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# Stepwise Projection: Toward Brane Setups for Generic Orbifold Singularities

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Abstract: The construction of brane setups for the exceptional series $E_{6,7,8}$ of $SU(2)$ orbifolds remains an ever-haunting conundrum. Motivated by techniques in some works by Muto on non-Abelian $SU(3)$ orbifolds, we here provide an algorithmic outlook, a method which we call stepwise projection, that may shed some light on this puzzle. We exemplify this method, consisting of transformation rules for obtaining complex quivers and brane setups from more elementary ones, to the cases of the $D$-series and $E_6$ finite subgroups of $SU(2)$. Furthermore, we demonstrate the generality of the stepwise procedure by appealing to Frøbenius' theory of Induced Representations. Our algorithm suggests the existence of generalisations of the orientifold plane in string theory.

Keywords: Brane setups, non-Abelian Orbifolds, Induced Representations, Orientifolds.

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## Contents



## 1. Introduction

It is by now a well-known fact that a stack of $n$ parallel coincident D3-branes has on its world-volume, an $\mathcal{N} = 4$, four-dimensional supersymmetric $U(n)$ gauge theory. Placing such a stack at an orbifold singularity of the form $\mathbb{C}^k/\{\Gamma \subset SU(k)\}$ reduces the supersymmetry to $\mathcal{N} = 2, 1$ and $0$, respectively for $k = 2, 3$ and 4, and the gauge group is broken down to a product of $U(n_i)$'s [1, 2, 5].

Alternatively, one could realize the gauge theory living on D-branes by the so-called Brane Setups [3, 4] (or "Comic Strips" as dubbed by Rabinovici [6]) where D-branes are stretched between NS5-branes and orientifold planes. Since these two methods of orbifold projections and brane setups provide the same gauge theory living on D-branes, there should exist some kind of duality to explain the connection between them.

Indeed, we know now that by T-duality one can map D-branes probing certain classes of orbifolds to brane configurations. For example, the two-dimensional orbifold $\mathbb{C}^2/\{\mathbb{Z}_k \subset SU(2)\}$, also known as an ALE singularity of type $A_{k-1}$, is mapped into a circle of $k$ NS-branes (the so-called elliptic model) after proper T-duality transformations. Such a mapping is easily generalized to some other cases, such as the three-dimensional orbifold $\mathbb{C}^3/\{\mathbb{Z}_k \times \mathbb{Z}_l \subset SU(3)\}$ being mapped to the so-named Brane Box Model [15, 16] or the four-dimensional case of $\mathbb{C}^4/\{\mathbb{Z}_k \times \mathbb{Z}_l \times \mathbb{Z}_m \subset SU(4)\}$

being mapped to the brane cube model [17]. With the help of orientifold planes, we can T-dualise $\mathbb{C}^2/\{D_k \subset SU(2)\}$ to a brane configuration with $ON$-planes [18, 19], or $\mathbb{C}^3/\{\mathbb{Z}_k \times D_l \subset SU(3)\}$ to brane-box-like models with $ON$-planes [9].

A further step was undertaken by Muto [10, 11, 12] where an attempt was made to establish the brane setup which corresponds to the three-dimensional non-Abelian orbifolds $\mathbb{C}^3/\{\Gamma \subset SU(3)\}$ with $\Gamma = \Delta(3n^2)$ and $\Delta(6n^2)$. The key idea was to arrive at these theories by judiciously quotienting the well-known orbifold $\mathbb{C}^3/\{\mathbb{Z}_k \times \mathbb{Z}_l \subset SU(3)\}$ whose brane configuration is the Brane Box Model. In the process of this quotienting, a non-trivial $\mathbb{Z}_3$ action on the brane box is required. Though mathematically obtaining the quivers of the former from those of the latter seems perfectly sound, such a $\mathbb{Z}_3$ action appears to be an unfamiliar symmetry in string theory. We shall briefly address this point later.

Now, with the exception of the above list of examples, there have been no other successful brane setups for the myriad of orbifolds in dimension two, three and four. Since we believe that the methods of orbifold projection and brane configurations are equivalent to each other in giving D-brane world-volume gauge theories, finding the T-duality mappings for arbitrary orbifolds is of great interest.

The present work is a small step toward such an aim. In particular, we will present a so-called **stepwise projection** algorithm which attempts to systematize the quotienting idea of Muto, and, as we hope, to give hints on the brane construction of generic orbifolds.

We shall chiefly focus on the orbifold projections by the $SU(2)$ discrete subgroups $D_k$ and $E_6$ in relation to $\mathbb{Z}_n$. Thereafter, we shall evoke some theorems on induced representations which justify why our algorithm of stepwise projection should at least work in general mathematically. In particular, we will first demonstrate how the algorithm gives the quiver of $D_k$ from that of $Z_{2k}$. We then interpret this mathematical projection physically as precisely the orientifold projection, whereby arriving at the brane setup of $D_k$ from that of $\mathbb{Z}_{2k}$, both of which are well-known and hence giving us a consistency check.

Next we apply the same idea to $E_6$. We find that one can construct its quiver from that of $\mathbb{Z}_6$ or $D_2$ by an appropriate $\mathbb{Z}_3$ action. This is slightly mysterious to us physically as it requires a $\mathbb{Z}_3$ symmetry in string theory which we could use to quotient out the $\mathbb{Z}_6$ brane setup; such a symmetry we do not know at this moment. However, in comparison with Muto's work, our $\mathbb{Z}_3$ action and the $\mathbb{Z}_3$ investigated by Muto in light of the $\Delta$ series of $SU(3)$, hint that there might be some objects in string theory which provide a $\mathbb{Z}_3$ action, analogous to the orientifold giving a $\mathbb{Z}_2$, and which we could use on the known brane setups to establish those yet unknown, such as those corresponding to the orbifolds of the exceptional series.

The organisation of the paper is as follows. In §2 we review the technique of orbifold projections in an explicit matrix language before moving on to §3 to present our stepwise projection algorithm. In particular, §3.1 will demonstrate how to obtain the $D_k$ quiver from the $\mathbb{Z}_{2k}$ quiver, §3.2 and §3.3

will show how to get that of $E_6$ from those of $D_2$ and $\mathbb{Z}_6$ respectively. We finish with comments on the algorithm in §4. We will use induced representation theory in §4.1 to prove the validity of our methods and in §4.2 we will address how the present work may be used as a step toward the illustrious goal of obtaining brane setups for the generic orbifold singularity.

During the preparation of the manuscript, it has come to our attention that independent and variant forms of the method have been in germination [20, 21]; we sincerely hope that our systematic treatment of the procedure may be of some utility thereto.

## Nomenclature

Unless otherwise stated we shall adhere to the convention that $\Gamma$ refers to a discrete subgroup of $SU(n)$ (i.e., a finite collineation group), that $\langle x_1, .., x_n \rangle$ is a finite group generated by $\{x_1, .., x_n\}$, that $|\Gamma|$ is the order of the group $\Gamma$, that $D_k$ is the binary dihedral group of order $4k$, that $E_{6,7,8}$ are the binary exceptional subgroups of $SU(2)$, and that $R^{\bullet}_{G(n)}(x)$ is a representation of the element $x \in G$ of dimension $n$ with $\bullet$ denoting properties such as regularity, irreducibility, etc., and/or simply a label. Moreover, $S^T$ shall denote the transpose of the matrix $S$ and $A \otimes B$ is the tensor product of matrices $A$ and $B$ with block matrix elements $A_{ij}B$. Finally we frequently use the Pauli matrices $\{\sigma_i, i = 1, 2, 3\}$ as well as $\mathbb{1}_N$ for the $N \times N$ identity matrix. We emphasise here that the notation for the binary groups differs from our other works in the exclusion of $\hat{}$ and in the convention for the sub-index of the binary dihedral group.

## 2. A Review on Orbifold Projections

The general methodology of how the finite group structure of the orbifold projects the gauge theory has been formulated in [5]. The complete lists of two and three dimensional cases have been treated respectively in [1, 2] and [7, 10] as well as the four dimensional case in [8]. For the sake of our forth-coming discussion, we shall not use the nomenclature in [5, 7, 9] where recourse to McKay's Theorem and abstractions to representation theory are taken. Instead, we shall adhere to the notations in [2] and explicitly indicate what physical fields survive the orbifold projection.

Throughout we shall focus on two dimensional orbifolds $\mathbb{C}^2/\{\Gamma \subset SU(2)\}$. The parent theory has an $SU(4) \cong Spin(6)$ R-symmetry from the $\mathcal{N} = 4$ SUSY. The $U(n)$ gauge bosons $A^{\mu}_{IJ}$ with $I, J = 1, ..., n$ are R-singlets. Furthermore, there are Weyl fermions $\Psi^{i=1,2,3,4}_{IJ}$ in the fundamental $\mathbf{4}$ of $SU(4)$ and scalars $\Phi^{i=1,..,6}_{IJ}$ in the antisymmetric $\mathbf{6}$.

The orbifold imposes a projection condition upon these fields due to the finite group $\Gamma$. Let $R^{reg}_{\Gamma}(g)$ be the regular representation of $g \in \Gamma$, by which we mean

$$R^{reg}_{\Gamma}(g) := \bigoplus_i \Gamma_i(g) \otimes \mathbb{1}_{\dim(\Gamma_i)}$$

where $\{\Gamma_i\}$ are the irreducible representations of $\Gamma$. In matrix form, $R_\Gamma^{reg}(g)$ is composed of blocks of irreps, with each of dimension $j$ repeated $j$ times. Therefore it is a matrix of size $\sum\limits_i \dim(\Gamma_i)^2 = |\Gamma|$. Let $\mathrm{Irreps}(\Gamma) = \{\Gamma_1^{(1)}, \ldots, \Gamma_{m_1}^{(1)}; \Gamma_1^{(2)}, \ldots, \Gamma_{m_2}^{(2)}; \ldots\ldots; \Gamma_1^{(n)}, \ldots, \Gamma_{m_n}^{(n)}\}$, consisting of $m_j$ irreps of dimension $j$, then

$$R_\Gamma^{reg} := \begin{pmatrix} \Gamma_1^{(1)} & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & \Gamma_{m_1}^{(1)} & & & & & & & \\ & & & \begin{pmatrix} \Gamma_1^{(2)} & \\ & \Gamma_1^{(2)} \end{pmatrix} & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \begin{pmatrix} \Gamma_{m_2}^{(2)} & \\ & \Gamma_{m_2}^{(2)} \end{pmatrix} & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \begin{pmatrix} \Gamma_1^{(n)} & & \\ & \ddots & \\ & & \Gamma_1^{(n)} \end{pmatrix}_{n\times n} & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \begin{pmatrix} \Gamma_{m_n}^{(n)} & & \\ & \ddots & \\ & & \Gamma_{m_n}^{(n)} \end{pmatrix}_{n\times n} \end{pmatrix}. \tag{2.1}$$

Of the parent fields $A^\mu, \Psi, \Phi$, only those invariant under the group action will remain in the orbifolded theory; this imposition is what we mean by *surviving the projection*:

$$\begin{aligned} A^\mu &= R_\Gamma^{reg}(g)^{-1} \cdot A^\mu \cdot R_\Gamma^{reg}(g) \\ \Psi^i &= \rho(g)^i_j \; R_\Gamma^{reg}(g)^{-1} \cdot \Psi^j \cdot R_\Gamma^{reg}(g) \\ \Phi^i &= \rho'(g)^i_j \; R_\Gamma^{reg}(g)^{-1} \cdot \Phi^j \cdot R_\Gamma^{reg}(g) \qquad \forall \;\; g \in \Gamma, \end{aligned} \tag{2.2}$$

where $\rho$ and $\rho'$ are induced actions because the matter fields carry R-charge (while the gauge bosons are R-singlets). Clearly if $\Gamma = \langle x_1, ..., x_n \rangle$, it suffices to impose (2.2) for the generators $\{x_i\}$ in order to find the matter content of the orbifold gauge theory; this observation we shall liberally use henceforth.

Letting $n = N|\Gamma|$ for some large $N$ and $n_i = \dim(\Gamma_i)$, the subsequent gauge group becomes $\prod\limits_i U(n_i N)$ with $a^4_{ij}$ Weyl fermions as bifundamentals $\left(\mathbf{n_i N}, \overline{\mathbf{n_j N}}\right)$ as well as $a^6_{ij}$ scalar bifundamentals. These bifundamentals are pictorially summarised in quiver diagrams whose adjacency matrices are the $a_{ij}$'s.

Since we shall henceforth be dealing primarily with $\mathbb{C}^2$ orbifolds, we have $\mathcal{N} = 2$ gauge theory in four dimensions [5]. In particular we choose the induced group action on the R-symmetry to be $\mathbf{4} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}$ and $\mathbf{6} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}^2$ in order to preserve the supersymmetry. For this reason we can specify the final fermion and scalar matter matrices by a single quiver characterised by the $\mathbf{2}$ of $SU(2)$ as the trivial $\mathbf{1}$'s give diagonal 1's. These issues are addressed at length in [7].

# 3. Stepwise Projection

Equipped with the clarification of notations of the previous section we shall now illustrate a technique which we shall call **stepwise projection**, originally inspired by [10, 11, 12], who attempted brane realisations of certain non-Abelian orbifolds of $\mathbb{C}^3$, an issue to which we shall later turn.

The philosophy of the technique is straight-forward[2]: say we are given a group $\Gamma_1 = \langle x_1, ..., x_n \rangle$ with quiver diagram $Q_1$ and $\Gamma_2 = \langle x_1, ..., x_{n+1} \rangle \supset \Gamma_1$ with quiver $Q_2$, we wish to determine $Q_2$ from $Q_1$ by the projection (2.2) by $\{x_1, ..., x_n\}$ followed by another projection by $x_{n+1}$.

We now proceed to analyse the well-known examples of the cyclic and binary dihedral quivers under this new light.

## 3.1 $D_k$ Quivers from $A_k$ Quivers

We shall concern ourselves with orbifold theories of $\mathbb{C}^2/\mathbb{Z}_k$ and $\mathbb{C}^2/D_k$. Let us first recall that the cyclic group $A_{k-1} \cong \mathbb{Z}_k$ has a single generator

$$\beta_k := \begin{pmatrix} \omega_k & 0 \\ 0 & \omega_k^{-1} \end{pmatrix}, \qquad \text{with} \quad \omega_n := e^{\frac{2\pi i}{n}}$$

and that the generators for the binary dihedral group $D_k$ are

$$\beta_{2k} = \begin{pmatrix} \omega_{2k} & 0 \\ 0 & \omega_{2k}^{-1} \end{pmatrix}, \qquad \gamma := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We further recall from [9] that $D_k/\mathbb{Z}_{2k} \cong \mathbb{Z}_2$.

Now all irreps for $\mathbb{Z}_k$ are 1-dimensional (the $k^{th}$ roots of unity), and (2.1) for the generator reads

$$R^{reg}_{\mathbb{Z}_k}(\beta_k) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega_k & 0 & 0 & 0 \\ 0 & 0 & \omega_k^2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \omega_k^{k-1} \end{pmatrix}.$$

[2] A recent work [21] appeared during the final preparations of this draft; it beautifully addresses issues along a similar vein. In particular, cases where $\Gamma_1$ is normal in $\Gamma_2$ are discussed in detail. However, our stepwise method is not restricted by normality.

On the other hand, $D_k$ has 1 and 2-dimensional irreps and (2.1) for the two generators become

$$R^{reg}_{D_k}(\beta_{2k}) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} \omega_{2k} & 0 \\ 0 & \omega_{2k}^{-1} \end{pmatrix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} \omega_{2k} & 0 \\ 0 & \omega_{2k}^{-1} \end{pmatrix} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} \omega_{2k}^{k-1} & 0 \\ 0 & \omega_{2k}^{-(k-1)} \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} \omega_{2k}^{k-1} & 0 \\ 0 & \omega_{2k}^{-(k-1)} \end{pmatrix} \end{pmatrix}$$

and

$$R^{reg}_{D_k}(\gamma) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i^{k \bmod 2} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} -1 & 0 \\ 0 & -i^{k \bmod 2} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} 0 & i^{k-1} \\ i^{k-1} & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} 0 & i^{k-1} \\ i^{k-1} & 0 \end{pmatrix} \end{pmatrix}.$$

In order to see the structural similarities between the regular representation of $\beta_{2k}$ in $\Gamma_1 = \mathbb{Z}_{2k}$ and $\Gamma_2 = D_k$, we need to perform a change of basis. We do so such that each pair (say the $j^{th}$) of the 2-dimensional irreps of $D_2$ becomes as follows:

$$\Gamma^{(2)}(\beta_{2k}) = \begin{pmatrix} \begin{pmatrix} \omega_{2k}^{j} & 0 \\ 0 & \omega_{2k}^{-j} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \omega_{2k}^{j} & 0 \\ 0 & \omega_{2k}^{-j} \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} \omega_{2k}^{j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \omega_{2k}^{-j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

where $j = 1, 2, \ldots, k-1$. In this basis, the 2-dimensionals of $\gamma$ become

$$\Gamma^{(2)}(\gamma) = \begin{pmatrix} \begin{pmatrix} 0 & i^{j} \\ i^{j} & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & i^{j} \\ i^{j} & 0 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & i^{j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i^{j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \end{pmatrix}.$$

Now for the 1-dimensionals, we also permute the basis:

$$\Gamma^{(1)}(\beta_{2k}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \Gamma^{(1)}(\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i^{k \bmod 2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i^{k \bmod 2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i^{k \bmod 2} & 0 \\ 0 & 0 & 0 & -i^{k \bmod 2} \end{pmatrix}.$$

Therefore, we have

$$
R_{D_k}^{reg}(\beta_{2k}) = \begin{pmatrix} 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & -1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & \omega_{2k} & 0 & & 0 & 0 \\ 0 & 0 & 0 & \omega_{2k}^{-1} & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & \omega_{2k}^{k-1} & 0 \\ 0 & 0 & 0 & 0 & & 0 & \omega_{2k}^{-(k-1)} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

which by now has a great resemblance to the regular representation of $\beta_{2k} \in \mathbb{Z}_{2k}$; indeed, after one final change of basis, by ordering the powers of $\omega_{2k}$ in an ascending fashion while writing $\omega_{2k}^{-j} = \omega_{2k}^{2k-j}$ to ensure only positive exponents, we arrive at

$$
\begin{aligned} R_{D_k}^{reg}(\beta_{2k}) &= \begin{pmatrix} 1 & 0 & 0 & & 0 \\ 0 & \omega_{2k} & 0 & & 0 \\ 0 & 0 & \omega_{2k}^2 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & \omega_{2k}^{2k-1} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= R_{\mathbb{Z}_{2k}}^{reg}(\beta_{2k}) \otimes \mathbb{1}_2, \end{aligned} \tag{3.1}
$$

the key relation which we need.

Under this final change of basis,

$$
R_{D_k}^{reg}(\gamma) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} i^{k-3} & 0 \\ 0 & i^{k-3} \end{pmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} i^{k-2} & 0 \\ 0 & i^{k-2} \end{pmatrix} & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \begin{pmatrix} i^{k-1} & 0 \\ 0 & i^{k-1} \end{pmatrix} \\ \vdots & & & & \begin{pmatrix} i^{k \bmod 2} & 0 \\ 0 & -i^{k \bmod 2} \end{pmatrix} & & & \vdots \\ 0 & \begin{pmatrix} i^{k-3} & 0 \\ 0 & i^{k-3} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} i^{k-2} & 0 \\ 0 & i^{k-2} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} i^{k-1} & 0 \\ 0 & i^{k-1} \end{pmatrix} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.2}
$$

Our strategy is now obvious. We shall first project according to (2.2), using (3.1), which is *equivalent to* a projection by $\mathbb{Z}_{2k}$, except with two identical copies (physically, this simply means we place twice as many D3-brane probes). Thereafter we shall project once again using (3.2) and the resulting theory should be that of the $D_k$ orbifold.

### An Illustrative Example

Let us turn to a concrete example, namely $\mathbb{Z}_4 \to D_2$. The key points to note are that $D_2 := \langle \beta_4, \gamma \rangle$ and $\mathbb{Z}_4 \cong \langle \beta_4 \rangle$. We shall therefore perform stepwise projection by $\beta_4$ followed by $\gamma$.

Equation (3.1) now reads

$$R^{reg}_{D_2}(\beta_4) = R^{reg}_{\mathbb{Z}_4}(\beta_4) \otimes \mathbb{1}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i^2 & 0 \\ 0 & 0 & 0 & i^3 \end{pmatrix} \otimes \mathbb{1}_2. \tag{3.3}$$

We have the following matter content in the parent (pre-orbifold) theory: gauge field $A^\mu$, fermions $\Psi^{1,2,3,4}$ and scalars $\Phi^{1,2,3,4,5,6}$ (suppressing gauge indices $IJ$). Projection by $R^{reg}_{D_2}(\beta_4)$ in (3.3) according to (2.2) gives a $\mathbb{Z}_4$ orbifold theory, which restricts the form of the fields to be as follows:

$$A^\mu, \Psi^{1,2}, \Phi^{1,2} = \begin{pmatrix} \square & & & \\ & \square & & \\ & & \square & \\ & & & \square \end{pmatrix}; \quad \Psi^3, \Phi^{3,5} = \begin{pmatrix} & \square & & \\ & & \square & \\ & & & \square \\ \square & & & \end{pmatrix}; \quad \Psi^4, \Phi^{4,6} = \begin{pmatrix} & & & \square \\ \square & & & \\ & \square & & \\ & & \square & \end{pmatrix} \tag{3.4}$$

where $\square$ are $2 \times 2$ blocks. We recall from the previous section that we have chosen the R-symmetry decomposition as $\mathbf{4} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}$ and $\mathbf{6} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}^2$. The fields in (3.4) are defined in accordance thereto: the fermions $\Psi^{1,2}$ and scalars $\Phi^{1,2}$ are respectively in the two trivial $\mathbf{1}$'s of the $\mathbf{4}$ and $\mathbf{6}$; $(\Psi^3, \Psi^4)$, $(\Phi^3, \Phi^4)$ and $(\Phi^5, \Phi^6)$ are in the doublet $\mathbf{2}$ of $\Gamma$ inherited from $SU(2)$. Indeed, the $R^{reg}_{\mathbb{Z}_4}(\beta_4)$ projection would force $\square$ to be numbers and not matrices as we do not have the extra $\mathbb{1}_2$ tensored to the group action, in which case (3.4) would be $4 \times 4$ matrices prescribing the adjacency matrices of the $\mathbb{Z}_4$ quiver. For this reason, the quiver diagram for the $\mathbb{Z}_4$ theory as drawn in part (I) of Figure 1 has the nodes labelled 2's instead of the usual Dynkin labels of 1's for the $A$-series. In physical terms we have placed twice as many image D-brane probes. The key point is that because $\square$ are now matrices (and (3.4) are $8 \times 8$), further projection internal thereto may change the number and structure of the product gauge groups and matter fields.

Having done the first step by the $\beta_4$ projection, next we project with the regular representation of $\gamma$:

$$R^{reg}_{D_2}(\gamma) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & 0 & 0 \end{pmatrix} := \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{1}_2 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

In accordance with (3.4), let the gauge field be

$$A^\mu := \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix},$$

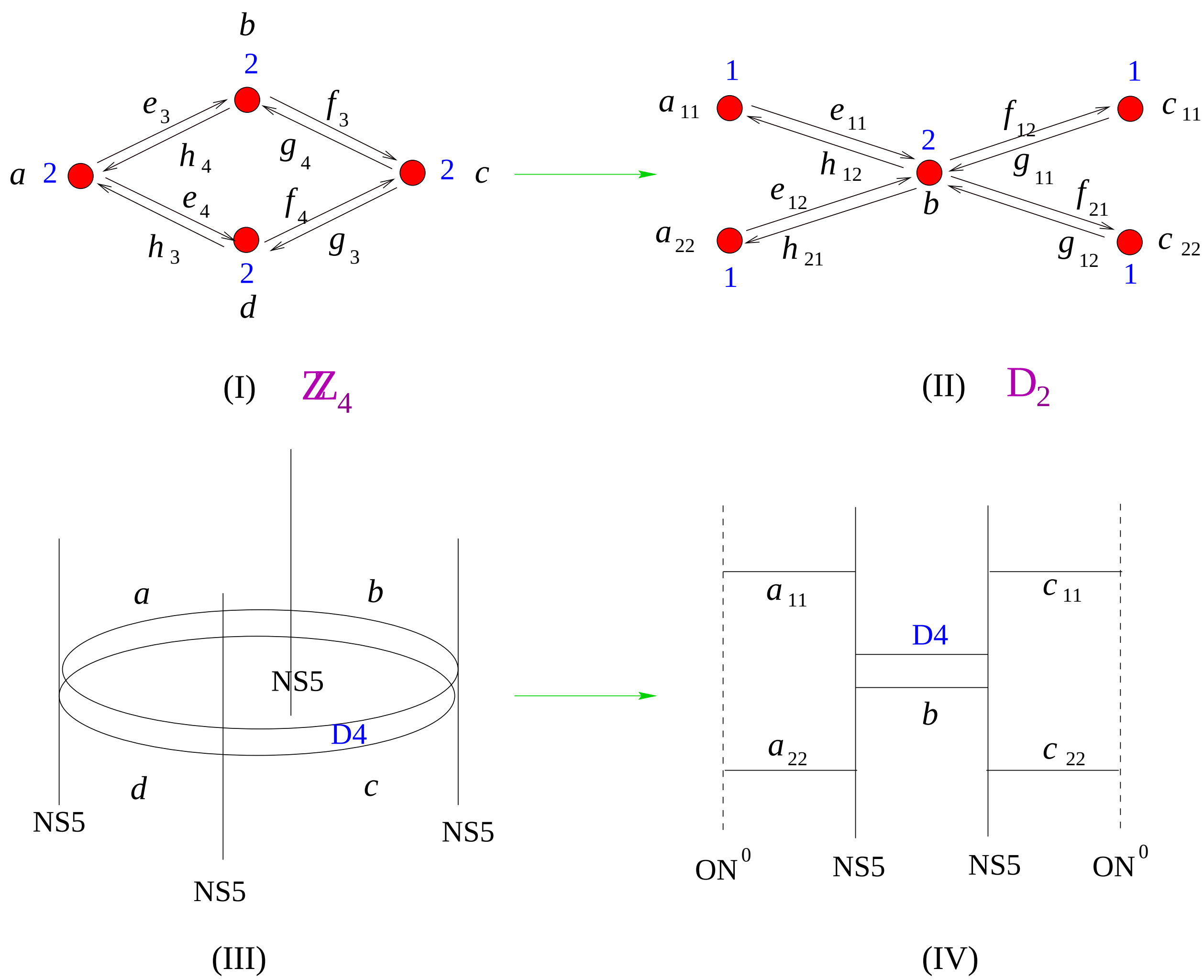


**Figure 1:** From the fact that $D_2 := \langle \beta_4, \gamma \rangle$ is generated by $\mathbb{Z}_4 = \beta_4$ together with $\gamma$, our stepwise projection, first by $\beta_4$, and then by $\gamma$, gives 2 copies of the $\mathbb{Z}_4$ quiver in Part (I) and then the $D_2$ quiver in Part (II) by appropriate joining/splitting of the nodes and arrows. The brane configurations for these theories are given in Parts (III) and (IV).

with $a, b, c, d$ denoting the $2 \times 2$ blocks $\square$, (2.2) for (3.5) now reads

$$A^\mu = R_{D_2}^{reg}(\gamma)^{-1} \cdot A^\mu \cdot R_{D_2}^{reg}(\gamma) \Rightarrow$$

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & -i\mathbb{1}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{1}_2 & 0 & 0 \end{pmatrix},$$

giving us a set of constraining equations for the blocks:

$$\sigma_3 \cdot a \cdot \sigma_3 = a; \qquad d = b; \qquad \sigma_3 \cdot c \cdot \sigma_3 = c. \tag{3.6}$$

Similarly, for the fermions in the **2**, viz.,

$$\Psi^3 = \begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix}, \qquad \Psi^4 = \begin{pmatrix} 0 & 0 & 0 & e_4 \\ f_4 & 0 & 0 & 0 \\ 0 & g_4 & 0 & 0 \\ 0 & 0 & h_4 & 0 \end{pmatrix},$$

the projection (2.2) is

$$\gamma \cdot \begin{pmatrix} \Psi^3 \\ \Psi^4 \end{pmatrix} = R_{D_2}^{reg}(\gamma)^{-1} \cdot \begin{pmatrix} \Psi^3 \\ \Psi^4 \end{pmatrix} \cdot R_{D_2}^{reg}(\gamma).$$

We have used the fact that the induced action $\rho(\gamma)$, having to act upon a doublet, is simply the $2 \times 2$ matrix $\gamma$ herself. Therefore, writing it out explicitly, we have

$$i \begin{pmatrix} 0 & 0 & 0 & e_4 \\ f_4 & 0 & 0 & 0 \\ 0 & g_4 & 0 & 0 \\ 0 & 0 & h_4 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & -i\mathbb{1}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{1}_2 & 0 & 0 \end{pmatrix}$$

and

$$i \begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & -i\mathbb{1}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & e_4 \\ f_4 & 0 & 0 & 0 \\ 0 & g_4 & 0 & 0 \\ 0 & 0 & h_4 & 0 \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{1}_2 & 0 & 0 \end{pmatrix},$$

which gives the constraints

$$f_4 = -h_3 \cdot \sigma_3; \qquad g_4 = \sigma_3 \cdot g_3; \qquad h_4 = -f_3 \cdot \sigma_3; \qquad e_4 = \sigma_3 \cdot e_3. \tag{3.7}$$

The doublet scalars $(\Phi^{3,5}, \Phi^{4,6})$ of course give the same results, as should be expected from supersymmetry.

In summary then, the final fields which survive both $\beta_4$ and $\gamma$ projections (and thus the entire group $D_2$) are

$$A^\mu = \begin{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} & & & \\ & b & & \\ & & \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix} & \\ & & & b \end{pmatrix}; \quad \begin{cases} e_3 = \begin{pmatrix} e_{11} & e_{12} \\ 0 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 & f_{12} \\ 0 & f_{22} \end{pmatrix}, \\ \\ g_3 = \begin{pmatrix} g_{11} & g_{12} \\ 0 & 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & h_{12} \\ 0 & h_{22} \end{pmatrix}, \end{cases} \tag{3.8}$$

$$\Psi^3 = \begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix}, \qquad \Psi^4 = \begin{pmatrix} 0 & 0 & 0 & \sigma_3 \cdot e_3 \\ -h_3 \cdot \sigma_3 & 0 & 0 & 0 \\ 0 & \sigma_3 \cdot g_3 & 0 & 0 \\ 0 & 0 & -f_3 \cdot \sigma_3 & 0 \end{pmatrix}.$$

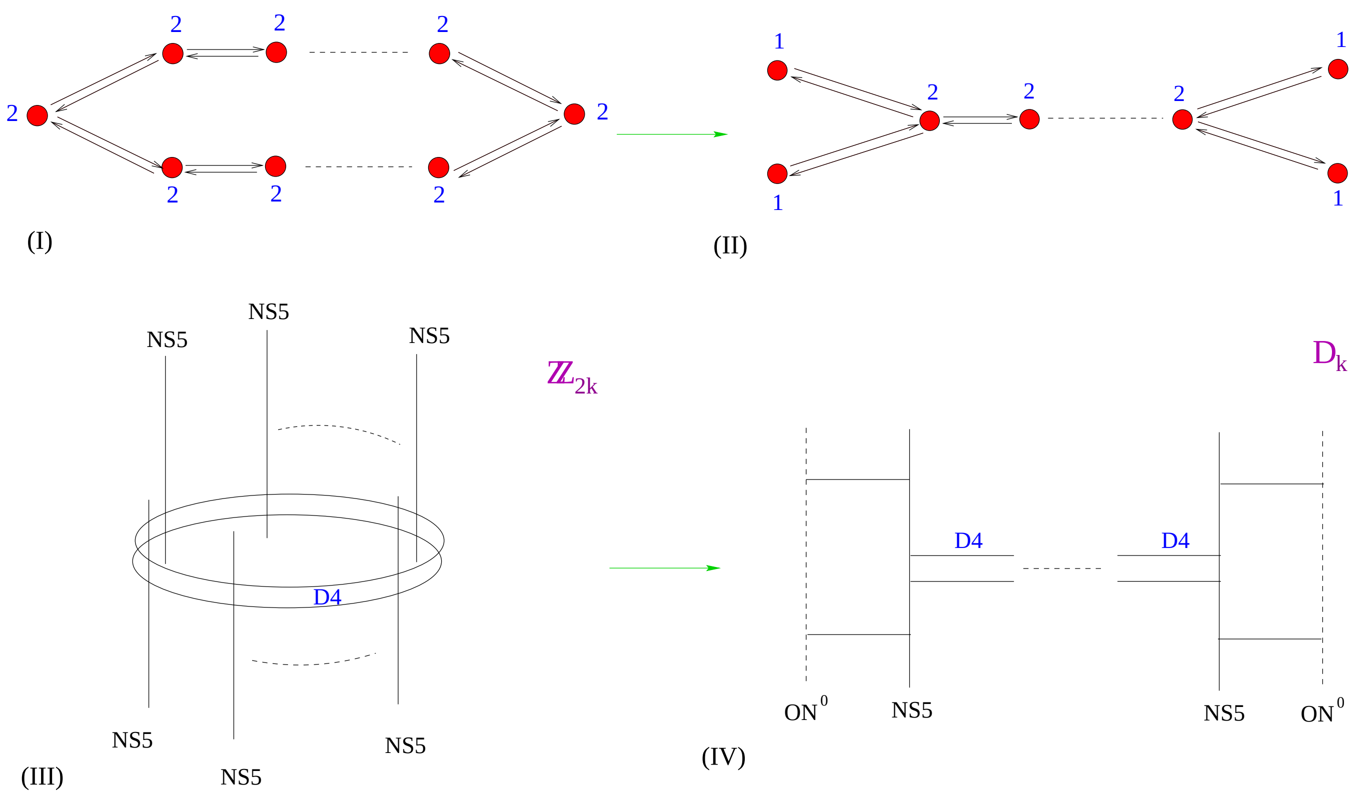


**Figure 2:** Obtaining the $D_k$ quiver (II) from the $\mathbb{Z}_{2k}$ quiver (I) by the stepwise projection algorithm. The brane setups are given respectively in (IV) and (III).

The key features to be noticed are now apparent in the structure of these matrices in (3.8). We see that the 4 blocks of $A^\mu$ in (3.4), which give the four nodes of the $\mathbb{Z}_4$ quiver, now undergo a metamorphosis: we have written out the components of $a, c$ explicitly and have used (3.6) to restrict both to diagonal matrices, while $b$ and $d$ are identified, but still remain blocks without internal structure of interest. Thus we have a total of 5 non-trivial constituents $a_{11}, a_{22}, c_{11}, c_{22}$ and $b$, precisely the 5 nodes of the $D_2$ quiver (see parts (I) and (II) of Figure 1). Thus *nodes of the quiver merge and split as we impose further projections*, as we mentioned a few paragraphs ago.

As for the bifundamentals, i.e., the arrows of the quiver, (3.4) prescribes the blocks $e_{3,4}$, $f_{3,4}$, $g_{3,4}$ and $h_{3,4}$ as the 8 arrows of Part (I) of Figure 1. After the projection by $\gamma$, and imposing the constraint (3.7) as well as the fact that all entries of matter matrices must be non-negative, we are left with the 8 fields $e_{11,12}, f_{12,22}, g_{11,12}$ and $h_{12,22}$, precisely the 8 arrows in the $D_2$ quiver (see Part (II) of Figure 1).

### The General Case

The generic situation of obtaining the $D_k$ quiver from that of $\mathbb{Z}_{2k}$ is completely analogous. We would always have two end nodes of the $\mathbb{Z}_{2k}$ quiver each splitting into two while the middle ones coalesce pair-wise, as is shown in Figure 2.

## 3.2 The $E_6$ Quiver from $D_2$

We now move on to tackle the binary tetrahedral group $E_6$ (with the relation that $E_6/D_2 \cong \mathbb{Z}_3$), whose generators are

$$\beta_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \delta := \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}.$$

We observe therefore that it has yet one more generator $\delta$ than $D_2$, hence we need to continue our stepwise projection from the previous subsection, with the exception that we should begin with more copies of $\mathbb{Z}_4$. To see this let us first present the irreducible matrix representations of the three generators of $E_6$:

| | $\beta_4$ | $\gamma$ | $\delta$ |
|---|---|---|---|
| $\Gamma_1^{(1)}$ | $1$ | $1$ | $1$ |
| $\Gamma_2^{(1)}$ | $1$ | $1$ | $\omega_3$ |
| $\Gamma_3^{(1)}$ | $1$ | $1$ | $\omega_3^2$ |
| $\Gamma_4^{(2)}$ | $\beta_4$ | $\gamma$ | $\delta$ |
| $\Gamma_5^{(2)}$ | $\beta_4$ | $\gamma$ | $\omega_3\delta$ |
| $\Gamma_6^{(2)}$ | $\beta_4$ | $\gamma$ | $\omega_3^2\delta$ |
| $\Gamma_7^{(3)}$ | $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} -\frac{i}{2} & \frac{i}{\sqrt{2}} & -\frac{i}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{2} & -\frac{i}{\sqrt{2}} & \frac{i}{2} \end{pmatrix}$ |

The regular representation for these generators is therefore a matrix of size $3 \cdot 1^2 + 3 \cdot 2^2 + 3^3 = 24$, in accordance with (2.1).

Our first step is as with the case of $D_2$, namely to change to a convenient basis wherein $\beta_4$ becomes diagonal:

$$R^{reg}_{E_6}(\beta_4) = R^{reg}_{\mathbb{Z}_4}(\beta_4) \otimes 1\!\!1_6. \tag{3.9}$$

The only difference between the above and (3.3) is that we have the tensor product with $1\!\!1_6$ instead of $1\!\!1_2$, therefore at this stage we have a $\mathbb{Z}_4$ quiver with the nodes labeled 6 as opposed to 2 as in Part (I) of Figure 1. In other words we have 6 times the usual number of D-brane probes.

Under the basis of (3.9),

$$R^{reg}_{E_6}(\gamma) = \begin{pmatrix} \Sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i1\!\!1_6 \\ 0 & 0 & \Sigma_3 & 0 \\ 0 & i1\!\!1_6 & 0 & 0 \end{pmatrix} \quad \text{where} \quad \Sigma_3 := \sigma_3 \otimes 1\!\!1_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.10}$$

Subsequent projection gives a $D_2$ quiver as in part (II) of Figure 1, but with the nodes labeled as $3, 3, 6, 3, 3$, three times the usual. Note incidentally that (3.9) and (3.10) can be re-written in terms of regular representations of $D_2$ directly: $R^{reg}_{E_6}(\beta_4) = R^{reg}_{D_2}(\beta_4) \otimes \mathbb{1}_3$ and $R^{reg}_{E_6}(\gamma) = R^{reg}_{D_2}(\gamma) \otimes \mathbb{1}_3$. To this fact we shall later turn.

To arrive at $E_6$, we proceed with one more projection, by the last generator $\delta$, the regular representation of which, observing the table above, has the form (in the basis of (3.9))

$$R^{reg}_{E_6}(\delta) = \begin{pmatrix} S_1 & 0 & S_2 & 0 \\ 0 & \omega_8^{-1} P & 0 & \omega_8^{-1} P \\ S_3 & 0 & S_4 & 0 \\ 0 & -\omega^8 P & 0 & \omega_8 P \end{pmatrix} \tag{3.11}$$

where

$$S_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes R^{reg}_{\mathbb{Z}_3}(\beta_3), \quad S_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$S_3 := -i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_4 := i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}_3$$

and

$$P := R^{reg}_{\mathbb{Z}_3}(\beta_3) \otimes \frac{1}{\sqrt{2}} \mathbb{1}_2; \quad \text{recalling that} \quad R^{reg}_{\mathbb{Z}_3}(\beta_3) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \end{pmatrix}.$$

The inverse of (3.11) is readily determined to be

$$R^{reg}_{E_6}(\delta)^{-1} = \begin{pmatrix} \tilde{S}_1 & 0 & -S_3 & 0 \\ 0 & \frac{1}{2}\omega_8 P^{-1} & 0 & -\frac{1}{2}\omega_8^{-1} P^{-1} \\ S_2^T & 0 & -S_4^T & 0 \\ 0 & \frac{1}{2}\omega_8 P^{-1} & 0 & \frac{1}{2}\omega_8^{-1} P^{-1} \end{pmatrix}, \qquad \tilde{S}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes R^{reg}_{\mathbb{Z}_3}(\beta_3)^{-1}.$$

Thus equipped, we must use (2.2) with (3.11) on the matrix forms obtained in (3.8) (other fields can of course be checked to have the same projection), with of course each number therein now being $3 \times 3$ matrices. The final matrix for $A^\mu$ is as in (3.8), but with

$$a_{11} = \begin{pmatrix} a_{11(1)} & 0 & 0 \\ 0 & a_{11(2)} & 0 \\ 0 & 0 & a_{11(3)} \end{pmatrix}_{3\times 3}; \quad c_{11} = c_{22} = a_{22}; \quad b = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}_{6\times 6}$$

where $a_{22}$, $c_{ii}$ are $3 \times 3$ while $b_{ii}$ are $2 \times 2$ blocks. We observe therefore, that there are 7 distinct gauge group factors of interest, namely $a_{11(1)}, a_{11(2)}, a_{11(3)}, a_{22}, b_{11}, b_{22}$ and $b_{33}$, with Dynkin labels $1, 1, 1, 3, 2, 2, 2$ respectively. What we have now is the $E_6$ quiver and the bifundamentals split and join accordingly; the reader is referred to Part (I) of Figure 3.

### 3.3 The $E_6$ Quiver from $\mathbb{Z}_6$

Let us make use of an interesting fact, that actually $E_6 = \langle \beta_4, \gamma, \delta \rangle = \langle \beta_4, \delta \rangle = \langle \gamma, \delta \rangle$. Therefore, alternative to the previous subsection wherein we exploited the sequence $\mathbb{Z}_4 = \langle \beta_4 \rangle \xrightarrow{+\gamma} D_2 \xrightarrow{+\delta} E_6$, we could equivalently apply our stepwise projection on $\mathbb{Z}_6 = \langle \delta \rangle \xrightarrow{+\beta_4} E_6$.

Let us first project with $\delta$, an element of order 6 and the regular representation of which, after appropriate rotation is

$$R^{reg}_{E_6}(\delta) = R^{reg}_{\mathbb{Z}_6}(\delta) \otimes \mathbb{1}_4. \tag{3.12}$$

Therefore at this stage we have a $\mathbb{Z}_6$ quiver with labels of six 4's due to the $\mathbb{1}_4$; this is drawn in Part (II) of Figure 3. The gauge group we shall denote as $A^\mu := \text{Diag}(a, b, c, d, e, f)_{24\times 24}$, with $a, b, \cdots, f$ being $4 \times 4$ blocks.

Next we perform projection by $R^{reg}_{E_6}(\beta_4)$ in the rotated basis, splitting and joining the gauge groups (nodes) as follows

$$A^\mu = \begin{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & \tilde{a} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} c_{11} & 0 \\ 0 & \tilde{c} \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 & \begin{pmatrix} e_{11} & 0 \\ 0 & \tilde{e} \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \end{pmatrix}; \quad s.\ t. \quad \begin{array}{l} \tilde{a} = \tilde{c} = \tilde{e}, \\ b_2 = d_1, \\ d_2 = f_1, \\ f_2 = b_1, \end{array}$$

which upon substitution of the relations, gives us 7 independent factors: $a_{11}, c_{11}$ and $e_{11}$ are numbers, giving 1 as Dynkin labels in the quiver; $b_1, b_2$ and $d_2$ are $2 \times 2$ blocks, giving the 2 labels; while $\tilde{a}$ is $3\times 3$, giving the 3. We refer the reader to Part (II) of Figure 3 for the diagrammatical representation.

## 4. Comments and Discussions

Our procedure outlined above is originally inspired by a series of papers [10, 11, 12], where the quivers for the $\Delta$ series of $\Gamma \subset SU(3)$ were observed to be obtainable from the $\mathbb{Z}_n \times \mathbb{Z}_n$ series after an appropriate identification. In particular, it was noted that $\Delta(3n^2) = \langle \left\{ \mathbb{Z}_n \times \mathbb{Z}_n := \begin{pmatrix} \omega_n^i & 0 & 0 \\ 0 & \omega_n^j & 0 \\ 0 & 0 & \omega_n^{-i-j} \end{pmatrix}_{i,j=0,\cdots,n-1} \right\}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle$ and subsequently the quiver for $\Delta(3n^2)$ is that of $\mathbb{Z}_n \times \mathbb{Z}_n$ modded out by a certain $\mathbb{Z}_3$ quotient. Similarly, the quiver for

$$\Delta(6n^2) = \langle \mathbb{Z}_n \times \mathbb{Z}_n, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \rangle$$

is that of $\mathbb{Z}_n \times \mathbb{Z}_n$ modded out by a certain $S_3$ quotient. In [12], it was further commented that the $\Sigma$ series could be likewise treated.

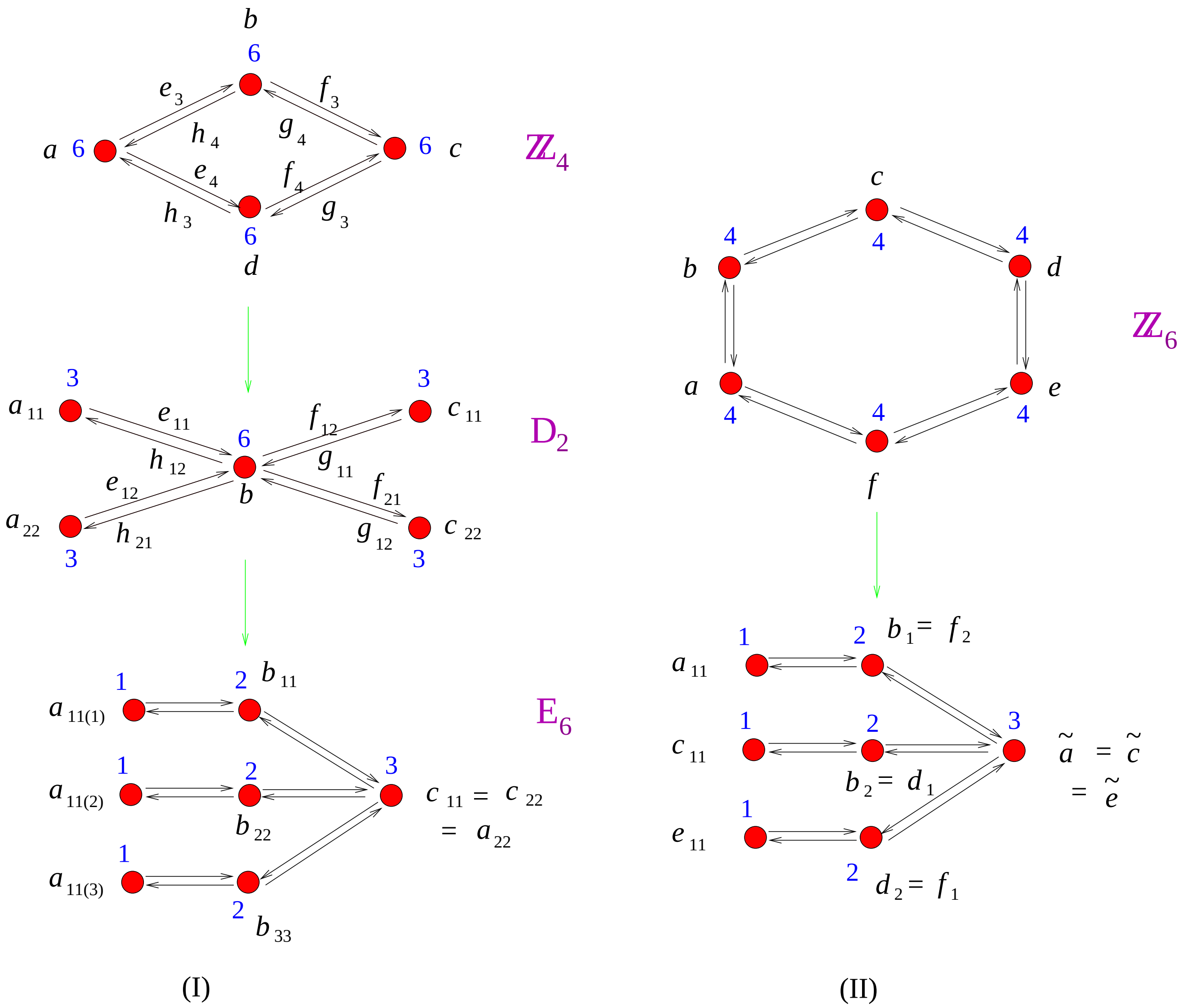


**Figure 3:** Obtaining the quiver diagram for the binary tetrahedral group $E_6$. We compare the two alternative stepwise projections: (I) $\mathbb{Z}_4 = \langle\beta_4\rangle \to D_2 = \langle\beta_4, \gamma\rangle \to E_6 = \langle\beta_4, \gamma, \delta\rangle$ and (II) $\mathbb{Z}_6 = \langle\delta\rangle \to E_6 = \langle\delta, \beta_4\rangle$.

The motivation for those studies was to realise a brane-setup for the non-Abelian $SU(3)$ orbifolds as geometrical quotients of the well-known Abelian case of $\mathbb{Z}_m \times \mathbb{Z}_n$, viz., the Brane Box Models. The key idea was to recognise that the irreducible representations of these groups could be labelled by a double index $(l_1, l_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$ up to identifications.

Our purpose here is to establish an algorithmic treatment along similar lines, which would be generalisable to arbitrary finite groups. Indeed, since any finite group $\Gamma$ is finitely generated, starting from the cyclic subgroup (with one single generator), our stepwise projection would give the quiver for $\Gamma$ as appropriate splitting and joining of nodes, i.e., as a certain geometrical action, of the $\mathbb{Z}_n$ quiver.

### 4.1 A Mathematical Viewpoint

To see why our stepwise projection works on a more axiomatic level, we need to turn to a brief review of the Theory of Induced Representations.

It was a fundamental observation of Frøbenius that the representations of a group could be constructed from an arbitrary subgroup. The aforementioned chain of groups, where we tried to relate the regular representations, is precisely in this vein. Though we shall largely follow the nomenclature of [13], we shall now briefly review this theory in the spirit of the above discussions.

Let $\Gamma_1 = \langle x_1, ..., x_n \rangle$ and $\Gamma_2 = \langle x_1, ..., x_{n+1} \rangle$. We see thus that $\Gamma_1 \subset \Gamma_2$. Now let $R_{\Gamma_1}(x)$ be a representation (not necessarily irreducible) of the element $x \in \Gamma_1$. Extending it to $\Gamma_2$ gives

$$R_{\Gamma_2}(y) = \begin{cases} R_{\Gamma_1}(x) \text{ if } & y = x \in \Gamma_1 \\ 0 \qquad\quad \text{ if } & y \notin \Gamma_1 \end{cases}$$

It follows then that if we decompose $\Gamma_2$ as (right) cosets of $\Gamma_1$,

$$\Gamma_2 = \Gamma_1 t_1 \cup \Gamma_1 t_2 \cup \cdots \cup \Gamma_1 t_m$$

we have an **Induced Representation** of $\Gamma_2$ as

$$R_{\Gamma_2}(y) = R_{\Gamma_1}(t_i y t_j^{-1}) = \begin{pmatrix} R_{\Gamma_1}(t_1 y t_1^{-1}) & R_{\Gamma_1}(t_1 y t_2^{-1}) & \cdots & R_{\Gamma_1}(t_1 y t_m^{-1}) \\ R_{\Gamma_1}(t_2 y t_1^{-1}) & R_{\Gamma_1}(t_2 y t_2^{-1}) & \cdots & R_{\Gamma_1}(t_2 y t_m^{-1}) \\ \vdots & \vdots & & \vdots \\ R_{\Gamma_1}(t_m y t_1^{-1}) & R_{\Gamma_1}(t_m y t_2^{-1}) & \cdots & R_{\Gamma_1}(t_m y t_m^{-1}) \end{pmatrix}. \tag{4.1}$$

A beautiful property of (4.1) is that it has only one member of each row or column non-zero and whereby it is essentially a generalised permutation (see e.g., 3.1 of [13]) matrix acting on the $\Gamma_1$-stable submodules of the $\Gamma_2$-module.

Now, for the case at hand the coset decomposition is simple due to the addition of a single new generator: the (right) transversals $t_1, \cdots, t_m$ are simply powers of the extra generator $x_{n+1}$ and $m$ is simply the index of $\Gamma_1 \subset \Gamma_2$, namely $|\Gamma_2|/|\Gamma_1|$, i.e.,

$$t_i = x_{n+1}^{i-1} \qquad i = 1, 2, \cdots, m; \quad m = \frac{|\Gamma_2|}{|\Gamma_1|}. \tag{4.2}$$

Now let us define an important concept for an element $x \in \Gamma_2$

DEFINITION 4.1 *We call a representation* $R_{\Gamma_2}(x)$ **factorisable** *if it can be written, up to possible change of bases, as a tensor product* $R_{\Gamma_2}(x) = R_{\Gamma_1}(x) \otimes \mathbb{1}_k$ *for some integer* $k$.

Factorisability of the element, in the physical sense, corresponds to the ability to initialise our stepwise projection algorithm, by which we mean that the orbifold projection by this element is performed on $k$ copies as in the usual sense, i.e., a stack of $k$ copies of the quiver. Subsequently we could continue with the stepwise algorithm to demonstrate how the nodes of these copies merge or split. In the corresponding D-brane picture this simply means that we should consider $k$ copies of each image D-brane probe in the covering space.

The natural question to ask is of course why our examples in the previous section permitted factorisable generators so as to in turn permit the performance of the stepwise projection. The following claim shall be of great assurance to us:

PROPOSITION 4.1 *Let $H$ be a subgroup of $G$, then the representation $R_G(x)$ for an element $x \in H \subset G$ induced from $R_H(x)$ according to (4.1) is factorisable and $k$ is equal to $|G|/|H|$, the index of $H$ in $G$.*

Proof: Take $R_H(x \in H)$, and tensor it with $\mathbb{1}_{k=|G|/|H|}$; this remains of course a representation for $x \in H$. It then remains to find the representations of $x \notin H$, which we supplement by the permutation actions of these elements on the $H$-cosets. At the end of the day we arrive at a representation $R'_G(x)$ of dimension $k$, such that it is factorisable for $x \in H$ and a general permutation for $x \notin H$. However by the uniqueness theorem of induced representations (q.v. e.g. [14] Thm 11) such a linear representation $R'_G(x)$ must in fact be isomorphic to $R_G(x)$. Thus by explicit construction we have shown that $R_G(x \in H) = R_H(x) \otimes \mathbb{1}_k$. $\square$

We can be more specific and apply Proposition 4.1 to our case of the two groups the second of which is generated by the first with one additional generator. Using the elegant property that the induction of a regular representation remains regular (q.v. e.g., 3.3 of [14]), we have:

COROLLARY 4.1 *Let $\Gamma_1$ and $\Gamma_2$ be as defined above, then*

$$R^{reg}_{\Gamma_2}(x_i) = R^{reg}_{\Gamma_1}(x_i) \otimes \mathbb{1}_{|\Gamma_2|/|\Gamma_1|} \qquad \text{for common generators} \qquad i = 1, 2, \ldots, n.$$

In particular, since any $G = \langle x_1, \ldots, x_n \rangle$ contains a cyclic subgroup generated by, say $x_1$ of order $m$, i.e., $\mathbb{Z}_m = \langle x_1 \rangle$, we conclude that

COROLLARY 4.2 $R^{reg}_G(x_1) = R^{reg}_{\mathbb{Z}_m}(x_1) \otimes \mathbb{1}_{|G|/m}$*, and hence the quiver for $G$ can always be obtained by starting with the $\mathbb{Z}_m$ quiver using the stepwise projection.*

Let us revisit the examples in the previous section equipped with the above knowledge. For the case of $\Gamma_1 = \mathbb{Z}_4 = \langle \beta_4 \rangle$ and $\Gamma_2 = D_2$ with the extra generator $\gamma$, (4.2) becomes $t_1 = \mathbb{1}$ and $t_2 = \gamma$ as the index of $\mathbb{Z}_4$ in $D_2$ is $\frac{|D_2|=8}{|\mathbb{Z}_4|=4} = 2$. The induced representation of $\beta_4$ according to (4.1) reads

$$R_{D_2}(\beta_4) = \begin{pmatrix} R^{reg}_{\mathbb{Z}_4}(\mathbb{1}\beta_4\mathbb{1}^{-1}) & R^{reg}_{\mathbb{Z}_4}(\mathbb{1}\beta_4\gamma^{-1}) \\ R^{reg}_{\mathbb{Z}_4}(\gamma\beta_4\mathbb{1}^{-1}) & R^{reg}_{\mathbb{Z}_4}(\gamma\beta_4\gamma^{-1}) \end{pmatrix} = \begin{pmatrix} R^{reg}_{\mathbb{Z}_4}(\beta_4) & 0 \\ 0 & R^{reg}_{\mathbb{Z}_4}(\beta_4^{-1}) \end{pmatrix}$$

using the fact that $\gamma\beta_k\gamma^{-1} = \beta_k^{-1}$ in $D_k$ for the last entry. Recalling that $R^{reg}_{\mathbb{Z}_4}(\beta_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i^2 & 0 \\ 0 & 0 & 0 & i^3 \end{pmatrix}$, this is subsequently equal to $R^{reg}_{\mathbb{Z}_4} \otimes \mathbb{1}_2$ after appropriate permutation of basis. Thus Corollary 4.1

manifests her validity as we see that the $R_{D_2}$ obtained by Frøbenius induction of $R^{reg}_{\mathbb{Z}_4}$ is indeed regular and moreover factorisable, as (3.3) dictates.

Similarly with the case of $\mathbb{Z}_6 \to E_6$, we see that Corollary 4.1 demands that for the common generator $\delta$, $R^{reg}_{E_6}(\delta)$ should be factorisable, as is indeed indicated by (3.12). So too is it with $\mathbb{Z}_4 \to E_6$, where $R^{reg}_{E_6}(\beta_4)$ should factorise, precisely as shown by (3.9).

The above have actually been special cases of Corollary 4.2, where we started with a cyclic subgroup; in fact we have also presented an example demonstrating the general truism of Proposition 4.1. In the case of $D_2 \to E_6$, we mentioned earlier that $R^{reg}_{E_6}(\beta_4) = R^{reg}_{D_2}(\beta_4) \otimes \mathbb{1}_3$ and $R^{reg}_{E_6}(\gamma) = R^{reg}_{D_2}(\gamma) \otimes \mathbb{1}_3$ for the common generators as was seen from (3.9) and (3.10); this is exactly as expected by the Proposition.

### 4.2 A Physical Viewpoint: Brane Setups?

Now mathematically it is clear what is happening to the quiver as we apply stepwise projection. However this is only half of the story; as we mentioned in the introduction, we expect T-duality to take D-branes at generic orbifold singularities to brane setups. It is a well-known fact that the brane setups for the $A$ and $D$-type orbifolds $\mathbb{C}^2/\mathbb{Z}_n$ and $\mathbb{C}^2/D_n$ have been realised (see [15, 16] and [19] respectively). It has been the main intent of a collective of works (e.g [9, 11, 12]) to establish such setups for the generic singularity.

In particular, the problem of finding a consistent brane-setup for the remaining case of the exceptional groups $E_{6,7,8}$ of the $ADE$ orbifold singularities of $\mathbb{C}^2$ (and indeed analogues thereof for $SU(3)$ and $SU(4)$ subgroups) so far has been proven to be stubbornly intractable. An original motivation for the present work is to attempt to formulate an algorithmic outlook wherein such a problem, with the insight of the algebraic structure of an appropriate chain of certain relevant groups, may be addressed systematically.

#### 4.2.1 The $\mathbb{Z}_2$ Action on the Brane Setup

Let us attempt to recast our discussion in Subsection 3.1 into a physical language. First we try to interpret the action by $R^{reg}_{D_k}(\gamma)$ in (3.2) on the $\mathbb{Z}_{2k}$ quiver as a string-theoretic action on brane setups to get the corresponding brane setup of $D_k$ from that of $\mathbb{Z}_{2k}$.

Now the brane configuration for the $\mathbb{Z}_{2k}$ orbifold is the well-known *elliptic model* consisting of $2k$ NS5-branes arranged in a circle with D4-branes stretched in between as shown in Part (III) of Figure 1. After stepwise projection by $\gamma$, the quiver in Part (I) becomes that in Part(II) (see Figure 2 also). There is an obvious $\mathbb{Z}_2$ quotienting involved, where the nodes $i$ and $2k - i$ for $i = 1, 2, ..., k - 1$ are identified while each of the nodes 0 and $k$ splits into two parts. Of course, this symmetry is not immediately apparent from the properties of $\gamma$, which is a group element of order 4. This phenomenon is true in general: the order of the generator used in the stepwise projection does not necessarily determine what symmetry the parent quiver undergoes to arrive at the resulting quiver; instead we must observe *a posteriori* the shapes of the respective quivers.

Let us digress a moment to formulate the above results in the language used in [10, 11]. Recalling from the brief comments in the beginning of Section 4, we adopt their idea of labelling the irreducible representations of $\Delta$ by $\mathbb{Z}_n \times \mathbb{Z}_n$ up to appropriate identifications, which in our terminology is simply the by-now familiar stepwise projection of the parent $\mathbb{Z}_n \times \mathbb{Z}_n$ quiver. As a comparison, we apply this idea to the case of $\mathbb{Z}_{2k} \to D_k$. Therefore we need to label the irreps of $D_k$ or appropriate tensor sums thereof, in terms of certain (reducible) 2-dimensional representations of $\mathbb{Z}_{2k}$. Motivated by the factorization property (3.3), we chose these representations to be

$$R^l_{\mathbb{Z}_{2k}(2)} := R^{l,irrep}_{\mathbb{Z}_{2k}(1)} \oplus R^{l,irrep}_{\mathbb{Z}_{2k}(1)} \tag{4.3}$$

where $l \in \mathbb{Z}_{2k}$, and amounts to precisely a $\mathbb{Z}_{2k}$-valued index on the representations of $D_k$ (since $\mathbb{Z}_{2k}$ is Abelian), which with foresight, we shall later use on $D_k$. We observe that such a labelling scheme has a symmetry

$$R^l_{\mathbb{Z}_{2k}(2)} \cong R^{-l}_{\mathbb{Z}_{2k}(2)},$$

which is obviously a $\mathbb{Z}_2$ action. Note that $l = 0$ and $l = k$ are fixed points of this $\mathbb{Z}_2$.

We can now associate the 2-dimensional irreps of $D_k$ with the non-trivial equivalence classes of the $\mathbb{Z}_{2k}$ representations (4.3), i.e., for $l = 1, 2, \ldots, k-1$ we have

$$R^l_{\mathbb{Z}_{2k}(2)} \cong R^{-l}_{\mathbb{Z}_{2k}(2)} \to R^{l,irrep}_{D_k(2)}. \tag{4.4}$$

These identifications correspond to the merging nodes in the associated quiver diagram. As for the fixed points, we need to map

$$\begin{aligned} R^0_{\mathbb{Z}_{2k}(2)} &\to R^{1,irrep}_{D_k(1)} \oplus R^{2,irrep}_{D_k(1)} \\ R^k_{\mathbb{Z}_{2k}(2)} &\to R^{3,irrep}_{D_k(1)} \oplus R^{4,irrep}_{D_k(1)}. \end{aligned} \tag{4.5}$$

These fixed points are associated precisely with the nodes that split.

This construction shows clearly how, in the labelling scheme of [10, 11], our stepwise algorithm derives the $D_k$ quiver as a $\mathbb{Z}_2$ projection of the $\mathbb{Z}_{2k}$ quiver. The consistency of this description is verified by substituting the representations $R^l_{\mathbb{Z}_{2k}(2)}$ in the $\mathbb{Z}_{2k}$ quiver relations $\mathcal{R} \otimes R^l_{\mathbb{Z}_{2k}(2)} = \bigoplus\limits_{\bar{l}} a^{\mathbb{Z}_{2k}(\mathcal{R})}_{l\bar{l}} R^{\bar{l}}_{\mathbb{Z}_{2k}(2)}$ using (4.4) and (4.5), which results exactly in the $D_k$ quiver relations. We can of course apply the stepwise projection for the case of $\mathbb{Z}_n \times \mathbb{Z}_n \to \Delta$, and would arrive at the results in [10, 11].

In the brane setup picture, the identification of the nodes $i$ and $2k-i$ for $i = 1, 2, ..., k-1$ corresponds to the identification of these intervals of NS5-branes as well as the D4-branes in between in the $X^{6789}$ directions (with direction-6 compact). Thus the $\mathbb{Z}_2$ action on the $\mathbb{Z}_{2k}$ quiver should include a space-time action which identifies $X^{6789} = -X^{6789}$. Similarly, the splitting of gauge fields in intervals 0 and $k$ hints the existence of a $\mathbb{Z}_2$ action on the string world-sheet. Thus the overall $\mathbb{Z}_2$ action should include two parts: a space-time symmetry which identifies and a world-sheet symmetry which splits respective gauge groups.

What then is this action physically? What object in string theory performs the tasks in the above paragraph? Fortunately, the space-time parity and string world-sheet $(-1)^{F_L}$ actions [18, 19] are precisely the aforementioned symmetries. In other words, the *ON-plane* is that which we seek. This is of great assurance to us, because the brane setup for $D_k$ theories, as given in [19], is indeed a configuration which uses the ON-plane to project out or identify fields in a manner consistent with our discussions.

### 4.2.2 The General Action on the Brane Setup?

It seems therefore, that we could now be boosted with much confidence: since we have proven in the previous subsection that our stepwise projection algorithm is a constructive method of arriving at *any* orbifold quiver by appropriate quotient of the $\mathbb{Z}_n$ quiver, could we not simply find the appropriate object in string theory which would perform such a quotient, much in the spirit of the orientifold prescribing $\mathbb{Z}_2$ in the above example, on the well-known $\mathbb{Z}_n$ brane setup, in order to solve our problem?

Such a confidence, as is with most in life, is overly optimistic. Let us pause a moment to consider the $E_6$ example. The action by $\delta$ in the case of $D_2 \to E_6$ in §3.2 and that of $\beta_4$ in the case of $\mathbb{Z}_6 \to E_6$ in §3.3 can be visualised in Parts (I) and (II) of Figure 3 to be an $\mathbb{Z}_3$ action on the respective parent quivers. In particular, the identifications $c_{11} \sim c_{22} \sim a_{22}$ and $\tilde{a} \sim \tilde{c} \sim \tilde{e}; b_1 \sim f_2, b_2 \sim d_1, d_2 \sim f_1$ respectively for Parts (I) and (II) are suggestive of a $\mathbb{Z}_3$ action on $X^{6789}$. The tripartite splittings for $b, a_{11}$ and $a, b, d$ respectively also hint at a $\mathbb{Z}_3$ action on the string world-sheet.

Again let us phrase the above results in the scheme of [10, 11], and manifestly show how the $E_6$ quiver results from a $\mathbb{Z}_3$ projection of the $D_2$ quiver. We define the following representations of $D_2$: $R^0_{D_2(6)} = R^{irrep}_{D_2(2)} \oplus R^{irrep}_{D_2(2)} \oplus R^{irrep}_{D_2(2)}$ and $R^l_{D_2(3)} = R^{l,irrep}_{D_2(1)} \oplus R^{l,irrep}_{D_2(1)} \oplus R^{l,irrep}_{D_2(1)}$ where $l \in \mathbb{Z}_4$ labels the four 1-dimensional irreducible representations of $D_2$. There is an identification

$$R^l_{D_2} \cong R^{f(l)}_{D_2}$$

where

$$f(l) = \begin{cases} 0, & l = 0 \\ 2, & l = 1 \\ 3, & l = 2 \\ 1, & l = 3 \end{cases}$$

Clearly this is a $\mathbb{Z}_3$ action on the index $l$. Note that we have two representations labelled with $l = 0$ which are fixed points of this action. In the quiver diagram of $D_2$ these correspond to the middle node and another one arbitrarily selected from the remaining four, both of which split into three. The remaining three nodes are consequently merged into a single one (see Figure 3). To derive the

$E_6$ quiver we need to map the nodes of the parent $D_2$ quiver as

$$\begin{array}{c} R^0_{D_2(6)} \to R^{1,irrep}_{E_6(2)} \oplus R^{2,irrep}_{E_6(2)} \oplus R^{3,irrep}_{E_6(2)} \\ R^0_{D_2(3)} \to R^{1,irrep}_{E_6(1)} \oplus R^{2,irrep}_{E_6(1)} \oplus R^{3,irrep}_{E_6(1)} \\ R^l_{D_2(3)} \cong R^{f(l)}_{D_2(3)} \to R^{irrep}_{E_6(3)}, \qquad l \in \mathbb{Z}_4 - \{0\}. \end{array}$$

Consistency requires that if we replace $R_{D_2}$ in the $D_2$ quiver defining relations and then use the above mappings, we get the $E_6$ quiver relations for $R^{irrep}_{E_6}$.

The origin of this $\mathbb{Z}_3$ analogue of the orientifold $\mathbb{Z}_2$-projection is thus far unknown to us. If an object with this property is to exist, then the brane setup for the $E_6$ theory could be implemented; on the other hand if it does not, then we would be suggested at why the attempt for $E_6$ has been prohibitively difficult.

The $\mathbb{Z}_3$ action has been noted to arise in [11] in the context of quotienting the $\mathbb{Z}_n \times \mathbb{Z}_n$ quiver to arrive at the quiver for the $\Delta$-series. Indeed from our comparative study in Section 4.2.1, we see that in general, labelling the irreps by a multi-index is precisely our stepwise algorithm in disguise, as applied to a product Abelian group: the $\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$ orbifold. Therefore in a sense we have explained why the labelling scheme of [10, 11] should work.

And the same goes with $E_7$ and $E_8$: we could perform stepwise projection thereupon and mathematically obtain their quivers as appropriate quotients of the $\mathbb{Z}_n$ quiver by the symmetry $S$ of the identification and splitting of nodes. To find a physical brane setup, we would then need to find an object in string theory which has an $S$ action on space-time and the string world-sheet. Note that the above are cases of the $\mathbb{C}^2$ orbifolds; for the $\mathbb{C}^k$-orbifold we should initialise our algorithm with, and perform stepwise projection on the quiver of $\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$ ($k-1$ times), i.e., the brane box and cube ($k = 2, 3$).

Though mathematically we have found a systematic treatment of constructing quivers under a new light, namely the "stepwise projection" from the Abelian quiver, much work remains. In the field of brane setups for singularities, our algorithm is intended to be a small step for an old standing problem. We must now diligently seek a generalisation of the orientifold plane with symmetry $S$ in string theory, that could perform the physical task which our mathematical methodology demands.

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