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SYMMETRIC GROUPS, WREATH PRODUCTS, MORITA EQUIVALENCES, AND BROUÉ'S ABELIAN DEFECT GROUP CONJECTURE

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Abstract. It is shown that for any prime p, and any non-negative integer w less than p, there exist p-blocks of symmetric groups of defect p^w, which are Morita equivalent to the principal p-block of the group $S_p \wr S_w$. Combined with work of J. Rickard, this proves that Broué's abelian defect group conjecture holds for p-blocks of symmetric groups of defect at most p^5.

1. Introduction

Broué has made some deep conjectures involving the derived categories of blocks of finite groups. One such is his Abelian Defect Group Conjecture [1]: a p-block of a finite group G with abelian defect group D and its Brauer correspondent in $N_G(D)$ should be derived equivalent over a complete discrete valuation ring with residue field of characteristic p.

We show that this conjecture is true for a family of blocks of symmetric groups; this family is given by a certain combinatorial criterion and contains blocks whose defect groups have arbitrarily large rank (as long as we allow p to vary). Our work is the construction of Morita equivalences of blocks in this family and blocks of wreath products of symmetric groups. These equivalences, which were conjectured to exist by Rouquier, appear below in Theorem 2. When composed with certain derived equivalences described by Andrei Marcus in [8] they produce derived equivalences predicted by Broué's conjecture.

We construct Morita equivalences only for blocks in the aforementioned family, but taken together with unpublished work of Jeremy Rickard they imply that Broué's conjecture holds for all p-blocks of symmetric groups whose defect groups have order less than or equal to p^5 (Corollary 3 below). This generalizes the main result of [3], in which the same was proved for blocks whose defect groups have order less than or equal to p^2.

2. Preliminaries

Let p a prime number and let $\mathcal{O}$ be a complete discrete valuation ring with maximal ideal $\mathcal{J}$ such that its residue field $k = \mathcal{O}/\mathcal{J}$ has characteristic p and is algebraically closed, and its fraction field K has characteristic 0. To indicate reduction modulo $\mathcal{J}$ of elements of $\mathcal{O}$ or of group algebras over $\mathcal{O}$, we use the conventional bar notation. We will write $\otimes$ in place of $\otimes_{\mathcal{O}}$.

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Let G be a finite group. We state the Abelian Defect Group Conjecture of Broué:

Conjecture 1. (Broué) Let b be a block idempotent of $\mathcal{O}G$ with abelian defect group D, and let c be the Brauer correspondent block idempotent of $\mathcal{O}N_G(D)$. Then $\mathcal{O}Gb$ and $\mathcal{O}N_G(D)c$ are derived equivalent, i.e., there is an equivalence of triangulated categories between their derived module categories.

We will be making use of the Brauer quotient and the Brauer homomorphism, which we briefly describe (see [2] for more details). Let M be an $\mathcal{O}G$-module and P a p-subgroup of G. The Brauer quotient

$$M(P) = M^P / \left(\sum_{Q<P} \mathrm{Tr}_Q^P(M^Q) + \mathcal{J} M^P \right)$$

is defined by taking the quotient of the P-fixed points of M by relative traces from proper subgroups and then passing to the residue field; $M(P)$ is a $kN_G(D)$-module. If M is a summand of a permutation module, then $M(P) \neq 0$ if and only if M has a direct summand which has a vertex containing P. In the case that $M = \mathcal{O}G$ and G acts by conjugation, we have $\mathcal{O}G(P) = kC_G(P)$, and the natural quotient map

$$\mathrm{Br}_P^G : (\mathcal{O}G)^P \to kC_G(P),$$

which may be given by the rule

$$\mathrm{Br}_P^G \left(\sum_{g\in G} a_g g \right) = \sum_{g \in C_G(P)} \overline{a_g} g,$$

is a homomorphism of $\mathcal{O}$-algebras, called the Brauer homomorphism with respect to P. If H is a subgroup of G containing P, and $i \in (\mathcal{O}G)^H$ is an idempotent, then $\mathcal{O}Gi$ is a summand of $\mathcal{O}G$ as an $\mathcal{O}(G \times H)$-module, and writing ΔP for the diagonally embedded subgroup $\{(x, x) \mid x \in P\} \leq G \times H$, we have that $(\mathcal{O}Gi)(\Delta P)$ and $kC_G(P)\,\mathrm{Br}_P^G(i)$ are isomorphic as $kN_{G\times H}(\Delta P)$-modules.

We introduce some combinatorics relating to partitions. Removing rim p-hooks from the diagram of a partition λ we obtain the diagram of a partition which is called the p-core of λ; it is independent of the manner in which the hooks are removed. The number of hooks removed is called the p-weight of λ. A partition with p-weight 0 is called a p-core. For our purposes it is useful to represent partitions on an abacus with p runners, following Gordon James [6, pages 77-78]. We label the runners of the abacus $0, \ldots, p-1$, from left to right, and label its rows $0, 1, \ldots$, from the top down. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition with less than or equal to m nonzero parts then we may represent λ on the abacus using m beads: for $i = 1, \ldots, m$, we write $\lambda_i + m - i = s + pt$, with $0 \leq s \leq p-1$, and place a bead on the s-th runner in the t-th row. Sliding a bead up one row on its runner into a previously vacant position corresponds to removing a p-hook. Thus sliding all the beads in an abacus representation of a partition up their runners as far as possible produces an abacus representation of the p-core of that partition.

Fix an abacus representation of λ, and for $i = 0, \ldots, p-1$, let $\lambda_1^{(i)}$ be the number of unoccupied positions on the i-th runner which occur above the bottommost bead on that runner, let $\lambda_2^{(i)}$ be the number of unoccupied positions on the i-th runner which occur above the next to bottommost bead on that runner, and so on. Then $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots)$ is a partition, and the p-tuple $(\lambda^{(0)}, \ldots, \lambda^{(p-1)})$ is called the p-quotient of λ. We note that it depends on the number of beads used in the abacus representation. The weight of λ is equal to $\left|\lambda^{(0)}\right| + \ldots + \left|\lambda^{(p-1)}\right|$.

Given a p-core κ and a nonnegative integer w, consider partitions with p-core κ and p-weight w. Choosing m so that any such partition has less than or equal to m nonzero parts and representing these partitions on an abacus with m beads, we have a p-quotient for each one. This gives a bijection between this set of partitions and the set of p-tuples $(\sigma_0, \ldots, \sigma_{p-1})$ of partitions satisfying $\left|\sigma^{(0)}\right| + \ldots + \left|\sigma^{(p-1)}\right| = w$.

3. Blocks of symmetric groups

For any nonnegative integer n denote by S_n the symmetric group of degree n, and for any finite set V denote by $S(V)$ the symmetric group on V. The irreducible characters χ^λ of KS_n (or of $KS(V)$, if $|V| = n$) are indexed by partitions λ of n. Nakayama's rule (see, e.g., [6, page 245, Theorem 6.1.20]) states that two characters are in the same p-block of $KS(V)$ if and only if the associated partitions have the same p-core. Hence the p-blocks of characters of $KS(V)$ are labelled by p-cores, and all the partitions associated to characters in a given p-block have the same p-weight. Thus it makes sense to speak of the weight of a p-block.

Let e be the primitive central idempotent of $\mathcal{O}S(V)$ corresponding to a p-block which has weight w and is labelled by a p-core κ. Let U be a subset of V of cardinality pw and let D be a Sylow p-subgroup of $S(U)$. Then D is a defect group of $\mathcal{O}S(V)e$ (see, e.g., [6, page 263, Theorem 6.2.39]). Thus $\mathcal{O}S(V)e$ has abelian defect groups if and only if $w < p$.

As our main interest is Conjecture 1, we will consider only blocks with abelian defect groups; we will therefore assume that $w < p$ for the remainder of this paper.

We have

$$C_{S(V)}(D) = D \times S(V - U)$$

and

$$N_{S(V)}(D) = N_{S(U)}(D) \times S(V - U).$$

A theorem of Puig [9] tells us that

$$\mathrm{Br}_D^{S(V)}(e) = 1_{kD} \otimes \overline{e_0}, \tag{1}$$

where e_0 is the block idempotent of $\mathcal{O}S(V - U)$ corresponding to the p-block labelled by the p-core κ. Thus the Brauer correspondent of $\mathcal{O}S(V)e$ in $\mathrm{N}_G(D)$ is

$$\mathcal{O}N_{S(U)}(D) \otimes \mathcal{O}S(V - U)e_0.$$

This block is Morita equivalent to $\mathcal{O}N_{S(U)}(D)$ because $\mathcal{O}S(V - U)e_0$ is a block of weight 0 and therefore defect 0. The block $\mathcal{O}N_{S(U)}(D)$ is in turn derived equivalent to the principal block of $\mathcal{O}(S_p \wr S_w)$, by a result of Andrei

Marcus [8, Example 5.7] based on the existence of derived equivalences for blocks with cyclic defect groups, which was proved by Rickard in [11]. Hence the Brauer correspondent of any p-block of weight w of a symmetric group is derived equivalent to the principal block of $\mathcal{O}(S_p \wr S_w)$. We now state our result, which was originally conjectured by Rouquier.

Theorem 2. *Given any $w < p$ there exist blocks of weight w of symmetric groups which are Morita equivalent to the principal block of $\mathcal{O}(S_p \wr S_w)$. In particular there exist blocks of weight w for which Broué's Abelian Defect Group Conjecture holds.*

Jeremy Rickard, in an important unpublished theorem (announced in [12]), which built on the work of Scopes in [15], constructed derived equivalences of blocks; he proved in particular that for $w \leq 5$ (no restriction on p), all p-blocks of symmetric groups of weight w are derived equivalent. In light of this result and the comments above, our theorem has the following corollary.

Corollary 3. *Broué's Abelian Defect Group Conjecture (Conjecture 1) is true for all blocks of symmetric groups of weight less than or equal to* 5.

Remark 3.1. The proof given here is an improvement over that in [3], in which the case $w = 2$ was handled using a more convoluted argument.

4. Proof of the theorem

Let ρ be a p-core which satifies the following property: ρ has an abacus representation in which each runner other than the leftmost one (the 0-th runner) has at least $w-1$ more beads than the runner to its immediate left. Having chosen such a ρ, we may assume that there are at least w beads on each runner; let m be the number of beads in such an abacus representation of ρ. We will be considering partitions with p-core ρ and p-weight $v \leq w$. Any such partition can be represented on an abacus with m beads and we do so. We remark that one candidate for ρ is the p-core which has an abacus display with $w + i(w-1)$ beads on the i-th runner, for $i = 0, \ldots, p-1$; this is the smallest ρ possible.

In the second part of the following lemma we give the key property of ρ. If μ and λ are partitions then by $\mu \subseteq \lambda$ we mean that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$, and so on.

Lemma 4.

1. *Let λ be a partition with p-core ρ and weight $v \leq w$. Suppose that in the (m-bead) abacus representation of λ there is a bead on the s-th runner in the t-th row. Then on any runner to the right of the s-th runner there are beads in all rows above the t-th row. If $v < w$, then on any runner to the right of the s-th runner there are beads in all rows on or above the t-th row.*
2. *Let λ be a partition with p-core ρ and weight $v \leq w$, and let μ be a partition with p-core ρ and weight $v-1$. If $\mu \subseteq \lambda$ then there exists s with $0 \leq s \leq p-1$ such that $\mu^{(i)} = \lambda^{(i)}$ for $i \neq s$ and $\mu^{(s)} \subseteq \lambda^{(s)}$ with $\left|\mu^{(s)}\right| = \left|\lambda^{(s)}\right| - 1$. Moreover the complement of the Young diagram of μ in that of λ is the Young diagram of the hook partition $(s+1, 1^{p-s-1})$.*

Proof. (1) Let $0 \leq s < s' \leq p-1$. Let m_s and $m_{s'}$ be the number of beads on the s-th and s'-th runners in the abacus representation of λ. All the beads on the s-th runner lie in the first $m_s + \left|\lambda^{(s)}\right|$ rows, while on the s'-th runner there is a bead in each the first $m_{s'} - \left|\lambda^{(s')}\right|$ rows. We have

$$\begin{aligned}\left(m_s + \left|\lambda^{(s)}\right|\right) - (m_{s'} - \left|\lambda^{(s')}\right|) &= m_s - m_{s'} + \left|\lambda^{(s)}\right| + \left|\lambda^{(s')}\right| \\ &\leq -(w-1) + v,\end{aligned}$$

which implies the desired result.

(2) For $i = 1, \ldots, m$ set $\alpha_i = \lambda_i + m - i$ and $\beta_i = \mu_i + m - i$. Choose n such that $\beta_n < \alpha_n$ and $\beta_i = \alpha_i$ for $i > n$. Write $\beta_n = s + pt$ where $0 \leq s \leq p-1$. Then in the abacus representation of μ there is a bead on the s-th runner in row t, while there is no bead in the corresponding spot in the abacus representation of λ. Because the abacus representations of λ and μ have the same number of beads in each runner (as they have the same p-core), there must be a bead on the s-th runner of the abacus representation of λ below the t-th row. By the first part of the lemma, we see that in the abacus representations of μ and λ, there is a bead in the t-th row on each runner to the right of the s-th runner, and that in the abacus representation of λ, there is no bead in the $(t+1)$-st row on each runner to the left of the s-th runner. Hence we have

$$\begin{array}{ll}\beta_n = s + pt, & \alpha_n = (s+1) + pt; \\ \beta_{n-1} = (s+1) + pt, & \alpha_{n-1} = (s+2) + pt; \\ \vdots & \vdots \\ \beta_{n-(p-s-2)} = (p-2) + pt, & \alpha_{n-(p-s-2)} = (p-1) + pt; \\ \beta_{n-(p-s-1)} = (p-1) + pt, & \alpha_{n-(p-s-1)} \geq s + p(t+1).\end{array}$$

Therefore

$$\begin{aligned}\mu_n &= \lambda_n - 1; \\ \mu_{n-1} &= \lambda_{n-1} - 1; \\ &\vdots \\ \mu_{n-(p-s-2)} &= \lambda_{n-(p-s-2)} - 1; \\ \mu_{n-(p-s-1)} &\leq \lambda_{n-(p-s-1)} - (s+1).\end{aligned}$$

As $\mu \subseteq \lambda$ and $|\mu| = |\lambda| - p$, the inequality must be equality, and we also must have $\mu_i = \lambda_i$ and $\beta_i = \alpha_i$ for $i < n - (p-s-1)$. Thus the abacus representation of λ is obtained from that of μ by moving a bead on the s-th runner from the t-th row to the $(t+1)$-th row, and all the statements in the lemma hold.

□

Let V be a set of cardinality $pw + r$, let $U_1, \ldots, U_w$ be disjoint subsets of V of cardinality p, and let U be the union of these subsets. In what follows, all groups we consider will be viewed as subgroups of $S(V)$ in an obvious way. For $i = 1, \ldots, w$, let D_i be a Sylow p-subgroup of $S(U_i)$, and let a_i be the principal block idempotent of $\mathcal{O}S(U_i)$. For $i = 0, \ldots, w$ let e_{w-i} be the block idempotent of $\mathcal{O}S(U_{i+1} \cup \ldots \cup U_w \cup (V-U))$ corresponding to the p-core ρ, let

$$G_i = S(U_1) \times \cdots \times S(U_i) \times S(U_{i+1} \cup \ldots \cup U_w \cup (V-U))$$

and let

$$b_i = a_1 \otimes \cdots \otimes a_i \otimes e_{w-i},$$

a block idempotent of $\mathcal{O}G_i$. We have

$$G_i \cong \underbrace{S_p \times \cdots \times S_p}_{i} \times S_{(w-i)p+r}.$$

We set $G = G_0$, $b = b_0$, $L = G_w$, and $f = b_w$. Letting $D = D_1 \times \cdots \times D_w$, we have

$$C_G(D) = D \times S(V-U) = D_1 \times \cdots \times D_w \times S(V-U).$$

Let $\widetilde{N}$ be the subgroup of $S(U)$ consisting of permutations sending each U_i into some U_j; we note that $\widetilde{N}$ is isomorphic to the wreath product $S_p \wr S_w$. Set $N = \widetilde{N} \times S(V-U)$, a subgroup of G containing $N_G(D)$ and L and normalizing L.

Lemma 5. (1) *For $0 \leq i \leq p-1$, we have that D is a defect group of $\mathcal{O}G_ib_i$;*

(2) *For $0 \leq i \leq p-1$, we have $\mathrm{Br}_D^G(b_i) = 1_{kD} \otimes \overline{e_0}$;*

(3) *We have that N stabilizes f, and $\mathcal{O}Nf$ is a block which is Morita equivalent to the principal block of $\mathcal{O}\left(S_p \wr S_w\right)$;*

(4) *As an $\mathcal{O}(N \times L)$-module, $\mathcal{O}Nf$ is indecomposable with vertex ΔD, and $\mathrm{rank}_{\mathcal{O}}(\mathcal{O}Nf) = w! \cdot \mathrm{rank}_{\mathcal{O}}(\mathcal{O}Lf)$.*

Proof. (1) We have that $D_{i+1} \times \cdots \times D_w$ is a defect group of $\mathcal{O}S(U_{i+1} \cup \ldots \cup U_w \cup V)e_{w-i}$, as this block has weight $w-i < p$, while for $0 \leq j \leq i$, we have that D_j is a defect group of $\mathcal{O}S(U_j)a_j$. Hence D is a defect group of $\mathcal{O}G_ib_i$.

(2) First we note that $G_i \geq C_G(D)$, so that $\mathrm{Br}_D^G(b_i) = \mathrm{Br}_D^{G_i}(b_i)$. Next, for $0 \leq j \leq i$, we have $\mathrm{Br}_{D_j}^{S(U_i)}(a_j) = 1_{kD_j}$, and by Puig's result (1), we have

$$\mathrm{Br}_{D_{i+1}\times\cdots\times D_w}^{S(U_{i+1}\cup\ldots\cup U_w\cup(V-U))}(e_{w-i}) = 1_{k(D_{i+1}\times\cdots\times D_w)} \otimes \overline{e_0}.$$

Hence

$$\begin{aligned} \mathrm{Br}_D^G(b_i) &= \mathrm{Br}_D^{G_i}(b_i) \\ &= \mathrm{Br}_{D_1}^{S(U_1)}(a_1) \otimes \cdots \otimes \mathrm{Br}_{D_i}^{S(U_i)}(a_i) \otimes \mathrm{Br}_{D_{i+1}\times\cdots\times D_w}^{S(U_{i+1}\cup\ldots\cup U_w\cup(V-U))}(e_{w-i}) \\ &= 1_{\mathcal{O}D} \otimes \overline{e_0}. \end{aligned}$$

(3) It is clear that N stabilizes f, and because $a_1 \otimes \cdots \otimes a_i$ is the principal block idempotent of $\mathcal{O}\widetilde{N}$ and e_0 is a block idempotent of $\mathcal{O}S(V-U)$ corresponding to a block of defect 0, the second statement holds.

(4) By part 1 of the Lemma, $\mathcal{O}Lf$ has vertex ΔD. An easy argument using the fact that $C_G(D) \subseteq L$ shows that a conjugate of ΔD by an element of $N \times L$ not in $L \times L$ is not conjugate to ΔD in $L \times L$. Consequently, the stabilizer of $\mathcal{O}Lf$ in $N \times L$ is just $L \times L$, so

$\mathcal{O}Nf = \mathrm{Ind}_{L\times L}^{N\times L}(\mathcal{O}Lf)$ is indecomposable and has vertex ΔD. As $[N : L] = w!$, the statement relating ranks is clear.

□

The lemma shows that the blocks $\mathcal{O}Gb$ and $\mathcal{O}Nf$ both have defect group D and are Brauer correspondents. Thus the $\mathcal{O}(G\times G)$-module $\mathcal{O}Gb$ and the $\mathcal{O}(N \times N)$ -module $\mathcal{O}Nf$ both have vertex ΔD and are Green correspondents. Let X be the Green correspondent of $\mathcal{O}Gb$ in $G \times N$, so that X is an indecomposable $\mathcal{O}(G\times N)$-module with vertex ΔD, and X is isomorphic to a direct summand of $\mathrm{Res}_{G\times N}^{G\times G}(\mathcal{O}Gb)$. Because $\mathcal{O}Nf$ is isomorphic to a direct summand of $\mathrm{Res}_{N\times N}^{G\times N}(X)$, we have $Xf \neq 0$, so $Xf = X$ and X is a $(\mathcal{O}Gb, \mathcal{O}Nf)$-bimodule.

Our goal is to show that X induces a Morita equivalence of $\mathcal{O}Gb$ and $\mathcal{O}Nf$. As $\mathcal{O}Gb$ is a block of a symmetric group of weight w, and $\mathcal{O}Nf$ is Morita equivalent to the principal block of $\mathcal{O}\left(S_p \wr S_w\right)$ by the lemma, this will imply that Theorem 2 is true.

As an $\mathcal{O}Gb$-module X is a progenerator; indeed, $\mathcal{O}Gb$ is isomorphic to a direct summand of $\mathrm{Ind}_{G\times N}^{G\times G}(X)$, and applying Mackey's formula we get that $\mathcal{O}Gb$ is isomorphic to a direct summand of a direct sum of copies of $\mathrm{Res}_{G\times 1}^{G\times N}(X)$. Hence by Morita theory, to prove that X induces an Morita equivalence, it suffices to show that the homomorphism

$$\mathcal{O}Nf \to \mathrm{End}_G(X)$$

induced by the action of $\mathcal{O}Nf$ on X is an isomorphism. Note that it is at least a split monomorphism of $(\mathcal{O}Nf, \mathcal{O}Nf)$-bimodules, because $\mathcal{O}Nf$ is isomorphic to a direct summand of $\mathrm{Res}_{N\times N}^{G\times N}(X)$. Thus we need only to show that

$$\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_G(X)) \leq \mathrm{rank}_{\mathcal{O}}(\mathcal{O}Nf)$$

or equivalently, by part 4 of Lemma 5,

$$\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_G(X)) \leq w! \cdot \mathrm{rank}_{\mathcal{O}}(\mathcal{O}Lf). \tag{2}$$

Now $\mathrm{Res}_{G\times L}^{G\times N}(X)$ is a direct sum of indecomposable modules whose vertices are conjugates of ΔD; indeed, ΔD is a vertex of X and $G \times L$ is a normal subgroup of $G \times N$ containing ΔD. On the other hand, $\mathrm{Res}_{G\times L}^{G\times N}(X) \cong X \otimes_N \mathcal{O}N \cong X \otimes_N \mathcal{O}Nf$ is a summand of $\mathrm{Ind}_{N\times L}^{G\times L}(\mathcal{O}Nf)$, and by the Green correspondence the latter is a direct sum of an indecomposable module with vertex ΔD and indecomposable modules with strictly smaller vertices, as $\mathcal{O}Nf$ has vertex ΔD and $N_{G\times L}(\Delta D) \subseteq N \times L$. Therefore $\mathrm{Res}_{G\times L}^{G\times N}(X)$ is an indecomposable module with vertex ΔD.

Each of the b_i is an idempotent contained in $(\mathcal{O}G)^L$; as these idempotents commute with each other, their product is also an idempotent contained in

$(\mathcal{O}G)^L$. Hence $Y = \mathcal{O}Gb_0 \cdots b_w$ is a summand of $\mathcal{O}Gb_0$ as an $\mathcal{O}(G \times L)$-module, and we have

$$\begin{aligned} Y(\Delta D) &= \mathcal{O}Gb_0 \cdots b_w(\Delta D) \\ &\cong kC_G(D)\,\mathrm{Br}_D^G(b_0 \cdots b_w) \\ &= kC_G(D)\,\mathrm{Br}_D^G(b_0) \cdots \mathrm{Br}_D^G(b_w) \\ &= kC_G(D)\,\mathrm{Br}_D^G(b) \\ &\cong \mathcal{O}Gb(\Delta D), \end{aligned}$$

where the last equality is a consequence of part 2 of Lemma 5. We have $\mathcal{O}Gb \cong Y \oplus E$, for some $\mathcal{O}(G \times L)$-module E. Now $\mathcal{O}Gb(\Delta D) \cong Y(\Delta D) \oplus E(\Delta D)$, and it follows that $E(\Delta D) = 0$. As E is a direct summand of the permutation module $\mathcal{O}G$, this implies that E does not have an indecomposable summand with vertex ΔD. Hence any indecomposable direct summand of $\mathcal{O}Gb$ as an $\mathcal{O}(G \times L)$-module which has vertex ΔD must be isomorphic to a direct summand of Y. Thus $\mathrm{Res}_{G\times L}^{G\times N}(X)$ is isomorphic to a direct summand of Y; consequently there exists an injective ring homomorphism

$$\mathrm{End}_G(X) \hookrightarrow \mathrm{End}_G(Y).$$

Hence in order to show that the inequality (2) holds it suffices to demonstrate that

$$\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_G(Y)) \leq w! \cdot \mathrm{rank}_{\mathcal{O}}(\mathcal{O}Lf). \tag{3}$$

This will be accomplished by making a calculations with characters.

The endomorphism ring $\mathrm{End}_G(Y)$ may be identified with $b_w \cdots b_0 \mathcal{O}G_0 b_0 \cdots b_w$. The K-algebra KG_0b_0 is semisimple, so by general theory $K \otimes_{\mathcal{O}} \mathrm{End}_G(Y) = b_w \cdots b_0 KG_0 b_0 \cdots b_w$ is semisimple, and the restriction of any irreducible character of KG_0b_0 to $b_w \cdots b_0 KG_0 b_0 \cdots b_w$ is either an irreducible character or the zero character. Moreover each irreducible character of $b_w \cdots b_0 KG_0 b_0 \cdots b_w$ arises exactly once in this way. Thus

$$\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_G(Y)) = \sum_{\chi \in \mathrm{Irr}(KG_0b_0)} \chi(b_w \cdots b_0)^2. \tag{4}$$

We note that

$$\chi(b_w \cdots b_0) = b_w \,\mathrm{Res}_{G_w}^{G_{w-1}}(\cdots b_2 \,\mathrm{Res}_{G_2}^{G_1}(b_1 \,\mathrm{Res}_{G_1}^{G_0}(\chi)) \cdots)(1). \tag{5}$$

The principal p-block of KS_p is the block labelled by the empty p-core. So for $i = 1, \ldots, w$, we have $\mathrm{Irr}(KS(U_i)a_i) = \{\nu_0, \cdots, \nu_{p-1}\}$, where for $j = 0, \ldots, p-1$, we write

$$\nu_j = \chi^{(j+1, 1^{p-j-1})}$$

for the sake of convenience. Then for $i = 0, \ldots, w$, we have

$$\mathrm{Irr}(KG_ib_i) = \left\{ \nu_{j_1} \times \cdots \times \nu_{j_i} \times \chi^\lambda \right\},$$

where the j_γ's run from 0 to $p-1$ and λ runs over the set of partitions with p-core ρ and p-weight $w - i$.

Let $\mathcal{C}$ be the set of p-tuples $c = (c_0, \ldots, c_{p-1})$ of nonnegative integers satisfying $c_0 + \cdots + c_{p-1} = w$. For $i = 0, \ldots, w$ and $c \in \mathcal{C}$, we define

$\operatorname{Irr}(KG_ib_i, c)$ to be the set of characters $\nu_{j_1} \times \cdots \times \nu_{j_i} \times \chi^\lambda \in \operatorname{Irr}(KG_ib_i)$ such that for $j = 0, \dots, p-1$, we have

$$|\lambda^{(j)}| + \#\{\gamma \mid j_\gamma = j\} = c_j.$$

For $i = 0, \dots, w$ we have a disjoint union

$$\operatorname{Irr}(KG_ib_i) = \coprod_{c \in \mathcal{C}} \operatorname{Irr}(KG_ib_i, c).$$

Note that for a fixed $c \in \mathcal{C}$, there are $\frac{w!}{c_0! \cdots c_{p-1}!}$ characters in $\operatorname{Irr}(KG_wb_w, c)$, and all have the same degree.

Suppose that $0 \le i \le w-1$ and $\chi = \nu_{j_1} \times \cdots \times \nu_{j_i} \times \chi^\lambda \in \operatorname{Irr}(KG_ib_i)$. Then using the Littlewood-Richardson rule (see, e.g., [6, page 93, Theorem 2.8.13]) and part 2 of Lemma 4 we get

$$e_{i+1} \operatorname{Res}^{G_i}_{G_{i+1}}(\chi) = \sum_{s=0}^{p-1} \sum_{\mu} \nu_{j_1} \times \cdots \times \nu_{j_i} \times \nu_s \times \chi^\mu, \tag{6}$$

where the second sum runs over all partitions μ with p-core ρ and p-weight $w-(i+1)$ such that $\mu^{(j)} = \lambda^{(j)}$ for $j \neq s$ and $\mu^{(s)} \subseteq \lambda^{(s)}$ with $\left|\mu^{(s)}\right| = \left|\lambda^{(s)}\right| - 1$. Note that if $\chi \in \operatorname{Irr}(KG_ib_i, c)$ then every constituent of $e_{i+1} \operatorname{Res}^{G_i}_{G_{i+1}}(\chi)$ is in $\operatorname{Irr}(KG_{i+1}b_{i+1}, c)$.

Now fix $\chi^\lambda \in \operatorname{Irr}(KG_0b_0, c)$. For each $\varphi \in \operatorname{Irr}(KG_wb_w, c)$, we may apply (6) repeatedly to deduce that the multiplicity of φ as a constituent of

$$b_w \operatorname{Res}^{G_{w-1}}_{G_w}(\cdots b_2 \operatorname{Res}^{G_1}_{G_2}(b_1 \operatorname{Res}^{G_0}_{G_1}(\chi)) \cdots)$$

is

$$f(\lambda^{(0)}) \cdot \ldots \cdot f(\lambda^{(p-1)}),$$

where for any partition σ of n we denote by $f(\sigma)$ the number of nested sequences $\tau_1 \subseteq \cdots \subseteq \tau_n$ of partitions such that $|\tau_i| = i$ for $i = 1, \dots, n$, and $\tau_n = \sigma$. It follows from the branching rule (see, e.g., [6, page 59, Theorem 2.4.3]) that $f(\sigma)$ is the dimension of the irreducible character χ^σ of S_n. It is now immediate that

$$\sum_{\sigma \vdash n} f(\sigma)^2 = n!,$$

and we will use this fact below. Now using equation (5) we have

$$\chi^\lambda(b_w \cdots b_0) = f(\lambda^{(0)}) \cdot \ldots \cdot f(\lambda^{(p-1)}) \cdot \sum_{\varphi \in \operatorname{Irr}(KG_wb_w, c)} \varphi(1). \tag{7}$$

Squaring both sides of this equation and summing over all characters in $\mathrm{Irr}(KG_0b_0, c)$ we get

$$\begin{aligned}
&\sum_{\chi^\lambda \in \mathrm{Irr}(KG_0b_0,c)} \chi^\lambda(b_w \cdots b_0)^2 \\
= &\sum_{\chi^\lambda \in \mathrm{Irr}(KG_0b_0,c)} f(\lambda^{(0)})^2 \cdot \ldots \cdot f(\lambda^{(p-1)})^2 \cdot \left(\sum_{\varphi \in \mathrm{Irr}(KG_wb_w,c)} \varphi(1) \right)^2 \\
= &\left(\sum_{\mu_0 \vdash c_0} f(\mu_0)^2 \right) \cdot \ldots \cdot \left(\sum_{\mu_{p-1} \vdash c_{p-1}} f(\mu_{p-1})^2 \right) \cdot \\
&\cdot \frac{w!}{c_0! \cdots c_{p-1}!} \cdot \left(\sum_{\varphi \in \mathrm{Irr}(KG_wb_w,c)} \varphi(1)^2 \right) \\
= &\; w! \cdot \sum_{\varphi \in \mathrm{Irr}(KG_wb_w,c)} \varphi(1)^2,
\end{aligned}$$

and in turn summing this over all $c \in \mathcal{C}$ and using equation (4) we get

$$\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_G(Y)) = w! \cdot \sum_{\varphi \in \mathrm{Irr}(KG_wb_w)} \varphi(1)^2.$$

Remembering that $L = G_w$ and $f = b_w$ we see that the right hand side of this equation is equal to $w! \cdot \mathrm{rank}_{\mathcal{O}}(\mathcal{O}Lf)$ and that therefore the inequality (3) holds. This completes the proof of Theorem 2.

5. Splendid Rickard Equivalences

For blocks of group algebras there is an important strengthening of the notion of derived equivalence which is called splendid Rickard equivalence. This notion was introduced by Rickard [13] for principal blocks and later extended to the case of arbitrary blocks by Harris [5] and also by Linckelmann [7]. In the situation of Conjecture 1 it is hoped that there exists this strengthened form of a derived equivalence. In this section we give some pertinent definitions and explain briefly why we have not only derived equivalences but splendid Rickard equivlances in Theorem 2.

Let G and H be a finite groups and let b and c be block idempotents of $\mathcal{O}G$ and $\mathcal{O}H$. The corresponding blocks $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are called Rickard equivalent if there exists a bounded complex C of $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodules, each projective as left $\mathcal{O}Gb$-module and as right $\mathcal{O}Hc$-module, such that $\mathrm{End}_{\mathcal{O}Gb}(C)$ and $\mathcal{O}Hc$ are homotopy equivalent as complexes of $(\mathcal{O}Hc, \mathcal{O}Hc)$-bimodules, and $\mathrm{End}_{\mathcal{O}Hc}(C)$ and $\mathcal{O}Gb$ are homotopy equivalent as complexes of $(\mathcal{O}Gb, \mathcal{O}Gb)$-bimodules. In this situation C is called a Rickard complex and the functors $C \otimes_{\mathcal{O}Hc} ?$ and $\mathrm{Hom}_{\mathcal{O}Gb}(C, ?)$ induce inverse equivalences of the derived bounded categories $D^b(\mathcal{O}Gb)$ and $D^b(\mathcal{O}Hc)$. Of course, if C has only one nonzero term then $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are Morita equivalent.

Suppose that $\mathcal{O}Gb$ and $\mathcal{O}Hc$ have a common defect group D and that each term of C, considered as an $\mathcal{O}(G \times H)$-module, is a direct summand of a permutation module and is relatively ΔD-projective, where ΔD is the diagonal subgroup $\{(x, x) \mid x \in D\} \leq G \times H$. Then we say that C is splendid and that $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are splendidly Rickard equivalent. If C has only one nonzero term we say that $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are splendidly Morita equivalent. Note that in this case, we get that a source algebra of $\mathcal{O}Gb$ is isomorphic as interior D-algebra to a source algebra of $\mathcal{O}Hc$ (see, e.g., [10, Remark 7.5]).

Here is a simple situation in which we have a splendid Morita equivalence:

Lemma 6. *Let G_1 and G_2 be finite groups, and let b_1 and b_2 be block idempotents of $\mathcal{O}G_1$ and $\mathcal{O}G_2$. Then if $\mathcal{O}G_2b_2$ has defect group 1, then $\mathcal{O}G_1b_1$ and $\mathcal{O}G_1b_1 \otimes \mathcal{O}G_2b_2$ (a block of $G_1 \times G_2$) are splendidly Morita equivalent.*

Proof. Let i be a primitive idempotent in $\mathcal{O}G_2b_2$. Then the $\mathcal{O}((G_1 \times G_2) \times G_1)$-module $\mathcal{O}G_1b_1 \otimes \mathcal{O}G_2i$ is a summand of a permutation module and has vertex ΔD, where D is a defect group of $\mathcal{O}G_1b_1$. Furthermore $\mathcal{O}G_1b_1 \otimes \mathcal{O}G_2i$ induces a Morita equivalence between $\mathcal{O}G_1b_1$ and $\mathcal{O}G_1b_1 \otimes \mathcal{O}G_2b_2$. □

In order to describe a situation in which a composition of two splendid Rickard equivalences produces again a splendid Rickard equivalence, we consider the control of fusion of p-subgroups. For D any p-subgroup of G, define $\mathcal{F}(G, D)$ to be the set of homomorphisms $\psi : P \to D$, where P is any subgroup of D for which there exists $g \in G$ such that $\psi(x) = gxg^{-1}$ for all $x \in P$.

Lemma 7. *Let G, H, and L be finite groups with a common p-subgroup D, and let b, c, and f be block idempotents of $\mathcal{O}G$, $\mathcal{O}H$, and $\mathcal{O}L$ with defect group D. Suppose that $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are splendidly Rickard equivalent and that $\mathcal{O}Hc$ and $\mathcal{O}Lf$ are splendidly Rickard equivalent. If either $\mathcal{F}(H, D) \subseteq \mathcal{F}(G, D)$ or $\mathcal{F}(H, D) \subseteq \mathcal{F}(L, D)$, then $\mathcal{O}Gb$ and $\mathcal{O}Lf$ are splendidly Rickard equivalent.*

Proof. See, e.g., [3, Lemma 8.3]. □

We now return to the notation used in section 4 and assume that $w \leq 5$. Consider an arbitrary p-block of the symmetric groups which has weight w. We may take this block to be $\mathcal{O}S(V')e$, where V' is a finite set containing U and e is a block idempotent of $\mathcal{O}S(V')$ corresponding to a p-core κ. Then D is a defect group of $\mathcal{O}S(V')e$. We have that

$$N_{S(V')}(D) = N_{S(U)}(D) \times S(V' - U),$$

and the Brauer correspondent of $\mathcal{O}S(V')e$ in $N_{S(V')}(D)$ is

$$\mathcal{O}N_{S(U)}(D) \otimes \mathcal{O}S(V' - U)e'_0,$$

where e'_0 is the block idempotent of $\mathcal{O}S(V' - U)$ corresponding to the p-core κ. We consider the following pairs of blocks:

(1) $\mathcal{O}S(V')e$ and $\mathcal{O}Gb$;
(2) $\mathcal{O}Gb$ and $\mathcal{O}Nf$;
(3) $\mathcal{O}Nf$ and the principal block of $\mathcal{O}\widetilde{N}$;
(4) the principal block of $\mathcal{O}\widetilde{N}$ and $\mathcal{O}N_{S(U)}(D)$;
(5) $\mathcal{O}N_{S(U)}(D)$ and $\mathcal{O}N_{S(U)}(D) \otimes \mathcal{O}S(V' - U)e'_0$.

In his unpublished work ([12]) Rickard proved that the blocks in the first pair are splendidly Rickard equivalent. Andrei Marcus [8] showed that the blocks in the fourth pair are splendidly Rickard equivalent based on the existence of splendid Rickard equivalences for blocks with cyclic defect groups, proved by Rouquier in [14]. The blocks in the third pair, and those in the 5th pair are splendidly Morita equivalent by Lemma 6. Finally the blocks in the second pair are splendidly Morita equivalent; indeed, in section 4 we showed that an equivalence is induced by a bimodule X which has vertex ΔD and is a summand of a permutation module, namely $\mathcal{O}G$.

We can use Lemma 7 to combine these splendid equivalences, as long as we can show that $\mathcal{F}(G_1, D) \subseteq \mathcal{F}(G_2, D)$ for any groups G_1 and G_2 appearing in the list above. We may assume that G_1 and G_2 are subgroups of $S(W)$ which contain $N_{S(U)}(D)$, where W is a finite set containing U. Suppose P is a p-subgroup of D and g_1 an element of G_1 such that $g_1 P g_1^{-1} \subseteq D$. Let T be the subset of U consisting of elements not fixed by P. Then because $g_1 P g_1^{-1} \subseteq D$ we have that $g_1(T) \subseteq U$. Thus there exists $\omega \in S(U)$ such that $\omega(t) = g_1(t)$ for all $t \in T$. Then we have $\omega x \omega^{-1} = g_1 x g_1^{-1}$ for all $x \in P$. Now as D is a Sylow p-subgroup of $S(U)$ and is abelian, a theorem of Burnside (see, e.g., [4, page 240, Theorem 1.1]) tells us that there exists $\tau \in N_{S(U)}(D)$ such that $\tau x \tau^{-1} = \omega x \omega^{-1}$ for all $x \in P$. Thus we have $\tau x \tau^{-1} = g_1 x g_1^{-1}$ for all $x \in P$, where $\tau \in G_2$. So indeed $\mathcal{F}(G_1, D) \subseteq \mathcal{F}(G_2, D)$.

We have shown then that $\mathcal{O}S(V')e$ and its Brauer correspondent are splendidly Rickard equivalent. So we indeed get a splendid Rickard equivalence in Theorem 1.

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