Cox, A. (2000). On the blocks of the infinitesimal Schur algebras. The Quarterly Journal of Mathematics, 50(1), 39 - 56. doi: 10.1093/qmathj/50.1.39 <http://dx.doi.org/10.1093/qmathj/50.1.39>

On the blocks of the infinitesimal Schur algebras

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For a reductive algebraic group scheme G, much can be learnt about its representations over a field k of characteristic $p > 0$ by studying the representations of a related group scheme, G_rT, associated to the rth Frobenius kernel G_r and a maximal torus T of G. In the case $G = \mathrm{GL}(n, k)$ one can also consider the polynomial representations, and reduce to the study of representations of the Schur algebras. In [8] these two approaches were combined, and gave rise to the construction of a monoid scheme M_rD whose representations are equivalent to the polynomial representations of G_rT. Just as in the ordinary case, this leads naturally to the study of certain finite dimensional algebras, the infinitesimal Schur algebras. In this paper we determine the blocks of these algebras when $n = 2$, which extends a result in [9] where the blocks were determined in the case $n = 2$ and $r = 1$. We conclude by defining a quantum version of the infinitesimal Schur algebras, and show that the corresponding result also holds in this case.

1 Preliminaries

In this section (based on [8]) we briefly review the basic results and notation that will be needed later. We set M to be the monoid of $n \times n$ matrices over k. This can be regarded as a monoid scheme over k, and taking F to be the usual Frobenius morphism on M we may consider $M_r = \ker(F^r)$, an infinitesimal sub-monoid. Let D be the submonoid of M corresponding to the diagonal matrices, and set $M_rD = (F^r)^{-1}(D)$.

More concretely, we have that $k[M] = k[c_{ij} : 1 \leq i, j \leq n]$ with bialgebra structure given by

$$\Delta(c_{ij}) = \sum_{k=1}^{n} c_{ik} \otimes c_{kj}; \qquad \epsilon(c_{ij}) = \delta_{ij}.$$

Fix r and let J_r be the ideal generated by $c_{ij}^{p^r}$ for $1 \leq i \neq j \leq n$. This is also a coideal, and $k[M]/J_r$ is naturally a graded bialgebra. Now denote by $A(n, d)_r$ the set of homogeneous polynomials of degree $d \geq 0$. We obtain the infinitesimal Schur algebras by setting $S(n, d)_r = A(n, d)_r^*$.

Throughout this paper, we will freely use standard notation from [14]. We denote by Φ the root system of G (=GL(n,k)), by Φ^+ the set of positive roots, and by Π the set of simple roots. The set of rational (respectively polynomial) weights will be denoted $X(T)$ (respectively $P(D)$) and be identified with $\mathbb{Z}^n$ (respectively $\mathbb{N}^n$).

Now the simple G_rT-modules correspond (by [14, II 9.5b)]) to the weights in $X(T)$. For $\lambda \in X(T)$ denote the corresponding simple module by $\hat{L}_r(\lambda)$ and its restriction to G_r by $L_r(\lambda)$. Then by [8, Corollary 3.2] the set of isomorphism classes of simple M_rD-modules is

$$\{\hat{L}_r(\lambda) : \lambda \in \Gamma_r(D)\},$$

where $\Gamma_r(D) = P_r(D) + p^r P(D)$, and

$$P_r(D) = \{\lambda \in P(D) : 0 \leq \lambda_i - \lambda_{i+1} \leq p^r - 1 \quad \text{for } 1 \leq i \leq n\}$$

with $\lambda_{n+1} = 0$. We also write $\Gamma_r^d(D)$ for the set of elements of $\Gamma_r(D)$ of degree d, which indexes the set of isomorphism classes of simple $S(n,d)_r$-modules. The one-dimensional module corresponding to the determinant representation will also be denoted by det.

For $\lambda \in X(T)$, let $\hat{Q}_r(\lambda)$ denote the injective hull of $\hat{L}_r(\lambda)$ in Mod(G_rT). Similarly for each $\lambda \in \Gamma_r(D)$, let $\hat{I}_r(\lambda)$ denote the injective hull of $\hat{L}_r(\lambda)$ in Mod(M_rD). We can define induction and restriction functors (denoted ind and res respectively), and we set $\hat{Z}_r(\lambda) = \text{ind}_{B_rT}^{G_rT}\lambda$ for $\lambda \in X(T)$.

Finally we define two functors from Mod(G_rT). For $V \in$ Mod(G_rT) we set $\mathcal{F}_{M_rD}(V) \in$ Mod(M_rD) to be equal to the unique maximal submodule of V that is an M_rD-module, and $\mathcal{O}_\pi(V) \in$ Mod(G_rT) to be equal to the unique maximal submodule of V all of whose composition factors are M_rD-modules. (Each functor takes morphisms to their corresponding restrictions.) Given an M_rD-module V, we write $\text{inf}_{G_rT}V$ for the G_rT module obtained via inflation. Then $\text{inf}_{G_rT}\mathcal{F}_{M_rD}$ and $\mathcal{O}_\pi$ are equivalent by the main result in [15, Appendix].

2 Infinitesimal blocks

In this section we begin to determine the blocks of the infinitesimal Schur algebras. This will use the description of the blocks of G_rT implicit in [13]. We will denote the block of G_r containing λ by $\mathcal{B}_r(\lambda)$, and the block of G_rT containing λ by $\hat{\mathcal{B}}_r(\lambda)$. Blocks will be identified with subsets of $\mathbb{Z}^n$ in the usual way, thus allowing us to consider the intersection of blocks for different categories of modules.

We begin by recalling various results from [14]. Define m $(=m(\lambda))$ to be the least integer such that there exists an $\alpha \in \Phi^+$ with $\langle \lambda+\rho, \alpha\check{\ } \rangle \notin \mathbb{Z}p^m$. Then, by [14, II 9.19(1)], we have

$$\mathcal{B}_r(\lambda) = W.\lambda + p^m\mathbb{Z}\Phi + p^r X(T). \tag{1}$$

By [14, II 9.16 Lemma (a)] we also have that

$$\hat{\mathcal{B}}_r(\lambda) \subseteq W.\lambda + p\mathbb{Z}\Phi. \tag{2}$$

We can relate the blocks of G_r and G_rT, as

$$\mathrm{Ext}^1_{G_r}(L_r(\lambda), L_r(\mu)) = \bigoplus_{\nu \in X(T)} \mathrm{Ext}^1_{G_rT}(\hat{L}_r(\lambda + p^r\nu), \hat{L}_r(\mu)), \tag{3}$$

(see [14, II 9.16(3)]). This, along with (1) and (2), gives that $\hat{\mathcal{B}}_r(\lambda) \subseteq \mathcal{B}_r(\lambda)$, and hence

$$\hat{\mathcal{B}}_r(\lambda) \subseteq W.\lambda + p^{\min(m,r)}\mathbb{Z}\Phi. \tag{4}$$

Proposition 2.1 *For all $r > 0$ and $\lambda \in X(T)$, we have*

$$\hat{\mathcal{B}}_r(\lambda) = \begin{cases} W.\lambda + p^m\mathbb{Z}\Phi & \text{if } m \leq r, \\ \{\lambda\} & \text{if } m > r, \end{cases}$$

where m is defined as above.

Proof: We first consider the case $m > r$. By [14, II 11.8], we have that for all $\mu \in W.\lambda + p^r\mathbb{Z}\Phi$, the module $\hat{Z}_r(\lambda)$ is simple. So the result in this case follows from the usual characterisation of blocks (see [14, II 11.4]). Now suppose that $m \leq r$, and $\mu \in W.\lambda + p^m\mathbb{Z}\Phi$. Then λ and μ are in the same G_r block, and so there exists a sequence $\lambda = {}_0\lambda, {}_1\lambda, \ldots, {}_t\lambda = \mu$ such that $\mathrm{Ext}^1_{G_r}(L_r({}_i\lambda), L_r({}_{i+1}\lambda)) \neq 0$. So by (3) there exist ${}_0\nu, \ldots, {}_{t-1}\nu \in X(T)$ such that

$$\mathrm{Ext}^1_{G_rT}(\hat{L}_r({}_i\lambda + p^r{}_i\nu), \hat{L}_r({}_{i+1}\lambda)) \neq 0.$$

Thus ${}_i\lambda + p^r{}_i\nu$ is in the same G_rT block as ${}_{i+1}\lambda$ for $0 \leq i \leq t-1$. As $\hat{L}_r(\tau + p^r\nu) \cong \hat{L}_r(\tau) \otimes p^r\nu$ by [14, II 9.5 Proposition], this implies that ${}_0\lambda + p^r({}_0\nu + \ldots + {}_{t-1}\nu)$ is in the same G_rT block as ${}_t\lambda = \mu$. So we will be done if we can show that λ is in the same G_rT block as ${}_0\lambda + p^r({}_0\nu + \ldots + {}_{t-1}\nu)$. But $\mu \in W.\lambda + p^m\mathbb{Z}\Phi$ implies that ${}_0\lambda + p^r({}_0\nu + \ldots + {}_{t-1}\nu) \in W.\lambda + p^m\mathbb{Z}\Phi$ by (4), and hence that $p^r({}_0\nu + \ldots + {}_{t-1}\nu) \in p^m\mathbb{Z}\Phi$. The result now follows by repeated use of the short exact sequence in [13, Section 5.5 before (2)].

For the polynomial case we will need the following lemma, which will enable us to proceed by induction on r.

Lemma 2.2 *For all $\lambda' \in P_r(D)$ and $\lambda'' \in P(D)$, we have*

$$\mathrm{res}_{M_rD}\hat{I}_{r+1}(\lambda' + p^r\lambda'') \leq \bigoplus_{\nu} \hat{I}_r(\lambda' + p^r\nu),$$

where the sum runs over the set of polynomial weights of $\hat{Q}_1(\lambda'')$, counted with multiplicities.

Proof: We first note that for any $G_{r+1}T$-module X, it is clear that

$$\mathrm{res}_{M_rD}\mathcal{F}_{M_{r+1}D}(X) \leq \mathcal{F}_{M_rD}\mathrm{res}_{G_rT}(X).$$

We also have, from [14, II 11.15 Lemma], that

$$\hat{Q}_{r+1}(\lambda' + p^r\lambda'') \cong_{G_rT} \hat{Q}_r(\lambda') \otimes \hat{Q}_1(\lambda'')^{F^r},$$

which implies, by [14, II 11.3 (2)], that

$$\hat{Q}_{r+1}(\lambda' + p^r\lambda'') \cong_{G_rT} \bigoplus_{\nu} \hat{Q}_r(\lambda' + p^r\nu),$$

where the sum runs over the set of weights of $\hat{Q}_1(\lambda'')$. The result nows follows from [8, 4.1 Proposition], which gives that $\mathcal{F}_{M_rD}(\hat{Q}_r(\lambda)) \cong \hat{I}_r(\lambda)$.

We will denote the block of $S(n,d)$ containing λ by $\mathcal{B}^d(\lambda)$ and the corresponding block of $S(n,d)_r$ by $\mathcal{B}^d_r(\lambda)$. We also use the notation from [8, Section 3] for various subsets of $X(T)$. We first note that, by [4, Theorem], we have

$$\mathcal{B}^d(\lambda) = (W.\lambda + p^m\mathbb{Z}\Phi) \cap \Lambda^+(n,d). \tag{5}$$

The main conjecture of this section is

Conjecture 2.3 *For all $r > 0$ and $\lambda \in \Gamma^d_r(D)$, we have*

$$\mathcal{B}^d_r(\lambda) = \hat{\mathcal{B}}_r(\lambda) \cap \Gamma^d_r(D).$$

This is already known to hold in the case $n = 2$ and $r = 1$, as shown in [9]. As a first step we can at least prove one of the inclusions.

Proposition 2.4 *For all $r > 0$ and $\lambda \in \Gamma^d_r(D)$ we have*

$$\mathcal{B}^d_r(\lambda) \subseteq \hat{\mathcal{B}}_r(\lambda) \cap \Gamma^d_r(D).$$

Proof: To show that our block is contained in this intersection, we first note that by [8, 4.1 Proposition] we have that $\mathcal{F}_{M_rD}(\hat{Q}_r(\lambda)) \cong \hat{I}_r(\lambda)$. But then if $\hat{L}_r(\mu)$ is a composition factor of $\hat{I}_r(\lambda)$, it is also one of $\hat{Q}_r(\lambda)$, and so the result now follows.

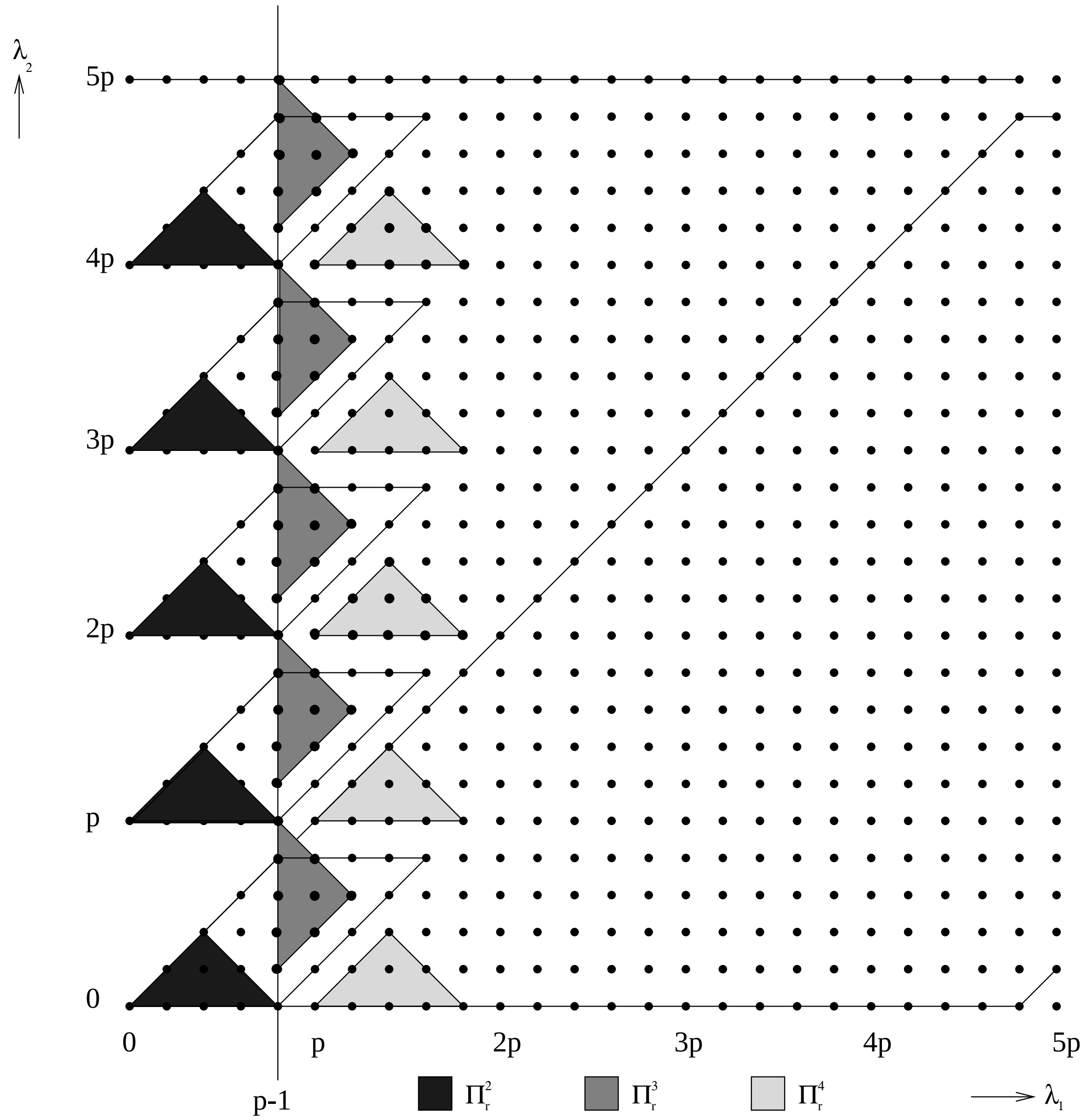

Figure 1: The case n=2, p=5, and r=1.

For convenience we will set $\mathcal{C}_r^d(\lambda) = \hat{\mathcal{B}}_r(\lambda) \cap \Gamma_r^d(D)$, and the rest of this section will be devoted to proving this equals $\mathcal{B}_r^d(\lambda)$ when $n = 2$. In this case there is one simple root $\alpha = (1, -1)$. Henceforth we will write $\lambda \sim \mu$ if λ and μ are linked as M_rD-weights.

We will also need to define various regions of the plane, for which it may be helpful to refer to Figure 1. We first set

$$\Pi_r^1 = \{\lambda \in P(D) : \lambda_1 \geq p^r - 1\}.$$

Writing $\lambda \in \Gamma_r(D)$ in the form $\lambda = \lambda' + p^r\lambda''$, with $\lambda' \in P_r(D)$ and $\lambda'' \in P(D)$, we also define

$$\Pi_r^2 = \{\lambda \in \Gamma_r(D) : \lambda_1' + \lambda_2' \leq p^r - 1 \text{ and } \lambda_1'' = 0\}.$$

Then our main result is

Theorem 2.5 *For $n = 2$ and $d \geq 0$ we have that, for all $\lambda \in \Gamma_r^d(D)$,*

$$\mathcal{B}_r^d(\lambda) = \mathcal{C}_r^d(\lambda).$$

The rest of this section is devoted to proving this result.

We will fix d and assume that we have proved the result for all $d' < d$. We first note that for $r >> 0$, we have $S(n,d)_r = S(n,d)$ (see [8, Section 2.3 Remark (2)]), so we will proceed by descending induction on r. So assume the result holds for $r+1$, that $d \geq p^r$ (as otherwise we are done by (5)) and that $m \leq r$ (as otherwise the result is clear from (2.4)). We first show

Lemma 2.6 *All weights in the set $\Pi_r^2 \cap \mathcal{C}_r^d(\lambda)$ are linked.*

Proof: Suppose λ and μ lie in this set. Then (using the usual notation) $\lambda'' = \mu''$. Now λ' is linked to μ' as these both have weight $d' < p^r$, for which the result is known from (5). So there is a chain of weights $\lambda = {}_0\lambda, \ldots, {}_t\lambda = \mu$ in $\Pi_r^2 \cap \mathcal{B}_r^{d'}(\lambda')$ such that, for each i, we have $\mathrm{Ext}_{M_rD}(\hat{L}_r({}_i\lambda), \hat{L}_r({}_{i+1}\lambda)) \neq 0$ or $\mathrm{Ext}_{M_rD}(\hat{L}_r({}_{i+1}\lambda), \hat{L}_r({}_i\lambda)) \neq 0$. Now as tensoring up with a one-dimensional module does not cause an extension to split, we get, in the category of G_rT-modules, a chain of non-trivial extensions by tensoring up with $p^r\lambda''$. But as these are all M_rD-modules by restriction, the equivalence of $\mathcal{F}$ and $\mathcal{O}_\pi$ (see [15, Appendix]) gives that this is still a chain of non-trivial extensions for M_rD (see [8, Section 6.2, Remark]). The result now follows, as $\hat{L}_r({}_i\lambda) \otimes p^r\lambda'' \cong \hat{L}_r({}_i\lambda + p^r\lambda'')$ for all i.

We will also need the following pair of lemmas.

Lemma 2.7 *For $\lambda \in \Gamma_r(D)$, if $\lambda_1 \in \Pi_r^1$ then*

$$\inf_{G_rT}\mathcal{F}_{M_rD}(\hat{Z}_r(\lambda)) \cong \hat{Z}_r(\lambda).$$

Proof: By [14, II 9.2 (6)], all weights μ of $\hat{Z}_r(\lambda)$ satisfy $\lambda - (p^r - 1)(1, -1) \leq \mu \leq \lambda$. So if $\lambda_1 \geq p^r - 1$, then all these weights are polynomial, and so, as $\mathcal{F}_{M_rD}$ is equivalent to $\mathcal{O}_\pi$, the result follows.

Lemma 2.8 *If λ, $\mu \in \Gamma_r^d(D) \cap \Pi_r^1$ and $\lambda - \mu \in p^m\mathbb{Z}\alpha$, then $\lambda \sim \mu$.*

Proof: The argument follows just as in [13, Section 5.5] as the exact sequence constructed there remains non-trivial when we apply $\mathcal{F}_{M_rD}$, by the last result.

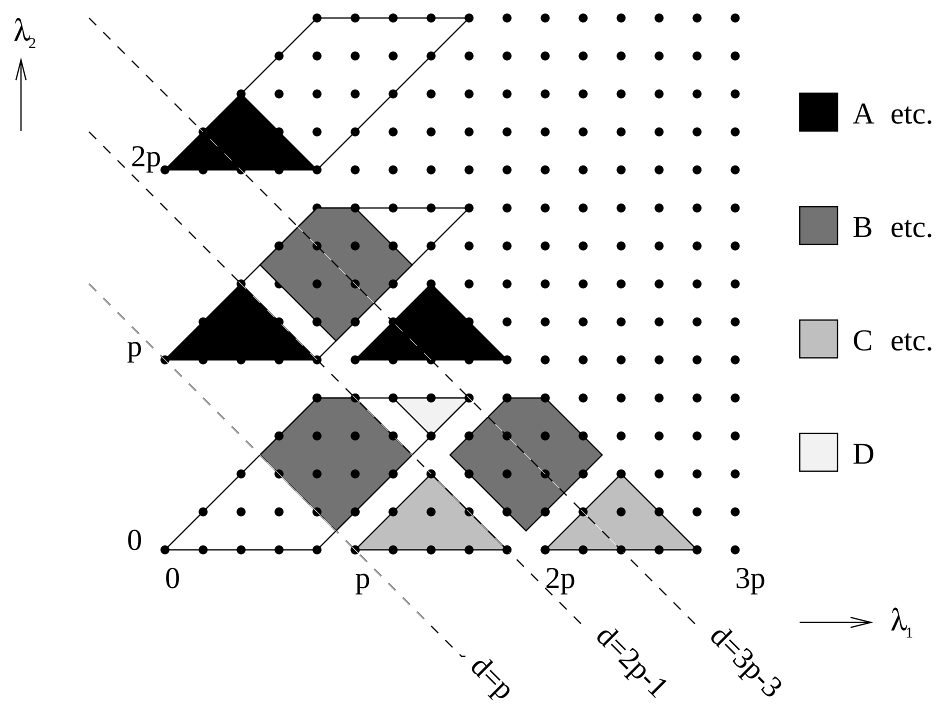

Figure 2: The case n=2, p=5, and r=1.

We now consider the case when $p^r \leq d \leq 2p^r - 1$. In this case it will be convenient to divide $\Gamma_r(D)$ into three regions; we set $A = \Pi_r^2 \cap \Gamma_r(D)$, $B = P_r(D) \cap \Gamma_r(D)$, and C to be the remainder (see Figure 2). Then

Lemma 2.9 *All the weights in $\mathcal{C}_r^d(\lambda) \cap B$ are linked.*

Proof: Consider $d' \in \{p^r - 1, p^r - 2\}$ such that $d - d'$ is even. Then we know that all weights in $\mathcal{C}_r^{d'}(\lambda - \frac{d-d'}{2}(1,1))$ are linked, as this reduces to the ordinary Schur algebra case. So for any two weights in this set there is a chain of simple modules with non-trivial extensions between consecutive terms. Tensoring up with $\det^{\frac{d-d'}{2}}$ then gives the result as above.

As $\Gamma^d_{r+1}(D) \subseteq \Gamma^d_r(D)$, we now consider the case where $\lambda \in \Gamma^d_{r+1}(D)$ and $\mu \in \mathcal{B}^d_{r+1}(\lambda)$. Then $\mathcal{B}^d_{r+1}(\lambda) = (W.\lambda + p^m\mathbb{Z}\Phi) \cap \Gamma^d_{r+1}(D)$, and there exists a chain $\lambda = {}_0\lambda, {}_1\lambda, \ldots, {}_t\lambda = \mu$ in $\Gamma^d_{r+1}(D)$ such that either $[\hat{I}_{r+1}({}_i\lambda) : \hat{L}_{r+1}({}_{i+1}\lambda)] \neq 0$ or $[\hat{I}_{r+1}({}_{i+1}\lambda) : \hat{L}_{r+1}({}_i\lambda)] \neq 0$ for $1 \leq i \leq t-1$. Now for all i set ${}_i\lambda = {}_i\lambda' + p^r{}_i\lambda''$, where ${}_i\lambda' \in P_r(D)$ and ${}_i\lambda'' \in P(D)$. By (2.2), and as $\hat{L}_{r+1}({}_i\lambda) \cong_{M_rD} \hat{L}_r({}_i\lambda') \otimes \hat{L}_1({}_i\lambda'')^{F^r}$, we have that for $2 \leq i \leq t$ there exists ${}_i\nu, {}_i\nu' \in P(D)$ such that either

$$[\hat{I}_r({}_i\lambda' + p^r{}_i\nu) : \hat{L}_r({}_{i+1}\lambda)] \neq 0,$$

or

$$[\hat{I}_r({}_{i+1}\lambda' + p^r{}_{i+1}\nu') : \hat{L}_r({}_i\lambda)] \neq 0.$$

Hence either ${}_i\lambda$ is linked to ${}_{i+1}\lambda' + p^r{}_{i+1}\nu'$ or ${}_{i+1}\lambda$ is linked to ${}_i\lambda' + p^r{}_i\nu$. With this we can now prove

Lemma 2.10 *For $p^r \leq d \leq 2p^r - 1$ we have that either $\mathcal{C}^d_r(\lambda)$ is a single block, or it is the union of the two blocks $\mathcal{C}^d_r(\lambda) \cap \Pi^2_r$ and $\mathcal{C}^d_r(\lambda) \backslash \Pi^2_r$.*

Proof: First note that if $\mathcal{C}^d_r(\lambda) \subseteq B$ then we are done by the previous lemma, so we may assume that this does not hold. Thus, as $\mathcal{C}^d_r(\lambda) \cap (A \cup C) \neq \emptyset$, and $C = \{a + p^r\alpha : a \in A\}$, we must have $\mathcal{C}^d_r(\lambda) \cap C \neq \emptyset$, and hence $\mathcal{C}^d_{r+1}(\lambda) \cap C \neq \emptyset$. Consider the sequence of linked weights introduced above, and assume — as by the last remark we may — that $\lambda \in C$. Suppose that $\mu \in B$. Now as the only weight equal to μ modulo $p^r\alpha$ is μ, and the only weights equal to those in C modulo $p^r\alpha$ lie in $A \cup C$, there exists some weight τ such that $\tau \in A \cup C$ and $\mu \sim \tau$. We will consider the following two sets of weights:

$$B_1 = \{\mu \in \mathcal{B}^d_{r+1}(\lambda) \cap B : \exists \tau \in A \text{ with } \mu \sim \tau\},$$

and

$$B_2 = \{\mu \in \mathcal{B}^d_{r+1}(\lambda) \cap B : \exists \tau \in C \text{ with } \mu \sim \tau\}.$$

By (2.6), all the weights in $\mathcal{B}^d_{r+1}(\lambda) \cap A$ are linked, and by tensoring up with $p^r(1, -1)$ we see that all the weights in $\mathcal{B}^d_{r+1}(\lambda) \cap C$ are linked also. So if $B_1 = B_2 = \emptyset$ we are done. Otherwise there are two possibilities: $B_1 = B_2 = B$, or $B_1 \cap B_2 = \emptyset$.

Choose a minimal weight $\tau \in \mathcal{B}^d_{r+1}(\lambda) \cap C$ (this exists by our initial assumption). By (2.7), $\hat{Z}_r(\tau)$ has polynomial weights, and so (as it is not simple by [14, II 11.8 Lemma]) we see by [14, II 9.1 (6)] that τ is linked to some lower weight. By minimality this weight lies in B or A. If it is in A then $B_1 = B_2 = B$, while if it is in B then $B_2 \neq \emptyset$. So by the previous lemma we either have $B_1 = B_2 = B$, or $B_1 = \emptyset$ as required.

Now we consider the case when $2p^r \leq d \leq 3p^r - 1$. Once again it will be convenient to divide our weights into regions. For a set of weights X, we will set $X' = \{x + p^r(0,1) : x \in X\}$ and $X'' = \{x + p^r(1,0) : x \in X\}$. We also denote by D the set of weights with $2p^r \leq d \leq 3p^r - 1$ that are not contained in $(A \cup B \cup C)' \cup (A \cup B \cup C)''$.

Lemma 2.11 *For $2p^r \leq d \leq 3p^r - 1$ we have that either $\mathcal{C}_r^d(\lambda)$ is a single block, or it is the union of the two blocks $\mathcal{C}_r^d(\lambda) \cap \Pi_r^2$ and $\mathcal{C}_r^d(\lambda) \backslash \Pi_r^2$.*

Proof: First consider $\mathcal{C}_r^d(\lambda) \cap (B' \cup B'' \cup C' \cup D)$. Let $d' \in \{2p^r - 1, 2p^r - 2\}$ be such that $d - d'$ is even. Then as all the weights in $A \cup B \cup C$ are linked by the induction hypothesis, we see by tensoring up by $\det^{\frac{d-d'}{2}}$ that all of these weights are linked also. If $\mathcal{C}_r^d(\lambda) \cap C'' \neq \emptyset$ then these weights can be linked to those in C' by (2.8).

We now show that $\mathcal{C}_r^d(\lambda)$ is in fact a single block for $p \leq 3p^r - 1$. For this we will need to define two further regions of the plane. Decomposing $\lambda = \lambda' + p^r \lambda''$ as usual, we set

$$\Pi_r^3 = \{\lambda + (0,1) \in \Gamma_r(D) : \lambda_1' \geq p^r - 1,\ \lambda_1' + \lambda_2' < 2p^r - 1,\ \text{and}\ \lambda_1'' = 0\}$$

and

$$\Pi_r^4 = \{\lambda + p^r(1,0) : \lambda \in \Pi_r^2\}.$$

We can now show

Lemma 2.12 *For $d \leq 3p^r - 1$ and $\lambda \in \Gamma_r^d$ we have $\mathcal{C}_r^d(\lambda) = \mathcal{B}_r^d(\lambda)$.*

Proof: First suppose that $\lambda \in \Pi_r^1$. By (2.7), the lowest weight in $\hat{A}_r(\lambda)$ is $\lambda - (p^r - 1)(1, -1)$. Now for $\tau = \tau' + p^r \tau''$, with $\tau' \in P_r(D)$ and $\tau'' \in P(D)$, we have $\hat{L}_r(\tau) \cong \hat{L}_r(\tau') \otimes p^r \tau''$, and by [14, II 3.15 Proposition] $\hat{L}_r(\tau') \cong L(\tau')$. Now the lowest weight in $L(\tau')$ is $w_0 \tau'$ (where w_0 is the non-trivial element of the Weyl group), and hence the lowest weight in $\hat{L}_r(\tau)$ is $w_0 \tau' + p^r \tau''$. Clearly $\hat{L}_r(\tau)$ and $\hat{L}_r(\mu)$ have the same lowest weight if, and only if, $\tau = \mu$, and so $\hat{A}_r(\lambda)$ has a composition factor $\hat{L}_r(\tau)$, where $w_0 \tau' + p^r \tau'' = \lambda - (p^r - 1)(1, -1)$.

Now we may assume that $\mathcal{C}_r^d(\lambda) \cap \Pi_r^2 \neq \emptyset$ (else the result holds by our earlier calculations). Then, as $\Pi_r^4 = \{\mu + p^r \alpha : \mu \in \Pi_r^2\}$, we have that $\mathcal{C}_r^d(\lambda) \cap \Pi_r^4 \neq \emptyset$. Modulo $p^r \alpha$, we have that

$\Pi_r^4 = w_0.\Pi_r^3$, and so $\mathcal{C}_r^d(\lambda) \cap \Pi_r^3 \neq \emptyset$. So we may assume that $\lambda \in \Pi_r^3$. Then, by considering Figure 1, along with the above remarks, we see that $\tau \in \Pi_r^2$, and we are done (by (2.6), (2.10) and (2.11)).

To complete the proof we require

Lemma 2.13 *If $d \geq 2p^r - 1$ then, for all $\lambda \in \Gamma_{r+1}^d$ and $w \in W$, there is some element of the form $w.\lambda + p^m z\alpha$ in $\Gamma_{r+1}^d \cap \Pi_r^1$.*

Proof: For each $w \in W$ there is one such representative in any chain of p^m consecutive weights in Γ_{r+1}^d. So it is enough to show that such a chain exists. But all (μ_1, μ_2) with $\mu_1 + \mu_2 = d$ and $\mu_1 \geq \mu_2$ lie in $\Gamma_{r+1}^d \cap \Pi_r^1$, so such a chain always exists if $d \geq 2p^m - 1$.

To conclude we suppose that $d \geq 3p^r - 1$. Then there exists integers a and d' such that $2p^r \leq d' \leq 3p^r - 1$ and $d = d' + p^r a$. By the previous lemma, $\mathcal{C}_r^d(\lambda) \cap \Pi_r^1 \cap \{\mu + p^r(0,a) : \mu \in \Gamma_r^{d'}\}$ contains a representative of each $w.\lambda$ class, and all the weights in $\mathcal{C}_r^d(\lambda) \cap \{\mu + p^r(0,a) : \mu \in \Gamma_r^{d'}\}$ are linked by tensoring up the corresponding chains from $\Gamma_r^{d'}$. All other weights in $\mathcal{C}_r^d(\lambda)$ are linked to these, as they are linked to their corresponding $w.\lambda$ class representative by (2.8).

3 The infinitesimal q-Schur algebra

In this and the following section we will define the infinitesimal q-Schur algebras, and develop some of their basic representation theory. Although all of the classical results in [8, Sections 1–5] can be replicated (by analogous methods) in the quantum setting, we shall restrict our attention to those results that are required to generalise the preceding block calculation. A more detailed development can be found in [1, Chapter 4].

We first recall the definition of the quantum general linear group due to Dipper and Donkin [3]. We fix $q \in k\backslash\{0\}$, and define $A_q(n)$ to be the k-algebra generated by the n^2 indeterminates c_{ij}, with $1 \leq i, j \leq n$, subject to the relations

$$\begin{array}{rll} c_{ij}c_{rs} & = qc_{rs}c_{ij} & \text{for } i > r \text{ and } j \leq s, \\ c_{ij}c_{rs} & = c_{rs}c_{ij} + (q-1)c_{rj}c_{is} & \text{for } i > r \text{ and } j > s, \\ c_{ij}c_{il} & = c_{il}c_{ij} & \text{for all } i, j, l. \end{array}$$

We note that when $q = 1$ these relations just say that the c_{ij} commute; in this case we will usually denote the c_{ij} by x_{ij}. As was shown in [3, 1.4.2 Theorem], $A_q(n)$ has the structure of a bialgebra with comultiplication and counit maps as in the classical case.

We shall often write $k[q\text{-M}(n,k)]$ for $A_q(n)$ and regard this as corresponding to a quantum monoid q-M(n,k). We can define (see [3]) a 'quantum determinant' d_q in $k[q\text{-M}(n,k)]$, and we denote the Hopf algebra obtained by localising at this by $k[q\text{-GL}(n,k)]$. This corresponds to the quantum group q-GL(n,k) of Dipper and Donkin, which we shall often denote just by G. Certain quantum subgroups of G have been defined in [6, Section 2]; in particular the torus q-T(n,k) and the (negative) Borel subgroup q-B(n,k). We shall denote the corresponding submonoids of q-M(n,k) by q-D(n,k) and q-L(n,k) respectively.

Henceforth, we restrict our attention to the case when q is a primitive lth root of unity, and k has characteristic $p>0$. In this case there is a Frobenius morphism $F : q\text{-GL}(n,k) \longrightarrow \text{GL}(n,k)$ whose associated comorphism takes x_{ij} to c_{ij}^l. We also have the usual Frobenius map F on GL(n,k) associated to the comorphism taking x_{ij} to x_{ij}^p. Henceforth we shall abuse notation and write F^r for $\mathrm{F}^{r-1}F$.

Let J_r be the ideal in $A_q(n)$ generated by all $c_{ij}^{lp^{r-1}}$ for $1 \leq i \neq j \leq n$. This is in fact a coideal; $\epsilon(J_r)=0$ is clear, while $\delta(c_{ij}^{lp^{r-1}}) = \sum_{k=1}^n c_{ik}^{lp^{r-1}} \otimes c_{kj}^{lp^{r-1}}$ by [10, 3.1] and the centrality of c_{ij}^l (see [3, 1.3.2]). Thus $A_q(n)/J_r$ is also a bialgebra, and gives rise to a quantum monoid which we denote by M_rD (or q-M_rD if we wish to emphasise the role of q).

A quantum analogue of the Janzten subgroup G_rT was defined in [2]. In fact, $k[M_rD]$ is the subbialgebra of $k[G_rT]$ generated by the c_{ij}, and $k[G_rT]$ is the localisation of $k[M_rD]$ at the quantum determinant. Thus $k[M_rD]$ is the polynomial part of $k[G_rT]$. We call objects in $\text{Mod}_{k[M_rD]}(G_rT)$ polynomial G_rT modules.

We have that $A_q(n)/J_r = \bigoplus_{d\geq 0} A_q(n,d)_r$, where $A_q(n,d)_r$ is the subspace consisting of the homogeneous polynomials of degree d in the c_{ij}. This subspace is clearly also a subcoalgebra of $A_q(n)/J_r$, for all d. Hence we may define the infinitesimal q-Schur algebra $S_q(n,d)_r = A_q(n,d)_r^*$. We will say that objects in $\text{Mod}_{A_q(n,d)_r}(G_rT)$ are homogeneous of degree d.

Proposition 3.1 *i) The category of polynomial G_rT-modules is equivalent to the category of M_rD-modules.*
ii) Every polynomial G_rT-module V has a direct sum decomposition $V = \bigoplus_{d\geq 0} V_d$ where V_d is homogeneous of degree d.
iii) The category of finite-dimensional $S_q(n,d)_r$-modules is equivalent to the category of homogeneous polynomial G_rT-modules of degree d.

Proof: This follows just as in the ordinary case (see [11, Section 1.6] and [12, pages 3–11]) as noted in [8, 2.1 Proposition].

We next classify the simple M_rD-modules, and hence the simple polynomial G_rT-modules. The result also classifies the simple $S_q(n,d)_r$-modules. We carry over the notation for the various subsets of the weights in $X(T)$ from Section 1, with the following modifications. We set $\Gamma_r(D) = P_r(D) + lp^{r-1}P(D)$, where

$$P_r(D) = \{\lambda \in P(D) : 0 \leq \lambda_i - \lambda_{i+1} \leq lp^{r-1} - 1 \quad \text{for } 1 \leq i \leq n\}$$

with $\lambda_{n+1} = 0$. This latter definition coincides with that of $X_r(T)$ in [2, Section 3], and the notation we use for this set will depend on the context in which it arises. We now obtain

Theorem 3.2 *Let V be a simple G_rT-module with all its weights polynomial. Then V is of the form $L(\lambda') \otimes lp^{r-1}\lambda''$ with $\lambda' \in P_r(D)$ and $\lambda'' \in P(D)$.*

Proof: This follows just as in the classical case (see [8, 3.2 Theorem]), using the fact that the character of a G-module is invariant under the Weyl group (see [6, Lemma 3.1(v)]).

By [7, 3.1(13)(iii)] (which generalises to the case $r > 1$) we have the following corollary, as in [8].

Corollary 3.3 *A complete set of non-isomorphic simple modules in* $\mathrm{Mod}(M_rD)$ *is given by* $\{\hat{L}_r(\lambda) : \lambda \in \Gamma_r(D)\}$.

From this it follows that every simple M_rD-module has a unique tensor product decomposition of the form

$$\hat{L}_r(\lambda) \cong L(\lambda') \otimes lp^{r-1}\lambda'',$$

for $\lambda' \in P_r(D)$ and $\lambda'' \in P(D)$. Further, if we set $\Gamma_r^d(D) = \{\lambda \in \Gamma_r(D) : |\lambda| = d\}$, then it is clear that the set of simple $S_q(n,d)_r$-modules is in one-to-one correspondence with $\Gamma_r^d(D)$. Henceforth, we will denote by $\hat{L}_r(\lambda)$ both the simple M_rD- and G_rT-modules corresponding to $\lambda \in \Gamma_r(D)$.

4 Truncation functors and induced modules

Just as in the classical case, we can define the two truncation functors $\mathcal{F}_{M_rD}$ and $\mathcal{O}_\pi$, and the inflation functor $\inf_{G_rT}$. Most of this section is devoted to considering

Conjecture 4.1 *We have an equivalence of functors between $\mathcal{F}_{M_rD}$ and $\mathcal{O}_\pi$; that is for all G_rT-modules V, we have*

$$\inf_{G_rT}\mathcal{F}_{M_rD}(V) \cong \mathcal{O}_\pi(V).$$

If this holds, then any G_rT-module, all of whose composition factors lift to M_rD, will itself lift. Unfortunately, we are not able to generalise the classical proof in [15, Appendix] to the quantum case, as it relies on an action of the symmetric group on the coordinate algebra (which does not exist in our setting). However, similar methods will at least give the result in the case $n = 2$.

We begin with a result relating the injective modules for M_rD and G_rT. For each $\lambda \in \Gamma_r(D)$, we denote the injective hull of $\hat{L}_r(\lambda)$ in $\mathrm{Mod}(M_rD)$ by $\hat{I}_r(\lambda)$, and in $\mathrm{Mod}(G_rT)$ by $\hat{Q}_r(\lambda)$. The basic properties of $\hat{Q}_r(\lambda)$ have been developed in [7] in the case $r = 1$, and it is straightforward to verify that similar arguments hold for $r > 1$. By (3.3) we have

$$\mathrm{soc}_{M_rD}\mathcal{F}_{M_rD}(V) \cong \mathcal{F}_{M_rD}(\mathrm{soc}_{G_rT}V),$$

as M_rD-modules, for every G_rT-module V.

Proposition 4.2 *For $\lambda \in \Gamma_r(D)$ we have $\mathcal{F}_{M_rD}(\hat{Q}_r(\lambda)) \cong \hat{I}_r(\lambda)$.*

Proof: This is immediate as $\mathcal{F}_{M_rD}$ takes injectives to injectives, and $\hat{Q}_r(\lambda)$ has the appropriate simple socle.

Returning to our conjecture, we note that we have an inclusion $k[M_rD] \subseteq \mathcal{O}_\pi(k[G_rT])$. Equivalence will follow if we can show this is in fact an equality, by the following lemma (an analogue of [8, 4.1 Lemma]).

Lemma 4.3 *With $\pi = \{\hat{L}_r(\lambda) : \lambda \in \Gamma_r(D)\}$, the following are equivalent:*

i) $\mathcal{O}_\pi$ is equivalent to $\mathcal{F}_{M_rD}$;

ii) $\mathcal{O}_\pi(k[G_rT]) \cong k[M_rD]$;

iii) for all d, if π_d is the set of simple $S_q(n,d)_r$-modules then $\mathcal{O}_{\pi_d}(k[G_rT]) \cong A_q(n,d)_r$;

iv) $\mathcal{O}_\pi(\hat{Q}_r(\lambda)) \cong \hat{I}_r(\lambda)$ for all $\lambda \in \Gamma_r(D)$.

Proof: The equivalence of i) and ii) is clear, as every G_rT-module embeds into a direct sum of copies of $k[G_rT]$, by [16, 2.4.4]. The equivalence of ii) and iii) is also immediate. For the equivalence of ii) and iv) we use that

$$k[M_rD] = \bigoplus_{\lambda \in \Gamma_r(D)} [\dim \hat{L}_r(\lambda)]\hat{I}_r(\lambda).$$

This follows (as $k[M_rD]$ is injective [16, 2.8.2(1)] and $\mathrm{Mod}(M_rD)$ has enough injectives [16, 2.8.1]) by the usual arguments (see [14, I.3.14–17]). From (3.3) and the definition of $\mathcal{O}_\pi$, we see that $\mathcal{O}_\pi(\hat{Q}_r(\lambda)) \neq 0$ if, and only if, $\lambda \in \Gamma_r(D)$. There is a similar decomposition to that of $k[M_rD]$ above for $k[G_rT]$, so applying $\mathcal{O}_\pi$ to each side gives

$$\mathcal{O}_\pi(k[G_rT]) = \bigoplus_{\lambda \in \Gamma_r(D)} [\dim \hat{L}_r(\lambda)]\mathcal{O}_\pi(\hat{Q}_r(\lambda)).$$

As $\hat{I}_r(\lambda) \cong \mathcal{F}_{M_rD}(\hat{Q}_r(\lambda)) \subseteq \mathcal{O}_\pi(\hat{Q}_r(\lambda))$, the result now follows.

The following pair of lemmas will allow us to prove the result for the case $n = 2$. The former is a modification of the main lemma used by Jantzen in his proof for the classical case.

In order to be able to state our next result we need another description of $k[G_rT]$. By [10, 3.1], we have that

$$d_q^{lp^{r-1}} = c_{11}^{lp^{r-1}} c_{22}^{lp^{r-1}} \cdots c_{nn}^{lp^{r-1}},$$

and hence, as in [15, Appendix], we obtain

$$k[G_rT] = k[c_{ij}, c_{ii}^{-1} : 1 \leq i,j \leq n]/\langle c_{ij}^{lp^{r-1}} : i \neq j \rangle,$$

with the usual relations.

Lemma 4.4 *Let V be a $k[G_rT]$-module with all weights polynomial. Then the coefficient space of V lies in*

$$k[c_{ij}, c_{tt}^{-1} : 1 \leq i,j \leq n, 1 \leq t \leq n-1]/\langle c_{ij}^{lp^{r-1}} : i \neq j \rangle.$$

Proof: Consider the natural map

$$\phi : k[G_rT] \longrightarrow k[B_rT] \otimes k[T] \otimes k[B_r^+T].$$

This is injective (by standard arguments — compare with [16, (8.1.1) Theorem]), and writing c_{ij} for the generators of all four quantum groups we see that

$$\phi(c_{ij}) = \sum_{t \leq i,j} c_{it} \otimes c_{tt} \otimes c_{tj}. \tag{6}$$

In particular, the only case in which any of the middle factors can contain a c_{nn} is when $i = j = n$.

Now take a basis of weight vectors for V, say $\{v_i : 1 \leq i \leq t\}$, with the corresponding set of coefficient functions $\{f_{ij}\}$. By assumption, the f_{ii} are polynomial for all i. As V is a comodule, we have that

$$(\mathrm{id} \otimes \delta)\delta(f_{ij}) = \sum_{s,t} f_{is} \otimes f_{st} \otimes f_{tj},$$

and as $\epsilon(f_{ij}) = \delta_{ij}$ this implies that

$$\phi(f_{ij}) = \sum_t \bar{f}_{it} \otimes \bar{f}_{tt} \otimes \bar{f}_{tj},$$

where the bars denote the appropriate restrictions. Thus we see that

$$\phi(f_{ij}) \subseteq k[B_rT] \otimes k[D] \otimes k[B_r^+T]. \tag{7}$$

Suppose now that there exists some f_{ij} involving c_{nn}^{-1}. We have that $f_{ij} = d_q^{-tlp^{r-1}} a$ with $a \in k[M_rD]$, and hence

$$\begin{aligned} \phi(f_{ij}) &= \phi(d_q^{-tlp^{r-1}})\phi(a) \\ &= \bar{d}_q^{-tlp^{r-1}} \otimes \bar{d}_q^{-tlp^{r-1}} \otimes \bar{d}_q^{-tlp^{r-1}} \phi(a). \end{aligned}$$

Writing ϕ_2 for the projection of ϕ onto the central factor of the tensor product we thus have that

$$\phi_2(f_{ij}) = c_{11}^{-tlp^{r-1}} \cdots c_{nn}^{-tlp^{r-1}} \phi_2(a).$$

By assumption, $a = c_{nn}^{tlp^{r-1}} b + e$, where e is non-zero and no term of e contains $c_{nn}^{tlp^{r-1}}$. So, again by [10, 3.1],

$$\phi(a) = (c_{nn}^{tlp^{r-1}} \otimes c_{nn}^{tlp^{r-1}} \otimes c_{nn}^{tlp^{r-1}})\phi(b) + \phi(e),$$

with $\phi(e)$ non-zero by the injectivity of ϕ. However, by (6), no term of $\phi_2(e)$ contains $c_{nn}^{tlp^{r-1}}$, and so $\phi_2(a) \notin c_{nn}^{tlp^{r-1}} k[D]$. Thus $\phi_2(f_{ij}) \notin k[D]$, which contradicts (7).

In the classical case, Jantzen now uses the invariance of the coefficient space under the action of the symmetric group to obtain the desired result. This action does not exist for non-trivial q, but we can at least prove the result for the case $n = 2$.

Lemma 4.5 *Let V be a $k[G_rT]$-module with all weights polynomial. Then the coefficient space of V lies in*

$$k[c_{ij}, c_{tt}^{-1} : 1 \leq i, j \leq n, 2 \leq t \leq n-1] / \langle c_{ij}^{lp^{r-1}} : i \neq j \rangle.$$

Proof: Consider $k[q\text{-GL}(n,k)]$ with the usual generators, and $k[q^{-1}\text{-GL}(n,k)]$ with generators e_{ij} and $d_{q^{-1}}$. We define a map

$$\phi : k[q\text{-M}(n,k)] \longrightarrow k[q^{-1}\text{-M}(n,k)]$$

by $\phi(c_{ij}) = e_{n+1-i,n+1-j}$. It is easy to check that this is a well-defined bialgebra homomorphism, and that it extends to a map of the corresponding quantum groups. Furthermore, it is also clear that it restricts to a map between the corresponding Jantzen subgroups, and so induces a map Φ from $\text{Mod}(q\text{-}G_rT)$ to $\text{Mod}(q^{-1}\text{-}G_rT)$. If V is a q-G_rT-module with polynomial weights, and its coefficient space contains terms involving c_{11}^{-1}, then $\Phi(V)$ is a q^{-1}-G_rT-module with polynomial weights whose coefficient space contains terms involving e_{nn}^{-1}. This gives a contradiction, as the previous lemma also holds for $\text{Mod}(q^{-1}\text{-}G_rT)$.

We conclude this section by defining certain important induced modules. First we define $k[L_rD] = k[q\text{-L}(n,k)]/J'_r$, where $J'_r = J_r \cap k[q\text{-L}(n,k)]$. Now for $\lambda \in P(D)$, we can consider the induced module $\hat{A}_r(\lambda) = \text{ind}_{L_rD}^{M_rD} k_\lambda$. (Here k_λ denotes the one-dimensional D-module of weight λ, which can be regarded as a module for L_rD in the usual way.) This is the analogue for M_rD of the G_rT module $\hat{Z}_r(\lambda)$, defined in [2]. When $r = 1$, the basic properties of $\hat{Z}_r(\lambda)$ have been determined in [7], and it is straightforward to verify that similar arguments hold for $r > 1$.

Proposition 4.6 *Let $\lambda \in P(D)$. Then*

i) $\hat{A}_r(\lambda) = 0$ *unless* $\lambda \in \Gamma_r(D)$*;*

ii) if $\lambda \in \Gamma_r(D)$*, then* $\hat{A}_r(\lambda) \cong \mathcal{F}_{M_rD}(\hat{Z}_r(\lambda))$*.*

Proof: Let $\lambda \in P(D)$. There exists an embedding $\hat{A}_r(\lambda) \longrightarrow k[G_rT]$, the composition of the natural inclusion of $\hat{A}_r(\lambda)$ in $k[M_rD]$ with the injection $\iota : k[M_rD] \longrightarrow k[G_rT]$. Consider induction from L_rD to M_rD. We have the obvious map $\hat{\phi} : k[M_rD] \longrightarrow k[L_rD]$ and, by definition,

$$\hat{A}_r(\lambda) = \{f \in |\lambda| \otimes k[M_rD] : f = e_\lambda \otimes g \text{ and } \tau(e_\lambda) \otimes g = \sum_i e_\lambda \otimes \hat{\phi}(g_i') \otimes g_i''\},$$

where $\delta(g) = \sum_i g_i' \otimes g_i''$ and e_λ is a basis element for λ. Now $\tau(e_\lambda) = e_\lambda \otimes c_{11}^{\lambda_1} \dots c_{nn}^{\lambda_n}$, so

$$\hat{A}_r(\lambda) \cong \{g \in k[M_rD] : c_{11}^{\lambda_1} \dots c_{nn}^{\lambda_n} \otimes g = \sum_i \hat{\phi}(g_i') \otimes g_i''\}.$$

Similarly,

$$\hat{Z}_r(\lambda) \cong \{g \in k[G_rT] : c_{11}^{\lambda_1} \dots c_{nn}^{\lambda_n} \otimes g = \sum_i \hat{\psi}(g_i') \otimes g_i''\},$$

where $\hat{\psi} : k[G_rT] \longrightarrow k[B_rT]$ is the obvious map and $\delta'(g) = \sum_i g_i' \otimes g_i''$. Clearly $\hat{\psi}\iota = \hat{\phi}$, and $\delta'\iota = \delta$, so by the embedding above we have that if $f \in k[M_rD]$ lies in $\hat{A}_r(\lambda)$, then f lies in $\hat{Z}_r(\lambda)$. Hence $\hat{A}_r(\lambda)$ injects into $\hat{Z}_r(\lambda)$. The proof now proceeds just as in the classical case (see [8, 5.1 Proposition]).

The following corollary is now an immediate consequence of the result above, along with the known structure of $\hat{Z}_r(\lambda)$ and the classification in (3.3).

Corollary 4.7 *Let $\lambda \in \Gamma_r(D)$.*

i) Let $\hat{A}_r(\lambda) = \sum_{\mu \in P(D)} \hat{A}_r(\lambda)^\mu$ be a D-weight space decomposition. Then we have $\dim \hat{A}_r(\lambda)^\lambda = 1$ *and* $\dim \hat{A}_r(\lambda)^\mu \neq 0$ *implies that $\mu \leq \lambda$ for all $\mu \in P(D)$.*

ii) The module $\hat{A}_r(\lambda)$ has simple socle $\hat{L}_r(\lambda)$.

Proof: See [7, 3.1(13)(i) and (20)(ii)].

5 The blocks in the quantum case

In this final section, we verify that the various infinitesimal results used in the block calculation of Section 2 generalise to the quantum setting (at least for the case $n = 2$). Having done this, we will obtain a description of the blocks of $S_q(2, d)_r$ just as in the classical case.

Lemma 5.1 *For all $i \geq 0$, $B(n,k)$-modules M and G-modules V, we have*

$$R^i \operatorname{ind}_{G_rB}^G (V \otimes M^{F^r}) \cong V \otimes (R^i \operatorname{ind}_{\mathrm{B}(n,k)}^{\mathrm{GL}(n,k)} M)^{F^r}.$$

Proof: See [2, Lemma 4.6].

With this lemma, we can now prove the following proposition, relating filtrations of $\hat{Z}_r(\lambda)$ and $\operatorname{ind}_B^G(\lambda)$. We shall denote this induced module by $\nabla(\lambda)$, and the corresponding classical module by $\bar{\nabla}(\lambda)$. We set $X(T)^+ = \{\lambda \in X(T) : \nabla(\lambda) \neq 0\}$, and note that this is described explicitly in [6, Lemma 3.2].

Proposition 5.2 *Given $\lambda \in X(T)^+$, suppose that each composition factor of $\hat{Z}_r(\lambda)$ has the form $\hat{L}_r(\mu' + lp^{r-1}\mu'')$, with $\mu' \in P(D)$ and $\mu'' \in X(T)$, such that $\langle \mu'' + \rho, \alpha\check{\ } \rangle \geq 0$ for all $\alpha \in \Pi$. Then $\nabla(\lambda)$ has a filtration with factors of the form $L(\mu') \otimes \bar{\nabla}(\mu'')^{F^r}$, with $\mu' \in P(D)$ and $\mu'' \in X(T)^+$. Each such module occurs as often as $\hat{L}_r(\mu' + lp^{r-1}\mu'')$ occurs in a composition series of $\hat{Z}_r(\lambda)$.*

Proof: We first note that $\nabla(\lambda) \cong \operatorname{ind}_{G_rB}^G \hat{Z}_r(\lambda)$, as in [14, 9.8 Lemma]. The result now follows, by the previous lemma and Kempf's Vanishing Theorem ([6, Theorem 3.4]), just as in the classical case (see [14, II 9.11 Proposition]).

Consider $\lambda \in X(T)$, not equal to $-\rho$. We define $m(\lambda)$ to be the least positive integer such that there exists an $\alpha \in \Phi^+$ with $\langle \lambda + \rho, \alpha\check{\ } \rangle \notin lp^{m(\lambda)}\mathbb{Z}$.

Corollary 5.3 *Let λ, $\mu \in X(T)$:*

i) if $\hat{L}_r(\mu)$ is a composition factor of $\hat{Z}_r(\lambda)$, then $\mu \in W.\lambda + lp^{\min(m,r-1)}\mathbb{Z}\Phi$;

ii) if $L_r(\mu)$ is a composition factor of $Z_r(\lambda)$ then $\mu \in W.\lambda + lp^m\mathbb{Z}\Phi + lp^{r-1}X(T)$.

Proof: This is a strengthened version of the classical result [14, II 9.12 Corollary], and follows from the previous proposition just as there, but replacing the appeal to the strong linkage principle with an application of the description of the blocks of G in [2, Theorem 5.14].

Lemma 5.4 *For all λ, $\mu \in X(T)$,*

$$\operatorname{Ext}^i_{G_r}(L_r(\lambda), L_r(\mu)) = \bigoplus_{\tau \in X(T)} \operatorname{Ext}^i_{G_rT}(\hat{L}_r(\lambda + lp^{r-1}\tau), \hat{L}_r(\mu)).$$

Proof: This follows just as in [14, I 6.9(5)], once we note that (by the remarks before [7, 3.1(9)]) G_r and G_rT satisfy the hypotheses of [6, Proposition 1.6], giving the required spectral sequence.

We can now give one of the desired inclusion of blocks.

Lemma 5.5 *For λ, $\mu \in X(T)$:*

i) if $\mathrm{Ext}^1_{G_rT}(\hat{L}_r(\lambda), \hat{L}_r(\mu)) \neq 0$, then $\mu \in W.\lambda + lp^{\min(m,r-1)}\mathbb{Z}\Phi$;

ii) if $\mathrm{Ext}^1_{G_r}(L_r(\lambda), L_r(\mu)) \neq 0$, then $\mu \in W.\lambda + lp^m\mathbb{Z}\Phi + lp^{r-1}X(T)$.

Proof: To define a contravariant duality as described before [16, (11.1.3)], we note that the coalgebra anti-automorphism used there translates via [10, Proposition 2.1 and Theorem 2.4] to one for the Dipper–Donkin quantisation. By considering the explicit description of this, it is clear that it now restricts to an anti-automorphism of G_rT. Then arguing as in [14, II 2.12] we see that for all $i \in \mathbb{N}$ and λ, $\mu \in X(T)$ we have

$$\mathrm{Ext}^i_{G_rT}(\hat{L}_r(\lambda), \hat{L}_r(\mu)) \cong \mathrm{Ext}^i_{G_rT}(\hat{L}_r(\mu), \hat{L}_r(\lambda)).$$

With this, the lemma now follows from the previous two results just as in [14, II 9.16 Lemma].

For the reverse inclusion we will need a few technical lemmas. The first of these is a straightforward adaptation of the corresponding calculation in [14, page 329].

Lemma 5.6 *For all $\lambda \in X(T)$ and $w \in W$, there exists a $\tau \in X(T)$ such that $\lambda - lp^{r-1}\tau$ and $w.\lambda - lp^{r-1}w\tau$ are linked as G_rT-weights.*

Proof: By [7, 3.1(20)ii)] we have

$$\mathrm{ch}\hat{Z}_r(\lambda) = e(\lambda - (lp^{r-1} - 1)\rho)\chi((lp^{r-1} - 1)\rho).$$

Hence, as $\chi((lp^{r-1} - 1)\rho) \in \mathbb{Z}[X(T)]^W$, we have

$$\begin{aligned} \mathrm{ch}\hat{Z}_r(w.\lambda + lp^{r-1}\rho) &= e(w(\lambda + \rho))\mathrm{ch}\hat{Z}_r(lp^{r-1}\rho) \\ &= w[e(\lambda + \rho)\mathrm{ch}\hat{Z}_r(lp^{r-1}\rho)], \end{aligned}$$

and so

$$\mathrm{ch}\hat{Z}_r(w.\lambda + lp^{r-1}\rho) = w\mathrm{ch}\hat{Z}_r(\lambda + lp^{r-1}\rho). \tag{8}$$

Now any $\mu \in X(T)$ can be written uniquely in the form $\mu = \mu' + lp^{r-1}\mu''$, with $\mu' \in P_r(D)$ and $\mu'' \in X(T)$, so for any finite dimensional module M we have

$$\mathrm{ch}M = \sum_{\mu' \in P_r(D)} \sum_{\mu'' \in X(T)} [M : \hat{L}_r(\mu)] e(lp^{r-1}\mu'') \mathrm{ch}L(\mu').$$

Taking $M = \hat{Z}_r(\lambda + lp^{r-1}\rho)$ and applying w, we see from (8) that

$$\mathrm{ch}\hat{Z}_r(w.\lambda + lp^{r-1}\rho) = \sum_{\mu' \in P_r(D)} \sum_{\mu'' \in X(T)} [\hat{Z}_r(\lambda + lp^{r-1}\rho) : \hat{L}_r(\mu)] e(lp^{r-1}w\mu'') \mathrm{ch}L(\mu').$$

Comparing coefficients for $M = \hat{Z}_r(w.\lambda + lp^{r-1}\rho)$ we see that

$$[\hat{Z}_r(\lambda + lp^{r-1}\rho) : \hat{L}_r(\mu)] = [\hat{Z}_r(w.\lambda + lp^{r-1}\rho) : \hat{L}_r(\mu' + lp^{r-1}w\mu'')]. \tag{9}$$

Hence, by tensoring up with suitable one-dimensional modules, we obtain that

$$[\hat{Z}_r(\lambda - lp^{r-1}\mu'') : \hat{L}_r(\mu' - lp^{r-1}\rho)] = [\hat{Z}_r(w.\lambda - lp^{r-1}w\mu'') : \hat{L}_r(\mu' - lp^{r-1}\rho)].$$

Now taking $\tau = \mu''$ for some μ for which the left hand side of (9) is non-zero gives the result.

Lemma 5.7 *For $\lambda \in X(T)$, if $\langle \lambda + \rho, \alpha\check{} \rangle \in \mathbb{Z}lp^{r-1}$ for all $\alpha \in \Pi$ then $\hat{Z}_r(\lambda)$ is simple.*

Proof: This follows just as in [14, II 11.8 Lemma], using [7, 3.1(22), 3.1(13)(i), and 3.1(20)(ii)].

For our next lemma, it is necessary to restrict to the case when $n = 2$. However, as our result on the classical blocks only holds in this case, this will be sufficient for our needs. Recall that we denote the unique simple root in this case by α. We will also set $\theta(m) = \begin{cases} lp^i & \text{if } i \geq 0, \\ 1 & \text{if } i = -1. \end{cases}$

Lemma 5.8 *For $\lambda \in X(T)$, if $\langle \lambda + \rho, \alpha\check{} \rangle = alp^{m-1} + b\theta(m-2)$ for some $1 \leq m \leq r$, $a \in \mathbb{Z}$ and $0 < b < p$ (or $0 < b < l$ if $m = 1$), then*

$$[\hat{Z}_r(\lambda) : \hat{L}_r(\lambda - b\theta(m-2)\alpha)] \neq 0.$$

Proof: We first note that, by [7, 3.1(20)(ii)], we have

$$\mathrm{ch}\hat{Z}_r(\lambda) = e(\lambda)[1 + e(-\alpha) + \cdots + e(-(lp^{r-1} - 1)\alpha)].$$

Now assume that $m > 1$. Then we have

$$\begin{aligned} e(\lambda_1, \lambda_2) &= e(\lambda_2 + lp^{m-2} - 1, \lambda_2)e(alp^{m-1} + blp^{m-2} - 1, 0) \\ &= e(\lambda_2 + lp^{m-2} - 1, \lambda_2)e(ap + b - 1, 0)^{F^{m-1}}. \end{aligned}$$

Similarly,

$$[1 + \cdots + e(-(lp^{r-1} - 1)\alpha)] = [1 + \cdots + e(-(lp^{m-2} - 1)\alpha)][1 + \cdots + e(-(p^{r+1-m} - 1)\alpha)]^{F^{m-1}},$$

and hence we obtain that

$$\mathrm{ch}\hat{Z}_r(\lambda) = [\mathrm{ch}\hat{Z}_{m-1}(\lambda_2 + lp^{m-2} - 1, \lambda_2)][\mathrm{ch}\bar{Z}_{r+1-m}(ap + b - 1, 0)]^{F^{m-1}},$$

where $\bar{Z}_s(\mu)$ is the classical induced module for the sth Jantzen subgroup of $GL(2, k)$. Now $\hat{L}_{m-1}(\lambda_2 + lp^{m-2} - 1, \lambda_2) \cong_{G_{m-1}T} L(\lambda_2 + lp^{m-2} - 1, \lambda_2)$, which has dimension lp^{m-2} by Steinberg's tensor product theorem [7, 3.2(5)]. Hence $\hat{Z}_{m-1}(\lambda_2 + lp^{m-2} - 1, \lambda_2) \cong \hat{L}_{m-1}(\lambda_2 + lp^{m-2} - 1, \lambda_2)$. Again by Steinberg's tensor product theorem, the result will now follow in this case if we can show that

$$[\bar{Z}_{r+1-m}(ap + b - 1, 0) : \bar{L}_{r+1-m}((a - 1)p + p - b - 1)] \neq 0,$$

where $\bar{L}(\mu)$ is the usual simple module for $GL(2, k)$. But this follows from the calculations in [13, Section 5.5].

We now consider the case $m = 1$. Now $[\hat{Z}_r(\lambda) : \hat{L}_r(\lambda)] = 1$ by [7, 3.1(13)(i) and (20)(ii)], so we consider $\mathrm{ch}\hat{Z}_r(\lambda) - \mathrm{ch}\hat{L}_r(\lambda)$. Writing $a = a' + p^{r-1}a''$ with $0 \leq a' < p^{r-1}$ we have that

$$\hat{L}_r(\lambda) \cong_{G_rT} L(\lambda_2 + b - 1, \lambda_2) \otimes \bar{L}(a')^F \otimes lp^{r-1}a'',$$

and so, as $b < l$, the highest remaining weight in $\mathrm{ch}\hat{Z}_r(\lambda) - \mathrm{ch}\hat{L}_r(\lambda)$ is

$$(\lambda_2 + (a - 1)l + l - 1, \lambda_2 + b) = \lambda - b\alpha$$

as required.

We are now able to determine the desired blocks. As in the classical case, we denote the blocks of G_rT and G_r containing λ by $\hat{\mathcal{B}}_r(\lambda)$ and $\mathcal{B}_r(\lambda)$ respectively.

Theorem 5.9 *For $n = 2$, $r > 0$ and $\lambda \in X(T)$, we have*

$$\hat{\mathcal{B}}_r(\lambda) = \begin{cases} W.\lambda + lp^m\mathbb{Z}\Phi & \text{if } m \leq r-1, \\ \{\lambda\} & \text{if } m > r-1, \end{cases}$$

and

$$\mathcal{B}_r(\lambda) = W.\lambda + lp^m\mathbb{Z}\Phi + lp^{r-1}X(T).$$

Proof: We first consider the G_rT case. For $m > r-1$ the result follows from (5.7). For $m \leq r-1$, one inclusion comes from (5.5). For the reverse inclusion, given two weights in $W.\lambda + lp^{m-1}\mathbb{Z}\Phi$, we use (5.6) and (5.8) to construct a chain of weights linking them in the G_rT case. Finally we deduce the G_r case from the G_rT result using (5.4).

The determination of the blocks of the infinitesimal q-Schur algebras (in the case $n = 2$) will now follow just as in the classical case described earlier, once we have verified a few remaining technical results. We first collect together those results whose proofs are just appropriate modifications of the G_1T results obtained in [7].

Lemma 5.10 *For $\lambda = \lambda' + lp^{r-1}\lambda'' \in X(T)$, with $\lambda' \in P_r(D)$ and $\lambda'' \in X(T)$, we have*

i) $\hat{Q}_r(\lambda) \cong \hat{Q}_r(\lambda') \otimes lp^{r-1}\lambda''$;

ii) all weights of $\hat{Z}_r(\lambda)$ satisfy $\lambda - 2(lp^{r-1} - 1)\rho \leq \mu \leq \lambda$.

Proof: See [7, 3.2(10)(ii) and 3.1(20)(ii)] respectively.

It now only remains to check

Lemma 5.11 *For $\lambda = \lambda' + lp^{r-1}\lambda'' \in X(T)$ with $\lambda' \in P_r(D)$ and $\lambda'' \in X(T)$ we have*

$$\hat{Q}_{r+1}(\lambda) \cong_{G_rT} \hat{Q}_r(\lambda') \otimes \hat{Q}_1(\lambda'')^{F^r}.$$

Proof: This follows just as in [14, II 11.15 Lemma], once we have shown that $\hat{Q}_{r+1}(\lambda)$ is injective as a G_rT module, and that the appropriate spectral sequence exists. Set $H = G_{r+1}T$, and denote by $\bar{H}$ the factor group generated by $d_q^{-lp^{r-1}}$ and the $c_{ij}^{lp^{r-1}}$, for all $1 \leq i, j \leq n$. It is routine to check that this is a sub-Hopf algebra, indeed $\bar{H} \cong GL(n,k)_1T$ under the map taking $c_{ij}^{lp^{r-1}} \longmapsto x_{ij}$ and $d_q^{-lp^{r-1}} \longmapsto d^{-1}$. The corresponding subgroup H_1 (in the notation of [6, Section 1]) has defining ideal generated by the elements $c_{ij}^{lp^{r-1}} - \delta_{ij}$ and $d_q^{-lp^{r-1}} - 1$, for all $1 \leq i, j \leq n$. Hence $H_1 \cong G_r$.

Arguing as in [3, (1.3.3)], we see that $k[H]$ is free (so certainly faithfully flat) as a $k[\bar{H}]$-module. So by [6, Proposition 1.6] we get the spectral sequence required in the proof of the lemma. Now by [6, Proposition 1.5], or the main theorem in [5], $\hat{Q}_{r+1}(\lambda)$ is an injective G_r-module. Also, by [7, 3.1(9)], $\mathrm{Ind}_{G_r}^{G_rT}$ is exact so, as σ_H and σ_{H_1} are anti-automorphisms (see [6, Remark 2.2]), we have that $\hat{Q}_{r+1}(\lambda)$ is an injective G_rT-module by [16, (2.9.1)].

Now the arguments from Section 2, along with the above results and [2, Theorem 5.3], gives

Theorem 5.12 *For $n = 2$ and $d \geq 0$ we have for all $\lambda \in \Gamma_r^d(D)$ that*

$$\mathcal{B}_r^d(\lambda) = \hat{\mathcal{B}}_r(\lambda) \cap \Gamma_r^d(D).$$

Acknowledgements: I would like to thank Stephen Donkin for a number of helpful discussions and comments. This work was supported by the EPSRC.

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