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Super spin chain coherent state actions and $AdS_5 \times S^5$ superstring

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Abstract

We consider a generalization of leading-order matching of coherent state actions for semiclassical states on the super Yang-Mills and superstring sides of the AdS/CFT duality to sectors with fermions. In particular, we discuss the $SU(1|1)$ and $SU(2|3)$ sectors containing states with angular momentum J in S^5 and spin in AdS_5. On the SYM side, we start with the dilatation operator in the $SU(2|3)$ sector having super spin chain Hamiltonian interpretation and derive the corresponding coherent state action which is quartic in fermions. This action has essentially the same "Landau-Lifshitz" form as the action in the bosonic $SU(3)$ sector with the target space CP^2 replaced by the projective superspace $CP^{2|2}$. We also discuss the complete $PSU(2,2|4)$ one-loop SYM spin chain coherent state sigma model action. We then attempt to relate it to the corresponding truncation of the full $AdS_5 \times S^5$ superstring action written in a light-cone gauge where it has simple quartic fermionic structure. In particular, we find that part of the superstring action describing $SU(1|1)$ sector reduces to an action of a massive 2d relativistic fermion, with the expansion in the effective coupling $\tilde{\lambda} = \frac{\lambda}{J^2}$ being equivalent to a non-relativistic expansion.

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1 Introduction

Recent progress in understanding the AdS/CFT duality beyond the supergravity sector was inspired by the suggestion to consider a subsector of semiclassical string states (and near-by fluctuations) that carry large quantum numbers [1, 2, 3, 4] and by the relation between the $\mathcal{N} = 4$ super Yang-Mills dilatation operator and integrable spin chains [5, 6, 7] (for reviews and further references see, e.g., [9, 10, 11]).

Here we shall concentrate on a particular approach (suggested in [12] and developed in [13, 14, 15, 16, 17, 18, 19, 20, 21]) to comparing gauge theory (spin chain) and string theory sides of the duality. It is based on considering a low-energy effective action for coherent states of the ferromagnetic spin chain and relating it to a "fast-motion" limit of the string action. In addition to explaining how a limit of string action "emerges" from the gauge theory dilatation operator, this approach also clarifies the identification of states on the two sides of the duality [17, 21] as well as matching the integrable structures. For example, in the $SU(3)$ sector containing states corresponding to operators $\mathrm{tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3})$ built out of 3 chiral combinations of $\mathcal{N} = 4$ SYM scalars one finds from the 1-loop spin chain Hamiltonian [5] the "Landau-Lifshitz" type action for coherent states defined on CP^2, and an equivalent action comes out (in the large J limit) of the bosonic part of the classical superstring action [14, 15].

The motivation behind the present work is to try to generalize previous discussions of the matching of coherent state actions in bosonic sectors to sectors with fermions. Previous interesting work in this direction appeared in [19, 20] where quadratic order in fermions was considered. The full one-loop SYM dilatation operator [22] is a Hamiltonian of the $PSU(2, 2|4)$ super spin chain [7], and it is of obvious importance to understand in general a relation between the corresponding coherent state action and the $AdS_5 \times S^5$ superstring action [8] based on the $PSU(2, 2|4)/[SO(1, 4) \times SO(5)]$ supercoset. That may help to further clarify the structure of superstring theory in this background and its connection to SYM theory.

Let us note that, in general, one should be comparing quantum gauge theory states on $R\times S^3$ to quantum string theory states in (global) $AdS_5 \times S^5$. In the "semiclassical" limit of large quantum numbers it is natural to consider coherent states on both sides of the duality. In the presence of fermions one cannot follow the bosonic pattern and directly compare classical string solutions to spin chain configurations: the classical solutions will dependent on Grassmann parameters and their energy and charges will be even elements of a Grassmann algebra.[1] To give an interpretation to such solutions one would need to assign some expectation values to the Grassmann elements so that they approximate the results found for the corresponding quantum states with some fermionic occupation numbers.

In order to by-pass this complication one may compare not the states/solutions

[1] Related issues were recently discussed in [23, 24].

but semiclassical effective actions with fermions that appear in the relevant limits both on the spin chain side and the string theory side. Indeed, one may reformulate the spin-chain dynamics in terms of a coherent-state path integral and then compare the fermionic action that appears there in a continuum limit (and describing a particular class of "semiclassical" states) to a limit of superstring action appearing in the string path integral. Such a comparison of fermionic effective actions is what we will be aiming at below, but we will also discuss some of their Grassmann-valued classical solutions.

On the SYM or spin chain side, we shall concentrate on the closed $SU(2|3)$ sector [25] which generalizes the scalar $SU(3)$ sector to include in the single-trace operators powers of two "gluino" fermionic components. We shall systematically derive the corresponding (1-loop) coherent state action which has a natural interpretation as a Landau-Lifshitz sigma model on the projective superspace $CP^{2|2}$ (an equivalent action was found independently in [20]).[2] On the string theory side, we shall start with an explicit form of the $AdS_5 \times S^5$ action in the light-cone κ-symmetry gauge of [27, 28]. This action is at most quartic in fermions and has manifest $SU(4)$ symmetry. We shall discuss how to truncate this action to the $SU(2|3)$ sector by first writing it in the $SU(3) \times U(1)$ invariant form and then isolating the singlet fermionic sector. Understanding the issue of consistent truncation to the $SU(2|3)$ sector and also attempting to including quartic fermionic terms are novel elements of the present work. The precise matching of the quartic fermionic terms appears to depend on a particular choice of field redefinitions that we did not succeed in finding.

To motivate the required truncation of the superstring action let us recall the contents of the $SU(2|3)$ sector on the SYM side [25, 29]. Starting with the $\mathcal{N} = 4$ SYM theory written in terms of the $\mathcal{N} = 1$ superfields we may consider the operator O=tr$(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3}\psi_1^{K_1}\psi_2^{K_2})$ built out of three chiral scalars of the "matter" supermultiplets and two spinor components of the "gaugino" supermultiplet ($W_\alpha = \psi_\alpha + ...$, $\alpha = 1, 2$). Then we will have the $SU(3) \subset SO(6)$ R-symmetry acting on the scalars (under which the fermions ψ_α are singlets) and the $SU(2) \subset SU(2, 2)$ symmetry acting on the fermions (under which the scalars are singlets). The latter symmetry is essentially the Lorentz spin symmetry, and ψ_1 may be thought of as a "spin-up", and ψ_2 as a "spin-down" state. The above operator O has canonical dimension $\Delta_0 = J + \frac{3}{2}(K_1 + K_2)$, with $J = J_1 + J_2 + J_3$ being the total R-charge. Then $S = \frac{1}{2}(K_1 - K_2)$ is the Lorentz spin and $L = J + K_1 + K_2$ is the total number of fields or the length of the corresponding spin chain.[3] One may also consider various subsectors of the $SU(2|3)$ sector, for example, $SU(1|3)$ (3 scalars and 1 fermion). The simplest subsector (which is closed to all orders [25, 11]) is the $SU(1|1)$ subsector containing the operators tr$(\Phi^J\psi^K)$ with $\Delta_0 = J + \frac{3}{2}K = L + S,\ L = J + K,\ S = \frac{1}{2}K$,

[2]See also a related discussion in the very recent paper which studies the $SU(1, 1|1)$ sector [26] and its relation to superstring action to quadratic order in fermions.

[3]Beyond one loop the length can fluctuate as one can trade a scalar $SU(3)$ singlet $\epsilon^{ijk}\Phi_i\Phi_j\Phi_k$ for the $SU(2)$ singlet $\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta$ which has the same canonical dimension [25].

with the $K = 0$ case being the BPS vacuum.

The corresponding string states should thus have both the S^5 angular momenta (carried by the bosonic coordinates) and one component of the AdS_5 spin (carried by the fermionic coordinates). Also, the two non-zero fermionic coordinates should be singlets under the $SU(3)$ R-symmetry. In particular, the closed $SU(1|1)$ sector should be described by an "extension" of the BMN point-like BPS state (carrying S^5 momentum J) by a single fermion. The associated coherent state action will then involve only a single fermionic variable.

We start in section 2 by deriving the coherent-state action corresponding to the 1-loop SYM dilatation operator in the $SU(2|3)$ sector. We emphasize its geometrical interpretation as a Landau-Lifshitz sigma model on the projective superspace $CP^{2|2}$ and show that there exists a field redefinition that makes the action quartic in fermions. We mention then a particular fermionic classical solution which generalizes a static bosonic $SU(2)$ Landau-Lifshitz solution (corresponding to a circular spinning string with two equal angular momenta [4, 30]). We also discuss the generalization to the full $PSU(2,2|4)$ spin chain coherent state sigma model action.

In section 3 we consider the $AdS_5 \times S^5$ superstring action [8] in the light-cone κ-symmetry gauge [27, 28] which contains two fermionic coordinates transforming in the fundamental representation of $SU(4)$. We choose an ansatz for the AdS_5 bosonic coordinates that describes strings localised at the center of AdS_5 in global coordinates, with the global time proportional to the world-sheet time. Then we rewrite the fermionic part of the action in the manifestly $SU(3) \times U(1)$ form that no longer involves gamma-matrices. That facilitates the truncation to the $SU(2|3)$ sector where only two $SU(3)$-singlet fermionic coordinates are present.

In section 4 we present some classical solutions of the superstring action that are fermionic generalizations of the bosonic spinning string solutions. That helps to clarify possible consistent truncations of the superstring equations of motion.

In section 5 we consider the matching of the Landau-Lifshitz spin chain action to the "fast-string" limit of the string action. In particular, we consider the $SU(1|1)$ subsector and relate the resulting fermionic action to that of a free *relativistic* 2d fermion.

In Appendix A we summarize our notation and give useful gamma-matrix relations. In Appendix B we present the expressions for $SU(4)$ charges of the string action.

2 From spin chains to sigma models: $SU(2|3)$ sector

In this section we shall find the continuum limit of the coherent state expectation value of the one-loop $\mathcal{N}$=4 SYM dilatation operator in the $SU(m|n)$ sub-sector of the full

$SU(2,2|4)$ dilatation operator. In our choice of the fundamental representation of $SU(n|m) = SU(m|n)$ $n = 0, 1, 2$ will be the number of chiral fermionic (grassmann) fields and $m = 1, 2, 3$ – the number of chiral scalar bosonic fields in the corresponding SYM single-trace operators [25].[4]

Our aim will be to determine the structure of the associated low-energy effective actions for the coherent state fields. The cases of the purely bosonic ($n = 0$) $SU(2)$ and $SU(3)$ sectors were discussed previously in [12, 13] and in [14, 15]. Related supergroup-type sigma models were considered, e.g., in [39]. Our final result for the coherent state action in the $SU(2|3)$ subsector will be the same as in [20] but we shall emphasize the simple geometrical structure of the action. We shall also mention some classical fermionic solutions and a generalization to the $PSU(2,2|4)$ case (see also [19]).

2.1 Coherent state expectation value of $SU(2|3)$ dilatation operator

The starting point will be the one-loop dilatation operator in the $SU(2|3)$ sector which can be put into the form of a spin chain Hamiltonian [25]

$$D = \frac{2\lambda}{(4\pi)^2} \sum_{l=1}^{L} (1 - P_{l,l+1}) \ . \tag{2.1}$$

Here $P_{l,l+1}$ is the graded permutation operator, which acts by permuting a fermion or boson assigned to a site l with a fermion or boson assigned to a site $l+1$ with an additional minus sign if both fields are fermionic. The key observation is that the permutation operator in the $SU(m|n)$ sector can be expressed in terms of the $SU(m|n)$ generators as [5]

$$P_{l,l+1} = \frac{1}{m-n} + \sum_{A,B} g_{AB} X_l^A X_{l+1}^B \,, \tag{2.2}$$

[4] In this section we shall often use the notation $SU(3|2)$ instead of $SU(2|3)$ used in [25]. While the notation $SU(2|3)$ (with $SU(2) \times SU(3)$ subgroups being the space-time spin acting on fermions and internal R-symmetry acting on bosons) is natural for a subgroup of the full symmetry supergroup $PSU(2,2|4)$ (with $SU(2,2) \times SU(4)$ bosonic subgroup being the product of the space-time conformal symmetry and the internal R-symmetry), the "reverse" notation $SU(3|2)$ seems more natural in discussing the coset superspaces we will be interested below. We will choose the fundamental representation of $SU(m|n)$ to contain m "bosons" and n "fermions". The superalgebra $SU(m|n)$ can be realised in fundamental representation as a set of $(m+n) \times (m+n)$ matrices $M = \begin{pmatrix} B & F \\ F' & B' \end{pmatrix}$, where the even $m \times m$ matrix B and $n \times n$ matrix B' are hermitian, with $\mathrm{Str} M = \mathrm{tr} B - \mathrm{tr} B' = 0$, and the odd matrices satisfy $F^\dagger = F'$.

[5] When proving this identity it is important to recall the definition of the tensor product on a super-vector space: $X^A \otimes X^f (v_f \otimes v_B) = -X^A(v_f) \otimes X^f(v_B)$, where X^f (or v_f) is a fermionic operator (or vector) and X^A (v_B) is any bosonic or fermionic operator (or vector).

where g_{AB} is the Cartan metric on the Lie superalgebra, i.e. the inverse of

$$g^{AB} = \mathrm{Str}(X^A X^B)\,. \tag{2.3}$$

For example, for the $SU(2)$ case ($m = 2,\ n = 0$) with X^A being the Pauli matrices $g_{AB} = \frac{1}{2}\delta_{AB}$ and $P_{l,l+1} = \frac{1}{2}(I + \sigma_l \cdot \sigma_{l+1})$.

Then the dilatation operator in the $SU(m|n)$ sector is given explicitly by

$$D = \frac{2\lambda}{(4\pi)^2} \sum_{l=1}^{L} \left(\frac{m-n-1}{m-n} - \sum_{A,B} g_{AB} X_l^A X_{l+1}^B \right)\,. \tag{2.4}$$

Let us first consider the simplest non-trivial $SU(2|1)$ sector where

$$D_{SU(2|1)} = -\frac{2\lambda}{(4\pi)^2} \sum_{l=1}^{L} \sum_{A,B} g_{AB} X_l^A X_{l+1}^B\,, \tag{2.5}$$

and derive the corresponding coherent state effective action (generalisations to other $SU(m|n)$ sectors will be straightforward). We may choose the generators of $SU(2|1)$ in the fundamental representation as

$$X^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$
$$X^5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ X^6 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},\ X^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\ X^8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix},$$

where $X^{1,2,3,4,8}$ are even and $X^{5,6,7}$ are odd.[6] X^3 and X^4 form Cartan subalgebra, and X^1, X^2 and X^3 form the bosonic $SU(2)$ subgroup. As usual, we can define a coherent state by a "rotation" of a "vacuum state" by the generators that do not preserve it [37]

$$|N\rangle \equiv \mathcal{N} e^{i(a_1 X^1 + a_2 X^2 + \theta_1 X^5 + \theta_2 X^6)}\,|0\rangle\,. \tag{2.6}$$

Here a_1, a_2 and θ_1, θ_2 are real even and odd parameters and $\mathcal{N}$ is a (Grassmann-even) normalisation.[7] We shall choose the vacuum state to be $|0\rangle = (1, 0, 0)$ which is an eigen-state of the Cartan generators (and is annihilated by X^7, X^8) – it corresponds to the BPS vacuum $\mathrm{Tr}(\Phi^L)$. The coherent states are thus parametrized by the elements of the supercoset G/H where H is the stability subgroup of the vacuum, i.e. by the

[6] We take the even (b) and odd (f) generators to satisfy $X^\dagger_{(b)} = X_{(b)}\,,\quad X^\dagger_{(f)} = -X_{(f)}\,.$

[7] Let us recall that in the case of a free fermionic oscillator (or Clifford algebra) the fermionic coherent states may be defined as $|\theta\rangle = e^{-\theta a^\dagger}\,|0\rangle\,,\ a\,|0\rangle = 0,\ aa^\dagger + a^\dagger a = 1,\ a\,|\theta\rangle = \theta\,|\theta\rangle\,.$

points in the projective superspace $CP^{1|1} = SU(2|1)/[SU(1|1)\times U(1)]$.[8] Similarly, we define

$$\langle N| \equiv \mathcal{N}^* \langle 0| e^{-i(aX^1+bX^2-\theta_1 X^5-\theta_2 X^6)} . \tag{2.7}$$

In order to satisfy $\langle N|N\rangle = 1$ we require

$$\mathcal{N} = \mathcal{N}^* = 1 - \frac{\sin 2\Delta}{\Delta}\theta^2 , \tag{2.8}$$

where

$$\Delta \equiv \sqrt{a_1^2+a_2^2} , \qquad \theta^2 \equiv \theta\bar\theta , \qquad \theta = \theta_1 + i\theta_2 , \qquad \bar\theta = \theta_1 - i\theta_2 , \tag{2.9}$$

Then

$$\langle N| X^A |N\rangle = x^A , \qquad\qquad x^A = x^A(a_1,a_2,\theta_1,\theta_2) , \tag{2.10}$$

where the explicit form of the functions x^A is

$$\begin{aligned}
x^1 &= \cos\varphi\left[-\sin 2\Delta + \frac{\theta^2}{\Delta}(\cos 2\Delta + \frac{\sin 2\Delta(4\sin^2\Delta - 1)}{2\Delta})\right] , \\
x^2 &= -\sin\varphi\left[-\sin 2\Delta + \frac{\theta^2}{\Delta}(\cos 2\Delta + \frac{\sin 2\Delta(4\sin^2\Delta - 1)}{2\Delta})\right] , \\
x^3 &= \cos 2\Delta + \frac{\theta^2}{\Delta}(\sin 2\Delta - \frac{\sin\Delta\sin 3\Delta}{\Delta}) , \qquad x^4 = 1 + \frac{\theta^2}{\Delta^2}\sin^2\Delta , \\
x^5 &= \frac{\sin 2\Delta}{2\Delta}\left(\bar\theta - \theta\right) , \qquad x^6 = i\frac{\sin 2\Delta}{2\Delta}\left(\bar\theta + \theta\right) , \qquad x^7 = \frac{\sin^2\Delta}{\Delta}\left(\theta e^{-i\varphi} - \bar\theta e^{i\varphi}\right) , \\
x^8 &= -i\frac{\sin^2\Delta}{\Delta}\left(\theta e^{-i\varphi} + \bar\theta e^{i\varphi}\right) , \qquad e^{i\varphi} \equiv \frac{a_2 + i a_1}{\Delta}
\end{aligned} \tag{2.11}$$

Let us define the $SU(2|1)$ matrix N belonging to the supercoset $SU(2|1)/[SU(1|1)\times U(1)]$ as

$$N = \sum_{A,B=1}^{8} g_{AB}\, x^A X^B , \qquad\qquad x^A = \mathrm{Str}(N X^A) . \tag{2.12}$$

It is then easy to show that N can be written as

$$N^q{}_p = \mathbf{V}^q \mathbf{V}_p - \delta^q_p , \qquad\qquad p,q = 1,2,3 , \tag{2.13}$$

where

$$\mathbf{V}^p = (V_1, V_2, \psi) , \qquad \mathbf{V}_p \equiv (\mathbf{V}^p)^\dagger = (V_1^*, V_2^*, \bar\psi) , \qquad \mathbf{V}_p\mathbf{V}^p = \mathbf{V}^p\mathbf{V}_p = 1 , \tag{2.14}$$

[8]Had we chosen instead the "fermionic" vacuum $(0,0,1)$ (which corresponds to a non-BPS state $\mathrm{Tr}(\psi^L)$) we would get the coset $SU(2|1)/[SU(2)\times U(1)]$.

and thus

$$N^\dagger = N\ , \qquad \mathrm{Str}N = 0\ , \qquad N^2 = -N\ . \tag{2.15}$$

It is assumed that the ${}_p{}^p$ summation (as in $A_p B^p$) is done with plus sign for the fermionic components, while the ${}^p{}_p$ summation (as in $B^p A_p$) – with minus sign (so it is consistent with the definition of the supertrace). The explicit form of the Grassmann-valued constraint on $\mathbf{V}^p$ is thus

$$|V_1|^2 + |V_2|^2 + \bar{\psi}\psi = 1\,. \tag{2.16}$$

Both N and $\mathbf{V}^p$ (the latter modulo $U(1)$ phase transformations) thus parametrise the supercoset $CP^{1|1} = SU(2|1)/[SU(1|1)\times U(1)]$. The components of $\mathbf{V}^p$ can be expressed in terms of Δ, φ and θ in (2.11) as

$$V_1 = -\cos\Delta - \frac{\theta^2}{2\Delta^2}\sin\Delta(\Delta - \sin 2\Delta)\,, \tag{2.17}$$

$$V_2 = e^{i\varphi}\Big[\sin\Delta + \frac{\theta^2}{4\Delta^2}(\sin 3\Delta - \sin\Delta - 2\Delta\cos\Delta)\Big]\ , \tag{2.18}$$

$$\psi = \frac{\sin\Delta}{\Delta}\theta\,. \tag{2.19}$$

The above construction is straightforward to generalise to the $SU(m|n)$ case where the matrix N should belong to (CP^{m-1} in the bosonic $n=0$ case [15])

$$CP^{m-1|n} = \frac{SU(m|n)}{SU(m-1|n)\times U(1)}$$

i.e.

$$N^q{}_p = (m-n)\mathbf{V}^q\mathbf{V}_p - \delta^q_p\ , \qquad \mathbf{V}_p\mathbf{V}^p = 1\ , \tag{2.20}$$

$$N^\dagger = N\ , \quad \mathrm{Str}N = 0\ , \qquad N^2 = (m-n-2)N + (m-n-1)I\,. \tag{2.21}$$

The components of $\mathbf{V}^p$ are V_i ($i = 1,...,m$) and ψ_α ($\alpha = 1,...,n$) with $\bar{\psi}_\alpha \equiv \psi^\dagger_\alpha$

$$\mathbf{V}^p = (V_i, \psi_\alpha)\ , \qquad V_i^* V_i + \bar{\psi}_\alpha\psi_\alpha = 1\ , \tag{2.22}$$

where we assume summation over repeated i and α index.

Returning back to $SU(2|1)$ case let us now define the coherent state for the whole spin chain as

$$|N\rangle \equiv \prod_{l=1}^{L} |N_l\rangle\ , \tag{2.23}$$

where $|N_l\rangle$ are given by (2.6). Computing the matrix element of (2.5) we get

$$\begin{aligned}
\langle N| D_{SU(2|1)} |N\rangle &= -\frac{\lambda}{(4\pi)^2}\sum_{l=1}^{L} g_{AB} \ \mathrm{Str}(N_l X^A) \ \mathrm{Str}(X^B N_{l+1}) \\
&= \frac{\lambda}{(4\pi)^2}\sum_{l=1}^{L} \frac{1}{2} g_{AB} \ \mathrm{Str}[(N_{l+1}-N_l)X^A] \ \mathrm{Str}[(N_{l+1}-N_l)X^B] \\
&= \frac{\lambda}{(4\pi)^2}\sum_{l=1}^{L} \mathrm{Str}(N_{l+1}-N_l)^2 \ . \qquad (2.24)
\end{aligned}$$

We used the completeness relation

$$\sum_{A,B} g_{AB} \ \mathrm{Str}(MX^A) \ \mathrm{Str}(X^B M) = 2 \ \mathrm{Str} M^2 \,, \qquad (2.25)$$

valid for any matrix M in the $SU(2|1)$ superalgebra and also that $\mathrm{Str}N_l^2 = -\mathrm{Str}N_l = 0$. Then taking the relevant [12, 13, 15] continuum limit describing semiclassical low-energy states of the spin chain, i.e. $L \to \infty$ with $\tilde\lambda \equiv \frac{\lambda}{L^2}$=fixed, we get

$$\langle N| D_{SU(2|1)} |N\rangle \quad \to \quad L\int_0^{2\pi} \frac{d\sigma}{2\pi} \ \frac{\tilde\lambda}{4} \ \mathrm{Str}(\partial_1 N \partial_1 N) \,. \qquad (2.26)$$

Rescaling $t \to \mathrm{t} = \tilde\lambda^{-1} t$, the total coherent state path integral action becomes

$$I = L \int d\mathrm{t} \int_0^{2\pi} \frac{d\sigma}{2\pi} \ \mathcal{L}_{SU(2|1)} \ , \qquad \mathcal{L}_{SU(2|1)} = \mathcal{L}_{\mathrm{WZ}}(N) - \frac{1}{4}\mathrm{Str}(\partial_1 N \partial_1 N) \,. \qquad (2.27)$$

Here $\mathcal{L}_{\mathrm{WZ}}(N)$ is the usual WZ type term (which can be computed as $\langle N| i\partial_0 |N\rangle$)

$$\mathcal{L}_{\mathrm{WZ}}(N) = \frac{i}{2}\int_0^1 dz \ \mathrm{Str}\left(N[\partial_z N, \partial_0 N\}\right) \,, \qquad (2.28)$$

where $[\ ,\ \}$ is the superbracket.

2.2 $SU(2|3)$ Landau-Lifshitz sigma model

The Lagrangian in (2.27) admits a simpler local representation in terms of the vector variable $\mathbf{V}^p$ with an additional $U(1)$ gauge symmetry which is a direct generalization of the CP^{m-1} "Landau-Lifshitz" Lagrangians in the bosonic $SU(2)$ and $SU(3)$ cases in the form given in [13, 15]:

$$\mathcal{L} = -iU_i^*\partial_0 U_i - \frac{1}{2}|D_1 U_i|^2 \ , \qquad |U_i|^2 = 1 \ , \qquad (2.29)$$

$$D_aU_i = \partial_aU_i - iC_aU_i\ , \qquad C_a = -iU_i^*\partial_aU_i\ , \qquad U_i^*D_aU_i = 0\ . \tag{2.30}$$

Here U_i $(i = 1, ..., m)$ belongs to $CP^{m-1} = SU(m)/[SU(m-1)\times U(1)]$: in addition to the unit modulus constraint the action has gauge $U(1)$ symmetry. In the supercoset case we get the Lagrangian (2.27) defined on the projective superspace (cf. (2.20))[9] $CP^{m-1|n} = SU(m|n)/[SU(m-1|n) \times U(1)]$ (here $\mathbf{V}_p = (\mathbf{V}^p)^\dagger,\ \mathbf{D}_1\mathbf{V}_p = (\mathbf{D}_1\mathbf{V}^p)^\dagger$)

$$\mathcal{L} = -i\mathbf{V}_p\partial_0\mathbf{V}^p - \frac{1}{2}\mathbf{D}_1\mathbf{V}_p\mathbf{D}_1\mathbf{V}^p\ , \qquad \mathbf{V}_p\mathbf{V}^p = 1\ , \tag{2.31}$$

$$\mathbf{D}_a\mathbf{V}^p = \partial_a\mathbf{V}^p - i\mathbf{C}_a\mathbf{V}^p\ , \qquad \mathbf{C}_a = -i\mathbf{V}_p\partial_a\mathbf{V}^p\ . \tag{2.32}$$

Written explicitly in terms of the component fields in (2.22) this becomes

$$\begin{aligned}\mathcal{L} &= -iV_i^*\partial_0V_i - i\bar\psi_\alpha\partial_0\psi_\alpha - \frac{1}{2}\left(|\mathcal{D}_1V_i|^2 + \mathcal{D}_1^*\bar\psi_\alpha\mathcal{D}_1\psi_\alpha\right)\\ &= -iV_i^*\partial_0V_i - i\bar\psi_\alpha\partial_0\psi_\alpha - \frac{1}{2}\left(|\partial_1V_i|^2 + \partial_1\bar\psi_\alpha\partial_1\psi_\alpha - \mathbf{C}_1^2\right)\ ,\end{aligned} \tag{2.33}$$

where

$$\mathcal{D}_a(V_i,\psi_\alpha) = (\partial_a - i\mathbf{C}_a)(V_i,\psi_\alpha)\ , \qquad \mathbf{C}_a = -iV_i^*\partial_aV_i - i\bar\psi_\alpha\partial_a\psi_\alpha\,. \tag{2.34}$$

It is convenient to decouple bosons and fermions in the constraint (2.22). Let us first consider the $SU(2|1)$ sector and define the new bosonic field U_i $(i = 1, 2)$ by

$$V_i = U_i(1 - \frac{1}{2}\bar\psi\psi)\,, \qquad U_i = V_i(1 + \frac{1}{2}\bar\psi\psi)\,. \tag{2.35}$$

Then the normalisation condition (2.16) becomes simply

$$|U_1|^2 + |U_2|^2 = 1\,, \tag{2.36}$$

so that U_i belongs to CP^1 (both U_i and ψ still transform under the $U(1)$ gauge symmetry). The gauge field in (2.34) is then

$$\mathbf{C}_a = C_a(1 - \bar\psi\psi) - \frac{i}{2}(\bar\psi\partial_a\psi + \psi\partial_a\bar\psi)\,, \qquad C_a = -iU_i^*\partial_aU_i\,. \tag{2.37}$$

and so the $SU(2|1)$ Lagrangian (2.33) takes the form

$$\mathcal{L}_{SU(2|1)} = -iU_i^*\partial_0U_i - i\bar\psi D_0\psi - \frac{1}{2}\left[(1-\bar\psi\psi)|D_1U_i|^2 + D_1^*\bar\psi D_1\psi\right]\ , \tag{2.38}$$

where $D_a \equiv \partial_a - iC_a$ is the "bosonic" covariant derivative. A remarkable feature of this Lagrangian is that there are no terms quartic in fermions: terms which come from

[9] Standard 2-d Lorentz-covariant sigma models with projective superspaces as target spaces were studied, e.g., in [31].

$\mathbf{C}_1^2$ and from $\partial_1 V_i \partial_1 V_i^*$ cancel. This $CP^{1|1}$ supercoset Landau-Lifshitz Lagrangian is a generalization of the CP^1 Lagrangian of the bosonic $SU(2)$ sector.

Another special case is the $CP^{0|1}$ model corresponding to the $SU(1|1)$ sector where we have only one component of U_i ($|U_1|^2 = 1$) which can be gauge-fixed to 1. Then (2.38) reduces to the abelian Landau-Lifshitz system

$$\mathcal{L}_{SU(1|1)} = -i\bar{\psi}\partial_0\psi - \frac{1}{2}\partial_1\bar{\psi}\partial_1\psi\,. \tag{2.39}$$

We may repeat the same transformation in the $SU(3|2)$ case (2.33) by solving the normalisation condition using the new bosonic fields U_i

$$V_i = U_i\sqrt{1-\bar{\psi}_\alpha\psi_\alpha} = U_i\,\left[1-\frac{1}{2}\bar{\psi}_\alpha\psi_\alpha - \frac{1}{8}(\bar{\psi}_\alpha\psi_\alpha)^2\right], \qquad\qquad |U_i|^2 = 1\ . \tag{2.40}$$

Then $\mathbf{C}_a$ in (2.34) becomes

$$\mathbf{C}_a = C_a\,\left(1-\bar{\psi}_\alpha\psi_\alpha\right) - \frac{i}{2}(\bar{\psi}_\alpha\partial_a\psi_\alpha + \psi_\alpha\partial_a\bar{\psi}_\alpha)\,, \qquad\qquad C_a = -iU_i^*\partial_a U_i\,, \tag{2.41}$$

and the Lagrangian (2.33) takes the form ($i = 1,2,3;\ \alpha = 1,2$)

$$\begin{aligned}\mathcal{L}_{SU(3|2)} \;=\;& -iU_i^*\partial_0 U_i - i\bar{\psi}_\alpha D_0\psi_\alpha - \frac{1}{2}\Big[|D_1U_i|^2(1-\bar{\psi}_\alpha\psi_\alpha) + D_1^*\bar{\psi}_\alpha D_1\psi_\alpha \\ & +\frac{1}{2}(\psi_\alpha D_1^*\bar{\psi}_\alpha)^2 + \frac{1}{2}(\bar{\psi}_\alpha D_1\psi_\alpha)^2 + \frac{1}{4}\bar{\psi}_\alpha\psi_\alpha\partial_1(\bar{\psi}_\beta\psi_\beta)\partial_1(\bar{\psi}_\gamma\psi_\gamma)\Big]\,.\end{aligned} \tag{2.42}$$

We may recover the $SU(2|1)$ Lagrangian (2.38) by setting $U_3 = 0$ and $\psi_2 = 0$: the last sixth-order term (which originated from $\partial_1 V_i\partial_1 V_i$) then vanishes, and the quartic fermionic terms also become zero since they are proportional to $\psi_\alpha\psi_\beta$ or $\bar{\psi}_\alpha\bar{\psi}_\beta$.

Since we would like to compare the spin chain action to the superstring action which (in a particular gauge) contains terms that are at most quartic in fermions, it is useful to notice that one can eliminate the sixth-order term by redefining the fermionic field

$$\psi_\alpha \to \psi_\alpha - \frac{1}{2}\,(\bar{\psi}_\beta\psi_\beta)\psi_\alpha\,. \tag{2.43}$$

Then

$$\begin{aligned}\mathcal{L}_{SU(3|2)} \;=\;& -iU_i^*\partial_0 U_i - i(1-\bar{\psi}_\alpha\psi_\alpha)\bar{\psi}_\alpha D_0\psi_\alpha \\ & -\frac{1}{2}\,\Big[(1-\bar{\psi}_\alpha\psi_\alpha + (\bar{\psi}_\alpha\psi_\alpha)^2)|D_1U_i|^2 + (1-\bar{\psi}_\beta\psi_\beta)D_1^*\bar{\psi}_\alpha D_1\psi_\alpha \\ & +(\psi_\alpha D_1^*\bar{\psi}_\alpha)(\bar{\psi}_\beta D_1\psi_\beta)\Big]\ .\end{aligned} \tag{2.44}$$

In this form our $CP^{2|2}$ Landau-Lifshitz Lagrangian agrees with the $SU(2|3)$ spin chain coherent state Lagrangian found earlier in [20].

2.3 An example of fermionic solution of the Landau-Lifshitz model

The bosonic $SU(2)$ Landau-Lifshitz model has a very simple static solution [13] which corresponds, in the string theory picture, to the circular string rotating in S^3 part of S^5 with two equal angular momenta (i.e. $\mathrm{X}_1 = \frac{1}{\sqrt{2}}e^{iw\tau+in\sigma}$, $\mathrm{X}_2 = \frac{1}{\sqrt{2}}e^{iw\tau-in\sigma}$): $U_1 = \frac{1}{\sqrt{2}}e^{in\sigma}$, $U_2 = \frac{1}{\sqrt{2}}e^{-in\sigma}$. Interestingly, the equations of motion that follow from the Lagrangian (2.38) (Λ is the Lagrange multiplier imposing $U_i^* U_i = 1$)

$$\begin{aligned} 0 &= -2i(1-\bar{\psi}\psi)\partial_0 U_i + (1-\bar{\psi}\psi)D_1^2 U_i + \Lambda U_i \,, && (2.45) \\ 0 &= -2iD_0\psi + |D_1 U_i|^2\psi + D_1^2\psi \,. && (2.46) \end{aligned}$$

admit the following generalization of the above bosonic static solution

$$\begin{aligned} U_1 &= \frac{1}{\sqrt{2}}\left(e^{in\sigma} - e^{-in\sigma}\bar{\zeta}\zeta\right) \,, && (2.47) \\ U_2 &= \frac{1}{\sqrt{2}}\left(e^{-in\sigma} + e^{in\sigma}\bar{\zeta}\zeta\right) \,, && (2.48) \\ \psi &= e^{im\sigma}\zeta \,, && (2.49) \end{aligned}$$

where ζ is a constant complex Grassmann parameter (U_i are even elements of the Grassmann algebra) and $n\,, m$ are integers. Note that since our action has local $U(1)$ symmetry (in addition to global $SU(2|1)$ symmetry) this solution is equivalent to the one with constant ψ field.

For the above ansatz

$$C_1 = -iU_i^*\partial_1 U_i = 0\,, \qquad \text{and} \qquad |\partial_1 U_i|^2 = n^2\,, \tag{2.50}$$

and then

$$\begin{aligned} 0 &= (1-\bar{\psi}\psi)\partial_1^2 U_i + \Lambda U_i \,, && (2.51) \\ 0 &= n^2\psi + \partial_1^2\psi \,. && (2.52) \end{aligned}$$

The latter equation is solved if $n = m$ while the former determines the Lagrange multiplier to be

$$\Lambda = -(1-\bar{\psi}\psi)U_i^*\partial_1^2 U_i = n^2(1-\bar{\zeta}\zeta)\,. \tag{2.53}$$

Since $\partial_1^2 U_i = -n^2 U_i$, our ansatz does indeed satisfy the U_i equation of motion.

Surprisingly, the corresponding energy density (determined by the spatial derivative part of (2.38)) $\mathcal{E} = \frac{1}{2}\left[(1-\bar{\psi}\psi)|D_1 U_i|^2 + D_1^*\bar{\psi}D_1\psi\right]$, evaluated on the above solution, is equal simply to $\frac{1}{2}n^2$, i.e. it does not depend on ζ. Indeed, one can show that this solution can be obtained from the bosonic $SU(2)$ subsector solution by means of a global $SU(1|2)$ rotation and a local $U(1)$ rotation.

2.4 On $PSU(2,2|4)$ Landau-Lifshitz model

As was found in [15, 16, 19, 17], the generalization of the CP^2 Landau-Lifshitz action of the $SU(3)$ sector to the case of the $SO(6)$ sector is a similar action on the Grassmanian $G_{2,6} = SO(6)/[SO(4)\times SO(2)]$ which is the same as a quadric in CP^5 defined by $U_iU_i = 0$ ($i = 1, ..., 6$) imposed in addition to $U_iU_i^* = 1$ and the $U(1)$ gauge invariance. The discussion in [19] suggests that a generalization of the above $CP^{2|2}$ supercoset action (2.31) for the $SU(2|3)$ sector to the coherent state action for the full $PSU(2,2|4)$ spin chain of [7] (with the vacuum chosen again to represent the BPS state $\mathrm{tr}\Phi^J$) should be defined on a super-Grassmanian which generalizes the product of the two bosonic Grassmanians $SO(2,4)/[SO(4)\times SO(2)]$ and $SO(6)/[SO(4)\times SO(2)]$ [19]:

$$G_{2|2,4|4} = SU(2,2|4)/[SU(2|2)\times SU(2|2)\times U(1)\times U(1)]\ .$$

The corresponding Lagrangian is a direct generalization of (2.31) where the analog of $\mathbf{V}^p$ is subject to an additional $(\mathbf{V}^p)^2 = 0$ constraint. To get an action with unconstrained fermions one would then need only to redefine the bosons in a fashion similar to equation (2.40).

It remains a challenge to directly relate this action to a limit of the superstring action [8] defined on the supercoset $SU(2,2|4)/[SO(1,4)\times SO(5)]$. As was argued in [19] (using a time-averaging procedure on the string side), the quadratic fermionic terms in the two actions should indeed match.

Below we would like to further clarify this relation by explaining the truncation of the superstring action that should correspond to a particular fermionic sector on the spin chain side and attempting to go beyond the quadratic level in fermions.

3 Superstring theory action

One would like to start with type IIB superstring action [8] and to show that a fermionic action equivalent to the one found from the spin chain in the previous section emerges from it in the "fast-string" limit, thus generalizing the observations made in the bosonic sectors [12, 13, 15, 14, 17, 18]. Since the spin chain side is sensitive only to physical degrees of freedom, we are free to choose any diffeomorphism and κ-symmetry gauge. We shall use the string action in the light-cone κ-symmetry gauge of [27, 28], i.e. $\Gamma^+\theta = 0$, where "+" direction is a light-cone direction in the Poincare coordinates of AdS_5 space. The advantage of this gauge is that the fermionic part of the action (and, in particular, its $SU(4)$ structure) becomes relatively simple and explicit.

We shall then fix the bosonic conformal gauge and make an ansatz for the bosonic AdS_5 fields that corresponds to the choice of the *global* AdS_5 time being proportional to the world-sheet time τ. The $t \sim \tau$ relation is needed to ensure that the resulting

2-d Hamiltonian will match the spin chain Hamiltonian whose eigen-values should be anomalous dimensions. Put differently, while we will be using the κ-symmetry gauge "adapted" to the Poincare coordinates, we may still replace the bosonic AdS_5 Poincare coordinates by the global AdS_5 ones[10] and then fix the latter (time, radial, and unit 4-vector representing angles of S^3) as $t = \nu\tau + ...,\ \rho = 0 + ...,\ n_i = 0 + ...$, where dots stand for possible fermionic terms.[11] The parameter ν will be related to the (large) angular momentum in S^5 and thus $1/\nu$ will be our expansion parameter.

Next, one is to try choose a consistent ansatz for the bosonic and fermionic fields which would restricts the string action to the same sector of states ($SU(2|3)$ or its subsectors) that we discussed above on the spin chain side. We shall only consider classical string configurations, i.e. semiclassical string states corresponding to coherent states of the spin chain; as in the previously discussed bosonic sectors, the string α' corrections should correspond to subleading $1/L$ corrections on the spin chain side [32, 33, 34]. While the truncation of the purely-bosonic string sigma model equations to particular subsectors is relatively straightforward [15, 17, 33], this is no longer so in the presence of superstring fermions, which, in particular, couple together the AdS_5 and S^5 parts of the bosonic action.

One may try to set part of fermionic fields to zero and make certain field redefinitions in order to show the existence of a truncation of classical string equations to a subsector with non-zero fermions.[12] Having found a consistent truncation of the classical string equations, one may then attempt to reconstruct (at least in a certain fast-string limit corresponding to the leading order approximation on the gauge theory side) an action that describes this subsector.

[10]The standard transformation is $e^{\phi}x^0 = \cosh\rho\ \sin t,\ e^{\phi}x_i = \sinh\rho\ n_i,\ e^{\phi} = \cosh\rho\ \cos t - n_4 \sinh\rho,\ n_i^2 + n_4^2 = 1.$

[11]Since we will be interested in the $SU(2|3)$ sector of states the corresponding string states should be rotating in S^5 (as in the $SU(3)$ sector), and, in addition, the fermionic degrees of freedom should carry a spin component in AdS_5.

[12]Consistent truncations of the *phase-space* equations of the light-cone superstring in $AdS_5 \times S^5$ were recently found in [24]. They involve setting to zero certain components of the generalized even momenta (depending on both bosonic and fermionic variables) and are equivalent to the truncations in the Lagrangian approach we discuss below.

3.1 Superstring Lagrangian in a light-cone gauge

Our starting point will be the κ-symmetry gauge fixed Lagrangian of [27][13]

$$
\begin{aligned}
\mathcal{L} \ = \ & -\frac{1}{2}\sqrt{g}g^{\mu\nu}\left[2e^{2\phi}(\partial_\mu x^+\partial_\nu x^- + \partial_\mu x\partial_\nu \bar{x}) + \partial_\mu\phi\partial_\nu\phi + \partial_\mu X^M\partial_\nu X^M\right] \\
& -\frac{i}{2}\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\mu x^+\left[\theta^A\partial_\nu\theta_A + \theta_A\partial_\nu\theta^A + \eta^A\partial_\nu\eta_A + \eta_A\partial_\nu\eta^A\right] \\
& -i\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\mu x^+ X^N\partial_\nu X^M\eta_A\rho^{MNA}{}_B\eta^B \\
& +\frac{1}{2}\sqrt{g}g^{\mu\nu}e^{4\phi}\partial_\mu x^+\partial_\nu x^+\left[(\eta^A\eta_A)^2 + (X^N\eta_A\rho^{MNA}{}_B\eta^B)^2\right] \\
& +\epsilon^{\mu\nu}e^{2\phi}\partial_\mu x^+ X^M\left(\eta^A\rho^M_{AB}\partial_\nu\theta^B + \eta_A\rho^{MAB}\partial_\nu\theta_B\right) \\
& +i\sqrt{2}\epsilon^{\mu\nu}e^{3\phi}\partial_\mu x^+ X^M\left(\partial_\nu\bar{x}\eta_A\rho^{MAB}\eta_B - \partial_\nu x\eta^A\rho^M{}_{AB}\eta^B\right) \ .
\end{aligned}
\tag{3.1}
$$

Here $\mu,\nu = 0,1$ and ϕ, x^0, x^i are the Poincare coordinates of AdS_5 with

$$
x^\pm = \frac{1}{\sqrt{2}}(x^3 \pm x^0)\,, \qquad x = \frac{1}{\sqrt{2}}(x^1 + ix^2)\,, \qquad \bar{x} = \frac{1}{\sqrt{2}}(x^1 - ix^2)\,, \tag{3.2}
$$

and X^M ($M,N = 1,...,6$) is a unit 6-vector parametrising S^5 (the constraint $X^M X^M = 1$ can be imposed with a Lagrange multiplier Λ). The 4+4 complex Grassmann fields θ_A, η_A (with $A = 1,2,3,4$ and $\theta_A = (\theta^A)^\dagger$, $\eta_A = (\eta^A)^\dagger$) transform in the fundamental representation of $SU(4)$. [14] The 4×4 matrices ρ^M are "off-diagonal" blocks of the $SO(6)$ gamma-matrices in the chiral representation (their properties are listed in Appendix A), and $\rho^{MN} = -\rho^{[M}\rho^{*N]}$.

Notice that θ^A enter the action only quadratically (all quartic fermionic terms involve only η_A) and thus it could be in principle "integrated out". We recall [27, 28] that θ^A correspond to the (linearly realised) supersymmetry generators of the superconformal algebra $PSU(4,4|4)$ while η^A – to the non-linearly realised superconformal generators.

In what follows we shall choose the conformal gauge for the 2-d metric and will make the following ansatz for the bosonic AdS_5 fields which corresponds to the global AdS_5 time $t = \nu\tau + ...$, namely,

$$
e^\phi = \cos\nu\tau\,, \qquad x^+ = \frac{\tan\nu\tau}{\sqrt{2}}\,, \qquad x^- = -\frac{\tan\nu\tau}{\sqrt{2}} + f(\tau,\sigma)\,, \qquad x = \bar{x} = 0\,, \tag{3.3}
$$

where f is to be determined. Then

$$
e^{2\phi}\partial_0 x^+ = \frac{\nu}{\sqrt{2}}\ , \tag{3.4}
$$

[13] We ignore the overall factor of string tension $\frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'}$.

[14] We mostly follow the notation of [27, 28] but use $A, B = 1,2,3,4$ instead of i, j as $SU(4)$ indices.

and the x^+ equation of motion (the one obtained by varying x^-) is automatically satisfied. Since we would like also to keep some fermions non-zero, it is not *a priori* clear if such an ansatz is consistent with all the equations of motion. Indeed, we expect that it will place restrictions on allowed fermions and on $f(\tau,\sigma)$. For example, setting $x=0$ in the equation of motion for x is possible as long as η satisfies

$$\partial_1(X^M\eta^A\rho^M_{AB}\eta^B)=0\,, \tag{3.5}$$

plus a similar complex conjugate relation coming from the $\bar{x}$-equation. The ϕ equation of motion gives

$$\begin{aligned}\partial_0 f &= -iX^N\partial_0X^M\eta_A\rho^{MNA}{}_B\eta^B-\frac{i}{2}\left(\theta^A\partial_0\theta_A+\theta_A\partial_0\theta^A+\eta^A\partial_0\eta_A+\eta_A\partial_0\eta^A\right)\\ &\quad -X^M\left(\eta^A\rho^M_{AB}\partial_1\theta^B+\eta_A\rho^{MAB}\partial_1\theta_B\right)\,,\end{aligned} \tag{3.6}$$

while the equation for x^- implies

$$\begin{aligned}\partial_1^2 f &= \partial_1\Big[-iX^N\partial_1X^M\eta_A\rho^{MNA}{}_B\eta^B-\frac{i}{2}\left(\theta^A\partial_1\theta_A+\theta_A\partial_1\theta^A+\eta^A\partial_1\eta_A+\eta_A\partial_1\eta^A\right)\\ &\quad -X^M\left(\eta^A\rho^M_{AB}\partial_0\theta^B+\eta_A\rho^{MAB}\partial_0\theta_B\right)\Big]\,.\end{aligned} \tag{3.7}$$

The θ equation of motion and its conjugate are

$$\partial_0\theta_A+i\partial_1(X^M\rho^M_{AB}\eta^B)=0\,,\qquad \partial_0\theta^A+i\partial_1(X^M\rho^{MAB}\eta_B)=0 \tag{3.8}$$

These relations may be used to eliminate the θ fermions from the action.

The conformal gauge constraints ($\frac{\delta S}{\delta g^{\mu\nu}}=0$ with $\sqrt{g}g^{\mu\nu}=\eta^{\mu\nu}$) place further restrictions on the allowed fermionic configurations. Using our ansatz (3.3) the one of the two constraints becomes

$$\begin{aligned}\nu^2 &= \partial_1X^M\partial_1X^M+\partial_0X^M\partial_0X^M+\sqrt{2}\nu\partial_0 f\\ &\quad +\frac{i\nu}{\sqrt{2}}\left[2X^N\partial_0X^M\eta_A\rho^{MNA}{}_B\eta^B+(\theta^A\partial_0\theta_A+\theta_A\partial_0\theta^A+\eta^A\partial_0\eta_A+\eta_A\partial_0\eta^A)\right]\\ &= \partial_1X^M\partial_1X^M+\partial_0X^M\partial_0X^M-\sqrt{2}\nu X^M\left(\eta^A\rho^M_{AB}\partial_1\theta^B+\eta_A\rho^{MAB}\partial_1\theta_B\right)\,.\end{aligned} \tag{3.9}$$

In the last line we have used equation (3.6). The other constraint implies

$$\begin{aligned}0 &= \partial_1X^M\partial_0X^M+i\frac{\nu}{\sqrt{2}}X^N\partial_1X^M\eta_A\rho^{MNA}{}_B\eta^B\\ &\quad +\frac{\nu}{\sqrt{2}}\left[\partial_1 f+\frac{i}{2}(\theta^A\partial_1\theta_A+\theta_A\partial_1\theta^A+\eta^A\partial_1\eta_A+\eta_A\partial_1\eta^A)\right]\,.\end{aligned} \tag{3.10}$$

This determines the value of $\partial_1 f$ that should be consistent with (3.7).

3.2 Fermionic action in $SU(3)$ notation

As already mentioned, we would like to consider a subspace of classical string configurations that should be dual to spin chain states from $SU(2|3)$ subsector. The corresponding gauge theory operators are built out of 3 chiral complex combinations of 6 scalars and the two spinor components of the gluino Weyl fermion. The fermions should carry Lorentz spin but should be singlets under the Cartan $[U(1)]^3$ subgroup of $SO(6)$ whose charges (S^5 angular momenta) are carried by the scalars. To identify the corresponding fermionic components on the string theory side we should thus do the 3+1 split of the $SU(4)$ fermionic components and at the end keep only the $SU(3)$ singlet fields. Thus a systematic procedure to isolate the $SU(2|3)$ sector should be based on:

(i) introducing 3 chiral bosonic fields X_i and isolating their common large phase factor α $(i = 1, 2, 3)$

$$\mathrm{X}_i = e^{i\alpha} U_i \,, \qquad \mathrm{X}_i \equiv X_{2i-1} + iX_{2i}, \qquad U_i U_i^* = 1 \,, \tag{3.11}$$

$$\alpha = \nu\tau + v(\tau, \sigma) \,, \tag{3.12}$$

and (ii) splitting the $SU(4)$ fermions in (3.1) in 3+1 way

$$\eta_A \equiv (\eta_i, \eta) \,, \qquad \theta_A \equiv (\theta_i, \theta) \,, \qquad i = 1, 2, 3 \,. \tag{3.13}$$

The two $SU(3)$ singlet fields $\eta \equiv \eta_4$ and $\theta \equiv \theta_4$ should be eventually related to the two fermionic variables ψ_1, ψ_2 of the spin chain action (2.42) which are singlets under $SU(3)$ but are rotated by an additional global $SU(2)$ symmetry.

Finally, (iii) one is to eliminate η_i and θ_i fields from the Lagrangian in the large ν approximation. That step may be facilitated by applying some proper $U(1)$ redefinitions of fermions by $e^{i\alpha}$ factors. While such rotations may not be necessary for the "dummy" $SU(3)$ variables η_i and θ_i, we may need them in order to relate the singlet fields η, θ to ψ_1, ψ_2 of the spin chain.

Using the specific representation of ρ^M matrices and relations given in Appendix A one can rewrite the fermionic part of the Lagrangian (3.1) in the following manifestly $SU(3)$ invariant form depending on $\mathrm{X}_i, \eta_i, \theta_i, \eta$ and θ (after using also the ansatz (3.3),(3.4))

$$\tilde{\mathcal{L}}_F \equiv \sqrt{2}\nu^{-1}\mathcal{L}_F = \tilde{\mathcal{L}}_{2F} + \tilde{\mathcal{L}}_{4F} \,, \tag{3.14}$$

where the quadratic terms are

$$\begin{aligned}
\tilde{\mathcal{L}}_{2F} \;=\;& i\eta^i\partial_0\eta_i + i\bar{\eta}\partial_0\eta + i\theta^i\partial_0\theta_i + i\bar{\theta}\partial_0\theta \\
&+ \epsilon_{ijk}\eta^i\partial_1\theta^j\mathrm{X}^k - \epsilon^{ijk}\eta_i\partial_1\theta_j\mathrm{X}_k \\
&+ \eta^i\partial_1\bar{\theta}\mathrm{X}_i - \eta_i\partial_1\theta\mathrm{X}^i + \partial_1\theta^i\bar{\eta}\mathrm{X}_i - \partial_1\theta_i\eta\mathrm{X}^i \\
&- i(\mathrm{X}^i\partial_0\mathrm{X}_j - \mathrm{X}_j\partial_0\mathrm{X}^i)\eta_i\eta^j - \; i\mathrm{X}^i\partial_0\mathrm{X}_i(\eta^j\eta_j - \bar{\eta}\eta) \\
&- \; i(\epsilon^{ijk}\mathrm{X}_j\partial_0\mathrm{X}_k\eta_i\bar{\eta} - \epsilon_{ijk}\mathrm{X}^j\partial_0\mathrm{X}^k\eta\eta^i) \,,
\end{aligned} \tag{3.15}$$

and the quartic terms are

$$\tilde{\mathcal{L}}_{4F} = -\frac{\nu}{\sqrt{2}}\big(3\eta^i\eta_i\bar{\eta}\eta \;\;-\;\; 4\mathrm{X}_i\eta^i\mathrm{X}^j\eta_j\bar{\eta}\eta + 4\eta_i\mathrm{X}^i\eta^j\mathrm{X}_j\eta_k\eta^k$$
$$+\;\; 2\epsilon_{ijk}\eta^i\eta^j\mathrm{X}^k\eta_l\mathrm{X}^l\eta + 2\epsilon^{ijk}\eta_i\eta_j\mathrm{X}_k\eta^l\mathrm{X}_l\bar{\eta}\big)\ , \qquad (3.16)$$

where

$$\mathrm{X}^i = \mathrm{X}_i^*\ , \qquad \eta^i = \eta_i^\dagger\ , \qquad \theta^i = \theta_i^\dagger\ , \qquad \bar{\eta} = \eta^\dagger\ , \qquad \bar{\theta} = \theta^\dagger\ . \qquad (3.17)$$

For completeness, the bosonic S^5 part of the Lagrangian (3.1) written in terms of X_i is

$$\mathcal{L}_B = -\frac{1}{2}\partial^\mu \mathrm{X}_i^*\partial_\mu \mathrm{X}_i + \frac{1}{2}\Lambda(\mathrm{X}_i^*\mathrm{X}_i - 1)\ . \qquad (3.18)$$

4 Some fermionic solutions to superstring equations of motion

In this section we present a number of simple classical rotating string solutions of the above action that have non-zero fermions. This will help to understand better which truncations of the superstring coordinates are consistent with equations of motion. The solutions we shall discuss are generalisations of the rotating circular string solutions found in [4, 30].

Our starting point will be the action (3.15),(3.16). There are a number of consistent truncations of the Lagrangian (3.15). One includes restricting the bosonic fields to AdS_3 inside AdS_5 and S^3 inside S^5 and also truncating the fermions in one of two possible ways, i.e.

$$(x, \mathrm{X}_3; \eta, \eta_3, \theta_1, \theta_2) = 0\,, \qquad \text{i.e.} \qquad (\mathrm{X}_1, \mathrm{X}_2; \theta, \theta_3, \eta_1, \eta_2) \neq 0\ , \qquad (4.1)$$

or

$$(x, \mathrm{X}_3; \theta, \theta_3, \eta_1, \eta_2) = 0\,, \qquad \text{i.e.} \qquad (\mathrm{X}_1, \mathrm{X}_2; \eta, \eta_3, \theta_1, \theta_2) \neq 0\ . \qquad (4.2)$$

We can also restrict to AdS_3 inside AdS_5 and S^1 inside S^5 and truncate fermions further in one of the two ways

$$(x, \mathrm{X}_2, \mathrm{X}_3; \eta, \eta_2, \eta_3, \theta_1, \theta_2, \theta_3) = 0\,, \qquad \text{i.e.} \qquad (\mathrm{X}_1; \theta, \eta_1) \neq 0\ , \qquad (4.3)$$

or

$$(x, \mathrm{X}_2, \mathrm{X}_3; \theta, \theta_2, \theta_3, \eta_1, \eta_2, \eta_3) = 0\,, \qquad \text{i.e.} \qquad (\mathrm{X}_1; \eta, \theta_1) \neq 0\ . \qquad (4.4)$$

It is natural to expect that after integrating out "extra" fermions (i.e. leaving only η or θ in each case) these subsectors may be related to the $SU(1|2)$ and $SU(1|1)$ gauge theory sectors. In this section we shall use the names "$SU(1|2)$" and "$SU(1|1)$" for the above superstring truncations. Similar truncations were also obtained in [24] using phase space formulation.

4.1 "$SU(1|1)$" fermionic string solution

Below we present a particular "circular" string solution for the ansatz (4.4). We shall take the AdS_5 fields to be of the form given in equation (3.3). The η equation of motion then reduces to

$$0 = \partial_0^2 \mathrm{X}_1 \eta - \mathrm{X}_1 \partial_0^2 \eta + \partial_1^2(\mathrm{X}_1 \eta)\,. \tag{4.5}$$

We will solve this by taking ($|\mathrm{X}_1|^2 = 1$)

$$\mathrm{X}_1 = e^{i\nu\tau}(1 - iC\tau\bar{\zeta}\zeta)\,, \qquad \eta = e^{i(n\sigma+\omega\tau)}\zeta\,, \qquad \theta = e^{i(n\sigma+(\omega+\nu)\tau)}\bar{\zeta}\,, \tag{4.6}$$

where ζ is a constant complex Grassmann number, C is a real constant and

$$\omega = \sqrt{n^2+\nu^2}\,. \tag{4.7}$$

The θ_1 equation of motion (3.8) then gives

$$\theta_1 = -i\frac{\omega+\nu}{n}e^{-i(n\sigma+(\omega-\nu)\tau)}\bar{\zeta} = -i\frac{\omega+\nu}{n}e^{i\nu\tau}\bar{\eta}\,, \tag{4.8}$$

while the η_1 equation implies that

$$\eta_1 = i\frac{\omega+\nu}{n}e^{-i(n\sigma+\omega\tau)}\zeta = i\frac{\omega+\nu}{n}e^{i\nu\tau}\bar{\theta}\,. \tag{4.9}$$

The X_1 equation of motion is satisfied then if the Lagrange multiplier is

$$\Lambda = -\nu^2 - A\bar{\zeta}\zeta\,, \qquad A = -2\sqrt{2}\nu^2 - \frac{2\sqrt{2}\nu^3}{n^2}(\nu-\omega_n) - 2\nu C\,. \tag{4.10}$$

The ϕ equation of motion gives $\partial_0 f = 0$, while the conformal gauge constraint (3.10) implies that $\partial_1 f = 0$. In other words, this solution has the *same* AdS_5 part as the bosonic solutions representing strings rotating on S^5. It is easy to see that eq.(3.7) is also satisfied. Finally, the conformal gauge constraint (3.9) is satisfied for

$$C = -2\sqrt{2}(\omega-\nu)\,. \tag{4.11}$$

The solution has energy ν, and is charged under the Cartan generators $J^A{}_A$ of $SU(4)$, with all other $SU(4)$ charges zero. Indeed, for this solution $J^1{}_1 = -J^2{}_2 = -J^3{}_3 = J^4{}_4 \equiv J$, where

$$J = -\frac{1}{2}\nu + \sqrt{2}\bar{\zeta}\zeta\left(\omega - \nu - \frac{\omega^2-\nu\omega}{2m^2}\right)\,. \tag{4.12}$$

We have thus obtained a formal classical superstring solution which generalizes the BMN geodesic solution ($t=\nu\tau,\ \mathrm{X}_1 = e^{i\nu\tau}$) to the presence of non-trivial σ-dependent fermions. Here the string is "spread" only in the odd directions of superspace and in the even x^- direction. Its charge depends on $\bar{\zeta}\zeta$, and hence appears to be Grassmann valued. This is an artifact of our semi-classical treatment of fermions; one may view $\bar{\zeta}\zeta$ in equation (4.12) as a real-valued expectation value $\langle\bar{\zeta}\zeta\rangle$.

4.2 "$SU(1|2)$" solution with $\theta \neq 0$

Let us now present a solution in the case of the truncation (4.1). Guided by analogy with the solution of the Landau-Lifshitz model in section 2.3, we will try the following ansatz

$$\mathrm{X}_1 = \frac{1}{\sqrt{2}}\left(e^{in\sigma} - e^{-in\sigma}\bar{\zeta}\zeta\right) e^{i\mathrm{w}\tau - F(\tau,\sigma)\bar{\zeta}\zeta - iG(\tau,\sigma)\bar{\zeta}\zeta}\,, \tag{4.13}$$

$$\mathrm{X}_2 = \frac{1}{\sqrt{2}}\left(e^{-in\sigma} + e^{in\sigma}\bar{\zeta}\zeta\right) e^{i\mathrm{w}\tau + F(\tau,\sigma)\bar{\zeta}\zeta - iG(\tau,\sigma)\bar{\zeta}\zeta}\,, \tag{4.14}$$

$$\theta = e^{im\sigma + i\omega\tau}\zeta\,, \tag{4.15}$$

together with $\theta_3 = 0$ and

$$\eta_1 = \frac{A_1}{\sqrt{2}} e^{i((n-m)\sigma + (\mathrm{w}-\omega)\tau)}\bar{\zeta}\,, \tag{4.16}$$

$$\eta_2 = \frac{A_2}{\sqrt{2}} e^{i(-(n+m)\sigma + (\mathrm{w}-\omega)\tau)}\bar{\zeta}\,. \tag{4.17}$$

As above, ζ is a constant complex Grassmann parameter, G and F are real σ-periodic function and the A_i are constants.[15] The equations of motion for η, θ_1, θ_2 and η_3 are then trivially satisfied. The θ, η_1, η_2 and θ_3 equations of motion reduce to the following constraints

$$0 = A_1 - A_2\,, \tag{4.18}$$

$$0 = (A_1 + A_2)m - 2i\omega\,, \tag{4.19}$$

$$0 = A_1(\omega - \mathrm{w}) - A_2\mathrm{w} - im\,, \tag{4.20}$$

$$0 = A_2(\omega - \mathrm{w}) - A_1\mathrm{w} - im\,. \tag{4.21}$$

The solution to these is

$$\omega = \mathrm{w} \pm \sqrt{\mathrm{w}^2 + m^2}\,, \qquad A_1 = A_2 = -i\frac{\mathrm{w} \pm \sqrt{\mathrm{w}^2 + m^2}}{m}\,. \tag{4.22}$$

Turning to the bosonic equations of motion it is easy to see that the X_3 equation of motion is trivial while the $\mathrm{X}_1, \mathrm{X}_2$ ones reduce to the condition

$$0 = 4n\partial_1 G - 2\partial_1^2 F + 4i\mathrm{w}\partial_0 F + 2\partial_0^2 F\,. \tag{4.23}$$

The equations of motion for the AdS_5 coordinates and the conformal gauge constraints give rise to further constraints

$$0 = \partial_0\partial_1 G + 2n\partial_0 F\,, \tag{4.24}$$

$$0 = \sqrt{2}(\mathrm{w} \pm \sqrt{m^2 + \mathrm{w}^2})\nu - \mathrm{w}\partial_0 G\,, \tag{4.25}$$

$$0 = \partial_1^2 G + 2n\partial_1 F\,, \tag{4.26}$$

[15] Of course, $e^{-iG\bar{\zeta}\zeta} = 1 - iG\bar{\zeta}\zeta$ but we prefer to use the exponential parametrization.

as well as

$$\mathrm{w} = \sqrt{\nu^2 - n^2}\,. \tag{4.27}$$

Given the equations (4.18)–(4.21), the equation of motion for ϕ (3.6) implies that

$$\partial_0 f = 0\,, \tag{4.28}$$

while the conformal gauge constraint (3.10) reduces to

$$\partial_1 f = \frac{1}{m\nu}\Big(2(m^2 + \mathrm{w}^2 \pm \mathrm{w}\sqrt{m^2+\mathrm{w}^2})\nu - 2\sqrt{2}mn\mathrm{w}F - \sqrt{2}m\mathrm{w}\partial_1 G\Big)\,\bar\zeta\zeta\,. \tag{4.29}$$

The simplest solution to these equations which ensures that $x^+ = -x^-$ is

$$\begin{aligned} F(\tau,\sigma) &= -\frac{\sqrt{2}(m^2+\mathrm{w}^2 \pm \mathrm{w}\sqrt{m^2+\mathrm{w}^2})\nu}{2mn\mathrm{w}}\,, &(4.30)\\ G(\tau,\sigma) &= \sqrt{2}\nu\ [\ 1 \pm \sqrt{1+(m/\mathrm{w})^2}\ ]\ \tau\,. &(4.31)\end{aligned}$$

One can check that for this solution the Cartan charges $J^A{}_A$ are the only non-zero components of the $SU(4)$ charges (see Appendix B).

To match this solution to the spin-chain sigma model one, we need to take $\nu \to 0$. In order to keep our solution finite in this limit we will consider the minus sign choice in the above relations. In this limit A_1 and A_2 tend to zero, and, as a result, the only non-zero fields are X_1, X_2 and θ, which can be matched to the spin chain variables. We should stress, however, that besides being rotated by a common phase $G\bar\zeta\zeta$, the X_i are also rescaled by factor $(1 \pm F\bar\zeta\zeta)$. This implies that while a rotation by a common phase, discussed below in section 5, can be used to relate the string and spin chain variables and actions to the leading order, at higher orders one will need more involved field redefinitions.

4.3 "$SU(1|2)$" solution with $\eta \neq 0$

Let us now consider the case of the truncation (4.2). In this sector it turns out that one needs to consider a more general ansatz for the bosons

$$\begin{aligned} \mathrm{X}_1 &= \frac{1}{\sqrt{2}}\left(e^{in\sigma} - e^{-in\sigma}\bar\zeta\zeta\right) e^{i\mathrm{w}\tau - F(\tau,\sigma)\bar\zeta\zeta - iG(\tau,\sigma)\bar\zeta\zeta}\,, &(4.32)\\ \mathrm{X}_2 &= \frac{1}{\sqrt{2}}\left(e^{-in\sigma} + e^{in\sigma}\bar\zeta\zeta\right) e^{i\mathrm{w}\tau + F(\tau,\sigma)\bar\zeta\zeta - iH(\tau,\sigma)\bar\zeta\zeta}\,, &(4.33)\end{aligned}$$

where F, G and H are real σ-periodic functions. For the fermions we shall choose

$$\begin{aligned} \eta &= e^{im\sigma + i\omega\tau}\zeta\,, & \eta_3 &= Be^{i(m\sigma + (\omega - 2\mathrm{w})\tau)}\zeta\,, &(4.34)\\ \theta_1 &= \frac{A_1}{\sqrt{2}}e^{i((n-m)\sigma + (\mathrm{w}-\omega)\tau)}\bar\zeta\,, & \theta_2 &= \frac{A_2}{\sqrt{2}}e^{i((-n-m)\sigma + (\mathrm{w}-\omega)\tau)}\bar\zeta\,. &(4.35)\end{aligned}$$

The fermionic equations of motion then reduce to

$$\begin{aligned}
0 &= A_1(m-n) + A_2(m+n) + 2i(\mathrm{w}+\omega)\,, &(4.36)\\
0 &= (m-n)(1-B^*) + iA_1(\mathrm{w}-\omega)\,, &(4.37)\\
0 &= (m+n)(1+B^*) + iA_2(\mathrm{w}-\omega)\,, &(4.38)\\
0 &= A_1(m-n) - A_2(m+n) + 2iB^*(3\mathrm{w}-\omega)\,. &(4.39)
\end{aligned}$$

These equations can be solved for A_i, B and ω in terms of n, m and w. The general solution is quite involved, but setting $n = m$ gives three simple solutions

$$I: \quad \omega = \mathrm{w}\,, \qquad A_1 = \text{free}\,, \qquad A_2 = -\frac{2i\mathrm{w}}{n}\,, \qquad B = -1\,, \tag{4.40}$$

$$\begin{aligned}
II_\pm: \quad \omega &= \mathrm{w} \pm 2\sqrt{\mathrm{w}^2+n^2}\,, \qquad A_1 = 0\,,\\
A_2 &= \frac{2(-i\mathrm{w} \mp i\sqrt{\mathrm{w}^2+n^2})}{n}\,, \quad B = \frac{n^2 + 2(\mathrm{w}^2 \pm \mathrm{w}\sqrt{\mathrm{w}^2+n^2})}{n^2}\,. \qquad (4.41)
\end{aligned}$$

For the solution II_-, the X_i equations of motion reduce to

$$\begin{aligned}
0 = {}& 8\sqrt{2}\mathrm{w}\nu + \frac{16\sqrt{2}\nu\mathrm{w}^2(\mathrm{w}-\nu)}{n^2} + 2n\partial_1 M - 2\partial_1^2 F - i\partial_1^2 N\\
&+4i\mathrm{w}\partial_0 F - 2w\partial_0 N + 2\partial_0^2 F + i\partial_0^2 N\,,
\end{aligned} \tag{4.42}$$

where we have defined

$$M(\tau,\sigma) = G(\tau,\sigma) + H(\tau,\sigma)\,, \qquad N(\tau,\sigma) = G(\tau,\sigma) - H(\tau,\sigma)\,. \tag{4.43}$$

The AdS_5 equations of motion and the conformal gauge constraints reduce to

$$\begin{aligned}
0 &= 4n\mathrm{w}\partial_1 F + \mathrm{w}\partial_1^2 M + n\partial_0^2 M\,, &(4.44)\\
0 &= 4n\mathrm{w}\partial_0 F + \mathrm{w}\partial_0\partial_1 M + n\partial_1^2 M\,, &(4.45)\\
0 &= 8\sqrt{2}\mathrm{w}\nu(n^2 + 2\mathrm{w}^2 - 2\mathrm{w}\nu) + n^3\partial_1 N + n^2\mathrm{w}\partial_0 M\,, &(4.46)
\end{aligned}$$

together with the condition

$$w = \sqrt{\nu^2 - n^2}\,, \tag{4.47}$$

and the following equations for f

$$\begin{aligned}
\partial_0 f &= 0\,, &(4.48)\\
\partial_1 f &= -8\nu^3(n^2 - 2\nu^2 + 2\nu\mathrm{w}) + \sqrt{2}n^3\left(4n\mathrm{w}F + \mathrm{w}\partial_1 M + n\partial_0 N\right)\,. &(4.49)
\end{aligned}$$

A simple solution to these equations is

$$\begin{aligned}
M &= -\frac{8\sqrt{2}\nu(n^2 + 2\mathrm{w}^2 - 2\mathrm{w}\nu)}{n^2}\tau\,, &(4.50)\\
N &= \frac{4\sqrt{2}\nu(n^2 + 2\mathrm{w}^2 - 2\mathrm{w}\nu)}{n^2}\tau\,, &(4.51)\\
F &= \frac{\sqrt{2}\nu(n^2+\nu^2)(n^2 + 2\mathrm{w}^2 - 2\mathrm{w}\nu)}{wn^4}\,, &(4.52)
\end{aligned}$$

and $f = 0$ which implies $x^+ = -x^-$. While the existence of this exact solution is quite remarkable, we should stress that its complexity (in particular, the fact that the phases of X_1 and X_2 are different) indicates again the need for some further field redefinitions to match the string and the spin chain variables. This solution also shows the difference between the $\eta = \eta_4$ and $\theta = \theta_4$ subsectors on the string side. Comparing to the solution in the previous subsection it is clear that some field redefinitions are needed to make explicit the $SU(2)$ symmetry between the two fermions θ and η.

5 Matching the string and spin-chain actions

Let us now discuss how to relate the Landau-Lifshitz action (2.44) representing the low-energy coherent states of the $SU(2|3)$ spin chain to a "fast-string" limit of the superstring action (3.1) or (3.15),(3.16),(3.18). We shall first consider the quadratic fermionic term in the general $SU(2|3)$ case (to "one-loop" or leading term in "fast-string" expansion) and then discuss the special case of $SU(1|1)$ sector (including also subleading terms).

5.1 $SU(2|3)$ case to leading order

As was mentioned in sect. 3.2, we should isolate the common large phase α of the S^5 bosons as in (3.11) and simplify the Lagrangian assuming that ν in $\alpha = \nu\tau + v$ large. In order to do that one may also redefine the two pairs of 3+1 fermions as follows

$$\eta_i \to \frac{1}{\nu} e^{i\nu\tau} \xi_i \ , \qquad \eta \to e^{-i\nu\tau}\psi \ , \qquad \theta_i \to \frac{1}{\nu} e^{2i\nu\tau}\zeta_i \ , \qquad \theta \to \theta \ , \tag{5.1}$$

where in general $\nu\tau$ should be replaced by $\alpha = \nu\tau + v$. Note that after these rotations the $\epsilon_{ijk}\eta^i\partial_1\theta^j X^k$ terms in (3.15) have large non-vanishing phases and thus average to zero as in [16, 19, 17].[16] The remaining terms in (3.15) become ($U^i \equiv U_i^*$)

$$\begin{aligned} \tilde{\mathcal{L}}_{2F} \ = \ & i\bar{\psi}\partial_0\psi + i\bar{\theta}\partial_0\theta + U_i\xi^i\partial_1\bar{\theta} - U^i\xi_i\partial_1\theta + U_i\partial_1\zeta^i\bar{\psi} - U^i\partial_1\zeta_i\psi \\ & + 2\zeta_i\zeta^i + 2U^iU_j\xi_i\xi^j + iU^i\partial_0U_i\bar{\psi}\psi \\ & + \frac{i}{\nu^2}\Big[\xi^i\partial_0\xi_i + \zeta^i\partial_0\zeta_i - (U^i\partial_0U_j - U_j\partial_0U^i)\xi_i\xi^j - U^i\partial_0U_i\xi^j\xi_j\Big] \ . \end{aligned} \tag{5.2}$$

Dropping the subleading $\frac{1}{\nu^2}$ term we observe that $U^i\xi_i$ and ζ_i become non-dynamical variables that can be easily solved for and then eliminated from the Lagrangian:

$$U^i\xi_i = \frac{1}{2}\partial_1\bar{\theta} \ , \qquad\qquad \zeta_i = -\frac{1}{2}\partial_1(U_i\bar{\psi}) \ . \tag{5.3}$$

[16] A possible alternative to using the averaging procedure may be to splitting η_i into the transverse and longitudinal part with respect to U^i $\eta_i = \eta_{\perp i} + U_i q$, $q = U^i\eta_i$, $U^i\eta_{\perp i} = 0$ and to try to decouple q and $\eta_{\perp i}$.

Then, to leading order in ν, the quadratic part of the Lagrangian (5.2) is

$$\tilde{\mathcal{L}}_{2F} = i\bar{\psi}\partial_0\psi + i\bar{\theta}\partial_0\theta - \frac{1}{2}\partial_1\bar{\theta}\partial_1\theta - \frac{1}{2}\partial_1(U_i\bar{\psi})\partial_1(U^i\psi) + iU^i\partial_0 U_i\bar{\psi}\psi\,. \tag{5.4}$$

Solving the conformal gauge constraints (3.9),(3.10) we obtain [17]

$$\partial_0 v \;=\; -C_0 - \frac{1}{2}|D_1U_i|^2 + \dots\,, \qquad \partial_1 v = -C_1 + \dots\,, \qquad C_a = -iU^i\partial_a U_i\,. \tag{5.5}$$

These relations will be modified by fermionic terms indicated by .... To determine the quadratic term in the Lagrangian it is, however, enough to ignore these terms. In order to match the spin-chain action (2.44) for (U_i, ψ_α) we need an extra redefinition of the fermions θ and ψ

$$(\theta, \psi) \to 2^{1/4}e^{iv}(\psi_1\,, \psi_2\,)\;. \tag{5.6}$$

Then the Lagrangian (5.4) takes the form

$$\tilde{\mathcal{L}}_{2F} = i\bar{\psi}_\alpha D_0\psi_\alpha + \frac{1}{2}|D_1U_i|^2\bar{\psi}_\alpha\psi_\alpha - \frac{1}{2}D_1^*\bar{\psi}_\alpha D_1\psi_\alpha\;. \tag{5.7}$$

Combined with the $SU(3)$ sector bosonic contribution [15] from (2.29) this is almost identical to the quadratic part of the spin chain Lagrangian (2.44) apart from the minus sign in the fermionic D_0 term. This sign can be matched by renaming $\tau \to -\tau$ and $U_i \to U_i^*$ in relating the string action to the spin chain action.

What remains is to show that (i) the ansatz (3.3) for the bosonic we used is consistent, and (ii) quartic fermionic terms also match. To demonstrate (i) one is to show, in particular, that the two equations for x^- implied by the ϕ and x^+ equations of motion following from (3.1) or (3.15) are indeed consistent with each another, and that the two equations for v given in (5.5) are also consistent. Since we have only worked to quadratic order these questions can be justifiably ignored in our treatment, but need to be addressed as part of understanding (ii).

Proving (ii) may involve an additional field redefinition which we did not find. We shall only mention that the structure of the quartic terms in (3.16) (where the ϵ_{ijk} terms should not contribute after time averaging) is, in principle, consistent with that of quartic fermionic terms in (2.44) (which contain spatial derivatives of fermions) after one uses (5.1),(5.3) to eliminate $U^i\xi_i$ in terms of $\partial_1\theta$.

Let us now look at subsectors of the superstring action and discuss their relation to the corresponding subsectors of the Landau-Lifshitz action.

[17] We use the identity $U^iD_1U_i = 0$. We also drop by "averaging" the same terms that we omitted in getting from eq. (3.15) to eq. (5.2). Equivalently, these terms should be dropped already in the action (3.1).

5.2 $SU(1|1)$ case

Let us specialize to the $SU(1|1)$ sector where $U_1 = 1,\ U_2, U_3 = 0$ and there is just one fermionic degree of freedom. The quadratic fermionic part of both the spin chain (2.44) and the string (5.7) Lagrangians reduces to the following leading-order term (here we rescale $\tau \to t$ and set $\tilde\lambda = \frac{\lambda}{J^2}$)

$$\mathcal{L} = -i\bar\psi\partial_t\psi - \frac{1}{2}\tilde\lambda\ \partial_1\bar\psi\partial_1\psi\ . \tag{5.8}$$

This may be interpreted as a Lagrangian for a non-relativistic fermion (see also [14]).[18]

One interesting question is how that action extends to higher orders in $\tilde\lambda$. The answer turns out to be that it is just a natural "relativistic" generalization (up to integration by parts):

$$\mathcal{L} = -i\bar\psi\partial_t\psi - \bar\psi\left(\sqrt{1-\tilde\lambda\partial_1^2} - 1\right)\psi\ . \tag{5.9}$$

This the action that reproduces the equation for the upper component of the 2-d massive 2-component Dirac fermion (upon elimination of the other component) with mass $m = \frac{1}{\sqrt{\tilde\lambda}} = \frac{J}{\sqrt{\lambda}}$; it is thus in agreement with the BMN spectrum to all orders in $\tilde\lambda$. This action is in agreement with recent results on the higher-order generalization of the Bethe ansatz in the $SU(1|1)$ sector: the above expression (or its direct discretization) reproduces the leading "BMN" terms in the corresponding Bethe ansatz expressions in [35, 29].

The bosonic analog of (5.9) appeared already in the discussion of the $SU(2)$ sector in [13]: there the sum of all terms in the string effective Hamiltonian that are of second order in the 3-vector $\vec n$ parametrising the semiclassical state of a fast string (or coherent state of the spin chain) was found to be

$$\mathcal{L} = \vec C(n)\partial_t\dot{\vec n} - \frac{1}{4}\vec n\left(\sqrt{1-\tilde\lambda\ \partial^2} - 1\right)\vec n + O(\vec n^4)\ . \tag{5.10}$$

This expression is in agreement with the few leading-order results for the coherent-state action derived from the $SU(2)$ sector dilatation operator and with the exact BMN spectrum [41].[19]

Let us explain now how the action (5.9) can be derived from the full superstring Lagrangian (we shall consider only the quadratic terms in the latter). One expects to

[18] The fact that a massive non-relativistic fermion action appears in the coherent state path integral of the XY spin chain in a magnetic field is well-known [40]. For a fine-tuned coefficient of the magnetic field the XY model has [29] hidden $SU(1|1)$ symmetry (a relation of this spin chain to free fermion was pointed out earlier in [35]).

[19] The coherent state analogs of the BMN states are small fluctuations near the vacuum state $\vec n_0 = (0, 0, 1)$. On the spin chain side these correspond (in the discrete version) to the microscopic spin wave excitations or magnons. Similar relation appears in the $SU(1|1)$ sector [35, 29].

reproduce the BMN-type massive fermion action for quadratic fermionic fluctuations in the case of the point-like bosonic background

$$\mathrm{X}_1 = e^{i\alpha} \,, \qquad \mathrm{X}_2, \mathrm{X}_3 = 0 \,. \tag{5.11}$$

Using (5.11) in the action (3.15) we get (here $a, b = 2, 3$)

$$\begin{aligned}\tilde{\mathcal{L}}_F &= i\eta^1\partial_0\eta_1 + i\eta^a\partial_0\eta_a + i\bar{\eta}\partial_0\eta + i\theta^1\partial_0\theta_1 + i\theta^a\partial_0\theta_a + i\bar{\theta}\partial_0\theta \\ &\quad + \epsilon_{ab}\eta^a\partial_1\theta^b e^{-i\alpha} - \epsilon^{ab}\eta_a\partial_1\theta_b e^{i\alpha} + \eta^1\partial_1\bar{\theta}e^{i\alpha} - \eta_1\partial_1\theta e^{-i\alpha} + \partial_1\theta^1\bar{\eta}e^{i\alpha} - \partial_1\theta_1\eta e^{-i\alpha} \\ &\quad -\partial_0\alpha(\eta^1\eta_1 + \bar{\eta}\eta - \eta^a\eta_a) + O(\eta^4) \,.\end{aligned} \tag{5.12}$$

It is clear now that θ_a and η_a decouple from the singlet sector and we can consistently set them to zero. The same conclusion remains after we include the quartic fermionic term (3.16) which reduces simply to $\frac{\nu}{\sqrt{2}}\eta^1\eta_1\bar{\eta}\eta$. Then we are left with

$$\begin{aligned}\tilde{\mathcal{L}}_F &= i\eta^1\partial_0\eta_1 + i\bar{\eta}\partial_0\eta + i\theta^1\partial_0\theta_1 + i\bar{\theta}\partial_0\theta \\ &\quad + \eta^1\partial_1\bar{\theta}e^{i\alpha} - \eta_1\partial_1\theta e^{-i\alpha} + \partial_1\theta^1\bar{\eta}e^{i\alpha} - \partial_1\theta_1\eta e^{-i\alpha} \\ &\quad -\partial_0\alpha(\eta^1\eta_1 + \bar{\eta}\eta) + \frac{\nu}{\sqrt{2}}\eta^1\eta_1\bar{\eta}\eta \,.\end{aligned} \tag{5.13}$$

Now we can further do one of the two possible truncations (or (4.2),(4.1))

$$\theta_1 = \eta = 0 \quad \mathit{or} \qquad \eta_1 = \theta = 0 \,.$$

Both are consistent choices, and in both cases the quartic fermionic term vanishes. In the first case we end up with

$$\tilde{\mathcal{L}}_F = i\eta^1\partial_0\eta_1 + i\bar{\theta}\partial_0\theta + \eta^1\partial_1\bar{\theta}e^{i\alpha} - \eta_1\partial_1\theta e^{-i\alpha} - \partial_0\alpha\eta^1\eta_1 \,, \tag{5.14}$$

while in the second

$$\tilde{\mathcal{L}}_F = i\bar{\eta}\partial_0\eta + i\theta^1\partial_0\theta_1 + \partial_1\theta^1\bar{\eta}e^{i\alpha} - \partial_1\theta_1\eta e^{-i\alpha} - \partial_0\alpha\bar{\eta}\eta \,. \tag{5.15}$$

What remains then is to integrate out η_1 in the first case or θ_1 in the second.

More precisely, one should ensure that the remaining singlet fields are kept "massless" and eliminate time-dependent exponential factors in the mixing terms. That means that in the first case one should first apply the same redefinition as in (5.1), i.e. $\eta_1 \to e^{i\alpha}\eta_1$, $\theta \to \theta$. Then the mass of η_1 doubles, and integrating it out we get

$$\tilde{\mathcal{L}}_F = i\bar{\theta}\partial_0\theta - \partial_1\bar{\theta}\frac{1}{2\nu - i\partial_0}\partial_1\theta = i\bar{\theta}\partial_0\theta - \frac{1}{2\nu}\partial_1\bar{\theta}\partial_1\theta + \dots \,. \tag{5.16}$$

In the second case, we should keep η as a "light" field and so should do a redefinition to absorb its mass term $\eta \to e^{-i\alpha}\eta$, and then do a compensating redefinition of $\theta_1 \to e^{2i\alpha}\theta_1$ to eliminate the exponential phase factors in the mixing terms. The resulting redefinition is then the same as in (5.1). as a result, we end up with the same action for redefined $(\eta, \theta_1) \equiv (\psi, \zeta)$ as for the redefined $(\theta, \eta_1) \equiv (\theta, \xi)$ in the first case, i.e.

$$\tilde{\mathcal{L}} = i\bar{\psi}\partial_0\psi + i\bar{\zeta}\partial_0\zeta - 2\nu\bar{\zeta}\zeta + \partial_1\bar{\zeta}\ \bar{\psi} - \partial_1\zeta\ \psi\,. \tag{5.17}$$

Eliminating the massive ζ field, we finish with the same action as in (5.16)

$$\tilde{\mathcal{L}} = i\bar{\psi}\partial_0\psi - \partial_1\bar{\psi}\frac{1}{2\nu - i\partial_0}\partial_1\psi\,. \tag{5.18}$$

This provides a justification for the redefinition used in (5.1).

The second term in (5.18) should be treated perturbatively in ∂_0/ν (assuming the large ν limit). An equivalent action that leads to the same equations of motion is then

$$\tilde{\mathcal{L}} = i\bar{\psi}\partial_0\psi - \nu\bar{\psi}\left(\sqrt{1-\nu^{-2}\partial_1^2} - 1\right)\psi\,. \tag{5.19}$$

Indeed, the equation of motion for (5.18) written in momentum space implies $p_0 = \frac{p_1^2}{2\nu+p_0}$, solved by $p_0 = -\nu + \sqrt{\nu^2 + p_1^2}$, which is the same relation that follows from (5.19). The overall factor of ν is finally absorbed by a redefinition of $\tau \to t = \nu\tau$, i.e. we finish with (5.9) where $J = \sqrt{\lambda}\ \nu$.

The Lagrangian (5.19) is also equivalent to the Dirac Lagrangian for a massive ($m = \nu = \frac{J}{\sqrt{\lambda}}$) relativistic 2d fermion with one component integrated out. This is not surprising given that the superstring action in the S^5 geodesic (BMN) background is known to contain free massive 2d fermions [36]. The usual 2d fermionic Lagrangian can be written as

$$\mathcal{L} = i\bar{\Psi}\rho^a\partial_a\Psi + m\bar{\Psi}\rho^3\Psi\ , \qquad \bar{\Psi} = \Psi^\dagger\rho^0\ , \quad \rho^0 = i\sigma_2\ , \quad \rho^1 = \sigma_1\ , \tag{5.20}$$

where $\rho^3 = \rho^0\rho^1 = \sigma_3$ and $\Psi = (\psi_1, \psi_2)$. Explicitly,

$$L = -i\psi_1^*(\partial_0 - \partial_1)\psi_1 - i\psi_2^*(\partial_0 + \partial_1)\psi_2 + m(\psi_1^*\psi_2 + \psi_2^*\psi_1) \tag{5.21}$$

This leads to the same dispersion relation as the one that follows from (5.19) (with only one solution chosen, as dictated by large mass expansion). That means that there should be a direct field redefinition that relates the two quadratic actions.

As for possible higher order fermionic terms in (5.19) (e.g. $\bar{\psi}\psi\partial_1\bar{\psi}\partial_1\psi$, etc.) we expect that there exists a field redefinition that completely eliminates them. As suggested by the form of the exact solution of sect. 4.1, such a field redefinition should involve shifting X_1 by fermionic terms.

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Appendix A The ρ-matrices

We follow the notation of [27, 28]. The six 4×4 matrices ρ^M_{AB} are blocks of the $SO(6)$ Dirac matrices γ^M in the chiral representation, i.e.

$$\gamma^M = \begin{pmatrix} 0 & (\rho^M)^{AB} \\ \rho^M_{AB} & 0 \end{pmatrix}, \qquad \rho^M_{AB} = -\rho^M_{AB}\,, \qquad (\rho^M)^{AB} \equiv -(\rho^M_{AB})^*\,, \tag{A.1}$$

$$(\rho^M)^{AC}\rho^N_{CB} + (\rho^N)^{AC}\rho^M_{CB} = 2\delta^{MN}\delta^A_B\,. \tag{A.2}$$

Note that since $X_M X^M = 1$

$$X_M\rho^M_{AB}X_N\rho^{NBC} = \delta^C_A\,. \tag{A.3}$$

In this paper we have chosen the following representation for the ρ^M_{AB} matrices

$$\rho^1_{AB} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \rho^2_{AB} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \rho^3_{AB} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\rho^4_{AB} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \rho^5_{AB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \rho^6_{AB} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

With this choice the following relations hold

$$X_M\rho^M_{ij} = \epsilon_{ijk}\mathrm{X}^k\,, \qquad X_M\rho^M_{i4} = \mathrm{X}_i\,, \qquad X_M\rho^M_{4j} = -\mathrm{X}_j\,, \tag{A.4}$$

where $i,\ j = 1,2,3$ are the $SU(3)$ indicies, $\epsilon_{123} = \epsilon^{123} = 1$, and we have defined

$$\mathrm{X}_j = X_{2j-1} + iX_{2j}\,, \qquad \mathrm{X}^i \equiv \mathrm{X}^*_i\,, \qquad \mathrm{X}^i\mathrm{X}_i = 1\,. \tag{A.5}$$

Similarly, we have

$$X_M \rho^{Mij} = -\epsilon^{ijk} \mathrm{X}_k \,, \qquad X_M \rho^{Mi4} = -\mathrm{X}^i \,, \qquad X_M \rho^{M4i} = \mathrm{X}^i \,. \tag{A.6}$$

We also define

$$\rho^{MNA}{}_B = \frac{1}{2} \left(\rho^{MAC} \rho^N{}_{CB} - \rho^{NAC} \rho^M{}_{CB} \right) \,. \tag{A.7}$$

With our choice of ρ^M the only diagonal matrices among ρ^{MN} are ρ^{12}, ρ^{34} and ρ^{56}. In the above $SU(3)$ notation we get

$$X_M \partial X_N \rho^{MNA}{}_B = \begin{pmatrix} \mathrm{X}^i \partial \mathrm{X}_l - \partial \mathrm{X}^i \mathrm{X}_l + \delta^i_l \mathrm{X}_m \partial \mathrm{X}^m & \epsilon^{ijk} \mathrm{X}_j \partial \mathrm{X}_k \\ \epsilon_{ljk} \partial \mathrm{X}^j \mathrm{X}^k & \mathrm{X}^j \partial \mathrm{X}_j \end{pmatrix} \,. \tag{A.8}$$

Here we have used that $\mathrm{X}_i \mathrm{X}^i = 1$ implies $\mathrm{X}^j \partial \mathrm{X}_j = -\mathrm{X}_j \partial \mathrm{X}^j$.

Other useful relations are (we always assume the sum over repeated M, N indices)

$$\begin{aligned} X^M (\rho^{Mi})^l{}_k &= 2\delta^i_k X^l - \delta^l_k X^i \,, & X^M (\rho^{Mi})^l{}_4 &= 2\epsilon^{lmi} X_m \,, \\ X^M (\rho^{Mi})^4{}_k &= 0 \,, & X^M (\rho^{Mi})^4{}_4 &= X^i \,, \end{aligned} \tag{A.9}$$

$$\begin{aligned} X^M (\rho^M{}_i)^l{}_k &= -2\delta^i_l X_k + \delta^l_k X_i \,, & X^M (\rho^M{}_i)^l{}_4 &= 0 \,, \\ X^M (\rho^M{}_i)^4{}_k &= 2\epsilon_{kin} X^n \,, & X^M (\rho^M{}_i)^4{}_4 &= -X_i \,. \end{aligned} \tag{A.10}$$

Here we defined

$$\rho^{Mi} \equiv \rho^{M,2i-1} - i\rho^{M,2i} \,, \qquad \rho^M_i \equiv \rho^{M,2i-1} + i\rho^{M,2i} \,. \tag{A.11}$$

We find also that

$$X^M \eta_A (\rho^{Mi})^A{}_B \eta^B = 2\eta^i \eta_j \mathrm{X}^j + \mathrm{X}^i (\bar\eta \eta - \eta^j \eta_j) - 2\epsilon^{ijk} \mathrm{X}_k \eta_j \bar\eta \,, \tag{A.12}$$

$$X^M \eta_A (\rho^M_i)^A{}_B \eta^B = 2\eta_i \eta^j \mathrm{X}_j - \mathrm{X}_i (\bar\eta \eta - \eta^j \eta_j) - 2\epsilon_{ijk} \mathrm{X}^k \eta^j \eta \,, \tag{A.13}$$

and

$$\begin{aligned} X^M X^K \rho^{MNi}{}_j \eta_C \rho^{NKC}{}_D \eta^D &= -2X_j X^i (\eta_k \eta^k - \eta^2) + 2\eta_j X^i X_k \eta^k - 2\eta^i X_j \eta_k X^k \\ &\quad -2\epsilon_{kjm} X^i \eta X^m \eta^k + 2\epsilon^{kmi} X_m X_j \eta_k \bar\eta \\ &\quad +\delta^i_j (\eta_k \eta^k - \eta^2 - 2X^k \eta_k X_m \eta^m \,, \end{aligned} \tag{A.14}$$

$$X^M X^K \rho^{MNi}{}_4 \eta_C \rho^{NKC}{}_D \eta^D = -2X_k \eta^k \epsilon^{kim} X_m \eta_k - 2\eta\eta^i + 2\eta X^i X_k \eta^k \,, \tag{A.15}$$

$$X^M X^K \rho^{MN4}{}_j \eta_C \rho^{NKC}{}_D \eta^D = 2X^k \eta_k \epsilon^{jkm} X^m \eta_k - 2\eta_j \bar\eta + 2X_j X^k \eta_k \bar\eta \,, \tag{A.16}$$

$$X^M X^K \rho^{MN4}{}_4 \eta_C \rho^{NKC}{}_D \eta^D = \eta^2 - \eta_k \eta^k + 2\eta_k X^k \eta^l X_l \,. \tag{A.17}$$

These formulæ in turn give the relation used in simplifying the quartic fermionic terms in the string action in section 3

$$\begin{aligned} & [X^M \eta_A (\rho^{MN})^A{}_B \eta^B][X^K \eta_C (\rho^{KN})^C{}_D \eta^D] \\ = \; & 4\bar\eta \eta \eta^i \eta_i - 8\mathrm{X}_i \eta^i \mathrm{X}^j \eta_j \bar\eta \eta + 8\eta_i \mathrm{X}^i \eta^j \mathrm{X}_j \eta_k \eta^k - \eta_i \eta^i \eta_j \eta^j \\ + \; & 4\epsilon_{ijk} \eta^i \eta^j \mathrm{X}^k \eta_l \mathrm{X}^l \eta + 4\epsilon^{ijk} \eta_i \eta_j \mathrm{X}_k \eta^l \mathrm{X}_l \bar\eta \,. \end{aligned} \tag{A.18}$$

Appendix B $SU(4)$ charges of the string action

Here we will express the string sigma model $SU(4)$ charges obtained in [27, 28] in $SU(3)$ notation. The $SU(4)$ charges are given by (using our AdS_5 ansatz (3.3))

$$\mathcal{J}^A{}_B = \int d\sigma\ \mathcal{J}^{0A}{}_B\,, \tag{B.1}$$

$$\begin{aligned}\mathcal{J}^{0A}{}_B &= \frac{i}{2}X^M\partial_0 X^n\rho^{MNA}{}_B \\ &\quad -\frac{\nu}{\sqrt{2}}\Big(\theta^A\theta_B+\eta^A\eta_B+\frac{1}{4}(\theta_C\theta^C+\eta_C\eta^C)-\frac{1}{2}X^M X^K\rho^{MNA}{}_B\eta_C\rho^{NKC}{}_D\eta^D\Big)\,.\end{aligned} \tag{B.2}$$

Using the expressions in Appendix A we can re-write $\mathcal{J}^{0A}{}_B$ in the $SU(3)$ notation

$$\begin{aligned}\mathcal{J}^{0i}{}_j &= \frac{i}{2}\Big(\mathrm{X}^i\partial_0\mathrm{X}_j-\mathrm{X}_j\partial_0\mathrm{X}^i+\delta^i_j\mathrm{X}_k\partial_0\mathrm{X}^k\Big) \\ &\quad +\frac{\nu}{\sqrt{2}}\Big(\theta^i\theta_j+\eta^i\eta_j+\mathrm{X}_j\mathrm{X}^i(\eta_k\eta^k+\bar\eta\eta)-2\eta_j\mathrm{X}^i\mathrm{X}_k\eta^k+2\eta^i\mathrm{X}_j\mathrm{X}^k\eta_k \\ &\qquad +2\epsilon_{kjm}\mathrm{X}^i\eta\mathrm{X}^m\eta^k-2\epsilon^{kmi}\mathrm{X}_j\eta_k\mathrm{X}_m\bar\eta \\ &\qquad +\frac{1}{4}\delta^i_j(\theta_k\theta^k-\bar\theta\theta-\eta_k\eta^k-2\bar\eta\eta+2\eta_k\mathrm{X}^k\mathrm{X}_m\eta^m)\Big)\,,\end{aligned} \tag{B.3}$$

$$\mathcal{J}^{0i}{}_4 = \frac{i}{2}\epsilon^{imk}\mathrm{X}_m\partial_0\mathrm{X}_k+\frac{\nu}{\sqrt{2}}\Big(\theta^i\theta+2\eta^i\eta+\mathrm{X}^i\eta\mathrm{X}_k\eta^k-\epsilon^{imk}\mathrm{X}_k\eta_m\mathrm{X}_l\eta^l\Big)\,, \tag{B.4}$$

$$\mathcal{J}^{04}{}_i = \frac{i}{2}\epsilon_{imk}\mathrm{X}^m\partial_0\mathrm{X}^k+\frac{\nu}{\sqrt{2}}\Big(\theta_i\bar\theta+2\bar\eta\eta_i+\mathrm{X}_i\mathrm{X}^k\eta_k\bar\eta-\epsilon_{imk}\mathrm{X}^k\eta^m\mathrm{X}^l\eta_l\Big)\,, \tag{B.5}$$

$$\mathcal{J}^{04}{}_4 = \frac{i}{2}\mathrm{X}^k\partial_0\mathrm{X}_k+\frac{\nu}{4\sqrt{2}}\Big(3\bar\theta\theta+5\bar\eta\eta+\theta_k\theta^k+3\eta_k\eta^k-\eta_k\mathrm{X}^k\eta^m\mathrm{X}_m\Big)\,, \tag{B.6}$$

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