HOMOMORPHISMS AND HIGHER EXTENSIONS FOR SCHUR ALGEBRAS AND SYMMETRIC GROUPS

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Abstract. This paper surveys, and in some cases generalises, many of the recent results on homomorphisms and the higher Ext groups for q-Schur algebras and for the Hecke algebra of type A. We review various results giving isomorphisms between Ext groups in the two categories, and discuss those cases where explicit results have been determined.

1. Introduction

Since 1901 and the work of Schur [Sch01] it has been known that the representation theories of the symmetric and general linear groups are intimately related. Following later work of Schur [Sch27] and Carter and Lusztig [CL74], Green [Gre80] showed how this relationship could be explained via passage through the Schur algebra using the Schur functor. This algebra has been an object of intense study ever since, and with the rise of quantum groups was generalised by Dipper and James [DJ89] to give the q-Schur algebra. This plays a corresponding role in relating the representation theories of type A Hecke algebras and the quantum general linear group.

On the level of cohomology however, there are some striking differences between the symmetric groups and the Schur algebras. For example, Schur algebras are quasi-hereditary and hence have finite global dimension, while the global dimension of the symmetric group is usually infinite. Thus we must expect some difficulties to arise in the passage of cohomological data from one category to the other.

The majority of this paper is concerned with the Ext-groups corresponding to pairs of induced (or Weyl) modules for the algebraic/quantum group, and to pairs of Specht modules for the symmetric group/Hecke algebra. In their respective categories these classes of modules play a key role, and correspond under the action of the Schur functor, so we might hope that their cohomology theories are closely related. We will review what is known, and show how some of the results can be extended and refined.

Carter and Lusztig [CL74] showed that at the level of homomorphisms these two theories could essentially be identified under the standard correspondence. Surprisingly however, consideration of the relationship between the higher Ext-groups has been a relatively recent phenomenon.

The first real results in that direction are due to Doty, Erdmann and Nakano [DEN04], who used certain spectral sequences arising in the general setting of finite dimensional algebras with idempotent functors. These gave rise to various relationships between certain Ext^1 and Ext^2 spaces. Using this framework, Kleshchev and Nakano [KN01] and Hemmer and Nakano [HN04] were able to refine these results further, and recently Donkin [Don] and Parshall and Scott [PS05] have generalised these results.

Donkin, and Parshall and Scott, show that there are equalities of Ext-groups in degrees greater than zero (for certain classes of modules) provided that the degree is small compared with the characteristic of the field (or the degree of the root of unity in the quantum case). This will be our key tool in relating the two categories.

There is an important reduction result due to Donkin [Don85] (surprisingly little-known), which can be used to reduce to lower rank calculations in certain special cases, and a similar result due to Cline, Parshall, and Scott [CPS04] where we allow one of the modules to be simple.

Unfortunately, relatively little is known about the actual cohomology for explicit modules in either category. We do have an upper bound on the degree of a non-zero Ext group between two non-zero modules given by 7.1 and 7.4, but this is the best we can do in general. On the algebraic group side the only setting where complete information is known is SL_2 (and q-$\mathrm{GL}_2(k)$), where the second author [Parb] has recently determined all Ext-groups between Weyl modules. This extends work of Erdmann and the first author [Erd95, Cox98, CE00] where the groups up to degree 2 where determined. For SL_3, the authors have determined all homomorphisms between Weyl modules in characteristic at least 3 [CP]. (This has recently been generalised to q-$\mathrm{GL}_3(k)$ in [Para].)

Apart from these low-rank calculations, the other main general results are either for weights which are close together (for example in [And81], [Kop86] and [Wen89]) or for weights which differ by a single reflection. In the latter case there is a theorem of Carter and Payne [CP80] which has recently been generalised by Fayers and Martin [FM04]. This generalisation gives a set of sufficient conditions for the existence of a homomorphism which can be difficult to verify in particular examples, and we will show how these conditions may be simplified.

Other results in this area consider extensions between simple modules. Kleschev and Sheth [KS99] consider Ext^1 between simple modules for the symmetric group. They show in some cases that an extension between two simple modules D^λ and D^μ must appear in the Specht module S^μ when λ and μ have at most $p-1$ parts and $p \geq 3$. We will give a generalisation of this result.

Wherever possible we have stated results using partition combinatorics rather than alcove geometry and the affine Weyl group, in order to simplify the exposition. However, we believe that the alcove-geometric approach has been under-utilised in the Hecke algebra setting and provides a natural context for many results in this area (see for example Section 4 and the references therein).

2. Notation

Let k be an algebraically closed field of characteristic $p \geq 0$ and $q \in k$ a root of unity. We define l to be p if $q = 1$, or the smallest non-zero exponent for which $q^l = 1$ if $q \neq 1$. We are interested in the relationship betwen the representation theories of two objects: $G = q$-$\mathrm{GL}_n(k)$, the quantum general linear group as defined by Dipper and Donkin [DD91], and $H = H_{q,n}(k)$, the Hecke algebra of type A. If $q = 1$ then G is the classical general linear group $\mathrm{GL}_n(k)$, the group of n by n invertible matrices over k, and H is the group algebra of the symmetric group $\mathfrak{S}_r$. In this case we will further assume that $p > 0$. For the rest of this section we will review the basic results which we require. This material can be found in [Don98] for the quantum group and [Mat99] for the Hecke algebra (which are respectively in the spirit of the classical expositions in [Jan87] and [JK81]).

Our main tool for relating the two module categories will be the q-Schur algebra $S(n,r)$, as defined in [DJ89] (generalising the classical definition of Green [Gre80]). This will often be abbreviated to S when the context is clear. The module category of $S(n,r)$ can be identified with the category of polynomial modules for G of degree r. If $r \leq n$ there exists a certain idempotent $e \in S$. This give rise to the Schur functor $f : S \longrightarrow eSe$ defined on modules by $f(M) = eM$, together with a partial inverse g given by $g(N) = Se \otimes_{eSe} N$. As $eSe \cong H$ this provides the desired connection between the representations of G and H.

We first consider the quantum group. Let $X \cong \mathbb{Z}^n$ be the set of weights for G. We chose a Borel B such that the dominant weights are given by $X^+ = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in X \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$. When a dominant weight λ satisfies $\lambda_n \geq 0$ we call it a *polynomial weight*, and identify it with the corresponding partition of $|\lambda| = \sum_i \lambda_i$. We denote by $\mathrm{Mod}(G)$ the rational modules for G and by $\mathrm{mod}(G) \subset \mathrm{Mod}(G)$ the finite dimensional rational modules for G. We have two dualities on $\mathrm{mod}(G)$; the usual linear dual *, and a contravariant duality $^\circ$ (which in the classical case is induced by the transpose map on matrices).

Given a dominant weight λ there exists a corresponding one-dimensional B-module k_λ. We let $\nabla(\lambda)$ denote the G-module obtained from this by induction. This has highest weight λ, and simple socle which we denote by $L(\lambda)$. These socles give a full set of inequivalent simple G-modules. The contravariant dual $\Delta(\lambda) = \nabla(\lambda)^\circ$ is called a *Weyl module*.

Let $\Lambda^+(r)$ denote the set of partitions of r, and $\Lambda^+(n,r)$ denote the subset of partitions of r with at most n parts. Given a partition λ we denote the conjugate partition by λ'. A partition λ of r is *column l-regular* if

$\lambda_i - \lambda_{i+1} < l$ for all i. A partition λ of r is *row l-regular* if and only if λ' is column l-regular. We let $\Lambda^+_{\mathrm{col}}(n,r)$ (respectively $\Lambda^+_{\mathrm{row}}(n,r)$) denote the subset of $\Lambda^+(n,r)$ consisting of the column (respectively row) l-regular partitions. Let μ and λ be two partitions of r. We say λ *dominates* μ if $|\lambda| = |\mu|$ and $\sum_{i=1}^{j} \mu_i \leq \sum_{i=1}^{j} \lambda_i$ for all $1 \leq j \leq n$ and write $\mu \leq \lambda$.

The simple modules for $S(n,r)$ are precisely the $L(\lambda)$ for G with $\lambda \in \Lambda^+(n,r)$. We define a symmetric power associated to each $\lambda \in \Lambda^+(n,r)$ by

$$S^\lambda E = S^{\lambda_1} E \otimes S^{\lambda_2} E \otimes \cdots \otimes S^{\lambda_n} E$$

where $S^i E$ is the ith symmetric power of the natural module E. The $S^\lambda E$ are injective $S(n,r)$-modules, but are not indecomposable in general. We let $I(\lambda)$ be the injective hull of $L(\lambda)$, regarded as an S-module. The module $I(\lambda)$ is a direct summand of $S^\lambda E$ and is the unique summand of $S^\lambda E$ with $L(\lambda)$ as its socle.

Although we have a labelling of the simples, they are not well understood. We do not even know their dimension in general. To determine their characters it would be enough to determine $[\nabla(\lambda) : L(\mu)]$ for all pairs $\mu \leq \lambda$, where $[M : L]$ denotes the multiplicity of the simple module L as a composition factor of M.

We next consider the Hecke algebra $H_{q,n}(k)$. For each $\lambda \in \Lambda^+(r)$ we can define corresponding H-modules S^λ, the *Specht module*, Y^λ, the *Young module*, and M^λ, the *permutation module*. If $\lambda \in \Lambda^+_{\mathrm{row}}(n,r)$ then S^λ has simple head, which we denote by D^λ. These simple heads provide a full set of inequivalent H-modules.

For our purposes these various modules can be defined in terms of the Schur functor f. In particular, if $r \leq n$ we have

$$fS^\lambda E \cong M^\lambda \qquad fI(\lambda) \cong Y^\lambda \qquad f\nabla(\lambda) \cong S^\lambda \tag{1}$$

We also know that $fL(\lambda) = 0$ unless $\lambda \in \Lambda^+_{\mathrm{col}}(n,r)$; however in order to describe the effect of f on the remaining simple modules we need a little more notation. We write sgn for the q-analogue of the one-dimensional sign representation of $\mathfrak{S}_r$. Then the module $D^\mu \otimes \mathrm{sgn}$ is again a simple H-module, denoted $D^{m(\mu)}$. The map $m(-)$ on $\Lambda^+_{\mathrm{row}}(n,r)$ is an involutory bijection and is known as the *Mullineux map.* It can be shown that $fL(\lambda) = D^{m(\lambda')}$ if $\lambda \in \Lambda^+_{\mathrm{col}}(n,r)$.

Since f is exact we also have $[\nabla(\lambda) : L(\mu)] = [S^\lambda : D^{m(\mu')}]$ for $\mu \in \Lambda^+_{\mathrm{col}}(n,r)$. We can also show that Y^λ is projective as an H-module and $I(\lambda)$ is projective as a $S(n,r)$-module if $\lambda \in \Lambda^+_{\mathrm{col}}(n,r)$.

3. Equating Ext groups for the Schur algebra and the symmetric group

We wish to investigate the extent to which Ext groups for our two categories can be identified. The following fundamental theorem will be our starting point. It was originally proved in the classical case by Carter and Lusztig [CL74, Theorem 3.7] and later generalised to the quantum case by Dipper and James [DJ91, 8.6 Corollary].

Theorem 3.1 (Carter-Lusztig, Dipper-James). *If $l \geq 3$ or if $l = 2$ and either $\lambda \in \Lambda^+_{\mathrm{row}}(n,r)$ or $\mu \in \Lambda^+_{\mathrm{col}}(n,r)$ then*

$$\mathrm{Hom}_{S(n,r)}(\nabla(\lambda), \nabla(\mu)) \cong \mathrm{Hom}_H(S^\lambda, S^\mu).$$

Remark 3.2. This theorem appears in a slightly different form in the literature: stated in terms of Weyl modules and without the case $\mu \in \Lambda^+_{\mathrm{col}}(n,r)$. The translation from Weyl to induced modules is clear by applying contravariant duality, while the additional case follows from the case $\lambda \in \Lambda^+_{\mathrm{row}}(n,r)$ and the following pair of results.

Lemma 3.3 (Donkin). *Suppose that $n \leq N$ and λ, $\mu \in \Lambda^+(n,r)$, which is considered in the usual way as a subset of $\Lambda^+(N,r)$. Then for all $i \geq 0$ we have*

$$\mathrm{Ext}^i_G(\nabla(\lambda), \nabla(\mu)) \cong \mathrm{Ext}^i_{q-\mathrm{GL}_N(k)}(\nabla(\lambda), \nabla(\mu)).$$

Proof. This follows using [Don98, 4.2 (17)]. □

Note that it is only the Ext's between induced modules that are stable; the extensions between simples, for example, are dependent on n. However if n is larger than the degree of the partition labelling the simples, then the Ext groups become stable (using a result of [Gre80]). We will also see that $\mathrm{Ext}^i_G(L(\lambda),\nabla(\mu)) \cong \mathrm{Ext}^i_{q-\mathrm{GL}_N(k)}(L(\lambda),\nabla(\mu))$, this follows from Theorem 3.12.

Lemma 3.4 (Donkin)**.** *Suppose $r \le n$. Then for all $m \in \mathbb{N}$ we have*

$$\mathrm{Ext}^m_{S(n,r)}(\nabla(\lambda),\nabla(\mu)) \cong \mathrm{Ext}^m_{S(n,r)}(\nabla(\mu'),\nabla(\lambda'))$$

and

$$\mathrm{Ext}^m_H(S^\lambda,S^\mu) \cong \mathrm{Ext}^m_H(S^{\mu'},S^{\lambda'}).$$

Proof. The result in the Schur algebra is proved using the Ringel dual, [Don93, Corollaries 3.8 and 3.9] and [Don98, Proposition 4.1.5]. If we distinguish the modules for the Ringel dual of $S(n,r)$ by primes then we have

$$\begin{aligned}\mathrm{Ext}^m_{S(n,r)}(\nabla(\lambda),\nabla(\mu)) &\cong \mathrm{Ext}^m_{S(n,r)'}(\Delta'(\lambda'),\Delta'(\mu'))\\ &\cong \mathrm{Ext}^m_{S(n,r)}(\Delta(\lambda'),\Delta(\mu'))\\ &\cong \mathrm{Ext}^m_{S(n,r)}(\nabla(\mu'),\nabla(\lambda'))\end{aligned}$$

where the first isomorphism follows by [Don98, A4.8 (i)] the second by $S(n,r)$ being Ringel self-dual if $r \le n$ and the third by taking contravariant duals.

The Hecke algebra version follows easily from the isomorphism [Don98, Proposition 4.5.9].

$$(S^\lambda)^* \cong S^{\lambda'} \otimes \mathrm{sgn}\,. \tag{2}$$

We have

$$\begin{aligned}\mathrm{Ext}^m_H(S^\lambda,S^\mu) &\cong \mathrm{Ext}^m_H((S^\mu)^*,(S^\lambda)^*)\\ &\cong \mathrm{Ext}^m_H(S^{\mu'}\otimes\mathrm{sgn},S^{\lambda'}\otimes\mathrm{sgn})\\ &\cong \mathrm{Ext}^m_H(S^{\mu'},S^{\lambda'})\end{aligned}$$

□

We would like to have an analogue of Theorem 3.1 for the higher Ext-groups. However, it is easy to see that they cannot always be equal. We know that H is a symmetric algebra and for $l \le r$ is usually not semi-simple. (It is semi-simple if $l > r$.) In general H has infinite global dimension: i. e. for all $m \in \mathbb{N}$, there exist M, $N \in \mathrm{mod}H$ such that $\mathrm{Ext}^m_{k\mathfrak{S}_r}(M,N) \neq 0$. On the other hand $S(n,r)$ is a quasi-hereditary algebra, which is usually not symmetric and not semi-simple. Quasi-heredity implies that $S(n,r)$ has finite global dimension. So there exists $m \in \mathbb{N}$ such that $\mathrm{Ext}^j_{S(n,r)}(M,N) = 0$ for all $j > m$ and all M, $N \in \mathrm{mod}S(n,r)$.

The relationship between the two cohomology theories has been studied in [DEN04], in a general setting. Based on this work, Theorem 3.1 has been generalised by Kleshchev and Nakano [KN01] and Hemmer and Nakano [HN04], and further generalised by Donkin [Don, Section 10, Propositions 2 and 3] and Parshall and Scott [PS05, Theorem 4.6]. We state here Donkin's version, that of Parshall and Scott is similar.

Theorem 3.5 (Donkin)**.** *Suppose that $X \in \mathrm{mod}H$ has a Specht series and that $Y \in \mathrm{mod}S(n,r)$. If $l \ge 4$ and $0 \le i \le l-3$ then*

$$\mathrm{Ext}^i_{S(n,r)}(gX,Y) \cong \mathrm{Ext}^i_H(X,fY).$$

and $gS^\lambda = \nabla(\lambda)$.

Combining Theorems 3.1 and 3.5 with the identifications in and after (1) we obtain

Corollary 3.6 (Donkin, Parshall-Scott)**.** *If λ, $\mu \in \Lambda^+(n,r)$ and $l \ge 3$ then*

$$\mathrm{Ext}^i_{S(n,r)}(\nabla(\lambda),\nabla(\mu)) \cong \mathrm{Ext}^i_H(S^\lambda,S^\mu)$$

and

$$\mathrm{Ext}^i_{S(n,r)}(\nabla(\lambda),L(\mu)) \cong \mathrm{Ext}^i_H(D^{\mu'},S^{\lambda'})$$

for $0 \le i \le l-3$.

Proof. The first isomorphism is immediate from the Theorem; for the second (which is not explicitly stated by Donkin) we have for $l > 3$ that

$$\begin{aligned}\mathrm{Ext}^i_{S(n,r)}(\nabla(\lambda), L(\mu)) &\cong \mathrm{Ext}^i_H(S^\lambda, D^{m(\mu')})\\ &\cong \mathrm{Ext}^i_H(D^{m(\mu')}, (S^\lambda)^*)\\ &\cong \mathrm{Ext}^i_H(D^{\mu'}, S^{\lambda'})\end{aligned}$$

where the second isomorphism follows as $D^{m(\mu')}$ is self-dual and the third from tensoring by the sign representation, using (2). For $l = 3$ and $i = 0$ see [PS05, Theorem 4.6]. □

It possible to extend the first part of the above corollary for $l = 3$ to Ext^1 between row three-regular three-part partitions, using Ringel duality. As $S(n,r)$ is quasihereditary, there is for each $\lambda \in \Lambda^+(n,r)$ a corresponding indecomposable tilting module $T(\lambda)$, and hence we can consider the Ringel dual of $S(n,r)$.

When $r \leq n$, Donkin has shown [Don98, Proposition 4.1.4] that $S(n,r)$ is Ringel self-dual (generalising the classical result first proved in [Don93]). If $l > n$ then for all r there exists an ideal $I(n)$ in $H_{q,r}$ such that the Ringel dual of $S(n,r)$ can be identified with $H_{q,r}/I(n)$. Further, the standard modules for this quotient algebra are precisely the Specht modules labelled by partitions with at most n parts. (See [Erd94] for the classical case, with [Don98, Section 4.7] for the general case.) This induces an equivalence of categories between the category of $S(n,r)$-modules with a filtration by induced modules and the category of $H_{q,r}$-modules with a filtration by Specht modules labelled by partitions with at most n parts. If we restrict attention to row regular partitions this equivalence also holds for $l = n$, (by applying [Don98, Section 4.7 (6)]) which implies

Proposition 3.7. *If λ, $\mu \in \Lambda^+_{\mathrm{row}}(3,r)$ and $l = 3$ then*

$$\mathrm{Ext}^1_{S(3,r)}(\nabla(\lambda), \nabla(\mu)) \cong \mathrm{Ext}^1_H(S^\lambda, S^\mu).$$

It is well-known [CPSvdK77] that

$$\mathrm{Ext}^i_{S(n,r)}(\nabla(\lambda), \nabla(\mu)) \cong \begin{cases} k & \text{if } \lambda = \mu \text{ and } i = 0\\ 0 & \text{if } \lambda = \mu \text{ and } i > 0, \text{ or if } \lambda \not< \mu.\end{cases} \tag{3}$$

Combining this with Corollary 3.6 we obtain

Corollary 3.8. *If λ, $\mu \in \Lambda^+(n,r)$ and $l \geq 3$ then*

$$\mathrm{Ext}^i_H(S^\lambda, S^\mu) \cong \begin{cases} k & \text{if } \lambda = \mu \text{ and } i = 0\\ 0 & \text{if } \lambda = \mu \text{ and } i > 0, \text{ or if } \lambda \not< \mu.\end{cases}$$

for $0 \leq i \leq l - 3$.

In particular $\mathrm{Ext}^1_H(S^\lambda, S^\lambda) = 0$, if $l \geq 4$, and so this Corollary can be regarded as a generalisation of results of Kleshchev and Nakano [KN01] and Hemmer and Nakano [HN04].

Let λ^1 and $\mu^1 \in \Lambda^+(n_1, r_1)$ and λ^2 and $\mu^2 \in \Lambda^+(n_2, r_2)$ be such that $\lambda^1_{n_1} \geq \lambda^2_1$ and $\mu^1_{n_1} \geq \mu^2_1$. We can form a new partition $\lambda \in \Lambda^+(n,r)$, where $n = n_1 + n_2$ and $r = r_1 + r_2$, by defining $\lambda_i = \lambda^1_i$ if $i \leq n_1$ and $\lambda_i = \lambda^2_{i-n_1}$ if $n_1 < i \leq n_1 + n_2$ (and similarly form a new partition μ). We will say that such a pair of partitions (λ, μ) has a *horizontal cut*. For such a pair we can reduce the calculation of Ext-groups between induced modules to the same calculation for lower rank groups using

Theorem 3.9 (Donkin)**.** *Suppose that (λ, μ) has a horizontal cut. Then for all $m \geq 0$ we have*

$$\mathrm{Ext}^m_{S(n,r)}(\nabla(\lambda), \nabla(\mu)) \cong \bigoplus_{m=m_1+m_2} \Big(\mathrm{Ext}^{m_1}_{S(n_1,r_1)}(\nabla(\lambda^1), \nabla(\mu^1)) \otimes \mathrm{Ext}^{m_2}_{S(n_2,r_2)}(\nabla(\lambda^2), \nabla(\mu^2))\Big).$$

Remark 3.10. (i) This is a result that really goes back to [Don85] (to a result about Levi subgroups), although it was not explicitly stated in this form. A more explicit version can be found in Erdmann [Erd95], and in the form above in [Don].
(ii) We have only stated a special case of Donkin's result for type A; the general version holds for any reductive group.

Combining this with Corollary 3.6 we obtain

Corollary 3.11 (Donkin). *Suppose that (λ,μ) has a horizontal cut. Then for all $0\leq m\leq l-3$ we have*

$$\mathrm{Ext}^m_{H_{q,n}(k)}(S^\lambda,S^\mu)\cong\bigoplus_{m=m_1+m_2}\Big(\mathrm{Ext}^{m_1}_{H_{q,n_1}(k)}(S^{\lambda^1},S^{\mu^1})\otimes\mathrm{Ext}^{m_2}_{H_{q,n_2}(k)}(S^{\lambda^2},S^{\mu^2})\Big).$$

In the classical case a proof of this last result when $m=0$ (given entirely in the context of the symmetric group) can be found in [FL03].

We say two partitions admit a *vertical cut* if their conjugates admit an horizontal cut. There are vertical cut analogues of Theorems 3.9 and 3.11, which follow from the above results and Lemma 3.4.

There is an analogue of Theorem 3.9 in [CPS04, Corollary 10].

Theorem 3.12 (Cline-Parshall-Scott). *Suppose that (λ,μ) has a horizontal cut as in Theorem 3.9. Then for all $m\geq 0$ we have*

$$\mathrm{Ext}^m_{S(n,r)}(L(\lambda),\nabla(\mu))\cong\bigoplus_{m=m_1+m_2}\Big(\mathrm{Ext}^{m_1}_{S(n_1,r_1)}(L(\lambda^1),\nabla(\mu^1))\otimes\mathrm{Ext}^{m_2}_{S(n_2,r_2)}(L(\lambda^2),\nabla(\mu^2))\Big).$$

Remark 3.13. (i) As for Theorem 3.9, we have only stated the type A version of [CPS04, Corollary 10]. The general version holds for all reductive groups.
(ii) Even in type A, the result in [CPS04] is actually stated rather differently, corresponding to the form of Theorem 3.9 given in [Erd95]. However a standard choice of Levi subgroups, together with an easy application of the Künneth formula, gives the result in the form stated above.
(iii) This theorem and Theorem 3.9 are both special cases of a theorem of Donkin's proved in [LM04, Theorem 4.2].
(iv) Note that we cannot combine Theorem 3.12 and the second part of Corollary 3.6 to obtain an analogue of Corollary 3.11, as the Ext-groups in each case are no longer of the same form. Indeed if we consider whether $\mathrm{Ext}^m_{H_{q,n}}(D^\lambda,S^\mu)$ is the same as

$$\bigoplus_{m=m_1+m_2}\Big(\mathrm{Ext}^{m_1}_{H_{q,n_1}}(D^{\lambda^1},S^{\mu^1})\otimes\mathrm{Ext}^{m_2}_{H_{q,n_2}}(D^{\lambda^2},S^{\mu^2})\Big).$$

where (λ,μ) has a v-cut or a h-cut, then this is false in general, as the following counter example shows.

We know that $\mathrm{Ext}^m_{H_{q,n}}(D^\lambda,S^\mu)\cong\mathrm{Ext}^m_{S(n,r)}(\nabla(\lambda'),L(\mu'))$ for $m\leq l-3$, using Corollary 3.6, so we give a counter example for algebraic groups. Let $l\geq 4$, $\lambda=(l-1,1)$, $\lambda^1=(2,1)$ and $\lambda^2=(l-3)$. So $\lambda'=(2,1^{l-2})$, $\lambda^{1'}=(2,1)$ and $\lambda^{2'}=(1^{l-3})$. The induced module $\nabla(2,1^{l-2})$ is not simple, it has two composition factors — namely $L(1^l)$ as its head and $L(2,1^{l-2})$ as its socle. Thus $\mathrm{Hom}(\nabla(\lambda'),L(\lambda'))=0$. But $\mathrm{Hom}(\nabla(\lambda^{1'}),L(\lambda^{1'}))=k$ and $\mathrm{Hom}(\nabla(\lambda^{2'}),L(\lambda^{2'}))=k$ as both these induced modules are simple and so the formula cannot hold in general.

Similarly if $\lambda'=(3,1^{l-3})$, $\lambda^{1'}=(1^{l-2})$ and $\lambda^{2'}=(2)$ then we also get $\mathrm{Hom}(\nabla(\lambda'),L(\lambda'))=0$, but $\mathrm{Hom}(\nabla(\lambda^{1'}),L(\lambda^{1'}))\otimes\mathrm{Hom}(\nabla(\lambda^{2'}),L(\lambda^{2'}))=k$.

We present one more general type of result before we look at which Ext-groups have been explicitly calculated. This is a generalisation of a recent result of Fayers [Fay].

We first define a function on X^+. Given an integer s, we define $\mathcal{D}_{(s,n)}\lambda$ for $\lambda\in X^+$ to be the dominant weight $(s-\lambda_n,s-\lambda_{n-1},\dots,s-\lambda_1)$. (This is the $\check{\lambda}$ of Fayers [Fay], but we want to emphasise the s and n.) Let $D=L(1,1,\dots,1)$ be the determinant module for GL_n. We interpret $D^{\otimes s}$ to be D tensored with itself s times if s is positive and to be the module $L(-1,-1,\dots,-1)\cong D^*$ tensored with itself $-s$ times if s is negative. We set $D^{\otimes 0}$ be the trivial module $L(0,0,\dots,0)\cong k$.

Lemma 3.14. *For all M, $N\in\mathrm{mod}(G)$ and $i\geq 0$, we have a canonical isomorphism*

$$\mathrm{Ext}^i_G(M,N)\cong\mathrm{Ext}^i_G((M^*)^\circ\otimes D^{\otimes s},(N^*)^\circ\otimes D^{\otimes s}).$$

Proof. We have that

$$\mathrm{Ext}^i_G((M^*)^\circ \otimes D^{\otimes s}, (N^*)^\circ \otimes D^{\otimes s}) \cong \mathrm{Ext}^i_G((M^*)^\circ, (N^*)^\circ)$$

as the module $D^{\otimes s}$ is one-dimensional. This latter Ext group is isomorphic to

$$\mathrm{Ext}^i_G(N^*, M^*) \cong \mathrm{Ext}^i_G(M, N)$$

after removing both duals. □

Thus we may define a functor $\mathcal{D}_{(s,n)} : \mathrm{mod}(G) \to \mathrm{mod}(G)$ which takes $M \in \mathrm{mod}(G)$ to the module $((M^*)^\circ) \otimes D^{\otimes s}$, and this is an equivalence on $\mathrm{mod}(G)$. The reason for abusing notation and reusing the notation $\mathcal{D}_{(s,n)}$ is made clear by the next lemma.

Lemma 3.15. *Let $s \in \mathbb{Z}$, and λ, $\mu \in X^+$. Then*

(i) $\nabla(\mathcal{D}_{(s,n)}(\lambda)) \cong \mathcal{D}_{(s,n)}(\nabla(\lambda))$
(ii) $\Delta(\mathcal{D}_{(s,n)}(\lambda)) \cong \mathcal{D}_{(s,n)}(\Delta(\lambda))$
(iii) $L(\mathcal{D}_{(s,n)}(\lambda)) \cong \mathcal{D}_{(s,n)}(L(\lambda))$
(iv) $T(\mathcal{D}_{(s,n)}(\lambda)) \cong \mathcal{D}_{(s,n)}(T(\lambda))$

Proof. We prove (i), the other statements are similar. We have $\nabla(\lambda)^* \cong \Delta(-w_0\lambda)$ where w_0 is the longest element of the associated Weyl group $\mathfrak{S}_n$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ then $-w_0\lambda = (-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1)$. We thus have $(\nabla(\lambda)^*)^\circ \cong \nabla(\mathcal{D}_{(0,n)}(\lambda))$. Hence $\mathcal{D}(\nabla(\lambda)) \cong \nabla(\mathcal{D}_{(0,n)}(\lambda)) \otimes D^{\otimes s} \cong \nabla(\mathcal{D}_{(s,n)}(\lambda))$. □

Corollary 3.16. *For all λ, $\mu \in X^+$, $s \in \mathbb{Z}$, and $i \geq 0$ we have*

$$\mathrm{Ext}^i_G(A(\lambda), B(\mu)) \cong \mathrm{Ext}^i_G(A(\mathcal{D}_{(s,n)}(\lambda)), B(\mathcal{D}_{(s,n)}(\mu)))$$

where $A(-)$ and $B(-)$ are any of $\nabla(-)$, $\Delta(-)$, $L(-)$ and $T(-)$.

When all the weights in the above corollary are polynomial we get the same isomorphism for the q-Schur algebra. Using Corollary 3.6 we obtain the following result for the corresponding Hecke algebra.

Theorem 3.17. *Suppose that λ, $\mu \in \Lambda^+(n, r)$ and that s is greater than λ_1 and μ_1. Then for $l \geq 3$ and $0 \leq i \leq l-3$ we have that*

$$\mathrm{Ext}^i_H(S^\lambda, S^\mu) \cong \mathrm{Ext}^i_H(S^{\mathcal{D}_{(s,n)}(\lambda)}, S^{\mathcal{D}_{(s,n)}(\mu)})$$

and

$$\mathrm{Ext}^i_H(D^{\lambda'}, S^{\mu'}) \cong \mathrm{Ext}^i_H(D^{\mathcal{D}_{(s,n)}(\lambda')}, S^{\mathcal{D}_{(s,n)}(\mu')})$$

We also get the first isomorphism for $i = 0$ and $l = 2$ when either λ is row 2-regular or μ is column 2-regular.

Proof. We combine Theorem 3.5 and Theorem 3.1 and apply them to the previous corollary. (Note that λ is row l-regular if and only if $\mathcal{D}_{(s,n)}(\lambda)$ is row l-regular) □

Remark 3.18. (i) In the case $i = 0$ and $p \geq 3$, a proof of the first part of this result located entirely in the symmetric group setting has been given by Fayers [Fay]. It is also known to be false in full generality for $p = 2$.
(ii) We have only stated the type A version of the equivalence of categories given by $\mathcal{D}_{(s,n)}$, and of Corollary 3.16. The general version holds for all reductive groups.
(iii) In type A, this equivalence has been realised via an isomorphism of generalised Schur algebras by Feng, Henke, and König [FHK].

4. Calculating Ext-groups I: Homomorphisms

In the remaining sections we will consider various results which give explicit values for certain Ext-groups. We begin with the homomorphism case. Very little is known about homomorphisms between Weyl (or Specht) modules in general. Almost all the general results are either for weights which are 'close together' for example those of [Kop86], or those where the calculation can be reduced to a low rank case using Theorem 3.9. The only exception to this, due to Carter and Payne, still only considers weights which are related by a single reflection (in the alcove-theoretic sense). As many of the results in this section are most naturally expressed in this geometric language we will switch freely between the partition and alcove contexts. In this section we take $q = 1$ unless explicitly indicated otherwise.

We say that a weight μ is a *Steinberg* weight if $\mu_i - \mu_{i+1} \equiv -1 \pmod p$ for all i. We let W_p be the affine Weyl group for G which acts on X via the dot action. This defines a system of hyperplanes and facets for X: details may be found in [Jan87, II, Section 1.5 and Chapter 6]. Two dominant weights can only be in the same block if they are in the same W_p-orbit. Koppinen [Kop86, Theorem 7.1] proves the following.

Theorem 4.1 (Koppinen)**.** *Let μ be a dominant Steinberg weight, and W_p^μ be the stabiliser of μ in W_p. Take λ lying in a facet whose closure contains μ and set $J = W_p^\mu \cdot \lambda$. Then*
(i) If ξ, $\xi' \in J$ and $\xi \geq \xi'$ then $\mathrm{Hom}_G(\nabla(\xi), \nabla(\xi')) \cong k$.
(ii) If ξ, ξ', $\xi'' \in J$ and $\nabla(\xi) \to \nabla(\xi') \to \nabla(\xi''')$ are non-zero homomorphisms then the composite map is nonzero.

This theorem has been slightly generalised by Wen [Wen89], who also has more to say about the higher Ext groups. We will return to these results in section 7.

Let $\lambda, \mu \in \Lambda^+$. Suppose that there exists a pair $i > j$, and integers $n > 0$ and m such that μ is obtained from λ by moving d boxes from the ith row to the jth row, with $d = \lambda_i - \lambda_j - i + j - mp^a < p^a$. In this case we say that μ is a *local p^a-reflection of* λ.

We have the following theorem due to Carter and Payne [CP80].

Theorem 4.2 (Carter-Payne)**.** *Let μ be a local p^a-reflection of λ for some $a \geq 0$. Then we have*

$$\mathrm{Hom}_{S(n,r)}(\nabla(\lambda), \nabla(\mu)) \neq 0$$

and if $p > 2$ then

$$\mathrm{Hom}_{k\mathfrak{S}_r}(S^\lambda, S^\mu) \neq 0.$$

This is proved first for SL_n by looking at the action of the hyperalgebra on Weyl modules, and then by applying Theorem 3.1 to get the result for the symmetric group. It has recently been generalised by Fayers and Martin [FM04], who work in the symmetric group setting by looking at semi-standard homomorphisms. We will return to this generalisation in Section 5.

Remark 4.3. (i) If $j = i + 1$ then the corresponding quantum result also holds, but replacing occurences of p^a by lp^{a-1}. Further, if $j = i + 1$ then these are the only homomorphisms for the Schur algebra, and for the Hecke algebra if $l \geq 3$. Thus in both cases the Hom-spaces are at most one-dimensional. This result is well-known; a proof can be found in [CE00, Theorem 5.1].
(ii) The reflection terminology is motivated by the alcove-geometric approach to weights. Using the standard dot action of the Weyl group for G, this condition corresponds to μ being the reflection of λ about some p^a-hyperplane P such that no parallel p^a-hyperplane lies between λ and P.
(iii) The condition $p > 2$ is necessary for the symmetric group result. For example when $p = 2$ we have that $S^{(1,1)}$ is isomorphic to $S^{(2)}$ and so the corresponding Hom-space will be non-zero.
(iv) We expect the corresponding quantum result to hold. However the proof in the classical case requires explicit manipulation of the action of the hyperalgebra, which will be much more complicated in the quantum case. The quantum result is known to hold if $j = i + 1$ as in (i) above or if $j = i + 2$, see [Para, theorem 9.1] and apply Theorem 3.9.

Kulkarni [Kula, Kulb] has calculated certain Ext groups between Weyl modules over the integers, and his results have consequences for the modular theory. In particular, he shows that

$$\dim \mathrm{Hom}(\nabla(r, 0, \ldots, 0), \nabla(\lambda)) \leq 1$$

and can describe precisely when the Hom-space is non-zero [Kula, Theorem 2.2 and the following Remark (3)].

Apart from the GL_2-result mentioned above, the only other classical case where homomorphisms have been completely determined is for GL_3, (with $p > 2$) where we have recently given a complete classification [CP]. This is given by a complicated recursive set of interlocking results, expressed in the language of alcove geometry, which is too lengthy to reproduce here. Instead we will outline the main features of the classification.

The basic idea is to use certain nice filtrations of induced modules, called good p-filtrations. These exist for all induced modules, and for GL_3 the structure of these filtrations has been completely determined in [Par01]. A *good p-filtration* is one where successive quotients are all of the form $\nabla_p(\mu) = \nabla(\mu'')^F \otimes L(\mu')$ where $\mu = \mu' + p\mu''$ for some dominant weight μ'' and p-restricted weight μ'. Given a pair of such modules $\nabla_p(\mu)$ and $\nabla_p(\tau)$ we have

$$\mathrm{Hom}_G(\nabla_p(\mu), \nabla_p(\tau)) \cong \begin{cases} \mathrm{Hom}_G(\nabla(\mu''), \nabla(\tau'')) & \text{if } \mu' = \tau' \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

which enables us to proceed by induction.

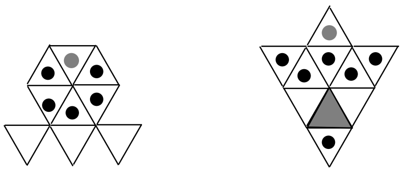

FIGURE 1.

When the weight λ is small then all terms in the good p-filtration of $\nabla(\lambda)$ are simple, and the homomorphisms from $\nabla(\lambda)$ can be classified using the results in [Par01]. For example, suppose that λ lies in an alcove labelled by a lightly shaded dot in Figure 1, and suppose it is such that all nine unshaded alcoves in the respective diagram are dominant. Then these alcoves label the terms in the good p-filtration for $\nabla(\lambda)$, and the marked alcoves indicate the possible homomorphisms in the case where these terms are all simple.

In the general case we use translation functor arguments to translate from a Steinberg weight θ just below λ — where we know all homomorphisms by induction, as $\nabla(\theta) \cong \nabla_p(\theta)$. (If no such λ exists we use an alternative argument based on an identification of certain induced modules with symmetric powers.) Each homomorphism from $\nabla(\theta)$ to some $\nabla(\tau)$ gives rise to the possibility of a map from $\nabla(\lambda)$ into certain induced modules labelled by weights 'near' to τ. This, together with the known block structure of the module category, provides a necessary condition for there to be a non-zero homomorphism from $\nabla(\lambda)$ to $\nabla(\mu)$. As $L(\mu)$ is the socle of $\nabla(\mu)$ another obvious necessary condition is that $L(\mu)$ is a composition factor of $\nabla(\lambda)$.

It is almost always the case that this pair of necessary conditions is also sufficient. We can also give an explicit description of the exceptions to this rule. All these cases are most easily described graphically in a series of diagrams similar in nature to those in Figure 1, with one such diagram for each homomorphism from $\nabla(\theta)$.

To show that these conditions are sufficient, it is enough to construct homomorphisms in each of the remaining cases. There are two obvious classes of homomorphisms already at our disposal. The first of these are the Carter-Payne maps coming from Theorem 4.2, and we begin by determining a necessary condition for composites of such maps to be non-zero. The second obvious class consists of those from the top term in the good p-filtration (which is uniquely defined). By (4) these are twists of maps from induced modules labelled by smaller weights, which are assumed known by induction.

By considering all the various possible cases we can show that almost all non-zero homomorphisms between induced modules are either composites of Carter-Payne maps or are twisted maps. Unfortunately, there are again exceptions to this rule, arising from certain explicit configurations of weights which we can completely classify. These exceptional maps can be constructed from pairs of maps on terms in the good p-filtration.

At several key stages during the proof we need the following result, which is also proved as part of the induction argument

Theorem 4.4. *Suppose that $p > 2$. For any pair λ, μ of dominant weights for GL_3, we have*

$$\dim \mathrm{Hom}(\nabla(\lambda), \nabla(\mu)) \leq 1.$$

Putting this together with the argument outlined above, and Theorem 3.1 we obtain

Theorem 4.5. *Suppose that $p > 2$. For any pair λ, μ of dominant weights for GL_3, we can classify all homomorphisms between $\nabla(\lambda)$ and $\nabla(\mu)$. If λ and μ are both polynomial weights then we can also classify all homomorphisms between S^λ and S^μ.*

Remark 4.6. (i) A similar result holds for the quantum case. The main obstacle to proving this in [CP] was the absence of quantum versions of the good p-filtration structure theorems from [Par01] and of the Carter-Payne theorem. The former, and a special case of the latter sufficient for this problem, have now been proved by the second author [Para].
(ii) It is hard to see how the result for GL_3 could be stated using combinatorics of partitions. We believe that the alcove-theoretic formalism has been under-utilised in the Hecke algebra setting; further applications of this formalism can be found in [Cox].
(iii) One might hope that all Hom-spaces were at most one dimensional, by analogy with the Verma module case. However, while we know of no examples of Hom-spaces for induced modules which are greater than one dimensional, this may simply be because such Hom-spaces are only known in certain very special cases. The methods used to date in such calculations provide no compelling indication either way. As we will see in Remark 5.4(iii), a potential source of large Hom-spaces has been given by Fayers and Martin [FM04]; however no actual examples have yet been found.

5. A generalisation of the Carter-Payne theorem

In this section we assume that $q = 1$. Fayers and Martin [FM04] have given a sufficient condition for the existence of a non-zero homomorphism from S^λ to S^μ when μ is obtained from λ by moving s boxes from one row of λ to a lower row. The condition is rather a complicated one; we will review the basic combinatorial framework needed to state the result, and then indicate a few simplifications that can be made. (In a similar vein, Künzer [Kün00, Kün] has considered the cases $s = 1$ and 2, giving an explicit construction of a morphism.) In what follows we assume standard facts about tableaux etc.

Let T be a λ-tableau of weight μ (i.e. shape λ and with μ_i entries equal to i). For a tableau T let T_j^i be the number of entries equal to i in row j, and $T_h = \sum_{j>h} T_h^j$. Given integers a and b we set $a^{b\downarrow} = \prod_{i=0}^{b-1}(a-i)$ and $a^{b\uparrow} = \prod_{i=0}^{b-1}(a+i)$.

Using Corollary 3.11 , we may reduce to the case where

$$\lambda = (l_0 + s, l_1^{m_1-1}, l_2^{m_2-m_1}, \ldots, l_r^{m_r-m_{r-1}}) \quad \text{and} \quad \mu = (l_0, l_1^{m_1-1}, l_2^{m_2-m_1}, \ldots, l_r^{m_r-m_{r-1}}, s).$$

Pick $\gamma_i \in \mathbb{Z}$ for $1 \leq i \leq r$, and set $m_0 = 1$. With these assumptions we define $n_i(T) = \sum_{m_{i-1}+1}^{m_i} T_h$ and

$$c_i(T) = \gamma_i^{(s-n_i(T))\downarrow} \prod_{h=m_{i-1}+1}^{m_i} T_h!.$$

We will need two functions f and g. The first is easy to define:

$$f(T) = \prod_{i=1}^{r} c_i(T).$$

Given a partition μ, and integers $d \geq 1$ and $0 \leq t \leq \mu_{d+1}$ we define a new *composition* $\nu = \nu(\mu, d, t)$ by setting

$$\nu_i = \begin{cases} \mu_i + t & \text{if } i = d \\ \mu_i - t & \text{if } i = d+1 \\ \mu_i & \text{otherwise.} \end{cases}$$

Given λ, we define $g_{\mu,b,t}(S)$ for a λ-tableau S of type $\nu(\mu, m_b, t)$ by

$$g_{\mu,b,t}(S) = \begin{cases} f(S)\frac{(l_0-l_1+m_1-1+s-\gamma_1)^{t\uparrow}}{t!} & \text{if } b = 0 \\ \frac{f(S)}{c_b(S)}\frac{(l_b-l_{b+1}+m_{b+1}-m_b+\gamma_b-\gamma_{b+1})^{t\uparrow}}{t!}(\gamma_b^{(s-n_b(S)-t)\downarrow}(S_{m_b}+t)!)\prod_{j=m_{b-1}+1}^{m_b-1} S_j! & \text{if } 0 < b < r \\ \frac{f(S)}{c_b(S)}\frac{(l_r-s+1+\gamma_r))^{t\uparrow}}{t!}(\gamma_b^{(s-n_b(S)-t)\downarrow}(S_{m_b}+t)!)\prod_{j=m_{b-1}+1}^{m_b-1} S_j! & \text{if } b = r. \end{cases} \tag{5}$$

Finally, we say a tableau is *pseudo-standard* if the entries weakly increase along rows and T_j^i is non-zero only if $i = j$ or $\lambda_i < \lambda_j$. A *nice* tableau is one where the number of entries greater than m_i in the set of all rows of length l_i is at most $l_i - l_{i+1}$, for each i.

The following result is due to Fayers and Martin [FM04, Theorem 22], and (although this is not obvious) generalises Theorem 4.2.

Theorem 5.1 (Fayers-Martin)**.** *Suppose that λ and μ are as above, and that we can pick integers $e > 0$ and γ_i such that:*

(i) *For some nice pseudo-standard λ-tableau T of type μ the coefficient $f(T)$ is not divisible by p^e.*
(ii) *For all $0 \leq i \leq r$ and $1 \leq t \leq \mu_{m_i+1}$, and all pseudo-standard λ-tableaux S of type $\nu(\mu, m_i, t)$, the coefficient $g_{\mu,i,t}(S)$ is divisible by p^e.*

Then there exists a non-zero homomorphism from S^λ to S^μ.

It will be useful to simplify the expression for $g_{\mu,b,t}(S)$. Let $\gamma_0 = s$, $m_{r+1} = m_r + 1$, $l_{r+1} = s$, and $\gamma_{r+1} = 0$. Then the expression above simplifies to

$$g_{\mu,m_b,t}(S) = f(S)\frac{(l_b - l_{b+1} + m_{b+1} - m_b + \gamma_b - \gamma_{b+1})^{t\uparrow}}{t!}(S_d + t)^{t\downarrow}\frac{1}{(\gamma_b + n_b(S) + t - s)^{t\downarrow}} \tag{6}$$

for all $0 \leq b \leq r$.

This theorem is rather difficult to work with. As has already been noted, it gives a generalisation of Theorem 4.2, but even this is not clear. (Fayers and Martin do not give Theorem 4.2 as a direct consequence of their main result, but instead show that it can be proved by methods which are then generalised to give Theorem 5.1.)

There are many freely chosen variables, and for a given λ and μ potentially many g's to calculate. It is not even clear as stated where the requirement that λ and μ are in the same block will play a role. We will show that the choice of the γ_i is essentially unique (modulo p), and how the block restriction then appears in the combinatorics.

Proposition 5.2. *Suppose that for given λ and μ the integers γ_i are such that conditions (i) and (ii) of Theorem 5.1 hold. Then we must have*

$$\gamma_i \equiv l_0 - l_i + m_i + s - 1 \pmod{p}$$

for $1 \leq i \leq r$, and

$$l_0 + m_r \equiv 0 \pmod{p}.$$

Proof. First suppose that $d = 1 (= m_0)$ and that T is as in Theorem 5.1. We will take $t = 1$ and construct a new tableau S. If there is a 2 in the first row of T let S be the tableau obtained by replacing this with a 1; then $f(S) = f(T)$. By assumption we know that p^e divides $f(S)$ but not $f(T)$, and hence from (5) we deduce that

$$\gamma_1 \equiv l_0 - l_1 + m_1 - 1 + s \pmod{p}$$

as required.

If there is not a 2 in the first row of T then there must be some entry greater than 2 in the first row: exchange this for a 2 from the second row (which must contain only 2s) and then replace the 2 by a 1. In this case we have $n_1(S) = n_1(T) + 1$ and $S_2! = 1! = 0! = T_2$. Therefore

$$f(T) = (\gamma_1 - s + n_1(T))f(S)$$

and from (5) we deduce that

$$g(S)(\gamma_1 - s + n_1(T)) = f(T)(l_0 - l_1 + m_1 - 1 + s - \gamma_1).$$

As p^e divides $g(S)$ but not $f(T)$ we must have that p divides $(l_0 - l_1 + m_1 - 1 + s - \gamma_1)$, as required.

The cases $d > 1$ are similar; we set $t = 1$ and indicate how to construct a suitable S from the initial choice of T.

Let $d = m_b$ with $1 \leq b < r$. We will proceed by induction on b; the case $b = 0$ having already been considered, and hence assume that γ_b has already been determined. First suppose that $T_d > 0$. Then there exists a d in T in a row above row d and an entry $a > d$ at the end of row d. If there exists a $d+1$ in one of the first d rows then replace it by d and swap this new entry with a (if it is not already the entry a)to form S. In this case $S_d = T_d - 1$ and $n_b(S) = n_b(T) - 1$ and we have

$$f(S) = f(T)\frac{(\gamma_b - s + n_b(T))}{T_d\delta} \tag{7}$$

where $\delta = 1$. If there is not a $d+1$ in one of the first d rows then $T_{d+1} = 0$, and we let S be the tableau obtained from T by replacing the final element of row $d+1$ by d and swapping it with the $a > d+1$ at the end of row d. Then $S_d = T_d - 1$ with $n_b(S) = n_b(T) - 1$, and $S_{d+1}! = T_{d+1}! = 1$ with $n_{b+1}(S) = n_{b+1}(T) + 1$ and $f(S)$ is given by equation (7) where $\delta = \gamma_{b+1} - s + 1 + n_{b+1}(T)$.

Thus when $T_d > 0$ we have from (7) and (6) that

$$g(S) = f(T)\frac{(l_b - l_{b+1} + m_{b+1} - m_b + \gamma_b - \gamma_{b+1})(S_d + 1)(\gamma_b + n_b(S) + 1 - s)}{(S_d + 1)\delta(\gamma_b + n_b(S) + 1 - s)}$$

and hence

$$\delta g(S) = f(T)(l_b - l_{b+1} + m_{b+1} - m_b + \gamma_b - \gamma_{b+1}).$$

Arguing as in the $d = 1$ case we deduce that

$$l_b - l_{b+1} + m_{b+1} - m_b + \gamma_b - \gamma_{b+1} \equiv 0 \pmod p$$

which by induction implies that γ_{b+1} is of the required form.

If $T_d = 0$ then there exists some entry $a \geq d+1$ in a row above row d. If $T_{d+1} > 0$ then we may chose $a = d+1$, and S is obtained by replacing a by d, and $f(S) = f(T)$ (and $S_d + 1 = 1$). If $T_{d+1} = 0$ then change the final $d+1$ to a d and swap this entry with the entry a. In either case it is now easy to verify as above that γ_{b+1} must be of the required form.

Finally, suppose that $d = m_r$ (and so λ has no row $d+1$). If there exists a $d+1$ in a row above row d in T let S be obtained by replacing this with a d. Otherwise there must be a $d+1$ in row d; again replace this by a d. In either case by arguing as above we see that we must have

$$\gamma_r + 1 \equiv l_0 - l_{r+1} + m_{r+1} + s - 1 \pmod p.$$

But we fixed $l_{r+1} = s$, $\gamma_{r+1} = 0$ and $m_{r+1} = m_r + 1$, which implies that

$$l_0 + m_r \equiv 0 \pmod p$$

as required. □

Remark 5.3. (i) Suppose that λ and μ are both partitions of a with at most n parts (for some a and n), such that the conditions in Theorem 5.1 are satisfied. Recall the description of the blocks of the symmetric group: D^λ and D^μ are in the same block of $k\Sigma_a$ if and only if there exists a permutation $\sigma \in \Sigma_n$ such that

$$\lambda_i - i \equiv \mu_{\sigma(i)} - \sigma(i) \pmod p.$$

By our initial assumptions on λ and μ we have $\lambda_i = \mu_i$ for $1 < i < m_r$ and hence to be in the same block we must have either $\lambda_i \equiv \mu_i \pmod p$ for all i or $\lambda_1 - 1 \equiv \mu_{m_r+1} - (m_r + 1) \pmod p$. But this latter condition

becomes $l_0 + s - 1 \equiv s - m_m - 1 \pmod p$, which is automatically satisfied by Proposition 5.2.
(ii) Fayers and Martin have already observed [FM04, Remark after Theorem 22] that their result is most useful when a stronger version of the conditions in this Proposition hold: taking all but one of the congruences for the γ_i to be equalities. This observation was based on empirical evidence; our result provides theoretical corroboration for this.

Unfortunately, the conditions in Theorem 5.1 are very difficult to work with, even after introducing the simplifications in Proposition 5.2. We conclude this section with a brief discussion of what is known, and what might be hoped to be true.

Remark 5.4. (i) The only new example given in [FM04] as an application of their main theorem is the case $\lambda = (7,3)$ and $\mu = (4,3^2)$, with $p = 3$. This case would be covered if we could relax the condition on d in the definition of local p^a-reflection from $d < p^a$ to $d \leq p^a$. (Of course there would have to be further conditions present, as we cannot relax the definition in this way for two-part partitions.) It would be interesting to know whether there are any examples given by Theorem 5.1 which are not of this form.
(ii) There certainly are more general pairs of weights related by a single reflection than would be obtained simply by relaxing the above inequality; an examples for three-part partitions is given by the homomorphism from the highest to lowest alcove in the righthand diagram in Figure 1. As stated Theorem 5.1 gives only a sufficient condition for the existence of a homomorphism; it would be interesting to know whether it is also necessary.
(iii) Fayers and Martin have also provided a potential source of greater than one-dimensional Hom-spaces. They introduce the notion of a *good* quasi-standard tableaux, and prove that homomorphisms corresponding to such tableaux from Theorem 5.1 are linearly independent [FM04, Proposition 23]. However, we know of no examples of pairs of such tableaux — and of course they cannot exist for any of the examples where Hom-spaces have already been calculated.

6. Calculating Ext-groups II: The q-GL_2 case

In this section we will consider the problem of determining all Ext groups between pairs of induced modules for q-GL_2. This is the only case in which a complete answer can be given.

The first work in this area was by Erdmann [Erd95], who determined all the Ext^1 between induced modules for GL_2. This result was quantised by the first author [Cox98], who with Erdmann refined the methods used in order to determine Ext^2 between induced modules [CE00]. This work was based on computations for the corresponding Frobenius kernel (where everything can be determined relatively easily), followed by an elementary application of the Lyndon-Hochschild-Serre spectral sequence to transfer these results to the algebraic group. Similar methods were used by De Visscher [deV02] to calculate Ext^1 between twisted tensor products of Weyl modules and induced modules.

Recent work of the second author [Parb] has determined Ext^i for all i between induced modules for q-GL_2 (including the classical case $q = 1$), as well as $\mathrm{Ext}^i(\nabla(\lambda), L(\mu))$, $\mathrm{Ext}^i(L(\lambda), \nabla(\mu))$ and $\mathrm{Ext}^i(L(\lambda), L(\mu))$, for λ, $\mu \in X^+$. By Corollary 3.6 this result for induced modules also calculates the Ext^i between Specht modules for 2-part partitions provided that $i \leq l - 3$. We will describe the main features of this result.

The main idea is to refine the method first used by Erdmann, and consider in detail the spectral sequence involved. By giving an alternative derivation of this sequence, it becomes possible to describe in detail various pages in the sequence. From this can be deduced various reduction theorems for Ext-groups, which allow any Ext-group to be computed by induction on the weights.

In order to state the main results explicitly we focus for the moment on two part partitions. We will distinguish the modules for classical $\mathrm{GL}_2(k)$ and the quantum group q-$\mathrm{GL}_2(k)$ by putting a bar on the modules for the classical groups. The two are related by the Frobenius morphism $\mathrm{F} : q\text{-}\mathrm{GL}_2(k) \to \mathrm{GL}_2(k)$, and we have an map of module categories which takes a module for classical $\mathrm{GL}_2(k)$ to a module for the quantum group q-$\mathrm{GL}_2(k)$, namely the map that sends $\overline{M}$ to the twisted module $\overline{M}^{\mathrm{F}}$.

Theorem 6.1. *For $a \geq b$ with $a-b$ odd, $0 \leq i \leq l-2$ and $m \in \mathbb{N}$ we have*

$$\begin{aligned}&\mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\nabla(la+i,0),\nabla(lb+l-2-i+d,d))\\&\quad\cong \mathrm{Ext}^{m-1}_{q-\mathrm{GL}_2(k)}(\nabla(la-1,i+1),\nabla(lb+l-i-2+d,d)) \oplus \mathrm{Ext}^m_{\mathrm{GL}_2(k)}(\overline{\nabla}(a-1,0),\overline{\nabla}(b+f,f))\end{aligned}$$

where Ext^{-1} is interpreted as the zero module, $f=\frac{a-b-1}{2}$ and $d=lf+i+1$.

Theorem 6.2. *For $a \geq b$ with $a-b$ even, $0 \leq i \leq l-2$ and $m \in \mathbb{N}$ we have*

$$\mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\nabla(la+i,0),\nabla(lb+i+d,d)) \cong \mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\nabla(l(a-b)+i,0),\nabla(i+d,d))$$

where $d=l(\frac{a-b}{2})$. If $m \geq 1$ then also

$$\mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\nabla(l(a-b)+i,0),\nabla(i+d,d)) \cong \mathrm{Ext}^{m-1}_{q-\mathrm{GL}_2(k)}(\nabla(l(a-b)-1,i+1),\nabla(i+d,d)).$$

Theorem 6.3. *For $a \geq b$ with $a-b$ even and $m \in \mathbb{N}$ we have*

$$\mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\nabla(la+l-1,0),\nabla(lb+l-1+d,d)) \cong \mathrm{Ext}^m_{\mathrm{GL}_2(k)}(\overline{\nabla}(a,0),\overline{\nabla}(b+f,f))$$

where $f=\frac{a-b}{2}$ and $d=lf$.

For simplicity we have stated these results in terms of a weight of the form $(c,0)$. We have for all $b>0$ that

$$\nabla(a,b) \cong \nabla(a-b,0) \otimes \det{}^b$$

(where det is the one dimensional module corresponding to the determinant), and so all other cases can be reduced to this one. By block considerations and equation (3) any non-zero Ext-group can be reduced to one of the forms considered in the above Theorems, and hence (as we already know the $m=0$ results and using Theorem 7.1) any Ext-group can be computed by induction. For $m \leq 2$ explicit closed forms for these results were given in [CE00]. Even for such small degrees the closed forms quickly become unmanageable, and in general a recursive formula seems to be the best that can be expected.

A recent calculation of Erdmann and Hannabuss [EH] using the result for $p=2$ has shown that in this case the dimension of the Ext group $\mathrm{Ext}^m_{\mathrm{GL}_2(k)}(\nabla(2a,0),\nabla(a,a))$ is the same as the number of partitions $(\eta_1,\eta_2,\ldots,\eta_{m+1})$ with $2^{\eta_1}+2^{\eta_2}+\cdots+2^{\eta_{m+1}}-1=a$.

We also have

Theorem 6.4. *For $a \geq b$ with $a-b$ odd, $0 \leq i \leq l-2$ and $m \in \mathbb{N}$ we have*

$$\begin{aligned}&\mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\nabla(la+i,0),L(lb+l-2-i+d,d))\\&\quad\cong \mathrm{Ext}^{m-1}_{q-\mathrm{GL}_2(k)}(\nabla(la-1,i+1),L(lb+l-i-2+d,d)) \oplus \mathrm{Ext}^m_{\mathrm{GL}_2(k)}(\overline{\nabla}(a-1,0),\bar{L}(b+f,f))\end{aligned}$$

where Ext^{-1} is interpreted as the zero module, $f=\frac{a-b-1}{2}$ and $d=lf+i+1$.

Theorem 6.5. *For $a \geq b$ with $a-b$ even, $0 \leq i \leq l-2$ and $m \in \mathbb{N}$ we have*

$$\mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\nabla(la+i,0),L(lb+i+d,d)) \cong \mathrm{Ext}^{m-1}_{q-\mathrm{GL}_2(k)}(\nabla(la-1,i+1),L(lb+i+d,d))$$

where $d=l(\frac{a-b}{2})$.

and

Theorem 6.6. *For $a \geq b$ with $a-b$ even, and $m \in \mathbb{N}$ we have*

$$\mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\nabla(la+l-1,0),L(lb+l-1+d,d)) \cong \mathrm{Ext}^m_{q-\mathrm{GL}_2(k)}(\overline{\nabla}(a,0),\bar{L}(b+f,f))$$

where $f=\frac{a-b}{2}$ and $d=lf$.

7. Calculating Ext-groups III: Higher ranks

In this section we will complete our survey of general results for Ext-groups between pairs of induced or Specht modules. The results here can all be regarded as corresponding to cases where the pair of indexing weights λ and μ are 'close together'. To make this notion more precise we introduce the following function of λ.

Given $\lambda \in \Lambda^+(n,r)$ we define

$$d(\lambda) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left\lfloor \frac{\lambda_i - \lambda_j - i + j - 1}{l} \right\rfloor .$$

We will consider two weights λ and μ to be close if the difference $|d(\lambda) - d(\mu)|$ is small.

In what follows, most of the results were originally stated using alcove combinatorics and the standard length function on elements of the corresponding affine Weyl group. The correspondence between this and the partition combinatorics together with the function d is given in [Par03, Lemmas 3.5 and 5.1].

The first part of the following result is due to Andersen [And81, Proposition 3.5] and the second to Ryom-Hansen [RH03, Theorem 2.4]. (Both stated and proved in the classical case, but the quantum version follows in the same way.) In each case the cited results are for the case where one module is simple; the statements involving two induced modules follows easily from this, confer [Par03, Proposition 4.6].

Theorem 7.1 (Andersen, Ryom-Hansen)**.** *Suppose λ, $\mu \in \Lambda^+$. If $m > d(\lambda) - d(\mu)$ we have*

$$\mathrm{Ext}^m_{S(n,r)}(L(\lambda), \nabla(\mu)) \cong \mathrm{Ext}^m_{S(n,r)}(\nabla(\lambda), \nabla(\mu)) \cong 0.$$

Futhermore, if for all i and $j > i$ neither $\lambda_i - \lambda_j - i + j$ nor $\mu_i - \mu_j - i + j$ is divisible by l, and λ and μ are in the same block of G, then

$$\mathrm{Ext}^{d(\lambda)-d(\mu)}_{S(n,r)}(L(\lambda), \nabla(\mu)) \cong \mathrm{Ext}^{d(\lambda)-d(\mu)}_{S(n,r)}(\nabla(\lambda), \nabla(\mu)) \cong k.$$

Remark 7.2. In this paper we have tried to avoid using alcove-geometric notation, for simplicity of exposition. However, the conditions in the second part of this theorem (and Theorem 7.4) can be restated as requiring λ (and μ) to lie in the interior of an alcove, with λ and μ in the same orbit of the affine Weyl group W_l (under the usual 'dot' action on weights). When thus stated both results are valid for an arbitrary reductive algebraic group.

If $l - 3 \geq d(\lambda) - d(\mu)$ then an application of Corollary 3.6 would give the corresponding result for the Hecke algebra — but this is only going to be true for partitions that are close together.

The following very nice result of Wen [Wen89, theorem 8.3, lemma 8.3.3] gives the value of the Ext groups (in small degrees) between induced modules when we have weights which are close together and related by a particular type of element in the affine Weyl group. Recall that a weight λ is *strictly dominant* if $\lambda - (l-1)\rho$ is dominant. (Again, this result was stated and proved for the classical case, but the quantum version follows in the same way.)

Theorem 7.3 (Wen)**.** *Let $\lambda \in \Lambda^+(n,r)$ be inside an alcove and strictly dominant. Let S be a commutative subset of the reflections which fix the walls in the closure of the alcove containing λ. Let w be in the subgroup of W_l generated by S such that $w \cdot \lambda < \lambda$. Set $d = d(\lambda) - d(w \cdot \lambda)$. We have for each i with $0 \leq i \leq d$ that*

$$\dim \mathrm{Ext}^i_G(L(\lambda), \nabla(w \cdot \lambda)) = \begin{cases} 1 & \text{if } d = i \\ 0 & \text{otherwise.} \end{cases}$$

$$\dim \mathrm{Ext}^i_G(\nabla(\lambda), \nabla(w \cdot \lambda)) = \binom{d}{i}.$$

This result gives a special case of Theorem 7.1 but also finds all the Ext groups for weights satisifying the special conditions above. The usefulness of the result is limited by the size of the subset S and this has an upper bound of $\frac{n}{2}$. As usual, we have only stated the type A version; the general result holds for any reductive group.

In a similar spirit to Theorem 7.1 we have [Par03, Corollaries 4.4 and 4.5]

Theorem 7.4. *Suppose $\lambda, \mu \in \Lambda^+$. If $m > d(\lambda) + d(\mu)$ then we have*

$$\mathrm{Ext}^m_{S(n,r)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^m_{S(n,r)}(\nabla(\lambda), \Delta(\mu)) \cong 0.$$

Furthermore, if $\lambda_i - \lambda_j - i + j$ is not divisible by l for all i and $j > i$ and if $\mu_i - \mu_j - i + j$ is not divisible by l for all i and $j > i$ then we have

$$\mathrm{Ext}^{d(\lambda)+d(\mu)}_{S(n,r)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^{d(\lambda)+d(\mu)}_{S(n,r)}(\nabla(\lambda), \Delta(\mu)) \cong k.$$

Using these results, the second author has determined [Par03, Theorems 5.8 and 5.9] the global dimension of the Schur algebra $S(n, r)$ when $l > n$ or $l = n$ and $r \equiv 0 \pmod{l}$:

Theorem 7.5. *If $l > n$ then the global dimension of $S(n, r)$ is*

$$2(n-1)\left\lfloor \frac{r}{l} \right\rfloor.$$

The global dimension of $S(l, ml)$ is

$$2(l-1)m.$$

When $r \leq n$ the global dimension of $S(n, r)$ has been calculated by Totaro [Tot97] in the classical case, and Donkin [Don98, Section 4.8] in the quantum case.

Another sometimes useful fact about Ext groups is the following.

Lemma 7.6. *If $\mu \neq \lambda$ then*

$$\sum(-1)^i \dim \mathrm{Ext}^i_G(\nabla(\mu), \nabla(\lambda)) = 0$$

This is a consequence of [Jan87, II, 6.21(6)]. We use this to show that there are two dimensional Ext^1 groups for G for $l \geq 3$ and $n \geq 3$, even for weights which are close together. Suppose $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a dominant weight for G with $l \geq 3$. We further suppose that $\lambda_1 - \lambda_2 \not\equiv -1 \pmod{l}$, $\lambda_2 - \lambda_3 \not\equiv -1 \pmod{l}$, and $\lambda_1 - \lambda_3 \not\equiv -2 \pmod{l}$, (i.e. that λ lies in the interior of an alcove), and that $d(\lambda) \geq 2$. Consider a dominant weight $\mu < \lambda$ which is in the same block as λ, does not satisfy $\lambda - \mu \in \mathbb{N}\alpha$ where α is a root, and satisfies $d(\lambda) - d(\mu) = 2$. (Such a weight exists as $d(\lambda) \geq 2$.) We then have that $\mathrm{Hom}_G(\nabla(\lambda), \nabla(\mu)) \cong k$, using the results of [CP] or Theorem 4.1, and that $\mathrm{Ext}^2_G(\nabla(\lambda), \nabla(\mu)) \cong k$ and $\mathrm{Ext}^i_G(\nabla(\lambda), \nabla(\mu)) \cong 0$ for $i \geq 3$ using Theorem 7.1. Thus using Lemma 7.6 we have that

$$\mathrm{Ext}^1_G(\nabla(\lambda), \nabla(\mu)) \cong k^2.$$

So there are two-dimensional Ext^1 groups for $l \geq 3$, even for weights which are close together.

The above example shows that calculating Ext groups for modules for groups of rank at least three is quite complicated in general. Indeed, this problem may be intractable for groups of large rank. There is still possibly some hope of calculating Ext^1 between induced modules for $\mathrm{GL}_3(k)$, but the dimension of the Ext^1 groups may be unbounded for $n = 3$. This is in contrast to the $n = 2$ case where the Ext^1 groups between induced modules have dimension at most one. Theorem 7.3 shows that the dimension of the Ext^1 group can be as large as we like, as long as we take n large enough.

8. Ext-groups involving simple modules

In the paper [KS99] Kleshchev and Sheth ask when

$$\mathrm{Ext}^1_H(D^\lambda, D^\mu) \cong \mathrm{Hom}_H(\mathrm{rad}\, S^\lambda, D^\mu)?$$

One sufficient condition for the equality to hold will be if $\mathrm{Ext}^1_H(S^\lambda, D^\mu) \cong 0$. Kleshchev and Sheth prove the following for the classical case.

Theorem 8.1 ([KS99, theorems 2.9 and 2.10])**.** *Suppose $p \neq 2$, $\lambda \not> \mu$ and λ and $\mu \in \Lambda^+_{\text{row}}(p-1, r)$ then*

$$\mathrm{Ext}^1_H(S^\lambda, D^\mu) \cong \begin{cases} k & \text{if } \lambda = \mu = \epsilon_r \text{ and } r \geq p \\ 0 & \text{otherwise.} \end{cases}$$

where ϵ_r is the Mullineax map applied to the partition $(r, 0, 0, \ldots)$ and

$$\mathrm{Ext}^1_H(D^\lambda, D^\mu) \cong \mathrm{Hom}_H(\mathrm{rad}\, S^\lambda, D^\mu).$$

This motivates the following lemma.

Lemma 8.2. *If λ, $\mu \in \Lambda^+_{\text{row}}(n, r)$ and $l \geq 4$ then $\mathrm{Ext}^1_H(S^\lambda, D^\mu) \not\cong 0$ implies that $m(\mu)' < \lambda$.*

Proof. We will apply the Schur functor to the following short exact sequence for the tilting module $T(\lambda)$:

$$0 \to K \to T(\lambda) \to \nabla(\lambda) \to 0.$$

By [Don98, 4.4(2)] we get the following short exact sequence

$$0 \to fK \to Y^{\lambda'} \otimes \mathrm{sgn} \to S^\lambda \to 0$$

where fK has a filtration by Specht modules S^{ν_i} with $\nu_i < \lambda$.

Now

$$\mathrm{Ext}^1_H(Y^{\lambda'} \otimes \mathrm{sgn}, D^\mu) \cong \mathrm{Ext}^1_H(Y^{\lambda'}, D^\mu \otimes \mathrm{sgn}) \cong 0$$

as $Y^{\lambda'}$ is projective if $\lambda \in \Lambda^+_{\text{row}}(n, r)$. Thus we get the following sequence

$$0 \to \mathrm{Hom}_H(S^\lambda, D^\mu) \to \mathrm{Hom}_H(Y^{\lambda'} \otimes \mathrm{sgn}, D^\mu) \to \mathrm{Hom}_H(fK, D^\mu) \to \mathrm{Ext}^1_H(S^\lambda, D^\mu) \to 0$$

Thus $\mathrm{Ext}^1_H(S^\lambda, D^\mu) \not\cong 0$ implies that $\mathrm{Hom}_H(fK, D^\mu)$ is non-zero. Now

$$\mathrm{Hom}_H(fK, D^\mu) \cong \mathrm{Hom}_S(gfK, L(m(\mu)'))$$

using Theorem 3.5. But $l \geq 4$ and fK has a Specht filtration with $(fK : S^\nu) = (K : \nabla(\nu))$ and so $gfK \cong K$. Thus $\mathrm{Hom}_S(gfK, L(m(\mu)'))$ can only be non-zero if $L(m(\mu)')$ is in the head of one of the $\nabla(\nu)$ (with $\nu < \lambda$) appearing in a filtration of $T(\lambda)$. Let $\nabla(\nu)$ be such a ν with head $L(m(\mu)')$. Thus $m(\mu)' \leq \nu < \lambda$. ☐

We can thus generalise the result of Kleschev and Sheth to the following

Corollary 8.3. *If λ, $\mu \in \Lambda^+_{\text{row}}(r)$, $l \geq 4$ and $m(\mu)' \not< \lambda$ then*

$$\mathrm{Ext}^1_H(D^\lambda, D^\mu) \cong \mathrm{Hom}_H(\mathrm{rad}\, S^\lambda, D^\mu).$$

Kleshchev and Martin have conjectured that for $l \geq 3$ we have

$$\mathrm{Ext}^1_{k\mathfrak{S}_r}(D^\lambda, D^\lambda) = 0.$$

Unfortunately Corollary 8.3 does not help us show that $\mathrm{Ext}^1_H(D^\lambda, D^\lambda) = 0$, as $m(\lambda)' \leq \lambda$. This can be seen by considering the Specht module $S^{m(\lambda)'}$. We claim that D^λ occurs once as a composition factor (indeed it is the socle of $S^{m(\lambda)'}$) and so $\lambda \geq m(\lambda)'$. Now $[S^{m(\lambda)'} : D^\lambda] = [S^{m(\lambda)'} \otimes \mathrm{sgn} : D^\lambda \otimes \mathrm{sgn}] = [(S^{m(\lambda)})^* : D^{m(\lambda)}] = [S^{m(\lambda)} : D^{m(\lambda)}] = 1$.

We can now deduce that the condition $\lambda \not> m(\mu)'$ implies $\lambda \not> \mu$ but that the reverse implication does not hold in general.

Hemmer [Hem01, Hem05] also gets results concerning extensions between simples. In particular, he determines the non-split extensions between simple modules when the simples involved are completely splittable.

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