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ROW AND COLUMN REMOVAL IN THE q-DEFORMED FOCK SPACE

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Abstract. Analogues of James's row and column removal theorems are proved for the q-decomposition numbers arising from the canonical basis in the q-deformed Fock space.

1. Introduction

Throughout we fix an integer $n \geq 2$. Lascoux, Leclerc, and Thibon [11, 7] used the representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ to introduce for every pair of partitions λ and σ a polynomial $d_{\lambda\sigma}(q)$ with integer coefficients (which depends on n). They conjectured these polynomials to be q-analogues of decomposition numbers for Hecke algebras and quantized Schur algebras at complex n-th roots of unity. These conjectures were proved by Ariki [1] and by Varagnolo-Vasserot [14] respectively. The $d_{\lambda\sigma}(q)$'s are also known to be parabolic affine Kazhdan-Lusztig polynomials.

Leclerc's lectures [10] are a good introduction to this subject as well as a convenient reference for the results we need here.

The purpose of this note is to prove the following theorem (see below for definitions).

Theorem 1. *Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be partitions.*

(1) *(Row removal) Suppose that $\lambda_1 + \ldots + \lambda_r = \mu_1 + \ldots + \mu_r$ for some r and let*

$$\lambda^{(0)} = (\lambda_1, \ldots, \lambda_r), \qquad \mu^{(0)} = (\mu_1, \ldots, \mu_r),$$
$$\lambda^{(1)} = (\lambda_{r+1}, \lambda_{r+2}, \ldots), \qquad \mu^{(1)} = (\mu_{r+1}, \mu_{r+2}, \ldots).$$

Then $d_{\lambda\mu}(q) = d_{\lambda^{(0)}\mu^{(0)}}(q) d_{\lambda^{(1)}\mu^{(1)}}(q)$.

(2) *(Column removal) Suppose that $\lambda'_1 + \ldots + \lambda'_r = \mu'_1 + \ldots + \mu'_r$ for some r and let*

$$\lambda^{(0)\prime} = (\lambda'_1, \ldots, \lambda'_r), \qquad \mu^{(0)\prime} = (\mu'_1, \ldots, \mu'_r),$$
$$\lambda^{(1)\prime} = (\lambda'_{r+1}, \lambda'_{r+2}, \ldots), \qquad \mu^{(1)\prime} = (\mu'_{r+1}, \mu'_{r+2}, \ldots).$$

Then $d_{\lambda\mu}(q) = d_{\lambda^{(0)}\mu^{(0)}}(q) d_{\lambda^{(1)}\mu^{(1)}}(q)$.

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Example. *Let $n = 2$. We have*

$$\begin{aligned} d_{(5,5),(6,4)}(q) &= q, \\ d_{(4,4),(3,2,2,1)}(q) &= q^3 + q, \\ d_{(5,5,4,4),(6,4,3,2,2,1)}(q) &= q^4 + q^2 \\ &= d_{(5,5),(6,4)}(q) d_{(4,4),(3,2,2,1)}(q). \end{aligned}$$

If we put $q = 1$ in Theorem 1 and use Ariki's and Varagnolo-Vasserot's theorems, we recover a result of James [5, Theorem 6.18 and Corollary 6.20] on the decomposition numbers of Hecke algebras and quantized Schur algebras, albeit only at complex roots of unity. James's result, which is valid for any Hecke algebra or quantized Schur algebra, generalizes earlier work of himself [4] and of Donkin [2]. The names 'row removal' and 'column removal' come from the original work [4], which dealt with the special case $r = 1$.

After introducing the polynomials $d_{\lambda\sigma}(q)$ as the coefficients of canonical basis vectors in the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$, we prove 'column removal' by a direct calculation in the Fock space. Then 'row removal' is deduced from this using a theorem of Leclerc.

2. Background

A composition is a sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$ of nonnegative integers with finite sum $|\gamma| = \gamma_1 + \gamma_2 + \cdots$. A nonincreasing composition is called a partition. Let $\mathcal{P}$ be the set of partitions. We identify a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ with its Young diagram

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq k \leq \lambda_j\}.$$

The standard lexicographic and dominance orders on $\mathcal{P}$ are denoted by $>$ and $\rhd$ respectively, and λ' is the partition conjugate to λ.

Given any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, write $l(\lambda)$ for the largest integer r such that $\lambda_r \neq 0$. We say λ is n-restricted if $\lambda_i - \lambda_{i+1} < n$ $(1 \leq i \leq l(\lambda))$ and is n-regular if λ' is n-restricted.

If $\gamma = (\gamma_1, \gamma_2, \ldots)$ and $\delta = (\delta_1, \delta_2, \ldots)$ are compositions, write $\gamma + \delta$ for the composition $(\gamma_1 + \delta_1, \gamma_2 + \delta_2, \ldots)$. If λ and μ are partitions then so is $\lambda + \mu$, and if in addition $\mu'_1 \leq \lambda'_{\lambda_1}$, we write $\lambda \oplus \mu$ instead of $\lambda + \mu$ to emphasize that the new partition is formed simply by putting λ and μ side by side. Also, if $\lambda_{l(\lambda)} \geq \mu_1$ we define (λ, μ) to be the partition $(\lambda_1, \ldots, \lambda_{l(\lambda)}, \mu_1, \mu_2, \ldots)$.

The algebra $U = U_q(\widehat{\mathfrak{sl}}_n)$ is the associative algebra over $\mathbb{C}(q)$ with generators e_i, f_i, k_i, k_i^{-1} $(0 \leq i \leq n-1)$, d, d^{-1} subject to certain relations for which the interested reader may refer to, for example, [10, §4]. The subalgebra obtained by omitting the generators d and d^{-1} is denoted U' and the subalgebra generated by the f_i's is denoted U^-.

For any integer r, we denote by θ_r the automorphism of U^- which sends f_i to f_{i+r}, where we read the subscripts modulo n.

An important U-module is the Fock space representation $\mathcal{F}$ [3, 13], which as a $\mathbb{C}(q)$-vector space has a basis $\{s(\lambda)\}_{\lambda \in \mathcal{P}}$. For our purposes an explicit description of the action of just the f_i's on $\mathcal{F}$ will suffice.

A node $\gamma = (j, k)$ in a Young diagram μ is called an i-node ($i \in \{0, 1, \ldots, n-1\}$) of μ if $k - j$ is congruent to i modulo n. If in removing γ we obtain a Young diagram λ then we call γ a removable i-node of μ or an indent i-node of λ. Let $N(\lambda, \mu)$ be the number of indent i-nodes of λ situated to the right of γ minus the number of removable i-nodes of λ situated to the right of γ. We have

$$f_i s(\lambda) = \sum_{\mu} q^{N(\lambda,\mu)} s(\mu),$$

where the sum is over all Young diagrams μ obtained from λ by adding an indent i-node.

The submodule of $\mathcal{F}$ generated by $s(\emptyset)$ is a highest weight module for U, called the basic representation. Following [8] we describe operators V_k ($k \in \mathbb{N}$) on $\mathcal{F}$ in terms of ribbon tableaux:

$$V_k s(\lambda) = \sum_{\mu} (-q)^{-\operatorname{spin}(\mu/\lambda)} s(\mu),$$

where the sum is over all partitions μ such that μ/λ is a horizontal n-ribbon strip of weight k, and $\operatorname{spin}(\mu/\lambda)$ is the spin of the strip. We will not need to define these terms here; for us the important observation is that the coefficient of $s(\mu)$ depends only on the skew diagram μ/λ. The operators V_k commute pairwise and also with the action of U'; in fact $\mathbb{C}(q)[V_k; k \geq 1]s(\emptyset)$ is equal to the whole space of U'-highest weight vectors in $\mathcal{F}$ [11, 12]. It follows that $\mathcal{F} = \mathcal{A}^- s(\emptyset)$ where $\mathcal{A}^- = U^- \otimes \mathbb{C}(q)[V_k; k \geq 1]$. The automorphisms θ_r extend uniquely to automorphisms of $\mathcal{A}^-$ fixing each V_k.

Let $\langle -, - \rangle$ denote the inner product on $\mathcal{F}$ for which $\{s(\lambda) \mid \lambda \in \mathcal{F}\}$ is orthonormal.

Lemma 2. *Let λ , ν, and α be partitions such that $\nu'_1 \leq \lambda'_{\lambda_1}$, and let $\xi \in \mathcal{A}^-$. Then $\langle \xi s(\lambda \oplus \nu), s(\alpha) \rangle \neq 0$ implies that either $\alpha'_1 + \ldots + \alpha'_{\lambda_1} > |\lambda|$ or $\alpha = \lambda \oplus \mu$ for some partition μ with $\mu'_1 \leq \lambda'_{\lambda_1}$. Moreover*

$$\langle \xi s(\lambda \oplus \nu), s(\lambda \oplus \mu) \rangle = \langle \theta_{-\lambda_1}(\xi) s(\nu), s(\mu) \rangle .$$

Proof. It's easy to check that the lemma holds when $\xi = f_i$ or $\xi = V_k$. Next, observe that given α with the property $\alpha'_1 + \ldots + \alpha'_{\lambda_1} > |\lambda|$, both $f_i s(\alpha)$ and $V_k s(\alpha)$ are linear combinations of $s(\beta)$'s for β's satisfying the same property. Thus the lemma holds when ξ is a monomial in the f_i's and V_k's (and therefore for all $\xi \in \mathcal{A}^-$) by induction on degree. □

In order to define the canonical basis in $\mathcal{F}$, Leclerc and Thibon [11] introduce an involution $x \to \overline{x}$ on $\mathcal{F}$ characterized by the following properties:

$$\begin{aligned}
\overline{s(\emptyset)} &= s(\emptyset), \\
\overline{\phi(q)x + y} &= \phi(q^{-1})\overline{x} + \overline{y}, \quad (\phi(q) \in \mathbb{C}(q); x, y \in \mathcal{F}), \\
\overline{f_i x} &= f_i \overline{x}, \quad (i = 0, \ldots, n-1; x \in \mathcal{F}), \\
\overline{V_k x} &= V_k \overline{x}, \quad (k = 1, 2, \ldots; x \in \mathcal{F}).
\end{aligned}$$

Theorem 3 (Leclerc-Thibon [11])**.** *For each $\sigma \in \mathcal{P}$ there is a unique element $G(\sigma) \in \mathcal{F}$ such that $\overline{G(\sigma)} = G(\sigma)$ and such that, defining $d_{\lambda\sigma}(q) :=$*

$\langle G(\sigma), s(\lambda)\rangle$, we have

$$
\begin{aligned}
&d_{\sigma\sigma}(q) = 1, \\
&d_{\lambda\sigma}(q) \in q\mathbb{Z}[q] \text{ for all } \lambda \in \mathcal{P},\ |\lambda| = |\sigma|,\ \lambda \lhd \sigma, \\
&d_{\lambda\sigma}(q) = 0 \text{ for all other } \lambda \in \mathcal{P}.
\end{aligned} \tag{2.1}
$$

Moreover $B = \{G(\sigma)\}_{\sigma\in\mathcal{P}}$ is a basis of $\mathcal{F}$.

Taking just the $G(\sigma)$'s for n-regular partitions one obtains a basis of the basic representation $Us(\emptyset)$ of U which coincides with Kashiwara's lower global basis [6].

3. Column removal

We reformulate part 2 of Theorem 1:

Theorem 4 (Column removal)**.** *Let λ, μ, σ, and τ be partitions such that $|\lambda| = |\sigma|$, $\lambda_1 = \sigma_1$ ($= r$, say), $\mu'_1 \le \lambda'_r$, and $\tau'_1 \le \sigma'_r$. Then*

$$d_{\lambda\oplus\mu,\sigma\oplus\tau}(q) = d_{\lambda\sigma}(q)d_{\mu\tau}(q).$$

Fix σ and τ as in the statement of Theorem 4. We of course may assume that $\tau \ne \emptyset$. Write $G(\tau) = \xi_\tau s(\emptyset)$, where $\xi_\tau \in \mathcal{A}^-$. We define $X := \theta_r(\xi_\tau)G(\sigma)$, regarding it as a sort of 'first approximation' to $G(\sigma\oplus\tau)$. We may assume that $\xi_\tau = \sum \xi_{\tau,j}$ where each $\xi_{\tau,j}$ is a monomial in the F_i's and V_k's and such that $\{\xi_{\tau,j}s(\emptyset)\}$ is a linearly independent set in $\mathcal{F}$. Then $\overline{G(\tau)} = G(\tau)$ implies that the coefficient $\phi_{\tau,j}(q)$ in each monomial $\xi_{\tau,j}$ must satisfy $\phi_{\tau,j}(q^{-1}) = \phi_{\tau,j}(q)$. It follows then that $\overline{X} = X$.

Let $\mathcal{P}_{bad}$ be the set of partitions α satisfying $\alpha'_1 + \cdots + \alpha'_r > |\sigma|$ and let $\mathcal{F}_{bad}$ be the span in $\mathcal{F}$ of the $\alpha \in \mathcal{P}_{bad}$. Note that if $\alpha \in \mathcal{P}_{bad}$ then $G(\alpha) \in \mathcal{F}_{bad}$.

We have

$$
\begin{aligned}
X &= \theta_r(\xi_\tau)G(\sigma) \\
&= \sum_{\substack{\lambda\in\mathcal{P} \\ |\lambda|=|\sigma|}} d_{\lambda\sigma}(q)\theta_r(\xi_\tau)s(\lambda).
\end{aligned}
$$

If $\lambda_1 > r$, then $\lambda \not\trianglelefteq \sigma$, and therefore $d_{\lambda\sigma}(q) = 0$. On the other hand if $\lambda_1 < r$, then $\theta_r(\xi_\tau)s(\lambda) \in \mathcal{F}_{bad}$. Hence working modulo $\mathcal{F}_{bad}$ we have, by Lemma 2, that

$$
\begin{aligned}
X &\equiv \sum_{\substack{\lambda\in\mathcal{P} \\ |\lambda|=|\sigma|,\lambda_1=r}} d_{\lambda\sigma}(q)\theta_r(\xi_\tau)s(\lambda) \\
&\equiv \sum_{\substack{\lambda\in\mathcal{P} \\ |\lambda|=|\sigma|,\lambda_1=r}} \sum_{\substack{\mu\in\mathcal{P} \\ \mu'_1\le\lambda'_r}} d_{\lambda\sigma}(q) \left\langle \xi_\tau s(\emptyset), s(\mu)\right\rangle s(\lambda\oplus\mu) \\
&\equiv \sum_{\substack{\lambda\in\mathcal{P} \\ |\lambda|=|\sigma|,\lambda_1=r}} \sum_{\substack{\mu\in\mathcal{P} \\ \mu'_1\le\lambda'_r}} d_{\lambda\sigma}(q)d_{\mu\tau}(q)s(\lambda\oplus\mu),
\end{aligned}
$$

In particular $\langle X, s(\sigma\oplus\tau)\rangle = 1$, and $\langle X, s(\alpha)\rangle \in q\mathbb{Z}[q]$ if $\alpha \notin \{\sigma\oplus\tau\} \cup \mathcal{P}_{bad}$.

Set $E = G(\sigma \oplus \tau) - X$. Write

$$E = \sum_{\alpha \in \mathcal{P}} b_\alpha(q) G(\alpha)$$

where $b_\alpha(q) \in \mathbb{C}(q)$; we have $b_\alpha(q^{-1}) = b_\alpha(q)$ because $\overline{E} = E$. Note that $b_\alpha(q) \neq 0$ implies $\alpha \in \mathcal{P}_{bad}$: otherwise, let γ be the largest partition in the lexicographic order such that $b_\gamma(q) \neq 0$ and $\gamma \notin \mathcal{P}_{bad}$; then

$$b_\gamma(q) = \langle E, s(\gamma) \rangle = \langle G(\sigma \oplus \tau) - X, s(\gamma) \rangle \in q\mathbb{Z}[q],$$

a contradiction. Consequently $E \in \mathcal{F}_{bad}$; hence for any $\lambda, \mu \in \mathcal{P}$ as in the statement of the theorem we have

$$\begin{aligned} d_{(\lambda \oplus \mu),(\sigma \oplus \tau)}(q) &= \langle G(\sigma \oplus \tau), s(\lambda \oplus \mu) \rangle \\ &= \langle X, s(\lambda \oplus \mu) \rangle \\ &= d_{\lambda\sigma}(q) d_{\mu\tau}(q). \end{aligned}$$

4. Row removal

Given a partition α with r parts or less, let $\mathbf{n}(\alpha)_i$ be the number of elements in $\{\alpha_1 + r - 1, \alpha_2 + r - 2, \dots, \alpha_r\}$ which are congruent to i modulo n. Define

$$\mathbf{n}(\alpha) = \sum_{i=0}^{n-1} \binom{\mathbf{n}(\alpha)_i}{2}.$$

It is easy to see that $\mathbf{n}(\alpha + (1^r)) = \mathbf{n}(\alpha)$.

We also define $\alpha^{\#} = (\alpha_r, \alpha_{r-1}, \dots, \alpha_1)$.

Keeping these notations, we have the following theorem.

Theorem 5 (Leclerc [9])**.** *Let λ and σ be two partitions such that $\lambda_1, \sigma_1 \leq r$. Let $\sigma' = n\sigma^{(1)} + \sigma^{(2)}$ be the unique decomposition of σ' with $\sigma^{(2)}$ being n-restricted. Then*

$$d_{\lambda\sigma}(q) = q^{\binom{r}{2} - \mathbf{n}(\lambda')} d_{\widetilde{\lambda}\widehat{\sigma}}(q^{-1}),$$

where $\tilde{\lambda} = (K^r) + \lambda'$, $K = (n-1)(r-1)$ and

$$\widehat{\sigma} = (2n-2)(r-1, r-2, \dots, 1, 0) + n\sigma^{(1)} + (\sigma^{(2)})^{\#}.$$

Using this theorem and the 'column removal' theorem proved in the last section, we obtain a special case of 'row removal':

Proposition 6. *Let λ and σ be two partitions such that $\lambda_1 = \sigma_1 = r$. Let $\mu = (\lambda_2, \lambda_3, \dots)$ and $\tau = (\sigma_2, \sigma_3, \dots)$. Then*

$$d_{\lambda\sigma}(q) = d_{\mu\tau}(q).$$

Proof. We keep the notations of Theorem 5. Note first that $\widetilde{\lambda} = (1^r) + \widetilde{\mu}$. Let $\tau' = n\tau^{(1)} + \tau^{(2)}$ be the unique decomposition of τ' with $\tau^{(2)}$ being n-restricted. Then $\sigma' = (1^r) + n\tau^{(1)} + \tau^{(2)} = n(\tau^{(1)} + \varepsilon(1^r)) + ((1^r) + \tau^{(2)} - \varepsilon(n^r))$ is the analogous unique decomposition of σ', where

$$\varepsilon = \begin{cases} 1, & \text{if } (1^r) + \tau^{(2)} \text{ is not } n\text{-restricted;} \\ 0, & \text{otherwise.} \end{cases}$$

It is then easy to check that $\widehat{\sigma} = (1^r) + \widehat{\tau}$. Hence,

$$\begin{aligned} d_{\lambda\sigma}(q) &= q^{\binom{r}{2} - \mathbf{n}(\lambda')} d_{\widetilde{\lambda}\widehat{\sigma}}(q^{-1}) && \text{(by Theorem 5)} \\ &= q^{\binom{r}{2} - \mathbf{n}((1^r)+\mu')} d_{(1^r)+\widetilde{\mu},(1^r)+\widehat{\tau}}(q^{-1}) \\ &= q^{\binom{r}{2} - \mathbf{n}(\mu')} d_{\widetilde{\mu}\widehat{\tau}}(q^{-1}) && \text{(by Theorem 4)} \\ &= d_{\mu\tau}(q) && \text{(by Theorem 5).} \end{aligned}$$

□

The general case of 'row removal' can now be deduced easily from Theorem 4 and Proposition 6 using an argument of James [5, Corollary 6.20]. The following is just a reformulation of part 1 of Theorem 1.

Theorem 7 (Row removal)**.** *Let λ, μ, σ, and τ be partitions such that $|\lambda| = |\sigma|$, $r = \lambda'_1 = \sigma'_1$, $\mu_1 \leq \lambda_r$, and $\tau_1 \leq \sigma_r$. Then*

$$d_{(\lambda,\mu),(\sigma,\tau)}(q) = d_{\lambda\sigma}(q) d_{\mu\tau}(q).$$

Proof. We only need to consider the case where $\lambda \trianglelefteq \sigma$ and $\mu \trianglelefteq \tau$ since the theorem holds trivially otherwise. Then,

$$\mu_1 \leq \tau_1 \leq \sigma_r \leq \lambda_r.$$

Let $s = \sigma_r$, and let partitions α and β be defined by $\lambda = (s^r) \oplus \alpha$ and $\sigma = (s^r) \oplus \beta$. Then $(\lambda, \mu) = (s^r, \mu) \oplus \alpha$ and $(\sigma, \tau) = (s^r, \tau) \oplus \beta$. Thus

$$\begin{aligned} d_{(\lambda,\mu),(\sigma,\tau)}(q) &= d_{(s^r,\mu)\oplus\alpha,(s^r,\tau)\oplus\beta}(q) \\ &= d_{(s^r,\mu),(s^r,\tau)}(q) d_{\alpha\beta}(q) && \text{(by Theorem 4)} \\ &= d_{\mu\tau}(q) d_{(s^r),(s^r)}(q) d_{\alpha\beta}(q) && \text{(by Proposition 6)} \\ &= d_{\mu\tau}(q) d_{(s^r)\oplus\alpha,(s^r)\oplus\beta}(q) && \text{(by Theorem 4)} \\ &= d_{\lambda\sigma}(q) d_{\mu\tau}(q). \end{aligned}$$

□

References

[1] S. Ariki. On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$. *J. Math. Kyoto Univ.*, 36(4):789–808, 1996.

[2] S. Donkin. A note on decomposition numbers for general linear groups and symmetric groups. *Math. Proc. Cambridge Philos. Soc.*, 97(1):57–62, 1985.

[3] T. Hayashi. q-analogues of Clifford and Weyl algebras—spinor and oscillator representations of quantum enveloping algebras. *Comm. Math. Phys.*, 127(1):129–144, 1990.

[4] G. D. James. On the decomposition matrices of the symmetric groups. III. *J. Algebra*, 71(1):115–122, 1981.

[5] G. D. James. The decomposition matrices of $\mathrm{GL}_n(q)$ for $n \leq 10$. *Proc. London Math. Soc. (3)*, 60(2):225–265, 1990.

[6] M. Kashiwara. On crystal bases of the Q-analogue of universal enveloping algebras. *Duke Math. J.*, 63(2):465–516, 1991.

[7] A. Lascoux, B. Leclerc, and J.-Y. Thibon. Hecke algebras at roots of unity and crystal bases of quantum affine algebras. *Comm. Math. Phys.*, 181(1):205–263, 1996.

[8] A. Lascoux, B. Leclerc, and J.-Y. Thibon. Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties. *J. Math. Phys.*, 38(2):1041–1068, 1997.

[9] B. Leclerc. Decomposition numbers and canonical bases. *Algebr. Represent. Theory*, 3(3):277–287, 2000.

[10] B. Leclerc. Symmetric functions and the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$. Lectures at the Newton Institute, Cambridge, 2001.

[11] B. Leclerc and J.-Y. Thibon. Canonical bases of q-deformed Fock spaces. *Internat. Math. Res. Notices*, (9):447–456, 1996.

[12] B. Leclerc and J.-Y. Thibon. Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials. In K. et al., editor, *Combinatorial methods in representation theory*, volume 28, pages 155–220, 2000.

[13] K. Misra and T. Miwa. Crystal base for the basic representation of $U_q(\mathfrak{sl}(n))$. *Comm. Math. Phys.*, 134(1):79–88, 1990.

[14] M. Varagnolo and E. Vasserot. On the decomposition matrices of the quantized Schur algebra. *Duke Math. J.*, 100(2):267–297, 1999.

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