Decomposition numbers for distant Weyl modules

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Consider a semisimple, connected, simply-connected algebraic group G over an algebraically closed field k of characteristic $p > 0$. One can construct for each dominant weight λ a Weyl module $\Delta(\lambda)$ with that highest weight whose character is given by Weyl's character formula. Although not in general simple, $\Delta(\lambda)$ has a simple head $L(\lambda)$, and all simple modules arise in this manner.

Knowledge of the decomposition numbers $d_{\lambda\mu} = [\Delta(\lambda) : L(\mu)]$ for λ and μ 'small' (i.e. p-restricted) is equivalent to calculating the characters of the corresponding simple modules — and hence by Steinberg's tensor product theorem to determining the characters of all the simples. Consequently, much work has been undertaken to try to determine these numbers, concentrating mainly on the case when p is large enough to be able to consider the Lusztig conjecture. Indeed, for sufficiently large primes the $d_{\lambda\mu}$ are now known by the work of Andersen, Jantzen and Soergel [1].

Although in principle all decomposition numbers can be determined from those for p-restricted weights — via character calculations using the tensor product theorem and Weyl's character formula — this is not straightforward in practice. Further, it is often more convenient to know decomposition numbers than characters; for example when relating representations of the general linear and symmetric groups via Ringel duality only the former can be translated between the two categories.

We shall consider the situation where λ is 'large', and give an elementary algorithm for calculating decomposition numbers given those for all p^2-restricted weights. If we regard Steinberg's tensor product theorem as an algorithm for determining large characters

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from smaller ones, then this is an analogous result for decomposition numbers. Our algorithm can be easily inverted, and we discuss an application of this to the representation theory of the symmetric group using Ringel duality.

There is another, similar, recursive character formula for Weyl modules due to Jantzen [13]. Away from the boundary of the dominant region this corresponds to a filtration of $\Delta(\lambda)$. This is obtained by considering representations of certain induced modules for infinitesimal subgroups G_rT of G related to the Frobenius kernels. Doty and Sullivan [9] have given an algorithm for determining decomposition numbers for these induced modules, and have shown how the corresponding result for Weyl modules can be deduced from this.

In order to describe our algorithm, we introduce certain sets of virtual decomposition factors with multiplicities. Although these arise naturally in our argument, this is essentially a combinatorial procedure — and hence it is not immediately clear that such sets have any representation-theoretic interpretation (or even that the associated multiplicities are non-negative). However, we shall show that they are precisely the composition factors (with multiplicities) of certain modules studied by Lin [16] arising as lifts of modules from corresponding quantum groups at a primitive p^2 root of unity. More generally, Lin considers the lifts of modules from quantum groups at p^r roots of unity, and our algorithm can also be used to determine the decomposition numbers for these modules.

In the light of these results, it is natural to ask if our algorithm corresponds to successive refinements of some filtration of the Weyl module. This seems to be related to a conjecture of Humphreys [11] concerning filtrations of Weyl modules, which we briefly discuss. We then consider evidence for such a structural interpretation, arising from results of Doty [8] on the submodule structure of the symmetric powers and of Kühne-Hausmann [15] on the structure of suitably 'generic' Weyl modules for SL_3.

Finally, we conclude by noting that an appropriate analogue of our algorithm can also be derived for the quantum general linear group at a root of unity in positive characteristic.

1. PRELIMINARIES

In this section we shall briefly review those basic results that will be required later, mainly so as to fix our notation. All of this material can be found in [14, II, Chapters 1–6]. Towards the end we shall also prove an elementary proposition on the geometry of lattice points in facets that will be needed in the next section.

We fix a maximal torus $T \subset G$, and hence the lattice of weights $X(T)$. The pair (G, T) determines a root system R, inside which we choose a set of positive roots R^+. The corresponding set of simple roots we denote by S. The Weyl group W and associated affine Weyl groups W_{p^i} act on the space $E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

More precisely, let $\alpha\check{}$ be the coroot associated to α in $X(T)^*$, and $\langle -, - \rangle$ the usual bilinear form on $X(T) \times X(T)^*$. For each $\alpha \in R$ we denote by s_α the reflection on $X(T)$ given by $s_\alpha\lambda = \lambda - \langle \lambda, \alpha\check{} \rangle\alpha$. This action extends to the whole of E. Then W is just the group generated by these reflections. For $i \geq 1$ we define W_{p^i}, the affine Weyl group, to be the semidirect product of W with the group $p^i\mathbb{Z}R$ (acting by translations on E).

Let $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$, an element of $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. It is easy to verify that the dot action $w.\lambda = w(\lambda + \rho) - \rho$ of W (or W_{p^i}) on E maps $X(T)$ into itself. Henceforth we shall use this action without further comment. As G is semisimple and simply-connected, the set $\{\alpha\check{} : \alpha \in S\}$ is a basis for $X(T)^*$, and there is a corresponding basis $\{\omega_\alpha : \alpha \in S\}$ of the fundamental weights for $X(T)$, such that $\langle \omega_\alpha, \beta\check{} \rangle = \delta_{\alpha\beta}$ for all simple roots α and β. This further implies that $\rho = \sum_{\alpha \in S} \omega_\alpha \in X(T)$.

The action of W_{p^i} on E defines a system of p^i-facets; these are sets of the form

$$F = \left\{ \lambda \in E : \begin{array}{rl} \langle \lambda + \rho, \alpha\check{} \rangle = n_\alpha p^i & \text{for all } \alpha \in R_0^+(F) \\ (n_\alpha - 1)p^i < \langle \lambda + \rho, \alpha\check{} \rangle < n_\alpha p^i & \text{for all } \alpha \in R_1^+(F) \end{array} \right\}$$

for suitable integers n_α and a disjoint decomposition $R^+ = R_0^+(F) \cup R_1^+(F)$. A facet F is called an alcove if $R_0^+(F) = \emptyset$, and a wall if $|R_0^+(F)| = 1$. The closure $\bar{F}$ of any alcove F is a fundamental domain for W_{p^i} on E, and W_{p^i} permutes the alcoves simply transitively. Similarly, $\bar{F} \cap X(T)$ is a fundamental domain for W_{p^i} on $X(T)$. Thus it will often be sufficient to study just the standard alcove C_i, where $n_\alpha = 1$ for all α. We will also need to consider the set of

p^i-restricted weights

$$X_i(T) = \{\lambda \in X(T) : 0 < \langle \lambda + \rho, \alpha\check{} \rangle \leq p^i \quad \text{for all } \alpha \in S\}.$$

Clearly, $X(T)$ is a disjoint union of translates of this set by the p^i-weight lattice $p^i X(T)$, and $X_i(T)$ is a union of sets of the form $X(T) \cap F$ for certain p^i-facets F of W_{p^i}. Any weight λ can be uniquely written in the form $\lambda = \lambda' + p^i \lambda''$ with $\lambda' \in X_i(T)$, and any decomposition of λ in this way is to be assumed to be of this form.

Key to our arguments will be the notion of scaling. For each weight λ and p^i-facet F, there is at most one element of $W_{p^i}.\lambda$ in $X(T) \cap F$. Thus it is enough to identify the p^i-facet in which a weight lies and its orbit under W_{p^i} to determine the weight itself. Let $\varepsilon_i : E \longrightarrow E$ be the map $x \longmapsto p^{i-1}(x + \rho) - \rho$. Note that ε_i is a bijection taking p-facets to p^i-facets, and that under this bijection the W_p-orbit of x corresponds to the W_{p^i}-orbit of $\varepsilon_i(x)$.

We may identify W_{p^i} and W_{p^j} via the isomorphism induced from the obvious isomorphism between $p^i \mathbb{Z}R$ and $p^j \mathbb{Z}R$. Given an element $w \in W_p$ we may denote its image in W_{p^i} under this identification by $w^{(i)}$. It is now easy to verify that $\varepsilon_i(w^{(j)}.\lambda) = w^{(i+j-1)}.\varepsilon_i(\lambda)$, and in particular that ε_i commutes with the dot action of W. We also have that $\varepsilon_i(\lambda + \mu) = \varepsilon_i(\lambda) + p^{i-1}\mu$ for all weights λ and μ.

We will often regard p^i-facets as though they are p-facets (by means of ε_i^{-1}), and use certain combinatorics of p-facets associated to W_p to determine a new family of p-facets, and hence of p^i-facets via ε_i. By the remarks above, given a weight λ in our original p^i-facet, this will unambiguously determine a corresponding set of weights in the p^i-facets thus obtained. We shall refer to the identification of p^i-facets with p-facets (and vice versa) via ε_i as *scaling*, and given a p-facet F shall call $\varepsilon_i(F)$ the *p^i-facet corresponding to F*.

For our later work, it will be important to know when the intersection of a p^i-facet with the weight lattice is non-empty. We shall abuse notation and say that such facets are *non-empty*. Set $h = \max\{\langle \rho, \beta\check{} \rangle + 1 : \beta \in R^+\}$. As G is connected this is just the Coxeter number of R. It is well known that a p^i-alcove contains a lattice point if and only if $p^i \geq h$, and as G is simply connected the same is also true for walls (see [14, II 6.3(1)]). We will need the following easy generalisation of these results (our proof follows that in [14, II 6.3(1)]).

PROPOSITION 1.1. *Suppose that $p \geq h$. A p-facet F is non-empty if and only if the corresponding p^i-facets (under scaling) are non-empty (for all i).*

Proof. It is clear that if F contains a lattice point, then so does $\varepsilon_i(F)$ for all i. Thus it is enough to show that if F does not contain a lattice point, then neither does $\varepsilon_i(F)$ for any i. The action of W_p on p-facets corresponds under scaling to the action of W_{p^i} on p^i-facets and hence, by the conjugacy of alcoves under W_p, we may assume that $F \subset \bar{C}_1$.

For each wall in $\bar{C}_1$, there is a unique reflection that fixes it. As noted in [14, II 6.3], the set of such reflections consists of all s_α with α simple, and $s_{\beta,p}$, where β is the longest short root of R. Here $s_{\beta,p}$ is the reflection that fixes those λ satisfying $\langle \lambda + \rho, \beta\check{}\rangle = p$. Now every facet F in $\bar{C}_1$ can be identified with a distinct subset of these reflections by setting $\mathrm{Fix}(F)$ to be the set of those reflections which fix F pointwise. For example, $\mathrm{Fix}(C_1)$ is the empty set.

Any element of $X(T)$ can be written in the form $\lambda = \sum_{\alpha \in S} m_\alpha \omega_\alpha$ and so

$$\langle \lambda + \rho, \alpha\check{}\rangle = m_\alpha + 1$$

for all simple roots α (by the explicit expression for ρ in terms of the ω_α). Also, if we write the coroot associated to the longest short root β in the form $\beta\check{} = \sum_{\alpha \in S} b_\alpha \alpha\check{}$ then for any other root γ with $\gamma\check{} = \sum_{\alpha \in S} c_\alpha \alpha\check{}$ we have $c_\alpha \leq b_\alpha$ for all $\alpha \in S$ (as $\beta\check{}$ is the maximal (long) root in the dual root system $R\check{}$).

First suppose that our facet is not fixed by $s_{\beta,p}$. Then to contain a lattice point we require that there exists $\lambda = \sum_{\alpha \in S} m_\alpha \omega_\alpha$ such that $\langle \lambda + \rho, \alpha\check{}\rangle$ is zero for all $\alpha \in \mathrm{Fix}(F)$ and strictly between 0 and p for all other roots not in the linear span of $\mathrm{Fix}(F)$. Consider $\lambda = \sum_{\alpha \in \mathrm{Fix}(F)} -\omega_\alpha$. Clearly $\langle \lambda + \rho, \alpha\check{}\rangle = 0$ for all $\alpha \in \mathrm{Fix}(F)$, and for all roots γ not in the linear span of $\mathrm{Fix}(F)$ we have

$$0 < \langle \lambda + \rho, \gamma\check{}\rangle \leq \langle \rho, \gamma\check{}\rangle \leq \langle \rho, \beta\check{}\rangle < h \leq p.$$

Thus every facet not fixed by $s_{\beta,p}$ contains a lattice point.

So it only remains to consider those facets F fixed by $s_{\beta,p}$. Let $\mathrm{SFix}(F)$ be the set of reflections in $\mathrm{Fix}(F)$ not equal to $s_{\beta,p}$. For F to contain a lattice point we require that there exists a $\lambda = \sum_{\alpha \in S} m_\alpha \omega_\alpha$ such that $\langle \lambda + \rho, \alpha\check{}\rangle$ is zero for all $\alpha \in \mathrm{SFix}(F)$, equal to p when

$\alpha = \beta$, and strictly between 0 and p for all other roots not in the linear span of $\mathrm{Fix}(F)$.

Arguing as in the last paragraph, it is easy to see that it is enough to solve the equation

$$\sum_{\alpha \in S \setminus \mathrm{SFix}(F)} (m_\alpha + 1) b_\alpha = p \tag{1}$$

for some integers m_α satisfying $0 \leq m_\alpha < p - 1$ for each $\alpha \in S \setminus \mathrm{SFix}(F)$, as setting $m_\alpha = -1$ for all $\alpha \in \mathrm{SFix}(F)$ will then give the desired λ.

The proof now reduces to a case by case examination of the possible values of the b_α for each root systems. Using the tables given in [2, Planches I–IX] it is easy to verify for each root system that there is a solution of (1) for $p \geq h$ whenever the highest common factor of the b_α for $\alpha \in S \setminus \mathrm{SFix}(F)$ is 1. Thus in these cases there is always a lattice point in F. By inspection, when the highest common factor is greater than 1 it must be less than h (and hence less than p), and so in this case there is no lattice point in F. However in this case exactly the same argument holds for the p^i-facets (as there is still no solution to (1) when we replace p by p^i), and so the result now follows.

We conclude this section by recalling the basic properties of simple and Weyl modules that we shall require. Given a Borel $T \subset B \subset G$ we can define the modules $H^i(\lambda) = R^i\mathrm{ind}_B^G k_\lambda$, where k_λ is the one-dimensional B-module of weight λ, and $R^i\mathrm{ind}_B^G$ is the ith right derived functor of induction. We set $\chi(\lambda) = \sum_{i \geq 0} (-1)^i \mathrm{ch} H^i(\lambda)$.

By choosing B appropriately, we may arrange that $H^0(\lambda)$ is non-zero precisely when λ is dominant, and for these weights $\chi(\lambda) = \mathrm{ch} H^0(\lambda)$ by Kempf's vanishing theorem. The Weyl module $\Delta(\lambda)$ is the contravariant dual of $H^0(\lambda)$, and has the same character, which is given by Weyl's character formula.

We will frequently use the following properties of χ (see [14, II 5.8–9]).

Lemma 1.2. *For all $\lambda \in X(T)$, $w \in W$ and $\sum_\mu a(\mu)e(\mu) \in \mathbb{Z}[X(T)]^W$ we have*

$$\chi(\lambda) \sum_\mu a(\mu) e(\mu) = \sum_\mu a(\mu) \chi(\lambda + \mu)$$

$$\chi(w.\lambda) = (-1)^{l(w)}\chi(\lambda).$$

Note that for each element λ, either $\chi(\lambda) = 0$, or there exists a unique element $w_\lambda \in W$ such that $w_\lambda.\lambda$ is a dominant weight. When $\chi(\lambda) = 0$, we set $w_\lambda = 1$.

Finally, we note that each Weyl module $\Delta(\lambda)$ has a simple head $L(\lambda)$ (whose character is W-invariant), and that all simple modules can be obtained in this manner. We will often abuse notation and refer to weights as composition factors by identifying λ with the module $L(\lambda)$. Any dominant weight λ can be uniquely written in the form $\lambda = \sum_{i\geq 0} \lambda_i p^i$ with $\lambda_i \in X_1(T)$ for all i. Then by Steinberg's tensor product theorem we have $L(\lambda) \cong \bigotimes_i L(\lambda_i)^{\mathrm{F}^i}$, where F is the Frobenius morphism.

2. THE MAIN THEOREM

Throughout this section we shall assume that the decomposition numbers for Weyl modules with highest weight in the set of p^2-restricted weights are known. In examples we shall only consider Weyl modules whose highest weight lies in the interior of an alcove — however our main result holds for all weights without restriction.

Henceforth we will assume that $p \geq h$. This will allow us to appeal to the translation principle [14, II 7.17 Corollary], and note that the facets containing composition factors of a Weyl module $\Delta(\lambda)$ depend only on the facet in which λ lies. Under this hypothesis, Proposition 1.1 will also ensure that the p-facet corresponding to a non-empty p^i-facet will also be non-empty. We will repeatedly make use of both of these properties without further comment. By considering the Steinberg weight $(p^i - 1)\rho$, it is easy to see that for $p \geq h$ we also have $X_i(T) \subseteq \bar{C}_{i+1}$, for all $i > 0$.

We will associate to each weight λ a set of i-virtual composition factors (with multiplicities). As this is a somewhat lengthy process, we will proceed in several stages. We begin by associating to each non-empty p-facet F containing a dominant weight a *decomposition diagram*. This is defined to be a set of facets H with multiplicities d_{FH} defined by picking an arbitrary weight $\lambda \in F$ and determining which facets contain composition factors of $\Delta(\lambda)$. These are the facets of the decomposition diagram, and their multiplicities are just those of the corresponding composition factors.

Next consider the set of p^2-facets inside the set of p^2-restricted weights. To each such facet $\varepsilon_2(F)$ that is non-empty, we define a p^2-*decomposition diagram* in the following manner. Under scaling, $\varepsilon_2(F)$ corresponds to the (non-empty) p-facet F, which has an associated decomposition diagram. The p^2-decomposition diagram associated to $\varepsilon_2(F)$ is just the set of p^2-facets (with multiplicities) corresponding to this diagram under scaling.

To each non-empty p-facet F in the set of p^2-restricted weights, we now associate a *virtual decomposition diagram.* We proceed by induction on the p^2-facets below the p^2-facet containing F (using the partial ordering on facets induced by the usual dominance ordering on weights). The set of p-facets in the virtual decomposition diagram for F are just those E for which the virtual decomposition number

$$c_{FE} = d_{FE} - \sum_{I<H} d_{HI} c_{JE} \neq 0 \tag{2}$$

where F lies in the p^2-facet $\varepsilon_2(H)$ and J is the image of F under W_{p^2} in $\varepsilon_2(I)$. The multiplicity of such a facet E is just c_{FE}.

It is possible for some of the facets J in (2) to lie outside $X_2(T)$. Thus for our inductive definition to make sense, we also need to define virtual decomposition numbers c_{JE} for such facets. Any such J can be uniquely written in the form $J = J' + p^2\tau$, where J' is a p-facet in $X_2(T)$ and $\tau \in X(T)$. The virtual decomposition numbers $c_{J'K}$ are already defined by induction, and we set $c_{JE} = c_{J'E'}$, where $E = E' + p^2\tau$.

Given a p^2-restricted weight λ, the set of virtual composition factors associated to λ is just the set of elements of $W_p.\lambda$ lying in some p-facet of the virtual decomposition diagram, with the corresponding multiplicities.

Before giving the definition of i-virtual composition factors, we shall illustrate the above definitions with a few examples concerning alcoves. For SL_2, there is only one p^2-alcove in the set of p^2-restricted weights. This corresponds under scaling to the unique p-restricted alcove C_1, whose associated decomposition diagram is just C_1. Thus the p^2-decomposition diagram associated to $\varepsilon_2(C_1)$ is just $\varepsilon_2(C_1)$, and the virtual decomposition diagram associated to a p-alcove F in the set of p^2-restricted weights consists of those E for which

$$c_{FE} = d_{FE} \neq 0.$$

More generally, for any group G, the p^2-alcove C_2 is its own p^2-decomposition diagram, and hence for any p-facet F in C_2 we have $c_{FE} = d_{FE}$. Thus for any weight in C_2, the virtual composition factors are just the usual composition factors of the associated Weyl module.

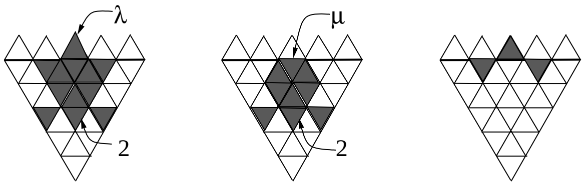

FIGURE 1. (a), (b) and (c)

For a non-trivial example, consider SL_3 with $p = 5$, and a weight λ in an alcove just above the lowest p^2-alcove (as shown in Figure 1(a)). Now λ lies in $\varepsilon_2(D)$, where D is the upper alcove in the set of p-restricted weights. The decomposition diagram associated to D is just D and C_1, each with multiplicity one. Thus for the p-alcove F containing λ, (and any p-facet E) we have

$$c_{FE} = d_{FE} - d_{DC_1} c_{JE} = d_{FE} - c_{JE} = d_{FE} - d_{JE}$$

where J is the image of F in $\varepsilon_2(C_1)$ under W_{p^2}. For λ in F as above, the p-alcove J is that containing μ in Figure 1(b). The decomposition diagrams for F and J are given in Figures 1(a) and (b) respectively, and so the virtual decomposition diagram associated to F is that given in Figure 1(c). (All multiplicities are 1 unless otherwise indicated.) The virtual composition factors associated to λ are just those weights in $W_p.\lambda$ lying in this final diagram.

Returning to our definitions, we next associate to each p^{i+1}-restricted weight λ a set of i-virtual composition factors (with multiplicities). When $i = 1$ these will just be the virtual composition factors defined above. Regard the p^i-facets as p-facets by scaling. Then the p^i-facet $\varepsilon_i(F)$ containing λ corresponds to the p-facet F in the set of p^2-restricted weights. We have already associated to F a virtual decomposition diagram. By scaling, we obtain a corresponding set of p^i-facets with multiplicities. This is the *i-virtual decomposition diagram* associated to $\varepsilon_i(F)$, and the *i-virtual composition factors* associated to λ are just those weights in $W_{p^i}.\lambda$ that lie in these p^i-facets. We shall denote the corresponding multiplicity of such a weight μ by $c^i_{\lambda\mu}$.

Finally, we shall associate a set of i-virtual composition factors to an arbitrary dominant weight λ. Any such weight can be uniquely written in the form $\lambda = \lambda' + p^{i+1}\lambda''$ with $\lambda' \in X_{i+1}(T)$. Now the i-virtual composition factors associated to λ are just those weights of the form $\mu = \mu' + p^{i+1}\lambda''$ where μ' runs over the set of i-virtual composition factors of λ', and $c^i_{\lambda\mu} = c^i_{\lambda'\mu'}$.

We will show that, for $p \geq h$, the following algorithm completely determines the decomposition numbers for a given Weyl module $\Delta(\lambda)$.

Algorithm 2.1.
(1) Let i be maximal such that λ does not lie in $\bar{C}_i$ and let $\mathrm{cf}(\lambda, i+1) = \{\lambda\}$. *(Thus λ lies in the set of p^{i+1}-restricted weights.)*
(2) If $i = 0$ then we are done, otherwise continue with step (3).
(3) For each weight μ in $\mathrm{cf}(\lambda, i+1)$ *(and keeping track of multiplicities) determine the set of i-virtual composition factors associated to μ.*
(4) Let $\mathrm{cf}(\lambda, i)$ *be the disjoint union of all the sets of virtual composition factors (with multiplicities) obtained during step (3).*
(5) Set $i = i - 1$ and repeat from step (2).

To illustrate the above algorithm we shall consider an example for SL_3. Let $p = 5$ and consider the Weyl module $\Delta(181, 0, 0)$ for GL_3 (where we use the usual partition labelling for polynomial dominant weights). This Weyl module is just the contravariant dual of a symmetric power of the natural module, and so its composition factors can be calculated using the results in [8].

The various iterations of the algorithm, and the final result, are shown in Figure 2. Note that the dotted lines indicate p-walls while the thicker lines indicate higher powers of p. As each p-alcove can contain a unique composition factor of any module, we merely indicate the alcove in which each factor lies. The small diagrams indicate

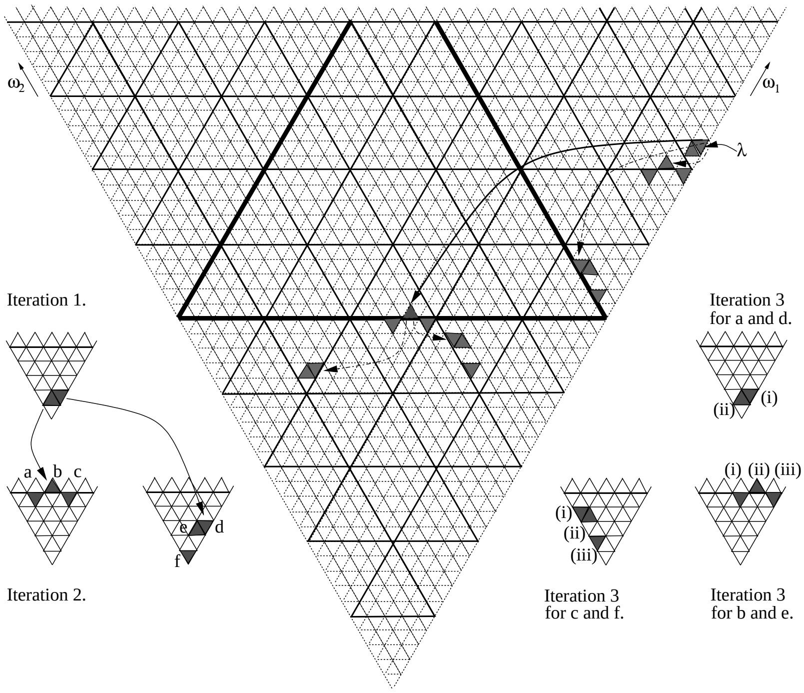

FIGURE 2.

the various virtual decomposition diagrams that arise during each iteration of the algorithm. For future reference we have labelled the composition factors that arise; for example, λ corresponds to the label $d(i)$.

We begin by considering the p^3-facets. After scaling, the p^3-facet containing λ corresponds to the upper shaded p-alcove shown in Iteration 1. Thus the first iteration of the algorithm produces λ and the element of $W_{p^3}.\lambda$ lying just above the lowest p^3-alcove (indicated by a solid arrow). After the second iteration, the virtual decomposition diagrams given in Iteration 2 arise, and after scaling (and translation for those associated to λ, since this is not a p^3-restricted weight) these give weights in the alcoves indicated by dotted arrows. The final iteration uses the virtual decomposition diagrams shown in Iteration 3 to produce the set of shaded p-alcoves shown in the central diagram.

Our main result is

Theorem 2.2. *Suppose that $p \geq h$. Given a dominant weight λ, the set of composition factors of $\Delta(\lambda)$ (counted with multiplicities) is precisely the set* $\mathrm{cf}(\lambda, 1)$ *obtained from Algorithm 2.1.*

Proof. First suppose that $\lambda \in X_2(T)$. We will show that the sum of the characters of the virtual composition factors obtained from the algorithm is $\chi(\lambda)$, as required. We first note that if $\lambda \in \overline{C}_2$, then the set of virtual composition factors associated to λ is precisely the full set of composition factors of $\Delta(\lambda)$ (by our earlier remarks), and hence the result is immediate.

If $\lambda \in X_2(T)$ does not lie in $\overline{C}_2$ then as noted at the start of the section we must have $\lambda \in \overline{C}_3$. Thus there are two iterations when the algorithm is applied to λ, and the set $\mathrm{cf}(\lambda, 2)$ is just the set of composition factors arising from the p^2-decomposition diagram. After the second iteration the multiplicity of μ in $\mathrm{cf}(\lambda, 1)$ is just

$$\sum_{\tau} d_{HI} c_{\tau\mu} = c_{\lambda\mu} + \sum_{\tau < \lambda} d_{HI} c_{\tau\mu}$$

where $\lambda \in \varepsilon_2(H)$, $\tau \in W_{p^2}.\lambda$, and $\tau \in \varepsilon_2(I)$. Comparing this with (2), we see that this equals $d_{\lambda\mu}$ as required.

Now suppose that λ is a p^{i+1}-restricted weight with $\lambda \notin \overline{C}_i$. We set $j = i - 1$ if $\lambda \in X_i(T)$, and $j = i$ otherwise. The first iteration of our algorithm begins at level i, and $\mathrm{cf}(\lambda, i+1) = \{\lambda\}$. We claim that, by Steinberg's tensor product theorem, it is enough to show that

$$\chi(\lambda) = \sum_{\mu \in \mathrm{cf}(\lambda, j)} \left(\mathrm{ch}(L(\mu'')^{\mathrm{F}^j} \right) \left(\chi(\mu') + \sum_{\tau < \mu'} a^j_{\mu'\tau} \chi(\tau) \right) \tag{3}$$

where $\mu = \mu' + p^j \mu''$ and the $a^j_{\mu'\tau}$ are defined in the following manner.

For E and F p-facets in $X_1(T)$, we define a^1_{FE} by choosing $\lambda \in F$ and solving

$$\mathrm{ch} L(\lambda) = \sum_{E \leq F} a^1_{FE} \chi(\mu_E)$$

where $\mu_E = (W_P.\lambda) \cap E$. For weights $\mu' \in \varepsilon_i(F)$ and $\tau \in \varepsilon_i(E)$ in $X_i(T)$, we define

$$a^i_{\mu'\tau} = \begin{cases} a^1_{FE} & \text{if } \tau \in W_{p^i}.\mu' \\ 0 & \text{otherwise.} \end{cases}$$

For the claim, note that for each μ on the right-hand side of (3) we have by our choice of j that $\mu' < \lambda$. Thus by induction our algorithm gives the character of $\chi(\mu')$ (and of $\chi(\tau)$ for all $\tau < \mu'$), but possibly starting at level j. The effect of starting at level $j-1$, as is the case when calculating for $\mu \in \mathrm{cf}(\lambda, j)$, is to lose the elements descended from those elements of $\mathrm{cf}(\mu, j)$ not equal to μ. After scaling, it can be seen that this corresponds to calculating decomposition numbers for the simple module in the corresponding p-alcove in $X_2(T)$ rather than of the Weyl module. Hence by the induction hypotheses and the definition of the $a^j_{\mu'\tau}$, it is enough to show (3).

Let $\mathrm{ch}\ L(\mu'') = \sum_\nu m_{\mu''\nu} e(\nu)$. Then we wish to show that

$$\begin{aligned}
&\textstyle\sum_\mu \left(\mathrm{ch}\ L(\mu'')^{\mathrm{F}^j}\right) \left(\sum_{\tau \leq \mu'} a^j_{\mu'\tau}\chi(\tau)\right) \\
&\qquad = \textstyle\sum_\mu \left(\sum_\nu m_{\mu''\nu} e(p^j\nu)\right) \left(\sum_{\tau \leq \mu'} a^j_{\mu'\tau}\chi(\tau)\right) \\
&\qquad = \textstyle\sum_\mu \sum_\nu m_{\mu''\nu} \left(\sum_{\tau \leq \mu'} a^j_{\mu'\tau}\chi(\tau + p^j\nu)\right) \\
&\qquad = \textstyle\sum_\mu \sum_\nu m_{\mu''\nu} \left(\sum_{\tau \leq \mu'} a^j_{\mu'\tau}(-1)^{l(w_{\tau\nu})}\chi(w_{\tau\nu}(\tau + p^j\nu))\right)
\end{aligned}$$

is equal to $\chi(\lambda)$ (where we write $w_{\tau\nu}$ for $w_{\tau+p^i\nu}$ for brevity). This expression is of the form

$$\chi(\lambda) + \sum_{\theta<\lambda} f_{\lambda\theta}\chi(\theta)$$

for some coefficients $f_{\lambda\theta}$ (where all the weights θ are dominant).

Now when $j = 1$ all these coefficients are zero by the calculation above, and the linear independence of the characters of Weyl modules. But in general the $f_{\lambda\theta}$ depend only on the $p^j\nu$, reflections about the boundaries of the dominant region, and the combinatorics of p^j-facets regarded as p-facets. Thus, after fixing an appropriate power of the Frobenius morphism, these coefficients depend only on the combinatorics of the p^j-facets regarded as p-facets and the weights of the $L(\mu'')$. As for $j = 1$ all the $f_{\lambda\theta}$ are zero, the same must be true for all j by scaling.

There is another recursive formula which can be used for determining the composition factors of Weyl modules, due to Jantzen [13, 3.1 Satz]. For any weight $\lambda \in X(T)$ we have

$$\chi(\lambda) = \sum_{\mu'' \in X(T)} \sum_{\mu' \in X_r(T)} [\hat{Z}_r(\lambda) : \hat{L}_r(p^r\mu'' + \mu')]\chi(\mu'')^{F^r}\mathrm{ch}(L(\mu')) \quad (4)$$

where the $\hat{Z}_r(\lambda)$ are certain coinduced modules for the Jantzen subgroup G_rT of G, and the $\hat{L}_r(\lambda)$ are the corresponding simple modules (see [14, II Chapter 9] for details). Suitably far away from the walls of the dominant region, all the weights in this sum are dominant, and the equality corresponds to a filtration of $H^0(\lambda)$ with factors of the form $L(\mu') \otimes H^0(\mu'')^{F^r}$ (see [14, II 9.11 Proposition]). However, near the boundary of the dominant region, the μ'' will not all be dominant, and although (4) can be modified using Lemma 1.2, the coefficients will now no longer all be positive.

Although the virtual composition factors associated to a single weight μ arising during Algorithm 2.1 may also (in principle) have negative multiplicities, we have

Lemma 2.3. *For any dominant weight λ, the multiplicities of the elements of* $\mathrm{cf}(\lambda, i)$ *obtained after each iteration of Algorithm 2.1 are all positive.*

Proof. Suppose there is some λ for which the lemma fails, and let i be as in Algorithm 2.1. Then there exists some $j \leq i$ such that the set F of j-virtual composition factors obtained when considering p^j-facets includes some negative multiplicities. Chose λ' such that it lies in the p-facet corresponding to the p^{i-j+1}-facet containing λ (under scaling). Then the set of composition factors of $H^0(\lambda')$ obtained using the algorithm will correspond (under scaling) to those in F, and have the same multiplicities. But these multiplicities are all positive, which gives the desired contradiction.

For there to be any possibility of obtaining a filtration of $H^0(\lambda)$ associated to our algorithm, it is clearly necessary that all the virtual composition multiplicities associated to a given weight are positive. We shall return to this question in Section 4. First however we shall exploit the easy invertibility of our algorithm.

3. RINGEL DUALITY AND REPRESENTATIONS OF THE SYMMETRIC GROUP

In this section we restrict our attention to the case where G is the general linear group GL_n. Although not itself semisimple, its representation theory can be easily deduced from that of SL_n, and so the results from the previous section apply. Associated to G are the

Schur algebras $S = S(n, d)$, and in [10] Erdmann showed how their representation theory can be related to that of the symmetric group Σ_d by Ringel duality. We shall very briefly review this relationship (details and further references can be found in [10]), and show how the invertibility of Algorithm 2.1 allows us to deduce certain results concerning representations of the symmetric group. Throughout this section we shall assume that $p > n$.

The category of $S(n, d)$-modules is naturally equivalent to the category of SL_n-modules all of whose composition factors $L(\lambda)$ satisfy $\lambda = \sum_i a_i\omega_i$ with $\sum_i ia_i = d - jn$ for some $j \geq 0$. As $S(n, d)$ is a quasi-hereditary algebra, there exists a certain characteristic module T for S, and we call the endomorphism algebra $S' = \mathrm{End}_S(T)$ the Ringel dual of S. In fact (up to Morita equivalence) we can identify S' precisely:

THEOREM 3.1. *Suppose that $p > n$. Then the Ringel dual of $S(n, d)$ is Morita equivalent to a certain (known) quotient of $k\Sigma_d$, the group algebra of the symmetric group on d symbols.*

Proof. This is a special case of the first part of [10, Theorem 4.4].

Let $\Lambda^+(n, d)$ be the set of n-part partitions of d. To each $\lambda \in \Lambda^+(n, d)$, we can associate a corresponding permutation module M^λ for $k\Sigma_d$, and certain explicitly defined submodules S^λ of M^λ, called Specht modules. The indecomposable direct summands of M^λ are called Young modules, and we define Y^λ to be the unique such summand of M^λ containing S^λ.

It is shown in [6, (2.6)] that Young modules have a Specht module filtration, and we shall denote the multiplicity of S^μ in some such filtration of Y^λ by $(Y^\lambda : S^\mu)$. It is easy to see (confer [20, 4.10 Corollary]) that if λ has at most r non-zero parts, so also must μ for any S^μ arising in such a filtration of Y^λ. Indeed, by general results on Ringel duals and the explicit identifications made in [10], we have

PROPOSITION 3.2. *Suppose that $p > n$. Then for all $\lambda, \mu \in \Lambda^+(n, d)$ we have*

$$(Y^\lambda : S^\mu) = [\Delta(\mu) : L(\lambda)].$$

Proof. See [10, 4.4 Theorem] and [7, Lemma A4.6].

Thus to determine the Specht modules arising in a given Young module Y^λ for $k\Sigma_d$, it is enough to determine all Weyl modules containing the simple module $L(\lambda)$ for SL_n with highest weight in a certain bounded set of weights. Here n can be taken to be the number of non-zero parts of λ. The advantage of our algorithm for computing decomposition numbers is that it can easily be run in reverse. Starting with a given weight λ and the initial data on Weyl modules corresponding to p^2-restricted weights, it is easy to determine those weights (with multiplicities) which give rise to λ after one iteration of Algorithm 2.1. Iterate this procedure by determining for each weight obtained at the ith stage a corresponding set (with multiplicities) of new weights in a similar way (by regarding p^i-facets as p-facets). Thus (as $h = n$) we obtain

PROPOSITION 3.3. *Suppose that $p > n$. Given a weight $\lambda \in \Lambda^+(n,d)$, we can invert Algorithm 2.1 to give an algorithm for determining $(Y^\lambda : S^\mu)$ for all $\mu \in \Lambda^+(n,d)$, from the decomposition numbers for Weyl modules for SL_n with p^2-restricted weights.*

4. LIFTING FROM THE QUANTUM GROUP

Although they arise naturally in the algorithm of Section 2, it is not yet clear that the sets of virtual composition factors have any representation-theoretic interpretation. In particular, it is not even clear that they have non-negative multiplicities. In this section we shall realize them as the sets of composition factors associated with G-modules obtained by lifting from a corresponding quantum group — at least when p is large enough for the Lusztig conjecture to hold. We then discuss a possible connection with a long-standing conjecture of Humphreys on the structure of Weyl modules.

The modules we require arise in the work of Lin [16], and are generalisations of certain modules considered by Lusztig. The constructions in this section are based on [16, Section 2], to which we refer the reader for further details.

We begin by defining the various quantum algebras that we require. Let U_q be the quantised enveloping algebra over $\mathbb{C}(q)$ corresponding to the root system R. If we set $\mathcal{A} = \mathbb{Z}[q, q^{-1}] \subset \mathbb{C}(q)$, then

Lusztig [19] has constructed a certain $\mathcal{A}$-form $U_{\mathcal{A}}$ of U_q. For ξ a fixed primitive p^r-th root of unity, $\mathbb{C}$ becomes an $\mathcal{A}$-algebra via the homomorphism taking q to ξ. We denote by U_ξ the corresponding algebra $\mathbb{C} \otimes_{\mathcal{A}} U_{\mathcal{A}}$.

Setting $\mathcal{B}_r$ to be the localisation of $\mathbb{Z}[\xi]$ at the ideal $(\xi - 1)$, we have that $\mathcal{B}_r$ is a discrete valuation ring, and $U_{\mathcal{B}_r} = \mathcal{B}_r \otimes_{\mathcal{A}} U_{\mathcal{A}}$ is a $\mathcal{B}_r$-form inside U_ξ. Finally, for k an algebraically closed field of characteristic p, the natural homomorphism $\mathcal{A} \to k$ taking q to 1 factors through the homomorphism $\mathcal{B} \to k$ taking ξ to 1. We obtain an isomorphism of k-algebras

$$U_k := k \otimes_{\mathcal{A}} U_{\mathcal{A}} \cong k \otimes_{\mathcal{B}_r} U_{\mathcal{B}_r}.$$

We next wish to define various modules for each of these algebras, following Lusztig [17, 18, 19]. For each dominant weight λ there exists a unique finite-dimensional irreducible U_q-module $L_q(\lambda)$ of type $\mathbf{1}$ with highest weight λ. If we fix some vector v_λ generating this module, then $L_{\mathcal{A}}(\lambda) = U_{\mathcal{A}} v_\lambda$ is a $U_{\mathcal{A}}$-invariant $\mathcal{A}$-lattice in $L_q(\lambda)$. We set $\Delta_{p^r}(\lambda) = \mathbb{C} \otimes_{\mathcal{A}} L_{\mathcal{A}}(\lambda)$, the quantum Weyl module for U_ξ. This has a unique simple quotient, which we denote by $L_{p^r}(\lambda)$. We denote the image of our generating vector v_λ in this quotient by $\bar{v}_\lambda$.

Now $L_{\mathcal{B}_r}(\lambda) = U_{\mathcal{B}_r} \bar{v}_\lambda$ is a $\mathcal{B}_r$-lattice in $L_{p^r}(\lambda)$, and so $\overline{L_{p^r}(\lambda)} = k \otimes_{\mathcal{B}_r} L_{\mathcal{B}_r}(\lambda)$ is a U_k-module. Indeed, Lusztig has shown that this is even a (rational) G-module. The Weyl module for G can also be constructed using U_q; we have

$$\Delta(\lambda) \cong k \otimes_{\mathcal{A}} L_{\mathcal{A}}(\lambda) \cong k \otimes_{\mathcal{B}_r} \Delta_{\mathcal{B}_r}(\lambda)$$

where $\Delta_{\mathcal{B}_r}(\lambda) = U_{\mathcal{B}_r} v_\lambda$ is a $\mathcal{B}_r$-lattice inside $\Delta_{p^r}(\lambda)$.

Our construction gives that $\overline{L_{p^r}(\lambda)}$ is a quotient of $\Delta(\lambda)$, and we wish to understand the structure of these modules. By [16, Theorem 2.7], we have that

$$\overline{L_{p^r}(\lambda)} \cong \overline{L_{p^r}(\lambda')} \otimes \Delta(\lambda'')^{\mathrm{F}^r}$$

where $\lambda = \lambda' + p^r \lambda''$ with $\lambda' \in X_r(T)$, and F denotes the usual Frobenius morphism. Thus we first need to understand the structure of $\overline{L_{p^r}(\lambda)}$ for all p^r-restricted weights.

In [16], Lin has investigated the decomposition patterns of these modules for suitably 'generic' weights λ. (Roughly, this means that

λ is suitably far away from any p^i-walls for certain i; see [16, pg 286] for the precise definition.) For arbitrary dominant weights λ and μ we have

$$[\Delta(\lambda) : L(\mu)] = \sum_{\nu} [\Delta_{p^r}(\lambda) : L_{p^r}(\nu)][\overline{L_{p^r}(\nu)} : L(\mu)]. \tag{5}$$

It is this identity that will allow us to relate our algorithm to the decomposition patterns of the $\overline{L_{p^r}(\lambda)}$.

For the rest of this section, we shall assume that Lusztig's conjecture (for algebraic groups) holds for our choice of G and p, and that $p \geq 2h - 2$. For fixed G this is known to be the case if we take p to be sufficiently large by the results in [1].

A dominant weight λ satisfies the Jantzen condition if $\langle \lambda+\rho, \alpha\check{}\rangle \leq p(p + 2 - h)$ for all $\alpha \in R^+$. For such λ we have

$$[\Delta_p(\lambda) : L_p(\mu)] = [\Delta(\lambda) : L(\mu)]$$

and we see by induction using (5) that $\overline{L_p(\lambda)} \cong L(\lambda)$ is irreducible.

More generally we shall say that a dominant weight λ satisfies the *ith Jantzen condition* if

$$\langle \lambda + \rho, \alpha\check{}\rangle \leq p^i(p + 2 - h)$$

for all $\alpha \in R^+$, and denote the set of such λ by $J_i(T)$. Note that λ satisfies the Jantzen condition if and only if $\varepsilon_i(\lambda)$ satisfies the ith Jantzen condition. For $\lambda \in J_i(T)$ we have

$$[\Delta_{p^i}(\lambda) : L_{p^i}(\mu)] = \begin{cases} d_{FE} & \text{if } \mu \in W_{p^i}.\lambda \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

where $\lambda \in \varepsilon_i(F)$ and $\mu \in \varepsilon_i(E)$. As $p \geq 2h-2$ we have that $X_i(T) \subseteq J_i(T)$, and hence for p^2-restricted weights, the p^2-decomposition diagrams defined in Section 2 are just the decomposition diagrams for the corresponding quantum group at a p^2 root of unity.

We next define some more sets of weights associated to Algorithm 2.1. It will be convenient to set $\mathrm{cf}(\lambda, i) = \{\lambda\}$ whenever $\lambda \in \bar{C}_i$, extending our earlier notation. Given a weight λ and an integer $i \geq 1$, we define $\mathrm{Desc}(\lambda, i)$, the set of *descendents of* λ *at level* i in the following manner. First note that for any weight λ and integer

i we have, by construction, $\lambda \in \mathrm{cf}(\lambda, i)$, with multiplicity one, and all other weights $\mu \in \mathrm{cf}(\lambda, i)$ satisfy $\mu < \lambda$. We define $\mathrm{Desc}(\lambda, i)$ by induction on λ via

$$\mathrm{cf}(\lambda, 1) = \dot{\bigcup}_{\mu \in \mathrm{cf}(\lambda, i)} \mathrm{Desc}(\mu, i)$$

where as usual we run over elements of the index set counted with multiplicities.

As an example, consider the weight denoted λ in Figure 2. For $i \geq 4$ we have $\mathrm{Desc}(\lambda, i) = \mathrm{cf}(\lambda, 1)$, the set of composition factors of $\Delta(\lambda)$. The set $\mathrm{Desc}(\lambda, 3)$ is precisely the set of those weights with labels of the form $d(-)$, $e(-)$, or $f(-)$, while $\mathrm{Desc}(\lambda, 2) = \{\lambda = d(i), d(ii)\}$ and $\mathrm{Desc}(\lambda, 1) = \{\lambda\}$.

The main result of this section is

Theorem 4.1. *Suppose that $p \geq 2h - 2$ is such that the Lusztig conjecture is satisfied for G. If $\lambda \in J_i(T)$ then the set of composition factors, with multiplicities, of $\overline{L_{p^i}(\lambda)}$ equals* $\mathrm{Desc}(\lambda, i)$.

Proof. We proceed by induction on λ. If λ lies in $\bar{C}_i$, then $\mathrm{cf}(\lambda, i) = \{\lambda\}$ and hence $\mathrm{Desc}(\lambda, i) = \mathrm{cf}(\lambda, 1)$. But in this case, as $[\Delta_{p^i}(\lambda) : L_{p^i}(\mu)] = \delta_{\lambda\mu}$ we see from (5) that $\overline{L_{p^i}(\lambda)} = \Delta(\lambda)$, and so we are done by Theorem 2.2.

Now suppose that $\lambda \notin \bar{C}_i$. As $J_i(T) \subseteq C_{i+1}$, we see that Algorithm 2.1 begins at level i, and $\mathrm{cf}(\lambda, i+1) = \{\lambda\}$. After scaling by ε_i, the p^i-facet containing λ corresponds to some p-facet F in C_2. By our remarks before Algorithm 2.1, this implies that the virtual decomposition diagram associated to F is just the usual decomposition diagram for F. Hence after scaling by ε_i we see that

$$c^i_{\lambda\mu} = \begin{cases} d_{FE} & \text{if } \mu \in W_{p^i}.\lambda \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda \in \varepsilon_i(F)$ and $\mu \in \varepsilon_i(E)$. But by (6) this equals $[\Delta_{p^i}(\lambda) : L_{p^i}(\mu)]$. Hence we have by Theorem 2.2 that the composition factors of $\Delta(\lambda)$ are given by $\mathrm{cf}(\lambda, 1)$, where

$$\mathrm{cf}(\lambda, 1) = \dot{\bigcup}_{\mu} [\Delta_{p^i}(\lambda) : L_{p^i}(\mu)] \mathrm{Desc}(\mu, i)$$

where μ runs over the set of dominant weights. By induction, we see by comparing with (5) that $\mathrm{Desc}(\lambda, i)$ is precisely the set of composition factors of $\overline{L_{p^i}(\lambda)}$, as required.

Recall that for $p \geq 2h-2$ we have $X_i(T) \subset J_i(T)$. Thus for suitably large p, our algorithm gives a means of calculating the composition factors of $\overline{L_{p^i}(\lambda)}$ for all p^i-restricted weights, given the decomposition numbers for $\Delta(\mu)$ for p^2-restricted weights μ. In particular, for any p^2-restricted weight λ we have that $\mathrm{Desc}(\lambda, 2)$ is just the set of virtual composition factors associated to λ, and hence obtain

Corollary 4.2. *Suppose that $p \geq 2h-2$ is such that the Lusztig conjecture is satisfied for G. Then for any p^2-restricted weight λ, the virtual decomposition numbers $c_{\lambda\mu}$ are all non-negative.*

Recall from Section 1 that for any weight λ we may define modules $H^j(\lambda)$ by considering the right derived functors of ind_B^G. In [11], Humphreys has conjectured that any Weyl module $\Delta(\lambda)$ should have a filtration with quotients of the form $H^j(\mu) \otimes L(\nu)^{\mathrm{F}^i}$, where μ is W-conjugate to a weight in $X_i(T)$ and ν is dominant. On the level of characters, this would imply that $\Delta(\lambda)$ has a filtration where each quotient has composition factors clustered around a certain translate of $X_i(T)$ (related to the weight ν defined above).

Now suppose that there exists a filtration of $\Delta(\lambda)$ associated to the ith level in Algorithm 2.1; that is, a filtration whose successive quotients have sets of composition factors of the form $\mathrm{Desc}(\mu, i)$ for elements $\mu \in \mathrm{cf}(\lambda, i)$. On the level of characters, this would imply that $\Delta(\lambda)$ has a filtration where each quotient has composition factors clustered around a certain p^i-alcove (related to the weight μ). In proving (5), Lin constructs an associated filtration of the Weyl module, which by Theorem 4.1 is compatible with the first iteration of Algorithm 2.1.

If we consider the case of SL_3, with λ suitably 'generic' in the lowest p^3-alcove (as in [11, Figure 1]), it is easy to verify that the clusters of composition factors arising from the final iteration of our algorithm correspond to the (conjectural) filtrations of $\Delta(\lambda)$ given by Humphreys in [11, pages 2672–4]. Indeed, we have

Theorem 4.3. *Suppose that λ is a dominant weight for* SL_3 *such that all composition factors of $\hat{Z}_1(\lambda)$ lie in the same p^2-alcove. Then there is a filtration of $\Delta(\lambda)$ corresponding to the final iteration of Algorithm 2.1.*

Proof. This is an easy consequence of a theorem of Kühne-Hausmann [15, Kapitel VI, Satz 2].

More generally, Kühne-Haussman has calculated the submodule structure of all Weyl modules for SL_3 which are multiplicity-free. For the examples given in [15, pages 174-6] — which do not satisfy the hypotheses of Theorem 4.3 — it is easy to verify that there is still a filtration associated to our algorithm.

If we assume that Humphreys' conjecture holds, the above remarks gives some evidence that there may be a refinement of his filtration associated to the corresponding level of Algorithm 2.1. In the next section we shall consider further evidence for the existence of such a filtration.

5. A SOCLE SERIES FOR CERTAIN INDUCED MODULES

In this section we shall show how the facet combinatorics introduced in Section 2 can be used to give a new description of certain results of Doty in [8]. In particular we shall give a filtration (and for weights in alcoves a description of the socle series) of symmetric powers of the natural module for SL_2 and SL_3. These are isomorphic to induced modules of the form $H^0(d\omega_1)$, which in turn are just the contravariant duals of the corresponding Weyl modules (and have the same composition factors). Thus this filtration will provide an interpretation in terms of module structure of the character-theoretic result in Section 2.

These results could also be deduced from [3] for SL_2, and [15, Kapitel VI, Satz 1] for SL_3. However, we prefer to consider the methods of Doty as these are both simpler and more accessible to generalisation to groups of higher rank.

We begin by recalling the combinatorial framework outlined in [8]. Let $S^d(V)$ denote the dth symmetric power of the natural representation V of GL_n. We write $d = \sum_{i\geq 0}^{M} d_i p^i$ with $0 \leq d_i \leq p-1$ for all i, and $d_M \neq 0$. Given a monomial $x^{\mathbf{b}} = x_1^{b_1} \cdots x_n^{b_n} \in S^d(V)$ we write $b_i = \sum_{j\geq 0} b_{ij}p^i$ with $0 \leq b_{ij} \leq p-1$ for all i and j. Then we can associate to each monomial $x^{\mathbf{b}}$ an M-tuple of non-negative integers $(c_1(\mathbf{b}), \ldots, c_M(\mathbf{b}))$, where the elements $c_k(\mathbf{b})$ are defined by the equations

$$\sum_{i\geq 0}\sum_{j<k} b_{ij}p^i = c_k(\mathbf{b})p^k + \sum_{j<k} d_j p^j.$$

An equivalent set of defining equations are given by

$$\sum_{i\geq 0} b_{ik} = d_k + pc_{k+1} - c_k. \tag{7}$$

We let $C(d)$ be the set of M-tuples arising from some $x^{\mathbf{b}} \in S^d(V)$. This set can be given a lattice structure by setting $\mathbf{c} \leq \mathbf{c}'$ if and only if $c_k \leq c'_k$ for all k. Then the main result in [8] is

Theorem 5.1. *There is a lattice isomorphism between the lattice of order closed subsets of $C(d)$ and the G-submodules of $S^d(V)$ (ordered by inclusion). In particular, the composition factors of $S^d(V)$ are in one-to-one correspondence with the elements of $C(d)$.*

Further, by considering (7), we can determine explicitly the highest weight vector of the composition factor corresponding to $\mathbf{c}$ as follows. Let $x^{\mathbf{b}}$ be the highest weight vector corresponding to $\mathbf{c}$. Then $\mathbf{b}$ is given by the set of equations

$$b_{ik} = \max\{\min\{d_k + p(c_{k+1} + 1 - i) - c_k + i - 1, p-1\}, 0\} \tag{8}$$

where the d_k and b_{ik} are as defined above. Thus, given the set $C(d)$ corresponding to d, we can determine the composition factors of $S^d(V)$. The final result from [8] that we shall need is the following criterion for recognising elements of $C(d)$.

Lemma 5.2. *An M-tuple $\mathbf{c}$ of non-negative integers is an element of $C(d)$ if and only if it satisfies the equations*

$$0 \leq c_k \leq \sum_{j\geq k} d_j p^{j-k}$$

and

$$0 \leq d_k + pc_{k+1} - c_k \leq n(p-1)$$

for all k, where we set $c_j = 0$ for all $j \leq 0$ and $j > M$.

We begin by analysing the structure of the lattice $C(d)$. First we note that by induction (using Lemma 5.2) it is clear that we must have $0 \leq c_k \leq n-1$ for each k. Given $\mathbf{c} \in C(d)$, we set $|\mathbf{c}| = \sum_{i \geq 0} c_i$. We shall say that such an element $\mathbf{c}$ lies in the jth socle layer $\mathrm{Soc}^j(C(d))$ of $C(d)$ if the longest chain of elements strictly below $\mathbf{c}$ is of length $j-1$.

Lemma 5.3. *An element $\mathbf{c} \in C(d)$ lies in the jth socle layer if and only if $|\mathbf{c}| = j$.*

Proof. Any chain of elements strictly below $\mathbf{c}$ must have length at most $|\mathbf{c}|$, and so it is enough to show that a chain of this length in fact exists. For this, it will be enough to show that for each $\mathbf{c} \neq \mathbf{0}$ there exists some $\mathbf{c}' < \mathbf{c}$ in $C(d)$ such that for some k_0 we have $c'_{k_0} = c_{k_0} - 1$, and $c'_i = c_i$ for all $i \neq k_0$.

We first claim that if $\mathbf{c} \neq \mathbf{0}$ then there exists some k such that $0 \leq d_k + pc_{k+1} - c_k < n(p-1)$ and $c_k \neq 0$. For otherwise take k_1 minimal such that $c_{k_0} \neq 0$. Then we must have $d_{k_1} + pc_{k_1+1} - c_{k_1} = n(p-1) \geq p$, and hence that $c_{k_1+1} > 0$. In a similar manner we deduce that $c_k > 0$ for all $k \geq k_1$ (and that for all such k we must have $d_k + pc_{k+1} - c_k = n(p-1)$). But this contradicts the condition that $c_{M+1} = 0$, and so the claim follows.

Taking k_0 to be minimal satisfying the above hypotheses, it is now an easy exercise using the inequalities in Lemma 5.2 to verify that the element $\mathbf{c}'$ defined above is in $C(d)$ as required.

Most of the rest of this section will be devoted to proving

Proposition 5.4. *For GL_2 and GL_3, the modules $S^d(V)$ have filtrations corresponding to the virtual decomposition factors obtained at any given stage in Algorithm 2.1. Moreover, for weights in the interior of alcoves, the socle series for $S^d(V)$ can be constructed in the course of implementing the algorithm.*

Proof. To each composition factor $L(\lambda)$ we have associated a carry pattern $\mathbf{c}(\lambda)$. We shall show that the iteration of Algorithm 2.1 corresponding to p^i-facets changes the value only of the first i elements of any $\mathbf{c}$ obtained so far, and that each virtual decomposition factor obtained at this stage from a given weight corresponds to a different value of c_i. In fact, for weights in the interior of an alcove we shall show that at this stage the first i elements of any carry pattern obtained so far are all equal. By Lemma 5.3, we shall thus obtain complete information on the socle series of the symmetric powers corresponding to weights in the interior of an alcove.

We begin with an easy example to illustrate this result for SL_2. (We postpone analysis of the socle series for the SL_3 example given in Figure 2 until later in this section.) Let $p = 3$ and $d = 43$. On applying Algorithm 2.1 we obtain the set of composition factors labelled a to f in Figure 3.

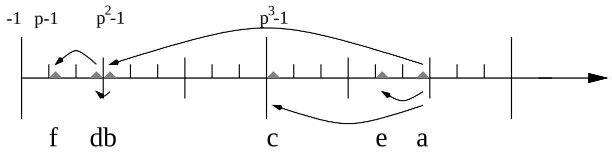

FIGURE 3.

For SL_2, the p^2-restricted weights are precisely those in the upper closure of the lowest p^2 alcove. Thus the virtual composition factors associated to a weight are just the ordinary composition factors of the corresponding Weyl module. This is simple for weights on walls or in the lowest p-alcove, and has two composition factors corresponding to the highest weight and its reflection about the p-wall immediately below it for weights in the remaining p-alcoves. As in this case the various c_i can only be 0 or 1, the corresponding socle series predicted by our result will be as shown in Figure 4(a) (with the actual submodule lattice obtained using the results in [8] given in Figure 4(b)).

FIGURE 4. (a) and (b)

We now return to the proof of Proposition 5.4. The submodule structure for the symmetric powers has already been given in [8], and we merely verify that the results given there can be converted into the required form. For this we need to know the relative positions of the virtual composition factors for certain p^2-restricted weights. It is for this reason that we restrict ourselves to considering SL_2 and SL_3, although we conjecture that the result should hold for SL_n without restriction on n.

In order to convert the results from [8] for GL_n into a form compatible with the facet geometry, we use the usual change of coordinates $\lambda = (\lambda_1, \ldots, \lambda_n) \longmapsto \bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{n-1})$, where we set $\bar{\lambda}_i = \lambda_i - \lambda_{i+1}$. This now gives the coordinates of the corresponding SL-weight in terms of the basis of fundamental weights.

We begin by considering the SL_2 case. Let $\mathbf{b} = (b_1, b_2)$ be the highest weight vector in some composition factor of $S^d(V)$. By the preceding remarks and Theorem 2.2 this is either $(d, 0)$ or $\bar{\mathbf{b}}$ is a reflection of some $\bar{\mathbf{b}}'$ about a p^i-wall for some weight $\mathbf{b}'$ generated at an earlier stage in the algorithm (where we take i to be maximal with this property). Let the corresponding elements of $C(d)$ be $\mathbf{c}$ and $\mathbf{c}'$ respectively.

It will be enough to show that $c_k = 1 - c'_k$ when $1 \leq k \leq i$ and $c_k = c'_k$ otherwise. Suppose that $\bar{\mathbf{b}}' = ap^i - 1 + b$ with $0 < b < p^i$ and $a \not\equiv 0 \bmod p$. Then it will suffice to show that the weight $\bar{\mathbf{x}}$ corresponding to the desired value of $\bar{\mathbf{c}}$ satisfies $\bar{\mathbf{x}} + \bar{\mathbf{b}}' = 2ap^i - 2$ (as then $\bar{\mathbf{x}}$ must equal $\bar{\mathbf{b}}$). By considering the various possible values of b'_{ik} arising from (8) we see that

$$\bar{b}'_k = b'_{1k} - b'_{2k} = \begin{cases} d_k & \text{if } (c_k, c_{k+1}) = (0,0) \\ p - d_k - 2 & \text{if } (c_k, c_{k+1}) = (0,1) \\ d_k - 1 & \text{if } (c_k, c_{k+1}) = (1,0) \\ p - 1 - d_k & \text{if } (c_k, c_{k+1}) = (1,1). \end{cases} \tag{9}$$

Clearly, we must have $\{(c_0, c_1), (c'_0, c'_1)\} = \{(0,0), (0,1)\}$, while for $1 \leq t \leq i-1$ we have $\{(c_t, c_{t+1}), (c'_t, c'_{t+1})\} = \{(0,0), (1,1)\}$. Further, by our inductive hypothesis we must have $\{(c_i, c_{i+1}), (c'_i, c'_{i+1})\} = \{(0,1), (1,1)\}$ or $\{(1,0), (0,0)\}$. Using (9) it is now easy to see that

$$\sum_k (\bar{b}'_k + \bar{x}_k) p^k = p - 2 + \sum_{t=1}^{i-1} (p-1)p^t + (2a-1)p^i = 2ap^i - 2$$

as required. This completes the proof in the SL_2 case.

The proof for SL_3 proceeds in a similar manner. We shall first consider the case where the initial weight lies in the interior of an alcove, and show that when considering p^t-alcoves, the set of virtual composition factors associated to a given weight contains at most one of each of the three types of alcove shown in Figure 5(a). Further, we show that the image of the given weight under W_{p^t} in each of these alcoves corresponds to the carry pattern whose first t elements are equal to the integer labelling that diagram, and that the remaining elements of the carry pattern are fixed for these alcoves.

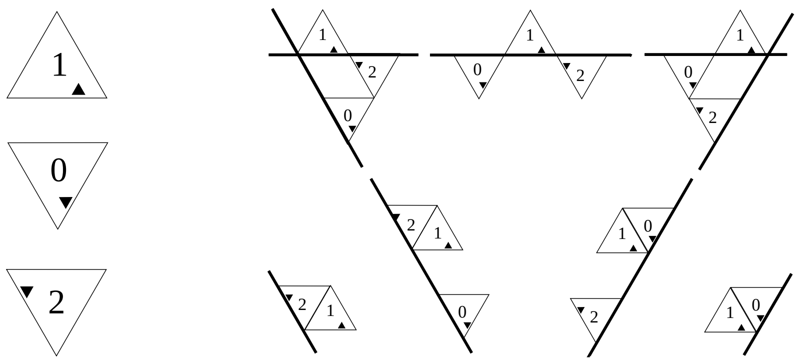

FIGURE 5. (a) and (b)

Using (4), and the known decomposition numbers for the $\hat{Z}_1(\lambda)$ for SL_3 [12], it is easy to show that the various possible patterns of virtual composition factors associated to weights outside the lowest p-alcove are as shown in Figure 5(b).

It just remains to consider the various possible cases, calculate the weights corresponding to the predicted carry patterns, and verify that the differences between them correspond to the relative positions of the various weights shown in each case. This is a routine (if lengthy) exercise using the expression for the b_{ik} given in (8).

If the initial weight does not lie in an alcove, then the strong version of the inductive hypothesis (asserting the precise form of the corresponding carry patterns) is no longer satisfied. However by similar arguments one shows that at every stage the virtual composition factors correspond to distinct values of an appropriate c_t, and that the $c_{t'}$ (for $t' > t$) remain constant. This completes the proof of Proposition 5.4.

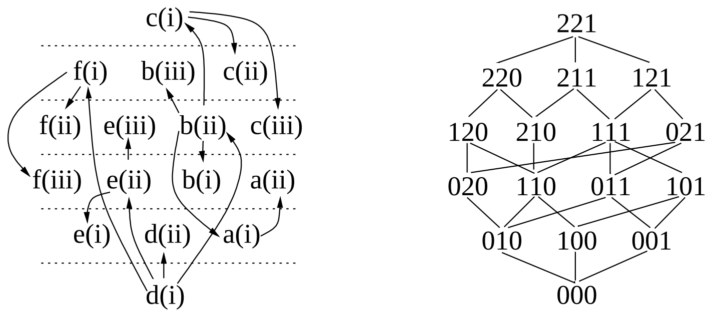

FIGURE 6. (a) and (b)

To illustrate the SL_3 case, we return to the example considered in Figure 2. In Figure 6(a) we give the socle series constructed in the proof of Proposition 5.4, while in Figure 6(b) we give the corresponding submodule lattice calculated using [8]. The labels on the left are those used in Figure 2, while on the right the corresponding elements of $C(d)$ are given.

6. THE QUANTUM MIXED CASE

To conclude, we consider the quantum general linear group q-$\mathrm{GL}(n,k)$ defined by Dipper and Donkin [5], in the case when q is a primitive lth root of unity and k has characteristic $p > 0$. Algorithm 2.1 can easily be modified to give a corresponding algorithm in this context, by replacing p^i-facets by lp^{i-1}-facets, and taking as the initial dataset the decomposition numbers for all lp-restricted weights.

Now the same arguments as in Sections 1 and 2 can be applied to show that this gives the composition factors of the quantum Weyl module corresponding to λ. For the results on the geometry of facets it is sufficient to require that both l and p are at least as big as the Coxeter number, which in this case is n. The general theory reviewed in Section 1 is given in the quantum case in [7] and [4].

Again, this has applications via Ringel duality; for the results in Section 3, we must replace the symmetric group by the Hecke algebra — the basic theory needed in this case can be found in [7] and [20]. Finally, the results on the submodule structure of the symmetric powers used in Section 5 have been generalised to the quantum setting in [21].

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