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On projective and injective polynomial modules

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1 Introduction

Let k denote a field. For each $n \geq 0$ we have a category $\mathrm{Pol}(n)$ of finite dimensional polynomial modules and a category $\mathrm{Rat}(n)$ of rational modules over k. For $r \geq 0$, we write $\mathrm{Pol}(n, r)$ for the full subcategory of $\mathrm{Pol}(n)$ whose objects are the polynomial modules of degree r. Each $V \in \mathrm{Pol}(n)$ has a unique decomposition $V = \bigoplus_{r=0}^{\infty} V(r)$, where $V(r)$ is polynomial of degree r. Furthermore, the category $\mathrm{Pol}(n, r)$ is naturally equivalent to the category of modules for the (finite dimensional, associative) Schur algebra $S(n, r)$. It follows that every polynomial module has a projective cover and an injective hull.

We here attempt to describe those polynomial modules which are both projective and injective. We give a complete description in the cases $n = 2, 3$ and make a conjecture which, if true, gives a complete description in the general case.

A finite dimensional polynomial module which is injective and projective must be a tilting module and, thanks to a natural duality on the category of polynomial modules, our problem is equivalent to that of determining which tilting modules are injective (equivalently projective). It is often in this form that we address the problem. The two main technical advantages enjoyed by tilting modules are a certain stability property, as n varies, and the fact that the theory of tilting modules exists in the larger category of rational modules.

We work with a quantum general linear group over an arbitrary base field with arbitrary parameter q. We do this not only in the interests of generality but also because our results, which are new even in the classical case, are obtained by arguments which involve reduction from the "ideal rational category" of representations of the quantum group over a field of characteristic 0 with parameter q a root of unity.

2 Preliminaries

2.1 Combinatorics

Much of polynomial representation theory is governed by the combinatorics of partitions. We now introduce some of the terminology. For a thorough treatment of most of the issues discussed here we refer the reader to Green's monograph [12]: useful additional references are [14] and [16].

By a partition we mean a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_i = 0$ for $i \gg 0$. The length of the zero partition is 0 and, if λ is non-zero, it has length m, where $\lambda_m \neq 0$ and $\lambda_{m+1} = 0$. We also call the length of λ its number of parts.

We denote by λ' the conjugate (or transpose) of a partition λ. We fix a positive integer l. A partition λ is called row l-regular if there is no $i \geq 0$ such that $\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+l} > 0$, and called column l-regular if λ' is row l-regular, i.e. if $\lambda_i - \lambda_{i+1} < l$ for all $i \geq 1$.

Let $\mathcal{P}$ denote the set of all partitions and let $\mathcal{P}_{\text{reg}}$ denote the set of row l-regular partitions. There is defined on $\mathcal{P}_{\text{reg}}$ an involution, known as the Mullineux map. For $\lambda \in \mathcal{P}_{\text{reg}}$ we write $\text{Mull}(\lambda)$ for the image of λ under this map and call $\text{Mull}(\lambda)$ the Mullineux conjugate (or transpose) of λ.

The standard terms "hook lengths, l-core, l-weight, rim, l-segment and l-

edge", in connection with partitions, will be used without further reference. For full explanations see [14] and [16].

We now fix a positive integer n. We write $\Lambda(n)$ for the set of all n-tuples of non-negative integers. We identify a partition having at most n parts with an element of $\Lambda(n)$ and write $\Lambda^+(n)$ for the set of partitions which have at most n parts. We also set $X(n) = \mathbf{Z}^n$ and write $X^+(n)$ for the set of $\lambda = (\lambda_1, \dots, \lambda_n) \in X(n)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We call elements of $X(n)$ weights, elements of $X^+(n)$ dominant weights, elements of $\Lambda(n)$ polynomial weights and elements of $\Lambda^+(n)$ dominant polynomial weights. For $\alpha = (\alpha_1, \dots, \alpha_n) \in X(n)$ we write $|\alpha|$ for $\alpha_1 + \cdots + \alpha_n$ and call this the degree of α. We write $\Lambda^+(n, r)$ for the set of dominant polynomial weights of degree r, i.e. partitions of degree r which have at most n parts. For $\lambda \in \Lambda^+(n)$ and $l \in \mathbf{N}$, we say that λ is l-restricted if $\lambda_i - \lambda_{i+1} < l$ for all $i = 1, \dots n-1$. Note that if λ is column l-regular then it is l-restricted but the converse is false.

We write ϵ_i for $(0, \dots, 0, 1, 0, \dots, 0)$ (where 1 appears in the ith place). We write Φ^+ for the set of elements $\epsilon_i - \epsilon_j$, with $1 \leq i < j \leq n$ and call Φ^+ the set of positive roots. We write Φ^- for the set of elements $\epsilon_j - \epsilon_i$, with $1 \leq i < j \leq n$ and call Φ^- the set of negative roots. We write Φ for $\Phi^+ \cup \Phi^-$, and call Φ the set of roots. There is a natural partial order on $X(n)$, called the dominance order: we have $\lambda \leq \mu$ if $\mu - \lambda$ is a sum of positive roots. For partitions λ, μ we write $\lambda \leq \mu$ if we have $\lambda \leq \mu$ when λ, μ are regarded as elements of $X(n)$, for n suitably large. Of particular importance to us are the elements $\delta = (n-1, n-2, \dots, 1, 0)$ and $\omega = (1, 1, \dots, 1)$ of $\Lambda^+(n)$.

The symmetric group $W = \mathrm{Sym}(n)$, of degree n, acts naturally on $X(n)$. For each positive root $\alpha = \epsilon_i - \epsilon_j \in \Phi$ we define the reflection $s_\alpha \in W$ to be the permutation which swaps the i-th and the j-th coordinates and leaves everything else fixed. Clearly, W is generated by $\{s_\alpha \,|\, \alpha \in \Phi\}$. Of particular importance to us is the longest element $w_0 = (1, n)(2, n-1) \dots$.

For an integer m, we define the affine reflection $s_{\alpha,m}$ by $s_{\alpha,m}(\lambda) = s_\alpha(\lambda) + m\alpha$. For a fixed positive integer l, we define the affine Weyl group W_l to be the group generated by $\{s_{\alpha,ml} \,|\, \alpha \in \Phi,\, m \in \mathbf{Z}\}$. We shall also use the dot action of the affine Weyl group W_l on $X(n)$ defined by $w \cdot \lambda = w(\lambda + \delta) - \delta$ for $\lambda \in X(n)$ and $w \in W_l$.

2.2 Global Representation Theory

We here establish notation relating to a quantum general linear group and its representation theory. For the most part this is a straightforward generalization of the theory for general linear groups, as described in [12]. For a fuller account we refer the reader to [9] (especially to the Introduction of [9] for the relationship with the ordinary general linear group).

Let k be a field. We shall use the expression "G is a quantum group over k" to indicate that we have in mind a Hopf algebra over k, denoted $k[G]$ and called the coordinate algebra of G. We use the expression "$\theta : G \to H$ is a morphism of quantum groups" to indicate that G and H are quantum groups and we have in mind a morphism of Hopf algebras from $k[H]$ to $k[G]$, denoted $\theta^\sharp$ and called the comorphism of θ. By the expression "H is a subgroup of the quantum group G" we indicate that we have in mind a quotient Hopf algebra $k[H] = k[G]/I_H$. The Hopf ideal I_H is called the defining ideal of $k[H]$. The morphism of quantum groups $i : H \to G$ such that $i^\sharp : k[G] \to k[H]$ is the natural map will be called (by abuse of notation) inclusion.

By a (left) G-module we mean a right $k[G]$-comodule. We write $\mathrm{Mod}(G)$ for the category of left G-modules and $\mathrm{mod}(G)$ for the category of finite dimensional left G-modules. For $V \in \mathrm{Mod}(G)$ we have the left exact functor $\mathrm{Hom}_G(V, -)$, from $\mathrm{Mod}(G)$ to the category of k-spaces. We write $\mathrm{Ext}^i_G(V, -)$ for the ith derived functor of $\mathrm{Hom}_G(V, -)$. Taking $V = k$, the trivial one dimensional G-module, we have the cohomology functors $H^i(G, -) = \mathrm{Ext}^i_G(k, -)$, from G-modules to k-spaces. In particular, for a G-module V with structure map $\tau : V \to V \otimes k[G]$ we have $H^0(G, V) = V^G$, the "fixed point space" $\{v \in V | \tau(v) = v \otimes 1\}$. For a finite dimensional G-module V the dual space $V^* = \mathrm{Hom}_k(V, k)$ has a natural G-module structure. For G-modules V, W the tensor product space $V \otimes W$ has a natural G-module structure. Moreover, if V is finite dimensional then $\mathrm{Hom}_k(V, W)$ has a natural G-module structure. We have, in general, $\mathrm{Hom}_G(V, W) \leq \mathrm{Hom}_k(V, W)^G$ and equality if the antipode $\sigma_G : k[G] \to k[G]$ is injective.

For a positive integer n, a commutative ring R and $q \in R$, we have the R-algebra $A(n) = A_{R,q}(n)$ given by generators $c_{ij} = c_{ij,R,q}$, $1 \leq i, j \leq n$, subject to the relations:

$$c_{ir}c_{is} = c_{is}c_{ir} \qquad \text{for all } 1 \leq i, r, s \leq n$$

$$c_{jr}c_{is} = qc_{is}c_{jr} \qquad \text{for all } 1 \leq i < j \leq n, 1 \leq r \leq s \leq n$$

$$c_{js}c_{ir} = c_{ir}c_{js} + (q-1)c_{is}c_{jr} \qquad \text{for all } 1 \le i < j \le n, 1 \le r < s \le n.$$

Now we assume that $R = k$, then $A(n)$ is a k-bialgebra with comultiplication $\delta : A(n) \to A(n) \otimes A(n)$ and augmentation $\epsilon : A(n) \to k$ satisfying $\delta(c_{ij}) = \sum_{r=1}^n c_{ir} \otimes c_{rj}$ and $\epsilon(c_{ij}) = \delta_{ij}$, for $1 \le i, j \le n$. The determinant $d \in A(n)$ is defined by $d = \sum_\pi \operatorname{sgn}(\pi) c_{1,1\pi} c_{2,2\pi} \dots c_{n,n\pi}$ where π runs over all permutations of $\{1, 2, \dots, n\}$ and where $\operatorname{sgn}(\pi)$ denotes the sign of a permutation π.

Now assume that $q \neq 0$. The bialgebra structure on $A(n)$ extends to the localization $A(n)_d$ of $A(n)$ at d (with $\delta(d^{-1}) = d^{-1} \otimes d^{-1}$ and $\epsilon(d^{-1}) = 1$). Furthermore, $A(n)_d$ is a Hopf algebra. We write $k[G(n)]$ for $A(n)_d$ and call $G = G(n)$ the quantum general linear group of degree n. We denote the antipode of $k[G(n)]$ by S. We have $S^2(f) = dfd^{-1}$, for $f \in k[G(n)]$, in particular, $S : k[G(n)] \to k[G(n)]$ is injective. Explicitly, we have

$$S^2(c_{ij}) = dc_{ij}d^{-1} = q^{j-i}c_{ij}.$$

Let $T(n)$ be the subgroup of $G(n)$ whose defining ideal is generated by $\{c_{ij} | 1 \le i, j \le n, i \neq j\}$. Then $k[T(n)]$ is the algebra of Laurent polynomial $k[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$, where $t_i = c_{ii} + I_{T(n)}$, for $1 \le i \le n$. We shall also need the subgroup $B(n)$ of $G(n)$, whose defining ideal is generated by $\{c_{ij} | 1 \le i < j \le n\}$.

We shall use the expression "V is a (left) rational $G(n)$-module" to indicate that V is a right $A(n)_d$-comodule. We write $\operatorname{Rat}(n)$ for the category of right $A(n)_d$-comodules. We write $\operatorname{Pol}(n)$ for the category of finite dimensional right $A(n)$-comodules. We shall also say that an object in $\operatorname{Pol}(n)$ is a polynomial $G(n)$-module. We have an algebra grading and coalgebra decomposition $A(n) = \bigoplus_{r=0}^\infty A(n,r)$ (obtained by giving each c_{ij} degree 1). Each $A(n,r)$ is finite dimensional and the Schur algebra $S(n,r)$ is by definition the dual algebra of $A(n,r)$. We write $\operatorname{Pol}(n,r)$ for the category of finite dimensional right $A(n,r)$-comodules (equivalently left $S(n,r)$-modules). We shall also say that an object in $\operatorname{Pol}(n,r)$ is a $G(n)$-module which is polynomial of degree r. For $V \in \operatorname{Pol}(n)$ we have a unique decomposition $V = \oplus_{r=0}^\infty V(r)$, with $V(r) \in \operatorname{Pol}(n,r)$. It follows that $\operatorname{Pol}(n)$ has enough injectives. For $V, W \in \operatorname{Pol}(n)$ we write $\operatorname{Hom}_{\operatorname{Pol}(n)}(V, W)$ for the space of comodule homomorphisms $\operatorname{Hom}_{A(n)}(V, W)$. Thus for $V \in \operatorname{Pol}(n)$ we have the left exact functor $\operatorname{Hom}_{\operatorname{Pol}(n)}(V, -)$, from $\operatorname{Pol}(n)$ to finite dimensional k-spaces, and derived functors $\operatorname{Ext}^i_{\operatorname{Pol}(n)}(V, -)$. However, for $W \in \operatorname{Pol}(n)$, we have $\mathrm{Ext}^i_G(V,W) = \mathrm{Ext}^i_{\mathrm{Pol}}(V,W)$, by [10], Section 4,(5). For G-modules V, W we shall often simply write $\mathrm{Ext}^i(V,W)$ for $\mathrm{Ext}^i_G(V,W)$, though occasionally write $\mathrm{Ext}^i_G(V,W)$ or $\mathrm{Ext}^i_{\mathrm{Pol}}(V,W)$, as appropriate, for emphasis. We shall also write simply $\mathrm{Ext}(V,W)$ for $\mathrm{Ext}^1_G(V,W)$.

In the case in which $q=1$ and k is algebraically closed, $\mathrm{Rat}(n)$ is naturally equivalent to the category of rational modules for the algebraic group $\mathrm{GL}_n(k)$ and $\mathrm{Pol}(n)$ is naturally equivalent to the category of finite dimensional polynomial $\mathrm{GL}_n(k)$-modules.

If V is a finite dimensional polynomial $G(n)$-module, and hence a rational module, then the dual rational module V^* is not, in general, polynomial. However, there is a duality which preserves polynomial modules. For V polynomial of degree r we denote by V° the polynomial module of degree r which is the contravariant dual of V. The construction is given in [9], Section 4.1, Remark (ii) (and in [12], Section 2.7 in the classical case).

For each $\lambda = (\lambda_1, \dots, \lambda_n) \in X(n)$ we have the one dimensional $T(n)$-module k_λ, with structure map taking $x \in k_\lambda$ to $x \otimes t^\lambda$, where $t^\lambda = t_1^{\lambda_1} t_2^{\lambda_2} \dots t_n^{\lambda_n}$. The modules $k_\lambda, \lambda \in X(n)$, form a complete set of pairwise non-isomorphic simple $T(n)$-modules and a $T(n)$-module V has a $T(n)$-module decomposition $V = \bigoplus_{\lambda \in X(n)} V^\lambda$, where V^λ is a sum of copies of k_λ. We say that $\mu \in X(n)$ is a weight of $V \in \mathrm{mod}(T(n))$ if $V^\mu \neq 0$. The (formal) character $\mathrm{ch}\, V$ of a $T(n)$-module is the element of the ring of Laurent polynomials $\mathbf{Z}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ defined by

$$\mathrm{ch}\, V = \sum_{\lambda \in X(n)} (\dim V^\lambda) X^\lambda$$

where $X^\lambda = X_1^{\lambda_1} \dots X_n^{\lambda_n}$, for $\lambda = (\lambda_1, \dots, \lambda_n) \in X(n)$.

For $\lambda \in X(n)$ we have

$$A_n(\lambda) = \sum_{w \in W} \mathrm{sgn}(w) X^{w\lambda} \in \mathbf{Z}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$$

where $\mathrm{sgn}(w)$ denotes the sign of $w \in W = \mathrm{Sym}(n)$. Moreover $A_n(\lambda + \delta)$ is divisible, in $\mathbf{Z}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ by $A_n(\delta)$ and we define $\chi_n(\lambda) = A_n(\lambda + \delta)/A_n(\delta)$. Usually we abbreviate $\chi_n(\lambda)$ to $\chi(\lambda)$.

For $\lambda \in X^+(n)$ there is an irreducible rational G-module $L(\lambda)$ with unique highest weight λ. Moreover, $\{L(\lambda) \mid \lambda \in X^+(n)\}$ is a complete set of pairwise non-isomorphic irreducible rational G-modules. Every irreducible rational module is absolutely irreducible. The module $L(\lambda)$ is polynomial if and only if λ is a polynomial dominant weight. Moreover, $\{L(\lambda) \,|\, \lambda \in \Lambda^+(n,r)\}$ is a complete set of pairwise non-isomorphic irreducible modules in $\mathrm{Pol}(n,r)$. It is easy to check that, for a finite dimensional polynomial module V, the contravariant dual V^{o} has the same weight multiplicities as V. It follows that $L(\lambda)^{\mathrm{o}}$ is isomorphic to $L(\lambda)$, for $\lambda \in \Lambda^+(n)$, and generally, that V and V^{o} have the same composition factors, counting multiplicities. We write $I(\lambda)$ for the injective hull and $P(\lambda)$ for the projective cover of $L(\lambda)$, in $\mathrm{Pol}(n)$, for $\lambda \in \Lambda^+(n)$.

We recall that a rational module V is polynomial if and only if all composition factors of V are polynomial (see, for example [10], Section 4,(5)). Let $\lambda \in X^+(n)$. There is a uniform bound on the dimension of rational modules V such that V has simple socle $L(\lambda)$ and all composition factors of $V/L(\lambda)$ come from the set $\{L(\mu) \,|\, \mu < \lambda\}$. We choose one of maximal dimension and denote it $\nabla(\lambda)$. Similarly there is a uniquely determined (up to isomorphism) finite dimensional rational module $\Delta(\lambda)$ which has simple head $L(\lambda)$, all other composition factors in $\{L(\mu) \,|\, \mu < \lambda\}$ and has maximal dimension subject to these conditions. For λ polynomial we have $\nabla(\lambda)^{\mathrm{o}} \cong \Delta(\lambda)$ (and $\Delta(\lambda)^{\mathrm{o}} \cong \nabla(\lambda)$). We have $\mathrm{ch}\nabla(\lambda) = \mathrm{ch}\Delta(\lambda) = \chi(\lambda)$.

Let V be a finite dimensional rational module. By a ∇-filtration of V we mean a filtration $0 = V_0 < V_1 < \cdots < V_t = V$ such that for each $1 \le i \le t$, we have $V_i/V_{i-1} \cong \nabla(\lambda(i))$, for some $\lambda(i) \in X^+(n)$. For $\lambda \in \Lambda^+(n)$, we identify $\nabla(\lambda)$ with a submodule of the injective envelope $I(\lambda)$ of $L(\lambda)$. The quotient $I(\lambda)/\nabla(\lambda)$ admits a ∇-filtration with all sections (up to isomorphism) from $\{\nabla(\mu) \,|\, \mu > \lambda\}$. This means that $\mathrm{Pol}(n,r)$ is a high weight category, in the terminology of Cline,Parshall and Scott, or equivalently that $S(n,r)$ is a quasi-hereditary algebra, for each $r \ge 0$. For a finite dimensional rational module V admitting a ∇-filtration and $\lambda \in X^+(n)$ we write $(V : \nabla(\lambda))$ for the filtration multiplicity of $\nabla(\lambda)$ in a ∇-filtration. We define Δ-filtrations and Δ-filtration multiplicities similarly.

For $\lambda, \mu \in \Lambda^+(n)$ we have that $(I(\lambda) : \nabla(\mu))$ is equal to the composition multiplicity $[\nabla(\mu) : L(\lambda)]$ and dually, writing $P(\lambda)$ for the projective cover of $L(\lambda)$, $\lambda \in \Lambda^+(n,r)$, we have that $P(\lambda)$ has a Δ-filtration and that $(P(\lambda) : \Delta(\mu)) = [\Delta(\mu) : L(\lambda)]$ (which is also $[\nabla(\mu) : L(\lambda)]$) for $\lambda, \mu \in \Lambda^+(n)$ (see e.g. [10], Section 4,(6)). If $\lambda, \mu \in X^+(n)$ and $\mathrm{Ext}^1(\nabla(\lambda), \nabla(\mu)) \neq 0$ then we must have $\lambda > \mu$. It follows that if V is a finite dimensional rational module which admits a ∇-filtration then there is a ∇-filtration $0 = V_0 < V_1 < \cdots < V_m = V$ with $V_i/V_{i-1} \cong \nabla(\mu_i)$ and $i < j$ whenever $\mu_i < \mu_j$.

We recall that, for $\lambda, \mu \in X^+(n)$, the tensor product $\nabla(\lambda) \otimes \nabla(\mu)$ has a

∇-filtration (see e.g. [9], Section 4,(3)) and hence in general a tensor product of modules admitting a ∇-filtration admits a ∇-filtration.

By a tilting module we mean a finite dimensional rational module V which admits a ∇-filtration and also a Δ-filtration. For $\lambda \in X^+(n)$ there exists an indecomposable tilting module $T(\lambda)$ with unique highest weight λ. Moreover $\{T(\lambda) \mid \lambda \in X^+(n)\}$ is a complete set of pairwise non-isomorphic indecomposable tilting modules. We have $\dim T(\lambda)^\lambda = 1$, for $\lambda \in X^+(n)$.

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in X(n)$ we define
$\lambda^* = -w_0\lambda = (-\lambda_m, \ldots, -\lambda_2, -\lambda_1)$. By considering highest weights one easily checks that, for $\lambda \in X^+(n)$, we have $L(\lambda)^* \cong L(\lambda^*)$, $\nabla(\lambda)^* \cong \Delta(\lambda^*)$, $\Delta(\lambda)^* \cong \nabla(\lambda^*)$, $T(\lambda)^* \cong T(\lambda^*)$.

We denote the natural polynomial module by E. Thus E has basis $e_1, \ldots, e_n$ and the structure map $\tau : E \to E \otimes A(n)$ satisfies $\tau(e_i) = \sum_{j=1}^n e_j \otimes c_{ji}$. We have a symmetric algebra $S(E)$ and exterior algebra $\bigwedge(E)$ which have natural G-module structures (see e.g. [10]). Moreover there are natural gradings $S(E) = \bigoplus_{r=0}^\infty S^r E$ and $\bigwedge(E) = \bigoplus_{r=0}^\infty \bigwedge^r E$ preserved by the G-action. We have, as G-modules, $\nabla(r, 0, \ldots, 0) = S^r E$, for $r \geq 0$, and $\nabla(1, 1, \ldots, 1, 0, \ldots, 0) = \bigwedge^r E$ (with 1 appearing r times) for $0 \leq r \leq n$. We write D for $\bigwedge^n E$ and sometimes call D the determinant module.

For a finite sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of non-negative integers, we have the modules $S^\alpha E = S^{\alpha_1} E \otimes \cdots \otimes S^{\alpha_m} E$ and $\bigwedge^\alpha E = \bigwedge^{\alpha_1} E \otimes \cdots \otimes \bigwedge^{\alpha_m} E$. We have a decomposition of polynomial modules $A(n, r) \cong \bigoplus_{\alpha \in \Lambda^+(n,r)} S^\alpha E$. It follows that a polynomial module of degree r is injective if and only if it is a direct summand of a direct sum of copies of modules $S^\alpha E$, $\alpha \in \Lambda^+(n, r)$.

We now discuss a truncation functor. We write E_n for E and, for $\lambda \in X^+(n)$, we write $L_n(\lambda)$, $I_n(\lambda)$, $\nabla_n(\lambda)$, $\Delta_n(\lambda)$, $T_n(\lambda)$ for $L(\lambda)$, $I(\lambda)$, $\nabla(\lambda)$, $\Delta(\lambda)$, $T(\lambda)$ when it is desirable to emphasize dependence on n. We choose $N \geq n$ and truncate from polynomial modules for $G(N)$ to polynomial modules for $G(n)$, as in [12], Chapter 6. We have a natural map $k[G(N)] \to k[G(n)]$, taking c_{ij} to c_{ij} for $1 \leq i, j \leq n$ and taking c_{ij} to 0 if $i > n$ or $j > n$. Thus a $G(N)$-module is naturally a $G(n)$-module by restriction.

We identify $\Lambda(n)$ with a subgroup of $\Lambda(N)$, in the obvious way. Let V be a polynomial $G(N)$-module. We define a subspace $fV = \bigoplus_{\theta \in \Lambda(n)} V^\theta$. One sees, for $\alpha = (\alpha_1, \alpha_2, \ldots)$ we have $fS^\alpha E_N = S^\alpha E_n$, which is a $G(n)$-submodule. Moreover, since one may embed any polynomial $G(N)$-module in a direct sum of copies of modules of the form $S^\alpha E$, one gets that fV is a $G(n)$-submodule of V, for every polynomial $G(N)$-module V. Thus fV is naturally a $G(n)$-module. If $\theta : V \to V'$ is a morphism of polynomial $G(N)$-

modules then the restriction $f\theta : fV \to fV'$ is a morphism of $G(n)$-modules. In this way we obtain an exact functor $f : \mathrm{Pol}(N) \to \mathrm{Pol}(n)$. We have $fE_N = E_n$ and, for $\lambda \in \Lambda^+(n)$, we have $fL_N(\lambda) = L_n(\lambda)$, $fI_N(\lambda) = I_n(\lambda)$, $f\nabla_N(\lambda) = \nabla_n(\lambda)$, $f\Delta_N(\lambda) = \Delta_n(\lambda)$ and $fT_N(\lambda) = T_n(\lambda)$.

Remark Note that if $q = 1$ and $\mathrm{char}\, k = 0$, or q is not a root of unity then all G-modules and hence all polynomial modules are semisimple (see e.g. [10], Section 4, (8)) and our problem becomes trivial. Moreover, in the construction of $A(n)$ we may even take $q = 0$. But in that case the algebras $S(n, r)$ (dual to the coalgebras $A(n, r)$) are quasi-Frobenius, by [9], Section 2.2, (5), so that the notions of projective and injective coincide, and again our problem is trivial.

Henceforth we shall therefore assume that q is a root of unity and that $\mathrm{char}\, k > 0$ if $q = 1$. Throughout the sequel l will be the smallest positive integer such that $1 + q + \cdots + q^{l-1} = 0$. Thus either $q = 1$ and k has finite characteristic l or $q \neq 1$ and is a primitive lth root of unity.

We shall make frequent use of the Steinberg module $\mathrm{St} = \nabla((l-1)\delta) = \Delta((l-1)\delta) = L((l-1)\delta)$.

2.3 Infinitesimal Representation Theory

The subalgebra of $k[G]$ generated c_{ij}^l, $1 \leq i, j \leq n$, and d^{-l} is a Hopf subalgebra of $k[G]$, which we denote $k[\overline{G}]$ and thereby define the quantum group $\overline{G}$. We identify $\overline{G}$ with the general linear group scheme GL_n over k, of degree n as follows. Let K be an algebraically closed field containing k. The linear algebraic group $\mathrm{GL}_n(K)$ has coordinate algebra $K[\mathrm{GL}_n(K)]$ generated by the coefficient functions $x_{11}, \ldots, x_{nn}$ and $\det(x_{ij})^{-1}$. The general linear group scheme GL_n over k has coordinate algebra $k[x_{11}, \ldots, x_{nn}, \det(x_{ij})^{-1}]$ (a Hopf k-form of $K[\mathrm{GL}_n(K)]$). We identify $\overline{G}$ with GL_n via the Hopf algebra isomorphism $k[\mathrm{GL}_n(K)] \to k[\overline{G}]$ taking x_{ij} to c_{ij}^l.

We have the morphism of quantum groups $F : G \to \overline{G}$ such that $F^\sharp : k[\overline{G}] \to k[G]$ is inclusion. We call F the Frobenius morphism. In addition, in the case in which k has characteristic $p > 0$ we have the usual Frobenius morphism $\bar{F} : \overline{G} \to \overline{G}$ satisfying $\bar{F}^\sharp(x_{ij}) = x_{ij}^p$, for all $1 \leq i, j \leq n$. We shall append $^-$ to the notations defined in section 2.2 when we consider the corresponding modules for $\overline{G}$. In particular, we shall write $\overline{L}(\lambda)$, $\overline{\nabla}(\lambda)$, $\overline{I}(\lambda)$ and $\overline{T}(\lambda)$ to denote simple, costandard, polynomial injective and tilting modules respectively, labelled by $\lambda \in X^+(n)$, for $\overline{G}$. We shall also write $\overline{D}$ to denote the one dimensional determinant module for $\overline{G}$.

The infinitesimal subgroup G_1 is the subgroup with defining ideal generated by all $c_{ij}^l - \delta_{ij}$. The Steinberg module St is injective (and projective) as G_1-module. Let $\lambda \in \Lambda^+(n)$ then $T(\lambda)$ is injective (and projective) as G_1-module if and only if $\lambda \in (l-1)\delta + \Lambda^+(n)$, see [7] (2.4).

For a G-module V we have the Lyndon-Hochschild-Serre spectral sequence with E_2-page $H^i(\overline{G}, H^j(G_1, V))$ converging to $H^*(G, V)$.

Setting $T = T(n)$, we also have the subgroup G_1T of G with defining ideal generated by all c_{ij}^l, $i \neq j$. For each $\lambda \in X(n)$ there is an irreducible G_1T-module $\hat{L}_1(\lambda)$ with unique highest weight λ (occurring with multiplicity one) and $\{\hat{L}_1(\lambda) \,|\, \lambda \in X(n)\}$ is a complete set of pairwise non-isomorphic irreducible G_1T-modules. (For further details see [9], especially Chapter 3.)

2.4 Statement of the Problem

After establishing the above notation we re-cap on the statement of our problem. The problem is to classify all finite dimensional polynomial modules which are both projective and injective. An indecomposable polynomial module is polynomial of degree r, for some r, so the problem is to classify indecomposable $S(n,r)$-modules M which are both projective and injective. Since M is injective we must have $M = I(\lambda)$, for some $\lambda \in \Lambda^+(n,r)$. Recall (for any $\lambda \in \Lambda^+(n,r)$) that $I(\lambda)$ has $\nabla(\lambda)$ occurring exactly once in a ∇-filtration and for all $\tau \in \Lambda^+(n,r)$ such that $\nabla(\tau)$ occurs, we must have $\tau > \lambda$. Hence $I(\lambda)$ is determined by its class in the Grothendieck group of finite dimensional $S(n,r)$-modules. Now if $I(\lambda)$ is projective then the contravariant dual $I(\lambda)^\circ$ is injective and hence isomorphic to $I(\tau)$ for some $\tau \in \Lambda^+(n,r)$. However, for any finite dimensional $S(n,r)$-module V the class represented by V in the Grothendieck group of finite dimensional modules is equal to the class in the Grothendieck group represented by V°. Hence we must have $\lambda = \tau$, i.e. $I(\lambda)^\circ = I(\lambda)$. Now $I(\lambda)^\circ$ has a ∇-filtration and hence $I(\lambda)$ has a Δ-filtration. So $I(\lambda)$ has both a ∇-filtration and a Δ-filtration. Thus $I(\lambda)$ is a tilting module, i.e. we have $I(\lambda) = T(\mu)$, for some $\mu \in \Lambda^+(n,r)$. Conversely, if $\mu \in \Lambda^+(n,r)$ is such that $T(\mu)$ is injective then we have $T(\mu) = I(\lambda)$ for some $\lambda \in \Lambda^+(n,r)$. Moreover, we have $T(\mu)^\circ = T(\mu)$ (since $T(\mu)^\circ$ is a tilting module representing the same class as $T(\mu)$ in the Grothendieck of finite dimensional $S(n,r)$-modules) and hence $I(\lambda)^\circ = I(\lambda)$. Since $I(\lambda)$ is injective,

$I(\lambda)^{\circ}$ is projective, i.e. $I(\lambda)$ is both injective and projective. Similarly if $T(\mu)$ is projective we deduce that $T(\mu) = P(\lambda)$ for some λ such that $P(\lambda) = I(\lambda)$. We have noted the following result.

(1) Let M be a (non-zero) finite dimensional indecomposable polynomial and let r be the degree of M. The following are equivalent:
(i) $M = I(\lambda) = P(\lambda)$ for some $\lambda \in \Lambda^+(n,r)$;
(ii) M is a tilting module and an injective module.

Problem Our problem is thus to find those $\lambda \in \Lambda^+(n)$ such that $I(\lambda) = T(\mu)$ for some $\mu \in \Lambda^+(n)$, equivalently to find those $\mu \in \Lambda^+(n)$ such that $T(\mu) = I(\lambda)$ for some $\lambda \in \Lambda^+(n)$. In the cases in which this happens we also seek to understand the relationship between λ and μ; and this we see as essentially a generalization of the Mullineux correspondence.

Prompted by this formulation we introduce the following notation: we set

$$\Lambda^+(n,r)^1_{k,q} = \{\lambda \in \Lambda^+(n,r) \,|\, I(\lambda) \text{ is a tilting module}\}$$

and

$$\Lambda^+(n,r)^2_{k,q} = \{\mu \in \Lambda^+(n,r) | T(\mu) \text{ is an injective module}\}.$$

Thus our problem is to describe these sets explicitly and the bijection $\Lambda^+(n,r)^1_{k,q} \to \Lambda^+(n,r)^2_{k,q}$, $\lambda \mapsto \mu$, defined by $I(\lambda) = T(\mu)$.

3 Truncation from the case $r \leq n$

Our first result in this section tells us where to look for the polynomial modules which are both projective and injective.

Lemma 3.1 *An indecomposable polynomial module is projective and injective if and only if it is an injective indecomposable summand of $E^{\otimes r}$, for some $r \geq 0$.*

Proof: An injective indecomposable polynomial module which is a summand of $E^{\otimes r}$ is an injective tilting module and hence an injective and projective module.

Now let M be an indecomposable polynomial module of degree r which is both injective and projective. Then $M = I(\lambda)$, for some $\lambda \in \Lambda^+(n,r)$. However, $I(\lambda)$ is a component of the tensor product of symmetric powers $S^\lambda E$. We have the natural epimorphism $E^{\otimes r} \to S^\lambda E$ and hence there is an epimorphism $E^{\otimes r} \to I(\lambda)$. But $I(\lambda)$ is projective so this epimorphism splits and $M = I(\lambda)$ is a direct summand of $E^{\otimes r}$.

We now review the known case $r \leq n$ (see [9], Section 4.3 and 4.4(14)) and derive two consequences of truncation from it. For a module V and a non-negative integer m we shall often write mV for the direct sum $V \oplus V \oplus \cdots \oplus V$ (m times).

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda(n)$, we have

$$S^\alpha E = \bigoplus_{\mu \in \Lambda(n)} d_\mu \, I(\mu) \tag{1}$$

where $d_\mu = \dim L(\mu)^\alpha$ (see [9] 2.1(8)). So first suppose that $r \leq n$. We have a decomposition

$$E^{\otimes r} = \bigoplus_{\lambda \in \Lambda^+(n,r)} d_\lambda I(\lambda)$$

where $d_\lambda = \dim L(\lambda)^{\omega_r}$, $\lambda \in \Lambda^+(n,r)$ and $\omega_r = (1,1,..,1,0,..,0) \in \Lambda^+(n,r)$. Now, $d_\lambda \neq 0$ if and only if λ is column l-regular. As E, and hence $E^{\otimes r}$, is a tilting module, we get that for λ column l-regular $I(\lambda)$ is projective. Moreover, in this case we have $I(\lambda) = T(\mathrm{Mull}(\lambda'))$, see [9] 4.4(14) and [2].

We now use truncation to deduce results for arbitrary n and r.

Lemma 3.2 *If $\mu \in \Lambda^+(n,r)^2_{k,q}$ then μ is row l-regular.*

Proof: We have that $T(\mu)$ is a component of $E^{\otimes r}$, where $r = |\mu|$. We choose $N \geq r$. Then we have $E^{\otimes r} = f E_N^{\otimes r}$, where $f : \mathrm{mod}(S(N,r)) \to \mathrm{mod}(S(n,r))$ is the truncation functor discussed in section 2.2. Hence $T(\mu)$ is a summand of $fT_N(\tau)$, for some row l-regular $\tau \in \Lambda^+(N,r)$. This implies $fT_N(\tau) \neq 0$ so that $\tau \in \Lambda^+(n,r)$ and indeed $fT_N(\tau) = T(\tau)$. Thus we get $\mu = \tau$, which is row l-regular.

Lemma 3.3 *Let $\lambda \in \Lambda^+(n)$ be a column l-regular partition. Then* $\mathrm{Mull}(\lambda')$ *has at most n parts and we have $I(\lambda) = T(\mathrm{Mull}(\lambda'))$. In particular $I(\lambda)$ is self dual, under contravariant duality, and hence projective.*

Proof: Let $|\lambda| = r$ and choose $N \geq r$. We use the truncation functor $f : \mathrm{Pol}(N) \to \mathrm{Pol}(n)$. We have $I_N(\lambda) = T_N(\mathrm{Mull}(\lambda'))$. In particular we have $\mathrm{Mull}(\lambda') \geq \lambda$ which implies that $\mathrm{Mull}(\lambda')$ has at most n parts. Now applying f to $I_N(\lambda) = T_N(\mathrm{Mull}(\lambda'))$ gives $I_n(\lambda) = T_n(\mathrm{Mull}(\lambda'))$, as required.

4 A Reduction Theorem

As mentioned in section 2.4, the problem of finding all projective injective modules is equivalent to the problem of finding all injective (and hence projective) tilting modules.

The next theorem gives an inductive description of which tilting modules $T(\lambda)$ with $\lambda \in (l-1)\delta + \Lambda^+(n)$ are injective provided we know all injective tilting modules $T(\mu)$ with $\mu \notin (l-1)\delta + \Lambda^+(n)$.

For a quantum group H and an H-module V the socle of V will be denoted $\mathrm{soc}_H(V)$ or simply $\mathrm{soc}(V)$. For a real number a we write $\lfloor a \rfloor$ for the integer part of a.

Theorem 4.1 *Let $\lambda \in \Lambda^+(n)$ be column l-regular and let $\mu \in \Lambda^+(n)$. Then $T((l-1)\delta + \lambda + l\mu)$ is injective as a polynomial G-module if and only if $\overline{T}(\mu) \otimes \overline{D}^{\otimes \lfloor \lambda_1/l \rfloor}$ is injective as a polynomial $\overline{G}$-module.*

In this case, we have $T((l-1)\delta + \lambda + l\mu) \cong I((l-1)\delta + w_0\lambda + l\eta)$ where η is defined by $\overline{T}(\mu) \cong \overline{I}(\eta)$.

Proof: We write $\lambda_1 = a + bl$ where $0 \leq a \leq l-1$. Suppose first that $\overline{T}(\mu) \otimes \overline{D}^{\otimes b}$ is injective. We will show that $\mathrm{St} \otimes T(\lambda) \otimes \overline{T}(\mu)^F$ is injective and hence, as $T((l-1)\delta + \lambda + l\mu)$ occurs as a summand, it must be injective as well. Suppose for a contradiction that $\mathrm{St} \otimes T(\lambda) \otimes \overline{T}(\mu)^F$ is not injective. Then for some $\theta \in \Lambda^+(n)$, we have

$$\mathrm{Ext}^1_G(L(\theta), \mathrm{St} \otimes T(\lambda) \otimes \overline{T}(\mu)^F) \neq 0$$

so that

$$\mathrm{Ext}^1_G(L(\theta) \otimes T(\lambda)^*, \mathrm{St} \otimes \overline{T}(\mu)^F) \neq 0$$

in other words

$$\mathrm{Ext}^1_G(L(\theta) \otimes T(-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1), \mathrm{St} \otimes \overline{T}(\mu)^F) \neq 0$$

and thus

$$\mathrm{Ext}^1_G(L(\theta) \otimes T(\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_2, 0), \mathrm{St} \otimes D^{\otimes a} \otimes (\overline{T}(\mu) \otimes \overline{D}^{\otimes b})^F) \neq 0.$$

Hence there is some simple composition factor L of $L(\theta) \otimes T(\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_2, 0)$ such that

$$\mathrm{Ext}^1_G(L, \mathrm{St} \otimes D^{\otimes a} \otimes (\overline{T}(\mu) \otimes \overline{D}^{\otimes b})^F) \neq 0.$$

But using [8] Section 4 Theorem (in the classical case) and [4] Proposition 5.4 (in the quantum case), we see that L must have the form $\mathrm{St} \otimes D^{\otimes a} \otimes \overline{L}(\eta)^F$, for some polynomial dominant weight η. Thus we have

$$\mathrm{Ext}^1_G(\mathrm{St} \otimes D^{\otimes a} \otimes \overline{L}(\eta)^F, \mathrm{St} \otimes D^{\otimes a} \otimes (\overline{T}(\mu) \otimes \overline{D}^{\otimes b})^F) \neq 0.$$

But then, since St is projective (and injective) as a G_1-module, the Lyndon-Hochschild-Serre spectral sequence of section 2.3 degenerates and we get

$$\mathrm{Ext}^1_G(\mathrm{St} \otimes D^{\otimes a} \otimes \overline{L}(\eta)^F, \mathrm{St} \otimes D^{\otimes a} \otimes (\overline{T}(\mu) \otimes \overline{D}^{\otimes b})^F)$$

$$\cong \mathrm{Ext}^1_{\overline{G}}(\overline{L}(\eta), \overline{T}(\mu) \otimes \overline{D}^{\otimes b}) \neq 0.$$

Thus $\overline{T}(\mu) \otimes \overline{D}^{\otimes b}$ is not injective. This is a contradiction.

Assume, conversely, that $T((l-1)\delta + \lambda + l\mu)$ is injective. Now it is injective as a G_1T-module and as it has highest weight $(l-1)\delta + \lambda + l\mu$, it follows from [9] 3.2(10)(ii) and (14)(iv) that $\hat{L}_1((l-1)\delta + w_0\lambda + l\xi)$ occurs in the G_1T-socle, for some $\xi \in \Lambda^+(n)$. Hence, $L((l-1)\delta + w_0\lambda)$ occurs in the G_1-socle and since the G-socle is simple, we must have

$$\mathrm{soc}_{G_1}(T((l-1)\delta + \lambda + l\mu)) = L((l-1)\delta + w_0\lambda) \otimes V^F$$

for some indecomposable $\overline{G}$-module V. Now $T((l-1)\delta + \lambda + l\mu)$ is a direct summand of $L(\lambda) \otimes \mathrm{St} \otimes \overline{T}(\mu)^F$ (see the argument of C. Pillen given in [7]

Theorem 2.5) and the socle of this module has $L((l-1)\delta + w_0\lambda)$-isotypic component $L((l-1)\delta + w_0\lambda) \otimes \overline{T}(\mu)^F$, which is indecomposable. Hence

$$\mathrm{soc}_{G_1} T((l-1)\delta + \lambda + l\mu) = L((l-1)\delta + w_0\lambda) \otimes \overline{T}(\mu)^F$$
$$= L((l-1)\delta + (\lambda_n - bl, \lambda_{n-1} - bl, \dots, a)) \otimes (\overline{T}(\mu) \otimes \overline{D}^{\otimes b})^F.$$

Suppose, for a contradiction, that $\overline{T}(\mu) \otimes \overline{D}^{\otimes b}$ is not injective. Then we have $\mathrm{Ext}^1_{\overline{G}}(\overline{L}(\theta), \overline{T}(\mu) \otimes \overline{D}^{\otimes b}) \neq 0$ for some $\theta \in \Lambda^+(n)$. Using the degeneracy of the Lyndon-Hochschild-Serre spectral sequence again, we see that

$$\mathrm{Ext}^1_G(L((l-1)\delta + (\lambda_n - bl, \lambda_{n-1} - bl, \dots, a)) \otimes \overline{L}(\theta)^F, T((l-1)\delta + \lambda + l\mu))$$
$$= \mathrm{Ext}^1_{\overline{G}}(\overline{L}(\theta), \overline{T}(\mu) \otimes \overline{D}^{\otimes b}) \neq 0.$$

This contradicts the fact that $T((l-1)\delta + \lambda + l\mu)$ is injective.

Remark: It follows immediately from Theorem 4.1 that if $\mathrm{char}\, k = 0$ and q is a primitive l-th root of unity then any $T(\lambda)$ with $\lambda \in (l-1)\delta + \Lambda^+(n)$ is injective. In fact, these modules are injective in the rational category $\mathrm{Rat}(n)$. Indeed it follows from the degeneracy of the Lyndon-Hochschild-Serre spectral sequence, that a rational G-module is injective (in $\mathrm{Rat}(n)$) if and only if it is injective as a module for G_1. Now the tilting module $T(\lambda)$ is injective if and only if $\lambda \in (l-1)\delta + \Lambda^+(n)$.

5 Conjecture

We make a conjecture which if true would determine the sets $\Lambda^+(n,r)^1_{k,q}$ and $\Lambda^+(n,r)^2_{k,q}$ and the bijection between them, see Theorem 5.1. We first give a couple of observations on addition and removal of columns.

Lemma 5.1 *Let M be a finite dimensional polynomial module such that $D \otimes M$ is projective (resp. injective). Then M is projective (resp. injective). In particular, if $\lambda \in \Lambda^+(n)$ is such that $D \otimes I(\lambda)$ is projective (resp. $D \otimes T(\lambda)$ is injective) then $I(\lambda)$ is projective (resp. $T(\lambda)$ is injective).*

Proof: Assume $D \otimes M$ is projective. For $V \in \mathrm{Pol}(n)$ we have $\mathrm{Ext}^1(M, V) \cong \mathrm{Ext}^1(D \otimes M, D \otimes V) = 0$. Hence M is projective. The other case is similar.

Results in the other direction hold rarely but we do have the following.

Lemma 5.2 *Suppose that $\lambda \in \Lambda^+(n)$ and that λ and $\omega + \lambda$ have the same l-weight.*
(i) We have $I(\omega+\lambda) = D \otimes I(\lambda)$ and if $I(\lambda)$ is projective then so is $I(\omega+\lambda)$.
(ii) If $T(\lambda)$ is injective then so is $T(\omega + \lambda)$.

Proof: Let γ be the l-core of λ. The hypothesis implies that $\omega + \gamma$ is the l-core of $\omega + \lambda$.
(i) Let $I = D \otimes I(\lambda)$. Then I has simple socle $L(\omega + \lambda)$ and hence I embeds in $I(\omega + \lambda)$. Now let $\tau \in \Lambda^+(n)$ be such that $(I(\omega + \lambda) : \nabla(\tau)) \neq 0$. Then τ lies in the block of $\omega + \lambda$, in particular it has the same l-core as $\omega + \lambda$, i.e. $\omega + \gamma$, and therefore τ has n rows. We may thus write $\tau = \omega + \xi$, for some partition ξ. Hence we have $(I(\omega + \lambda) : \nabla(\tau)) = [\nabla(\tau) : L(\omega + \lambda)] = [\nabla(\omega + \xi) : L(\omega + \lambda)] = [\nabla(\xi) : L(\lambda)] = (I(\lambda) : \nabla(\xi))$.
We thus get

$$\begin{aligned}\dim I(\omega + \lambda) &= \sum_{\xi} (I(\omega + \lambda) : \nabla(\omega + \xi)) \dim \nabla(\omega + \xi) \\ &= \sum_{\xi} (I(\lambda) : \nabla(\xi)) \dim \nabla(\xi) = \dim I(\lambda).\end{aligned}$$

Hence we have $I(\omega + \lambda) = D \otimes I(\lambda)$.
If $I(\lambda)$ is projective then it is a tilting module and hence $I(\omega+\lambda) = D \otimes I(\lambda)$ is an injective tilting module and hence also projective.
(ii) If $T(\lambda)$ is injective then we have $T(\lambda) = I(\mu)$ for some μ. Now λ and μ lie in the same block and hence μ also has l-core γ. Now $\omega + \mu$ has core $\omega + \gamma$ and so $I(\omega + \mu) = D \otimes I(\mu) = D \otimes T(\lambda) = T(\omega + \lambda)$ is injective.

Lemma 5.3 *Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition. The following are equivalent:*
(i) the rim of λ is equal to its l-edge and has length not divisible by l;
(ii) no first row hook length $h_{11}, h_{12}, \ldots$ is divisible by l.

Proof: We assume that the rim of λ is equal to its l-edge and its length is not divisible by l. If $\lambda = (1^t)$ for some t then t is not divisible by l and the only hook length h_{11} is t. So we now assume that λ has more than one column. Thus we have $\lambda = (\lambda_1, \ldots, \lambda_m, 1^a)$ for some m with $\lambda_m > 1$, $a \geq 0$. Removing the first column from λ we get a partition $\mu = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_m - 1)$ with rim equal to l-edge. Moreover the length of the rim of μ can not be divisible by l for otherwise the rim node $(1, m)$ of λ would not be on its l-edge. Thus we may assume inductively that the first row hook lengths of μ, i.e. $h_{12}, h_{13}, \ldots$, are not divisible by l. But h_{11} is the length of the rim of λ and this is not divisible by l. Thus no first row hook length of λ is divisible by l.

We leave it to the reader to reverse this argument.

Lemma 5.4 *Let $\mu = (\mu_1, \ldots, \mu_n) \in \Lambda^+(n)$ be column l-regular and suppose that $\mu_n < l - 1$. The following are equivalent:*
(i) $I(\omega + \mu) \cong D \otimes I(\mu)$;
(ii) for all $1 \leq i \leq n$ we have $\mu_i + (n - i) \not\equiv -1$ mod l.
(iii) μ and $\omega + \mu$ have the same l-weight.

Proof: (i) implies (ii). Suppose that $I(\mu) \otimes D \cong I(\omega + \mu)$. As $\mu_n < l - 1$, $\omega + \mu$ is column l-regular, so the highest weight of $I(\omega + \mu)$ is given by $\mathrm{Mull}((\omega + \mu)')$. But the highest weight of $I(\mu) \otimes D$ is given by $\omega + \mathrm{Mull}(\mu')$. Hence $\mathrm{Mull}((\omega+\mu)') = \omega + \mathrm{Mull}(\mu')$ and in particular, it has exactly n parts.

Let $\lambda = (\omega + \mu)'$. Denote by $a(\lambda)$ the length of its l-edge, by $e(\lambda)$ the length of its rim and by $r(\lambda)$ its number of parts. Define ϵ by $\epsilon = 0$ if l divides $a(\lambda)$ and $\epsilon = 1$ otherwise. Then, by definition of the Mullineux map (see [16]) we have

$$n = a(\lambda) - r(\lambda) + \epsilon \leq e(\lambda) - r(\lambda) + 1 = n + r(\lambda) - 1 - r(\lambda) + 1 = n.$$

Thus we have equality and the rim of λ is equal to its l-edge and has length not divisible by l. Thus no first row hook length is divisible by l. Hence no first column hook length of $\omega + \mu$ is divisible by l, in other words for each $1 \leq i \leq n$ we have that $\mu_i + (n - i) + 1$ is not divisible by l.
(ii) implies (iii). Display the partition μ on an l-runner abacus (labelled $0, 1, \ldots, l - 1$) containing n beads. By definition, the bead corresponding to μ_i lies on runner r_i if and only if $\mu_i + n - i \equiv r_i \bmod l$. By assumption, the $(l - 1)$-runner is empty. So $\omega + \mu$ is represented on the abacus simply

by shifting all the beads one step to the right. Recall that the l-core of a partition is obtained by pushing all the beads up as far as possible along the runners. We denote the l-core of μ by γ. Then the l-core of $\omega + \mu$ is equal to $\omega + \gamma$. Thus μ and $\omega + \mu$ have the same l-weight.
(iii) implies (i). See Lemma 5.2.

We now make our main conjecture.

Conjecture 5.1 *Suppose $\lambda \in \Lambda^+(n)$ is such that $I(\lambda)$ is projective. Then either $I(\lambda)$ is injective as a G_1-module or λ is column l-regular.*

Before seeing how this implies a complete solution to our problem we refine the conjecture above by describing the intersection between the two conditions. Recall from section 2.3 that a tilting module $T(\nu)$ is injective as a G_1-module if and only if $\nu \in (l-1)\delta + \Lambda^+(n)$. We claim that if $\lambda = (\lambda_1, \ldots, \lambda_n)$ is column l-regular then $I(\lambda)$ is injective as a G_1-module if and only if $\lambda_1 \geq (n-1)(l-1)$. Suppose that $I(\lambda)$ is injective as a G_1-module. Then $I(\lambda) \cong T((l-1)\delta + \eta + l\mu)$ for some η column l-regular and some $\mu \in \Lambda^+(n)$. So (as in the proof of Theorem 4.1) we have $\lambda = (l-1)\delta + w_0\eta + l\theta$ for some $\theta \in \Lambda^+(n)$ and so $\lambda_1 \geq (n-1)(l-1)$.

Conversely, suppose $\lambda_1 \geq (n-1)(l-1)$ then as λ is column l-regular, we see that $\lambda = (l-1)\delta + w_0\eta$ where η is a column l-regular partition. Hence, using Theorem 4.1, we see that $T((l-1)\delta + \eta)$ is injective and we must have $I(\lambda) \cong T((l-1)\delta + \eta)$ which is injective as a G_1-module.

We can now reformulate our conjecture as follows.

Conjecture 5.2 *Let $\lambda \in \Lambda^+(n)$ be such that $I(\lambda) \cong P(\lambda)$. Then exactly one of the following conditions holds:*
(i) λ is column l-regular and $\lambda_1 < (n-1)(l-1)$;
(ii) $I(\lambda)$ is injective as a G_1-module.

For $\mu \in \Lambda^+(n)$ column l-regular, we define $t_i(\mu)$ to be the least non-negative integer congruent to $\mu_i + (n-i)$ modulo l. We set
$t(\mu) = \max\{t_1(\mu), \ldots, t_n(\mu)\}$.

Theorem 5.1 *Assume Conjecture 5.2 holds. An indecomposable polynomial $G(n)$-module M is projective and injective in* $\mathrm{Pol}(n)$ *in precisely the following cases.*

(i)

$$M \cong I(\lambda) \cong T(\mathrm{Mull}(\lambda'))$$

for $\lambda \in \Lambda^+(n)$ column l-regular with $\lambda_1 < (n-1)(l-1)$.

(ii) (a) $\mathrm{char}\, k = 0$ *and*

$$M \cong I((l-1)\delta + w_0\lambda + l\mu) \cong T((l-1)\delta + \lambda + l\mu)$$

for $\lambda, \mu \in \Lambda^+(n)$ and λ column l-regular.

(b) $\mathrm{char}\, k = p$ *and*

$$M \cong I((lp^m - 1)\delta + w_0\lambda + lp^m\mu) \cong T((lp^m - 1)\delta + \lambda + lp^m\mathrm{Mull}(\mu'))$$

for $m \geq 0$, $\lambda, \mu \in \Lambda^+(n)$ with λ column lp^m-regular, μ column p-regular with $\mu_1 < (n-1)(p-1)$ and $\lambda_1 + lp^m t(\mu) < lp^{m+1}$.

Proof: Any projective injective module must be tilting so let us start with a tilting module $T(\eta)$, $\eta \in \Lambda^+(n)$, and assume that it is injective in $\mathrm{Pol}(n)$. Now $T(\eta)$ is injective as a G_1-module if and only if $\eta \in (l-1)\delta + \Lambda^+(n)$.

Suppose first that $\eta \notin (l-1)\delta + \Lambda^+(n)$. Then Conjecture 5.2 implies that $T(\eta) \cong I(\lambda)$ for some column l-regular λ with $\lambda_1 < (n-1)(l-1)$. Moreover, using Lemma 3.3, we must have $\eta = \mathrm{Mull}(\lambda')$.

Now suppose that $\eta \in (l-1)\delta + \Lambda^+(n)$. If $\mathrm{char}\, k = 0$ then using Theorem 4.1 we see that $T(\eta)$ is always injective and if we write $\eta = (l-1)\delta + \lambda + l\mu$ for $\lambda, \mu \in \Lambda^+(n)$ with λ column l-regular, then we have

$$T((l-1)\delta + \lambda + l\mu) \cong I((l-1)\delta + w_0\lambda + l\mu).$$

Now assume that $\mathrm{char}\, k = p > 0$. Define $m \geq 0$ by $\eta \in (lp^m - 1)\delta + \Lambda^+(n) \setminus (lp^{m+1} - 1)\delta + \Lambda^+(n)$. So η can be written as $\eta = (lp^m - 1)\delta + \lambda + lp^m\xi$ with λ column lp^m-regular and $\xi \notin (p-1)\delta + \Lambda^+(n)$. It is easy to see, using Theorem 4.1 inductively, that $T(\eta)$ is injective if and only if $\overline{T}(\xi) \otimes \overline{D}^{\otimes\lfloor\lambda_1/lp^m\rfloor}$ is injective and in this case we have $T(\eta) \cong I((lp^m - 1)\delta + w_0\lambda + lp^m\mu)$ where μ is defined by $\overline{T}(\xi) \cong \overline{I}(\mu)$. Now, as $\xi \notin (p-1)\delta + \Lambda^+(n)$, Conjecture 5.2 implies that μ must be column p-regular with $\mu_1 < (n-1)(p-1)$ and using Lemma 3.3 we have $\xi = \mathrm{Mull}(\mu')$. Moreover, we must have $\overline{I}(\mu + \lfloor\lambda_1/lp^m\rfloor\omega) \cong$

$\overline{I}(\mu) \otimes \overline{D}^{\otimes \lfloor \lambda_1/lp^m \rfloor}$. Using Lemma 5.4, we see that this is the case if and only if $\lfloor \lambda_1/lp^m \rfloor + t(\mu) < p$ i.e. $\lambda_1 + lp^m t(\mu) < lp^{m+1}$.

6 Another Reduction Theorem

We here reduce our problem to the case in which the base field k has characteristic zero.

Definition By a quantum modular reduction system, we mean a quintuple (K, R, F, Q, q), where R is a local, complete discrete valuation ring with field of fractions K and residue field F, and where Q is a unit in R with image q in F.

(Note that such a system is determined by the complete, discrete, valuation domain R and unit $Q \in R$.)

Examples (i) Let p be a prime, F the field of p elements and let $q = 1$. Let K be a splitting field of $X^p - 1$ over $\mathbf{Q}_p$, the field of p-adic numbers. Let R denote the ring of integers in K, i.e. the integral closure of the ring of p-adic integers $\mathbf{Z}_p$ in K. Then K is completely ramified over $\mathbf{Q}_p$ (see e.g. [3], Chapter 8, Lemma 4.2(i)). Hence the residue field of K is F. Let Q be any primitive pth root of 1 in K. Then (K, R, F, Q, q) is a quantum modular reduction system.
(ii) Let p be a prime and F a finite field with p^m elements, say, and let q be a non-zero element of F with multiplicative order $l > 1$. Let K be a splitting field of $X^{p^m} - X$ over $\mathbf{Q}_p$. Let R be the ring of integers in K. Then K is an unramified extension of $\mathbf{Q}_p$ of degree m. The residue field of K is F. Moreover, reduction modulo the maximal ideal of R determines a group isomorphism between the group of units of K of exponent $p^m - 1$ and the multiplicative group of the field F. (See [3] pp147,148). Hence if Q is an element of exponent $p^m - 1$ mapping to q then Q has order l and (K, R, F, Q, q) is a quantum modular reduction system.

Lemma 6.1 *If F is a subfield of the field L then we have $\Lambda^+(n,r)^1_{F,q} = \Lambda^+(n,r)^1_{L,q}$ and $\Lambda^+(n,r)^2_{F,q} = \Lambda^+(n,r)^2_{L,q}$.*

Proof: The absolute irreducibility of $L(\lambda)$, for all $\lambda \in \Lambda^+(n,r)$ implies that the tilting modules $T(\lambda)$, $\lambda \in \Lambda^+(n,r)$, are all absolutely indecomposable, see [9] Theorem A4.2(iii). The result follows and similarly for the statement concerning injective modules.

Thus, in the case in which $q = 1$ and k has characteristic p we may replace k by the field of p element, and in the case $q \neq 1$ (a root of unity) we may replace k by the finite subfield generated by q.

In order to proceed we need to consider modules over an R-form of the Schur algebra $S(n,r)$ over K with parameter Q.

For a commutative ring R and $Q \in R$ we have the R-algebra $A_{R,Q}(n)$ constructed by generators and relations as in section 2.2. According to [5] 2.4.1 Lemma, a ring homomorphism $\theta : R \to R'$, taking $Q \in R$ to $Q' \in R'$, induces an R-algebra homomorphism $\tilde{\theta} : A_{R,Q}(n) \to A_{R',Q'}(n)$, taking $c_{ij,R,Q}$ to $c_{ij,R'Q'}$, for all $1 \leq i,j \leq n$, and hence inducing an R-module homomorphism $A_{R,Q}(n,r) \to A_{R',Q'}(n,r)$ taking $c_{ij,R,Q}$ to $c_{ij,R',Q'}$, for all $i,j \in I(n,r)$, $r \geq 0$. Furthermore, the natural R'-module map $R' \otimes_R A_{R,Q}(n,r) \to A_{R',Q'}(n,r)$ is an isomorphism. In particular, if R is a domain and K is the field of fractions of R then the induced map $A_{R,Q}(n,r) \to A_{K,Q}(n,r)$ is injective. We thus identify $A_{R,Q}(n,r)$ with an R-submodule of $A_{K,Q}(n,r)$.

Then $A_{R,Q}(n,r)$ is a R-form of $A_{K,Q}(n,r)$. We write $S_{R,Q}(n,r)$ for the R-submodule $\{\xi \in S_{K,Q}(n,r) | \xi(A_{R,Q}(n,r)) \leq R\}$. Note that $\xi \in S_{K,Q}(n,r)$ belongs to $S_{R,Q}(n,r)$ if, and only if, $\xi(c_{ij,R,Q}) \in R$, for all $i,j \in I(n,r)$. Thus, for $\xi_1, \xi_2 \in S_{R,Q}(n,r)$ and $i,j \in I(n,r)$, we have
$(\xi_1 * \xi_2)(c_{ij,R,Q}) = \sum_{h \in I(n,r)} \xi_1(c_{ih,R,Q})\xi_2(c_{hj,R,Q}) \in R$. Thus the R-lattice $S_{R,Q}(n,r)$ of $S_{K,Q}(n,r)$ is in fact an R-subalgebra.

We now fix a quantum modular reduction system (K,R,F,Q,q). We further fix $n \geq 1$ and $r \geq 0$. We identify $A_{F,q}(n,r)$ with $F \otimes_R A_{R,Q}(n,r)$ and the F-algebra $S_{F,q}(n,r)$ with $F \otimes_R S_{R,Q}(n,r)$. We write simply S_K for $S_{K,Q}(n,r)$, write S_R for $S_{R,Q}(n,r)$ and write S_F for $S_{F,q}(n,r)$. Further, for $i,j \in I(n,r)$, we write c_{ij} for $c_{ij,K,Q}$ and $c_{ij,Q,R}$ and $c_{ij,F,q}$, hoping that the context makes it clear which is intended.

Let $\lambda \in \Lambda^+(n,r)$. For $U = K$ or F, to emphasize dependence on the field, we now write $L_U(\lambda)$ (resp. $\Delta_U(\lambda)$, resp. $\nabla_U(\lambda)$) for the simple (resp. standard, resp. costandard) module labelled by λ. We shall also need distinguished R-forms of the standard and costandard modules. Before defining these, we recall the definition of a certain involutory anti-automorphism on a Schur algebra.

We write E_K (resp. E_F) for the natural module for the quantum group $G(n)$ defined over K with parameter Q (resp. defined over F with parameter q). However, for $i \in I(n,r)$, we write simply e_i for the corresponding standard basis element of $E_K^{\otimes r}$ or $E_F^{\otimes r}$ (again hoping that the context will make it clear which is intended).

Recall that we have a non-singular bilinear form on the K-space $E_K^{\otimes r}$, defined by $(e_i, e_j) = \delta_{ij} Q^{d(i)}$, where $d(i)$ is the number of pairs (a,b) with $1 \leq a < b \leq r$ and $i_a < i_b$. We have an involutory, anti-automorphism J_K of the K-algebra S_K satisfying $(\xi x, y) = (x, J_K(\xi) y)$, for all $\xi \in S_K$, $x, y \in E_K^{\otimes r}$. One also has the involutory anti-automorphism J_F of S_F, defined in the same way. Let $\xi \in S_R$. For $i \in I(n,r)$, we have $\xi e_i = \sum_j \xi(c_{ij}) e_j \in E_R^{\otimes r}$. Hence $\xi(E_R^{\otimes r}) \leq E_R^{\otimes r}$. We note that if $x \in E_K^{\otimes r}$ and $(x,y) \in R$ for all $y \in E_R^{\otimes r}$ then $x \in E_R^{\otimes r}$. For $\xi \in S_R$ and $x \in E_R^{\otimes r}$ we have $(J_K(\xi)x, y) = (x, J_K(\xi) y) \in R$. Hence we have $J_K(\xi) \in S_R$ and so J_K restricts to an R-algebra anti-automorphism J_R of S_R. We note that, by construction, we have $J_F = 1 \otimes_R J_R : S_F \to S_F$.

Let $0 \neq v^+ \in \Delta_K(\lambda)$. We define $\Delta_R(\lambda) = S_R v^+$ and observe that (since $\dim \Delta_K(\lambda)^\lambda = 1$) the isomorphism type of $\Delta_R(\lambda)$ is independent of the choice of v^+. For an S_R-module M, which is an R-lattice (i.e. finitely generated and free as an R-module) we have the S_K-module $M_K = K \otimes_R M$ and the S_F-module $M_F = F \otimes_R M$. We recall that M_K and M_F have the same character, see [5], 2.4.3 Lemma. Thus, $F \otimes_R \Delta_R(\lambda)$ is generated by a highest weight vector of weight λ and has the same character as $\Delta_F(\lambda)$. Hence we must have $F \otimes_R \Delta_R(\lambda) \cong \Delta_F(\lambda)$.

For an S_R-lattice M we define M° to be the R-module $\mathrm{Hom}_R(M, R)$, regarded as an S_R-module via the action given by $(\xi\theta)(m) = \theta(J_R(\xi) m)$. We define $\nabla_R(\lambda) = \Delta_R(\lambda)^\circ$, for $\lambda \in \Lambda^+(n,r)$.

By a Δ-filtration of an S_R-lattice M we mean an S_R-module filtration $0 = M_1 < M_1 < \cdots < M_t = M$ such that, for each $1 \leq i \leq t$, we have $M_i / M_{i-1} \cong \Delta_R(\lambda)$, for some $\lambda \in \Lambda^+(n,r)$ (which may depend on i). We define ∇-filtration of an S_R-lattice similarly.

Lemma 6.2 *(i) For an S_R-lattice M we have $(M_F)^\circ \cong (M^\circ)_F$.*
(ii) An S_R-lattice M has a Δ-filtration (resp. ∇-filtration) if and only if the S_F-module M_F has a Δ-filtration (resp. ∇-filtration).

Proof: (i). This follows directly from the definitions.

(ii) If suffices to prove the result on Δ-filtrations : the result for ∇-filtrations follows by duality. Moreover, if M has a Δ-filtration then so has M_F, since $F \otimes_R \Delta_R(\lambda) \cong \Delta_F(\lambda)$, for each $\lambda \in \Lambda^+(n,r)$, as noted above. We now consider the converse. The proof is a refinement of the argument of [6], (11.5.3) Lemma, dealing with Weyl filtrations for modules for the hyperalgebra of a semisimple algebraic group.

So suppose now that M is an S_R-lattice such that M_F has a Δ-filtration. Let λ be a highest weight of M_K. Then $M^\lambda = M \cap M_K^\lambda$ is a free R-module so we may write $M^\lambda = Ry \oplus N$, for some $0 \neq y \in M$ and R-submodule N of M^λ. Let $Z = S_R y$.

Now inclusion $Z \to M$ induces an injective map $Z_K \to M_K$ (since K is torsion free, hence flat, as an R-module - or, because localization is exact) and we identify Z_K with a S_K-submodule of M_K. Now Z_K is generated by a vector of weight λ, and λ is a highest weight of M_K so that Z_K is a homomorphic image of $\Delta_K(\lambda)$. In particular, we have $\dim_K Z_K \leq \dim_K \Delta_K(\lambda)$. Thus the R-rank of Z, which is equal to the K-dimension of Z_K is at most $\dim_K \Delta_K(\lambda) = \dim_F \Delta_F(\lambda)$.

Now the short exact sequence $0 \to Z \to M \to M/Z \to 0$, of S_R-modules, gives rise to an exact sequence

$$0 \to \mathrm{Tor}_1^R(M/Z, F) \to Z_F \to M_F \to (M/Z)_F \to 0. \tag{2}$$

Let $y_F = 1 \otimes_R y \in Z_F$. Note that the image of y_F in M_F is non-zero (since Ry is an R-module summand of M). Since M_F has a Δ-filtration any weight vector of the highest weight λ generates a copy of $\Delta_F(\lambda)$. Thus the map $\phi : Z_F \to M_F$ has image of dimension $\dim_F \Delta_F(\lambda)$ and since $\dim Z_F \leq \dim_F \Delta_F(\lambda)$, we must have that $\phi : Z_F \to M_F$ is injective and that Z_F is isomorphic to $\Delta_F(\lambda)$.

Now we get that $\dim_K Z_K = \dim_F Z_F = \dim_F \Delta_F(\lambda) = \dim_K \Delta_K(\lambda)$ and, since Z_K is generated by a vector of weight λ, we must have $Z_K \cong \Delta_K(\lambda)$. Since Z is generated by a (non-zero) highest weight vector inside the module $\Delta_K(\lambda)$ we have $Z \cong \Delta_R(\lambda)$, by construction. But also, from (2) and the injectivity of ϕ, we get that $\mathrm{Tor}_1^R(F, M/Z) = 0$, i.e. M/Z is torsion free, i.e. M/Z is an S_R-lattice. Moreover, we have a short exact sequence $0 \to Z_F \to M_F \to (M/Z)_F \to 0$ and M_F has a Δ-filtration and $Z_F \cong \Delta_F(\lambda)$ so that $(M/Z)_F$ also has a Δ-filtration. Now we may assume, by induction on rank, that M/Z has a Δ-filtration and since $Z \cong \Delta_R(\lambda)$, we get that M has a Δ-filtration, as required.

Proposition 6.1 *Let M an S_R-lattice.*
(i) If M_F is a tilting module then so is M_K.
(ii) If M is a projective S_R-lattice (i.e. a finitely generated projective S_R-module) such that M_F is injective then M_K is injective.

Proof: (i) Since M_F has both a Δ-filtration and a ∇-filtration so too does M_R, by Lemma 6.2. Hence M_K has a Δ-filtration and a ∇-filtration, i.e. M_K is a tilting module.
(ii) The S_F-module M_F is both projective and injective, hence has both a Δ-filtration and a ∇-filtration. Hence M_F is tilting and so, by (i), M_K is too. Now M_K is a projective tilting module and hence also injective.

Theorem 6.1 *(i) If $\lambda \in \Lambda^+(n,r)$ is such that $T_F(\lambda)$ is injective then $T_K(\lambda)$ is injective.*
(ii) If $\mu \in \Lambda^+(n,r)$ is such that $I_F(\mu)$ is projective then $I_K(\mu)$ is also projective.

Proof: (i) If $T_F(\lambda)$ is injective then it is also projective. Since R is complete, there exists some projective S_R-lattice T such that $T_F \cong T_F(\lambda)$. Now T_K is a tilting module by Proposition 6.1. Moreover, T_K and T_F have the same character. Hence T_K is a projective tilting module with highest weight λ. Thus $T_K(\lambda)$ is a summand of T_K and hence also is a projective tilting module and hence also injective.
(ii) We have $I_F(\mu) = T_F(\lambda)$, for some $\lambda \in \Lambda^+(n,r)$ and $T_F(\lambda) = T_F$ for some projective R-lattice T. The rest of the argument is as in (i).

Corollary 6.1 *If the conjecture holds over all characteristic zero base fields then it holds over all fields.*

Proof: Suppose F is a field of characteristic p and $I_F(\mu)$ is projective. We may assume that F is the field generated by q. We have $I_F(\mu) = T_F(\lambda)$ for some λ. By Examples (i) and (ii) above we have a quantum modular reduction system (K, R, F, Q, q) where K is a p-adic number field and Q has finite multiplicative order l. By Theorem 6.1, $T_K(\lambda)$ is injective and so, by the conjecture for fields of characteristic 0, either $\lambda \in (l-1)\delta + \Lambda^+(n)$ or $\lambda = \mathrm{Mull}(\tau')$, for some column l-regular $\tau \in \Lambda^+(n)$. Hence either $I_F(\mu) = T_F(\lambda)$ is injective as a G_1-module or $\lambda = \mathrm{Mull}(\tau')$, which implies $T_F(\lambda) = I_F(\tau) = I_F(\mu)$ and hence $\mu = \tau$ is column l-regular.

7 The cases $n = 2$ and $n = 3$

In this section we will prove our conjecture when $n = 2, 3$. Note that by the result of the previous section, it is enough to prove it in characteristic zero. Throughout this section we assume that $\operatorname{char} k = 0$ and q is a primitive l-th root of unity.

Let us start with some general observations. Recall that for a subset π of the set of dominant weights $X^+(n)$ we have a functor O_π on the category $\mathrm{Rat}(n)$ defined on an object V by setting $O_\pi(V)$ to be the largest submodule of V with all composition factors $L(\theta)$ satisfying $\theta \in \pi$. Note that if π is saturated in $X^+(n)$ and V has a ∇-filtration then so does $O_\pi(V)$, see [9] Lemma A.3.1.

We now consider a family of subsets of $X^+(n)$. For an integer r, we define $\pi(r) = \{\theta = (\theta_1, \ldots, \theta_n) \in X^+(n) \,|\, \theta_n \geq r\}$. It is easy to check that $\pi(r)$ is saturated. For a rational module V, we call $O_{\pi(r)}(V)$ the r-th **level** of V. Note that $O_{\pi(0)}(V)$ is the largest polynomial submodule of V.

This definition was motivated by the following proposition.

Proposition 7.1 *Let $\eta \in \Lambda^+(n)$ and write $\eta = \lambda + l\mu$ where $\lambda, \mu \in \Lambda^+(n)$ with λ l-restricted and $\mu_n = 0$. Then*

$$I(\eta) \otimes D^{\otimes(n-1)(l-1)} \cong O_{\pi((n-1)(l-1))}(T(2(l-1)\delta + w_0\lambda + l\mu)).$$

Moreover,

$$I(\eta) \otimes D^{\otimes(n-1)(l-1)} \cong T(2(l-1)\delta + w_0\lambda + l\mu)$$

if and only if $\lambda_1 \geq (n-1)(l-1)$.

Proof: Set $T = T(2(l-1)\delta + w_0\lambda + l\mu)$. Using the remark following Theorem 4.1, we see that $T \otimes D^{\otimes -(n-1)(l-1)}$ is the injective hull of $L(\lambda + l\mu)$ in the category $\mathrm{Rat}(n)$. So the injective hull of $L(\lambda + l\mu)$ in $\mathrm{Pol}(n)$, namely $I(\lambda + l\mu)$, is equal to $O_{\pi(0)}(T \otimes D^{\otimes -(n-1)(l-1)})$. So we have

$$I(\lambda + l\mu) \otimes D^{\otimes(n-1)(l-1)} = O_{\pi((n-1)(l-1))}(T).$$

Now the last part of $2(l-1)\delta + w_0\lambda + l\mu$ is at least $(n-1)(l-1)$ if and only if $\lambda_1 \geq (n-1)(l-1)$ and in this case we have $O_{\pi((n-1)(l-1))}(T) = T$.

Recall that we have a natural W-invariant $\mathbf{Z}$-bilinear inner product $(\ ,\ ) : X(n) \times X(n) \to \mathbf{Z}$ for which $\epsilon_1, \ldots, \epsilon_n$ form an orthonormal basis. We put $\beta_0 = \epsilon_1 - \epsilon_n$ and note that $\mu \leq \lambda$ implies $(\mu, \beta_0) \leq (\lambda, \beta_0)$ for all $\mu, \lambda \in X(n)$.

Proposition 7.2 *Let $\lambda, \mu \in \Lambda^+(n)$ with λ l-restricted. Assume $\mu_i - \mu_{i+1} \geq n-1$ for all $1 \leq i \leq n-1$. Then no proper submodule of $T((l-1)\delta + \lambda + l\mu)$ can be tilting.*

Proof: Note that $T = T((l-1)\delta + \lambda + l\mu)$ is isomorphic to $T((l-1)\delta + \lambda) \otimes \overline{\nabla}(\mu)^F$. Now the character of $T((l-1)\delta + \lambda)$ is divisible by $\chi((l-1)\delta)$ and has highest weight $(l-1)\delta + \lambda$. It follows that the character of T is a sum of terms $\chi((l-1)\delta + w\nu + l\mu)$ where $w \in W$, $\nu \in \Lambda^+(n)$, $\nu \leq \lambda$. We claim that under our assumption every weight of this form belongs to $(l-1)\delta + \Lambda^+(n) \subset \Lambda^+(n)$. Now assume that the claim holds and suppose for a contradiction that Y is a proper submodule of T and that Y is tilting. Then since both Y and T admit ∇-filtrations, T/Y also admits a ∇-filtration. It follows that $\mathrm{ch}\, Y$ is a sum of terms $\chi((l-1)\delta + w\nu + l\mu)$, with $w \in W$, $\nu \in \Lambda^+(n)$, $\nu \leq \lambda$ and hence a sum of terms $\chi((l-1)\delta + \tau)$, $\tau \in \Lambda^+(n)$. In particular, the highest weight of Y belongs to $(l-1)\delta + \Lambda^+(n)$ so Y is injective and hence is a direct summand of the indecomposable module T. This is a contradiction.

Let us prove the claim. We need to show that for $w \in W$, $\nu \in \Lambda^+(n)$, $\nu \leq \lambda$ we have

$$((l-1)\delta + w\nu + l\mu, \alpha_i) \geq l-1$$

where $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for all $i = 1, ..., n-1$. So we have to prove that

$$(w\nu, \alpha_i) + l(\mu, \alpha_i) \geq 0$$

for all $i = 1, ..., n-1$. By assumption we have $(\mu, \alpha_i) \geq n-1$. Now we have

$$(w\nu, \alpha_i) = (\nu, w^{-1}\alpha_i) \geq -(\nu, \beta_0) \geq -(\lambda, \beta_0) \geq -(n-1)(l-1)$$

as λ is l-restricted. So

$$(w\nu, \alpha_i) + l(\mu, \alpha_i) \geq l(n-1) - (l-1)(n-1) \geq 0.$$

Corollary 7.1 *Let $\eta = \lambda + l\mu \in \Lambda^+(n)$ with λ l-restricted and $\mu_i - \mu_{i+1} \geq n-1$ for all $0 \leq i \leq n-1$. If $I(\eta) \cong P(\eta)$ then $I(\eta)$ is injective in* $\mathrm{Rat}(n)$.

Proof: Combine Proposition 7.1 with Proposition 7.2.

Proposition 7.3 *Conjecture 5.1 holds for* $n = 2$

Proof: Let $\lambda \in \Lambda^+(2)$ and suppose that $I(\lambda) \cong P(\lambda)$ but it is not injective in $\mathrm{Rat}(2)$. Then using Corollary 7.1 we see that λ must be l-restricted. Now if λ is not column l-regular then $\lambda_1 \geq \lambda_2 \geq l$. Using Proposition 7.1, we get that $I(\lambda)$ is injective in $\mathrm{Rat}(2)$. This is a contradiction.

Before proving the conjecture for $n = 3$, we shall prove and recall a few more general results. For an integer r, define the r-th **slice** of a rational module V by $\mathrm{Slice}_r(V) = O_{\pi(r)}(V)/O_{\pi(r+1)}(V)$. We set $\pi'(r) = \pi(r) \setminus \pi(r+1) = \{\theta = (\theta_1, \ldots, \theta_n) \in X^+(n) \,|\, \theta_n = r\}$. Let f be the truncation functor from $\mathrm{Pol}(n)$ to $\mathrm{Pol}(n-1)$ defined in section 2.2.

Lemma 7.1 *Let* $V \in \mathrm{Pol}(n)$ *with a* ∇*-filtration. Then for any* $r \geq 0$ *we have*

$$\mathrm{ch}(\mathrm{Slice}_r(V)) = \sum_{\lambda \in \pi'(r)} (V : \nabla(\lambda))\chi_n(\lambda)$$

and

$$\mathrm{ch}(fV) = \sum_{\lambda \in \pi'(0)} (V : \nabla(\lambda))\chi_{n-1}(\lambda).$$

Proof: This follows from the definitions of Slice_r and properties of f given in section 2.2.

Remark: Take V to be the indecomposable tilting module $T_n(\lambda)$. By tensoring it with some negative power of the determinant if necessary, we can assume that $\lambda_n = 0$. Then, since $fT_n(\lambda) = T_{n-1}(\lambda)$ (see section 2.2), Lemma 7.1 tells us that $\mathrm{Slice}_0(T_n(\lambda))$ gives the character of $T_{n-1}(\lambda)$. In particular, for $n = 3$ we know that the tilting modules in $\mathrm{Pol}(n-1) = \mathrm{Pol}(2)$ have at most two ∇-factors, see [9] 3.4(4).

We recall some well-known facts about the translation functors in $\mathrm{Rat}(n)$ (details for the analogous case of rational modules for a reductive group can be found in [15]II.7and [1] section 5). For $\lambda, \mu \in X^+(n)$ with $\mu - \lambda = w\nu$ where $w \in W$ and $\nu \in X^+(n)$ satisfies $(\nu+\delta, \beta_0) \leq l$, we define the translation functor T_λ^μ on $\mathrm{Rat}(n)$ given on an object $V \in \mathrm{Rat}(n)$ by

$$T_\lambda^\mu(V) = \mathrm{pr}_\mu(\mathrm{pr}_\lambda(V) \otimes L(\nu))$$

where, for $\eta \in X^+(n)$ we denote by pr_η the functor projecting a rational module onto the summand belonging to the (union of) blocks corresponding to the orbit of η under the dot action of the affine Weyl group W_l. Now if V has a ∇-filtration (resp. a Δ-filtration) then so does $T_\lambda^\mu(V)$. So if V is a tilting module then so is $T_\lambda^\mu(V)$.

Now suppose that μ belongs to the closure of the facet containing λ. Let $w \in W_l$ with $w \cdot \lambda$ and $w \cdot \mu \in \Lambda^+(n)$. Assume that the stabiliser of $w \cdot \mu$ in the affine Weyl group W_l is given by $\{1, s\}$ where s is an affine reflection and that $sw \cdot \lambda < w \cdot \lambda$. Then we have

$$T_\lambda^\mu \nabla(w \cdot \lambda) = T_\lambda^\mu \nabla(sw \cdot \lambda) = \nabla(w \cdot \mu) \tag{3}$$

and we have the following short exact sequence

$$0 \longrightarrow \nabla(sw \cdot \lambda) \longrightarrow T_\mu^\lambda \nabla(w \cdot \mu) \longrightarrow \nabla(w \cdot \lambda) \longrightarrow 0. \tag{4}$$

Moreover, the translates of the corresponding tilting modules are given by

$$T_\mu^\lambda T(w \cdot \mu) = T(w \cdot \lambda) \tag{5}$$

$$T_\lambda^\mu T(w \cdot \lambda) = 2T(w \cdot \mu). \tag{6}$$

Let us quote one last result proved by Humphreys and Jantzen for arbitrary semisimple simply connected algebraic groups. The proof is based on Weyl's dimension formula so the result clearly applies to our situation.

Theorem 7.1 [13] *If $\lambda \in \Lambda^+(n)$ and $\nabla(\lambda) = L(\lambda)$ then either λ is l-restricted or $\lambda \in (l-1)\delta + l\Lambda^+(n)$*

Proposition 7.4 *Assume $n = 3$. Let $\lambda, \mu \in \Lambda^+(3)$ with λ l-restricted and $\mu \neq 0$. Then $T((l-1)\delta+\lambda+l\mu)$ has no proper level which is a tilting module.*

Proof: Let us consider the character of the tilting module $T((l-1)\delta+\lambda+l\mu)$. For $\lambda_1 - \lambda_3 \leq l$ we have $\operatorname{ch} T((l-1)\delta + \lambda) = \chi((l-1)\delta)s(\lambda)$ where $s(\lambda)$ denotes the orbit sum of λ under the action of the Weyl group W, see e.g. [11] 5.Proposition. As $T((l-1)\delta + \lambda + l\mu) \cong T((l-1)\delta + \lambda) \otimes \overline{\nabla}(\mu)^F$ we have

$$\begin{aligned} \operatorname{ch} T((l-1)\delta + \lambda + l\mu) &= \chi((l-1)\delta)\chi(\mu)^F s(\lambda) \\ &= \chi((l-1)\delta + l\mu)s(\lambda) \\ &= \sum_{\theta \in W\lambda} \chi((l-1)\delta + \theta + l\mu), \end{aligned}$$

by Brauer's formula, see [6] (2.2.3).

Using (4) and (5) we see that for $\lambda_1 - \lambda_3 > l$, we have

$$\operatorname{ch} T((l-1)\delta+\lambda+l\mu) = \sum_{\nu \in W\lambda} \chi((l-1)\delta+\nu+l\mu) + \sum_{\tau \in W(s_{\beta_0,l}\cdot\lambda)} \chi((l-1)\delta+\tau+l\mu).$$

As $T(\eta + a\omega) \cong T(\eta) \otimes D^{\otimes a}$ for all $\eta \in \Lambda^+(3)$ and positive integer a, we shall write the character of tilting modules as "SL_3-characters". Namely, for a weight $\theta = (\theta_1, \theta_2, \theta_3) \in \Lambda(3)$ we shall consider instead the corresponding "SL_3-weight" $(\theta_1 - \theta_2, \theta_2 - \theta_3)$. We shall represent the character of a tilting module T as a set of dots in the dominant region, each dot representing the highest weight of a ∇-factor of T. A dot with a circle around it means that this ∇ occurs twice in a ∇-filtration of T. The straight lines represent the walls of the dot action of the affine Weyl group W_l on $X(n)$.

Note that the Morita equivalences given by the translation functors, see [15] II.7.9 (or Scopes equivalences, see [8] section 5 in this context) for two weights in the same facet implies that it is enough to consider one tilting module for each facet.

So we get the following pictures for the characters of the $T((l-1)\delta+\lambda+l\mu)$ where λ is l-restricted. We set $(a,b) = (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3)$ and $(c,d) = (\mu_1 - \mu_2, \mu_2 - \mu_3)$. Then $0 \leq a, b \leq l-1$ and $c, d \geq 0$.

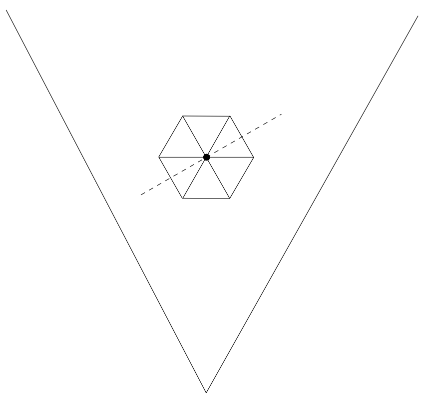


Figure 0
$a = b = 0$

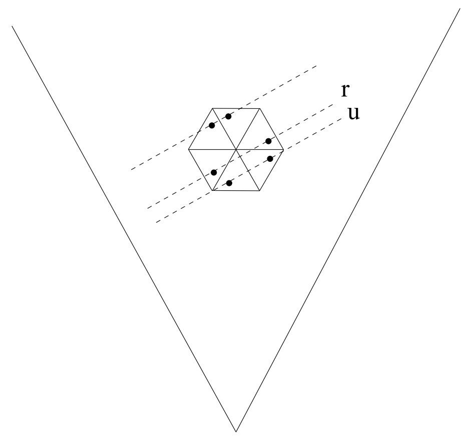


Figure 1
$a, b \neq 0,\ a + b < l$

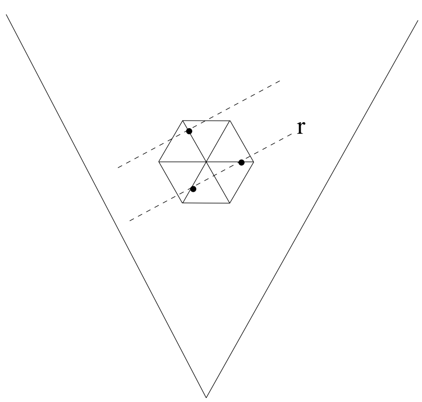


Figure 2
$a = 0,\ b \neq 0,\ a + b < l$

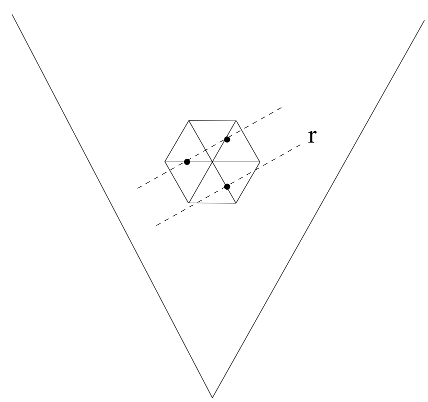


Figure 3
$a \neq 0,\ b = 0,\ a + b < l$

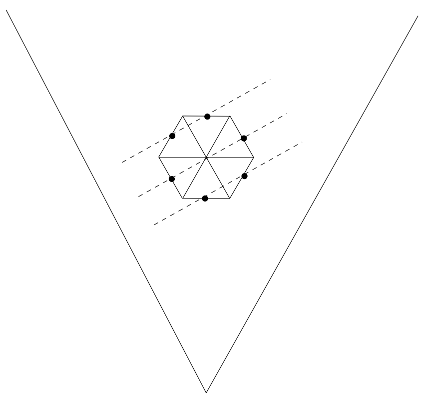


Figure 4
$a + b = l,\ c, d \geq 1$

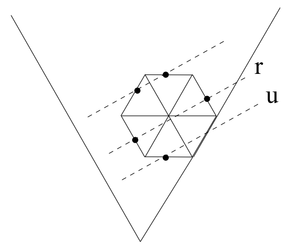


Figure 5
$a + b = l,\ c \geq 1,\ d = 0$

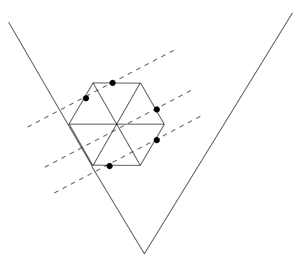


Figure 6
$a + b = l,\ c = 0,\ d \geq 1$

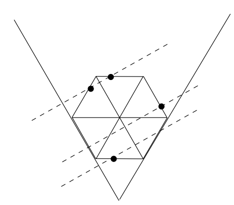


Figure 7
$a + b = l,\ c = d = 0$

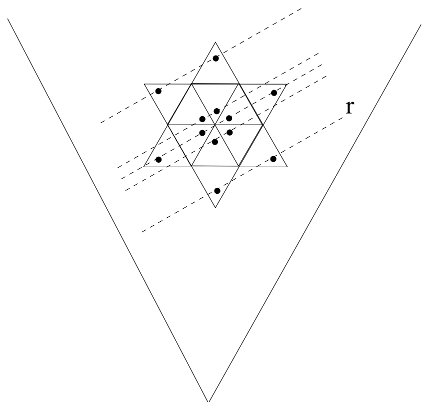


Figure 8
$a+b>l$, $c,d\geq 1$

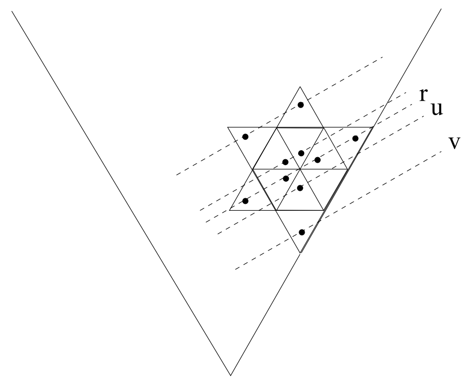


Figure 9
$a+b>l$, $c\geq 1$, $d=0$

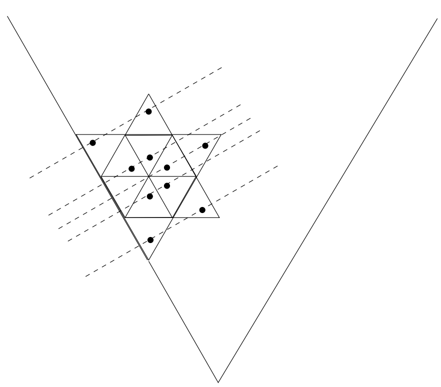


Figure 10
$a+b>l$, $c=0$, $d\geq 1$

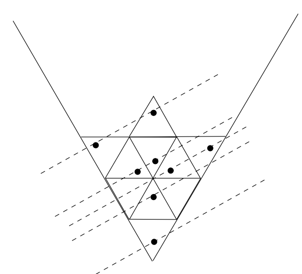


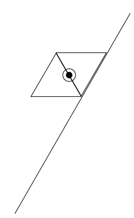
Figure 11
$a+b>l$, $c=d=0$

The dashed lines in the pictures represent the character of the slices of the tilting module, and the character of the levels are given by all the dots on and below a given slice.

First observe that if a level has a highest weight contained in $(l-1)\delta + X^+(n)$ then we are done using the same argument as in the proof of Proposition 7.2. So we are left with the following cases (note that we have assumed that $\mu \neq 0$): Figures 1,2,3 and 5: $c \geq 1$, $d = 0$ (so the hexagon is against the right wall), levels r and u; Figure 8: $c \geq 1$, $d = 1$ (so the star is touching the right wall), level r; Figure 9: levels r, u and v.

For Figure 3 level r, Figure 5 level u and Figure 9 level v, we see using Theorem 7.1 that they cannot be tilting.

In Figure 1, if level u were tilting, then translating onto the wall and using (3), we see that



would give the character of a tilting module, i.e. we would have a tilting module whose character is $2\chi(\nu)$ for some $\nu \in \Lambda^+(n)$ with ν not l-restricted and $\nu \notin (l-1)\delta + l\Lambda^+(n)$. But then using Theorem 7.1, we see that $\nabla(\nu)$ cannot be simple and so $2\chi(\nu)$ is not the character of a tilting module. The same argument shows that in Figure 9, level v cannot be tilting.

In Figure 1, if level r were tilting then using (3), the left hand side of Figure 12 would give the character of a tilting module.

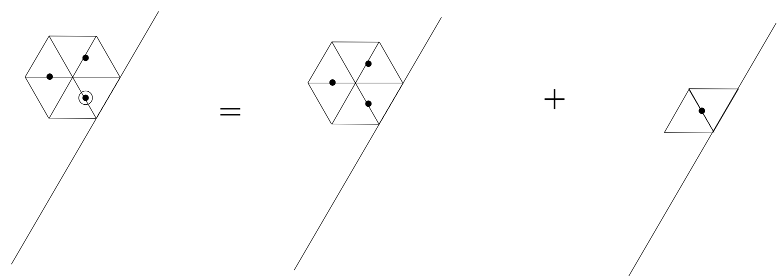


Figure 12

Now the character of the indecomposable tilting module with the same highest weight is given in Figure 3. So the tilting module whose character is given by the left hand side of Figure 12 would decompose as a direct sum of tilting modules whose characters are given by the right hand side of Figure 12. But using Theorem 7.1, the second summand cannot be tilting.

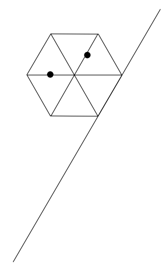
When $l \geq 3$ (i.e. the alcoves are non-empty), we can use (4) to show that if level r of Figure 2 were tilting then so would level r of Figure 1. But we have just shown that it was not. In fact, when $l = 2$, exactly the same argument works. Consider the r-th level Y, say, in Figure 2. Suppose that Y is tilting. We assume without loss of generality that $Y \otimes D^*$ is not polynomial. It has two ∇-quotients, namely $\nabla(2t, 0, 0)$ and $\nabla(2t-1, 1, 0)$. Apply the translation functor T_λ^μ where $\lambda = (2t, 0, 0)$ and $\mu = (2t+1, 1, 0)$, so we get

$$T_\lambda^\mu(Y) = \mathrm{pr}_\mu(Y \otimes \Lambda^2 E).$$

Since Y is tilting, so is $T_\lambda^\mu(Y)$ and it's easy to see that it has character $\chi(2t+1, 1, 0) + 2\chi(2t, 1, 1) + \chi(2t, 2, 0)$. This is exactly the character represented in Figure 12 and we have just seen that it cannot be the character of a tilting module.

In Figures 5 and 9, consider the r-th level Y, say. If Y were tilting then so is $Y \otimes D^{\otimes -r}$. But this would contradict Lemma 7.1 and the remark following it.

Finally, in Figure 8, if level r were tilting then using (3), the following picture would give the character of a tilting module



But it cannot be as the character of the indecomposable tilting module with the same highest weight is given in Figure 3. This completes the proof of Proposition 7.4.

Proposition 7.5 *Assume $n = 3$. Let $\lambda \in \Lambda^+(3)$ be l-restricted. If $I(\lambda) \cong P(\lambda)$ then either $I(\lambda)$ is injective in* Rat(3) *or λ is column l-regular.*

Proof: Write $T = T(2(l-1)\delta + w_0\lambda)$. Using Proposition 7.1, we have that

$$I(\lambda) \otimes D^{\otimes 2(l-1)} \cong O_{\pi(2(l-1))}(T)$$

Moreover, $I(\lambda) \otimes D^{2(l-1)}$ is isomorphic to T if and only if $\lambda_1 \geq 2(l-1)$. Now suppose that $\lambda_1 < 2(l-1)$ and that λ is not column l-regular, i.e. $\lambda_3 \geq l$. Then T has a ∇-quotient with highest weight

$$\begin{aligned} & s_{\alpha_1+\alpha_2,3l} \cdot (4(l-1) + \lambda_3, 2(l-1) + \lambda_2, \lambda_1) \\ & = (3l - 2 + \lambda_1, 2(l-1) + \lambda_2, l - 2 + \lambda_3) \end{aligned}$$

and $l - 2 + \lambda_3 \geq 2(l-1)$ as $\lambda_3 \geq l$. Moreover, it's easy to see that T has no factor $\nabla(\mu)$ with $(3l - 2 + \lambda_1, 2(l-1) + \lambda_2, l - 2 + \lambda_3) < \mu < (4(l-1) + \lambda_3, 2(l-1) + \lambda_2, \lambda_1)$. So $(3l - 2 + \lambda_1, 2(l-1) + \lambda_2, l - 2 + \lambda_3)$ is a highest weight of $I(\lambda) \otimes D^{\otimes 2(l-1)}$, but it belongs to $(l-1)\delta + \Lambda^+(3)$, this is a contradiction.

Corollary 7.2 *Conjecture 5.1 holds for $n = 3$.*

Proof: This follows from Propositions 7.4 and 7.5.

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