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# Rational sequences for the conductance in quantum wires from affine Toda field theories

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ABSTRACT: We analyse the expression for the conductance of a quantum wire which is decribed by an integrable quantum field theory. In the high temperature regime we derive a simple formula for the filling fraction. This expression involves only the inverse of a matrix which contains the information of the asymptotic phases of the scattering matrix and the solutions of the constant thermodynamic Bethe ansatz equations. Evaluating these expressions for minimal affine Toda field theory we recover several sequences of rational numbers, which are multiples of the famous Jain sequence for the filling fraction occurring in the context of the fractional quantum Hall effect. For instance we obtain $\nu = 4m/(2m+1)$ for $A_{4m-1}$-minimal affine Toda field theory. The matrices involved have in general non-rational entries and are not part of previous classification schemes based on integral lattices.

## 1. Introduction

The quantum [1] and in particular the fractional [2] quantum Hall effect have attracted an enormous amount of attention both from theorist [3] and experimentalists (for some very recent experiments see e.g. [4]). The key observation is that when subjecting an electron gas confined to two space dimensions to a strong uniform magnetic field, the transverse (Hall) conductance takes on preferably certain characteristic values $G = e^2/h\nu$, whereas the longitudinal conductance vanishes at these plateaux in complete analogy to the classical Hall effect [5]. The filling fractions $\nu$ are distinct universal, in the sense that they are independent of the geometry or type of the material, rational numbers, which can be determined experimentally to an extremely high precision. Many, but not all, of the experimentally observed filling fractions are part of Jain's famous sequence (see [6] and references therein)

$$\nu = \frac{m}{mp \pm 1} \qquad m, p/2 = 1, 2, 3, \ldots \tag{1.1}$$

which results as a theoretical prediction from a composite Fermion theory.

In the following we will show that remarkably multiples of these universal numbers also quantize the conductance of quantum wires when described by minimal affine Toda field theories (ATFT) [7]. However, no claims are made here that the systems studied actually correspond to any concrete description of the real quantum Hall effect. Nonetheless, one may speculate as there is a well defined way to reduce from a Chern-Simons type theory (an established description of the quantum Hall effect) to ATFT, see e.g. [8].

## 2. Conductance in the high temperature regime

Let us briefly recall [9] how to compute the conductance $G$ within the framework of the Landauer-Büttiker transport theory [10] as a function of the temperature $T$ and elaborate on that expression. Let us consider a one dimensional quantum wire within the Landauer-Büttiker transport theory. In order to compute $G$ we simply have to determine the difference of the static charge distribution at the left and right constriction of the wire, which we assume to be at the potentials $\mu_i^l$ and $\mu_i^r$, respectively. Then, to obtain the direct current $I_i$ for each particle of type $i$ with charge $q_i$, we have to integrate the density distribution functions $\rho_i^r(\theta, T, \mu_i)$ of occupied states over the full range of the rapidities $\theta$ and the total conductance simply reads

$$G(1/T) = \sum_i G_i = \sum_i \lim_{\Delta\mu_i \to 0} \frac{1}{\Delta\mu_i} I_i(1/T, \Delta\mu_i = \mu_i^l - \mu_i^r) \tag{2.1}$$

$$= \sum_i \lim_{\Delta\mu_i \to 0} \frac{q_i}{2\Delta\mu_i} \int_{-\infty}^{\infty} d\theta \left[ \rho_i^r(\theta, T, \mu_i^l) - \rho_i^r(\theta, T, \mu_i^r) \right]. \tag{2.2}$$

where $G_i$ denotes the contribution to the conductance of each particle $i$, and the sums above run both over particles and antiparticles. That explains the factor of $1/2$ in (2.2) wich accounts for the double counting. Hence, the main task in this approach is to determine the density distribution functions $\rho_i^r(\theta, T, \mu_i)$ of occupied states. It is remarkable that in the context of integrable models, despite the fact that these functions are neither Fermi-Dirac nor Bose-Einstein, there exist approaches in which they can be computed non-perturbatively, i.e. the thermodynamic Bethe ansatz (TBA) [11].

We briefly recall how this is possible. The central equations of the TBA relate the total density of available states $\rho_i(\theta, r)$ for particles of type $i$ with mass $m_i$ as a function of the inverse temperature $r = 1/T$ to the density of occupied states $\rho_i^r(\theta, r)$

$$\rho_i(\theta, r) = \frac{m_i}{2\pi} \cosh\theta + \sum_j [\varphi_{ij} * \rho_j^r](\theta). \tag{2.3}$$

By $(f * g)(\theta) := 1/(2\pi) \int d\theta' f(\theta - \theta') g(\theta')$ we denote as usual the convolution of two functions. There are only two inputs into the entire TBA analysis: first the dynamical interaction, which enters via the logarithmic derivative of the scattering matrix $\varphi_{ij}(\theta) = -id \ln S_{ij}(\theta)/d\theta$ and an assumption on the statistical interaction $g_{ij}$ amongst the particles $i$ and $j$ on which we comment further below. For the moment we chose this interaction to

be of fermionic type. The mutual ratio of the two types of densities serves as the definition of the so-called pseudo-energies $\varepsilon_i(\theta, r)$

$$\frac{\rho_i^r(\theta, r)}{\rho_i(\theta, r)} = \frac{e^{-\varepsilon_i(\theta, r)}}{1 + e^{-\varepsilon_i(\theta, r)}}\,, \tag{2.4}$$

which have to be positive and real. At thermodynamic equilibrium they can be computed from the non-linear integral equations

$$r m_i \cosh \theta = \varepsilon_i(\theta, r, \mu_i) + r \mu_i + \sum_j [\varphi_{ij} * \ln(1 + e^{-\varepsilon_j})](\theta)\,, \tag{2.5}$$

where $r = m/T$, $m_l \to m_l/m$, $\mu_i \to \mu_i/m$, with $m$ being the mass of the lightest particle in the model and chemical potential $\mu_i < 1$. As pointed out already in [11] (here just with the small modification of a chemical potential), the comparison between (2.5) and (2.3) leads to the useful relation

$$\rho_i(\theta, r, \mu_i) = \frac{1}{2\pi} \left( \frac{d\varepsilon_i(\theta, r, \mu_i)}{dr} + \mu_i \right) \sim \frac{1}{2\pi r} \epsilon(\theta) \frac{d\varepsilon_i(\theta, r, \mu_i)}{d\theta}. \tag{2.6}$$

Here $\epsilon(\theta) = \Theta(\theta) - \Theta(-\theta)$ is the unit step function, i.e. $\epsilon(\theta) = 1$ for $\theta > 0$ and $\epsilon(\theta) = -1$ for $\theta < 0$. In equation (2.4), we assume that in the large rapidity regime the density $\rho_i^r(\theta, r, \mu_i)$ is dominated by the last expression in (2.6) and in the small rapidity regime by the Fermi distribution function. Therefore, from (2.4) follows

$$\rho_i^r(\theta, r, \mu_i) = \frac{e^{-\varepsilon_i(\theta, r, \mu_i)}}{1 + e^{-\varepsilon_i(\theta, r, \mu_i)}} \rho_i(\theta, r, \mu_i) \tag{2.7}$$

$$\sim \frac{1}{2\pi r} \epsilon(\theta) \frac{d}{d\theta} \ln\left[1 + \exp(-\varepsilon_i(\theta, r, \mu_i))\right]\,. \tag{2.8}$$

Using this expression in equation (2.2), we can approximate the direct current in the ultraviolet by

$$\lim_{r \to 0} I_i(r, \Delta\mu_i) \sim \frac{q_i}{4\pi r} \int\limits_{-\infty}^{\infty} d\theta \ln\left[\frac{1 + \exp(-\varepsilon_i(\theta, r, \mu_i^l))}{1 + \exp(-\varepsilon_i(\theta, r, \mu_i^r))}\right] \frac{d\epsilon(\theta)}{d\theta}\,, \tag{2.9}$$

after a partial integration. Taking now the potentials at the end of the wire to be $\mu_i^r = -\mu_i^l = \mu_i/2$ we carry out the limit $\Delta\mu_i \to 0$ in (2.2) with the help l'Hôpital rule and the conductance becomes

$$\lim_{r \to 0} G_i(r) \sim \frac{q_i}{2\pi r} \int\limits_{-\infty}^{\infty} d\theta \frac{1}{1 + \exp[\varepsilon_i(\theta, r, 0)]} \left.\frac{d\varepsilon_i(\theta, r, \mu_i/2)}{d\mu_i}\right|_{\mu_i = 0} \frac{d\epsilon(\theta)}{d\theta}\,. \tag{2.10}$$

Noting that $d\epsilon(\theta)/d\theta = 2\delta(\theta)$, we obtain

$$\lim_{r \to 0} G_i(r) \sim \frac{q_i}{\pi r} \frac{1}{1 + \exp \varepsilon_i(0, r, 0)} \left.\frac{d\varepsilon_i(0, r, \mu_i/2)}{d\mu_i}\right|_{\mu_i = 0}\,. \tag{2.11}$$

The derivative $d\varepsilon_i(0, r, \mu_i/2)/d\mu_i$ can be obtained by solving

$$\frac{d\varepsilon_i(0, r, \mu_i/2)}{d\mu_k} = -\frac{r}{2}\delta_{ik} + \sum_j N_{ij} \frac{1}{1 + \exp \varepsilon_j(0, r, \mu_i/2)]} \frac{d\varepsilon_j(0, r, \mu_j/2)}{d\mu_k} ,$$ (2.12)

which results from performing a constant TBA analysis on the $\mu_k$-derivative of (2.5) in the spirit of [11]. At this point only the asymptotic phases of the scattering matrix enter via

$$N_{ij} = \frac{1}{2\pi i} \lim_{\theta \to \infty} [\ln[S_{ij}(-\theta)/S_{ij}(\theta)]] .$$ (2.13)

In principle we have now all quantities needed to compute the conductance, but to solve (2.12) for the derivatives of the pseudo-energies is somewhat cumbersome, see [9] for such a computation. Nonetheless, we can elaborate more on equation (2.12) and simplify the procedure further. For this purpose we introduce the quantity

$$Y_{ij} := \frac{1}{r(1 + e^{\varepsilon_i})} \frac{d\varepsilon_i}{d\mu_j} ,$$ (2.14)

such that we can re-write equation (2.12) equivalently as

$$M_{ij}Y_{jk} = \frac{\delta_{ik}}{2} \qquad \text{with} \qquad M_{ij} := N_{ij} - (1 + e^{\varepsilon_i})\delta_{ij}$$ (2.15)

where the pseudoenergies satisfy the constant TBA equations

$$e^{-\varepsilon_i} = \prod_j (1 + e^{-\varepsilon_j})^{N_{ij}} .$$ (2.16)

Returning now to dimensionful variables, i.e. replacing $1/2\pi \to e^2/h$, the conductance at high temperature in terms of the filling fraction $\nu$ then simply results to

$$G(0) = \frac{e^2}{h}\nu \qquad \text{with} \qquad \nu = 2 \sum_{i,j} q_i (M^{-1})_{ij} .$$ (2.17)

This means we have reduced the entire problem to compute filling fractions simply to the task of finding and inverting the matrix $M$. This is done in two steps: First from the asymptotic phases of the scattering matrix we compute $N_{ij}$ and subsequently we solve the constant TBA equations (2.16). Then it is a simple matter of inverting the matrix (2.15) and performing the sums in (2.17).

In the context of the fractional quantum Hall effect one encounters very often particles which obey some exotic (anyonic) statistics. So far we have assumed our particles to obey fermionic type statistics as this choice is most natural for the investigated theories [11]. However, one can easily implement more general statistics by adding a matrix $g_{ij}$ to the $N$-matrix [12].

The formula (2.17) reminds of course on the well-known expressions for the conductance as may be found for instance in [13, 14]. In that context it was found [13, 15] that Jain's sequence (1.1) can be obtained simply from the $(m \times m)$-matrix

$$M_{ij} = p \pm \delta_{ij} .$$ (2.18)

For this we have to take $q_i = 1/2 \ \forall \ i$ in our expression (2.17). We will now demonstrate that a sequence closely related to (1.1) can also be obtained in a more surprising way from fairly complicated matrices, even with non-rational entries, which result directly in the way indicated above, namely from a TBA analysis of minimal affine Toda field theories [7]. Each Toda theory is associated to a Lie algebra $\mathbf{g}$ of rank $\ell$ and it is well known [16] that in that case $N$ is an $(\ell \times \ell)$-matrix which is of the general form

$$N_{ij} = \delta_{ij} - 2(K_{\mathbf{g}}^{-1})_{ij} \ , \tag{2.19}$$

where $K_{\mathbf{g}}$ is the Cartan matrix related to $\mathbf{g}$ (see e.g. [17]). The solutions to the constant TBA equations are also known [18, 16] for most cases. In the ultraviolet limit these theories possess Virasoro central charge $c = 2\ell/(H+2)$, with $H$ being the Coxeter number of the Lie algebra $\mathbf{g}$.

## 3. Fractional filling fractions from minimal affine Toda field theory

### 3.1 The $4m/(2m+1)$-sequence

Let us start with some concrete examples to illustrate the working of our formulae. Specializing the general expression (2.19) to the $A_3$-case, the solutions to the constant TBA equations (2.16) are simply

$$e^{\varepsilon_1} = e^{\varepsilon_3} = 2, \qquad e^{\varepsilon_2} = 3 \ . \tag{3.1}$$

Then, the inverse of the $M$-matrix

$$M_{ij} = \delta_{ij} - 2(K_{A_3}^{-1})_{ij} - \delta_{ij}(1 + e^{\varepsilon_i}) \tag{3.2}$$

is computed to

$$M^{-1} = \frac{1}{36} \begin{pmatrix} 11 & -2 & -1 \\ -2 & 8 & -2 \\ -1 & -2 & 11 \end{pmatrix} \ . \tag{3.3}$$

From the fact that the $A_\ell$-minimal affine Toda field theories can also be viewed as complex sine-Gordon models [19], we know [20] that the charges in this theory are $q_1 = q_3 = 1$, $q_2 = 2$, such that (2.17) yields

$$\nu_{A_3} = 4/3 \ . \tag{3.4}$$

The next example, i.e. $A_5$-minimal affine Toda field theory, yields a less expected answer, even more since the $M$-matrix contains non-rational entries. With (2.19) for $A_5$ the solutions to the constant TBA equations are [18, 16]

$$e^{\varepsilon_1} = e^{\varepsilon_5} = 1 + \sqrt{2}, \qquad e^{\varepsilon_2} = e^{\varepsilon_4} = 2 + 2\sqrt{2}, \qquad e^{\varepsilon_3} = 3 + 2\sqrt{2} \ . \tag{3.5}$$

Assembling this into the $M$-matrix, it is clear that it will contain non-rational entries. Evidently this matrix is not of the form (2.18) and certainly falls out of the classification

scheme based on integral lattices [21]. Nonetheless, it will lead to a distinct rational value for $\nu$. We compute the inverse of $M$ to

$$
M^{-1} = \begin{pmatrix}
\left(\frac{35}{4} - 6\sqrt{2}\right) & \left(\frac{31}{2\sqrt{2}} - 11\right) & \left(\frac{7-5\sqrt{2}}{4}\right) & \left(6 - \frac{17}{2\sqrt{2}}\right) & \left(3\sqrt{2} - \frac{17}{4}\right) \\
\left(\frac{31}{2\sqrt{2}} - 11\right) & \left(15 - \frac{21}{\sqrt{2}}\right) & \left(\frac{7\sqrt{2}-10}{4}\right) & \left(6\sqrt{2} - \frac{17}{2}\right) & \left(6 - \frac{17}{2\sqrt{2}}\right) \\
\left(\frac{7-5\sqrt{2}}{4}\right) & \left(\frac{7\sqrt{2}-10}{4}\right) & \left(\frac{9}{4} - \frac{3}{\sqrt{2}}\right) & \left(\frac{7\sqrt{2}-10}{4}\right) & \left(\frac{7-5\sqrt{2}}{4}\right) \\
\left(6 - \frac{17}{2\sqrt{2}}\right) & \left(6\sqrt{2} - \frac{17}{2}\right) & \left(\frac{7\sqrt{2}-10}{4}\right) & \left(15 - \frac{21}{\sqrt{2}}\right) & \left(\frac{31}{2\sqrt{2}} - 11\right) \\
\left(3\sqrt{2} - \frac{17}{4}\right) & \left(6 - \frac{17}{2\sqrt{2}}\right) & \left(\frac{7-5\sqrt{2}}{4}\right) & \left(\frac{31}{2\sqrt{2}} - 11\right) & \left(\frac{35}{4} - 6\sqrt{2}\right)
\end{pmatrix} . \tag{3.6}
$$

Remarkably when taking into account that [20] $q_1 = q_5 = 1$, $q_2 = q_4 = 2$, $q_3 = 3$, we obtain by evaluating (2.17) for the matrix (3.6) the simple ratio

$$
\nu_{A_5} = 3/2 . \tag{3.7}
$$

We will now turn to the generic case. Taking the general solutions of the constant TBA equations into account [18, 16] and using a generic expression for the inverse of the Cartan matrix $K_{A_\ell}^{-1} = \min(i,j) - ij/(\ell+1)$ in (2.19), the M-matrix for an $A_{2\ell+1}$-minimal affine Toda field theory can be written generically as

$$
M_{ij} = \frac{ij}{\ell+1} - 2\min(i,j) - \delta_{ij} \frac{\sin\left(\frac{i\pi}{2\ell+4}\right)\sin\left(\frac{(i+2)\pi}{2\ell+4}\right)}{\sin^2\left(\frac{\pi}{2\ell+4}\right)} . \tag{3.8}
$$

As already indicated by the previous example this matrix is not of the form (2.18) and does not fit into the classification scheme proposed in [21]. According to [20] we have the charges

$$
q_i = q_{2\ell+2-i} \qquad \text{and} \qquad q_i = i \quad \text{for } i \leq \ell + 1 . \tag{3.9}
$$

As can be guessed from (3.6), it is not evident how to express the inverse in terms of a simple closed expression. We can, however, invert (3.8) case-by-case up to very high rank and we obtain from (2.17) together with (3.9) the sequence

$$
\nu_{A_{2\ell+1}} = \frac{2\ell + 2}{\ell + 2} . \tag{3.10}
$$

In view of (3.8), it is remarkable that the outcome is rational. Note for $\ell = 0$, that is $A_1$ we recover the free case with $\nu = 1$. Taking now $\ell = 2m - 1$, we obtain as a subsequence of this four times the most stable part of Jain's sequence (1.1) with $p = 2$

$$
\nu_{A_{4m-1}} = \frac{4m}{2m + 1} . \tag{3.11}
$$

In summary: *The conductance of a quantum wire which is described by a massive $A_{2\ell+1}$-minimal affine Toda field theory possesses in the high temperature regime, in which the model turns into a conformal field theory with Virasoro central $c = (2\ell + 1)/(\ell + 2)$, a filling fraction equal to (3.10). In particular for $\ell = 2m - 1$, we obtain the sequence (3.11).*

### 3.2 The $2m/(2m+1)$-sequence

We proceed now similarly as in the preceding section, but now for the $D_{2\ell+1}$-minimal affine Toda field theories, which all possess Virasoro central charge $c = 1$ in the ultraviolet limit. We label the particles in consecutive order along the Dynkin diagram (see e.g. [17] for more properties), starting from the not splitted end. Taking in (2.19) $\mathbf{g}=D_{2\ell+1}$ the solutions to the constant TBA equations are simply [18, 16]

$$e^{\varepsilon_i} = i(i+2) \qquad 1 \le i \le 2\ell - 1 \tag{3.12}$$

$$e^{\varepsilon_{2\ell+1}} = e^{\varepsilon_{2\ell}} = 2\ell . \tag{3.13}$$

Since these entries are all integer valued, we are not very surprised when we obtain rational values for the filling fraction, but what is not obvious is that the outcome is one of Jain's sequences. The $M$-matrix is computed to

$$M_{ij} = -2(K^{-1}_{D_{2\ell+1}})_{ij} - \delta_{ij} e^{\varepsilon_i} , \tag{3.14}$$

with the values (3.12) and (3.13). From these data we evaluate a simple expression for the determinant

$$\det M = \frac{(2\ell+1)^2 \, (2\ell+1)! \, (2\ell)!}{2} , \tag{3.15}$$

and the inverse of this matrix

$$(M^{-1})_{ij} = (M^{-1})_{ji} = \frac{2}{3j(1+j)(2+j)} \qquad 2 \le i < j \le 2\ell - 1 \tag{3.16}$$

$$(M^{-1})_{ii} = \frac{-(3i+1)}{3i(1+i)(2+i)} \qquad 1 \le i \le 2\ell - 1 \tag{3.17}$$

$$(M^{-1})_{i(2\ell+1)} = (M^{-1})_{i(2\ell)} = \frac{1}{6\ell(2\ell+1)} \qquad 1 \le i \le 2\ell - 1 \tag{3.18}$$

$$(M^{-1})_{(2\ell+1)i} = (M^{-1})_{(2\ell)i} = \frac{1}{6\ell(2\ell+1)} \qquad 1 \le i \le 2\ell - 1 \tag{3.19}$$

$$(M^{-1})_{(2\ell+1)(2\ell+1)} = (M^{-1})_{(2\ell)(2\ell)} = -\frac{10\ell+1}{12\ell(2\ell+1)} \tag{3.20}$$

$$(M^{-1})_{(2\ell+1)(2\ell)} = (M^{-1})_{(2\ell)(2\ell+1)} = \frac{2\ell-1}{12\ell(2\ell+1)} \tag{3.21}$$

Taking then the charges of the particles to be

$$q_{2\ell+1} = q_{2\ell} = \ell/2 \qquad \text{and} \qquad q_i = i \quad \text{for } i \le 2\ell - 1 , \tag{3.22}$$

the computation of (2.17) yields

$$\nu_{D_{2\ell+1}} = \frac{2\ell}{2\ell+1} . \tag{3.23}$$

Similarly as in the previous subsection, the sequence (3.23) gives twice the Jain sequence (1.1) with $p = 2$.

In summary: *The conductance of a quantum wire which is described by a massive $D_{2\ell+1}$-minimal affine Toda field theory possesses in the high temperature regime, in which all models turn into conformal field theories with Virasoro central charge $c = 1$, a filling fraction equal to (3.23) which is twice the principal Jain sequence (1.1).*

### 3.3 The $4m/(6m + 1)$-sequence

This sequence can be obtained similarly just by altering the values of the two charges at the very end of the Dynkin diagram. Considering now the $D_{6m+2}$-minimal affine Toda field theories we can employ the same $M$-matrix as in the previous subsection, but we take the charges of the particles to be

$$q_{6m+2} = q_{6m+1} = m/(2m + 1) \qquad \text{and} \qquad q_i = i \quad \text{for } i \leq 6m . \tag{3.24}$$

Evaluating then the expression for the filling fractions (2.17) gives

$$\nu_{D_{6m+2}} = \frac{4m}{6m + 1} , \tag{3.25}$$

which is four times the Jain's sequence (1.1) with $p = 6$.

## 4. Conclusions

Within a Landauer-Büttiker transport theory picture we have analyzed the expression for the conductance of a quantum wire which is described by an integrable quantum field theory. The final expression for the conductance in the high temperature regime is very simple (2.17) and involves the sum over the entries of the inverse of a certain matrix $M$ as defined in (2.15). This matrix is constructed from the knowledge of the asymptotic phases of the scattering matrix and the solutions of the constant TBA equations (2.16).

When evaluating this matrix for some concrete minimal affine Toda field theories, we obtain values for the filling fraction which coincide with multiples of several subsequences of Jain's series (1.1) and are therefore rational numbers. The fact that we obtain this special rational values is extremely surprising, in particular as for the $A_{2\ell+1}$-minimal affine Toda theories the related $M$-matrix has non-rational entries. One should note, however, that one does not always get these nice rational values. We did not report all examples here which we have computed, but for instance in general the $A_{2\ell}$ and the $D_{2\ell}$-minimal affine Toda theories lead to non-rational values for $\nu$.

Our findings pose several interesting questions: As it is clear that the $M$-matrices obtained are beyond the classification scheme carried out in [21] on the basis of integral lattices, one may attempt a new type of classification based on the Lie algebraic systematics which underlies the formulation of integrable quantum field theories. In order to do this we have to enlarge our considerations [22] to other algebras such as the $E$-series, non-simply laced Lie algebras and also to theories which are related to a pair of Lie algebras. It would also be interesting to perform an analysis based on a different expression from (2.2) for the conductance, such as the Kubo formula, and compare the findings similar as in [9].

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