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# The Euclidean Division as an Iterative ERES-based Process 

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#### Abstract

Considering the Euclidean Division of two real polynomials, we present an iterative process based on the ERES method to compute the remainder of the division and we represent it using a simple matrix form.


## Introduction

The representation of the Euclidean algorithm process is presented using the matrix-based methodology of Extended-Row-Equivalence and Shifting operations (ERES) $[3,4]$. This allows the use of numerical methodologies for algebraic computation problems with the additional advantage of being able to handle uncertain coefficients and numerical errors.

We consider two real polynomials:

$$
\begin{equation*}
P(x)=\sum_{i=0}^{m} p_{i} x^{i}, p_{m} \neq 0 \quad \text { and } \quad Q(x)=\sum_{i=0}^{n} q_{i} x^{i}, q_{n} \neq 0, \quad m, n \in \mathbb{N} \tag{0.1}
\end{equation*}
$$

with degrees $\operatorname{deg}\{P(x)\}=m, \operatorname{deg}\{Q(x)\}=n$ respectively, and $m \geq n$.
Definition 1. We define the set
$\mathcal{D}_{m, n}=\{(P(x), Q(x)): P(x), Q(x) \in \mathbb{R}[x], m=\operatorname{deg}\{P(x)\} \geq \operatorname{deg}\{Q(x)\}=n\}$
For any pair $\mathcal{D}=(P(x), Q(x)) \in \mathcal{D}_{m, n}$, we define a vector representative $\underline{D}(x)$ and a basis matrix $D_{m}$ represented as :

$$
\underline{D}(x)=[P(x), Q(x)]^{t}=[\underline{p}, \underline{q}]^{t} \cdot \underline{e}_{m}(x)=D_{m} \cdot \underline{e}_{m}(x)
$$

where $D_{m} \in \mathbb{R}^{2 \times(m+1)}, \underline{e}_{m}(x)=\left[x^{m}, x^{m-1}, \ldots, x, 1\right]^{t}$. The matrix $D_{m}$ is formed directly from the coefficients of the given polynomials $P(x)$ and $Q(x)$.

Definition 2. Given a pair $\mathcal{D}_{m, n}$ of real polynomials with a basis matrix $D_{m}$ the following operations are defined [3, 4]:
a) Elementary row operations with scalars from $\mathbb{R}$ on $D_{m}$.
b) Addition or elimination of zero rows on $D_{m}$.
c) If $\underline{a}^{t}=\left[0, \ldots, 0, a_{l}, \ldots, a_{k}\right] \in \mathbb{R}^{k}, a_{l} \neq 0$ then we define as the Shifting operation

$$
\operatorname{shf}: \operatorname{shf}\left(\underline{a}^{t}\right)=\left[a_{l}, \ldots, a_{k}, 0, \ldots, 0\right] \in \mathbb{R}^{k}
$$

By $\operatorname{shf}\left(\mathcal{D}_{m, n}\right) \equiv \mathcal{D}_{m, n}^{*}$, we shall denote the pair obtained from $\mathcal{D}_{m, n}$ by applying shifting on the rows of $D_{m}$. Type (a), (b) and (c) operations are referred to as Extended-Row-Equivalence and Shifting (ERES) operations.

The following theorem shows the relation between a matrix and its shifted form [1] .

Theorem 1 (Matrix representation of Shifting). If $D \in \mathbb{R}^{2 \times k}, k>2$, is an upper trapezoidal matrix with rank $\rho(D)=2$ and $D^{*} \in \mathbb{R}^{2 \times k}$ is the matrix obtained from $D$ by applying shifting on its rows, then there exists a matrix $S \in \mathbb{R}^{k \times k}$ such that: $D^{*}=D \cdot S$.

Corollary 1. If $D_{m} \in \mathbb{R}^{2 \times(m+1)}$ is the basis matrix of a pair of real polynomials $\mathcal{D}=(P(x), Q(x)) \in \mathcal{D}_{m, n}$, then $D_{m}^{*} \in \mathbb{R}^{2 \times(m+1)}$ is the basis matrix of the pair $\mathcal{D}^{*}=\left(P(x), x^{m-n} Q(x)\right) \in \mathcal{D}_{m, m}$ and there exists a matrix $S_{\mathcal{D}} \in \mathbb{R}^{(m+1) \times(m+1)}$ such that:

$$
\begin{equation*}
D_{m}^{*}=D_{m} \cdot S_{\mathcal{D}} \tag{0.2}
\end{equation*}
$$

## The ERES representation of the Euclidean Division

If we have a pair of polynomials $\mathcal{D}=(P(x), Q(x)) \in \mathcal{D}_{m, n}$, then, according to Euclid's division algorithm, it holds:

$$
\begin{equation*}
P(x)=\frac{p_{m}}{q_{n}} x^{m-n} Q(x)+R_{1}(x) \tag{0.3}
\end{equation*}
$$

This is the first and basic step of the Euclidean Division algorithm. The polynomial $R_{1}(x) \in \mathbb{R}[x]$ is given by:

$$
\begin{equation*}
R_{1}(x)=\sum_{i=m-n}^{m-1}\left(p_{i}-\frac{p_{m}}{q_{n}} q_{i-(m-n)}\right) x^{i}+\sum_{i=0}^{m-n-1} p_{i} x^{i} \tag{0.4}
\end{equation*}
$$

In the following, we will show that the remainder $R_{1}(x)$ can be computed by applying ERES operations to the basis matrix $D_{m}$ of the pair $\mathcal{D}$.

Proposition 1 (Matrix representation of the first remainder of the Euclidean Division). Applying the algorithm of the Euclidean Division to a pair $\mathcal{D}=(P(x), Q(x)) \in \mathcal{D}_{m, n}$ of real polynomials, there exists a polynomial $R_{1}(x) \in \mathbb{R}[x]$ with degree $0 \leq \operatorname{deg}\left\{R_{1}(x)\right\}<m$ such that:

$$
P(x)=\frac{p_{m}}{q_{n}} x^{m-n} Q(x)+R_{1}(x)
$$

Then, the remainder $R_{1}(x)$ can be represented in matrix form as:

$$
R_{1}(x)=\underline{v}^{t} \cdot E_{1} \cdot \underline{e}_{m}(x)
$$

where $E_{1} \in \mathbb{R}^{2 \times(m+1)}$ is the matrix, which occurs from the application of the ERES operations on the basis matrix $D_{m}$ of the pair $\mathcal{D}$ and $\underline{v}=[0,1]^{t}$.

Proof. If we consider the division $P(x) / Q(x)$, then, according to Euclid's algorithm, there is a polynomial $R_{1}(x)$ with degree $0 \leq \operatorname{deg}\left\{R_{1}(x)\right\}<m$ such that:

$$
R_{1}(x)=P(x)-\frac{p_{m}}{q_{n}} x^{m-n} Q(x)=[0,1] \cdot\left[\begin{array}{cc}
0 & 1  \tag{0.5}\\
1-\frac{p_{m}}{q_{n}}
\end{array}\right] \cdot\left[\begin{array}{c}
P(x) \\
x^{m-n} Q(x)
\end{array}\right]
$$

If we take into account the result in corollary 1 , we will have:

$$
R_{1}(x)=[0,1] \cdot\left[\begin{array}{cc}
0 & 1  \tag{0.6}\\
1 & -\frac{p_{m}}{q_{n}}
\end{array}\right] \cdot D_{m} \cdot S_{\mathcal{D}} \cdot \underline{e}_{m}(x)=\underline{v}^{t} \cdot C \cdot D_{m} \cdot S_{\mathcal{D}} \cdot \underline{e}_{m}(x)
$$

where $\underline{v}^{t}=[0,1], C=\left[\begin{array}{cc}0 & 1 \\ 1 & -\frac{p_{m}}{q_{n}}\end{array}\right], D_{m}$ is the basis matrix of the polynomials $P(x)$ and $Q(x)$ and $S_{\mathcal{D}}$ the respective shifting matrix. Therefore, there exists a matrix $E_{1} \in \mathbb{R}^{2 \times(m+1)}$ such that:

$$
\begin{equation*}
E_{1}=C \cdot D_{m} \cdot S_{\mathcal{D}} \quad \text { and } \quad R_{1}(x)=\underline{v}^{t} \cdot E_{1} \cdot \underline{e}_{m}(x) \tag{0.7}
\end{equation*}
$$

We consider now the basis matrix $D_{m}$ of the polynomials $P(x)$ and $Q(x)$ :

$$
D_{m}=\left[\begin{array}{c}
P(x)  \tag{0.8}\\
Q(x)
\end{array}\right]=\left[\begin{array}{ccccccc}
p_{m} & \ldots & p_{n+1} & p_{n} & p_{n-1} & \ldots & p_{0} \\
0 & \ldots & 0 & q_{n} & q_{n-1} & \ldots & q_{0}
\end{array}\right] \cdot \underline{e}_{m}(x)
$$

and we will show that the above matrix $E_{1}$ is produced by applying the ERES operations to the basis matrix $D_{m}$ of the polynomials $P(x)$ and $Q(x)$. We follow the next methodology:

1. We apply shifting on the rows of $D_{m}$. Let $S_{\mathcal{D}} \in \mathbb{R}^{(m+1) \times(m+1)}$, be the proper shifting matrix: $D_{m}^{(1)}=D_{m} \cdot S_{\mathcal{D}}=\left[\begin{array}{cccccc}p_{m} & \ldots & p_{m-n+1} & p_{m-n} & p_{m-n-1} & \ldots \\ q_{n} & p_{0} \\ q_{0} & \ldots & q_{1} & q_{0} & 0 & \ldots\end{array}\right]$
2. We reorder the rows of the matrix $D_{m}^{(1)}$. If $J=\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$ is the permutation matrix, then: $D_{m}^{(2)}=J \cdot D_{m}^{(1)}=\left[\begin{array}{ccccccc}q_{n} & \ldots & q_{1} & q_{0} & 0 & \ldots & 0 \\ p_{m} & \ldots & p_{m-n+1} & p_{m-n} & p_{m-n-1} & \ldots & p_{0}\end{array}\right]$
3. We apply stable row operations on $D_{m}^{(2)}$ (LU factorization). If $L=\left[\begin{array}{cc}1 & 0 \\ \frac{p_{m}}{q_{n}} & 1\end{array}\right]$ then $L^{-1}=\left[\begin{array}{cc}1 & 0 \\ -\frac{p_{m}}{q_{n}} & 1\end{array}\right]$ and therefore:

$$
\left.\begin{array}{rl}
D_{m}^{(3)} & =L^{-1} \cdot D_{m}^{(2)}=\left[\begin{array}{cc}
1 & 0 \\
-\frac{p_{m}}{q_{n}} & 1
\end{array}\right] \cdot\left[\begin{array}{cccccc}
q_{n} & \ldots & q_{1} & q_{0} & 0 & \ldots \\
p_{m} & \ldots & p_{m-n+1} & p_{m-n} & p_{m-n-1} & \ldots
\end{array} p_{0}\right.
\end{array}\right] .
$$

We notice that the term $\frac{p_{m}}{q_{n}}$ emerges from the LU factorization.
The above process can be described by the following equation:

$$
\begin{equation*}
D_{m}^{(3)}=L^{-1} \cdot J \cdot D_{m} \cdot S_{\mathcal{D}} \tag{0.9}
\end{equation*}
$$

which represents the ERES methodology. Obviously $L^{-1} \cdot J=C$ and therefore, we conclude that $D_{m}^{(3)} \equiv E_{1}$.

The following theorem establishes the connection between the ERES method and the Euclidean Division of two real polynomials.

Theorem 2 (Matrix representation of the remainder of the Euclidean Division). Applying the algorithm of the Euclidean Division to a pair $\mathcal{D}=$ $(P(x), Q(x)) \in \mathcal{D}_{m, n}$ of real polynomials, there are polynomials $G(x), R(x) \in$ $\mathbb{R}[x]$ with degrees $\operatorname{deg}\{G(x)\}=m-n$ and $0 \leq \operatorname{deg}\{R(x)\}<n$ respectively, such that:

$$
P(x)=G(x) Q(x)+R(x)
$$

Then, the final remainder $R(x)$ can be represented in matrix form as:

$$
R(x)=\underline{v}^{t} \cdot E_{N} \cdot \underline{e}_{m}(x)
$$

where $E_{N} \in \mathbb{R}^{2 \times(m+1)}$ is the matrix, which occurs from the successive application of the ERES operations on the basis matrix $D_{m}$ of the pair $\mathcal{D}$ and $\underline{v}=[0,1]^{t}$.

The proof of the previous theorem is based on the iterative application of the result from proposition 1 to the sequence $\left\{(P(x), Q(x)),\left(R_{i}(x), Q(x)\right)\right\}$, for $1 \leq i \leq(m-n)$. Therefore, we get a sequence of matrices $E_{i}=L_{i}^{-1} \cdot E_{i-1} \cdot S_{i}$, for $i=1,2, \ldots, N<m-n$, where the final matrix $E_{N}$ gives the total remainder $R(x)$ and every matrix $L_{i}$ gives a specific coefficient of the quotient $G(x)$.

## References

1. Christou, D., Karcanias, N., Mitrouli, M.: The matrix representation of the Euclidean Algorithm using the ERES methodology. Systems \& Control Engineering Centre. Research Report (May 2008). City University, London, U.K.
2. Gantmacher, F.R.: The Theory of Matrices. Volume I \& II. Chelsea Publishing Company. New York, N.Y. (1959)
3. Karcanias, N.: Invariance properties and characterisation of the greatest common divisor of a set of polynomials. Int. Journ. Control 46 (1987) 1751-1760
4. Mitrouli, M., Karcanias, N.: Computation of the GCD of polynomials using Gaussian transformation and shifting. Int. Journ. Control 58 (1993) 211-228
