\Lambda v_1 \supset [h,\lambda' v_1] \supset [h,i]$ and $\Lambda v_2 \supset [\lambda' v_2,k] \supset [j,k]$. Therefore $[h,k] = [h,i] \cup [i,j] \cup [j,k] = \Lambda V$.

Part (2.17):

Suppose $p,q,r \in P$ are such that $p < q < r$ and $p,r \in \Lambda^{-1}I$. Let $\Lambda p = [\alpha_1,\alpha_2]$, $\Lambda q = [\beta_1,\beta_2]$, $\Lambda r = [\gamma_1,\gamma_2]$ and $I = [i_1,i_2]$. We have $|\text{above }(p)| > |\text{above }(q)| > |\text{above }(r)|$ and $|\text{below}(p)| < |\text{below}(q)| < |\text{below}(r)|$. Hence by Theorem 2.14, $\alpha_2 < \beta_2 < \gamma_2$ and $\alpha_1 < \beta_1 < \gamma_1$. By the hypothesis on p,r, $I \cap \Lambda p \neq \emptyset$ and $I \cap \Lambda r \neq \emptyset$. Now suppose $I \cap \Lambda q = \emptyset$. Then either $i_2 < \beta_1 < \gamma_1$ which contradicts $I \cap \Lambda r \neq \emptyset$, or $\alpha_2 < \beta_2 < i_1$ which contradicts $I \cap \Lambda p \neq \emptyset$.

Part (2.18):

Suppose $p,q,r \in P$ satisfy $p < q < r$ and $p,r \in \lambda^{-1}I$. Now λ must have $\lambda p < \lambda q < \lambda r$ and so $\lambda q \in I$. □

Theorem 2.16:

Suppose that $V \subset P$ *is convex, and put* $k = |V|$, $m = |\text{below } V|$, $n = |\text{above } V|$. *If the interval* $I \subset [1 + m-k, |P| - n+k]$ *has length* $|I| = k$ *then there is a* $\lambda \in \Lambda$ *with* $\lambda V = I$.

Proof:

Starting with the λ of Theorem 2.13 push up and down repeatedly over V. □

We end this section with an analogous result to Theorem 2.9 for order-preserving injections from P into C. Therefore we assume here that $|C| > |P|$.

Theorem 2.17:

If $p < q$ *in* P *then* $(\Lambda^I p) \cap (\Lambda^I q) = \emptyset$ *iff* $|C| \prec |\text{above } \{p\}| + |\text{below } \{q\}| - 2$.

Proof:

(i) Suppose $(\Lambda^I p) \cap (\Lambda^I q) = \emptyset$. We deduce from Theorem 2.14 that $\Lambda^I p = [|\text{below } \{p\}|\,,\ |C|+1-|\text{above}\{p\}|]$ and $\Lambda^I q = [|\text{below}\{q\}|,\ |C|+1-|\text{above}\{q\}|]$. For any $\lambda^I \in \Lambda^I$ we must have $\lambda^I p < \lambda^I q$ and hence $|C|+1-|\text{above } \{p\}| < |\text{below } \{q\}|$.

(ii) Suppose $|C| \prec |\text{above } \{p\}| + |\text{below } \{q\}| - 2$. From Theorem 2.14 we infer that $\Lambda^I p \subset \{1,\ldots,|C| + 1 - |\text{above } \{p\}|\}$ and $\Lambda^I q \subset \{|\text{below } \{q\}|,\ldots,|C|\}$, and so the ranges of p and q must be disjoint. □

In the special case when p is covered by q, Theorem 2.17 shows that a necessary and sufficient condition for $(\Lambda^I p) \cap (\Lambda^I q) = \emptyset$ is that $|C| \prec |\text{above } \{p\}| + |\text{below } \{q\}| - 2 \prec |P| \prec |C|$. Thus in this case $(\Lambda^I p) \cap (\Lambda^I q) = \emptyset$ iff $\Lambda^I = \Lambda$, the set of linear extensions.

2.5. COMPLETION OF STRICT ORDER-PRESERVING MAPS

There is typically an iterative process involved in producing certain combinatorial structures. For example when colouring a graph, vertices are successively coloured until none remain to be coloured. At any stage of such an algorithm we have a partial structure. We here consider necessary and sufficient conditions for the completion of partial monotonic mappings. First we deal with "partial-ω" and in the next section with "partial-λ".

The concept of *intricacy* [B] can be described as follows. Given a combinatorial construction problem, determine the minimum number of parts which the structure can be partitioned into, so that each part may be completed independently. The intricacy of latin squares has been studied, where an $n \times n$ latin square is such that each row and each column is a permutation of $1,\ldots,n$. As an example the partial latin square:

Example 2.7:

1 2 ... n-1	
	n

can never be completed. On the other hand each of the $n \times n$ parts in Example 2.8 can be.

Example 2.8:

Moreover the concept of *delicacy* concerns the removal of entries from such a structure to enable completion. In Example 2.7 removing any one entry will suffice.

Motivated by intricacy we then give a partition of a partial-ω so that each part may be completed. Similarly for a partial-λ in the following section.

Let $S \subset P$ and $Y \subset \mathbb{Z}$ and $\psi : S \to Y$ be strict order-preserving. We say ψ *extends* if there is a strict order-preserving map $\xi : P \to Y$ with $\xi s = \psi s$ for all $s \in S$. We first give the conditions for ψ to extend when $Y = \mathbb{Z}$. Note that for any $p < q$ in S the function $\mathrm{ct}(p,q) = \mathrm{ht}(\overline{\{p,q\}})$ is defined in P.

Theorem 2.18:

Let $S \subset P$ *and* $\psi : S \to \mathbb{Z}$ *be strict order-preserving. Then* ψ *extends iff*

(2.19) $\quad \mathrm{ct}(s,t) - 1 + \psi s \leq \psi t$ *for all* $s < t$ *in* S.

Proof:

Clearly condition (2.19) is necessary for ψ to extend, and it implies that ψ is strict order-preserving. So assume that (2.19) holds.

We let p be any element of $P \setminus S$ and proceed to define ψp so that (2.19) still holds in $S \cup \{p\}$. Since P is finite, repetition will yield an extension of ψ.

<u>Case 1</u>:

We have $s < p$ for some $s \in S$. Here we put

$$\psi p = \max\{ct(s,p) - 1 + \psi s : s \in S,\ s < p\}.$$

We must show that if $p < t \in S$ then

$$ct(p,t) - 1 + \psi p \leq \psi t.$$

Now there is an $r \in S$ with $r < p$ and

$$\psi p = ct(r,p) - 1 + \psi r.$$

Also

$$ct(r,p) + ct(p,t) - 1 \leq ct(r,t) \leq 1 - \psi r + \psi t,$$

so the required inequality follows.

<u>Case 2</u>:

We have $p < s$ for some $s \in S$ but not $s' < p$, $s' \in S$. Here we put

$$\psi p = \min\{1 - ct(p,s) + \psi s : s \in S,\ p < s\}.$$

<u>Case 3</u>:

We have $p | s$ for all $s \in S$. Here we give ψp any value in $\mathbb{Z}$, which completes the proof of the theorem. $\square$

Recall that C denotes the finite interval $[1,|C|]$ and $ht(P) \leq |C|$. From the above theorem we deduce the conditions for ψ to extend when $Y = C$.

Theorem 2.19:

Let $S \subset P$ *and* $\psi : S \to C$ *be strict order-preserving. Then* ψ *extends to* $\omega \in \Omega$ *iff both* (2.19) *holds and*

(2.20) $\quad ht(s) \leq \psi s \leq |C| + 1 - dp(s)$ *for all* $s \in S$.

Proof:

In view of Theorem 2.18 and (2.1) the conditions are clearly necessary for ψ to extend. To prove the sufficiency suppose that (2.19) and (2.20) hold. Take r,t to be two new elements not belonging to P. Define a new poset $P' = P \cup \{r,t\}$ by taking the existing relations of P and adding the additional relations $r < p < t$ for all $p \in P$. Similarly extend both S to $S' = S \cup \{r,t\}$ and C to $C' = [0, |C| + 1]$. Then define $\psi r = 0$ and $\psi t = |C| + 1$. Now ψ extends from S' to P' iff (2.19) holds in S'. We will show that (2.19), (2.20) holding in S imply that (2.19) holds in S'. So let $s,s' \in S$.

Case 1:

When $s < s'$ then (2.19) holding in S implies that it holds in S'.

Case 2:

For $s < t$ we need to show $ct(s,t) - 1 + \psi s \leq \psi t$ in S'. Now there exists a $p \in P$ satisfying in P

$$ct(s,p) = dp(s) \leq |C| + 1 - \psi s.$$

Also in P' we have

$$ct(s,p) = ct(s,t) - 1.$$

Hence in P'

$$ct(s,t) - 1 + \psi s = ct(s,p) + \psi s = dp(s) + \psi s \prec |C| + 1,$$

as required.

Case 3:

For $r < s$ we must show $ct(r,s) - 1 + \psi r \prec \psi s$ in S'. Now there exists a $p \in P$ satisfying in P

$$ct(p,s) = ht(s) \prec \psi s.$$

Also in P' we have

$$ct(r,s) = 1 + ct(p,s).$$

Thus in P'

$$ct(r,s) - 1 + \psi r = ct(p,s) \prec \psi s.$$

Case 4:

For $r < t$ we will show $ct(r,t) - 1 + \psi r \prec \psi t$ in S'. We have by Theorem 2.1,

$$1 + ht(P) \prec |C| + 1 = \psi t.$$

We conclude that (2.19) holds in S' and then apply Theorem 2.18 to P',S'. Notice that only Case 1 of Theorem 2.18 will apply to any element of $P' \setminus S'$. □

With $Y = \mathbb{Z}$ again, we show how to partition S so that ψ extends from each of the parts of S independently.

Theorem 2.20:

Let $S \subset P$ *and* $\psi : S \to \mathbb{Z}$ *be strict order-preserving. If* $k = \lceil ht(P)/2 \rceil$ *then there is a partition* $S = S_1 \cup S_2 \cup \ldots \cup S_k$ *such that* ψ *extends from any one* S_i.

Proof:

For $1 \leq i \leq k$ put

$$S_i = \{p \in S : 2i-1 \leq ht_P(p) \leq 2i\}.$$

In each S_i a chain has at most two vertices, so (2.19) holds, and ψ extends from S_i by Theorem 2.18. □

Consider the example where P is the chain C_m and S is the odd numbered vertices and for $1 \leq i \leq \lceil m/2 \rceil$, $\psi(2i-1) = i$. This shows that the k in Theorem 2.20 cannot be reduced. For particular choices of S a smaller k would suffice.

We now prove the corresponding result for $Y = C$.

Theorem 2.21:

Let $S \subset P$ *and* $\psi : S \to C$ *be a strict order-preserving map satisfying* (2.20). *If* $k = \lceil ht(P)/2 \rceil$ *then there are* $\omega_1, \omega_2, \ldots, \omega_k \in \Omega$ *and a partition* $S = S_1 \cup S_2 \cup \ldots \cup S_k$ *such that* ψ *extends to* ω_i *from any one* S_i.

Proof:

For $1 \leq i \leq k$ define S_i as in the proof of Theorem 2.20. In each S_i a chain has at most two vertices, so (2.19) holds, and ψ extends from S_i to $\omega_i \in \Omega$ by Theorem 2.19. □

2.6 LINEAR EXTENSIONS FROM ORDER-PRESERVING INJECTIONS

This section is parallel to Section 2.5. Again we have $S \subset P$ and let $\psi: S \to \mathbb{Z}$ be strict order-preserving. For $R \subset S$ define

$$\max R = \max \{\psi r : r \in R\}, \ \min R = \min \{\psi r : r \in R\}.$$

The main result is:

Theorem 2.22: (D.E. Daykin [DD]).

If $S \subset P$ and $\psi: S \to \mathbb{Z}$ is strict order-preserving then ψ extends to an order-preserving injection iff

(2.21) $\quad |\bar{V}| \leq \max V - \min V + 1$ *for all* $\emptyset \neq V \subset S$.

See [DD] for a proof of the sufficiency of this result, which employs Hall's Marriage Theorem [H].

To prove the necessity of (2.21). Suppose that ψ extends to $\nu: P \to \mathbb{Z}$ as required. Since ν is injective, if $V \subset S$ then

$$|\bar{V}| = |\{\nu q : q \in \bar{V}\}| \leq \max \{\nu q : q \in \bar{V}\} - \min \{\nu q : q \in \bar{V}\} + 1.$$

Now ν is strict order-preserving so

$$\max \{\nu q : q \in \bar{V}\} = \max \{\nu q : q \in V\} = \max V,$$

and similarly for min V. $\square$

The fact that ψ is strict order-preserving does not imply that (2.21) holds and vice versa. To see this, let V be the union of k chains C_m, and for every chain $\psi i = i$ where $1 \leq i \leq m$. Then ψ is strict order-preserving, however

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The fact that ψ is strict order-preserving does not imply that (2.21) holds and vice versa. To see this, let V be the union of k chains C_m, and for every chain $\psi i = i$ where $1 \leq i \leq m$. Then ψ is strict order-preserving, however

$$|\bar{V}| = |V| = km \not< m = \max V - \min V + 1.$$

Conversely, if V is C_m with $\psi 1 = m$, $\psi m = 1$ and $\psi i = i$ for $1 < i < m$, then (2.21) holds but ψ is not order-preserving.

However, both (2.21) and ψ being strict order-preserving together imply that ψ is injective. For let $V = \{s_1, s_2\}$ and $\psi s_1 = \psi s_2 = \alpha$. If $s_1 < s_2$ then this contradicts ψ being strict order-preserving. Otherwise $s_1 | s_2$ and $|\bar{V}| = |V| = 2 \not< \alpha - \alpha + 1 = \max V - \min V + 1$. Hence we must have $\psi s_1 \neq \psi s_2$.

Theorem 2.23:

Let $S \subset P$ *and* $\psi : S \to C_{|P|}$ *be strict order-preserving. Then* ψ *extends to* $\lambda \in \Lambda$ *iff for all* $\emptyset \neq V \subset S$ *we have* (2.21) *and both*

(2.22) $\quad |\text{above } V| < |P| - \min V + 1,$

(2.23) $\quad |\text{below } V| < \max V.$

Proof:

By Theorem 2.22 condition (2.21) is necessary. Also in view of (2.13) conditions (2.22), (2.23) are clearly necessary for ψ to extend. To prove the sufficiency suppose (2.21), (2.22) and (2.23) all hold. Now define P' and S' as in the proof of Theorem 2.19. Similarly let $C' = [0, |P| + 1]$ and define $\psi r = 0$ and $\psi t = |P| + 1$. Now ψ will extend from S' to P' by Theorem 2.22 iff (2.21) holds in S'. We now show that (2.21), (2.22) and (2.23) holding in S imply that (2.21) holds in S'. So let $s, s' \in S$.

Case 1:

(2.21) holding for $\{s,s'\}$ in S implies it still holds in S'.

Case 2:

For $s < t$ we need to show $|\{\overline{s,t}\}| \leq |P| + 1 - \psi s + 1$ in S'. Now by (2.22),

$$|\{\overline{s,t}\}| = |\{p \geq s : p \in P\}| + 1 \leq |P| - \psi s + 2,$$

as required.

Case 3:

For $r < s$ we will show $|\{\overline{r,s}\}| \leq \psi s - 0 + 1$ in S'. From (2.23) we have

$$|\{\overline{r,s}\}| = |\{p \leq s : p \in P\}| + 1 \leq \psi s + 1.$$

Case 4:

For $r < t$ we must show $|\{\overline{r,t}\}| \leq |P| + 1 - 0 + 1$ in S'. Now the L.H.S. is $|P'|$ which is $|P| + 2$.

We conclude that (2.21) holds in S', and then apply Theorem 2.22 to $S' \subset P'$. □

Theorem 2.24: *Let* $S \subset P$ *and* $\psi : S \to \mathbb{Z}$ *be an order-preserving injection. If* $k = \lceil ht(P)/2 \rceil$ *then there is a partition* $S = S_1 \cup S_2 \cup \ldots \cup S_k$ *such that* ψ *extends to an order-preserving injection from any one* S_i.

Proof:

For $1 \leq i \leq k$ put

$$S_i = \{p \in S : 2i-1 \leq ht_P(p) \leq 2i\}.$$

In each S_i a chain has at most two vertices, so (2.21) holds, and ψ extends from S_i by Theorem 2.22. □

It is not sufficient for the ψ in Theorem 2.24 to be strict order-preserving. To see this define P to be the chain $A_1 < A_2 < \ldots < A_m$ where each A_i is an antichain. If S_i is A_i for some i and $\psi A_i = i$ then clearly ψ cannot extend to an order-preserving injection.

Further consider the example where P is the chain $[1,\ldots,m]$ and S is $[1,\ell]$, $\ell < m$ and $\psi(i+1) = \psi i + n$ for large n. This shows that the above ψ will not extend to an order-preserving bijection. The example in the previous section shows the minimality of k in general.

<u>Theorem 2.25</u>:

Let $S \subset P$ *and* $\psi : S \to C_{|P|}$ *be an order-preserving injection satisfying* (2.22) *and* (2.23). *If* $k = \lceil ht(P)/2 \rceil$ *then there are* $\lambda_1, \lambda_2, \ldots, \lambda_k \in \Lambda$ *and a partition* $S = S_1 \cup S_2 \cup \ldots \cup S_k$ *such that* ψ *extends to* λ_i *from any one* S_i.

<u>Proof</u>:

Define S_i as in the proof of Theorem 2.24, and similarly with (2.21) holding in each S_i then ψ extends from S_i by Theorem 2.23. □

We end this chapter by remarking that if $S \subset P$ and $\psi : S \to \mathbb{Z}$ is order-preserving, then ψ will always extend to an order-preserving map. For we can let $\psi(\text{above } S) = \max S$ and $\psi(\text{below } S) = \min S$ and set $\psi(\text{incomp } S)$ arbitrarily to any value in $[\min S, \max S]$.

CHAPTER 3 : CORRELATION INEQUALITIES FOR MONOTONIC FUNCTIONS

3.1 INTRODUCTION

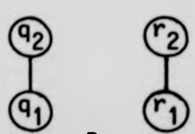
In 1980 R.L. Graham, A.C. Yao and F.F. Yao [GYY] established some monotonicity properties of partial orders. Contributions followed by D.E. Daykin, Kleitman, Shearer, Shepp, Winkler and others.

The results show that if all maps of the poset are equally likely, then the probabilities of maps satisfying sets of comparabilities in the poset are positively correlated or mutually favourable. The maps considered are usually linear extensions and strict order-preserving maps.

One motivation for these monotonicity properties is algorithmically based. In the theory of sorting a typical operation is to make a binary comparison x ? y between elements x,y. Most algorithms for sorting a set proceed by using such comparisons to build successively stronger partial orders on the set until a linear order can be deduced. A fundamental quantity in analyzing the expected efficiency of such algorithms is prob $(x < y)$, i.e., the probability that the result of x ? y is $x < y$ assuming all linear extensions of the poset are equally likely. Such quantities are also important in establishing the complexity of selecting the kth largest element.

So for any class of monotonic maps we define prob (α) to be the proportion of the set of maps for which α holds. By way of illustration consider:

Example 3.1:



P

Then for $\Lambda(P)$,

$$\text{prob}(q_1 < r_1) = 3/6, \qquad \text{prob}(q_1 < r_2) = 5/6,$$

$$\text{prob}(q_1 < q_2) = 1, \qquad \text{prob}\ (r_2 < r_1) = 0.$$

Conditional probabilities show

$$\text{prob}(q_1 < r_1 | P \cup \{q_1 < r_2\}) = \frac{|\{\Lambda: q_1 < r_1 \text{ and } q_1 < r_2\}|}{|\{\Lambda: q_1 < r_2\}|} = 3/5,$$

and $\quad \text{prob}(q_1 < r_2 | P \cup \{r_1 < q_2\}) = 4/5.$

Various intuitive but nontrivial properties of prob$(x < y)$ and related quantities are presented here. Some counter-intuitive features are also illustrated.

We begin with a motivating example. Suppose that the players in two tennis teams have been linearly ranked by skill, namely $Q = \{q_1 < q_2 < \ldots < q_m\}$ and $R = \{r_1 < r_2 < \ldots < r_n\}$. So player x will lose to player y when $x < y$. If the two teams have never met it is reasonable to assume that all relative rankings among the players of $P = Q \cup R$ are equally likely, provided they respect the linear orders Q,R. Now if $q_1 \,?\, r_1$ is the initial match, then

$$\text{prob}(q_1 < r_1 | P) = \binom{m-1+n}{n} \Big/ \binom{m+n}{n} = \frac{m}{m+n}\ .$$

Consider the different situation where the teams have previously competed, with results

$$S = \{q_{i_1} < r_{j_1},\ q_{i_2} < r_{j_2}, \ldots, q_{i_s} < r_{j_s}\}.$$

As we would expect

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$$\mathrm{prob}(q_1 < r_1) = 3/6, \qquad \mathrm{prob}(q_1 < r_2) = 5/6,$$
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Conditional probabilities show

$$\mathrm{prob}(q_1 < r_1 | P \cup \{q_1 < r_2\}) = \frac{|\{\Lambda : q_1 < r_1 \text{ and } q_1 < r_2\}|}{|\{\Lambda : q_1 < r_2\}|} = 3/5,$$

and $$\mathrm{prob}(q_1 < r_2 | P \cup \{r_1 < q_2\}) = 4/5.$$

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Consider the different situation where the teams have previously competed, with results

$$S = \{q_{i_1} < r_{j_1}, q_{i_2} < r_{j_2}, \ldots, q_{i_s} < r_{j_s}\}.$$

As we would expect

$$\text{prob}(q_1 < r_1 | P) < \text{prob}(q_1 < r_1 | P \cup S),$$

since the additional information indicates that the team R has stronger players.

Moreover, if Ψ and Φ are each sets of matches where various q's lost to r's, then our intuition is correct that Ψ and Φ are mutually favourable, i.e.,

$$\text{prob}(\Psi | P \cup S) < \text{prob}(\Psi | P \cup S \cup \Phi)$$

or equivalently

$$\text{prob}(\Psi | P \cup S)\text{prob}(\Phi | P \cup S) < \text{prob}(\Psi \text{ and } \Phi | P \cup S).$$

We demonstrate several general theorems concerning such monotone properties. Existing proofs have mainly employed the classical FKG lattice inequality. New results are introduced on corresponding negatively correlated properties of posets. For both the positive and negative correlations we show how it is simpler to use a lattice inequality of D.E. Daykin rather than that of FKG.

Graham [G1] made a conjecture for linear extensions involving the concept of ranges of elements. We characterize such posets and discuss supporting results for the conjecture along with obtaining weak forms of his required correlations. It is shown that the proposed range conditions have fundamental importance in this area.

Shepp [Sh2] established an important transitivity inequality and we derive a generalization of this via direct products of lattices.

Throughout this chapter we generally assume that the poset P has been partitioned as $P = Q \cup R$, and consider subposets defined on the elements of P whose relations are all of the form $q < r$ where $q \in Q$, $r \in R$, though it is not assumed that either $q \sim r$ or $q|r$ in P. We shall deal with results of the form "$P = Q \cup R$ has the **PCP** or **NCP** for M", where M is $\Omega^0, \Omega, \Lambda^I$ or Λ, and so we define these terms.

Definition 3.1:

The partition $P = Q \cup R$ has the **PCP** *positive correlation property* for M if, whenever both Ψ and Φ are a disjunction of conjunctions of inequalities in which each inequality has the form $q < r$ with $q \in Q$, $r \in R$, we have

$$(3.1) \qquad |\{M:\Psi\}|\,|\{M:\Phi\}| \leq |M|\,|\{M:\Psi \text{ and } \Phi\}|.$$

To see that we are dealing with probability results, many authors divide (3.1) by $|M|^2$ and express the result as

$$\text{prob}(\Psi)\ \text{prob}(\Phi) \leq \text{prob}(\Psi \text{ and } \Phi).$$

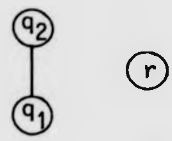
Definition 3.2:

The partition $P = Q \cup R$ has the **NCP** *negative correlation property* for M if, whenever Ψ (respectively Φ) is a disjunction of conjunctions of inequalities in which each inequality has the form $q < r$ (respectively $q > r$) with $q \in Q$, $r \in R$, we have

$$(3.2) \qquad |M|\,|\{M:\Psi \text{ and } \Phi\}| \leq |\{M:\Psi\}|\,|\{M:\Phi\}|.$$

Note that **PCP** can fail for Λ when Ψ and Φ are as in Definition 3.2.

<u>Example 3.2</u> :



Q R

Given $\Lambda(Q \cup R)$,

$$\left(\frac{2}{3}\right)^2 = \mathrm{prob}(q_1 < r)\mathrm{prob}(r < q_2) \nless \mathrm{prob}(q_1 < r < q_2) = \frac{1}{3}.$$

Similarly $\mathbf{NCP}$ can fail for Λ when Ψ and Φ are as in Definition 3.1: with the above $\Lambda(Q \cup R)$, take $\Psi = (q_1 < r)$ and $\Phi = (q_2 < r)$.

We first present some fundamentally important inequalities from the literature, commencing with a classical result of Chebyshev.

<u>Theorem 3.1</u>: (Chebyshev (see [HLP])).

Let $\mu(1),\ldots,\mu(n) \in \mathbb{R}^+$ *and* $f(1),\ldots,f(n),g(1),\ldots,g(n) \in \mathbb{R}$. *If* $f(1) \leq \ldots \leq f(n)$ *and* $g(1) \leq \ldots \leq g(n)$ *or* $f(1) \geq \ldots \geq f(n)$ *and* $g(1) \geq \ldots \geq g(n)$ *then*

$$(3.3) \qquad \left(\sum_x \mu(x)f(x)\right)\left(\sum_x \mu(x)g(x)\right) \leq \left(\sum_x \mu(x)\right)\left(\sum_x \mu(x)f(x)g(x)\right).$$

If $f(1) \leq \ldots \leq f(n)$ and $g(1) \geq \ldots \geq g(n)$ then the inequality (3.3) is reversed.

A result which has just begun to be exploited in combinatorics is the following FKG inequality, named after its authors. This inequality represents a way of extending (3.3) to the situation in which the ground set is only partially ordered, as opposed to the total order $1 < 2 < \ldots < n$ occurring in (3.3).

If L is a lattice then a function $f:L \to \mathbb{R}$ is *increasing* if $x < y$ in L implies $f(x) < f(y)$; f is *decreasing* if $-f$ is increasing.

<u>Theorem 3.2</u>: (Fortuin, Kasteleyn and Ginibre [FKG]).

Let L *be a finite distributive lattice and let* $\mu:L \to \mathbb{R}^+$ *satisfy*

(3.4) $\quad \mu(x)\mu(y) < \mu(x \vee y)\mu(x \wedge y)$ *for all* $x,y \in L$.

Let f,g *be both increasing (or decreasing) functions on* L. *Then*

(3.5) $\quad (\sum_{x\in L} \mu(x)f(x))(\sum_{x\in L} \mu(x)g(x)) < (\sum_{x\in L} \mu(x))(\sum_{x\in L} \mu(x)f(x)g(x)).$

A function μ satisfying (3.4) is called *log supermodular*, and if the inequality is reversed it is *log submodular*. When f is increasing and g is decreasing the inequality (3.5) is reversed.

So we have here a type of monotonicity property for distributive lattices. This 1971 result was derived by the authors in their work on the statistical mechanics of correlation properties of Ising ferromagnet spin systems. In turn the FKG inequality has stimulated research in various directions including generalizations, and applications particularly to computer science, the theory of posets and statistics. Applications to computer science of the FKG and related lattice inequalities include the important area of analysis of sorting algorithms based on partial orders.

A characterization of distributive lattices is given by:

<u>Theorem 3.3</u>: (D.E. Daykin [D1]).

A finite or infinite lattice is distributive iff $|A||B| < |A \vee B||A \wedge B|$, *where* A,B *are subsets of the lattice elements.*

The above three inequalities belong to a group of special cases (see [D2]) of the following result:

Theorem 3.4: (R. Ahlswede and D.E. Daykin [AD]).

Let L *be the family of all subsets of the set* $\{1,2,\ldots,n\}$. *If* $\alpha,\beta,\gamma,\delta : L \to \mathbb{R}^+$ *satisfy*

$$\alpha(a)\beta(b) \leq \gamma(a \vee b)\ \delta(a \wedge b) \text{ for all } a,b \in L,$$

then

$$\alpha(A)\beta(B) \leq \gamma(A \vee B)\ \delta(A \wedge B) \text{ for all } A,B \subset L.$$

(Recall from page 4, $\alpha(A) = \Sigma(a \in A)\ \alpha(a)$ etc.)

Using Theorem 1.2 we immediately have:

Corollary 3.1: (R. Ahlswede and D.E. Daykin [AD]).

Theorem 3.4 *holds for any distributive lattice* L. *Here* L, A, B *may be infinite.*

We introduce the use of Theorem 3.3 for proving that P has the **PCP** or **NCP** for M. Existing proofs using Theorem 3.2 typically take L to be a lattice including the set M (possibly in an encoded form - see Theorems 3.5, 3.7 and 3.13 below), and let μ be characteristic for M, f characteristic for functions respecting ψ and g characteristic for functions respecting ϕ. The natural correspondence between (3.5) and (3.1) yields **PCP**. In applying Theorem 3.3 to L for **PCP**, the sets A and B will respectively denote $\{M:\psi\}$ and $\{M:\phi\}$, whilst $A \vee B$ will correspond to M and $A \wedge B$ will correspond to $\{M:\psi \text{ and } \phi\}$, or vice versa. The inequality $|A||B| \leq |A \vee B|\ |A \wedge B|$ then implies (3.1). **NCP** is established analogously. Notice that A,B are arbitrary subsets whereas f,g are required to be increasing or decreasing functions. It will be seen that in using Theorem 3.3 a similar convexity property to (3.4) is necessary for all x in A and y in B rather than for all x,y in L.

With fewer conditions to satisfy it may be easier to find a suitable lattice to prove a result, which tends to be the critical part of the proof.

3.2 LINEAR EXTENSIONS OF TWO CHAINS

All the work in this chapter stems from the result of R.L. Graham, A.C. Yao and F.F. Yao [GYY] stating that, if $P = Q \cup R$ and Q,R are disjoint chains then P has the $\mathfrak{PCP}$ for Λ. Their proof used explicit combinatorial pairings of certain maps in Λ. This entailed defining lattice paths in $\mathbf{Z}^2$ to represent linear extensions, and "barriers" for any relations between the chains. Also they indicated that stronger monotonicity theorems may be established via the FKG inequality.

Indeed soon after [GYY] appeared contributions followed by Graham, Kleitman, Shearer, Shepp and others all involving the FKG. Included among these were rather short proofs by Kleitman and Shearer [KlSh] and independently Shepp [Sh1] of the above result.

Lattice paths in $\mathbf{Z}^2$ have been used in other areas when the poset is a union of two linear orders (an example is mentioned in Chapter 4.2). Since it is a useful technique we shall employ the construction given in [KlSh], whilst using Theorem 3.3 instead of the FKG inequality to show:

<u>Theorem 3.5</u>: *If* $P = Q \cup R$ *and* Q,R *are disjoint chains then* P *has the* $\mathfrak{PCP}$ *and the* $\mathfrak{NCP}$ *for* Λ.

<u>Proof</u>:

Let $|Q| = m$ and $|R| = n$. The proof proceeds by assigning a subset $S(\lambda)$ of the unit squares in an $m \times n$ rectangle to each linear extension λ of the partial order. Subsets of squares that correspond to such extensions will have the weighting $\alpha = 1$; all other subsets of squares will have $\alpha = 0$. We have that the set of 2^{mn} subsets of squares forms a distributive lattice. A and B will correspond to subsets of squares that obey the restrictions given by Ψ, Φ and P. Then D.E. Daykin's inequality yields the theorem. The proof is thus completed by supplying the following three steps:

(i) defining $S(\lambda)$;

(ii) verifying the inequality $\alpha(D)\alpha(E) \leq \alpha(D \cap E)\alpha(D \cup E)$ for subsets D,E of the rectangle;

(iii) defining A and B.

Given a linear extension of P, $\lambda_1 < \lambda_2 < \dots < \lambda_{m+n}$, we draw a lattice path in the plane starting from the origin consisting of m + n unit steps, the ith step being vertical if the ith element in the extension is an $r \in R$, horizontal if it is a $q \in Q$. Then $S(\lambda)$ consists of the squares of the $m \times n$ rectangle that are below and to the right of this path, which ends at the point (m,n).

Any inequalities $q_i < r_j$, in Ψ or Φ, can be interpreted as stating that in each extension λ obeying it there are in the corresponding set no squares in the jth row before the (i+1)th. A conjunction of these involves several such restrictions, and a disjunction of such conjunctions has the obvious meaning.

Similarly any relation $q_i \sim r_j$, $q_i \in Q$, $r_j \in R$ in P can be viewed as a vertical or horizontal *barrier* in the rectangle, through which the lattice path must not pass. Also given a pair of lattice paths suppose $i < j$ and λ_i, λ_j are common to both of them, with no λ_k common to both where $i < k < j$. In [GYY] the closed region bounded by these two path segments is called an *olive*.

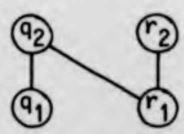
Suppose λ,μ are extensions of P with corresponding sets $D(\lambda)$, $E(\mu)$. Then $\alpha(D)\alpha(E) = 1$. This means that the lattice paths for λ,μ do not cross any of the barriers. Now $D \cap E$ is formed by choosing the squares below and to the right of the lower segment of every olive, which clearly respects all barriers, and so $\alpha(D \cap E) = 1$. Similarly $D \cup E$ chooses the upper segments of olives which also defines a subset with $\alpha(D \cup E) = 1$. This is step (ii).

First we will establish $\mathcal{PCP}$. Let $A = \{\Lambda:\Psi\}$ and $B = \{\Lambda:\Phi\}$. That is, A and B correspond to sets of subsets of squares with $\alpha = 1$ satisfying some conjunctions of horizontal barriers determined by Ψ and Φ respectively. Then clearly $A \wedge B \subset \{\Lambda : \Psi \text{ and } \Phi\}$ and we have already shown that $A \vee B \subset \Lambda$. The result follows from Theorem 3.3, i.e., we have proved the theorem of R.L. Graham, A.C. Yao and F.F. Yao.

For the $\mathcal{NCP}$ case let $A = \Lambda$ and $B = \{\Lambda : \Psi \text{ and } \Phi\}$. Then any set of squares in B obeys some conjunctions of the horizontal barriers specified by Ψ, along with some conjunctions of vertical barriers given by Φ. Hence $A \wedge B \subset \{\Lambda : \Psi\}$, while $A \vee B \subset \{\Lambda : \Phi\}$. Again we apply Theorem 3.3, giving $|\Lambda||\{\Lambda:\Psi \text{ and } \Phi\}| < |\Lambda:\Psi\}||\{\Lambda:\Phi\}|$. □

Although we may expect an immediate analogue of Theorem 3.5 for strict order-preserving maps we do not get one.

<u>Example 3.3</u>:



P

With reference to **PCP**, given $\Omega(P,C_3)$,

$$\left(\frac{1}{8}\right)^2 = \text{prob}(q_1 < r_1)\text{prob}(q_2 < r_2) \nless \text{prob}(q_1 < r_1 \text{ and } q_2 < r_2) = 0.$$

3.3 MONOTONICITY FOR TWO POSETS

A natural extension of Theorem 3.5 would be to weaken the restrictions on Q and R so that they are only partially ordered rather than totally ordered, and **PCP** for Λ was conjectured for a particular case of this situation in [GYY, preprint]. However, examples followed by Graham, Kleitman, Shearer, Shepp, A.C. Yao and F.F. Yao showing that this conjecture is not true when the poset may contain just one relation $q \sim r$, $q \in Q$, $r \in R$, even if it is of the type $q < r$. This is surprising since intuitively **PCP** is expressing that it is more likely that the elements of Q are generally lesser than those of R, as indicated by the tennis example, and a relation $q < r$ in $Q \cup R$ would reinforce this likelihood. Also as it was shown, when Q, R are each chains, **PCP** holds with any relations $q < r$ or $q > r$ in $Q \cup R$.

Soon followed:

Theorem 3.6: (Shepp [Sh1]).

If $P = Q \cup R$ *and* Q,R *are disjoint posets such that* $Q|R$ *then* P *has the* **PCP** *for* Ω *and for* Λ.

The proportion of members of $\Omega(P,C)$ which are injective tends to 1 as $|C| \to \infty$, that is if P,Q are posets then

$$\frac{|\Lambda(Q \cap P)|}{|\Lambda(P)|} = \lim_{|C|\to\infty} \frac{|\Omega(Q \cap P,C)|}{|\Omega(P,C)|} .$$

Thus Shepp proved that his result for Ω implied the result for Λ.

This limiting process is very useful since as yet no lattices have been defined which capture bijectivity directly, except for when the poset can be covered by two chains.

As $|C|$ increases it can be seen that $\Lambda(P)$ and $\Omega(P,C)$ appear to be very similar.

Using the same reasoning, the sets Ω and Ω^0 behave the same in the limit $|C| \to \infty$.

Graham then suggested as a sufficient condition that the poset $P = Q \cup R$ satisfy a range condition for Ω, namely

$$(3.6) \qquad q \in Q,\ r \in R,\ q \sim r \Rightarrow (\Omega q) \cap (\Omega r) = \emptyset.$$

An incomplete proof for the **PCP** situation of the following theorem appears in [G1]; a special case also for **PCP** due to J.W. Daykin and R.L. Graham is in [G2]; the full version given here for **PCP** and **NCP** is presented in [DD]. Again we apply D.E. Daykin's inequality as opposed to the FKG, and mention that the lattice was defined in [Sh1].

Theorem 3.7:

If $P = Q \cup R$ *and* Q,R *are disjoint posets satisfying*

$$q \in Q,\ r \in R \textit{ with } q \sim r \textit{ implies } (\Omega q) \cap (\Omega r) = \emptyset$$

then P *has the* **PCP** *and the* **NCP** *for* Ω.

Proof:

Let Q,R be disjoint sets and Θ be the set of all maps $\theta: Q \cup R \to C$. For $\theta_1,\theta_2 \in \Theta$ define $\theta_1 \vee \theta_2, \theta_1 \wedge \theta_2$ for $q \in Q$, $r \in R$ by

$$(\theta_1 \vee \theta_2)q = \max\{\theta_1 q, \theta_2 q\},\ (\theta_1 \wedge \theta_2)q = \min\{\theta_1 q, \theta_2 q\},$$

$$(\theta_1 \vee \theta_2)r = \min\{\theta_1 r, \theta_2 r\},\ (\theta_1 \wedge \theta_2)r = \max\{\theta_1 r, \theta_2 r\},$$

and then it follows that

(3.7) $\theta_1 q < \theta_1 r \Rightarrow (\theta_1 \wedge \theta_2)q < (\theta_1 \wedge \theta_2)r$,

(3.8) $\theta_1 q > \theta_1 r \Rightarrow (\theta_1 \vee \theta_2)q > (\theta_1 \vee \theta_2)r$.

Clearly $\theta_1 \vee \theta_2$, $\theta_1 \wedge \theta_2 \in \Theta$.

Let $q_1, q_2, \ldots, q_m$ and $r_1, r_2, \ldots, r_n$ be the elements of Q and R respectively. With each $\theta \in \Theta$ associate the vector $(\theta q_1, \ldots, \theta q_m, \theta r_1, \ldots, \theta r_n)$ considered as an element of the lattice $L = C^m(C^*)^n$, where C^* is the dual of C. It follows that $|L| = |C|^{m+n}$. Now C is a chain, and since L is a direct product of copies of C and C^* it is distributive. The order relation of L is defined by $\theta_1 < \theta_2$ iff $\theta_1 q_i < \theta_2 q_i$ and $\theta_1 r_j > \theta_2 r_j$ for $1 < i < m$, $1 < j < n$. Also the join and meet in L are given by $\theta_1 \vee \theta_2$ and $\theta_1 \wedge \theta_2$ respectively.

Claim 3.1:

Ω is a sublattice of L.

Proof:

It is required to show that if $\omega_1, \omega_2 \in \Omega \subset \Theta$ *then* $\omega_1 \vee \omega_2$, $\omega_1 \wedge \omega_2 \in \Omega$.

So first suppose that there exists a relation in Q, $q_i < q_j$ say, and $\theta_1, \theta_2 \in \Theta$ satisfy $\theta_1 q_i < \theta_1 q_j$ and $\theta_2 q_i < \theta_2 q_j$. Now

$$(\theta_1 \vee \theta_2)q_i = \max\{\theta_1 q_i, \theta_2 q_i\} < \max\{\theta_1 q_j, \theta_2 q_j\} = (\theta_1 \vee \theta_2)q_j,$$

$$(\theta_1 \wedge \theta_2)q_i = \min\{\theta_1, q_i, \theta_2 q_i\} < \min\{\theta_1 q_j, \theta_2 q_j\} = (\theta_1 \wedge \theta_2)q_j.$$

Since $q_i < q_j$ was arbitrary then this means that $\vee, \wedge$ preserve strict order-preserving mapping for relations in Q. Using the symmetry $\vee \leftrightarrow \wedge$, $q \leftrightarrow r$ we deduce that these operators also respect any relations existing in R.

Now suppose $q \in Q$, $r \in R$ and $q < r$ in P. Also assume $\theta_1 q < \theta_1 r$ and $\theta_2 q < \theta_2 r$. By Theorem 2.3 we know that Ωq, Ωr are intervals in C. Using this fact along with (3.6) we have

$$(\theta_1 \vee \theta_2)q = \max\{\theta_1 q, \theta_2 q\} < \min\{\theta_1 r, \theta_2 r\} = (\theta_1 \vee \theta_2)r,$$

further by (3.7),

$$(\theta_1 \wedge \theta_2)q < (\theta_1 \wedge \theta_2)r,$$

and likewise for any relations in $Q \times R$. The argument for relations in $R \times Q$ is similar, but requires (3.8), along with Theorem 2.3 and (3.6).

Therefore $\vee, \wedge$ preserve monotonicity for all relations in P and we conclude that $\Omega \subset L$ is closed under these operators. □

We are ready to apply Theorem 3.3. For the **PCP** case let $A = \{\Omega:\Psi\}$ and $B = \{\Omega:\Phi\}$. Then $A \wedge B \subset \{\Omega:\Psi \text{ and } \Phi\}$ by (3.7), and $A \vee B \subset \Omega$ by Claim 3.1. Hence

$$|\{\Omega:\Psi\}|\,|\{\Omega:\Phi\}| < |\{\Omega(P,C)\}|\,|\{\Omega:\Psi \text{ and } \Phi\}|.$$

Now let $A = \{\Omega:\Psi \text{ and } \Phi\}$ and $B = \Omega$. Then $A \wedge B \subset \{\Omega:\Psi\}$ by (3.7), while $A \vee B \subset \{\Omega:\Phi\}$ by (3.8). Thus we get **NCP**. □

The hypothesis of Theorem 3.6 satisfies (3.6) vacuously. So Theorem 3.6 follows from Theorem 3.7 by letting $|C| \to \infty$.

It is important to mention that we cannot let $|C| \to \infty$ to deduce the Λ case from that of Ω in Theorem 3.7. Since Ω depends on C, the assumption (3.6) must depend on C, and by Theorem 2.9 it is not satisfied when $|C|$ is large.

However the proof of Theorem 3.7 yields an extended form of Theorem 3.6: If $P = Q \cup R$ and Q,R are disjoint posets such that $Q|R$ then P has the **PCP** and the **NCP** for both Ω and Λ.

Graham asked [G1] whether condition (3.6) is necessary for $\mathcal{PCP}$ in Theorem 3.7. Looking back at Example 3.3 we see that $(\Omega q_2) \cap (\Omega r_1) \neq \emptyset$, i.e., (3.6) is not satisfied, and $\mathcal{PCP}$ fails. Hence the range condition is necessary here. This also shows that the closest analogue of Theorem 3.5 for Ω is given by Theorem 3.7, in which case Theorem 2.10 is relevant.

In relation to Theorem 3.7, Graham [G2] was interested in $|C|$ for $\Omega(P,C)$. In particular he considered $|C| = |P|$. We have taken $|C|$ to be arbitrary, and it is not difficult to find examples to show that the range condition is necessary when any of the following are true: $|C| < |P|$, $|C| = |P|$, $|C| > |P|$ or $|C| = ht(P)$.

3.4 RANGE POSETS

Once Graham had proposed condition (3.6) it was natural for him to require that the poset $P = Q \cup R$ satisfies a range condition for Λ, namely

$$(3.9) \qquad q \in Q,\ r \in R,\ q \sim r \rightarrow (\Lambda q) \cap (\Lambda r) = \emptyset.$$

Upon studying (3.9) we will prove in Theorem 3.8 that the only posets which satisfy this condition are what we now define to be range posets.

Definition 3.3:

$P = Q \cup R$ is a *range poset* if there are partitions

$$Q = Q_1 \cup Q_2 \cup \ldots \cup Q_m, \quad R = R_1 \cup R_2 \cup \ldots \cup R_n$$

such that

$$(3.10) \qquad Q_i < Q_j \text{ for } 1 \le i < j \le m,$$

$$(3.11) \qquad R_i < R_j \text{ for } 1 \le i < j \le n,$$

$$(3.12) \qquad \text{either } Q_i | R_j \text{ or } Q_i < R_j \text{ or } R_j < Q_i \text{ for } 1 \le i \le m,\ 1 \le j \le n.$$

Lemma 3.1:

We can assume that $m \le 2n + 1$ *and* $n \le 2m + 1$ *in Definition* 3.3.

Proof:

The intersection of R with above Q_i, below Q_i and incomp Q_i yields three linearly ordered components in R. Repeating this for $1 \le i \le m$ shows that $n \le 2m+1$. Similarly we get $m \le 2n+1$. □

Graham suggested [G1] that if $P = Q \cup R$ satisfies (3.9) then it has the **PCP** for Λ. We extend this interesting proposal into the Range Poset Conjecture:

Conjecture 3.1:

A range poset P has the **PCP** and the **NCP** for $\Omega(P,C)$ if $|C|$ large, which implies it has the **PCP** and the **NCP** for $\Lambda(P)$.

A simple calculation verifies that Example 3.3 does not break the conjecture.

We now give a complete characterization for range posets.

Theorem 3.8:

If $P = Q \cup R$ *then the following are equivalent*

$$q \in Q,\ r \in R,\ q \sim r \Rightarrow (\Lambda q) \cap (\Lambda r) = \emptyset, \tag{3.13}$$

$$\begin{cases} q \in Q,\ r \in R,\ q < r \Rightarrow P = (\text{above } \{q\}) \cup (\text{below } \{r\}), \\ q \in Q,\ r \in R,\ q > r \Rightarrow P = (\text{below } \{q\}) \cup (\text{above } \{r\}), \end{cases} \tag{3.14}$$

(3.15) P *is a range poset as in Definition* 3.3.

Proof Part (3.13) $\Rightarrow$ (3.14).

Suppose that $q \in Q$, $r \in R$ and $q < r$. We assume the worst case, namely r covers q. Let $U = \text{above } \{q\}$, $D = \text{below } \{r\}$ and then $|U \cap D| = |\{q,r\}| = 2$. Since $|P| = |Q| + |R|$, by Theorem 2.14 we get $\Lambda q = [|\text{below } \{q\}|, |P| + 1 - |U|]$ and $\Lambda r = [|D|, |P| + 1 - |\text{above } \{r\}|]$. From (3.13) we deduce that $|P| + 2 - |U| \leq |D|$. On the other hand $|U| + |D| = 2 + |U \cup D| \leq 2 + |P|$, so we have equality throughout, and the first of the conditions in (3.14) follows. The second then follows in turn by the symmetry $q \leftrightarrow r$.

Part (3.14) $\Rightarrow$ (3.15).

For each $q \in Q$ put

$$U(q) = R \cap \text{above } \{q\} \text{ and } D(q) = R \cap \text{below } \{q\}.$$

Claim 3.2:

If $q_1, q_2 \in Q$ *and either* $U(q_1) \setminus U(q_2) \neq \emptyset$ *or* $D(q_2) \setminus D(q_1) \neq \emptyset$ *then* $q_1 < q_2$.

Proof:

Suppose that $r \in U(q_1) \setminus U(q_2)$. Then $q_1 < r$ and so by (3.14) we have $q_2 \in P = (\text{above } \{q_1\}) \cup (\text{below } \{r\})$. Now $r \notin U(q_2)$ so $q_2 \nless r$ and hence $q_2 \notin \text{below } \{r\}$. Therefore $q_2 \in \text{above } \{q_1\}$ and $q_1 < q_2$. The rest of the claim follows in the same way. □

Next we define an equivalence relation σ in Q by putting $q_1 \sigma q_2$ iff both $U(q_1) = U(q_2)$ and $D(q_1) = D(q_2)$.

Claim 3.3:

If $q_1, q_2 \in Q$ *and* $q_1 | q_2$ *then* $q_1 \sigma q_2$.

Proof:

Suppose that $r \in U(q_1)$. Then $q_1 < r$ and by (3.14) we deduce that $q_2 \in \text{below } \{r\}$, and likewise for any such r. Hence $U(q_1) \subset U(q_2)$ and reversing the argument yields $U(q_1) \supset U(q_2)$. Similarly we get $D(q_1) = D(q_2)$. □

Let Q_1, Q_2 be different non-empty equivalence classes of σ and let $q_1 \in Q_1$, $q_2 \in Q_2$. By Claim 3.3 we have $q_1 \sim q_2$. Since $Q_1 \neq Q_2$, by definition of σ we have $U(q_1) \neq U(q_2)$ or $D(q_1) \neq D(q_2)$. We may assume either $U(q_1) \setminus U(q_2) \neq \emptyset$ or $D(q_2) \setminus D(q_1) \neq \emptyset$. Then Claim 3.2 shows that $q_1 < q_2$. If $q_3 \in Q_1$ then $q_1 \sigma q_3$, and the same argument shows that $q_3 < q_2$. Hence we have proved that the equivalence classes of σ are totally ordered. In other words we have obtained:

Claim 3.4:

We can let $Q_1, Q_2, \ldots, Q_m$ *be the equivalence classes of* σ *numbered so that* (3.10) *holds.*

We now repeat this whole procedure for R. So for each $r \in R$ put

$$U(r) = Q \cap \text{above } \{r\} \text{ and } D(r) = Q \cap \text{below } \{r\}.$$

Then let τ be the equivalence relation on R with $r_1 \tau r_2$ iff both $U(r_1) = U(r_2)$ and $D(r_1) = D(r_2)$. Finally we can assume that $R_1, R_2, \ldots, R_n$ are the equivalence classes of τ numbered so that (3.11) holds.

To see that (3.12) holds let $q < r$ with $q \in Q_i$, $r \in R_j$. If q_1 is also in Q_i then $r \in U(q) = U(q_1)$ so $q_1 < r$, and it follows that $Q_i < R_j$. In a similar way we can see how $Q_i | R_j$ and $R_j < Q_i$ also hold.

Part (3.15) $\Rightarrow$ (3.13).

Let $q \in Q_i$, $r \in R_j$ with $q < r$. In view of Theorem 2.14, if $\Lambda q = [k, \ell]$ and $\Lambda r = [k', \ell']$ then

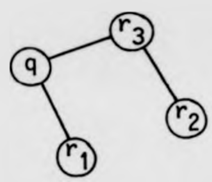
$$\begin{aligned} \ell &= |P| + 1 - |\text{above } \{q\}| \\ &< |P| + 1 - (|\{q\}| + |Q_{i+1}| + \ldots + |Q_m| + |R_j| + \ldots + |R_n|) \\ &= |Q_1| + \ldots + |Q_i| + |R_1| + \ldots + |R_{j-1}| \\ &< |\text{below } \{r\}| = k'. \end{aligned}$$

Hence $(\Lambda q) \cap (\Lambda r) = \emptyset$. If we now suppose $q > r$ then using symmetry we have again that these ranges are disjoint, and (3.13) follows.

This ends the proof of Theorem 3.8. □

We will show that the two parts of (3.14) are independent, i.e., only one of the conditions in (3.14) holding is not sufficient to ensure (3.13).

Example 3.4:



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We have $q < r_3 \Rightarrow P = (\text{above } \{q\}) \cup (\text{below } \{r_3\})$,
while $r_1 < q \not\Rightarrow P = (\text{above } \{r_1\}) \cup (\text{below } \{q\})$.

Now $\Lambda q = [2,3]$, $\Lambda r_1 = [1,2]$ and $\Lambda r_3 = 4$. This means $(\Lambda q) \cap (\Lambda r_3) = \emptyset$ but $(\Lambda q) \cap (\Lambda r_1) \neq \emptyset$.

To see that condition (3.9) is necessary for the case of Λ in the Range Poset Conjecture consider:

Example 3.5:

P

For $\Lambda(P)$ we have $(\Lambda q_2) \cap (\Lambda r_1) \neq \emptyset$.

Regarding $\mathbf{PCP}$,

$$\left(\frac{8}{12}\right)^2 = \text{prob}(q_1 < r_1)\text{prob}(q_2 < r_2) \not\leq \text{prob}(q_1 < r_1 \text{ and } q_2 < r_2) = \frac{5}{12},$$

and $\mathbf{NCP}$,

$$\left(\frac{3}{12}\right) = \text{prob}(q_1 < r_1 \text{ and } q_2 > r_2) \not\leq \text{prob}(q_1 < r_1)\text{prob}(q_2 > r_2) = \left(\frac{8}{12}\right)\left(\frac{4}{12}\right).$$

Example 3.5 had been given in [GYY], and various related examples appeared in the literature. Their purpose was to summarize the necessity that if $P = Q \cup R$ has the $\mathbf{PCP}$ for Λ then, any relation $q \sim r$, $q \in Q$, $r \in R$ in P implies Q,R must be disjoint chains (as in Theorem 3.5); whilst if Q,R are posets we must have $Q|R$ (as in Theorem 3.6). However, none of the posets in the examples is a range poset and hence they do not satisfy (3.9). So we reinterpret the observations to be that (3.9), and in particular the range poset structure, is necessary for $\mathbf{PCP}$ with Λ in Conjecture 3.1.

A special case of the Range Poset Conjecture is given by:

<u>Theorem 3.9</u>:

If P is a range poset with $m = 1$ then P has the $\mathbf{PCP}$ and the $\mathbf{NCP}$ for both Ω and Λ.

<u>Proof</u>:

If $P = Q \cup R$ is such that $Q|R$ then the extended form of Theorem 3.6 on page 60 applies. So assume there exists a $q \sim r$ with $q \in Q$, $r \in R$, and by Lemma 3.1 we have $n \leq 3$. Suppose that Ψ contains an inequality $q_i < r_j$ with $q_i \in Q$ and $r_j \in R_k$, $1 \leq k \leq 3$. Then $\text{prob}(q_i < r_j)$ is 1 if $Q < R_k$ and 0 if $Q > R_k$. Hence without loss of generality we can assume all inequalities in Ψ involve Q and R_k where $Q|R_k$, and similarly for any inequalities $q_i < r_j$ or $q_i > r_j$ in Φ. For $1 \leq j \leq k$ and $Q < R_j$ or $Q > R_j$ we let $P' = P \setminus R_j$, since R_j is independent by changing the probabilities only by a multiplicative factor. Finally the result is already established for P'. □

Notice that we have Theorems 3.5, 3.6 and 3.9 supporting the Range Poset Conjecture, where the range poset of Theorem 3.5 is the special type in which all Q_i and all R_j are singletons. Regarding this conjecture, if none of the relations in either Ψ or Φ involve some Q_i (or R_j), then we can assume that this Q_i (or R_j) is linearly ordered, as this will only affect the probabilities by a multiplicative factor.

We will establish a weak form of Conjecture 3.1. So let P be the range poset in Definition 3.3, and call $Q_1, Q_2, \ldots, Q_m$ the *blocks* of Q and $R_1, R_2, \ldots, R_n$ the blocks of R.

<u>Definition 3.4</u>:

We say that P has the *weak* **PCP** if Definition 3.1 holds except that the inequalities for Φ are between blocks instead of elements. Thus they now have the form $Q_i < R_j$ with $1 \leq i \leq m$, $1 \leq j \leq n$ instead of the form $q < r$ with $q \in Q$, $r \in R$.

<u>Definition 3.5</u>:

P has the *weak* **NCP** if Definition 3.2 holds except that now the inequalities for Φ are between blocks.

In both instances the inequalities for Ψ are still between elements, and those for Φ can be expressed in terms of large numbers of inequalities between elements. Our result is:

<u>Theorem 3.10</u>:

A range poset P *has the weak* **PCP** *and the weak* **NCP** *for* Λ.

<u>Proof</u>:

If $\lambda \in \Lambda$ and $S \subset P$ then there is a unique ordering $s_1, s_2, \ldots, s_{|S|}$ of the elements of S such that $\lambda s_1 < \lambda s_2 < \ldots < \lambda s_{|S|}$. We call $s_1 < s_2 < \ldots < s_{|S|}$ the chain which λ makes out of S. For all $\lambda, \mu \in \Lambda$ we write $\lambda \varepsilon \mu$ if λ and μ make the same chain out of S for every block S of P. Thus ε is an equivalence relation for Λ and we let $\mathbf{E}$ denote the set of equivalence classes of ε.

Let $E, F \in \mathbf{E}$ be fixed initially. Given a block S of P let $s_1 < \ldots < s_{|S|}$ be the chain which every $\lambda \in E$ makes out of S. Also let $t_1 < \ldots < t_{|S|}$ be the chain which every $\mu \in F$ makes out of S.

We now use the concept of 1-1 mapping to relate the cardinalities of the sets E and F. Further detail of this proof technique is given in the next chapter.

Given a $\lambda \in E$ we will construct a unique $\mu' \in F$ as follows. For a block S of P define $\sigma_S : S \to S$ by $\sigma_S s_i = t_i$ for $1 \leq i \leq |S|$. Since $t_1 < \ldots < t_{|S|}$ is monotonic for S then so is the map $\lambda\sigma_S$. Notice also that $\lambda s_i = \mu'\sigma_S s_i$.

Next we define $\pi : P \to P$ by $\pi p = \sigma_S p$ for all p in each block S. Then π is a permutation of P because each σ_S permutes its block S. Also $\lambda S = \lambda\pi S$ for every $\lambda \in E$ and block S. We showed that $\lambda\pi$ is monotonic on each block. Then since P is a range poset and $\lambda \in E \subset \Lambda$ it easily follows that $\lambda\pi$ is a linear extension. Further the map $\lambda \to \lambda\pi$ produces $\mu' \in F$. Hence the relative ranking of blocks is given by λ, whereas the order of elements within blocks is given by the chains in μ. Clearly given μ' we can reconstruct λ by substituting the chains for those of λ. This means μ' is unique. This in turn implies that $|E| \leq |F|$ and thus $|E| = |F|$ by symmetry.

Since Φ is defined in terms of inequalities between blocks we see that $\lambda \in E$ respects Φ iff $\lambda\pi \in F$ respects Φ. Therefore we have $|\{E:\Phi\}| \leq |\{F:\Phi\}|$ and so $|\{E:\Phi\}| = |\{F:\Phi\}|$.

Recall that any $\lambda \in E$ makes a chain $s_1 < \ldots < s_{|S|}$ out of each block S. Let P_E be the poset obtained by adjoining to P all the relations in each of these chains. Thus P_E is a union of linear orders, namely $Q_E \cup R_E$, where Q_E, R_E are the chains which every $\lambda \in E$ makes out of Q,R respectively. Also E is simply the set of all linear extensions of P_E.

Hence we can apply Theorem 3.5 to P_E to get

$$|\{E:\Psi\}|\,|\{E:\Phi\}| \leq |E|\,|\{E:\Psi \text{ and } \Phi\}|,$$

and so

$$|\{E:\Psi\}|\,|\{F:\Phi\}| \leq |F|\,|\{E:\Psi \text{ and } \Phi\}|. \tag{3.16}$$

Forming the double sum of (3.16) over all E, F $\in \mathbb{E}$ gives

$$|\{\Lambda:\Psi\}|\,|\{\Lambda:\Phi\}| \leq |\Lambda|\,|\{\Lambda:\Psi \text{ and } \Phi\}| ,$$

which is the weak $\mathfrak{PCP}$.

When the inequalities in Φ are of the form $Q_i > R_j$, with $Q_i, R_j \in P$, then we deduce from Theorem 3.5 and P_E that

$$|E|\,|\{E:\Psi \text{ and } \Phi\}| \leq |\{E:\Psi\}|\,|\{E:\Phi\}|,$$

from which we obtain the weak $\mathfrak{NCP}$, and the proof is completed. $\square$

Given the disjoint union of two chains of singletons satisfying some correlation properties, upon replacing each singleton by an arbitrary poset, we have proved partial results for the same correlations.

This naturally suggests a technique worth exploring. Namely to take an existing partial order with an established property or structure. Then replace all or some elements by arbitrary posets, or special posets like chains, antichains, to be ordered in the same way as the elements. Then does the known property or structure apply to the extended partial order?

As an easy example suppose $\Omega(P,C)$ satisfies the range condition (3.6). Define P' by repeatedly replacing singletons p,q in P by antichains A,B, where $A < B$ iff $p < q$. Then (3.6) still holds in $\Omega(P',C)$ by Theorem 2.9.

3.5 THE xyz INEQUALITY, AND UNIVERSAL CORRELATION

To date perhaps the most important result in this area is the xyz inequality. The 1981 conjecture was originally due to Rival and Sands, and extended by Winkler, Graham and others.

One motivation concerns transitivity. For Λ or Ω and elements x,y,z,

$$\text{prob}(z < x \mid P \cup \{z < y \text{ and } y < x\}) = 1.$$

However given partial information, $z < y$ say, then it seems more likely that z is a "small" element. It is tempting to conjecture correctly that for any poset P,

$$\text{prob}(z < x \mid P) \leq \text{prob}(z < x \mid P \cup \{z < y\}).$$

Rival indicated that "transitivity inequality" is a more apt name for this result, which demanded much effort by combinatorialists.

The conjecture had appeared less tempting when it was known that for Λ and elements u,v,x,y the following analogous inequality is false,

$$\text{prob}(x < u < y \mid P) \leq \text{prob}(x < u < y \mid P \cup \{x < v < y\}).$$

Using the reasoning that x is "small" and y is "large" it had seemed reasonable for $(x<u<y)$ and $(x<v<y)$ to be positively correlated.

In 1982 appeared:

<u>Theorem 3.11</u>: (Shepp [Sh2]).

If $x,y,z \in P$ *then given* Λ

$$\text{prob}(z < x)\text{prob}(z < y) \leq \text{prob}(z < x \text{ and } z < y).$$

Embedded in Shepp's FKG proof is the more general statement:

Theorem 3.12: (Shepp [Sh2]).

If $P = \{z\} \cup R$ *where* $z \notin R$ *then* P *has the* **PCP** *for* Ω *and for* Λ.

He first proved the Ω case and then deduced the Λ case by letting $|C| \to \infty$ as previously described.

We propose an extension to Theorem 3.12 involving direct products of the lattice defined by Shepp. With each lattice in the product we associate a poset, the **PCP** for a single poset being Shepp's result. As usual we will apply Theorem 3.3 instead of Theorem 3.2.

Theorem 3.13:

Let the elements of the poset P *be the disjoint union* $Z \cup R_1 \cup \ldots \cup R_n$ *where* $Z = \{z_1,\ldots,z_n\}_{\neq}$. *Assume further that if* $i \neq j$ *then both* $z_i | R_j$ *and* $R_i | R_j$. *For* $1 \leq i \leq n$ *let* $x_i, y_i \in R_i$. *Then for both* $\Omega(P,C)$ *and* $\Lambda(P)$ *we have,*

$$\text{prob}(z_i<x_i \text{ for all } i)\text{prob}(z_i<y_i \text{ for all } i) \leq \text{prob}(\text{both } z_i<x_i \text{ and } z_i<y_i \text{ for all } i).$$

Proof:

With $n = 1$ we prove the Ω case and then deduce the Λ case. Afterwards we indicate the proof for when $n > 1$.

Case $n = 1$: Let $m = |R_1|$ and $\Gamma_1 = \{1,2,\ldots,|C|\}^{m+1}$ be the set of $\gamma = (\gamma_0,\gamma_1,\ldots,\gamma_m)$ where each $\gamma_i \in \{1,2,\ldots,|C|\}$. We order Γ_1 by $\gamma \leq \gamma'$ iff

$$\gamma_0 \geq \gamma_0' \text{ and } \gamma_i - \gamma_0 \leq \gamma_i' - \gamma_0' \text{ for } 1 \leq i \leq m.$$

It is easy to verify that $\leq$ is a partial ordering relation on Γ_1. We define meet and join operators on Γ_1 componentwise by

$$(\gamma_i \wedge \gamma_i') = \min(\gamma_i - \gamma_0, \gamma_i' - \gamma_0') + \max(\gamma_0, \gamma_0') \text{ for } 0 \leq i \leq m,$$

$$(\gamma_i \vee \gamma_i') = \max(\gamma_i - \gamma_0, \gamma_i' - \gamma_0') + \min(\gamma_0, \gamma_0') \text{ for } 0 \leq i \leq m.$$

Since the components $(\gamma_i \wedge \gamma_i')$, $(\gamma_i \vee \gamma_i')$ each belong to $\{1,2,\ldots,|C|\}$, it follows that $\gamma \wedge \gamma'$, $\gamma \vee \gamma' \in \Gamma_1$, which thus determines a lattice.

In order to show that Γ_1 is distributive, consider the map defined by Winkler in [Sh2], namely

$$(\gamma_0, \gamma_1, \ldots, \gamma_m) \to (-\gamma_0, \gamma_1 - \gamma_0, \ldots, \gamma_m - \gamma_0) = \delta.$$

Now $-\gamma_0 \in \{-|C|,\ldots,-1\}$ and $\gamma_i - \gamma_0 \in \{1-|C|,\ldots,|C|-1\}$ for $1 \leq i \leq m$. The set Δ of all such δ is a direct product of chains, and thus a distributive lattice. The meet and join are componentwise min and max respectively in Δ. Clearly Γ_1 is isomorphic to a sublattice of Δ and so must also be distributive.

Let P_1 denote the elements of z_1 and R_1 that is the set $\{z_1, x_1, y_1, r_3, \ldots, r_m\}$. We associate each vector in Γ_1 with the elements of P_1 as follows. If Θ is the set of all maps $\theta : P_1 \to C$ then $\gamma_0 = \theta z_1$, $\gamma_1 = \theta x_1$, $\gamma_2 = \theta y_1$ and $\gamma_i = \theta r_i$ when $3 \leq i \leq m$.

<u>Claim 3.5</u>:

$\Omega(P_1, C)$ *is a sublattice of* Γ_1 .

<u>Proof</u>:

It is required to prove that if $\gamma, \gamma' \in \Omega(P_1, C) \subset \Gamma_1$ then $\gamma \vee \gamma', \gamma \wedge \gamma' \in \Omega$. So suppose $p_i < p_j$ is a relation of P_1 and that γ, γ' each respect this relation, namely $\gamma_i < \gamma_j$ and $\gamma_i' < \gamma_j'$. Then

$$(\gamma_i \wedge \gamma_i') = \min\{\gamma_i - \gamma_0, \gamma_i' - \gamma_0'\} + \max\{\gamma_0, \gamma_0'\} <$$

$$\min\{\gamma_j - \gamma_0, \gamma_j' - \gamma_0'\} + \max\{\gamma_0, \gamma_0'\} = (\gamma_j \wedge \gamma_j'),$$

similarly $(\gamma_i \vee \gamma_i') < (\gamma_j \vee \gamma_j')$; and likewise for all relations of P_1, implying that these operators preserve the necessary monotonicity. □

In order to apply Theorem 3.3, define $A = \{\gamma \in \Omega(P_1,C): z_1 < x_1\}$ and $B = \{\gamma' \in \Omega(P_1,C): z_1 < y_1\}$. Let $\gamma \vee \gamma' = \delta = (\delta_0, \delta_1, \ldots, \delta_m)$. Then $0 < \gamma_k - \gamma_0 \leq \delta_k - \delta_0$ when $k = 1$ and $0 < \gamma_k' - \gamma_0' \leq \delta_k - \delta_0$ when $k = 2$. This means $A \vee B \subset \{\gamma \in \Omega(P_1,C): z_1 < x_1 \text{ and } z_1 < y_1\} \subset \Gamma_1$. From Claim 3.5 we get $A \wedge B \subset \Omega(P_1,C) \subset \Gamma_1$. Accordingly by Theorem 3.3, for $\Omega(P_1,C)$

$$\text{prob}(z_1 < x_1)\text{prob}(z_1 < y_1) \leq \text{prob}(z_1 < x_1 \text{ and } z_1 < y_1).$$

To deduce the Λ case we let $|C| \to \infty$. Then the probability that $\gamma_i = \gamma_j$ for some $i \neq j$ tends to zero. Thus the above correlation holds for permutations induced by the variables $\gamma_0, \gamma_1, \ldots, \gamma_m$.

We mention that by varying the index variable k above, the proof goes through for disjunctions of conjunctions of the form $z_1 < r_i$ where $r_i \in R_1$. Hence we have proved Shepp's Theorem 3.12.

Further, in the usual way, we can extend the proof to establish that for both Ω and Λ with respect to P_1,

$$\text{prob}(z_1 < x_1)\text{prob}(z_1 > y_1) \geq \text{prob}(z_1 < x_1 \text{ and } z_1 > y_1).$$

This follows from setting $A = \{\gamma \in \Omega(P_1,C): z_1 < x_1 \text{ and } z_1 > y_1\}$ and $B = \Omega(P_1,C)$. In the above notation, if $\delta = \gamma \wedge \gamma'$ then

$0 > \gamma_2' - \gamma_0' \geq \delta_2 - \delta_0$. Thus $A \wedge B \subset \{\gamma \in \Omega(P_1,C): z_1 > y_1\}$ while $A \vee B \subset \{\gamma \in \Omega(P_1,C): z_1 < x_1\}$.

Case $n > 1$: Suppose $2 \leq j \leq n$, then in a similar way we define Γ_j to correspond to $P_j = \{z_j\} \cup R_j$. Further, define the distributive lattice $\Gamma^* = \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n$.

Claim 3.6:

$\Omega(P,C)$ *is a sublattice of* Γ^*.

Proof:

Any relation in P, between posets P_i and P_j when $i \neq j$, must be of the form $z_i < z_j$. So suppose $\gamma^*, \delta^* \in \Gamma^*$, where $\gamma^* = \{\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(n)}\}$ and $\delta^* = \{\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(n)}\}$, both respect this relation. Let $\gamma_0^{(i)} \in \gamma^{(i)} = (\gamma_0^{(i)}, \gamma_1^{(i)}, \ldots, \gamma_\ell^{(i)}) \in \Gamma_i$ where $\ell = |R_i|$, and similarly let $\gamma_0^{(j)} \in \gamma^{(j)}$, $\delta_0^{(i)} \in \delta^{(i)}$ and $\delta_0^{(j)} \in \delta^{(j)}$. Then we have both $\gamma_0^{(i)} < \gamma_0^{(j)}$ and $\delta_0^{(i)} < \delta_0^{(j)}$. Now $\gamma^* \wedge \delta^*$ and $\gamma^* \vee \delta^*$ are each defined componentwise. Regarding the ith and jth components of the meet, that is $\gamma^{(i)} \wedge \delta^{(i)}$ and $\gamma^{(j)} \wedge \delta^{(j)}$,

$$\max \{\gamma_0^{(i)}, \delta_0^{(i)}\} < \max \{\gamma_0^{(j)}, \delta_0^{(j)}\},$$

and of the join,

$$\min \{\gamma_0^{(i)}, \delta_0^{(i)}\} < \min \{\gamma_0^{(j)}, \delta_0^{(j)}\}.$$

It follows that any relations in Z will be preserved under the meet and join operators in Γ^*, and by Claim 3.5 we deduce that each $\Omega \subset \Gamma_i \in \Gamma^*$ is likewise closed under these operators, establishing this claim. □

Similarly to the previous case define $A^* = \{\gamma^* \in \Omega(P,C): z_i < x_i \text{ for all } i\} \subset \Gamma^*$ and $B^* = \{\gamma^* \in \Omega(P,C): z_i < y_i \text{ for all } i\} \subset \Gamma^*$ and apply Theorem 3.3, completing the case.

The proof can be seen to extend to yield the following for Ω and Λ with respect to P,

$$\text{prob}(z_i<x_i \text{ for all } i)\text{prob}(z_i>y_i \text{ for all } i) \geq \text{prob}(\text{both } z_i<x_i \text{ and } z_i>y_i \text{ for all } i),$$

and

$$\text{prob}(\alpha)\text{prob}(\beta) \leq \text{prob}(\alpha \text{ and } \beta),$$

where α,β are each disjunctions of conjunctions of inequalities in which each inequality has the form $z_i < r_{i_j}$ with $r_{i_j} \in R_i$ for any i. □

Theorem 3.4 has recently been applied by Fishburn [Fi2] to establish the following. Suppose x,y,z are pairwise incomparable elements in P. Let $\mathbb{N}(xyz)$ be the number of linear extensions of the poset in which x precedes y and y precedes z. Define $n = |P|$ and

$$k = \frac{\mathbb{N}(yxz)\mathbb{N}(zxy)}{(\mathbb{N}(xyz) + \mathbb{N}(xzy))(\mathbb{N}(yzx) + \mathbb{N}(zyx))}.$$

Then

$$k \leq \left(\frac{n-1}{n+1}\right)^2 \text{ if n is odd,}$$

$$k \leq \frac{n-2}{n+2} \quad \text{if n is even,}$$

where these bounds are best-possible. Fishburn also showed that these bounds on k yield a simple proof of strict inequality in the nontrivial cases of the xyz inequality, which is when the elements x,y,z form an antichain.

We mention some consequences of Theorems 3.11, 3.12. For $x \in P$, let

$$H_p(x) = \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \lambda x$$

denote the *average height* of x in P. It seems reasonable that $H_{P\cup\{x>y\}}(x) \geq H(x)$, and in fact more is true.

Theorem 3.14: (Winkler [Wi1]).

If $x|y$ *in* P *then*

$$H_{P\cup\{x>y\}}(x) \geq 1 + H_{P\cup\{x<y\}}(x).$$

Proof:

We have

$$H_p(x) = \Sigma\ \{\text{prob}(x > z) : z \neq x\} + 1.$$

Similarly

$$H_{P\cup\{x<y\}}(x) = \Sigma\ \{\text{prob}(x > z | x < y) : z \neq x\} + 1,$$

$$H_{P\cup\{x>y\}}(x) = \Sigma\ \{\text{prob}(x > z | x > y) : z \neq x\} + 1.$$

For any two events E and F, $\text{prob}(E|F) \geq \text{prob}(E)$ iff $\text{prob}(E \text{ and } F) \geq \text{prob}(E)\text{prob}(F)$ iff $\text{prob}(E|F) \geq \text{prob}(E|\text{not } F)$. The dual transitivity inequality states

$$\text{prob}(x > z | x > y) \geq \text{prob}(x > z),$$

and thus it follows that

$$\text{prob}(x > z|x > y) \geq \text{prob}(x > z|x < y)$$

for each $z \neq x$. Moreover in the case $z = y$ we get

$$\text{prob}(x > z|x > y) = \text{prob}(x > y|x > y) = 1$$

and

$$\text{prob}(x > z|x < y) = \text{prob}(x > y|x < y) = 0.$$

By comparing sums, we have the strong average height result:

$$H_{P\cup\{x>y\}}(x) > H_{P\cup\{x<y\}}(x).$$

This implies also the weak average height result:

$$H_{P\cup\{x>y\}}(x) > H_P(x). \quad \square$$

If E and F are random variables, then E will be said to *majorize* F if for any $r \in \mathbb{R}$, $\text{prob}(E > r) \geq \text{prob}(F > r)$. Theorem 3.12 yields:

<u>Theorem 3.15</u>: (Winkler [Wi1]). *If* $x|y$ *in* P *then* $\lambda x|P \cup \{x > y\}$ *majorizes* λx (*which in turn majorizes* $\lambda x|P \cup \{x < y\}$), *where* $\lambda \in \Lambda$.

Shepp and Mallows [Sh2] posed the general question, for which posets Q, R is it true that if the poset P contains the elements of Q and R, then given $\Lambda(P)$

$$\text{(3.17)} \qquad \text{prob}(Q|P)\text{prob}(R|P) \leq \text{prob}(Q \text{ and } R|P).$$

Such Q and R will be called *universally positively correlated*, and it is clearly necessary that they be *consistent*, i.e., the transitive closure of all the covering relations in $Q \cup R$ is a poset. For example, the xyz inequality says that the pair of posets $Q = \{z < x\}$ and $R = \{z < y\}$ are universally positively correlated.

Winkler [Wi2] gave a complete characterization of the universally positively correlated pairs of posets, which showed that all nontrivial cases (a trivial case being Q or R is empty) are ultimately deducible from the xyz inequality. The theorem states that, the necessary and sufficient condition is that at least one of the following diagrams hold, which are unique up to duality and exchange of Q and R. In the diagrams the set of all edges labelled Q' is the set of covering relations of Q, and similarly for R' and R.

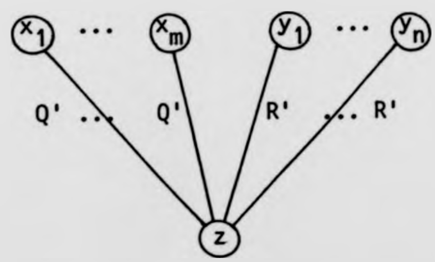


Figure 3.1

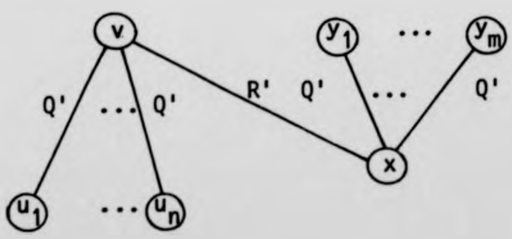


Figure 3.2

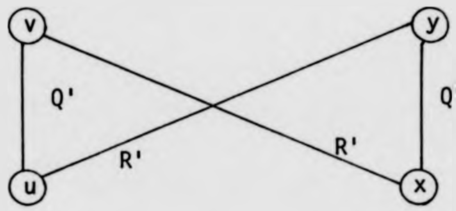


Figure 3.3

A corollary of Winkler showed that if Q and R are not universally positively correlated, and $|Q \cup R| = n$, then there is a poset P having at most $n + 1$ elements for which (3.17) fails.

Suppose x,y,z are elements of P and X,Y,Z are subposets of P. From Winkler's characterization, each of the pairs $\{z < x\}$, $\{z < y\}$ and $\{z < X\}$,$\{z < Y\}$ are universally positively correlated. However the reasonable-looking pair $\{Z < x\}$, $\{Z < y\}$ are not, as illustrated by $\{z_1 < x \text{ and } z_2 < x\}$, $\{z_1 < y \text{ and } z_2 < y\}$ not being covered by Figures 3.1, 3.2 or 3.3.

More recently D.E. Daykin [D3] has established pairs of posets Q,R which are *universally negatively correlated*, that is the reverse inequality in (3.17). Here it is obviously not required that Q and R are consistent.

For the statement of the theorem we need the two diagrams obtained by taking the dual of all the relations in Q' in Figures 3.1 and 3.2. In attempting this for Figure 3.3 we find that Q and R become inconsistent. The theorem states that when at least one of these diagrams holds and both Q and R have at least one covering relation, then Q and R are universally negatively correlated.

The impact of any characterizations is such that in proceeding to study for Λ a wider class of pairs of correlated posets, e.g., our Ψ and Φ, the underlying poset must be constrained. We have investigated the poset being partitioned into two sets and satisfying a range condition. Also the form of the poset in Theorem 3.13 means that the result does not violate the known constraints for universal correlation.

Outstanding is the following:

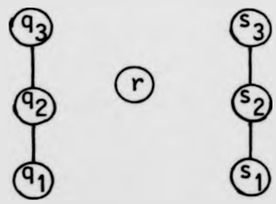
Problem 3.1:

Characterize the pairs of posets Q,R which are universally positively and negatively correlated for Ω.

3.6 MONOTONICITY FOR THREE POSETS, AND ORDER-PRESERVING MAPS

A natural direction to investigate for further monotonicity properties of partial orders is the partition $P = Q \cup R \cup S$ into disjoint posets Q,R,S. We could then specify that Ψ and Φ in Definition 3.1 are both a disjunction of conjunctions of inequalities in which each inequality has the form $q < r < s$ with $q \in Q$, $r \in R$, $s \in S$. Would we then get the corresponding $\mathcal{PCP}$ for Λ, say? No, for consider:

Example 3.6:



Q R S

Let $m = |Q| = |S|$, then given $m = 3$ and $\Lambda(Q \cup R \cup S)$,

$$\left(\frac{15}{140}\right)^2 = \left(\frac{\binom{2m}{m-1}}{\binom{2m+1}{m}\binom{m+1}{1}}\right)^2 = \text{prob}\,(q_3 < r < s_3)\text{prob}(q_1 < r < s_1) \not< \text{prob}(q_3 < r < s_1) = \frac{1}{\binom{2m+1}{m}\binom{m+1}{1}} = \frac{1}{140}\,.$$

The poset in this example would satisfy any range condition analogous to (3.9) by virtue of Q, R, S being pairwise incomparable.

Alternatively we could allow the relations in Ψ and Φ to be of the form $q < r$ or $q < s$ or $r < s$ in this context. The following proved to be useful for breaking any likely conjectures.

Example 3.7:

q r s

P

For $\Lambda(P)$,

$$(\tfrac{3}{6})^2 = \text{prob}(q < r)\text{prob}(r < s) \not\leq \text{prob}(q < r < s) = \tfrac{1}{6}.$$

In attempting an analogous characterization to range posets for the case $P = Q \cup R \cup S$, we were led to:

Conjecture 3.2:

Let P be covered by three non-empty disjoint chains C_1, C_2, C_3. Suppose that if $p, q \in P$ are in different chains and $p < q$ then $P = (\text{above } \{p\}) \cup (\text{below } \{q\})$. Then there is a partition $P = R_1 \cup \ldots \cup R_n$ such that (3.11) holds, and further for $1 \leq i \leq n$, either $R_i \cap C_j = \emptyset$ for some j, or if $p,q \in R_i$ are in distinct chains then $p|q$.

If this conjecture is true,* then with the help of Theorems 3.5 and 3.6 we obtain probability results based on comparabilities $c_i < c_j$, $c_i \in C_i$, $c_j \in C_j$ with $1 \leq i < j \leq 3$.

Lastly we consider correlation inequalities for order-preserving maps of the partition $P = Q \cup R$. Since the range of any element under these maps is the entire chain we can never have $(\Omega^0 q) \cap (\Omega^0 r) = \emptyset$ when $q < r$ in P. But this condition is nonetheless necessary for PCP here, which translates to: if $P = Q \cup R$ has the PCP for Ω^0 then $Q|R$.

---

* *Whilst examining this thesis, J.M. Robson (Australian National University) proved Conjecture 3.2 using an inductive argument.*

Example 3.8:

Let P be defined as in Example 3.3. Then given $\Omega^0(P,C_2)$, with reference to **PCP**,

$$(\tfrac{1}{8})^2 = \text{prob}(q_1 < r_1)\text{prob}(q_2 < r_2) \nleq \text{prob}(q_1 < r_1 \text{ and } q_2 < r_2) = 0.$$

Let *weak* Ψ, *weak* Φ denote the situation where the relations in Definitions 3.1, 3.2 for **PCP**, **NCP** are all of the form $q \prec r$, $q \succ r$ as appropriate. Perhaps it is more natural to specify weak Ψ, weak Φ for correlation results with order-preserving maps, so that we would count the proportion of maps satisfying sets of inequalities of the type $\omega^0 q \prec \omega^0 r$ (or $\omega^0 q \succ \omega^0 r$), $\omega^0 \in \Omega^0$. Again we have that $Q|R$ is necessary for **PCP** with these weaker forms.

Example 3.9:

Let P be defined as in Example 3.5. Then given $\Omega^0(P,C_2)$,

$$(\tfrac{10}{12})^2 = \text{prob}(q_1 \prec r_1)\text{prob}(q_2 \prec r_2) \nleq \text{prob}(q_1 \prec r_1 \text{ and } q_2 \prec r_2) = \tfrac{8}{12}.$$

Theorem 3.16:

Theorems 3.7 *and* 3.12 *hold with* Ω *replaced by* Ω^0 *for both* Ψ, Φ *and weak* Ψ, *weak* Φ.

Proof:

The proofs follow a parallel course to those for strict order-preserving maps, and in either case they are based on the same lattices.

Consider first Theorem 3.7. For example, to show that Ω^0 is a sublattice of L, suppose $q_i < q_j$ in Q and $\theta_1, \theta_2 \in \Theta$ satisfy $\theta_1 q_i \prec \theta_1 q_j$ and $\theta_2 q_i \prec \theta_2 q_j$. Then

$$(\theta_1 \vee \theta_2)q_i = \max\{\theta_1 q_i, \theta_2 q_i\} \prec \max\{\theta_1 q_j, \theta_2 q_j\} = (\theta_1 \vee \theta_2)q_j,$$

$$(\theta_1 \wedge \theta_2)q_i = \min\{\theta_1 q_i, \theta_2 q_i\} \prec \min\{\theta_1 q_j, \theta_2 q_j\} = (\theta_1 \wedge \theta_2)q_j.$$

For $\mathbf{PCP}$ set $A = \{\Omega^0 : \alpha\}$ and $B = \{\Omega^0 : \beta\}$, while for $\mathbf{NCP}$ set $A = \{\Omega^0 : \alpha$ and $\beta\}$ and $B = \Omega^0$: in either case, for the strong inequalities $\alpha = \Psi$, $\beta = \Phi$, whereas for the weak inequalities α = weak Ψ, β = weak Φ. The correlations for weak Ψ and weak Φ follow from

$$\theta_1 q \prec \theta_1 r \Rightarrow (\theta_1 \wedge \theta_2)q \prec (\theta_1 \wedge \theta_2)r,$$

$$\theta_1 q \succ \theta_1 r \Rightarrow (\theta_1 \vee \theta_2)q \succ (\theta_1 \vee \theta_2)r.$$

Consider now Theorem 3.12 for which we refer to the proof of Theorem 3.13, Case n = 1. It is straightforward to modify Claim 3.5 to demonstrate that Ω^0 is a sublattice of Γ_1.

For $\mathbf{PCP}$ set $A = \{\gamma \in \Omega^0(P,C):\alpha\}$ and $B = \{\gamma \in \Omega^0(P,C):\beta\}$, while to get $\mathbf{NCP}$ put $A = \{\gamma \in \Omega^0(P,C):\ \alpha$ and $\beta\}$ and $B = \Omega^0(P,C)$: in either situation, for the strong inequalities $\alpha = \Psi$ and $\beta = \Phi$, while for the weak inequalities α = weak Ψ and β = weak Φ.

To show $\mathbf{PCP}$ with weak Ψ, weak Φ let $\gamma \in A$ and $\gamma' \in B$. Suppose $z \prec r_i$ is among the set of relations in weak Ψ, and similarly $z \prec r_j$ in weak Φ, where $r_i, r_j \in R$. If $\gamma \vee \gamma' = \delta = (\delta_0, \delta_1, \ldots, \delta_m)$ then $0 \prec \gamma_i - \gamma_0 \prec \delta_i - \delta_0$ and $0 \prec \gamma_j' - \gamma_0' \prec \delta_j - \delta_0$. Hence $A \vee B \subset \{\gamma \in \Omega^0(P,C):$ weak Ψ and weak $\Phi\}$, while $A \wedge B \subset \Omega^0(P,C)$ since Ω^0 is a sublattice.

We point out that Theorem 3.13 will likewise hold for Ω^0 with both strong and weak forms of the relations $z_i < x_i$, $z_i < y_i$. $\square$

We conclude this chapter with some comments. Suppose that the poset $P = Q \cup R$ has the **PCP** and **NCP** for Ω or Λ. Under what conditions may we add a new relation β to the set of existing relations in P in order that these properties still hold in $P \cup \{\beta\}$?

If $P = \{q_1$ and q_2 and r_1 and $r_2\}$, according to Theorem 3.6 on page 60, P has the **PCP** and **NCP** for Ω and Λ. Putting $\beta = \{q_2 < r_1\}$ it follows from Example 3.5 that **PCP** and **NCP** fail for $\Lambda(P \cup \{\beta\})$. Similarly we can use Example 3.3 to illustrate for Ω.

Correlation inequalities for Λ^I will follow from those for Λ using the binomial coefficients, as indicated in the proof of Theorem 4.14.

A challenging area for further research would be to investigate correlation properties for monotonic functions from one partial order S, into another partial order T. By considering whether the preceeding inequalities also hold for these more general functions, we have for instance:

<u>Question 3.1</u>:

We say that $\omega^*:S \to T$ is strict order-preserving if for all $x,y \in S$, $x < y$ implies $\omega^*x < \omega^*y$; $\lambda^*:S \to T$ is an order-preserving bijection if $|S| = |T|$ and λ^* is 1-1 and λ^* is strict order-preserving. Then does the analogue of the transitivity inequality hold for either of these classes of maps? In other words is it true that for the poset $P = Q \cup R$ where $|Q| = 1$, the **PCP** and **NCP** hold for $\Omega^*(P,T)$, the set of all such ω^* or for $\Lambda^*(P,T)$, the set of all such λ^*?

Notice that $|\Lambda^*(S,T)| < |\Lambda(S,C_{|S|})|$, and unlike the set Λ of linear extensions of a poset, Λ^* may be empty: trivially let S be a chain and T an antichain.

CHAPTER 4 : LOG CONCAVITY FOR MONOTONIC FUNCTIONS

4.1 INTRODUCTION

In this chapter we consider the absolute values of the images of elements in a partial order. The main interest is in counting how many functions of some class map the element x to each of the points in the chain.

This is in contrast to the previous chapter where we were looking at the relative rankings of elements. The number of monotonic maps having x less than y was counted, regardless of the actual ranks of these elements.

The sequences of numbers of monotonic functions, according to the image of a fixed element, exhibit some interesting qualitative properties.

In 1981 Stanley [St] proved that, if N_i^L is the number of linear extensions of a poset P mapping an element x to rank i, then the sequence $N_1^L, N_2^L, \ldots, N_{|P|}^L$ is log concave. The main contribution in this section is to establish the corresponding results for both strict order-preserving and order-preserving maps. This requires a different proof technique from Stanley's. We also strengthen Stanley's Theorem to bring it into line with our results by showing that it holds in more general circumstances.

The FKG inequality led to a new inequality involving log convex sequences due to Seymour and Welsh [SW]. By studying mappings of minimal and dually maximal poset elements, we apply their inequality to sequences related to partial orders. We observe that the new inequality is a special case of Chebyshev's Theorem 3.1.

New results are presented on log concavity for the total numbers of non-bijective functions of the whole poset into a sequence of increasing length chains.

Finally we give probability inequalities of a new kind, which entail counting the number of functions taking an element in the poset to a fixed point in the chain. Counterexamples show limitations for this type of result.

All the proofs in this chapter construct explicit injections.

Let $W_0, W_1, \ldots$ be a sequence of non-negative real numbers. The sequence is called *logarithmically concave* if

$$W_{k-1}\, W_{k+1} < W_k^2 \text{ for } 1 < k.$$

Furthermore, the definition implies that

$$\frac{W_k}{W_{k+1}} < \frac{W_{k+1}}{W_{k+2}} < \ldots < \frac{W_{k+j}}{W_{k+1+j}} \quad \text{for } j > 0.$$

Hence

$$\frac{W_k}{W_{k+j}} < \frac{W_{k+1}}{W_{k+1+j}} < \ldots < \frac{W_{k+\ell}}{W_{k+\ell+j}} \text{ for } \ell, j > 0,$$

thus

$$\text{(4.1)} \qquad \frac{W_k}{W_{k+\ell}} < \frac{W_{k+j}}{W_{k+\ell+j}} \quad \text{for all } k, \ell, j > 0.$$

The sequence is *unimodal* if for some j we have $W_0 < W_1 < \ldots < W_j$ and $W_j > W_{j+1} > \ldots$. Log concavity implies unimodality but not vice versa. The sequence is called *logarithmically convex* if

$$W_k^2 < W_{k-1} W_{k+1} \text{ for } 1 < k.$$

Well known examples of log concave sequences which frequently occur in combinatorics are the binomial coefficients, the Gaussian coefficients, the Eulerian numbers, the Stirling numbers of both kinds; for a proof see Kurtz [Ku]. Moreover, there are many other sequences of integers arising naturally in combinatorial structures which it is conjectured are log concave (or log convex) and hence unimodal. Examples are:

(i) the absolute values of the coefficients of the chromatic polynomial of a graph or the characteristic polynomial of a geometric lattice;

(ii) the Whitney numbers (of both the first and second type), of a geometric lattice.

In both this chapter and Chapter 5 we will adopt the following notation. Let P be a poset with n elements and C be the chain $1 < 2 < \ldots < c$. If $x_1,\ldots,x_k$ is a fixed subset in P and $t_1,\ldots,t_k \in Z^+$ then define $N^O(t_1,\ldots,t_k)$ to be the number of order-preserving maps $\omega^O:P \to C$ such that $\omega^O(x_h) = t_h$ for $1 < h < k$; and define $N^S(t_1,\ldots,t_k)$ to be the number of strict order-preserving maps $\omega:P \to C$ such that $\omega(x_h) = t_h$ for $1 < h < k$; also define $N^I(t_1,\ldots,t_k)$ to be the number of order-preserving injections $\lambda^I:P \to C$ such that $\lambda^I(x_h) = t_h$ for $1 < h < k$; finally define $N^L(t_1,\ldots,t_k)$ to be the number of linear extensions $\lambda:P \to C_n$ such that $\lambda(x_h)=t_h$ for $1 < h < k$. Further, if $t_h \notin C$ for any h then set $N^O(t_1,\ldots,t_k) = 0$ and similarly for N^S, N^I and N^L. Throughout we will put $i,i_1,\ldots,i_k,j_1,\ldots,j_k \in C$.

When $k = 1$ we use the abbreviations: $x = x_1$, $N_i^O(x)$, $N_i^S(x)$, $N_i^I(x)$ and $N_i^L(x)$. At times it will be convenient to let M_i denote N_i^O, N_i^S, N_i^I or N_i^L, and M denote $\Omega^O,\Omega,\Lambda^I$ or Λ.

The development of log concavity results for partial orders was stimulated by an (unpublished) conjecture of R. Rivest that the sequence $N_1^L, N_2^L, \ldots, N_n^L$ is unimodal. In 1980 Chung, Fishburn and Graham [CFG] conjectured the stronger statement that this sequence is log concave and proved this for the case that P is a union of two linear orders. Soon afterwards Stanley established the conjecture and proved the following more general version of this fundamental result.

Theorem 4.1: (Stanley [St]).

Let $x_1 < \ldots < x_k$ *be a fixed chain in* P. *Suppose* $1 \leq j \leq k$, *then*

$$N^L(i_1,\ldots,i_{j-1}, i_j-1, i_{j+1},\ldots,i_k)N^L(i_1,\ldots,i_{j-1},i_j+1,i_{j+1},\ldots,i_k) \leq N^L(i_1,\ldots,i_k)^2. \tag{4.2}$$

In particular, the case $k = 1$ *yields* $N_{i-1}^L\ N_{i+1}^L \leq (N_i^L)^2$, *which is the conjecture of Chung et al.*

Graham [G1] then asked whether the analogue of Stanley's Theorem is true for strict order-preserving maps, and noted that the FKG inequality can be used very naturally to prove the log concavity of various sequences of a combinatorial nature. He further suggested [G2] that Stanley's result, and the analogue conjectured result for strict order-preserving maps, might have proofs based on the FKG inequality or the more general AD inequality (Theorem 3.4). But these had as yet eluded discovery.

Recently D.E. Daykin, J.W. Daykin and M.S. Paterson [DDP] established the analogue of Stanley's result for both strict order-preserving and order-preserving maps. That is (4.2) holds with each N^L replaced by N^S, and with each N^L replaced by N^O, also under more general conditions. These results were obtained by constructing an explicit injection, and to date neither Stanley's result nor these analogues have been proved using lattice inequalities.

4.2 LOG CONCAVITY FOR NON-BIJECTIVE MAPS

This chapter was mainly motivated by the log concavity result of Stanley. We may easily strengthen his theorem by removing the condition that the x's form a chain in P. For suppose $\{x_1,\dots,x_k\}$ is an arbitrary subset of P, then using bijectivity, without loss of generality assume $i_1 < \dots < i_k$. If we augment P with the new relations $x_1 < \dots < x_k$ then N^L is unchanged and Stanley's Theorem applies to the new partial order. Implementing (4.1) we can now write:

Theorem 4.2: (Stanley [St]).

Let $x_1,\dots,x_k$ *be a fixed subset in* P. *If* $r,s,t \in \mathbb{Z}^+$ *and* $i_h \notin [r,r+s+t]$ *for* $2 \leq h \leq k$, *then*

$$N^L(r,i_2,\dots,i_k)N^L(r+s+t,i_2,\dots,i_k) \leq N^L(r+s,i_2,\dots,i_k)N^L(r+t,i_2,\dots,i_k).$$

This inequality implies that the range of an element under linear extensions is an interval. In connection with this, it is straightforward to show the following, using the push up and down functions. Suppose $N^L(i_1,j_1) \neq 0$ and $N^L(i_1+\ell,j_1) \neq 0$ and $0 \leq h \leq \ell$, then if $i_1 + h \neq j_1$ we have $N^L(i_1 + h,j_1) \neq 0$; and similarly for N^S via the rake up and down functions, although the restriction $i_1 + h \neq j_1$ is not required in this case.

Stanley used the Aleksandrov-Fenchel inequalities to prove that certain sequences of combinatorial interest, including that in Theorem 4.2, are log concave (and therefore unimodal). These inequalities guarantee the logarithmic concavity of coefficients arising from the volume of weighted sums of m-dimensional polytopes.

Consider convex bodies $K_1, K_2, \ldots, K_s$ in $\mathbb{R}^m$. If $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{R}^+$ then define the convex body

$$K = \{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_s v_s : v_\ell \in K_\ell\}.$$

As a function of the variables $\alpha_1, \ldots, \alpha_s$, the volume $V(K)$ of the set K is a homogeneous polynomial of degree m,

$$V(K) = \sum_{\ell_1=1}^{s} \sum_{\ell_2=1}^{s} \cdots \sum_{\ell_m=1}^{s} V_{\ell_1 \ldots \ell_m} \lambda_{\ell_1} \cdots \lambda_{\ell_m},$$

where the coefficients $V_{\ell_1 \ldots \ell_m}$ are uniquely determined by requiring that they are symmetric in their subscripts. The coefficient $V_{\ell_1 \ldots \ell_m}$, also written as $V(K_{\ell_1}, \ldots, K_{\ell_m})$, is called the *mixed volume* of $K_{\ell_1}, \ldots, K_{\ell_m}$. The following inequality was proved independently by Alexandrov [Al] and Fenchel [Fe]

$$(V_{\ell_1 \ell_2 \ldots \ell_{m-1} \ell_m})^2 > (V_{\ell_1 \ldots \ell_{m-1} \ell_{m-1}})(V_{\ell_1 \ldots \ell_{m-2} \ell_m \ell_m}).$$

Surveys of mixed volumes appear in [BF, Bu, E]. It suffices to mention that these inequalities are naturally applicable to the bijective property of linear extensions, but not so for the other monotonic functions. Like the FKG and related lattice inequalities, the Alexandrov-Fenchel inequalities have just begun to be exploited in combinatorics.

Currently only two other applications of these inequalities are known in the literature. A reformulation of the inequalities led to the

Alexandrov-Fenchel inequality on permanents, recently used by Egoritsjev [Eg] and independently by Falikman [F] to prove the infamous van der Waerden permanent conjecture (see [L,W]). Kahn and Saks [KS] have employed the inequalities to show that every poset that is not a chain contains a pair of elements x and y such that for Λ

$$3/11 < \text{prob}(x < y) < 8/11.$$

Explicitly defined injective functions may be used to prove inequalities on numbers of combinatorial structures. In this case we first wish to show $N_r^S(x)N_{r+s+t}^S(x) \leq N_{r+s}^S(x)N_{r+t}^S(x)$. The injection consists of constructing, for each pair of strict order-preserving maps with $\omega_1(x)=r$ and $\omega_2(x)=r+s+t$, a unique pair of maps with $\omega_3(x)=r+s$ and $\omega_4(x)=r+t$. That is if two ordered pairs of the form (ω_1,ω_2) are distinct, then their two associated (ω_3,ω_4) pairs are distinct. This ensures the required inequality, for otherwise two (ω_3,ω_4) pairs associated with distinct (ω_1,ω_2) pairs would have to be identical.

An injective proof technique was used in [CFG] to derive log concavity for linear extensions. Since the case considered was when P can be covered by two chains, the linear extensions could be represented by lattice paths in $\mathbb{Z}^2$, as described in the proof of Theorem 3.5.

For log concavity of strict order-preserving and order-preserving maps we present here slightly more general forms than those appearing in [DDP].

<u>Theorem 4.3:</u>

Let $x_1,\ldots,x_k$ *be a fixed subset in* P. *If* $r,s,t \in \mathbb{Z}^+$ *and* $j_h \leq i_h + t$

for $2 < h < k$, *then*

$$N^S(r,i_2,\ldots,i_k)N^S(r+s+t,j_2,\ldots,j_k) < N^S(r+s,i_2,\ldots,i_k)N^S(r+t,j_2,\ldots,j_k).$$

In particular, the case $k = 1$ *and* $s = t = 1$ *yields* $N_r^S N_{r+2}^S < (N_{r+1}^S)^2$.

We first make some informal comments on the method of proof. The construction proceeds by iteratively defining a subset D of P. Now D is a function of the pairs of monotonic maps to be associated with each of the poset elements. Initially an operation is defined on the given fixed element $\{x_1\}= D$ in P. Using monotonicity, an element adjacent to any member of D may in turn become a member of this set, in which case it is operated on in the same manner. This is repeated until D can attract no more members, whereupon the process halts.

By way of explanation, D was chosen to stand for "Disease" in view of the "contagious" nature of the injection: "weak" elements succumbing to D whereas "strong" ones do not. We feel it is likely that there are other applications of such a contagious process on posets worth investigating.

As the injection considers only adjacent elements to D at each step, it does not retain a global consideration of the monotonic functions assigned to all of the elements. Hence if we commence with 1-1 mappings, this property may not be preserved.

So we have clear examples here of techniques for linear extensions being unsuitable for strict order-preserving maps and vice versa. Moreover, it will be seen that as is usual minor modifications to the techniques for strict order-preserving maps yield analogous results for order-preserving maps. Corresponding results for order-preserving injections are quite straightforward from the bijective case.

The following special case of Theorem 4.3 was established using the binomial coefficients, which lent supporting evidence to Graham's conjecture.

Theorem 4.4:

Let P *be a chain and* $x \in P$. *Then* $N_1^S, N_2^S, \ldots, N_c^S$ *is log concave.*

Proof:

Without loss of generality assume $c > ht(P) = n$, and put $1 < i < c-1$. Let $ht(x) = u+1$, $dp(x) = v+1$, $ht(i+1) = r+1$ and $dp(i+1) = s+1$. Now $\Omega(P,C)$ behaves here like order-preserving injections, and so

$$N_i^S = \binom{r-1}{u}\binom{s+1}{v}, \quad N_{i+1}^S = \binom{r}{u}\binom{s}{v} \text{ and } N_{i+2}^S = \binom{r+1}{u}\binom{s-1}{v}.$$

Log concavity of the binomial coefficients shows that $N_i^S\, N_{i+2}^S < (N_{i+1}^S)^2$. □

Proof of Theorem 4.3:

Suppose that the L.H.S. of the inequality is not equal to zero, and that $s > 0$ for otherwise the result clearly holds. We will first prove the result for $k = 1$, namely $N_r^S N_{r+s+t}^S < N_{r+s}^S\, N_{r+t}^S$, and then show how it easily extends to $k > 1$.

Given any pair of strict order-preserving maps $\omega_1, \omega_2 : P \to C$ with $\omega_1(x) = r$ and $\omega_2(x) = r+s+t$, we will construct a unique pair of strict order-preserving maps $\omega_3, \omega_4 : P \to C$ with $\omega_3(x) = r+s$ and $\omega_4(x) = r+t$.

Since C is $1 < 2 < \ldots < c$ we will write $C + t$ for $1+t < 2+t < \ldots < c+t$. Now the pair ω_1, ω_2 may equally be regarded as a strict order-preserving map B into the direct product $(C+t) \times C = \{(\gamma, \gamma') : \gamma \in C+t,\ \gamma' \in C\}$, with the partial ordering: $(\gamma_1,\gamma_2) < (\delta_1,\delta_2)$ if both $\gamma_1 < \delta_1$ in $C + t$ and $\gamma_2 < \delta_2$ in C. Thus $B = B_1 \times B_2 : P \to (C+t) \times C$ where, for $p \in P$, $B_1(p) = t + \omega_1(p)$ and $B_2(p) = \omega_2(p)$. In particular we have $B(x) = (r+t,\ r+s+t)$.

Now define the operation flip $(j,k) = (k,j)$. We will say p *forces* q for $p,q \in P$ if

either $\quad p < q$ and $\mathrm{flip}(B(p)) \nless B(q)$, i.e., $B_2(p) > B_1(q)$ or $B_1(p) > B_2(q)$

or $\quad p > q$ and $\mathrm{flip}(B(p)) \ngtr B(q)$.

Also define $D_B = \{p : p \in P,\ x\ (\text{forces})^* p\}$, where "(forces)*" is the reflexive and transitive closure of "forces". That is, $x \in D_B$ and the forcing procedure propagates from x to form the subset D_B of P. Since P is finite the propagation must halt (possibly with $D_B = P$), and then we let $\delta(B) : P \to \mathbf{Z}^+ \times \mathbf{Z}^+$ be defined by

$$\delta(B)(p) = \begin{cases} \mathrm{flip}(B(p)) & \text{if } p \in D_B \\ B(p) & \text{if } p \notin D_B. \end{cases}$$

<u>Lemma 4.1</u>:

$\delta(B)$ *is strict order-preserving.*

<u>Proof</u>:

Let $p,q \in P$ with $p < q$ and so $B(p) < B(q)$.

<u>Case</u>:

$p,q \notin D_B$. Then $\delta(B)(p) = B(p) < B(q) = \delta(B)(q)$ as required.

Case:

$p, q \in D_B$. Then $\delta(B)(p) = \text{flip}(B(p)) = (B_2(p), B_1(p)) <$ $(B_2(q), B_1(q)) = \text{flip}\ (B(q)) = \delta(B)(q)$ as required.

Case:

$p \in D_B$, $q \notin D_B$. Then p does not force q and so $\delta(B)(p) =$ $\text{flip}(B(p)) < B(q) = \delta(B)(q)$.

Case:

$p \notin D_B$, $q \in D_B$. Then q does not force p and so $\delta(B)(p) = B(p) <$ $\text{flip}(B(q)) = \delta(B)(q)$. □

Lemma 4.2:

$B_1(d) < B_2(d)$ *for* $d \in D_B$.

Proof:

We have $B_1(x) < B_2(x)$ since $s > 0$. So it is sufficient to show that if $B_1(d) < B_2(d)$ for some $d \in D_B$, then this relation holds for any p in D_B forced by d. Suppose first that $d < p$, and so
$B_1(p) \prec B_2(d) < B_2(p)$ or $B_2(p) \prec B_1(d) < B_2(d) < B_2(p)$.
The latter is impossible and the former establishes the claim. The proof if $d > p$ is similar. □

Lemma 4.3:

$\delta(B)(p) \in (C+t) \times C$ *for* $p \in P$.

Proof:

If $p \notin D_B$ then it is clearly true. Now for $p \in D_B$ we have $1 \prec 1 + t \prec B_1(p) < B_2(p) \prec c \prec c + t$. Hence $\text{flip}(B(d)) \in (C+t) \times C$. □

<u>Lemma 4.4:</u>

$\delta(\delta(B)) = B$.

<u>Proof:</u>

It is sufficient to show that $D_{\delta(B)} = D_B$. Suppose $d \in D_B$, then flip $(\delta(B)(d)) = B(d)$ by the definition of $D_{\delta(B)}$. Therefore d forces p *with respect to* $\delta(B)$ if

either $\quad d < p$ and $B(d) \nless \delta(B)(p)$,

or $\quad d > p$ and $B(d) \ngtr \delta(B)(p)$.

If $p \notin D_B$ then $\delta(B)(p) = B(p)$ and so, since B is strict order-preserving, d does not force p.

If $p \in D_B$ then $\delta(B)(p) = \text{flip}\ (B(p))$ and in this case d forces p with respect to $\delta(B)$ iff d forces p with respect to B.

Hence $D_{\delta(B)} = D_B$. $\square$

<u>Corollary 4.1:</u>

δ is injective. $\square$

Now ω_3 and ω_4 are given by $\delta(B) = (t + \omega_3, \omega_4)$ concluding the case $k = 1$.

Finally, we show how the result extends to a subset $\{x_1, \ldots, x_k\} \subset P$ where $k > 1$. For x_i with $2 \leq i \leq k$, we have $\omega_2(x_i) \leq \omega_1(x_i) + t$ and $B_1(x_i) = t + \omega_1(x_i) \geq \omega_2(x_i) = B_2(x_i)$. From Lemma 4.2 we deduce that $x_i \notin D_B$ giving $\delta(B)(x_i) = B(x_i)$ as required, which completes the proof. $\square$

In [DDP], it was required that each $j_h = i_h$, in the context of Theorem 4.3.

Clearly the subset D generated by the injection in Theorem 4.3 consists of a connected set of elements in the Hasse diagram, the set of covering pairs of P. However D is in general neither an up-set, a down-set nor a convex set. To see this let C_h be a long chain in P and set $D = \{1\}$. Also let $1 < p < h$ but $p \notin C_h$. It is easy to see that the elements of C_h can each be consecutively adjoined to D, whereas p may not be.

Notice that D is independent of the order in which elements are adjoined to it. That is if $d_1, d_2 \in D$, then having commenced with either d_1 or d_2 would yield the set D when the propagation halts. In other words D is uniquely defined.

It might be expected that if $i_2 < j_2$ we get $N^S(r,i_2)N^S(r+s+t,j_2) < N^S(r+s,i_2)N^S(r+t,j_2)$. However that this is not true is shown by:

<u>Example 4.1</u>:

P

With $r = s = 1$, $t = 3$, $i_2 = 4$, $j_2 = 8$ where in view of Theorem 4.3 $8 = j_2 \nless i_2 + t = 7$, then

$$2^2 = N^S(1,4)N^S(5,8) \nless N^S(2,4)N^S(4,8) = (1)(3).$$

This example can be readily adapted to hold for order-preserving maps and linear extensions.

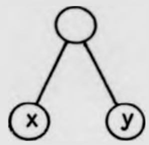
The particular case t = 0 of Theorem 4.3 is given by

$$N^S(r,i_2,\ldots,i_k)N^S(r+s,j_2,\ldots,j_k)$$

$$< N^S(r+s,i_2,\ldots,i_k)N^S(r,j_2,\ldots,j_k) \text{ when } j_h < i_h \text{ for all } h.$$

As expected we do not necessarily get equality here, verified by:

Example 4.2:



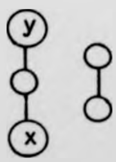
P

With $r = s = j_2 = 1$ and $i_2 = 2$ then

$$(c-2)^2 = N^S(1,2)N^S(2,1) < N^S(2,2)N^S(1,1) = (c-2)(c-1).$$

We see from Example 4.2 that the above inequality with t = 0 does not hold for linear extensions, while the next example shows that it is not always false.

Example 4.3:



P

If $r = s = 1$, $i_2 = 5$ and $j_2 = 4$ then

$$(3)(1) = N^L(1,5)N^L(2,4) \leq N^L(2,5)N^L(1,4) = 2^2.$$

By employing a corresponding injection to δ we can show log concavity for order-preserving maps.

<u>Theorem 4.5</u>:

The analogue of Theorem 4.3 *holds for* $N^O(i_1,\ldots,i_k)$.

<u>Proof</u>:

The proof follows a parallel course to that of Theorem 4.3, but $(C+t) \times C$ now takes the usual *product* ordering, and p *forces* q if $p < q$ and $B_2(p) > B_1(q)$ or $B_1(p) > B_2(q)$ and similarly when $p > q$. □

From Stanley's result we can immediately deduce:

<u>Theorem 4.6</u>:

Let $x_1,\ldots,x_k$ *be a fixed subset in* P. *If* $r,s,t \in \mathbf{Z}^+$ *and* $i_h \notin [r,r+s+t]$ *for* $2 \leq h \leq k$, *then*

$$N^I(r,i_2,\ldots,i_k)N^I(r+s+t,i_2,\ldots,i_k)$$
$$\leq N^I(r+s,i_2,\ldots,i_k)N^I(r+t,i_2,\ldots,i_k).$$

<u>Proof</u>:

Without loss of generality assume $c = |C| \geq |P| = n$. Define the poset Q by $P \cup C_{c-n}$. Then N^I with respect to P equals N^L with respect to Q. The result follows by log concavity of the function N^L. □

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By employing a corresponding injection to δ we can show log concavity for order-preserving maps.

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The analogue of Theorem 4.3 *holds for* $N^O(i_1,\dots,i_k)$.

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The proof follows a parallel course to that of Theorem 4.3, but $(C+t) \times C$ now takes the usual *product* ordering, and p *forces* q if $p < q$ and $B_2(p) > B_1(q)$ or $B_1(p) > B_2(q)$ and similarly when $p > q$. □

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Let $x_1,\dots,x_k$ *be a fixed subset in* P. *If* $r,s,t \in \mathbf{Z}^+$ *and* $i_h \notin [r,r+s+t]$ *for* $2 \leq h \leq k$, *then*

$$N^I(r,i_2,\dots,i_k)N^I(r+s+t,i_2,\dots,i_k)$$
$$\leq N^I(r+s,i_2,\dots,i_k)N^I(r+t,i_2,\dots,i_k).$$

Proof:

Without loss of generality assume $c = |C| \geq |P| = n$. Define the poset Q by $P \cup C_{c-n}$. Then N^I with respect to P equals N^L with respect to Q. The result follows by log concavity of the function N^L. □

4.3 LOG CONCAVE AND MONOTONE SEQUENCES

Clearly, if x is an isolated element in the poset P, the log concave sequence M_i also satisfies $0 \neq M_1 = M_2 = \ldots = M_c = |M(P \setminus x), C|$. We will show that for a minimal element in the poset, M_i achieves its maximum value at the minimal element in the chain.

Theorem 4.7:

Let x *be a minimal element of* P. *Then* $N_1^L, N_2^L, \ldots, N_n^L$ *is log concave and decreasing.*

Proof:

Log concavity of the sequence follows from Theorem 4.2. It is enough to show that

$$N_1^L > N_2^L$$

and monotonicity will follow from

$$\frac{N_1^L}{N_2^L} < \frac{N_2^L}{N_3^L} < \ldots < \frac{N_{n-1}^L}{N_n^L} .$$

So suppose that there exists a $\lambda \in \Lambda$ such that $\lambda x = 2$. Since $2 > |\text{below } \{x\}| = 1$ we can let $\mu \in \Lambda$ be the push down of λ over x.

By showing that this construction is injective we have proved the inequality. So let $\lambda^{-1} 1 = p$ and since x is minimal we have $x|p$ and thus $\mu p = 2$. This therefore means that

$$\lambda = \text{push up of (push down of } \lambda \text{ over } x) \text{ over } x. \quad \square$$

Modifying the proof for Λ yields similar results for Ω and Ω^0.

Theorem 4.8:

Let x *be a minimal element of* P. *Then* $N_1^S, N_2^S, \ldots, N_c^S$ *is log concave and decreasing.*

Proof:

Suppose $\omega \in \Omega$ satisfies $\omega x = 2$. Now $2 > ht(x) = 1$ and so let $\pi \in \Omega$ be the rake down of ω over x. That π is injective follows from the minimality of x and

$$\omega = \text{rake up of } \pi \text{ over } x. \quad \square$$

Theorem 4.9:

Let x *be a minimal element of* P. *Then* $N_1^0, N_2^0, \ldots, N_c^0$ *is log concave and decreasing.*

Proof:

If $\omega^0 \in \Omega^0$ with $\omega^0 x = 2$ then define

$$\tau^0 p = \begin{cases} 1 & \text{if } p = x \\ \omega^0 p & \text{otherwise.} \end{cases}$$

The construction of τ^0 is clearly unique, and since x is minimal then $\tau^0 \in \Omega^0$. $\square$

Using duality it follows that if x is maximal in P then the sequence M_i is log concave and increasing. Examples show that if

$x \in P$ with $ht(x) = 2$ or $|\text{below } \{x\}| = 2$ then the log concave sequences $M_2, M_3, \ldots, M_c$ are not decreasing.

<u>Example 4.4</u>:

P

For $\Lambda(P \cup C_2)$, $\langle N_1^L, N_2^L, N_3^L, N_4^L, N_5^L \rangle = \langle 0,3,4,3,0 \rangle$,

and for

$\Omega(P, C_5)$, $\langle N_1^S, N_2^S, N_3^S, N_4^S, N_5^S \rangle = \langle 0,3,4,3,0 \rangle$,

while for

$\Omega^0(P, C_5)$, $\langle N_1^0, N_2^0, N_3^0, N_4^0, N_5^0 \rangle = \langle 5,8,9,8,5 \rangle$.

The FKG inequality introduced in Chapter 3.1 has several applications in combinatorial theory (see [SW]). For instance it has led to new properties of log convex sequences, log supermodular functions and Bernstein polynomials. It is possible to deduce a result of Kleitman [Kl] about families of sets (which is also a special case of the AD inequality) from the FKG. Also various new properties of the Tutte polynomial of a geometric lattice, or matroid have been established. This polynomial introduced by Crapo is intimately related to the well-known colouring problems of graph theory, and the more general critical problem of combinatorial geometry. Further, the roles of statistical mechanics and combinatorics can be reversed so that the supply demand theorem of transport networks leads to a new proof of a theorem of Holley [Hol] (again a special case of AD), which includes as a special case the original FKG inequality.

A lattice is *modular* when it satisfies the identity

If $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Thus every distributive lattice is modular, but not vice versa.

Due to the power of the FKG inequality it would be nice to have a similar result for a wider class of lattices. The FKG itself is known to fail in the lattice of flats of a projective geometry. The determination of conditions which make (3.5) true for modular lattices would therefore seem to be the most important unsolved problem in this field.

The FKG inequality led to the following result involving log convex sequences, which is a considerable extension of the well-known inequality of Chebyshev, although both are clearly a special case of Theorem 3.1.

<u>Theorem 4.10</u>: (Seymour and Welsh [SW]).

If $\mu(1),\ldots,\mu(m)$ *is log convex and non-negative, and* $f(1),\ldots,f(m)$, $g(1),\ldots,g(m)$ *are both increasing or both decreasing real sequences, then*

$$(4.3) \qquad \Big(\sum_i \mu(i)f(i)\Big)\Big(\sum_i \mu(i)g(i)\Big) \leq \Big(\sum_i \mu(i)\Big)\Big(\sum_i \mu(i)f(i)g(i)\Big).$$

Note that by taking $\mu(i) = 1$ for all i we get the well-known result. Also with the sequence $(f(i):1 \leq i \leq m)$ increasing and the sequence $(g(i):1 \leq i \leq m)$ decreasing we reverse the inequality (4.3).

The theorem of Seymour and Welsh can be applied to our sequences for elements of partial orders.

Theorem 4.11:

Let p,q,r $\in$ P *where* p *is arbitrary and* q,r *are minimal. Then*

$$(4.4)\quad \left(\sum_{1<i<c} \frac{N_i^S(q)}{N_i^S(p)}\right)\left(\sum_{1<i<c} \frac{N_i^S(r)}{N_i^S(p)}\right) < \left(\sum_{1<i<c} \frac{1}{N_i^S(p)}\right)\left(\sum_{1<i<c} \frac{N_i^S(q)N_i^S(r)}{N_i^S(p)}\right).$$

Proof:

From the definitions we have that a sequence $(W_k : k > 0)$ is log convex if the sequence $(W_k^{-1} : k > 0)$ is log concave. By Theorem 4.3 $N_i^S(p)$ is log concave, while by Theorem 4.8 $N_i^S(q)$, $N_i^S(r)$ are both decreasing sequences. Now apply Theorem 4.10. □

When p is arbitrary, q is minimal and r is maximal in P, the inequality (4.4) is reversed. Analogous inequalities to (4.4) hold for both linear extensions and order-preserving maps.

4.4 LOG CONCAVITY FOR TOTAL NUMBERS OF MAPS

Consider sequences for numbers of monotonic maps of the entire poset. Since P is fixed while $|C|$ will be variable here, results do not arise for linear extensions.

Theorem 4.12:

Let ν_r^S *be the total number of strict order-preserving maps* $\omega : P \to C_r$. *Then* $\nu_1^S, \nu_2^S, \ldots$ *is log concave.*

Proof:

Given P and a new element q, define the poset Q by $P < q$. Then note that ν_i^S equals N_{i+1}^S for $x = q$ in Q and $1 \leq i$. The result follows by log concavity of the sequence $N_{i+1}^S, N_{i+2}^S, \ldots$. □

Theorem 4.13:

Let ν_r^O *be the total number of order-preserving maps* $\omega^O : P \to C_r$. *Then* $\nu_1^O, \nu_2^O, \ldots$ *is log concave.*

Proof:

The proof follows that of Theorem 4.12, with ν_i^O equal to N_i^O for $x = q$ in Q and $1 \leq i$. □

Theorem 4.14:

Let ν_r^I *be the total number of order-preserving injections* $\lambda^I : P \to C_r$. *Then* $\nu_1^I, \nu_2^I, \ldots$ *is log concave.*

Proof:

Without loss of generality assume $r \geq n$. Let k be the total number of linear extensions of P. Then for $r = n + j$ with $j \geq 0$

we have $\nu_r^I = k\binom{r}{n}$. The result follows by log concavity of the sequence of binomial coefficients $\binom{r}{n}, \binom{r+1}{n}, \dots$. □

Clearly ν_r^S, ν_r^0 and ν_r^I are all strict increasing sequences.

4.5 PROBABILITY RESULTS

If x,y are elements in the poset P, and i,j are elements in the chain C, when is it true that

$$\text{prob}(mx = i)\text{prob}(my = j) \leq \text{prob}(mx = i \text{ and } my = j),$$

where $m \in M$? Obviously when $x \nmid y$ and $i < j$, we require $|\overline{\{x,y\}}| \leq j-i+1$ for $M = \Lambda$, and similarly $ct(x,y) \leq j-i+1$ is necessary when $M = \Omega$.

Theorem 4.15:

Let Q,R *be posets and* x,y *be singletons. Suppose* P *is defined by* $Q < x < y < R$. *If* $j = i + 1$ *then for* $\omega \in \Omega(P,C_c)$,

$$\text{prob}(\omega x = i)\text{prob}(\omega y = j) \leq \text{prob}(\omega x = i \text{ and } \omega y = j).$$

Proof:

Suppose that the L.H.S. of the inequality is not equal to zero. Then let $\omega, \pi \in \Omega$ satisfy $\omega x = i$ and $\pi y = j$. Define the maps ω', π' by

$$\omega' p = \begin{cases} \omega p & \text{if } \omega p \leq i \\ \pi p & \text{if } \pi p \geq j, \end{cases}$$

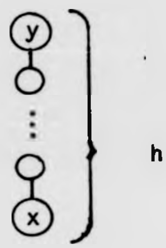
$$\pi' p = \begin{cases} \pi p & \text{if } \pi p < j \\ \omega p & \text{if } \omega p > i. \end{cases}$$

Notice that ω' satisfies $\omega' x = i$ and $\omega' y = j$. Also since $\omega:(Q < x) \to C_i$ and $\pi:(y < R) \to [j,\ldots,c]$ are each strict order-preserving, then so is $\omega':P \to C_c$. Similarly $\pi' \in \Omega$.

Given ω', π' we can immediately reconstruct ω, π. In other words we have an injection, establishing the inequality. □

We will demonstrate that certain constraints are necessary in Theorem 4.15.

Example 4.5:



P

Defining $k = |\Omega(P,C_c)| = \binom{c}{h}$ we find that h, i and j can be chosen so that

$$\frac{\binom{c-i}{h-1}}{k}\;\frac{\binom{j-1}{h-1}}{k} = \text{prob}(\omega x=i)\text{prob}(\omega y=j) \nless \text{prob}(\omega x=i \text{ and } \omega y=j) = \frac{\binom{j-i-1}{h-2}}{k}$$

when any of the following hold:

(i) $h = 2$ and $j > i+1$,

(ii) $2 < h = j - i + 1$,

(iii) $2 < h < j - i + 1$.

Therefore we require both that y covers x in P and j covers i in C_c in the above theorem. Setting $P' = P \cup C_{c-h}$ we get similar conclusions for $\Lambda(P',C_n)$.

Theorem 4.16:

Let Q,R *be posets,* S *a chain and* x,y *be singletons. Suppose* P *is defined by* $(Q < x < y < R) \cup S$. *If* $j = i + 1$ *then for* $\lambda \in \Lambda(P,C_n)$,

$$\text{prob}(\lambda x=i)\text{prob}(\lambda y=j) \leq \text{prob}(\lambda x=i \text{ and } \lambda y=j).$$

Proof:

The proof follows that of Theorem 4.15. Given $\lambda,\ \mu \in \Lambda$ we define the injection

$$\lambda'p = \begin{cases} \lambda p & \text{if } \lambda p \leq i \\ \mu p & \text{if } \mu p \geq j, \end{cases}$$

$$\mu'p = \begin{cases} \mu p & \text{if } \mu p < j \\ \lambda p & \text{if } \lambda p > i. \end{cases}$$

Suppose σ is the subset of S such that $\lambda\sigma \subset C_i$. Since S is totally ordered and j covers i in C, then $\mu(S\setminus\sigma) \subset [j,\ldots,n]$. It follows that $\lambda',\mu' \in \Lambda$. □

The proof techniques of Theorems 4.15 and 4.16 easily extend to yield the following results, which concludes this chapter.

Theorem 4.17:

Let Q,R,X,Y *be posets, and* P *be defined by* $Q < X < Y < R$. *Let the intervals* $[\alpha,\beta],\ [\gamma,\delta] \subset C_c$. *If* $\gamma = \beta + 1$ *then for* $\omega \in \Omega(P,C_c)$,

$$\text{prob}(\alpha \leq \omega X \leq \beta)\text{prob}(\gamma \leq \omega Y \leq \delta) \leq \text{prob}(\alpha \leq \omega X \leq \beta \text{ and } \gamma \leq \omega Y \leq \delta).$$

<u>Theorem 4.18</u>:

Let Q,R,X,Y *be posets,* S *a chain and* P *be defined by* $(Q < X < Y < R) \cup S$. *Let the intervals* $[\alpha,\beta]$, $[\gamma,\delta] \subset C_n$. *If* $\gamma = \beta + 1$ *then for* $\lambda \in \Lambda(P,C_n)$,

$$\text{prob}(\alpha \leq \lambda X \leq \beta)\text{prob}(\gamma \leq \lambda Y \leq \delta) \leq \text{prob}(\alpha \leq \lambda X \leq \beta \text{ and } \gamma \leq \lambda Y \leq \delta).$$

CHAPTER 5 : INEQUALITIES FOR THE NUMBER OF MONOTONIC FUNCTIONS

5.1 INTRODUCTION

The results in this chapter were motivated by the log concave sequences established in the previous chapter, and appear in [JD]. We are concerned here not with a single element $x \in P$ but with a pair of elements $x, y \in P$. We later consider larger subsets of the poset. The injection technique developed in Theorem 4.3 is extended to this more general situation, to obtain many results of a new kind.

We commence with *Theorem* 5.1. *If* $r,s,t,u,v,w \in \mathbf{Z}^+$ *then* $N^S(r,u + v + w)N^S(r + s + t,u) < N^S(r + s, u + w)N^S(r + t, u + v)$. The special case $v = 0$ yields Theorem 4.3 of D.E. Daykin, J.W. Daykin and M.S. Paterson which is an analogue of Stanley's Theorem 4.2 for linear extensions.

An inequality is derived for the numbers of strict order-preserving maps taking each of the poset elements x and y to various intervals in the chain.

The proofs given here for strict order-preserving maps are by means of injective construction, and as we may begin to expect by now, they can be modified to give the corresponding results for order-preserving maps.

We give related results for linear extensions which establish the existence of linear extensions satisfying certain properties.

The inequality in Theorem 5.1 is illustrated in Figure 5.1.

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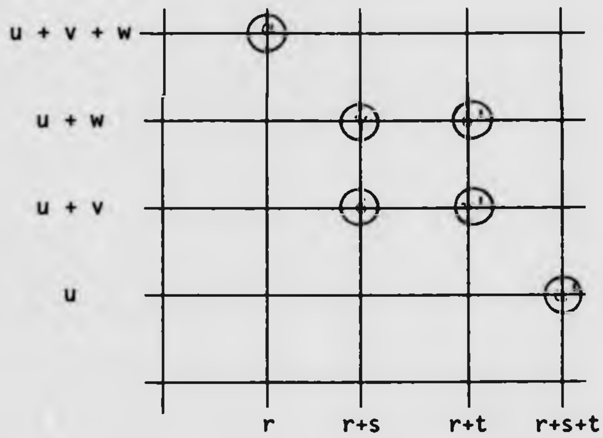


$\alpha = N^S(r, u + v + w)$

$\alpha' = N^S(r + s + t, u)$

$\vdots$

$\delta' = N^S(r + t,\ u + w).$

Theorem 5.1 says:

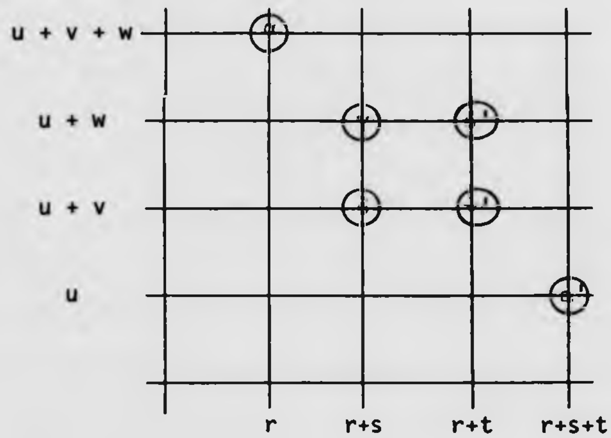
$\alpha\alpha' < \min\ \{\gamma\gamma', \delta\delta'\}.$

Figure 5.1

To see our motivation for this inequality, suppose for the moment that N^S was a smooth real function $N^S(x,y)$ of the real variables x,y. Then N^S defines a smooth surface above the x-y plane. Obviously the inequality implies that the section of this surface above any straight line drawn on the x-y plane will be a log concave curve.

An application of some of the theorems given here arises in the next chapter where we derive a lower bound for the computational complexity of comparison problems.

The notation used throughout this section was defined in Chapter 4.1. Also we will write x for x_1, y for x_2, i for i_1, j for i_2 and N for N^S.



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5.2 STRICT ORDER-PRESERVING MAPS

We will make use of the following process which produces certain mappings and sets from existing mappings and sets.

Given the poset P and a chain C, let $\omega_1, \omega_2 \in \Omega(P,C) \neq \emptyset$ and $x, y \in P$. Suppose $a, b \in \mathbb{Z}^+$, $\omega_1(x) < \omega_2(x) - a$ and $\omega_2(y) < \omega_1(y) - b$. We will construct a unique pair of strict order-preserving maps $\omega_3, \omega_4 : P \to C$ from ω_1, ω_2. Now ω_3, ω_4 are defined in terms of disjoint subsets D, E of P which will be defined below. For fixed x,y,a,b we define $\gamma : (\omega_1, \omega_2, x, y, a, b) \to (\omega_3, \omega_4, D, E)$ and $\Gamma : (\omega_1, \omega_2) \to (\omega_3, \omega_4)$ by

$$\omega_3(p) = \begin{cases} \omega_2(p) - a & \text{if } p \in D \\ \omega_2(p) + b & \text{if } p \in E \\ \omega_1(p) & \text{if } p \in P \setminus (D \cup E), \end{cases}$$

and

$$\omega_4(p) = \begin{cases} \omega_1(p) + a & \text{if } p \in D \\ \omega_1(p) - b & \text{if } p \in E \\ \omega_2(p) & \text{if } p \in P \setminus (D \cup E). \end{cases}$$

The subsets D and E are defined iteratively by:

$$D = \{x\}.$$

If $p \notin D$ and for some $d \in D$ either

$p < d$ and $\omega_2(p) > \omega_1(d) + a$

or $p > d$ and $\omega_1(p) < \omega_2(d) - a$ then $D = D \cup \{p\}$.

$$E = \{y\}.$$

If $p \notin E$ and for some $e \in E$ either

$p < e$ and $\omega_1(p) > \omega_2(e) + b$

or $p > e$ and $\omega_2(p) < \omega_1(e) - b$ then $E = E \cup \{p\}$.

The motivation for these definitions is that we want $x \in D$ and $y \in E$ and need to extend both sets as above to ensure ω_3 and ω_4 are strict order-preserving. Since P is finite the iterative construction of D and E must halt (possibly with $D \cup E = P$). To establish that ω_3 and ω_4 are well defined and satisfy the required properties we will prove:

<u>Lemma 5.1</u>:

(5.1) *For* $d \in D$, $\omega_1(d) < \omega_2(d) - a$.

(5.2) *For* $e \in E$, $\omega_2(e) < \omega_1(e) - b$.

An immediate consequence of Lemma 5.1 is:

Corollary 5.1:

$$D \cap E = \emptyset$$

which implies that ω_3 and ω_4 are well defined. Also Lemma 5.1 implies that for $d \in D$, $\omega_3(d), \omega_4(d) \in C$ since

$$1 < \omega_1(d) < \omega_2(d) - a = \omega_3(d) \text{ and}$$

$$\omega_4(d) = \omega_1(d) + a < \omega_2(d) < c.$$

Similarly for $e \in E$, $\omega_3(e), \omega_4(e) \in C$ and so ω_3 and ω_4 map P to C.

Next we prove:

Lemma 5.2:

ω_3 and ω_4 are strict order-preserving

and finally:

Lemma 5.3:

Γ is injective.

Proof of Lemma 5.1:

Since x satisfies (5.1) and y satisfies (5.2) we will proceed by induction. So suppose $d \in D$ satisfies (5.1) and $d > p \notin D$ where $\omega_2(p) > \omega_1(d) + a$. Then $D = D \cup \{p\}$ and

$$\omega_1(p) + a < \omega_1(d) + a < \omega_2(p).$$

While if $p > d$ and $\omega_1(p) < \omega_2(d) - a$ then $D = D \cup \{p\}$ and

$$\omega_1(p) < \omega_2(d) - a < \omega_2(p) - a ,$$

which establishes invariant (5.1). Similarly each new element added to E by the construction process must satisfy (5.2). □

Proof of Lemma 5.2:

We prove that ω_3 is strict order-preserving; the proof for ω_4 is analogous. Let $p, q \in P$ with $p < q$. We show that $\omega_3(p) < \omega_3(q)$ by considering nine cases, depending on which of the subsets D, E and $P \setminus (D \cup E)$ each of p and q belong. Clearly if p and q belong to the same subset, then since ω_1 and ω_2 are strictly order-preserving $\omega_3(p) < \omega_3(q)$. We proceed with the other six cases.

Case 1 $\quad p \in P \setminus (D \cup E)$ *and* $q \in D$.

$$\omega_3(p) = \omega_1(p) < \omega_1(q) < \omega_2(q) - a = \omega_3(q), \text{ using (5.1).}$$

Case 2 $\quad p \in P \setminus (D \cup E)$ *and* $q \in E$.

$$\omega_3(p) = \omega_1(p) < \omega_2(q) + b = \omega_3(q), \text{ using the definition of E.}$$

Case 3 $\quad p \in D$ *and* $q \in E$.

$$\omega_3(p) = \omega_2(p) - a < \omega_2(q) + b = \omega_3(q).$$

<u>Case 4</u> $p \in D$ *and* $q \in P \setminus (D \cup E)$.

$$\omega_3(p) = \omega_2(p) - a < \omega_1(q) = \omega_3(q), \text{ using the definition of } D.$$

<u>Case 5</u> $p \in E$ *and* $q \in P \setminus (D \cup E)$.

$$\omega_3(p) = \omega_2(p) + b < \omega_1(p) < \omega_1(q) = \omega_3(q), \text{ using (5.2).}$$

<u>Case 6</u> $p \in E$ *and* $q \in D$.

$$\omega_3(p) = \omega_2(p) + b < \omega_1(p) < \omega_1(q) < \omega_2(q) - a = \omega_3(q),$$

using (5.1), (5.2).

<u>Proof of Lemma 5.3</u>:

Suppose $\omega_1, \omega_2, \omega_1', \omega_2' \in \Omega$ are such that $(\omega_1, \omega_2) \neq (\omega_1', \omega_2')$, $\omega_1'(x) = \omega_1(x) < \omega_2(x) - a = \omega_2'(x) - a$ and $\omega_2'(y) = \omega_2(y) < \omega_1(y) - b$ $= \omega_1'(y) - b$.

<u>Case 1</u> $\gamma(\omega_1, \omega_2, x, y, a, b) = (\omega_3, \omega_4, D, E) = \gamma(\omega_1', \omega_2', x, y, a, b)$.

This is clearly contradictory from the definitions.

<u>Case 2</u> $\gamma(\omega_1, \omega_2, x, y, a, b) = (\omega_3, \omega_4, D, E) \neq (\omega_3', \omega_4', D', E') =$ $\gamma(\omega_1', \omega_2', x, y, a, b)$, while $(\omega_3, \omega_4) = (\omega_3', \omega_4')$. Assume $D \neq D'$. Without loss of generality p and d can be chosen so that $p \in D \setminus D'$ and $d \in D \cap D'$.

<u>Case</u> $p < d$.

Assume $\omega_2(p) > \omega_1(d) + a$. Using Lemma 5.1 and Corollary 5.1 we know that we adjoined p only to D and hence

$$\omega_3(p) = \omega_2(p) - a = \omega_1'(p).$$

We may now deduce

$$\omega_1'(d) = \omega_1(d) \leq \omega_2(p) - a = \omega_1'(p) < \omega_1'(d),$$

giving a contradiction.

Case $p > d$ follows similarly.

If we now assume $E \neq E'$ then this case follows by symmetry. We conclude that Γ is injective. □

Now suppose $\omega_1(x) < \omega_2(x) - a$, then ω_3,ω_4 can be constructed from ω_1,ω_2 by setting $E = \emptyset$ in the definition of the function γ. This case is denoted for fixed x,a by

$$\delta:(\omega_1,\omega_2,x,a) \to (\omega_3,\omega_4,D).$$

Similarly if $\omega_2(y) < \omega_1(y) - b$ then let $D = \emptyset$ in γ, resulting in

$$\varepsilon:(\omega_1,\omega_2,y,b) \to (\omega_3,\omega_4,E),$$

where y,b are fixed. It easily follows that ω_3,ω_4 are well and uniquely defined for these simpler cases.

One of the main results in this section is the following generalization of Theorem 4.3.

Theorem 5.1:

Let $x_1,\ldots,x_k$ *be a fixed subset in* P. *If* $r,s,t,u,v,w \in Z^+$ *and* $i_h - w \leq j_h \leq i_h + t$ *for* $3 \leq h \leq k$, *then*

$N(r,u+v+w,i_3,\ldots,i_k)N(r+s+t,u,j_3,\ldots,j_k) \leq N(r+s,u+w,i_3,\ldots,i_k)N(r+t,u+v,j_3,\ldots,j_k)$.

Each map ω counted by the function N has $\omega(x_h)$ fixed for $3 \leq h \leq k$ in the respective factors. From now on we will simplify such expressions to omit any i_h, j_h. Hence the statement of this theorem abbreviates to:

Theorem 5.1:

If $r,s,t,u,v,w \in \mathbb{Z}^+$ *and* $i_h - w \leq j_h \leq i_h + t$ *for* $3 \leq h \leq k$, *then*

$$N(r,u+v+w)N(r+s+t,u) \leq N(r+s,u+w)N(r+t,u+v).$$

Proof:

Suppose that the L.H.S. of the inequality is not zero. This implies $r,u > 0$. Given any pair of strict order-preserving maps $\omega_1,\omega_2 : P \to C$ with $\omega_1(x,y) = (r,u+v+w)$ and $\omega_2(x,y) = (r+s+t,u)$, we will construct a unique pair of strict order-preserving maps $\omega_3,\omega_4 : P \to C$ with $\omega_3(x,y) = (r+s,u+w)$ and $\omega_4(x,y) = (r+t,u+v)$. Initially ignore the elements $x_3,\ldots,x_k$.

Case 1 $s,v > 0$. We have $\omega_1(x) < \omega_2(x) - t$ and $\omega_2(y) < \omega_1(y) - w$, and so ω_3,ω_4 are constructed using

$$\gamma(\omega_1,\omega_2,x,y,t,w) = (\omega_3,\omega_4,D,E).$$

Then $\omega_3(x,y) = (\omega_2(x) - t,\ \omega_2(y) + w) = (r+s,u+w)$ and $\omega_4(x,y) = (\omega_1(x) + t,\ \omega_1(y) - w) = (r+t,u+v)$, as required.

<u>Case 2</u> Not $s,v > 0$. When $s = v = 0$ the result is trivial.
If $s = 0$ and $v > 0$ then $\omega_1(x,y) = (r,u+v+w)$ and $\omega_2(x,y) = (r+t,u)$.
So let

$$\varepsilon(\omega_1,\omega_2,y,w) = (\omega_3,\omega_4,E).$$

Since $\omega_2(x) \nleq \omega_1(x) - w$ then $x \notin E$ by (5.2). Therefore $\omega_3(x,y) = (\omega_1(x), \omega_2(y) + w) = (r,u+w)$ and $\omega_4(x,y) = (\omega_2(x),\omega_1(y) - w) = (r+t,u+v)$.
Otherwise $s > 0$ and $v = 0$, and similarly we use

$$\delta(\omega_1,\omega_2,x,t) = (\omega_3,\omega_4,D)$$

and (5.1) to establish ω_3,ω_4 with $\omega_3(x,y) = (r+s,u+w)$ and $\omega_4(x,y) = (r+t,u)$.

Finally consider elements x_h where $3 < h < k$. We have

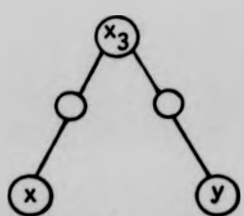
$$\omega_1(x_h) = i_h > j_h - t = \omega_2(x_h) - t,$$

$$\omega_2(x_h) = j_h > i_h - w = \omega_1(x_h) - w.$$

From Lemma 5.1 we deduce that $x_h \notin D$ and $x_h \notin E$ in both Cases 1 and 2. Hence $\omega_3(x_h) = i_h$ and $\omega_4(x_h) = j_h$ as required, which completes the proof. □

To see that the condition $i_h-w < j_h < i_h+t$ is necessary in Theorem 5.1 consider the following extension of Example 4.1.

<u>Example 5.1</u>:



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If $r = s = 1$, $t=3$, $u=2$, $v=w = 0$, $i_3 = 4$ and $j_3 = 8$ where $i_3 - w < j_3 \not< i_3 + t$, then

$$(2)(10) = N(1,2,4)N(5,2,8) \not< N(2,2,4)N(4,2,8) = (1)(15);$$

whilst if $r = 2$, $s = t = 0$, $u = v = 1$, $w = 3$, $i_3 = 8$ and $j_3 = 4$ where $i_3 - w \not< j_3 < i_3 + t$, then

$$(10)(2) = N(2,5,8)N(2,1,4) \not< N(2,4,8)N(2,2,4) = (15)(1).$$

From now on the strict order-preserving maps ω_1, ω_2 will be defined in terms of the L.H.S. of the inequality, analogous to the proof of Theorem 5.1, and so we will also assume the L.H.S. to be non-zero.

We next extend each of the elements $x, y \in P$ in Theorem 5.1 to subsets of P, by iterating the process of producing the sets D and E.

<u>Theorem 5.2</u>:

Let $k', k'', k \in \mathbb{Z}^+$ *with* $k' < k'' < k$. *If* $r_1, \ldots, r_{k'}, s_1, \ldots, s_{k'}, t, u_{k'+1}, \ldots, u_{k''}, v_{k'+1}, \ldots, v_{k''}, w \in \mathbb{Z}^+$ *and* $i_h - w < j_h < i_h + t$ *for* $k'' < h < k$, *then*

$$N(r_1, \ldots, r_{k'}, u_{k'+1} + v_{k'+1} + w, \ldots, u_{k''} + v_{k''} + w)$$
$$N(r_1 + s_1 + t, \ldots, r_{k'} + s_{k'} + t, u_{k'+1}, \ldots, u_{k''})$$
$$< N(r_1 + s_1, \ldots, r_{k'} + s_{k'}, u_{k'+1} + w, \ldots, u_{k''} + w)$$
$$N(r_1 + t, \ldots, r_{k'} + t, u_{k'+1} + v_{k'+1}, \ldots, u_{k''} + v_{k''}).$$

Proof:

Consider first the elements $x_1,\ldots,x_{k'}$. Let $\omega_3,\omega_4 \in \Omega$ be given by

$$\delta(\omega_1,\omega_2,x_1,t) = (\omega_3,\omega_4,D).$$

Thus $\omega_3(x_1) = r_1 + s_1$ and $\omega_4(x_1) = r_1+t$. Suppose that $y = x_2 \notin D$, for otherwise $\omega_3(y) = r_2+s_2$ and $\omega_4(y) = r_2+t$ as required. Then let $\omega_5,\omega_6 \in \Omega$ be given by

$$\delta(\omega_3,\omega_4,y,t) = (\omega_5,\omega_6,D').$$

We must show that $x = x_1 \notin D'$.

If $d \in D$ then $\omega_4(d) < \omega_3(d) + t$ follows from Lemma 5.1.

Then if $d \in D'$, $\omega_3(d) + t < \omega_4(d)$ by Lemma 5.1.

The contradiction shows that $x \notin D'$ and also $D \cap D' = \emptyset$. This process is iterated for elements $x_3,\ldots,x_{k'}$.

Next consider the elements $x_{k'+1},\ldots,x_{k''}$. From (5.1) we know that each x_h does not belong to any set D. As with the previous subset we repeatedly apply the injective function

$$\varepsilon:(\omega_{\ell+1},\omega_{\ell+2},x_h,w) \to (\omega_{\ell+3},\omega_{\ell+4},E).$$

Let $\mathcal{D} = \{D,D',\ldots,D''\}$ be the set of disjoint subsets of P generated by δ, and similarly $\mathcal{E} = \{E,E',\ldots,E''\}$ for ε. By Corollary 5.1, $D \cap E = \emptyset$, and so we deduce that $\mathcal{D} \cup \mathcal{E}$ is a set of pairwise disjoint sets.

Using Lemma 5.1 we see that $x_h \notin D$, $x_h \notin E$ for $k'' < h \leq k$, and any $D \in \mathcal{D}$, $E \in \mathcal{E}$. Hence $\omega_{2\ell+1}(x_h) = \omega_1(x_h) = i_h$ and $\omega_{2\ell+2}(x_h) = \omega_2(x_h) = j_h$ as required. $\square$

We consider again the singletons $x,y \in P$.

Theorem 5.3:

Suppose $r,s,t,u,v,w \in \mathbb{Z}^+$ *satisfy* $s < v$ *and* $t < w$. *If*

$$t < v \text{ and } i_h - v + t < j_h < i_h + t$$

or

$$t > v \text{ and } i_h < j_h < i_h + s$$

for $3 < h < k$, *then*

$$N(r,u)N(r+s+t,u+v+w) < N(r+s,u+v)N(r+t,u+w).$$

Proof:

If $v = 0$ then $s = 0$ and the result is trivial. When $v > 0$ and $s = 0$ we let

$$\delta(\omega_1,\omega_2,y,w) = (\omega_3,\omega_4,D).$$

Since $\omega_1(x) = r \nleq r+t-w = \omega_2(x)-w$ it follows from (5.1) that $x \notin D$. Therefore $\omega_3(x,y) = (r+s,u+v)$ and $\omega_4(x,y) = (r+t,u+w)$.

From now on assume $s,v > 0$.

Case 1 $t < v$. From $\delta(\omega_1,\omega_2,x,t)$ we get $\omega_3(x) = r+s$ and $\omega_4(x) = r+t$ as required, and a set $D \subset P$.

Case 1.1 $y \in D$. This implies $\omega_3(y) = u+v+w-t$ and $\omega_4(y) = u+t$. Now if $t = w$ we have the necessary maps ω_3,ω_4. So suppose $t < w$, and due to $\omega_4(y) < \omega_3(y)-(v-t)$, we can let

$$\varepsilon(\omega_3,\omega_4,y,v-t) = (\omega_5,\omega_6,E).$$

Also $\omega_4(x) \nless \omega_3(x) - (v-t)$ and hence by (5.2), $x \notin E$. This yields $\omega_5(x,y) = (r+s,u+v)$ and $\omega_6(x,y) = (r+t,u+w)$.

<u>Case 1.2</u> $y \notin D$. This implies $\omega_3(y) = u$ and $\omega_4(y) = u+v+w$. Let

$$\delta(\omega_3,\omega_4,y,w) = (\omega_5,\omega_6,D').$$

Now $\omega_3(x) \nless \omega_4(x) - w$ and so using (5.1), $x \notin D'$. Thus $\omega_5(x,y) = (r+s,u+v)$ and $\omega_6(x,y) = (r+t,u+w)$.

<u>Case 2</u> $t > v$. From $\delta(\omega_1,\omega_2,x,s)$ we get $\omega_3(x) = r+t$ and $\omega_4(x) = r+s$, and a set $D \subset P$.

<u>Case 2.1</u> $y \in D$. This implies $\omega_3(y) = u+v+w-s$ and $\omega_4(y) = u+s$. When $s = v$ then $\omega_3(y) = u+w$ and $\omega_4(y) = u+v$. So consider the case $s < v$, and note also that $w - s > 0$ here. We have $\omega_4(y) < \omega_3(y) - (w-s)$ and so we may let

$$\varepsilon(\omega_3,\omega_4,y,w-s) = (\omega_5,\omega_6,E).$$

Now $\omega_4(x) \nless \omega_3(x) - (w-s)$ and so by (5.2), $x \notin E$. It follows that $\omega_5(x,y) = (r+t,u+w)$ and $\omega_6(x,y) = (r+s,u+v)$.

So far we have shown that for $3 < h < k$,

$$N(r,u,i_h)N(r+s+t,u+v+w,j_h) < N(r+t,u+w,i_h)N(r+s,u+v,j_h).$$

When either $s = v$ or $s < v$ we have

$$r+t > r+s \text{ and } u+w > u+v.$$

Moreover, all $i_h \leq j_h$, and so we can use Theorem 5.2 to obtain

$$N(r+t,u+w,i_h)N(r+s,u+v,j_h) \leq N(r+s,u+v,i_h)N(r+t,u+w,j_h).$$

<u>Case 2.2</u> $y \notin D$. This implies $\omega_3(y) = u$ and $\omega_4(y) = u+v+w$. Let

$$\delta(\omega_3,\omega_4,y,v) = (\omega_5,\omega_6,D').$$

Now $\omega_3(x) \nleq \omega_4(x) - v$ and so using (5.1), $x \notin D'$. Thus $\omega_5(x,y) = (r+t,u+w)$ and $\omega_6(x,y) = (r+s,u+v)$. As in Case 2.1 we use Theorem 5.2 to get

$$N(r+t,u+w,i_h)N(r+s,u+v,j_h) \leq N(r+s,u+v,i_h)N(r+t,u+w,j_h),$$

completing this case.

For x_h with $3 \leq h \leq k$ we have $\omega_1(x_h) = i_h$ and $\omega_2(x_h) = j_h$. If $t \leq v$ then

$$i_h - (v-t) \leq j_h \leq i_h + t \leq i_h + w;$$

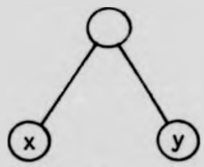
whilst if $t > v$ then

$$i_h - (w-s) < i_h \leq j_h \leq i_h + s \leq i_h + v < i_h + w.$$

Hence by Lemma 5.1, for any of the applications of δ or ε, the mappings of x_h remain fixed as is necessary. $\square$

We will establish that the conditions $s < v$ and $t < w$ are both necessary in Theorem 5.3.

<u>Example 5.2</u>:



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With $r = u = 1$, $s = 5$, $t = 2$, $v = 4$, $w = 3$ where

$$s > v \text{ and } t < w,$$

then

$$(c-1)(c-8) = N(1,1)N(8,8) \nless N(6,5)N(3,4) = (c-6)(c-4).$$

Further by Theorem 5.1, $N(6,5)N(3,4) > N(3,5)N(6,4)$. Thus for $r = u = 1$, $s = 2$, $t = 5$, $v = 4$, $w = 3$ where

$$s < v \quad \text{and} \quad t > w,$$

then

$$N(1,1)N(8,8) \nless N(3,5)N(6,4).$$

Special cases of Theorem 4.3 along with Theorems 5.1 and 5.3 can be stated as:

<u>Theorem 5.4</u>:

Suppose $r,s,t,u,v,w \in \mathbb{Z}^+$ *satisfy* $s < t, v < w$. *If* $i_h < j_h < i_h + s$ *for* $3 < h < k$, *then*

$$N(r,u)N(r+s+t,u+v+w) \le N(r+s,u+v)N(r+t,u+w)$$
$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$
$$N(r,u+v+w)N(r+s+t,u) \le N(r+s,u+w)N(r+t,u+v).$$

An application of Theorem 5.2 to the poset elements $x_1,\ldots,x_{k'}$ is shown by:

<u>Theorem 5.5</u>:

Let $k', k \in \mathbf{Z}^+$ *with* $k' \le k$. *Suppose* $r_1,\ldots,r_{k'}, s_1,\ldots,s_{k'} \in \mathbf{Z}^+$ *satisfy both*

(5.3) $\quad 0 \le s_1 \le s_2 \le \ldots \le s_{k'},$

(5.4) $\quad i_h - \beta \le j_h \le i_h + \alpha$ *for* $k' < h \le k$,

where $\quad \alpha = \min\ \{s_\ell - s_{\ell-1} : 1 \le \ell \le k',\ \ell\ odd\}$,

$\quad \beta = \min\ \{s_\ell - s_{\ell-1} : 2 \le \ell \le k',\ \ell\ even\}$.

Then

$$N(r_1,\ldots,r_{k'})N(r_1+2s_1,\ldots,r_{k'}+2s_{k'})$$
$$\le \quad N(r_1+s_1,\ldots,r_{k'}+s_{k'})N(r_1+s_1,\ldots,r_{k'}+s_{k'}).$$

<u>Proof</u>:

We will make k' applications of Theorem 5.2 to the fixed subset $x_1,\ldots,x_{k'}$ in P. Without loss of generality assume $s_1 > 0$. Putting $t = s_1$ in Theorem 5.2 we get

$$N(r_1,r_2,\ldots,r_{k'})N(r_1+2s_1,r_2+2s_2,\ldots,r_{k'}+2s_{k'})$$
$$\le \quad N(r_1+s_1,r_2+2s_2-s_1,\ldots,r_{k'}+2s_{k'}-s_1)N(r_1+s_1,r_2+s_1,\ldots,r_{k'}+s_1),$$

to which we associate the map

$$\omega_1,\omega_2 \to \omega_3,\omega_4,D_1.$$

Now $\omega_1,\omega_2,\omega_3,\omega_4$ are respectively associated with the four factors in the inequality; the set D_1 represents those elements whose image points have varied, namely $x_1,\ldots,x_{k'}$.

Subsequently if $k' > 2$, set $t = s_2-s_1,\ s_3-s_2,\ldots,s_{k'}-s_{k'-1}$, and we associate the sequence of mappings and sets

$$\omega_1,\omega_2 \to \omega_3,\omega_4,D_1 \to \cdots \to \omega_{2k'+1},\omega_{2k'+2},D_{k'}$$

with these applications.

By (5.3), for $1 \leq h,\ \ell \leq k'$ if ℓ is odd then

(5.5) $$\omega_{2\ell+1}(x_h) \geq r_h + s_h \geq \omega_{2\ell+2}(x_h),$$

and if ℓ is even then

(5.6) $$\omega_{2\ell+1}(x_h) \leq r_h + s_h \leq \omega_{2\ell+2}(x_h).$$

Equality in (5.5) or (5.6) implies from Theorem 5.2 that $x_h \notin D_\ell$ when $1 \leq h < \ell \leq k'$.

For elements x_h with $k' < h \leq k$ and $1 \leq \ell \leq k'$, using (5.4), if ℓ is odd then

$$j_h = \omega_2(x_h) \leq \omega_1(x_h) + \alpha = i_h + \alpha \leq i_h + s_\ell - s_{\ell-1},$$

and if ℓ is even then

$$j_h = \omega_2(x_h) > \omega_1(x_h) - \beta = i_h - \beta > i_h - (s_\ell - s_{\ell-1}).$$

Hence by Theorem 5.2, in either case $x_h \notin D_\ell$. □

We now give a higher order inequality.

<u>Theorem 5.6</u>:

Let $h, r_1, \ldots, r_h, u_1, \ldots, u_h$ *be integers, where* $h > 1$. *Suppose*

(5.7) $\Sigma(1 < \ell < h) r_\ell = 0$ *and* $\Sigma(1 < \ell < h) u_\ell = 0$.

Then

(5.8) $N(i+r_1, j+u_1) \ldots N(i+r_h, j+u_h) < N(i,j)^h$,

with $\langle \omega(x_3), \ldots, \omega(x_k) \rangle = \langle i_3, \ldots, i_k \rangle$ *in every factor.*

<u>Proof</u>:

Assume that some r_ℓ in (5.8) is negative. Then (5.7) implies that there exists a distinct pair $N(i+r_\ell, j+u_t)$, $N(i+r_{\ell'}, j+u_{t'})$ with $1 < \ell, \ell', t, t' < h$ in (5.8), such that $r_{\ell'}$ is positive. In view of (5.1), $r_\ell < r_{\ell'} - 1$.

<u>Case 1</u>:

$u_t > u_{t'} - 1$. Applying Theorem 5.1 to this pair shows

(5.9) $N(i+r_\ell, j+u_t) N(i+r_{\ell'}, j+u_{t'}) < N(i+r_{\ell'} - \alpha, j+u_t) N(i+r_\ell + \alpha, j+u_{t'})$, where $\alpha = \min\{|r_\ell|, r_{\ell'}\}$. Hence $r_\ell + \alpha = 0$ or $r_{\ell'} - \alpha = 0$, and $r_\ell + \alpha < r_{\ell'} - \alpha$. We make the substitution of (5.9) in the L.H.S. of (5.8) and note that (5.7) still holds.

Case 2:

$u_t < u_{t'} - 1$. Applying Theorem 5.2 to the distinct pair yields

$$(5.10)\quad N(i+r_\ell, j+u_t)N(i+r_{\ell'}, j+u_{t'}) < N(i+r_{\ell'}-\alpha, j+u_{t'}-\alpha)N(i+r_\ell+\alpha, j+u_t+\alpha),$$

where $\alpha = \min\{|r_\ell|, r_{\ell'}, u_{t'} - u_t - 1\}$. If α is $|r_\ell|$ or $r_{\ell'}$ then $r_\ell + \alpha = 0$ or $r_{\ell'} - \alpha = 0$, and by substituting (5.10) in the L.H.S. of (5.8), it follows that (5.7) is still satisfied. Otherwise with $\alpha = u_{t'} - u_t - 1$ we have $u_{t'} - \alpha < u_t + \alpha$, while $r_\ell + \alpha$ is negative and $r_{\ell'} - \alpha$ is positive, in which case we can apply Case 1.

After Case 1 or 2 we proceed to find a new distinct pair if one exists. This iteration results in all of the x components of (5.8) being equal to i. If at this stage some y components of (5.8) are not equal to j, then we can analogously apply Case 1, where now the mapping of the x component will remain fixed.

Moreover, Theorems 5.1 and 5.2 show that the mappings of x_ℓ remain equal to i_ℓ, for $3 < \ell < k$, throughout the iterations. □

Using the ideas developed here we obtain a result for elements $x, y, z \in P$.

Theorem 5.7:

Suppose $r, r', s, s', t', u, v \in \mathbb{Z}^+$ *satisfy* $s < v$ *and* $v - s < s', t'$. *If* $i_h - v + s < j_h < i_h + \min\{s, s - v + s'\}$ *for* $4 < h < k$, *then*

$$N(r,u,r'+s'+t')N(r+2s,u+2v,r') < N(r+s,u+v,r'+t')N(r+s,u+v,r'+s').$$

Proof:

If any of r, u or r' equals zero then the result is trivial because the L.H.S. is zero. If $s = 0$ the theorem reduces to Theorem 5.1. If $v = 0$ the theorem reduces to Theorem 4.3. Assume $s < v$ for otherwise this follows by Theorem 5.2.

Given a pair of maps ω_1, ω_2 representing the L.H.S. of the inequality, by a series of applications of δ, ε, we will construct a pair of maps ω_7, ω_8 representing the R.H.S.

Case $r, r', s, u, v > 0$:

We have $\omega_1(x) < \omega_2(x) - s$ and so $\omega_3, \omega_4 \in \Omega$ are constructed from

$$\delta(\omega_1, \omega_2, x, s) = (\omega_3, \omega_4, D).$$

Then $\omega_3(x) = \omega_4(x) = r+s$. Now for any element $p \in P$ and $\ell \in \mathbf{Z}^+$, if $\omega_{2\ell+1}(p) = \omega_{2\ell+2}(p)$ then Lemma 5.1 implies that $\omega_{2\ell+3}(p) = \omega_{2\ell+4}(p)$. So from now on, for any application of the injections, the image of x will remain fixed. Further from Lemma 5.1, $z \notin D$.

Case 1:

$y \in D$. Therefore $\omega_3(y) = u + 2v-s$ and $\omega_4(y) = u+s$. Since $\omega_4(y) < \omega_3(y) - (v-s)$ and $v-s > 0$ we can let

$$\varepsilon(\omega_3, \omega_4, y, v-s) = (\omega_5, \omega_6, E).$$

This results in $\omega_5(y) = \omega_6(y) = u+v$.

Case 1.1:

$z \in E$. This means that $\omega_5(z) = r' + v-s$ and $\omega_6(z) = r' + s' + t' -v+s$. By the hypothesis $\omega_5(z) < \omega_6(z) - (s-v+s')$, also $s-v + s' > 0$ and so we may define

$$\delta(\omega_5,\omega_6,z,s-v+s') = (\omega_7,\omega_8,D').$$

Hence $\omega_7(x,y,z) = (r+s,\ u+v,\ r'+t')$ and $\omega_8(x,y,z) = (r+s,u+v,r'+s')$ as required.

Case 1.2:

$z \notin E$. Then $\omega_5(z) = r'+s'+t'$ and $\omega_6(z) = r'$ and since we can assume that $s',\ t' > 0$, we may let

$$\varepsilon(\omega_5,\omega_6,z,t') = (\omega_7,\omega_8,E').$$

This results in the necessary ω_7,ω_8 since $x,y \notin E'$.

Case 2:

$y \notin D$. This implies $\omega_3(x,y,z) = (r+s,u,r'+s'+t')$ and $\omega_4(x,y,z) = (r+s,u+2v,r')$. Clearly we can set

$$\delta(\omega_3,\omega_4,y,v) = (\omega_5,\omega_6,D').$$

Using Lemma 5.1, $z \notin D'$, and since s' is assumed positive we let

$$\varepsilon(\omega_5,\omega_6,z,t') = (\omega_7,\omega_8,E').$$

This produces the required ω_7,ω_8 since $x,y \notin E'$.

Finally for x_h with $4 < h < k$ we have $\omega_1(x_h) = i_h$, $\omega_2(x_h) = j_h$. Let $\alpha = \min\{s, s-v+s'\}$. Now

$$i_h - t' < i_h - (v-s) < j_h < i_h + \alpha < i_h + s < i_h + v.$$

It follows from Lemma 5.1 that the mappings of x_h remain fixed for any of the above applications of the injections δ, ε. Hence $\omega_{2\ell+1}(x_h) = i_h$ and $\omega_{2\ell+2}(x_h) = j_h$. □

We next combine two concepts, firstly log concavity for monotonic functions of singletons, secondly the order between two elements, that is their relative rankings.

So if $x, y \in P$ and $i \in C$, then define $N_i^O(x < y)$ to be the number of $\omega^O \in \Omega^O$ such that $\omega^O x = i$ and $\omega^O x < \omega^O y$. Similarly define $N_i^S(x < y)$, $N_i^I(x < y)$ and $N^L(x < y)$ for the other classes Ω, Λ^I and Λ respectively.

Intuitively we would expect these functions to be decreasing. However putting $x < y$ and noting that $N^S = N^I$ in Example 4.4 demonstrates that these are not monotone sequences.

<u>Claim 5.1</u>:

Let $x, y \in P$. *Then* $N_i^S(x < y)$ *is a log concave sequence.*

<u>Proof</u>:

As we will only count maps with $\omega x < \omega y$ we can assume $x < y$ in P. Then

$$N_i^S(x < y) = N(i,i+1) + \ldots + N(i,c) = N_i.$$

Thus the result follows by log concavity of the sequence $N_i, N_{i+1}, \ldots$. □

We make an extension of this claim, based on the fact that here we have let y map to the interval $[i+1,c]$. In the following inequality we let each of the elements $x,y \in P$ map to intervals in C. Hence define $N([i_1,i_2], [j_1,j_2])$ to be the number of strict order-preserving maps $\omega:P \to C$ such that $\omega x \in [i_1,i_2]$ and $\omega y \in [j_1,j_2]$; and likewise define $N^0([i_1,i_2], [j_1,j_2])$ for order-preserving maps.

<u>Theorem 5.8</u>:

If $r,r',s,t,t',u,v,w,w' \in \mathbb{Z}^+$ *and* $w' < v$, *then*

$$(5.11)\quad N([r',r], [u,v])N([r+s+t,t'], [u+t+w,w'])$$

$$< N([r+s,t'-t], [u+w,v])N([r'+t,r+t], [u+t,w']).$$

<u>Proof</u>:

Suppose that the L.H.S. of the inequality is not zero. Thus we assume that the intervals on the L.H.S. are non-empty, i.e., $r' < r$, $u < v$, $r+s+t < t'$ and $u+t+w < w'$. Clearly on the R.H.S. we then have $r+s < t'-t$, $r'+t < r+t$, $u+t < w'$ and also $u+w < u+t+w < w' < v$.

Suppose $r' < h < r$ and $r + s + t < \ell < t'$, then we must show that

$$(5.12)\quad (N(h,u) + \dots + N(h,v))(N(\ell,u+t+w) + \dots + N(\ell,w'))$$

$$< (N(\ell-t,u+w) + \dots + N(\ell-t,v))(N(h+t,u+t) + \dots + N(h+t,w')).$$

First we will establish that

(5.13) $\quad N(h,j')N(\ell,j'') \leq N(\ell-t,j''-t)N(h+t,j'+t)$

when $u \leq j' < u+w$ and $u+t+w \leq j'' \leq w'$.

The inequality (5.13) follows from applying Theorem 5.2. Let the maps ω_1,ω_2 and ω_3,ω_4 represent the left and right side of (5.13) respectively. Concerning the element x, when $\omega_1(x) = h < \ell-t=\omega_2(x)-t$ then we can show that x is mapped into the required intervals with respect to the R.H.S. of (5.11):

$$r+s \leq \omega_3(x) = \ell-t \leq t'-t,$$

$$r'+t \leq \omega_4(x) = h+t \leq r+t.$$

Otherwise $h \geq \ell-t$, which implies that $\omega_1(x) = r$, $\omega_2(x) = r+s+t$ and $s = 0$. Hence we get $\omega_3(x) = \omega_1(x)$ and $\omega_4(x) = \omega_2(x)$, so x will clearly belong to the correct ranges. Concerning the element y, we always have $\omega_1(y) = j' < j'' - t = \omega_2(y) - t$, and again we map into the correct intervals:

$$u+w \leq \omega_3(y) = j''-t \leq w'-t \leq v,$$

$$u+t \leq \omega_4(y) = j'+t < u+t+w \leq w'.$$

We will prove that

(5.14) $\quad N(h,[u+w,v])N(\ell,[u+t+w,w']) \leq N(\ell-t,[u+w,v])N(h+t,[u+t+w,w'])$.

Summing (5.13) over j', j'' and adding (5.14) gives (5.12). Then summing (5.12) over h,ℓ gives (5.11) as required.

We prove (5.14) as follows. Given any ordered pair (ω_1, ω_2) of maps counted by the L.H.S. we construct a unique pair (ω_3, ω_4) counted by the R.H.S. So we have

$$\omega_1(x) = h, \qquad \omega_2(x) = \ell,$$

$$\omega_3(x) = \ell - t, \quad \omega_4(x) = h+t,$$

$$u+w < \omega_1(y), \omega_3(y) < v,$$

$$u+t+w < \omega_2(y), \omega_4(y) < w'.$$

If $\omega_1(x) = r$, $\omega_2(x) = r+s+t$ and $s = 0$ then let $\omega_3 = \omega_1$ and $\omega_4 = \omega_2$. Hence (ω_3, ω_4) will be unique.

Otherwise with $h < \ell - t$, we may let

$$\delta(\omega_1, \omega_2, x, t) = (\omega_3, \omega_4, D),$$

and so $\omega_3(x) = \ell - t$, $\omega_4(x) = h+t$.

Consider now the element y with respect to the set D.

<u>Case 1</u>:

$y \notin D$. By definition we have $\omega_3(y) = \omega_1(y)$ and $\omega_4(y) = \omega_2(y)$ which is enough to show that y is mapped to the required intervals in (5.14).

Case 2:

$y \in D$. From the invariant (5.1) we know that $\omega_1(y) < \omega_2(y)-t$. This relation is used to establish that the images of y will belong to the specified intervals, namely

$$u+w < \omega_2(y)-t = \omega_3(y) < w'-t < v-t < v,$$

$$u+t+w < \omega_1(y)+t = \omega_4(y) < \omega_2(y) < w'.$$

Suppose in Case 1 that $\delta(\omega_1,\omega_2,x,t) = (\omega_3,\omega_4,D)$ with $y \notin D$ and in Case 2 that $\delta(\omega_1',\omega_2',x,t) = (\omega_3',\omega_4',D')$ with $y \in D'$. The injective property of δ ensures that $(\omega_1,\omega_2) \neq (\omega_1',\omega_2')$ and $\omega_3(x,y) = \omega_3'(x,y)$, $\omega_4(x,y) = \omega_4'(x,y)$ implies that $(\omega_3,\omega_4) \neq (\omega_3',\omega_4')$.

We remark that similarly to the previous theorems we may extend this result to a fixed subset $x_1,\ldots,x_k$ in P. For $3 < h < k$ let $\alpha_h,\beta_h,\gamma_h,\delta_h \in C$. Then in (5.11) we put $\omega(x_h) \in [\alpha_h,\beta_h]$ in the first and third factors, and $\omega(x_h) \in [\gamma_h,\delta_h]$ in the second and fourth factors. With $\delta_h < \alpha_h + t$ this follows by Theorem 5.2 and (5.1). □

The following shows the necessity for the condition $w' < v$ in Theorem 5.8.

Example 5.3:

Let P be defined as in Example 5.2. Setting $r=r'=s=t=u=w=1$, $t'=3$, $v=2$ and $w'=4$ where $w' > v$, then

$$(2c-3)(2c-7) = N(1,[1,2])N(3,[3,4]) \nless N(2,2)N(2,[2,4]) = (c-2)(3c-9).$$

5.3 ORDER-PRESERVING MAPS

We will employ corresponding injections to γ, δ and ϵ in order to show that the preceeding inequalities also hold for order-preserving maps. The constructions follow a parallel course to those for strict order-preserving maps, whilst substituting ω^0 for ω where $\omega^0 \in \Omega^0$. For example, the subset D is now defined iteratively by:

$$D = \{x\}.$$

If $p \notin D$ and for some $d \in D$ either

$$p < d \text{ and } \omega_2^0(p) > \omega_1^0(d) + a$$

or $p > d$ and $\omega_1^0(p) < \omega_2^0(d) - a$ then $D = D \cup \{p\}$.

Similarly Cases 1 and 3 in the proof of Lemma 5.2 are modified as follows, to establish that $\omega_3^0(p) < \omega_3^0(q)$.

<u>Case 1</u> $p \in P\setminus(D \cup E)$ *and* $q \in D$.

$$\omega_3^0(p) = \omega_1^0(p) < \omega_1^0(q) < \omega_2^0(q) - a = \omega_3^0(q),$$

using (5.1) with ω replaced by ω^0.

Case 3 $p \in D$ *and* $q \in E$.

$$\omega_3^0(p) = \omega_2^0(p) - a < \omega_2^0(q) + b = \omega_3^0(q).$$

Hence by using analogous proofs we obtain:

Theorem 5.9:

Theorems 5.1 - 5.8 *hold with* $N = N^S$ *replaced by* N^0.

Notice that Examples 5.1 - 5.3 serve the same purpose in this section as for strict order-preserving maps, because the result in each example is the same although some numerical values are different.

In a similar way to Claim 5.1 we get:

Claim 5.2:

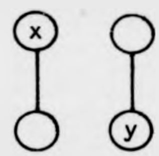
Let $x,y \in P$. *Then* $N_i^0(x < y)$ *is a log concave sequence.*

5.4 LINEAR EXTENSIONS

For $x,y \in P$ bijectivity implies that $N^L(i,j) = 0$ when $i = j$. If we plot $N^L(i,j)$ on the plane then by Stanley's Theorem 4.2 every row and every column is log concave.

We would not expect an immediate analogue of Theorem 5.1 for linear extensions and the following example establishes this view.

Example 5.4:



P

With $r = 2$ and $s = t = u = v = w = 1$ then

$$(1)(2) = N^L(2,3)N^L(4,1) \not\leq N^L(3,2)^2 = 1^2.$$

Existence theorems for certain types of linear extensions are next established. Recall that $n = |P|$.

Theorem 5.10:

If $x \nless y$ *in* P *and* $i < j$ *then* $N^L(i,j) > 0$ *iff all of*

$|\text{above } \{x,y\}| < n-i+1,$

$|\text{below } \{x,y\}| < j,$

$|\text{above } \{y\}| < n-j+1,$

$|\text{below } \{x\}| < i,$

and $|\text{above } \{x\} \cap \text{below } \{y\}| < j-i+1.$

Some easy corollaries now follow. As an analogy to Theorem 5.1 consider:

<u>Corollary 5.2</u>:

Let $x,y \in P$ *and* $h, \ell \in \mathbb{Z}^+$. *Suppose* $N^L(i,j+\ell) > 0$ *and* $N^L(i+\ell,j) > 0$. *If* $0 < h < \ell$ *and* $i + \ell < j + 1$ *or* $j + \ell < i + 1$ *then* $N^L(i + h, j + \ell - h) > 0$.

As an analogy to Theorem 5.2 we give:

<u>Corollary 5.3</u>:

Let $x,y \in P$ *and* $h,\ell \in \mathbb{Z}^+$. *Suppose* $N^L(i,j) > 0$ *and* $N^L(i+\ell, j+\ell) > 0$. *If* $0 < h < \ell$ *then* $N^L(i+h,j+h) > 0$.

These two corollaries can also be proved using the push up and down functions.

Example 5.1 is easily modified to illustrate the same conclusions for linear extensions. This is achieved by including C_4 in the definition of P, and incrementing i_3, j_3 so that

$$(2)(12) = N^L(1,2,5)N^L(5,2,9) \nleq N^L(2,2,5)N^L(4,2,9) = 0,$$

$$(12)(2) = N^L(2,5,9)N^L(2,1,5) \nleq N^L(2,4,9)N^L(2,2,5) = 0.$$

With respect to the conditions $s < v$ and $t < w$ in Theorem 5.3, we can similarly adapt Example 5.2 for linear extensions by adding C_7 to the definition of P. Setting $r = 1$, $s = 5$, $t = u = 2$, $v = 4$ and $w = 3$ where $s > v$ and $t < w$, then

$$(8)(1) = N^L(1,2)N^L(8,9) \nleq N^L(6,6)N^L(3,5) = 0.$$

Interchanging the values of s and t and of v and w, so that $s < v$ and $t > w$, produces the same results.

The following example shows that Theorem 5.8 is not true for linear extensions.

<u>Example 5.5:</u>

P

We have

$$(4)(2) = N^L(1,[2,5])N^L(6,[4,5]) \nleq N^L(5,[3,5])N^L(2,[3,5]) = (2)(3).$$

where $r = r' = t = w = 1$, $s = 4$, $t' = 6$, $u = 2$ and $v = w' = 5$.

We complete this section with:

Claim 5.3:

Let $x,y \in P$. *Then* $N_i^I(x < y)$ *is a log concave sequence.*

CHAPTER 6 : A LOWER BOUND TECHNIQUE FOR COMPARISON PROBLEMS

6.1 INTRODUCTION

We describe here a new lower bound technique in Complexity Theory for some comparison problems such as selection or sorting, (see [DP]). Each of these problems can be defined as producing a particular partitioned poset from the given set of elements.

For the lower bound we construct a set of partitions of the poset, to be represented by a distributive lattice Γ. Equivalently we can use the structure provided by taking order-preserving maps from the poset to a chain of length two.

The procedure requires *Lemma* 6.4. *If* $A, B, A \vee B, A \wedge B$ *are disjoint subsets in* Γ, *then min* $\{|A|, |B|\} \leq |\Gamma|/4$. To prove this we use the lattice inequality $|A||B| \leq |A \vee B||A \wedge B|$ detailed in Chapter 3.1. Corresponding results for strict order-preserving and order-preserving maps are established with the aid of inequalities for monotonic functions from Chapters 5.2, 5.3 or 4.2.

We apply the technique to find a lower bound for the equi-partition problem. It is shown that to form the partition of n elements into two equal sized parts, which respects their linear order, requires at least 1.2n comparisons.

6.2 COMPARISON PROBLEMS

Many of the well-known selection and sorting problems can be understood as the production of certain partial orders using binary comparisons. For example, the posets in Figure 6.1 correspond to the problems: select the first 3 elements in order, without order and select the 3rd element, respectively, out of a chain on n elements.

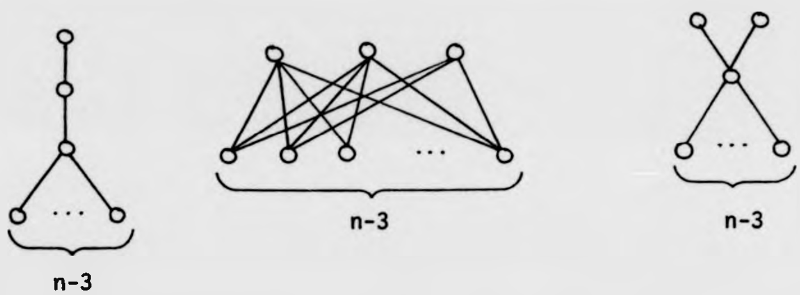


Figure 6.1

Let $\Phi(n)$ be the set of all (non-isomorphic) posets on n elements. We order $\Phi(n)$ by $P \prec Q$ in Φ iff there exists a monotone injection from P to Q. With this order relation $\Phi(n)$ itself becomes a poset with the antichain A_n on n elements as unique minimal element and the chain C_n as unique maximal element.

On the ground-set, or reservoir of n elements, we are given a fixed total order, unknown to us. Let $P \in \Phi(n)$. Our goal is to determine P with certainty by a sequence of comparisons between pairs of elements. By branching on the outcome of each comparison, any such algorithm T corresponds to a rooted binary tree with A_n as the root and with the nodes of the tree corresponding to the posets that have been determined up to then. The condition that T should determine P with certainty shall mean that P can be embedded monotonically into all the posets associated with the end-nodes of the tree. Hence we give the following definition, suggested by Schönhage [S].

Definition 6.1:

Let $P \in \Phi(n)$. An algorithm T *produces* P iff for all end-posets Q_i of T we have $P < Q_i$. The *length* $\ell(T)$ of T is the height of the corresponding tree.

Definition 6.2:

Let $P \in \Phi(n)$. The (serial) *cost* $\mathfrak{C}(P)$ of P is

$$\mathfrak{C}(P) = \min_{T} \ell(T)$$

where the minimum is extended over all algorithms T which produce P.

Since $\ell(T)$ is the maximal path length of the tree, $\mathfrak{C}(P)$ is the number of comparisons required to produce P in the worst case, by any optimal algorithm.

An important class of posets called partitions, considered by Aigner [A1] and Yap [Ya1], includes all selection problems treated in the literature so far.

Definition 6.3:

Let $n = k_1 + \ldots + k_d$ be an ordered partition of n into d positive integers k_i. To this partition there corresponds a unique poset in $\Phi(n)$ consisting of d groups $G_1, \ldots, G_d$ with $|G_i| = k_i$, $i = 1, \ldots, d$, such that

$G_i > G_j$ iff $i < j$. This poset is called the *partition of type* $(n;k_1,\ldots,k_d)$ and we denote its complexity by $\mathfrak{C}(n;k_1,\ldots,k_d)$.

Three types have received special attention in the theory of sorting:

(6.1) The partition $(n;\underbrace{1,\ldots,1}_{t},n-t)$ - sorting the first t elements in order.

(6.2) The partition $(n;t-1,1,n-t)$ - selecting the t-th element.

(6.3) The partition $(n;t,n-t)$ - selecting the first t elements without regard to order.

The cost functions for these types are commonly denoted by

$$W_t(n) = \mathfrak{C}(n;1,\ldots,1,n-t),$$

$$V_t(n) = \mathfrak{C}(n;t-1,1,n-t),$$

$$U_t(n) = \mathfrak{C}(n;t,n-t).$$

Clearly we have $U < V < W$.

In [A2] Aigner has suggested that the sequence $(U_t(n):t = 1,\ldots,n)$ is unimodal. That is, the problem of determining the t top elements as an unordered set becomes steadily harder right up to $t = \frac{n}{2}$. However it is not thought so likely that the sequence $(V_t(n):\ t = 1,\ldots,n)$ is unimodal, since there is no reason to believe that to select the 5th element is more difficult than to select the 4th element, for example.

The median problem, namely $(2m + 1;m,1,m)$, is the hardest selection problem, in the sense that if the median has complexity $V_{m+1}(n) > n$, with $n = 2m + 1$, then selecting the t-th largest element also has complexity $O(V_{m+1}(n))$. It is known [Ya2,SPP] that $\frac{11}{6}n \lesssim V_{m+1}(n) < 3n$, although the lower bound is as yet unpublished. Very recently John [J] has established a lower bound asymptotic to 2n for the median, by means of a relatively straightforward counting argument.

Efficient median algorithms find applications in many computational situations including sorting, minimum spanning tree and geometry problems.

To determine the partition $(2m;m,m)$ is known as the equi-partition or bipartition problem. Kirkpatrick [K] observed the importance of this problem for obtaining lower bounds to the median problem. That is all known proofs are unable to exploit any property which is peculiar to the median but which is not already available to the bipartition. Hence median lower bounds, which usually employ the notion of an oracle or adversary, along with case analysis of posets, also work for the bipartition problem. So we have $U_m(2m) > 4m$. With the oracle technique a procedure is defined to determine the outcome of each comparison in such a way as to force the algorithm to make many comparisons.

Kirkpatrick has shown [K] that the lower bound for the median follows from a lower bound for the bipartition as follows

$$\mathcal{C}(2m+1;m,1,m) > \mathcal{C}(2m+2;m+1,m+1) + 1.$$

Further by using induction on the bipartition case, it is possible to get a lower bound for any poset in the class of partitions (see [Ya2]). However the bound produced in this way will always be less than 2n, but using information theoretic arguments has shown $\mathcal{C}(n;n/4,n/4,n/4,n/4) > 2n$.

Sorting of n elements can be understood as the task of producing the total order, C_n, starting from n singletons, C_1. Accordingly in this case, production of C_n means the transition from $n(C_1)$ to C_n. The following information theoretical argument yields the lower bound for $S(n) = \mathfrak{C}(C_n)$.

There are exactly n! external nodes in a comparison tree which sorts n elements with no redundant comparisons. If the tree has k+1 ranks or levels, it follows that there can be at most 2^k external nodes in the tree. Hence, letting $k = S(n)$, we have $n! \leq 2^{S(n)}$ and so $S(n) \geq \lceil \log_2 n! \rceil$. Using Stirling's approximation we deduce that about $n \log_2 n$ comparisons are needed for this problem.

The best known upper bound of $S(n) \leq \sum_{1 \leq k \leq n} \lceil \log_2 (\frac{3}{4} k) \rceil$ comes from the Ford and Johnson algorithm [FJ]. Thus there is still a gap of order n between the two bounds.

No general method for obtaining upper bounds applicable to any given partition $(n;k_1,\ldots,k_d)$ is in sight as yet, but we can make a few observations about some precise bounds of special interest.

Using duality we get

$$\mathfrak{C}(n;k_1,\ldots,k_d) = \mathfrak{C}(n;k_d,\ldots,k_1),$$

and it is obvious that

$$U_1(n) = V_1(n) = W_1(n) = n-1,$$

$$W_n(n) = W_{n-1}(n) = S(n).$$

Further (see [A1]),

$$U_2(n) = n-2 + \lceil \log_2(n-1) \rceil,$$

$$V_2(n) = W_2(n) = n-2 + \lceil \log_2 n \rceil,$$

and $$\mathbf{C}(n;1,n-2,1) = \lceil \tfrac{3}{2} n \rceil - 2.$$

Merging of m elements with n elements means the transition from $C_m + C_n$ to C_{m+n}. In some cases the corresponding cost function $M(m,n)$ is known explicitly:

$$M(m,n) = m + n - 1 \text{ for } |m-n| < 1,$$

$$M(1,n) = \lceil \log_2(n+1) \rceil,$$

$$M(2,n) = \lceil \log_2(\tfrac{7}{12}(n+1)) \rceil + \lceil \log_2(\tfrac{14}{17}(n+1)) \rceil.$$

The latter formula gives some idea how intricate the answer to fairly simple problems of this type can be. Here the merging problem serves as an example, where the algorithms start from some prescribed partial order, for which the underlying total order is still unknown.

Further details and references for minimum-comparison sorting, selection and merging are given in [Kn], Sections 5.3.1,2,3.

6.3 A SET OF PARTITIONS OF A POSET

For any poset P let $\{S \cup T\}$ be a partition of P satisfying $s \in S$, $t \in T$ implies $s \nmid t$ in P. Then let $\mathcal{F}$ denote the set of all such partitions of P.

The lower bound technique commences by representing P, whose elements are $q_1,\ldots,q_n$, by this set $\mathcal{F}$. Now we have two equivalent ways of regarding $\mathcal{F}$.

Let $\mathcal{L}$ be the distributive lattice of subsets of $N = \{1,2,\ldots,n\}$ ordered by inclusion. Then $\mathcal{F} \subset \mathcal{L}$. If $\nu \in \mathcal{L}$ we think of ν as the set T and $N\setminus\nu$ as the set S. It follows that the elements of $\mathcal{L}$ of cardinality α, $0 \leq \alpha \leq n$, are all the partitions of $q_1,\ldots,q_n$ of the form $(n;\alpha,n-\alpha)$. For example when P is an antichain, $|\mathcal{F}| = |\mathcal{L}| = 2^n$.

For each element ν in $\mathcal{L}$ we can associate a unique map θ as follows. For $1 \leq i \leq n$ define

$$\theta_\nu(q_i) = \begin{cases} 2 & \text{if } i \in \nu \\ 1 & \text{if } i \notin \nu. \end{cases}$$

Any function $\omega^0 : P \to C_2$ defines a partition $\{S \cup T\}$ by $S = \omega^{0^{-1}}(1)$ and $T = \omega^{0^{-1}}(2)$. Clearly θ_ν is an order-preserving map if $\nu = \{i : q_i = \omega^{0^{-1}}(2)\}$ and $N\setminus\nu = \{i : q_i = \omega^{0^{-1}}(1)\}$ for some ω^0. Now notice that $\mathcal{F}$ is also given by the set of order-preserving maps from P to C_2.

We will show that $\mathcal{F}$ is an equivalent representation of P, that is, given $\mathcal{F}$ we can construct P.

So let $\mathcal{P}$ be the set of all labelled posets with n elements. We can assume that the labels belong to the set N. If ν is a subset of N, with $j \in \nu$ and $i \in N\setminus\nu$, then for some $P \in \mathcal{P}$ either $q_j > q_i$ or $q_j | q_i$ in P.

Let $\mathcal{S}$ be the set of all sets of subsets of N; hence $|\mathcal{S}| = 2^{2^n}$.

Define $f:\mathcal{P} \to \mathcal{S}$ as follows. If $P \in \mathcal{P}$, then $f(P) = \{\nu : \nu \subset N$, for all $i,j \in N$, if $q_j > q_i$ in P then $i \in \nu$ implies $j \in \nu\} = \mathcal{F}$. Since f(P) is a set of subsets of N we know f is well-defined.

Note that f is not surjective. For example, neither $\mathcal{L} \setminus \emptyset$ nor $\mathcal{L} \setminus N$ belongs to $f(\mathcal{P})$. Also if $P = \{q\}$ then $f(P) = \{\emptyset, \{1\}\}$.

<u>Lemma 6.1</u>:

f is injective.

<u>Proof</u>:

Let $P,Q \in \mathcal{P}$ with $P \neq Q$. This means that P,Q differ on some elements $x = q_i$ and $y = q_j$. Without loss of generality assume $x < y$ in P and $x \nless y$ in Q. Let $\nu = \{k : q_k \in Q,\ q_k \in \text{above } \{x\}\} \in f(Q)$. Then $i \in \nu$, $j \notin \nu$ and hence $\nu \notin f(P)$. Therefore $f(P) \neq f(Q)$. □

$\mathcal{F}$ is defined to be contained in $\mathcal{L}$. Further we have:

<u>Claim 6.1</u>:

$\mathcal{F}$ is a sublattice of $\mathcal{L}$.

<u>Proof</u>:

In $\mathcal{L}$ we have $\vee = \cup$ and $\wedge = \cap$. So it suffices to show that $\mathcal{F}$ is closed under union and intersection.

Let $\gamma, \delta \in \mathcal{F}$ with $\nu = \gamma \cup \delta$. Now suppose $\nu \notin \mathcal{F}$, then there exist $i \in \nu$, $j \notin \nu$ while $q_i < q_j$ in P. Without loss of generality assume $i \in \gamma$ and then we have $j \notin \gamma$. This implies $\gamma \notin \mathcal{F}$ giving a contradiction, and thus $\nu \in \mathcal{F}$.

The proof for intersection is similar. □

It is easy to see that a sublattice of a distributive lattice is itself distributive. Thus $\mathcal{F}$ is also distributive.

Lemma 6.2:

Let F *be a sublattice of* $\mathcal{L}$ *containing a maximal chain* C *of* $\mathcal{L}$. *Suppose* $0 < k < n$ *and that* F *contains exactly one element* ν *of cardinality* k. *Then* $F \subset \text{above } \{\nu\} \cup \text{below } \{\nu\}$, *and hence* $|F| < 2^{n-k} + 2^k$.

Proof:

Without loss of generality assume C is $\emptyset \subset \{1\} \subset \{1,2\} \subset \cdots \subset N$. Then $\nu = C_k$. Suppose there exists a $\delta \in F$ with $\delta | \nu$, then we will use the fact that F is closed under union and intersection to get a contradiction.

If $|\delta| > k$ let t be the kth smallest member of δ. We have $C_t \in C$ and $|C_t \cap \delta| = k$. Since $\nu|\delta$ it follows that $C_t \cap \delta \neq \nu$ giving a contradiction.

When $|\delta| < k$ let t be the (n-k+1)th largest member of $N \setminus \delta$. Then $C_t \in C$ and $|C_t \cup \delta| = k$, where $C_t \cup \delta \neq \nu$.

Finally $|\text{above } \{\nu\}| = 2^{n-k}$ and $|\text{below } \{\nu\}| = 2^k$ and ν is a common member of both sets. □

To show that we need the maximal chain C in F above, consider the example where $n = 3$ and $F = \{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}\}$.

We associate a linear extension with each maximal chain C_n belonging to f(P) as follows. Let δ cover γ in C_n, thus for some i, $\delta = \gamma \cup i$. Define $\lambda i = n - |\gamma|$ for each such i, and then $\lambda \in \Lambda(P)$.

<u>Lemma 6.3</u>:

The number of linear extensions of P *is equal to the number of maximal chains in* $\mathcal{F}$.

<u>Proof</u>:

Clearly $f(C_n)$ is a unique maximal chain in $\mathcal{L}$. Suppose λ,μ are distinct linear extensions of P. Then set $\lambda = C$, $\mu = D$ for chains $C,D \in \mathcal{P}$. By Lemma 6.1, $f(C) \neq f(D)$.

Now suppose C and D are distinct maximal chains in $\mathcal{F}$. For some $0 < k < n$ there are different elements $\gamma \in C$, $\delta \in D$ with $|\gamma|$, $|\delta| = k$. Let $\gamma = \alpha \cup i$ and $\delta = \beta \cup j$ for elements α,β, and so $i \neq j$. Then for linear extensions λ,μ we have $\lambda i = \mu j = n-k+1$ and hence $\lambda \neq \mu$. □

Note that not every collection of maximal chains in $\mathcal{L}$ constitutes $f(P)$ for some $P \in \mathcal{P}$. Let $n = 3$ and P be an antichain. Then $\mathcal{F}$ is given by the 2^3 elements of $\mathcal{L}$, whereas the chains $\emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\}$, $\emptyset \subset \{3\} \subset \{2,3\} \subset \{1,2,3\}$ suffice to determine P. That is the order dimension of P is 2.

6.4 A LATTICE INEQUALITY

Lemma 6.4:

If A,B *are subsets of the elements of a distributive lattice* Γ *and* $A|B$ *then*

$$\min\{|A|,|B|\} \leq |\Gamma|/4.$$

Proof:

Suppose $|\Gamma|/4 < |A|, |B|$. Then notice that $A|B$ is equivalent to the condition that $A, B, A \vee B, A \wedge B$ are disjoint in Γ. By disjointness $|A| + |B| + |A \vee B| + |A \wedge B| \leq |\Gamma|$ and so $|A \vee B| + |A \wedge B| < |\Gamma|/2$. According to Theorem 3.3, $(|\Gamma|/4)^2 < |A|\ |B| \leq |A \vee B|\ |A \wedge B|$. Eliminating $|A \wedge B|$ we have $(|\Gamma|/4)^2 < |A \vee B|(|\Gamma|/2 - |A \vee B|)$ which is $(|\Gamma|/4 - |A \vee B|)^2 < 0$ and the lemma is proved. □

It is well known that the down-sets of a finite poset P ordered by inclusion form a distributive lattice Γ. The join and meet in this lattice are given by union and intersection respectively of the down-sets in P. Further every distributive lattice may be formed in this way.

Notice that for each order-preserving map $\omega^0 : P \to C_2$, $\omega^{0^{-1}}(1)$ is a down-set of P. Hence we can construct Γ from $\Omega^0(P, C_2)$. Thus we derive the following specialization of Lemma 6.4.

Lemma 6.5:

If X,Y *are disjoint fixed subsets in* P *and* Ω^0 *is the set of order-preserving maps* $P \to C_2$, *then*

$$\min\{|a|, |b|\} \prec |\Omega^0|/4,$$

where $a = \{\omega^0 \in \Omega^0 : \omega^0 X = 1, \omega^0 Y = 2\},$

$b = \{\omega^0 \in \Omega^0 : \omega^0 X = 2, \omega^0 Y = 1\}.$

Proof:

Put

$c = \{\omega^0 \in \Omega^0 : \omega^0 X = 1, \omega^0 Y = 1\},$

and $d = \{\omega^0 \in \Omega^0 : \omega^0 X = 2, \omega^0 Y = 2\}.$

Now $|\Omega^0|$ is given by the number of down-sets in P. Let

$$A = \{\omega^{0^{-1}}(1) : \omega^0 \in a\} \subset \Gamma \text{ , so } |A| = |a|.$$

Similarly define B,C,D for b,c,d. Clearly A,B,C,D are disjoint and $A \vee B = C$ and $A \wedge B = D$. □

We now give the corresponding results for both strict order-preserving and order-preserving maps using a proof based on our inequalities for monotone functions. So let $M = \Omega^0$ or Ω and $m \in M$.

Lemma 6.6:

If $X = x_1,\ldots,x_j$, $Y = y_1,\ldots,y_k$ *are disjoint fixed subsets in* P *and* $M = M(P,C_2)$, *then*

$$\min\{|m : mX < mY|, |m : mX > mY|\} \prec |M|/4.$$

Proof:

From Theorem 5.2 we have for $\Omega(P,C_2)$,

$$ab = N^S(\underbrace{1,\dots,1}_{j},\underbrace{2,\dots,2}_{k})N^S(\underbrace{2,\dots,2}_{j},\underbrace{1,\dots,1}_{k}) \leq N^S(\underbrace{1,\dots,1}_{j+k})N^S(\underbrace{2,\dots,2}_{j+k}) = cd.$$

Further, by Theorem 5.9 we get for $\Omega^0(P,C_2)$,

$$\alpha\beta = N^0(1,\dots,1,2,\dots,2)N^0(2,\dots,2,1,\dots,1) \leq N^0(1,\dots,1)N^0(2,\dots,2) = \gamma\delta.$$

Now $\quad 4\min\{\alpha,\beta\} = (2\min(\sqrt{\alpha},\sqrt{\beta}))^2$

$\leq (\sqrt{\alpha}+\sqrt{\beta})^2$

$\leq \alpha+\beta+2\sqrt{\gamma\delta}$ (using $\alpha\beta \leq \gamma\delta$)

$\leq \alpha+\beta+\gamma+\delta$ (using geometric mean $\leq$

arithmetic mean) $\quad \leq |\Omega^0|$.

Likewise $4\min\{a,b\} \leq |\Omega|$.

Note that when $j = k = 1$ we can alternatively apply Theorems 4.3, 4.5 here. □

We will require the following corollary of Lemma 6.5 for the lower bound. Using the map θ we get:

<u>Corollary 6.1</u>:

If X,Y *are disjoint fixed subsets in* P *and* $\mathcal{F} = f(P)$, *then*

$$\min\{\alpha,\beta\} \leq |\mathcal{F}|/4,$$

where $\quad \alpha = |\{\nu \in \mathcal{F}: q_i \in X \text{ implies } i \notin \nu,\ q_j \in Y \text{ implies } j \in \nu\}|,$

$\beta = |\{\nu \in \mathcal{F}: q_i \in X \text{ implies } i \in \nu,\ q_j \in Y \text{ implies } j \notin \nu\}|.$

We mention that since $\mathcal{F}$ forms a distributive lattice we are able to prove this directly with the lattice inequality Theorem 3.3.

A motivation for this corollary concerns the partition $\{\frac{j}{i} \cup \frac{i}{j} \cup \frac{ij}{} \cup \frac{}{ij}\}$ of $\mathcal{F}$, according to the elements q_j, q_j, where $\frac{j}{i} = \{\nu \in \mathcal{F} : i \notin \nu,\ j \in \nu\}$ and so on. Theorem 3.3 shows that

$$\left|\frac{j}{i}\right| \left|\frac{i}{j}\right| < \left|\frac{ij}{}\right| \left|\frac{}{ij}\right|.$$

6.5 THE LOWER BOUND TECHNIQUE

We will illustrate the lower bound technique by applying it to demonstrate a bound for $U_{\lfloor tn \rfloor}(n)$.

Let P be the antichain $q_1,\ldots,q_n$ where these elements determine an unknown fixed linear order L. Let $\mathfrak{C}$ be the set of all algorithms for computing the t-partition $(n;\lfloor tn \rfloor, n-\lfloor tn \rfloor)$ of P, where $0 < t < \frac{1}{2}$.

Theorem 6.1:

If T $\in \mathfrak{C}$ *then in the worst case* T *requires at least* $2.4 \lfloor tn \rfloor$ *binary comparisons* $q_i \; ? \; q_j$ *where* $q_i, q_j \in P$.

Proof:

Let an algorithm $T \in \mathfrak{C}$ be chosen. Then T proceeds to perform successive pair-wise comparisons between the elements of P, each result being of the form $q_i < q_j$. Suppose T makes k comparisons. Then the results yield a subposet P_k of L.

Let $\mathcal{F}_k = f(P_k)$. The information $q_i < q_j$ shows that, if $\nu \in \mathcal{L}$ with $i \in \nu$ and $j \notin \nu$ then θ_ν is not order-preserving, and in particular ν is not the required unique element corresponding to the t-partition.

After any comparison we are interested only in those $\nu \in \mathcal{L}$ which respect the resulting subposet. So each of the k comparisons in effect deletes from $\mathcal{L}$ precisely those elements for which θ_ν is not order-preserving. The remaining elements define the subset $\mathcal{F}_k$ of $\mathcal{L}$.

From Lemma 6.1 and the definition of θ_ν we deduce that $\mathcal{F}_k$ is exactly the set $\Omega^0(P_k, C_2)$. Further by Claim 6.1 we have that $\mathcal{F}_k$ is a sublattice of $\mathcal{L}$.

Now L is $q_{\pi(1)} < q_{\pi(2)} < \cdots < q_{\pi(n)}$ for some permutation $\pi(1), \pi(2), \ldots, \pi(n)$ of N. This means that for $1 \leq i \leq n+1$ the map $\theta_{\{\pi(i),\pi(i+1),\ldots,\pi(n)\}}$ is order-preserving, and hence the maximal chain $\emptyset \subset \{\pi(n)\} \subset \ldots \subset \{\pi(1),\ldots,\pi(n)\}$ is not deleted from $\mathcal{L}$. Thus there exists at least one maximal chain of $\mathcal{L}$ in $\mathcal{F}_k$.

Suppose that after k comparisons we have determined the t-partition. Since it is unique we have deleted all but one element, ν say, of cardinality $\lfloor tn \rfloor$ from $\mathcal{L}$. Setting $\alpha = \lfloor tn \rfloor$ it now follows by Lemma 6.2 that $|\mathcal{F}_k| < 2^{n-\alpha} + 2^{\alpha}$. In the worst case $\theta_\nu^{-1}(1)$ corresponds to an antichain in P, and hence $|\text{above } \{\nu\}| = 2^{n-\alpha}$, and likewise for $\theta_\nu^{-1}(2)$.

<u>Claim 6.2:</u>

For any such comparison algorithm there is a sequence of worst case outcomes such that when the algorithm halts we have

$$|\mathcal{F}_k| \geq (\tfrac{3}{4})^k 2^n.$$

Assume for the moment that Claim 6.2 is true. Then

$$(\tfrac{3}{4})^k 2^n \leq |\mathcal{F}_k| < 2^{n-\alpha} + 2^{\alpha} \leq 2^{n-\alpha+1},$$

and $\qquad (\alpha-1)\log_{4/3}2 < k.$

Hence $\qquad \lfloor tn \rfloor 2.4 \lesssim k.$

It remains to prove Claim 6.2. The case k = 0 is trivial so we proceed with induction on k. Making a new (k+1)th comparison between q_i and q_j say, will delete from $\mathcal{F}_k$ either

$$A = \{\nu \in \mathcal{F}_k : i \notin \nu,\ j \in \nu\}$$

or

$$B = \{\nu \in \mathcal{F}_k : i \in \nu,\ j \notin \nu\},$$

where clearly $A|B$.

Without loss of generality assume $|A| < |B|$. From Corollary 6.1 we have that $|A| < |\mathcal{F}_k|/4$ and therefore in the worst case $\frac{3}{4}|\mathcal{F}_k| < |\mathcal{F}_{k+1}|$.

The claim follows and the proof is complete. □

Suppose the posets P,Q satisfy $P < Q$. Then to use this lower bound technique to derive the cost of producing Q from P, it is required to solve

$$(\tfrac{3}{4})^k \ |\Omega^0(P,C_2)| < |\Omega^0(Q,C_2)|.$$

From the quantity $\lceil \log_2 \binom{n}{\lfloor tn \rfloor} \rceil$, information theory and the entropy function yield a lower bound of at most n, where the case n corresponds to the equi-partition problem.

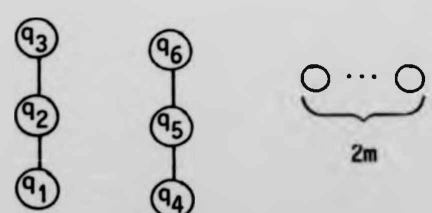
For the equi-partition problem it would seem natural to modify our counting technique so as to consider only the equator elements of $\mathcal{L}$, namely the set $\{\nu \in \mathcal{L} : |\nu| = n/2\}$. The advantage in the representation of $\mathcal{E} = \{\nu \in \mathcal{F} : |\nu| = n/2\}$, is that we commence with $\binom{n}{n/2}$ elements, which using Stirling's approximation is about 2^n, and always halt with only one equator element existing. However, with $\mathcal{F}$ we proceed to reduce the number of given elements, 2^n, until completion when there may be roughly as many as $2^{n/2}$ elements remaining. With the help of an analogue of Lemma 6.4 for the equator, the number of comparisons required via $\mathcal{E}$ must therefore yield a better lower bound.

Suppose that the first comparison of an equi-partition algorithm shows $q_i < q_j$. Then we would eliminate all equator elements having $i > j$, which amounts to the proportion $\binom{n-2}{\frac{n-2}{2}} / \binom{n}{\frac{n}{2}} = \frac{n}{4(n-1)}$. As n increases this proportion for the first comparison tends to $\frac{1}{4}$. Simple examples show that as the algorithm approaches finding the equi-partition, or similarly the median, a last (k+1)th comparison will delete $\frac{1}{2}|\mathcal{E}_k|$, where $\mathcal{E}_k = \{\nu \in \mathcal{F}_k : |\nu| = \frac{n}{2}\}$.

Clearly, the probability of an element in $\mathcal{F}$ being deleted increases the closer the element is to the equator. We may therefore expect that until the algorithm gets fairly close to halting, at each stage in the worst case we have $|\mathcal{E}_{k+1}| \gtrsim \frac{3}{4} |\mathcal{E}_k|$. However, we give the following counterexample due to A.C. Yao [Y].

Suppose that a comparison algorithm at some initial stage has produced the poset in:

Example 6.1:



P

We can define a partition of $\mathcal{E}(P)$ according to these two chains C_3, as described below.

$\frac{q_4q_5q_6}{}$	$\binom{2m}{m}$	$\binom{2m}{m-1}$	$\binom{2m}{m-2}$	$\binom{2m}{m-3}$
$\frac{q_5q_6}{q_4}$	$\binom{2m}{m+1}$	$\binom{2m}{m}$	$\binom{2m}{m-1}$	$\binom{2m}{m-2}$
$\frac{q_6}{q_4q_5}$	$\binom{2m}{m+2}$	$\binom{2m}{m+1}$	$\binom{2m}{m}$	$\binom{2m}{m-1}$
$\frac{}{q_4q_5q_6}$	$\binom{2m}{m+3}$	$\binom{2m}{m+2}$	$\binom{2m}{m+1}$	$\binom{2m}{m}$
	$\frac{}{q_1q_2q_3}$	$\frac{q_3}{q_1q_2}$	$\frac{q_2q_3}{q_1}$	$\frac{q_1q_2q_3}{}$

Figure 6.2

In Figure 6.2 the binomial coefficient at $(\frac{q_3}{q_1q_2}, \frac{q_5q_6}{q_4})$,

for example, determines $|\{\nu \in E: \{3,5,6\} \subset \nu, \{1,2,4\} \not\subset \nu\}|$.
Also $N_1 = 2\binom{2m}{m} + 2\binom{2m}{m+1}$, and similarly for N_2, M.

Now suppose the next comparison by the algorithm is q_2 ? q_5. The result $q_2 > q_5$ causes the set of partitions associated with N_1 in Figure 6.2 to be deleted from E; the result $q_2 < q_5$ causes those with N_2 to be deleted. Clearly in either case M does not get reduced.

We claim to show that this comparison has deleted more than $\frac{1}{4}|E|$. Now $|E| = 2M + 2N'$, where $N' = N_1 = N_2$. Using the unimodality of the binomial coefficients, as indicated in Figure 6.2, we deduce that $N' > M$. Hence it follows that $N' > \frac{1}{4}|E| = \frac{M}{2} + \frac{N'}{2}$.

Notice that as the algorithm proceeds, if the two chains C_3 in Example 6.1 increase, then so do the numbers of rows and columns in Figure 6.2. When comparing the middle elements of the longer chains, the counterexample still holds by the monotonicity in the sequence of the binomial coefficients. This ends Chapter 6.

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