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Lax pair and super-Yangian symmetry of the nonlinear super-Schrödinger equation

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We consider a version of the nonlinear Schrödinger equation with M bosons and N fermions. We first solve the classical and quantum versions of this equation, using a super-Zamolodchikov–Faddeev (ZF) algebra. Then we prove that the hierarchy associated to this model admits a super-Yangian $Y(gl(M|N))$ symmetry. We exhibit the corresponding (classical and quantum) Lax pairs. Finally, we construct explicitly the super-Yangian generators, in terms of the canonical fields on the one hand, and in terms of the ZF algebra generators on the other hand. The latter construction uses the well-bred operators introduced recently. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1625078]

I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation is one of the most studied systems in quantum integrable systems (for a review, see, e.g., Ref. 1), and its simplest (scalar) version played an important role in the development of the (quantum) inverse scattering method.[2] As usual in quantum integrable systems, its integrability relies on the existence of an infinite-dimensional symmetry algebra. In integrable systems, natural candidates for such algebras are the celebrated quantum groups associated to (affine) Lie algebras, or the Yangians. Indeed, it is known[3] that the quantum NLS model with spin 1/2 fermions and repulsive interaction on the line has a Yangian symmetry $Y(sl(2))$. More generally, its vectorial version, based on N-component bosons or on N-component fermions, was shown to possess a $Y(gl(N))$ symmetry.[4] The integrability can also be grounded on the existence of an infinite series of mutually commuting Hamiltonians, which thus generates a whole hierarchy of equations. In the case of scalar NLS equation, the hierarchy contains well-known models, such as the modified KdV equation.

It was natural to seek a supersymmetric version (including both bosons and fermions) of these models which admits the super-Yangian based on superalgebras $gl(M|N)$ as symmetry algebra. Different versions of such a generalization were already proposed, from the simple boson-fermion systems related to NLS,[5,6] or superfields formulation[7,8] of NLS, up to more algebraic studies of these models.[9,10] The difficulty with such generalizations is to keep the fundamental notion of integrability while allowing for the existence of supersymmetry. Even when some of the suggested supersymmetric systems were shown to pass some integrability conditions,[11] the status of such models remained not clearly established, and one is still looking for, e.g., their Lax presentation or their underlying infinite-dimensional symmetry algebra.

Another $\mathbb{Z}_2$-graded version of NLS was introduced by Kulish,[12] the fields being super-matrix valued and thus associated to both fermions and bosons. However, only the finite interval was studied, using the thermodynamical Bethe ansatz (see also Ref. 13), and the explicit quantum solutions are not known. The symmetry (super) algebra is also lacking in this presentation.

The aim of this article is to present a ''super-vectorial'' version (close to the matricial version introduced by Kulish) of the NLS model on the infinite line which includes M bosons **and** N

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fermions fields. The advantage of this version relies on its manifest integrability and the existence of quantum canonical solutions, which we will explicitly construct using a super-ZF algebra (Sec. II). Indeed, these solutions can be associated to a whole hierarchy of mutually commuting Hamiltonians, as it should be for an integrable model. It also admits, as we will show (Sec. III), a Lax presentation both at classical and quantum level (without using a superfield formalism). As usual, the Lax pair presentation allows us to recover the hierarchy of our super-NLS equation. Finally, this super-NLS hierarchy possesses a super-Yangian symmetry and we will construct it, both using the quantum canonical solutions or the super-ZF generators (Sec. IV).

II. NONLINEAR SUPER-SCHRÖDINGER EQUATION

A. The usual nonlinear Schrödinger equation

The NLS equation reads

$$(i\partial_t+\partial_x^2)\phi_i(x,t)=2g\phi^{\dagger j}(x,t)\phi_j(x,t)\phi_i(x,t),\quad i=1,...,N,\quad \text{with}\ \ g>0, \tag{2.1}$$

where summation over repeated indices is understood. It is obtained from the (time-independent) Hamiltonian

$$H(\phi_i,\phi_j^\dagger)=\int_{-\infty}^{\infty}dx(\partial_x\phi^{\dagger j}(x)\partial_x\phi_j(x)+g\phi^{\dagger i}(x)\phi^{\dagger j}(x)\phi_j(x)\phi_i(x)) \tag{2.2}$$

using the Hamiltonian equation of motion $\partial_t F=\{H,F\}$, valid for any functional $F(\phi_i,\phi_j^\dagger)$, where the Poisson bracket (PB) is canonically associated to ϕ and $\phi^\dagger$.

A solution *à la Rosales*[14] can be written as follows:

$$\phi_i(x,t)=\sum_{n=0}^{\infty}(-g)^n\phi_i^{(n)}(x,t),\quad g>0, \tag{2.3}$$

with

$$\begin{gathered}
\phi_i^{(n)}(x,t)=\int_{\mathbb{R}^{2n+1}}d^n\mathbf{p}d^{n+1}\mathbf{q}\lambda^{k_1}(p_1)\cdots\lambda^{k_n}(p_n)\lambda_{k_n}(q_n)\cdots\lambda_{k_1}(q_1)\lambda_i(q_0)\frac{e^{i\Omega_n(x,t;\mathbf{p},\mathbf{q})}}{Q_n(\mathbf{p},\mathbf{q},0)},\\
\Omega_n(x,t;\mathbf{p},\mathbf{q})=\sum_{j=0}^{n}(q_jx-q_j^2t)-\sum_{i=1}^{n}(p_ix-p_i^2t),\\
Q_n(\mathbf{p},\mathbf{q},\varepsilon)=\prod_{i=1}^{n}(p_i-q_{i-1}+i\varepsilon)(p_i-q_i+i\varepsilon),\\
d^n\mathbf{p}d^{n+1}\mathbf{q}=\prod_{\substack{i=1\\ j=0}}^{n}\frac{dp_i}{2\pi}\frac{dq_j}{2\pi},
\end{gathered} \tag{2.4}$$

where we have denoted $\mathbf{p}=(p_1,\ldots,p_n)$, $\mathbf{q}=(q_0,\ldots,q_n)$.

The Rosales solution is fundamental since its structure is preserved upon quantization[15] and we shall see below that this result survives when one includes fermions. The NLS equation and its hierarchy admit the Yangian $Y(gl(N))$ as symmetry, and the explicit construction of its generators was given in Ref. 3 [for $sl(2)$, in terms of canonical fields] and Ref. 4 [for $sl(N)$, in terms of the ZF generators]. A Lax pair formulation can be found in Refs. 16 and 17 (for NLS equation) and in Refs. 18 and 19 (for its vectorial generalization).

B. Classical nonlinear super-Schrödinger equation

We consider a generalized version of the NLS equation which includes both bosons and fermions. Due to the use of auxiliary spaces (see the Appendix), the corresponding equation will formally look like the original one, but let us insist that the present version is a ‘‘supersymmetric’’ version of it. While the similarities allow us to build the solution of the nonlinear super-Schrödinger equation, the differences will appear, for instance, in the nature of the symmetry algebra (see below).

We define $\Phi(x)=\Sigma_{j=1}^{M+N}\phi_j(x)e_j$, where e_j is an $(M+N)$-column vector in the auxiliary space and summation is understood for repeated indices. Here ϕ_j, $j=1,...,M$, and ϕ_j, $j=M+1,...,M+N$, are the bosonic and fermionic components, respectively. By fermionic functions, we mean Grassmann-valued functions depending on the real variable x, the integrations throughout the article being always in real (or complex) variables. For convenience, we set $K=M+N$. We shall also need adjoints of the fields

$$\Phi^\dagger(x)=\phi_i^\dagger(x)e_i^\dagger, \quad x\in\mathbb{R}. \tag{2.5}$$

The Hamiltonian reads

$$H(\Phi,\Phi^\dagger)=\int_{-\infty}^{\infty}dx(\partial_x\Phi^\dagger(x)\partial_x\Phi(x)+g(|\Phi(x)|^2)^2), \tag{2.6}$$

or, in components,

$$H(\Phi,\Phi^\dagger)=\int_{-\infty}^{\infty}dx(\partial_x\phi^{\dagger j}(x)\partial_x\phi_j(x)+g\,\phi^{\dagger j}(x)\,\phi^{\dagger k}(x)\,\phi_k(x)\,\phi_j(x)). \tag{2.7}$$

The canonical Poisson brackets for the basic fields $\Phi(x)$, $\Phi^\dagger(y)$ with corresponding components $\phi_i(x)$, $\phi_j^\dagger(y)$ take the following form:

$$\{\Phi_1(x),\Phi_2^\dagger(y)\}=i\,\delta_{12}\delta(x-y)=-\{\Phi_2^\dagger(y),\Phi_1(x)\} \quad \text{(globally)}, \tag{2.8}$$

$$\{\phi_j(x),\phi_k^\dagger(y)\}=i\,\delta_{jk}\delta(x-y)=-(-1)^{[j][k]}\{\phi_k^\dagger(y),\phi_j(x)\} \quad \text{(in components)}. \tag{2.9}$$

The field $\Phi(x,t)$ of components $\phi_i(x,t)$ satisfies the following Hamiltonian equation of motion which we call the classical nonlinear super-Schrödinger (NLSS) equation:

$$i\,\partial_t\Phi(x,t)=-\partial_x^2\Phi(x,t)+2g|\Phi(x,t)|^2\Phi(x,t) \quad \text{(globally)}, \tag{2.10}$$

$$i\,\partial_t\phi_j(x,t)=-\partial_x^2\phi_j(x,t)+2g(\phi_k^\dagger(x,t)\,\phi_k(x,t))\,\phi_j(x,t) \quad \text{(in components)}. \tag{2.11}$$

These equations are simply derived from the Hamiltonian equations of motion $\partial_t\Phi(x,t)=\{H,\Phi(x,t)\}$ and $\partial_t\phi_i(x,t)=\{H,\phi_i(x,t)\}$. The equations of motion are (formally) the same as the usual ones and the solution *à la Rosales* (2.3) and (2.4) is still valid in our case:

Theorem 2.1: *The solution of the classical NLSS equation (2.11) is given by*

$$\phi_j(x,t)=\sum_{n=0}^{\infty}(-g)^n\phi_j^{(n)}(x,t) \quad \textit{where} \tag{2.12}$$

$$\phi_j^{(n)}(x,t)=\int_{\mathbb{R}^{2n+1}}d^n\mathbf{p}d^{n+1}\mathbf{q}\sum_{k_1,\ldots,k_n=1}^{K}\lambda_{k_1}^\dagger(p_1)\cdots\lambda_{k_n}^\dagger(p_n)\lambda_{k_n}(q_n)\cdots\lambda_{k_1}(q_1)\lambda_j(q_0)\frac{e^{i\Omega_n(x,t;\mathbf{p},\mathbf{q})}}{Q_n(\mathbf{p},\mathbf{q},0)}, \tag{2.13}$$

using the same notations as in (2.4).

Proof: Substituting into the NLSS equation, it amounts to the following identity being satisfied,

$$\sum_{j=0}^{n} q_j^2 - \sum_{i=1}^{n} p_i^2 - \left(\sum_{j=0}^{n} q_j - \sum_{i=1}^{n} p_i \right)^2 = -2 \sum_{c=1}^{n-1} \sum_{a=1}^{c} (p_{a+1} - q_a)(p_{c+1} - q_{c+1}),$$

which is readily seen to hold. ■

Note that, due to the $\mathbb{Z}_2$-graded tensor product, the ordering of the $\lambda^\dagger$'s and of the λ's, respectively, matters.

C. Quantizing NLSS

1. Graded ZF algebra

We write a graded version of the ZF algebra,[20,21] using auxiliary spaces and entities containing bosonic and fermionic components (see the Appendix):

$$\mathbf{A}(k) = a_i(k) e_i \quad \text{and} \quad \mathbf{A}^\dagger(k) = a_i^\dagger(k) e_i^\dagger, \quad k \in \mathbb{R}. \tag{2.14}$$

Definition 2.2: The graded ZF algebra reads

$$\mathbf{A}_1(k_1) \mathbf{A}_2(k_2) = R_{21}(k_2 - k_1) \mathbf{A}_2(k_2) \mathbf{A}_1(k_1), \tag{2.15}$$

$$\mathbf{A}_1^\dagger(k_1) \mathbf{A}_2^\dagger(k_2) = \mathbf{A}_2^\dagger(k_2) \mathbf{A}_1^\dagger(k_1) R_{21}(k_2 - k_1), \tag{2.16}$$

$$\mathbf{A}_1(k_1) \mathbf{A}_2^\dagger(k_2) = \mathbf{A}_2^\dagger(k_2) R_{12}(k_1 - k_2) \mathbf{A}_1(k_1) + \boldsymbol{\delta}_{12} \delta(k_1 - k_2), \tag{2.17}$$

where

$$R_{12}(k) = \frac{k \mathbb{1} \otimes \mathbb{1} - igP_{12}}{k + ig} \tag{2.18}$$

is the R-matrix for the super-Yangian $Y(gl(M|N)) \equiv Y(M|N)$, and P_{12} is the super-permutation operator:

$$P_{12} = \sum_{i,j=1}^{K} (-1)^{[j]} E_{ij} \otimes E_{ji}. \tag{2.19}$$

Note that for even vectors u, v and even matrices B, C (as defined in the Appendix), one has $P_{12}(u \otimes v) = v \otimes u$ and $P_{12}(B \otimes C) P_{12} = C \otimes B$.

The R-matrix has the following useful properties:

$$R_{21}(k) = R_{12}(k), \tag{2.20}$$

$$R_{12}(k_1 - k_2) R_{21}(k_2 - k_1) = \mathbb{1} \otimes \mathbb{1}, \tag{2.21}$$

$$R_{12}^\dagger(k_1 - k_2) = R_{21}(k_2 - k_1). \tag{2.22}$$

For quantities of definite $\mathbb{Z}_2$-grade, we define their super-commutator by

$$[\![B, C]\!] = BC - (-1)^{[B][C]} CB. \tag{2.23}$$

Then, after some calculations, one shows that the component version of the ZF algebra reads $(j,k = 1,...,K)$

$$[\![a_j(k_1),a_k(k_2)]\!]=\frac{-ig}{k_2-k_1+ig}(a_j(k_2)a_k(k_1)+(-1)^{[j][k]}a_k(k_2)a_j(k_1)), \tag{2.24}$$

$$[\![a_j^\dagger(k_1),a_k^\dagger(k_2)]\!]=\frac{-ig}{k_2-k_1+ig}(a_j^\dagger(k_2)a_k^\dagger(k_1)+(-1)^{[j][k]}a_k^\dagger(k_2)a_j^\dagger(k_1)), \tag{2.25}$$

$$[\![a_j(k_1),a_k^\dagger(k_2)]\!]=\frac{-ig}{k_1-k_2+ig}\left((-1)^{[j][k]}a_k^\dagger(k_2)a_j(k_1)+\delta_{jk}\sum_{\ell=1}^{K}a_\ell^\dagger(k_2)a_\ell(k_1)\right)+\delta_{jk}\delta(k_1-k_2). \tag{2.26}$$

Note that these relations ensure the existence of a PBW basis, generated by the monomials having $a^\dagger$'s on the left of the a's, the a's on one hand, and the $a^\dagger$'s on the other hand, being ordered according to the magnitude of the "impulsions" k_j.

2. Fock representation

The previous algebra can be represented on a Fock space, which is most useful for our quantization of NLSS, and we follow the basic ideas of Ref. 15 (further developed in, e.g. Refs. 22 and 23). A detailed presentation of the graded version when $M=N=1$ has been given in Ref. 24. The general case follows the same lines, so that we just sketch the results, referring to Ref. 24 for more details about the $\mathbb{Z}_2$-graded case.

We introduce $\mathcal{F}_R=\oplus_{n=0}^{\infty}\mathcal{H}_R^n$ where $\mathcal{H}_R^0=\mathbb{C}$,

$$\mathcal{H}_R^1=\left\{\boldsymbol{\varphi}(p)=\sum_{j=1}^{K}\varphi_j(p)e_j \;\text{ s.t. }\; \varphi_j\in L^2(\mathbb{R}), j=1,...,K\right\}\equiv KL^2(\mathbb{R}),$$

and for $n\geqslant 2$

$$\begin{aligned}\mathcal{H}_R^n=\Bigg\{&\boldsymbol{\varphi}_{1\dots n}(p_1,\dots,p_n)\\ &=\sum_{i_1,\dots,i_n=1}^{K}\varphi_{i_1,\dots,i_n}(p_1,\dots,p_n)(e_{i_1}\otimes\cdots\otimes e_{i_n}) \;\text{ s.t. }\; \varphi_{i_1,\dots,i_n}\in L^2(\mathbb{R}^n),\\ &i_1,\dots,i_n=1,...,K, \quad\text{and}\quad \boldsymbol{\varphi}_{1\dots i,i+1\dots n}(p_1,\dots,p_i,p_{i+1},\dots,p_n)\\ &\qquad=R_{i,i+1}(p_i-p_{i+1})\boldsymbol{\varphi}_{1\dots i+1,i\dots n}(p_1,\dots,p_{i+1},p_i,\dots,p_n).\end{aligned}$$

There exists a (vacuum) vector $\Omega\in\mathcal{D}$ which is cyclic with respect to $\mathbf{A}^\dagger(k)$ and annihilated by $\mathbf{A}(k)$.

The scalar product which we define below on $\mathcal{H}_R^n$ provides the usual L^2 topology and $\mathcal{F}_R$ is the completed vector space over $\mathbb{C}$ for this topology.

The sesquilinear form $\langle,\rangle$ defined on $\mathcal{H}_R^n\times\mathcal{H}_R^n$, $n\geqslant 1$, by

$$\langle\boldsymbol{\varphi},\boldsymbol{\psi}\rangle=\int_{\mathbb{R}^n}d^np\;\boldsymbol{\varphi}_{1\dots n}^\dagger(p_1,\dots,p_n)\boldsymbol{\psi}_{1\dots n}(p_1,\dots,p_n), \tag{2.27}$$

$$\boldsymbol{\varphi}_{1\dots n}^\dagger(p_1,\dots,p_n)=(-1)^{\sum_{k=1}^{n-1}([i_1]+\cdots+[i_k])[i_{k+1}]}\overline{\varphi}^{i_1\cdots i_n}(e_{i_1}^\dagger\otimes e_{i_2}^\dagger\otimes\cdots\otimes e_{i_n}^\dagger) \tag{2.28}$$

is a (Hermitian) scalar product.

We introduce the finite particle space $\mathcal{F}_R^0\subset\mathcal{F}_R$, spanned by the sequences $(\varphi,\boldsymbol{\varphi}_1,\dots,\boldsymbol{\varphi}_{1\cdots n},\dots)$ with $\boldsymbol{\varphi}_{1\cdots n}\in\mathcal{H}_R^n$ and $\boldsymbol{\varphi}_{1\cdots n}=0$ for n large enough. As (2.27) is defined for all n, it extends naturally to $\mathcal{F}_R^0$. In this context, the vacuum state is $\Omega=(1,0,...,0,...)$, so that it is normalized to 1.

We are now able to define the (smeared) creation and annihilation operators $A(\mathbf{f})$ and $A^\dagger(\mathbf{f})$ on $\mathcal{F}_R^0$ through their action: $A(\mathbf{f})\Omega=0$ and for $\boldsymbol{\varphi}_{0\cdots n}\in\mathcal{H}_R^{n+1}$,

$$[A(\mathbf{f})\boldsymbol{\varphi}]_{1\cdots n}(p_1,\ldots,p_n)=\sqrt{n+1}\int_{\mathbb{R}} dp_0\, \mathbf{f}_0^\dagger(p_0)\boldsymbol{\varphi}_{0\cdots n}(p_0,p_1,\ldots,p_n). \tag{2.29}$$

Similarly, for $\boldsymbol{\varphi}_{1\ldots n}\in\mathcal{H}_R^n$:

$$\begin{aligned}[A^\dagger(\mathbf{f})\boldsymbol{\varphi}]_{0\cdots n}(p_0,\ldots,p_n)=&\frac{1}{\sqrt{n+1}}\boldsymbol{\varphi}_{1\cdots n}(p_1,\ldots p_n)f_0(p_0)+\frac{1}{\sqrt{n+1}}\sum_{k=1}^{n} R_{k-1,k}(p_{k-1}\\ &-p_k)\cdots R_{0k}(p_0-p_k)\boldsymbol{\varphi}_{0\cdots\hat{k}\cdots n}(p_0,\ldots,\widehat{p_k},\ldots,p_n)\mathbf{f}_k(p_k),\end{aligned} \tag{2.30}$$

where the hatted symbols are omitted.

It is easily checked that (2.29) and (2.30) are indeed elements of $\mathcal{H}_R^n$ and $\mathcal{H}_R^{n+1}$, respectively. Therefore, we have operators acting on $\mathcal{F}_R^0$ (linearity in $\boldsymbol{\varphi}$ obvious) with the additional property that they are bounded (i.e., continuous) on each finite particle sector $\mathcal{H}_R^n$. Another essential feature is the adjointness of these operators with respect to $\langle,\rangle$:

$$\forall\boldsymbol{\varphi}\in\mathcal{H}_R^n,\ \ \forall\boldsymbol{\psi}\in\mathcal{H}_R^{n+1},\quad \forall\mathbf{f}\in\mathcal{H}_R^1,\quad \langle\boldsymbol{\varphi},A(\mathbf{f})\boldsymbol{\psi}\rangle=\langle A^\dagger(\mathbf{f})\boldsymbol{\varphi},\boldsymbol{\psi}\rangle. \tag{2.31}$$

At this stage, the Fock representations $\mathbf{A}(p)$, $\mathbf{A}^\dagger(p)$ of the generators of the ZF algebra appear as operator-valued distributions through the definition

$$A(\mathbf{f})=\int_{\mathbb{R}} dp\, \mathbf{f}^\dagger(p)\mathbf{A}(p),\quad A^\dagger(\mathbf{f})=\int_{\mathbb{R}} dp\, \mathbf{A}^\dagger(p)\mathbf{f}(p). \tag{2.32}$$

It is readily shown from these definitions that $\mathbf{A}(p)$ and $\mathbf{A}^\dagger(p)$ satisfy the exchange relations (2.15)–(2.17), thus providing the desired representation.

We now have all the ingredients to deduce results for the whole Fock space $\mathcal{F}_R$ while working on smaller and more intuitive spaces dense in $\mathcal{F}_R$, using the continuity of the operators. In our case, one has to define such a "state space" $\mathcal{D}\subset\mathcal{F}_R$ in the sense of distributions as follows: $\mathcal{D}^0=\mathbb{C}$ and

$$\mathcal{D}^n=\left\{\int_{\mathbb{R}^n} d^n p\, \mathbf{A}_1^\dagger(p_1)\cdots\mathbf{A}_n^\dagger(p_n)\Omega\mathbf{f}(p_1,\ldots,p_n);\ \ \mathbf{f}\in K^nL^2(\mathbb{R}^n)\right\},\quad n\geqslant 1.$$

Then, $\mathcal{D}$ is spanned by the sequences $\boldsymbol{\chi}=(\chi,\boldsymbol{\chi}_1,\ldots,\boldsymbol{\chi}_{1\cdots n},\ldots)$, where $\boldsymbol{\chi}_{1\cdots n}\in\mathcal{D}^n$ and $\boldsymbol{\chi}_{1\cdots n}=0$ for n large enough. We also define

$$\mathcal{D}_0^0=\mathbb{C},\quad \mathcal{D}_0^n=\{\widetilde{A}_1^\dagger(\mathbf{f}_1,t)\cdots\widetilde{A}_n^\dagger(\mathbf{f}_n,t)\Omega,\mathbf{f}_1>\cdots>\mathbf{f}_n\}\subset\mathcal{H}_R^n,\quad n\geqslant 1, \tag{2.33}$$

where

$$\left.\begin{aligned}\widetilde{A}^\dagger(\mathbf{f},t)&=\int_{\mathbb{R}} dx\widetilde{\mathbf{A}}^\dagger(x,t)\mathbf{f}(x),\\ \widetilde{\mathbf{A}}^\dagger(x,t)&=\int_{\mathbb{R}} dp\mathbf{A}^\dagger(p)e^{iqx-iq^2t},\end{aligned}\right. \quad x,t\in\mathbb{R}, \tag{2.34}$$

and the space $\mathcal{D}_0$ is the linear span of sequences $\boldsymbol{\chi}=(\chi,\boldsymbol{\chi}_1,\ldots,\boldsymbol{\chi}_{1\cdots n},\ldots)$, where $\boldsymbol{\chi}_{1\cdots n}\in\mathcal{D}_0^n$ and $\boldsymbol{\chi}_{1\cdots n}=0$ for n large enough. We also introduce the following partial ordering relation:

$$\mathbf{f}>\mathbf{g}\Leftrightarrow\forall i,j=1,\ldots,K,\quad \forall x\in\mathrm{supp}(\mathbf{f}_i),\quad \forall y\in\mathrm{supp}(g_j),\quad x>y,$$

which is just the extension of the ordering of the momenta k_i in the definition of a state space basis $|k_1,\ldots,k_n\rangle$. Then, one shows that $\mathcal{D}$ and $\mathcal{D}_0$ are dense in $\mathcal{F}_R$.

Summarizing, we have constructed a graded ZF algebra and its Fock representation $\mathcal{F}_R$ and, inspired by earlier works,[15,16,25–27] we shall see that this allows us to construct the quantum version of NLSS and its solution.

3. Quantization of the fields

Following Refs. 15 and 27, we simply write the quantum version of $\phi_j^{(n)}(x,t)$ as

$$\phi_j^{(n)}(x,t)=\int_{\mathbb{R}^{2n+1}} d^n\mathbf{p}\, d^{n+1}\mathbf{q} \sum_{k_1,\dots,k_n=1}^{K} a_{k_1}^\dagger(p_1)\cdots a_{k_n}^\dagger(p_n) a_{k_n}(q_n)\cdots a_{k_1}(q_1) a_j(q_0) \frac{e^{i\Omega_n(x,t;\mathbf{p},\mathbf{q})}}{Q_n(\mathbf{p},\mathbf{q},\varepsilon)} \tag{2.35}$$

using the same notations as in (2.4) and an $i\epsilon$ contour prescription. The global field reads

$$\Phi(x,t)=\sum_{n=0}^{\infty}(-g)^n\Phi^{(n)}(x,t) \quad \text{with } \Phi^{(n)}(x,t)=\phi_j^{(n)}(x,t)e_j. \tag{2.36}$$

From (2.31), we deduce

$$\Phi^\dagger(x,t)=\sum_{n=0}^{\infty}(-g)^n\Phi^{\dagger(n)}(x,t) \tag{2.37}$$

with

$$\Phi^{\dagger(n)}(x,t)=\int_{\mathbb{R}^{2n+1}} d^n\mathbf{p}\, d^{n+1}\mathbf{q}\mathbf{A}^\dagger(q_0)\mathbf{A}_1^\dagger(q_1)\cdots\mathbf{A}_n^\dagger(q_n)\mathbf{A}_n(p_n)\cdots\mathbf{A}_1(p_1)\frac{e^{-i\Omega_n(x,t;\mathbf{p},\mathbf{q})}}{Q_n(\mathbf{p},\mathbf{q},-\varepsilon)}. \tag{2.38}$$

Just like we dealt with $A(\mathbf{f})$ and $A^\dagger(\mathbf{f})$, we are naturally led to introduce

$$\Phi(\mathbf{f},t)=\int_{\mathbb{R}}\mathbf{f}^\dagger(x)\Phi(x,t), \quad \Phi^\dagger(\mathbf{f},t)=\int_{\mathbb{R}}\Phi^\dagger(x,t)\mathbf{f}(x). \tag{2.39}$$

And just like we did in Ref. 24, one shows that $\Phi(\mathbf{f},t)$ and $\Phi^\dagger(\mathbf{f},t)$ are indeed well-defined operators on a common invariant domain which turns out to be $\mathcal{D}_0$. These fields also satisfy the following fundamental requirement.

Theorem 2.3: *The quantum fields $\Phi(\mathbf{f},t)$, $\Phi^\dagger(\mathbf{g},t)$ satisfy the equal time canonical commutation relations as operators on $\mathcal{F}_R^0$*

$$[\Phi(\mathbf{f},t),\Phi(\mathbf{g},t)]=[\Phi^\dagger(\mathbf{f},t),\Phi^\dagger(\mathbf{g},t)]=0, \tag{2.40}$$

$$[\Phi(\mathbf{f},t),\Phi^\dagger(\mathbf{g},t)]=\langle\mathbf{f},\mathbf{g}\rangle. \tag{2.41}$$

Proof: The proof is the same as in the ordinary NLS equation, see Ref. 15 or 23 for details. ■

One then deduces the equal time CCR in components for the operator-valued distributions $\phi_j(x,t)$, $\phi_k^\dagger(y,t)$:

$$[\![\phi_j(x,t),\phi_k(y,t)]\!]=[\![\phi_j^\dagger(x,t),\phi_k^\dagger(y,t)]\!]=0, \tag{2.42}$$

$$[\![\phi_j(x,t),\phi_k^\dagger(y,t)]\!]=\delta_{jk}\delta(x-y). \tag{2.43}$$

Let us remind that for j, $k=M+1,\dots K$, the above CCR correspond to anticommutator, consistent with the fermionic nature of these fields.

4. Time evolution

We first wish to emphasize that the form of the Hamiltonian (2.7) cannot be reproduced here owing to the nature of the fields (products of distributions are not defined). Fortunately, the power of the ZF algebra and the quantum inverse method [leading to (2.35) and (2.36)] rescues us by delivering a simple, freelike Hamiltonian in terms of oscillators. Indeed, one easily checks that the Hamiltonian defined by

$$H=\int_{\mathbb{R}} dp\, p^2 \mathbf{A}^\dagger(p)\mathbf{A}(p) \tag{2.44}$$

is self-adjoint, i.e., $H^\dagger = H$. Moreover,

$$\forall \boldsymbol{\varphi}\in\mathcal{D}, \quad [H\boldsymbol{\varphi}]_{1\cdots n}(p_1,\ldots,p_n)=(p_1^2+\cdots+p_n^2)\boldsymbol{\varphi}_{1\cdots n}(p_1,\ldots,p_n), \tag{2.45}$$

which shows that $\mathcal{D}$ is also an invariant domain for H and that this operator has the correct eigenvalues. Finally, H generates the time evolution of the field:

$$\Phi(f,t)=e^{iHt}\Phi(f,0)e^{-iHt}. \tag{2.46}$$

Therefore, H, so defined, is the Hamiltonian of our quantum system.

Note that (2.45) and (2.46) have to be understood as operator equalities and must be evaluated on $\mathcal{D}$.

The freelike expression for H in terms of creation and annihilation oscillators may be surprising at first glance, but it is actually a mere consequence of the rather complicated exchange relations (2.15)–(2.17). One can say that the effect of the nonlinear term has been encoded directly in the oscillators instead of the Hamiltonian (or equivalently the Lagrangian) of the field theory, yielding a (possibly misleading) simple expression for H. One may finally wonder about the coupling constant which seems to disappear. Once again, it is actually present through the R-matrix in the exchange relations.

Besides, the quantum nonlinear super-Schrödinger equation holds in the following form:

$$\forall \boldsymbol{\varphi},\boldsymbol{\psi}\in\mathcal{D}, \quad (i\partial_t+\partial_x^2)\langle \boldsymbol{\varphi},\Phi(x,t)\boldsymbol{\psi}\rangle = 2g\langle \boldsymbol{\varphi},:\Phi\Phi^\dagger\Phi:(x,t)\boldsymbol{\psi}\rangle. \tag{2.47}$$

5. Correlation functions

Again following the case of NLS, one shows that for $\boldsymbol{\varphi},\boldsymbol{\psi}\in\mathcal{D}$, one has

$$\mathbf{f}>\mathbf{g}, \quad \langle \boldsymbol{\varphi},\Phi^\dagger(\mathbf{g},t)\widetilde{A}^\dagger(\mathbf{f},t)\boldsymbol{\psi}\rangle = \langle \boldsymbol{\varphi},\widetilde{A}^\dagger(\mathbf{f},t)\Phi^\dagger(\mathbf{g},t)\boldsymbol{\psi}\rangle, \tag{2.48}$$

for $\mathbf{g}>\mathbf{f}_i$, $i=1,\ldots,n$,

$$\langle \boldsymbol{\varphi},\Phi^\dagger(\mathbf{g},t)\widetilde{A}^\dagger(\mathbf{f}_1,t)\cdots\widetilde{A}^\dagger(\mathbf{f}_n,t)\Omega\rangle = \langle \boldsymbol{\varphi},\widetilde{A}^\dagger(\mathbf{g},t)\widetilde{A}^\dagger(\mathbf{f}_1,t)\cdots\widetilde{A}^\dagger(\mathbf{f}_n,t)\Omega\rangle \tag{2.49}$$

and for any $\mathbf{f}_1>\mathbf{f}_2>\cdots>\mathbf{f}_n$,

$$\langle \boldsymbol{\varphi},\Phi(\mathbf{g},t)\widetilde{A}^\dagger(\mathbf{f}_1,t)\cdots\widetilde{A}^\dagger(\mathbf{f}_n,t)\Omega\rangle = \sum_{j=1}^{n}\langle \mathbf{g},\mathbf{f}_j\rangle\langle \boldsymbol{\varphi},\widetilde{A}^\dagger(f_1,t)\cdots\widehat{\widetilde{A}^\dagger}(\mathbf{f}_j,t)\cdots\widetilde{A}^\dagger(\mathbf{f}_n,t)\Omega\rangle. \tag{2.50}$$

This proves that the correlation functions of the NLSS model are completely determined, e.g.,

$$\langle \Omega, \Phi(\mathbf{g}_1,t)\cdots\Phi(\mathbf{g}_m,t)\Phi^\dagger(\mathbf{f}_1,t)\cdots\Phi^\dagger(\mathbf{f}_n,t)\Omega\rangle = \delta_{m,n}\sum_{\sigma\in S_n}\prod_{i=1}^{n}\langle \mathbf{g}_{\sigma(i)},\mathbf{f}_i\rangle,$$

$$\langle \boldsymbol{\varphi}_{1\cdots p}, \Phi(\mathbf{g}_1,t)\cdots\Phi(\mathbf{g}_n,t)\Phi^\dagger(\mathbf{f}_1,t)\cdots\Phi^\dagger(\mathbf{f}_m,t)\Omega\rangle$$
$$= \delta_{m,n+p}\sum_{\sigma\in S_{n+p}}\left(\prod_{i=1}^{n}\langle \mathbf{g}_{\sigma(i)},\mathbf{f}_i\rangle\right)\langle \boldsymbol{\varphi}_{1\cdots p}, \mathbf{g}_{\sigma(n+1)}\cdots\mathbf{g}_{\sigma(n+p)}\rangle.$$

Similar expressions can be obtained when dealing with the fields $\Phi(x,t)$ and $\Phi^\dagger(x,t)$.

III. LAX PAIR AND SUPER-YANGIAN SYMMETRY FOR NLSS

Let us stress once again that we aim at generalizing known results of integrability and symmetry for the nonlinear Schrödinger equation to the case of an arbitrary number of bosons **and** fermions. This physical motivation can be carried out by using appropriately the graded formalism presented in the Appendix. Furthermore, we also want to transport our results to the quantum case, which leads us to adopt the convenient Hamiltonian form of our model.

A. Classical Lax pairs

We define the Lax even super-matrix in $gl(M+1|N)$

$$L(\lambda;x) = \frac{i\lambda}{2}\Sigma + \Omega(x) \quad \text{with } \Sigma = \mathbb{I}_{K+1,K+1} - 2E_{K+1,K+1} \tag{3.1}$$

and

$$\Omega(x) = i\sqrt{g}\sum_{j=1}^{K}(\phi_j(x)E_{j,K+1} - \phi_j^\dagger(x)E_{K+1,j}). \tag{3.2}$$

Let us stress that, as above, the elementary matrices E_{jk} (with 1 at position j, k) are $\mathbb{Z}_2$-graded, with $[E_{jk}]=[j]+[k]$, $[j]=[K+1]=0$ for $1\leqslant j\leqslant M$ and $[j]=1$ for $M<j\leqslant K$. With this convention, the $gl(M+1|N)$ superalgebra has the unusual matrix form

$$\begin{pmatrix} M\times M & & M\times 1 \\ & N\times N & \\ 1\times M & & 1\times 1 \end{pmatrix},$$

where the size of the submatrices corresponding to bosonic generators have been explicitly written.

Using the PB of the ϕ's, it is easy to compute that

$$\{L_1(\lambda;x), L_2(\mu;y)\} = i\,\delta(x-y)[r(\lambda-\mu), L_1(\lambda;x)+L_2(\mu;y)] \tag{3.3}$$

with

$$r(\lambda-\mu) = \frac{g}{\lambda-\mu}\Pi_{12}, \tag{3.4}$$

where we have introduced the $(K+1)\times(K+1)$ super-permutation

$$\Pi_{12} = \sum_{i,j=1}^{K+1}(-1)^{[j]}E_{ij}\otimes E_{ji}.$$

Definition 3.1: We define the transition matrix by

$$\partial_x T(\lambda;x,y) = L(\lambda;x)T(\lambda;x,y), \quad x>y, \tag{3.5}$$

with the "initial condition" $T(\lambda;x,x)=\mathbb{I}$.

$T(\lambda;x,y)$ obeys the iterative equation

$$T(\lambda;x,y) = E(\lambda;x-y) + E(\lambda;x)\int_y^x dz\,\Omega(z)E(\lambda;z)T(\lambda;z,y), \tag{3.6}$$

where we have introduced

$$E(\lambda;x) = \exp\left(\frac{ix\lambda}{2}\Sigma\right) = e^{ix\lambda/2}\mathbb{I}_{K+1} + (e^{-ix\lambda/2} - e^{ix\lambda/2})E_{K+1,K+1}. \tag{3.7}$$

Property 3.2:

$$\{T_1(\lambda;x,y), T_2(\mu;x,y)\} = [r(\lambda-\mu), T(\lambda;x,y)\otimes T(\mu;x,y)]. \tag{3.8}$$

Proof: The equation (3.6) implies that

$$T(\lambda;x,y) = \sum_{n=0}^{\infty} T^{(n)}(\lambda;x,y), \tag{3.9}$$

$$\begin{aligned} T^{(n)}(\lambda;x,y) = \int_{\mathbb{R}^n} d^n z\,\theta(x>z_1>z_2>\cdots>z_n>y)E(\lambda;x-z_1)\Omega(z_1) \\ \times E(\lambda;z_1-z_2)\Omega(z_2)\cdots\Omega(z_n)E(\lambda;z_n-y). \end{aligned} \tag{3.10}$$

It is then simple to show that

$$\{\Phi_1(w), T_2(\lambda;x,y)\} = \sqrt{g}\,\theta(x>w>y)T_2(\lambda;x,w)\sigma_{12}^- T_2(\lambda;w,y), \tag{3.11}$$

$$\{\Phi_2(w), T_1(\lambda;x,y)\} = \sqrt{g}\,\theta(x>w>y)T_1(\lambda;x,w)\sigma_{21}^- T_1(\lambda;w,y), \tag{3.12}$$

$$\{\Phi_1^\dagger(w), T_2(\lambda;x,y)\} = \sqrt{g}\,\theta(x>w>y)T_2(\lambda;x,w)\sigma_{12}^+ T_2(\lambda;w,y), \tag{3.13}$$

$$\{\Phi_2^\dagger(w), T_1(\lambda;x,y)\} = \sqrt{g}\,\theta(x>w>y)T_1(\lambda;x,w)\sigma_{21}^+ T_1(\lambda;w,y), \tag{3.14}$$

where we have defined

$$\sigma_{12}^- = \sum_{j=1}^{K} e_j \otimes E_{K+1,j}; \quad \sigma_{12}^+ = \sum_{j=1}^{K} (-1)^{[j]} e_j^\dagger \otimes E_{j,K+1}. \tag{3.15}$$

From the form (A5) one also computes

$$\{\Phi_1(w), T_2(\lambda;x,y)\} = i(e_j\otimes\mathbb{I})\frac{\delta T_2(\lambda;x,y)}{\delta\phi_j^\dagger(w)}, \tag{3.16}$$

$$\{\Phi_2(w), T_1(\lambda;x,y)\} = i(\mathbb{I}\otimes e_j)\frac{\delta T_1(\lambda;x,y)}{\delta\phi_j^\dagger(w)}, \tag{3.17}$$

$$\{\Phi_1^\dagger(w), T_2(\lambda;x,y)\} = -i(-1)^{[j]}(e_j^\dagger\otimes\mathbb{I})\frac{\delta T_2(\lambda;x,y)}{\delta\phi_j(w)}, \tag{3.18}$$

$$\{\Phi_2^\dagger(w),T_1(\lambda;x,y)\}=-i(-1)^{[j]}(\mathbb{I}\otimes e_j^\dagger)\frac{\delta T_1(\lambda;x,y)}{\delta\phi_j(w)}. \tag{3.19}$$

This shows that the PB can be rewritten as

$$\{T_1(\lambda,x,y),T_3(\mu,x,y)\}=i\int_{\mathbb{R}}dw(\{\Phi_2^\dagger(w),T_1(\lambda;x,y)\}\{\Phi_2(w),T_3(\mu;x,y)\}$$
$$-\{\Phi_2^\dagger(w),T_3(\mu;x,y)\}\{\Phi_2(w),T_1(\lambda;x,y)\}). \tag{3.20}$$

Inserting (3.11) and (3.13) in this expression, one gets

$$\{T_1(\lambda;x,y),T_2(\mu;x,y)\}=ig\int_y^x dw\, T_1(\lambda;x,w)T_2(\mu;x,w)(\pi_{12}-\pi_{21})T_1(\lambda;w,y)T_2(\mu;w,y),$$
$$\text{where}\quad \pi_{12}=\sum_{j=1}^K E_{j,K+1}\otimes E_{K+1,j}\,. \tag{3.21}$$

Finally, a direct calculation shows that

$$\frac{\partial}{\partial w}(T_1(\lambda;x,w)T_2(\mu;x,w)\Pi_{12}T_1(\mu;w,y)T_2(\lambda;w,y))$$
$$=i\,\frac{\lambda-\mu}{2}T_1(\lambda;x,w)T_2(\mu;x,w)(\pi_{12}-\pi_{21})T_1(\lambda;w,y)T_2(\mu;w,y), \tag{3.22}$$

so that we get (3.8). ■

Property 3.3: The following limits are well defined:

$$T^-(\lambda;x)=\lim_{y\to-\infty}T(\lambda;x,y)E(\lambda;y), \tag{3.23}$$

$$T^+(\lambda;y)=\lim_{x\to\infty}E(\lambda;-x)T(\lambda;x,y), \tag{3.24}$$

$$T(\lambda)=T^+(\lambda;z)T^-(\lambda;z)=\lim_{\substack{x\to\infty\\ y\to-\infty}}E(\lambda;-x)T(\lambda;x,y)E(\lambda;y). \tag{3.25}$$

$T(\lambda)$ *is called the monodromy matrix.*

Proof: Using the equality $E(\lambda;x)\Omega(z)=\Omega(z)E(\lambda;-x)$, valid for any x, z, $T^{(n)}(\lambda;x,y)$ can be conveniently rewritten as

$$T^{(n)}(\lambda;x,y)=E(\lambda;x)\int_{\mathbb{R}^n}d^nz\,\theta(x>z_1>\cdots>z_n>y)$$
$$\times E\left(\lambda;2\sum_{j=1}^n(-1)^j z_j\right)\left(\prod_{k=1}^n\Omega(z_k)\right)E(\lambda;-y), \tag{3.26}$$

which shows that the limits are well defined. ■

Property 3.4:

$$\{T_1(\lambda),T_2(\mu)\}=r_+(\lambda-\mu)T(\lambda)\otimes T(\mu)-T(\lambda)\otimes T(\mu)r_-(\lambda-\mu) \tag{3.27}$$

with

$$r_+(\lambda-\mu)=\frac{g}{\lambda-\mu}(P_{12}+E_{K+1,K+1}\otimes E_{K+1,K+1})+i\pi g\,\delta(\lambda-\mu)(\pi_{12}-\pi_{21}), \tag{3.28}$$

$$r_-(\lambda-\mu)=\frac{g}{\lambda-\mu}(P_{12}+E_{K+1,K+1}\otimes E_{K+1,K+1})-i\pi g\,\delta(\lambda-\mu)(\pi_{12}-\pi_{21}), \tag{3.29}$$

where P_{12} is the super-permutation in the space of $K\times K$ matrices.

Proof: Direct calculation, plugging (3.25) into (3.8), and using the Cauchy principal value $\lim_{\lambda\to\infty} p.v.(e^{\pm i\lambda x}/x)=\pm i\pi\delta(x)$. ■

Introducing $t(\lambda)$, the $K\times K$ submatrix of $T(\lambda)$ with the last row and column removed, and $D(\lambda)=T_{K+1,K+1}(\lambda)$, one finally computes:

$$\{t_1(\lambda),t_2(\mu)\}=\frac{g}{\lambda-\mu}[P_{12},t(\lambda)\otimes t(\mu)], \tag{3.30}$$

$$\{D(\lambda),t(\mu)\}=0, \tag{3.31}$$

$$\{D(\lambda),D(\mu)\}=0. \tag{3.32}$$

Equation (3.30) shows that $t(\lambda)$ defines a classical version of the super-Yangian $Y(gl(M|N))$. Equation (3.32) shows that $D(\lambda)$ can be taken as a generating function for a hierarchy, and (3.31) proves that the super-Yangian is a symmetry of this hierarchy. It remains to identify this hierarchy.

Lemma 3.5: Only $T^{(2n)}(\lambda)$, $n\in\mathbb{Z}_+$, contribute to the super-Yangian generators $t(\lambda)$ and to the Hamiltonian generating function $D(\lambda)$.

Expanding $t(\lambda)$ and $D(\lambda)$ *as series in λ^{-1}, one has $T^{(2n)}(\lambda)=o(\lambda^{-n})$.*

Proof: It is clear that $T^{(n)}(\lambda)$ contains the product of exactly n matrices Ω, the other matrices entering in its definition being diagonal. Due to the form of Ω, only products of an even number of such matrices will contribute to $t(\lambda)$ and $D(\lambda)$.

To show the λ dependence, we consider the integration on z_{2j} and z_{2j+1}, and perform an integration by part, assuming that the fields Φ and $\Phi^\dagger$ are vanishing at infinity:

$$\begin{aligned}&\int_{-\infty}^{z_{2j-1}}dz_{2j}\int_{-\infty}^{z_{2j}}dz_{2j+1}\,E(\lambda;2z_{2j}-2z_{2j+1})\Omega(z_{2j})\Omega(z_{2j+1})I_{j,n}(z_{2j+1},\dots,z_{2n})\\&=\frac{i}{\lambda}\Sigma\int_{-\infty}^{z_{2j-1}}dz_{2j}\bigg[\Omega(z_{2i})^2I_{j,n}(z_{2j+1},\dots,z_{2n})-\int_{-\infty}^{z_{2j}}dz_{2j+1}\,E(\lambda;2z_{2j}-2z_{2j+1})\\&\quad\times\Omega(z_{2j})\partial_{2j+1}(\Omega(z_{2j+1})I_{j,n}(z_{2j+1},z_{2j+2},\dots,z_{2n}))\bigg].\end{aligned}$$

Above, ∂_k stands for $\partial/\partial z_k$, and $I_{j,n}(z_{2j+1},z_{2j+2},\dots,z_{2n})$ denotes the other integrals (depending on z_k, $k\geqslant 2j$) which enters into the definition of $T^{(n)}(\lambda)$.

It is clear that one can do this integration for all z_{2j}, $j=1,...,n$, and any number of times, so that the lowest power of λ^{-1} is n. ■

Property 3.6: The first Hamiltonians generated by $D(\lambda)$ read

$$D^{(1)}=igN \quad \text{with } N=\int_{-\infty}^{\infty}dx\,\Phi^\dagger(x)\Phi(x), \tag{3.33}$$

$$D^{(2)}=-\frac{1}{2}g^2N^2+gP \quad \text{with } P=\int_{-\infty}^{\infty}dx\,\Phi^\dagger(x)\partial\Phi(x), \tag{3.34}$$

$$D^{(3)}=-\frac{ig^3}{6}N^3+ig^2NP+igH, \tag{3.35}$$

$$H=\int_{-\infty}^{\infty}dx\ \partial\Phi^\dagger(x)\partial\Phi(x)+g\int_{-\infty}^{\infty}dx(\Phi^\dagger(x)\Phi(x))^2. \tag{3.36}$$

This shows that $D(\lambda)$ generates the Hamiltonians of the NLSS hierarchy, so that (3.31) proves that $Y(gl(M|N))$ is a symmetry of this hierarchy.

Proof: We use the techniques given in the above proof, focusing on the $(K+1,K+1)$ matrix element. The bounds in the integrals are simplified using the property

$$(\Omega(x_1)\partial^k\Omega(x_2)\Omega(x_3)\partial^l\Omega(x_4))_{K+1,K+1}=(\Omega(x_1)\partial^k\Omega(x_2))_{K+1,K+1}(\Omega(x_3)\partial^l\Omega(x_4))_{K+1,K+1}. \tag{3.37}$$

■

B. Time evolution

Strictly speaking, we have, up to now, constructed only the linear operator $L(\lambda;x)$ introduced in the Zakharov–Shabbat scheme.[2] This operator is only the first element of the Lax pair (L,M). It is sufficient to solve the problem, but for completeness, we now introduce M, the second element of the Lax pair.

The Lax pair is a reformulation of the equations of motion as the commutativity of two differential operators:

$$\left[\frac{\partial}{\partial x}-L(\lambda;x,t),\frac{\partial}{\partial t}-M(\lambda;x,t)\right]=0, \tag{3.38}$$

which amounts to the compatibility condition of the auxiliary system

$$\begin{aligned}\partial_x u&=L(\lambda;x,t)u,\\ \partial_t u&=M(\lambda;x,t)u.\end{aligned} \tag{3.39}$$

Starting from the definitions (3.1) and (3.2), it is a straightforward calculation to show that for

$$M(\lambda;x,t)=-\frac{i\lambda^2}{2}\Sigma+ig\Omega(x,t)\Sigma\Omega(x,t)-\sqrt{g}(\Sigma\partial_x+i\lambda)\Omega(x,t) \tag{3.40}$$

the condition (3.38) is equivalent to

$$(i\Sigma\partial_t+\partial_x^2)\Omega(x,t)=2g|\Phi(x,t)|^2\Omega(x,t), \tag{3.41}$$

which just reproduces the equations of motion (2.10) and their counterpart for $\Phi^\dagger(x,t)$.

As it should be clear from the system (3.39), $M(\lambda;x,t)$ is associated to time evolution in the same way $L(\lambda;x,t)$ is associated to spacial translation. This is confirmed by the following:

Property 3.7: The time evolution of the transfer and monodromy matrices is given by

$$\partial_t T(\lambda;x,y,t)=M(\lambda;x,t)T(\lambda;x,y,t)-T(\lambda;x,y,t)M(\lambda;y,t), \tag{3.42}$$

$$\partial_t T_+(\lambda;y,t)=-\frac{i\lambda^2}{2}\Sigma T(\lambda;y,t)-T(\lambda;y,t)M(\lambda;y,t), \tag{3.43}$$

$$\partial_t T_-(\lambda;x,t)=M(\lambda;x,t)T(\lambda;x,t)+\frac{i\lambda^2}{2}T(\lambda;x,t)\Sigma, \tag{3.44}$$

$$\partial_t T(\lambda;t) = -\frac{i\lambda^2}{2}[\Sigma, T(\lambda;t)]. \tag{3.45}$$

Proof: The first equation is proven showing that

$$Z(\lambda;x,y,t) = \partial_t T(\lambda;x,y,t) - M(\lambda;x,t)T(\lambda;x,y,t) + T(\lambda;x,y,t)M(\lambda;y,t)$$

obeys the differential equations $(x > y)$:

$$\partial_x Z(\lambda;x,y) = L(\lambda;x)Z(\lambda;x,y),$$

$$\partial_y Z(\lambda;x,y) = -Z(\lambda;x,y)L(\lambda;y),$$

together with the initial condition $Z(\lambda;x,x,t) = 0$.

The other equations are proved through the limits $x \to \infty$, $y \to -\infty$ using $\lim_{|x|\to\infty} M(\lambda;x,t) = -(i\lambda^2/2)\Sigma$. ■

To conclude this section, let us remark that the time-evolution (3.45) shows that we have

$$T(\lambda;x,y,t) = e^{it\lambda^2\Sigma/2}T(\lambda;x,y,0)e^{-it\lambda^2\Sigma/2} \tag{3.46}$$

in accordance with the ZF formulation of the Hamiltonian.

C. Quantum Lax pair

Following Sklyanin,[28] we define the following.

Definition 3.8: The quantum transition matrix $\mathcal{T}(\lambda;x,y)$ is the Wick (normal)-ordered classical transition matrix $T(\lambda;x,y)$ regarded as a functional of the quantum canonical fields $\Phi(x)$, $\Phi^\dagger(x)$:

$$\mathcal{T}(\lambda;x,y) = :T(\lambda;x,y):. \tag{3.47}$$

Here and below the normal ordering is defined as

$$:\phi_j(x)\phi_k^\dagger(y): = (-1)^{[j][k]}\phi_k^\dagger(y)\phi_j(x), \quad \forall x,y,$$

and extended to monomials in ϕ, $\phi^\dagger$ in the usual way, i.e., with all the ϕ's on the right of the $\phi^\dagger$'s, keeping the original order between the ϕ's and between the $\phi^\dagger$'s.

For convenience, we also define a symbol ‡‡ which acts on operators and is not to be confused with the symbol : :. It simply guarantees the ordering of Φ, $\Phi^\dagger$ in an expression containing $L(\lambda;x)$ and other (normal-ordered) functionals of the quantum fields without changing the internal ordering of the functionals. For example, if $A = :a:$ and $B = :b:$, then

$$\ddagger A L(\lambda;x) B \ddagger = \frac{i\lambda}{2}A\Sigma B + i\sqrt{g}\sum_{j=1}^{K}((-1)^{[j][A]}\phi_j(x)AE_{K+1,j}B - (-1)^{[j][B]}AE_{j,K+1}B\phi_j^\dagger(x)).$$

The previous definition gives rise to many questions dealing with operator theory and functional analysis which were answered for the bosonic case in the very detailed review[1] by Gutkin. But for the sake of brevity, we mimic the compact, albeit more formal, approach of Sklyanin since it contains all the fundamental and physical ideas, bearing in mind that everything is well defined.

In this sense, the quantum transition matrix is the fundamental solution of the quantum auxiliary problem

$$\partial_x \mathcal{T}(\lambda;x,y) = \ddagger L(\lambda;x)\mathcal{T}(\lambda;x,y)\ddagger \quad \text{with } \mathcal{T}(\lambda;x,x) = \mathbb{1} \tag{3.48}$$

and satisfies

$$\partial_y \mathcal{T}(\lambda;x,y) = -\ddagger \mathcal{T}(\lambda;x,y) L(\lambda;y) \ddagger,$$

$$\mathcal{T}(\lambda;x,y)\mathcal{T}(\lambda;y,z) = \mathcal{T}(\lambda;x,z) \quad \text{for } x<y<z \text{ or } x>y>z,$$

where $L(\lambda;x)$ is the Lax even super-matrix defined in (3.1) and (3.2).

This system of first-order differential equations together with the given initial condition is equivalent to the following Volterra integral representations:

$$\mathcal{T}(\lambda;x,y) = \mathbb{1} + \int_y^x d\omega \ddagger L(\lambda;\omega)\mathcal{T}(\lambda;\omega,y)\ddagger, \tag{3.49}$$

$$\mathcal{T}(\lambda;x,y) = \mathbb{1} + \int_y^x d\omega \ddagger \mathcal{T}(\lambda;x,\omega) L(\lambda;\omega)\ddagger. \tag{3.50}$$

In order to reach our final goal there are several steps which all rely on one simple idea extensively used in the inverse problem literature, that is two quantities are equal if and only if they satisfy the same first-order differential equation with the same initial condition. This is what is called ‘‘the differential equation approach’’ by Gutkin in Ref. 1. He criticized this approach but showed that it gives the correct answer using the ‘‘discrete approximation approach’’ which amounts to the same line of argument but deals with finite differences on subintervals of $[x,y]$ instead of a true derivative.

The first step is to obtain the commutation relations of matrix elements of the transition matrix and we need two preliminary lemmas.

Lemma 3.9: $\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)$ satisfies the following differential system:

$$\partial_x \{\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)\} = \ddagger \mathcal{L}_{12}(\lambda,\mu;x)\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)\ddagger, \tag{3.51}$$

$$\mathcal{T}_1(\lambda;x,x)\mathcal{T}_2(\mu;x,x) = \mathcal{T}_2(\mu;x,x)\mathcal{T}_1(\lambda;x,x) = \mathbb{1}\otimes\mathbb{1}, \tag{3.52}$$

where

$$\mathcal{L}_{12}(\lambda,\mu;x) = L_1(\lambda;x) + L_2(\mu;x) + g\pi_{12}. \tag{3.53}$$

Proof: The idea is once again to use the equivalence between the differential problem and the Volterra integral representation of the solution. Indeed, taking care of the ordering of the fields when using (3.49) and (3.50), one gets

$$\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y) = \mathbb{1}\otimes\mathbb{1} + \int_y^x d\omega \ddagger \mathcal{L}_{12}(\lambda,\mu;\omega)\mathcal{T}_1(\lambda;\omega,y)\mathcal{T}_2(\mu;\omega,y)\ddagger.$$

■

Lemma 3.10: The operator $\mathcal{L}_{12}(\lambda,\mu;x)$ satisfies the following relation:

$$\mathcal{R}_{12}(\lambda-\mu)\mathcal{L}_{12}(\lambda,\mu;x) = \mathcal{L}_{21}(\mu,\lambda;x)\mathcal{R}_{12}(\lambda-\mu), \tag{3.54}$$

where $\mathcal{R}_{12}(\lambda-\mu) = \mathbb{1} - ir(\lambda-\mu)$, and $r(\lambda-\mu)$ is given by (3.4).

Proof: Direct calculation using

$$[\Pi_{12}, L_1(\lambda;x) + L_2(\mu;x)] = i(\lambda-\mu)(\pi_{12} - \pi_{21}),$$

where π_{12} has been defined in (3.21). ■

We can now formulate the basic result of this paragraph.

Theorem 3.11: *The quantum transition matrix $\mathcal{T}(\lambda;x,y)$ satisfies the following finite volume commutation relations:*

$$\mathcal{R}_{12}(\lambda-\mu)\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)=\mathcal{T}_2(\mu;x,y)\mathcal{T}_1(\lambda;x,y)\mathcal{R}_{12}(\lambda-\mu). \tag{3.55}$$

Proof: Using the fact that $\mathcal{R}_{12}(\lambda)$ is a numerical, invertible (for λ real and nonzero) matrix, Lemmas 3.9 and 3.10 imply that the quantities $\mathcal{T}_2(\mu;x,y)\mathcal{T}_1(\lambda;x,y)$ and $\mathcal{R}_{12}(\lambda-\mu)\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)\mathcal{R}_{12}^{-1}(\lambda-\mu)$ satisfy the same first-order differential equation with the same initial condition. ■

Let us remark that if we restore the Planck constant in the canonical commutation relations, then $\mathcal{R}_{12}(\lambda-\mu)=\mathbb{1}-i\hbar r(\lambda-\mu)$ and we recover the relation (3.8) for the classical transition matrix, given that as $\hbar\to 0$, $\mathcal{T}(\lambda;x,y)\to T(\lambda;x,y)$ and $[,]\to i\hbar\{,\}$ and keeping the terms of order $\hbar$.

We are now in position to define the quantum monodromy matrix as an appropriate limit of the quantum transition matrix to obtain the infinite volume commutation relations corresponding to (3.55). The crucial difference with respect to the classical case comes from the nontrivial commutation relations of the quantum fields, which produces the term proportional to g in $\mathcal{L}_{12}(\lambda,\mu;x)$.

Therefore, one cannot define the limit as in (3.25) and insert it directly in the finite volume commutation relations. Instead, we are led to compare the asymptotic behavior of $\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)$, for which we have information with that of $\mathcal{T}_1(\lambda;x,y)$, $\mathcal{T}_2(\mu;x,y)$ separately, whose commutation relations in the infinite interval limit we are looking for.

Definition 3.12: The quantum equivalents of (3.23)–(3.25) are defined by

$$\mathcal{T}^-(\lambda;x)=:T^-(\lambda;x):,\quad \mathcal{T}^+(\lambda;y)=:T^+(\lambda;y):,\quad \mathcal{T}(\lambda)=:T(\lambda):, \tag{3.56}$$

and $\mathcal{T}(\lambda)=\mathcal{T}^+(\lambda;z)\mathcal{T}^-(\lambda;z)$ *is the quantum monodromy matrix.*

$E(\lambda;x)$ being a numerical matrix, one immediately deduces

$$\partial_x\mathcal{T}^-(\lambda;x)=\ddagger L(\lambda;x)\mathcal{T}^-(\lambda;x)\ddagger, \tag{3.57}$$

$$\partial_x\mathcal{T}^+(\lambda;x)=-\ddagger\mathcal{T}^+(\lambda;x)L(\lambda;x)\ddagger. \tag{3.58}$$

As a first step, we look for information on $\mathcal{T}_1^-(\lambda;x)\mathcal{T}_2^-(\mu;x)$ from what we know of $\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)$. This is gathered in the following lemma.

Lemma 3.13:

$$\lim_{y\to-\infty}\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)\xi_{12}(\lambda,\mu;y)=\mathcal{T}_1^-(\lambda;x)\mathcal{T}_2^-(\mu;x)C_{12}(\lambda,\mu), \tag{3.59}$$

where, π_{12} *being defined as in (3.21), we have introduced*

$$\xi_{12}(\lambda,\mu;y)=\exp\left[\left(\frac{i\lambda}{2}\Sigma_1+\frac{i\mu}{2}\Sigma_2+g\pi_{12}\right)y\right], \tag{3.60}$$

$$C_{12}(\lambda,\mu)=\mathbb{1}\otimes\mathbb{1}-\frac{ig}{\lambda-\mu+i\varepsilon}\pi_{12}. \tag{3.61}$$

Proof: Let

$$\Lambda(\lambda,\mu;x)=\lim_{y\to-\infty}\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)\xi_{12}(\lambda,\mu;y), \tag{3.62}$$

$$\Lambda^-(\lambda,\mu;x)=\mathcal{T}_1^-(\lambda;x)\mathcal{T}_2^-(\mu;x). \tag{3.63}$$

Rewriting $\mathcal{L}_{12}(\lambda,\mu;x)=\mathcal{L}_0(\lambda,\mu)+\Omega_1(x)+\Omega_2(x)$ with $\mathcal{L}_0(\lambda,\mu)=(i\lambda/2)\Sigma_1+(i\mu/2)\Sigma_2+g\pi_{12}$, one easily gets from (3.51) the integral representation

$$\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)=\xi_{12}(\lambda,\mu;x-y)+\int_y^x d\omega \ddagger\mathcal{T}_1(\lambda;x,\omega)\mathcal{T}_2(\mu;x,\omega)(\Omega_1(\omega)+\Omega_2(\omega))\ddagger\xi_{12}(\lambda,\mu;\omega-y), \tag{3.64}$$

which shows that $\Lambda(\lambda,\mu;x)$ is well defined and also satisfies

$$\partial_x\Lambda(\lambda,\mu;x)=\ddagger\mathcal{L}_{12}(\lambda,\mu;x)\Lambda(\lambda,\mu;x)\ddagger.$$

Now following the same line of argument as in Lemma 3.9, we get

$$\partial_x\Lambda^-(\lambda,\mu;x)=\ddagger\mathcal{L}_{12}(\lambda,\mu;x)\Lambda^-(\lambda,\mu;x)\ddagger.$$

Consequently,

$$\Lambda(\lambda,\mu;x)=\Lambda^-(\lambda,\mu;x)C_{12}(\lambda,\mu), \quad \forall x, \tag{3.65}$$

and we can determine $C_{12}(\lambda,\mu)$ from the asymptotic behavior as $x\to-\infty$. From the physical requirement that

$$\lim_{x\to\pm\infty}|\Phi(x)|=0$$

and Eq. (3.64), we see that

$$\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)\underset{\substack{y\to-\infty\\ x\to y}}{\sim}\xi_{12}(\lambda,\mu;x-y),$$

implying

$$\Lambda(\lambda,\mu;x)\underset{x\to-\infty}{\sim}\xi_{12}(\lambda,\mu;x). \tag{3.66}$$

On the other hand, from (3.57), $\Lambda^-(\lambda,\mu;x)$ can be represented as

$$\Lambda^-(\lambda,\mu;x)=E_1(\lambda;x)E_2(\mu;x)+\int_{-\infty}^x d\omega \ddagger\mathcal{T}_1(\lambda;x,\omega)\mathcal{T}_2(\mu;x,\omega)\times(\Omega_1(\omega)+\Omega_2(\omega)+g\pi_{12})\ddagger E_1(\lambda;\omega)E_2(\mu;\omega),$$

so that

$$\Lambda^-(\lambda,\mu;x)\underset{x\to-\infty}{\sim}E_1(\lambda;x)E_2(\mu;x)+I(\lambda,\mu;x),$$

where

$$I(\lambda,\mu;x)=g\int_{-\infty}^x d\omega\, \xi_{12}(\lambda,\mu;x-\omega)\pi_{12}E_1(\lambda;\omega)E_2(\mu;\omega)$$

can be evaluated from the knowledge of

$$\xi_{12}(\lambda,\mu;x)=E_1(\lambda;x)E_2(\mu;x)+2g\frac{\sin([(\lambda-\mu)/2]x)}{\lambda-\mu}\pi_{12}$$

and an $i\varepsilon$ prescription to get

$$I(\lambda,\mu;x)=\frac{ig}{\lambda-\mu+i\varepsilon}e^{-i[(\lambda-\mu)/2]x}\pi_{12}.$$

Now, adopting the regularization

$$2g\frac{\sin([(\lambda-\mu)/2]x)}{\lambda-\mu}=\frac{-ig}{\lambda-\mu+i\varepsilon}[e^{i[(\lambda-\mu)/2]x}-e^{-i[(\lambda-\mu)/2]x}],$$

we see that (3.65) holds for $C_{12}(\lambda,\mu)$ given in (3.61). ■

Theorem 3.14: *The commutation relations for the quantum matrices $\mathcal{T}^{\pm}(\lambda;x)$ and $\mathcal{T}(\lambda)$ for real λ and μ take the following form:*

$$\mathcal{R}_{12}(\lambda-\mu)\mathcal{T}_1^-(\lambda;x)\mathcal{T}_2^-(\mu;x)C_{12}(\lambda,\mu)=\mathcal{T}_2^-(\mu;x)\mathcal{T}_1^-(\lambda;x)C_{21}(\mu,\lambda)\mathcal{R}_{12}(\lambda-\mu),$$

$$\mathcal{R}_{12}(\lambda-\mu)C_{12}(\mu,\lambda)\mathcal{T}_1^+(\lambda;x)\mathcal{T}_2^+(\mu;x)=C_{21}(\lambda,\mu)\mathcal{T}_2^+(\mu;x)\mathcal{T}_1^+(\lambda;x)\mathcal{R}_{12}(\lambda-\mu), \tag{3.67}$$

$$\mathcal{R}_{12}^+(\lambda-\mu)\mathcal{T}_1(\lambda)\mathcal{T}_2(\mu)=\mathcal{T}_1(\mu)\mathcal{T}_2(\lambda)\mathcal{R}_{12}^-(\lambda-\mu),$$

where, defining $\mathbb{1}_K=\Sigma_{i=1}^K E_{ii}$,

$$\mathcal{R}_{12}^{\pm}(\lambda-\mu)=\frac{-ig}{(\lambda-\mu)}\mathbb{1}_K\otimes\mathbb{1}_K+P_{12}+\pi_{21}+\frac{(\lambda-\mu)^2+g^2}{(\lambda-\mu+i\varepsilon)^2}\pi_{12}+\frac{\lambda-\mu-ig}{\lambda-\mu}E_{K+1,K+1}\otimes E_{K+1,K+1}$$
$$\pm\pi g\,\delta(\lambda-\mu)(\mathbb{1}_K\otimes E_{K+1,K+1}-E_{K+1,K+1}\otimes\mathbb{1}_K).$$

Proof: We start with the proof of the first equality. Lemma 3.13 gives

$$\lim_{y\to-\infty}\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)\xi_{12}(\lambda,\mu;y)=\mathcal{T}_1^-(\lambda;x)\mathcal{T}_2^-(\mu;x)C_{12}(\lambda,\mu),$$

which in turn yields

$$\lim_{y\to-\infty}\mathcal{T}_2(\mu;x,y)\mathcal{T}_1(\lambda;x,y)\xi_{21}(\mu,\lambda;y)=\mathcal{T}_2^-(\mu;x)\mathcal{T}_1^-(\lambda;x)C_{21}(\mu,\lambda).$$

Multiplying (3.55) on the right by $\xi_{12}(\lambda,\mu;y)$ and using the property

$$\mathcal{R}_{12}(\lambda-\mu)\xi_{12}(\lambda,\mu;y)=\xi_{21}(\mu,\lambda;y)\mathcal{R}_{12}(\lambda-\mu)$$

we get

$$\mathcal{R}_{12}(\lambda-\mu)\mathcal{T}_1(\lambda;x,y)\mathcal{T}_2(\mu;x,y)\xi_{12}(\lambda,\mu;y)=\mathcal{T}_2(\mu;x,y)\mathcal{T}_1(\lambda;x,y)\xi_{21}(\mu,\lambda;y)\mathcal{R}_{12}(\lambda-\mu),$$

which gives the first equality in the limit $y\to-\infty$. The second equality is proved along the same line of argument. Now, combining the two equations and using the properties

$$\mathcal{T}(\lambda)=\mathcal{T}^+(\lambda;x)\mathcal{T}^-(\lambda;x)\quad\text{and}\quad\mathcal{T}_2^+(\mu;x)\mathcal{T}_1^-(\lambda;x)=\mathcal{T}_1^-(\lambda;x)\mathcal{T}_2^+(\mu;x),$$

we get

$$\mathcal{R}_{12}(\lambda-\mu)C_{12}(\mu,\lambda)\mathcal{T}_1(\lambda)\mathcal{T}_2(\mu)C_{12}(\lambda,\mu)=C_{21}(\lambda,\mu)\mathcal{T}_2(\mu)\mathcal{T}_1(\lambda)C_{21}(\mu,\lambda)\mathcal{R}_{12}(\lambda-\mu),$$

which take the form (3.67) if we define

$$\mathcal{R}_{12}^+(\lambda-\mu)=C_{12}^{-1}(\lambda,\mu)\Pi_{12}\mathcal{R}_{12}(\lambda-\mu)C_{12}(\mu,\lambda), \tag{3.68}$$

$$\mathcal{R}^{-}_{12}(\lambda-\mu)=C_{12}(\mu,\lambda)\Pi_{12}\mathcal{R}_{12}(\lambda-\mu)C_{12}^{-1}(\lambda,\mu), \tag{3.69}$$

whose explicit calculation we leave to the reader. ■

Let us extract the information contained in (3.67). We start by particularizing some entries of the quantum monodromy matrix $(i,j=1,...,K)$:

$$t_{ij}(\lambda)=(\mathcal{T}(\lambda))_{ij}, \tag{3.70}$$

$$b_j(\lambda)=(\mathcal{T}(\lambda))_{j,K+1}, \tag{3.71}$$

$$D(\lambda)=(\mathcal{T}(\lambda))_{K+1,K+1}. \tag{3.72}$$

Theorem 3.15: *The exchange relations of the entries of the quantum monodromy matrix read as follows:*

$$[\![t_{ij}(\lambda),t_{kl}(\mu)]\!]=ig(-1)^{[j][k]+[i]([j]+[k])}\frac{t_{kj}(\lambda)t_{il}(\mu)-t_{kj}(\mu)t_{il}(\lambda)}{\lambda-\mu}, \tag{3.73}$$

$$t_{ij}(\lambda)D(\mu)=D(\mu)t_{ij}(\lambda), \tag{3.74}$$

$$D(\lambda)D(\mu)=D(\mu)D(\lambda), \tag{3.75}$$

$$b_j(\lambda)b_k(\mu)=\frac{\mu-\lambda}{\mu-\lambda-ig}(-1)^{jk}b_k(\mu)b_j(\lambda)-\frac{ig}{\mu-\lambda-ig}b_j(\mu)b_k(\lambda), \tag{3.76}$$

$$b_j(\lambda)D(\mu)=\frac{\lambda-\mu-ig}{\lambda-\mu-i\varepsilon}D(\mu)b_j(\lambda). \tag{3.77}$$

Proof: By direct calculation. ■

Relations (3.73)–(3.75) are the quantum counterparts of Eqs. (3.30)–(3.32) and the same interpretation holds but for the quantum hierarchy here. As such, the super-Yangian $Y(gl(M|N))$ is a quantum symmetry of the hierarchy generated by $D(\lambda)$, which is just the quantum analog of Property 3.6 as can be seen from

$$D(\lambda)=1+\frac{ig}{\lambda}N+\frac{g}{\lambda^2}\left(P-\frac{g}{2}N(N-1)\right)+\frac{ig}{\lambda^3}\left(H+g(N-1)P-\frac{g^2}{6}N(N-1)(N-2)\right)+O\left(\frac{1}{\lambda^4}\right).$$

D. ZF algebra from Lax pair

The two relations (3.76) and (3.77) will allow us to recover the ZF algebra. Indeed, all the quantities of Theorem 3.15 are functionals of Φ, $\Phi^\dagger$, themselves involving the ZF generators [cf. (2.35)], and one can get the ZF algebra out of them as follows.

Property 3.16: Defining $a_j(\lambda)=(1/\sqrt{\pi g})b_j(\lambda)D(\lambda)^{-1}$, *Eqs. (3.76) and (3.77) give*

$$a_j(\lambda)a_k(\mu)=\frac{\mu-\lambda}{\mu-\lambda+ig}(-1)^{jk}a_k(\mu)a_j(\lambda)-\frac{ig}{\mu-\lambda+ig}a_j(\mu)a_k(\lambda). \tag{3.78}$$

Proof: Direct calculation from Theorem 3.15. ■

To complete our algebra, we need the exchange relations between $a_j(\lambda)$ and $a_k^\dagger(\mu)$. Contrary to the original one (bosonic) component case, this is not directly obtained from what we already have since there is no simple conjugate relationship for the entries of the monodromy matrix. We are naturally led to introduce a conjugate Lax super-matrix defined by

$$\overline{L}(\lambda;x) = -\frac{i\lambda}{2}\Sigma - i\sqrt{g}\phi_j^\dagger(x)E_{K+1,j} + i\sqrt{g}\phi_j(x)E_{j,K+1} \tag{3.79}$$

and the associated transition matrix

$$\partial_x \overline{\mathcal{T}}(\lambda;x,y) = \ddagger\overline{\mathcal{T}}(\lambda;x,y)\overline{L}(\lambda;x)\ddagger. \tag{3.80}$$

Now, to obtain information between the entries of $\mathcal{T}(\lambda;x,y)$ and $\overline{\mathcal{T}}(\mu;x,y)$ following the same steps as in Lemmas 3.9 and 3.10 and Theorem 3.11, one sees that we actually need to work with the super-transposed Lax matrix. The corresponding operation on an even super-matrix $A = \Sigma_{i,j=1}^{K+1} A_{ij}E_{ij}$ reads

$$A^t = \sum_{i,j=1}^{K+1} A_{ij}E_{ij}^t = \sum_{i,j=1}^{K+1} (-1)^{[i]([i]+[j])}A_{ji}E_{ij}. \tag{3.81}$$

It satisfies $(A^t)^t = A$ and $(AB)^t = B^tA^t$ for any even super-matrices A and B. We get

$$L^t(\lambda;x) = \frac{i\lambda}{2}\Sigma + i\sqrt{g}(-1)^{[j]}\phi_j(x)E_{K+1,j} - i\sqrt{g}\phi_j^\dagger(x)E_{j,K+1} \tag{3.82}$$

and the associated transition matrix

$$\partial_x \mathcal{T}^t(\lambda;x,y) = \ddagger\mathcal{T}^t(\lambda;x,y)L^t(\lambda;x)\ddagger. \tag{3.83}$$

Therefore, instead of (3.51) we get

$$\partial_x\{\overline{\mathcal{T}}_1(\lambda;x,y)\mathcal{T}_2^t(\mu;x,y)\} = \ddagger\overline{\mathcal{T}}_1(\lambda;x,y)\mathcal{T}_2^t(\mu;x,y)\Gamma_{12}(\lambda,\mu;x)\ddagger, \tag{3.84}$$

$$\partial_x\{\mathcal{T}_1^t(\mu;x,y)\overline{\mathcal{T}}_2(\lambda;x,y)\} = \ddagger\mathcal{T}_1^t(\mu;x,y)\overline{\mathcal{T}}_2(\lambda;x,y)\Gamma'_{12}(\lambda,\mu;x)\ddagger, \tag{3.85}$$

with

$$\Gamma_{12}(\lambda,\mu;x) = \overline{L}_1(\lambda;x) + L_2^t(\mu;x) + g\pi_{12}^{t_2},$$

$$\Gamma'_{12}(\lambda,\mu;x) = L_1^t(\mu;x) + \overline{L}_2(\lambda;x) + g\pi_{12}^{t_1}.$$

Now the key point is to find an invertible numerical matrix $\mathcal{R}'_{12}(\lambda)$ solution of the new Yang–Baxter equation

$$\mathcal{R}'_{12}(\lambda,\mu)\Gamma_{12}(\lambda,\mu;x) = \Gamma_{21}(\lambda,\mu;x)\mathcal{R}'_{12}(\lambda,\mu).$$

It is given by

$$\mathcal{R}'_{12}(\lambda,\mu) = \frac{ig}{\lambda-\mu}\Pi_{12}^{t_1} + \frac{\lambda-\mu-ig(M-N)}{\lambda-\mu}\Pi_{12}. \tag{3.86}$$

Following the same procedure as above, we finally deduce the infinite volume commutation relations under the form

$$\mathcal{R}'^{+}_{12}(\lambda-\mu)\overline{\mathcal{T}}_1(\lambda)\mathcal{T}_2^t(\mu) = \mathcal{T}_1^t(\mu)\overline{\mathcal{T}}_2(\lambda)\mathcal{R}'^{-}_{12}(\lambda-\mu) \tag{3.87}$$

with

$$\mathcal{R}'^{\pm}_{12}(\lambda-\mu)=\frac{ig}{\lambda-\mu}P_{12}^{t_1}+\frac{\lambda-\mu-ig(M-N)}{\lambda-\mu}(P_{12}+\pi_{12}+\pi_{21})$$
$$+\frac{(\lambda-\mu-ig)(\lambda-\mu-ig(M-N))}{(\lambda-\mu+i\varepsilon)^2}E_{K+1,K+1}\otimes E_{K+1,K+1}$$
$$\mp\pi g\delta(\lambda-\mu)(\pi_{21}^{t_1}-\pi_{21}^{t_2}).$$

All these results are the generalization to the graded case of Ref. 19 (K, the total number of bosonic or fermionic particles is replaced in our case by $M-N$, the difference of bosonic and fermionic particles). Accordingly, we get the same conclusions collected in the following proposition.

Property 3.17: Let $a_i^\dagger(\lambda)=(1/\sqrt{\pi g})(D^{-1})^\dagger(\lambda)b_j^\dagger(\lambda)$. *Then*

$$a_i(\lambda)a_j^\dagger(\mu)=\frac{\lambda-\mu}{\lambda-\mu+ig}(-1)^{[i][j]}a_j^\dagger(\mu)a_i(\lambda)-\delta_{ij}\frac{ig}{\lambda-\mu+ig}\sum_{\ell=1}^{K}a_\ell^\dagger(\mu)a_\ell(\lambda)+\delta_{ij}\delta(\lambda-\mu), \tag{3.88}$$

$$a_i^\dagger(\lambda)a_j^\dagger(\mu)=\frac{\mu-\lambda}{\mu-\lambda+ig}(-1)^{[i][j]}a_j^\dagger(\mu)a_i^\dagger(\lambda)-\frac{ig}{\mu-\lambda+ig}a_i^\dagger(\mu)a_j^\dagger(\lambda). \tag{3.89}$$

Proof: Noting that

$$b_j(\lambda)=\mathcal{T}^t(\lambda)_{K+1,j},\quad D(\lambda)=\mathcal{T}^t(\lambda)_{K+1,K+1},$$
$$b_j^\dagger(\lambda)=\overline{\mathcal{T}}(\lambda)_{K+1,j},\quad D^\dagger(\lambda)=\overline{\mathcal{T}}(\lambda)_{K+1,K+1},$$

(3.87) gives

$$D^\dagger(\lambda)D(\mu)=D(\mu)D^\dagger(\lambda),$$

$$D(\mu)b_i^\dagger(\lambda)=\frac{\lambda-\mu-ig}{\lambda-\mu+i\varepsilon}b_i^\dagger(\lambda)D(\mu),\quad b_j(\mu)D^\dagger(\lambda)=\frac{\lambda-\mu-ig}{\lambda-\mu+i\varepsilon}D^\dagger(\lambda)b_j(\mu),$$

$$b_i(\lambda)b_j^\dagger(\mu)=\frac{\mu-\lambda-ig}{\mu-\lambda+i\varepsilon}(-1)^{[i][j]}b_j^\dagger(\mu)b_i(\lambda)+\delta_{ij}\frac{ig(\mu-\lambda-ig)}{(\mu-\lambda+i\varepsilon)^2}\sum_{\ell=1}^{K}b_\ell^\dagger(\mu)b_\ell(\lambda)$$
$$+\delta_{ij}\pi g\delta(\lambda-\mu)D^\dagger(\mu)D(\lambda),$$

which in turn yields (3.88). The proof of (3.89) is similar. ■

IV. EXPLICIT CONSTRUCTION OF THE SUPER-YANGIAN GENERATORS

A. Super-Yangian generators in terms of canonical fields

We consider the classical case. The quantum case can be done in a similar way, with correction terms due to the noncommutativity of the fields Φ, $\Phi^\dagger$.

For any $K\times K$-matrix $\sigma\in gl(M|N)$, we introduce

$$Q_\sigma^{(0)}=\int dx\,\Phi^\dagger(x)\sigma\Phi(x)=\int dx\sum_{j,k=1}^{K}\phi_j^\dagger(x)\sigma^{jk}\phi_k(x), \tag{4.1}$$

$$Q_\sigma^{(1)}=\int dx\,\Phi^\dagger(x)\sigma\partial\Phi(x)-\frac{g}{2}\int dxdy\,sg(x-y)\Phi^\dagger(x)\sigma\Phi(y)\cdot\Phi^\dagger(y)\Phi(x), \tag{4.2}$$

$$Q_\sigma^{(2)}=\int dx\,\Phi^\dagger(x)\sigma\partial^2\Phi(x)-\frac{g}{2}\int dxdy\,sg(x-y)(\Phi^\dagger(x)\sigma\partial\Phi(y)-\partial\Phi^\dagger(x)\sigma\Phi(y))\Phi^\dagger(y)\Phi(x)$$
$$+\frac{g^2}{4}\int dxdydz\,sg(x-y)sg(y-z)\Phi^\dagger(y)\Phi(x)\cdot\Phi^\dagger(x)\sigma\Phi(z)\cdot\Phi^\dagger(z)\Phi(y). \tag{4.3}$$

The coefficients in (4.2) and (4.3) are fixed in such a way that

$$\{H,Q_\sigma^{(n)}\}=0,\quad n=0,1,2, \tag{4.4}$$

so that $Q_\sigma^{(n)}$ are indeed symmetry generators of the NLSS equation. With these definitions, it is a simple calculation to show

$$\{Q_\sigma^{(0)},Q_\omega^{(n)}\}=iQ^{(n)}_{[\![\sigma,\omega]\!]},\quad n=0,1,2, \tag{4.5}$$

$$\begin{aligned}\{Q_\sigma^{(1)},Q_\omega^{(1)}\}&=iQ^{(2)}_{[\![\sigma,\omega]\!]}-i\left(-\frac{g}{2}\right)^2\int dxdydt\,S(x,y,t)(\Phi^\dagger(x)\sigma\Phi(y)\cdot\Phi^\dagger(y)\omega\Phi(t)\\&\quad-\Phi^\dagger(x)\omega\Phi(y)\cdot\Phi^\dagger(y)\sigma\Phi(t))\Phi^\dagger(t)\Phi(x),\\ S(x,y,t)&=sg(t-x)sg(x-y)+sg(x-y)sg(y-t)+sg(y-t)sg(t-x).\end{aligned} \tag{4.6}$$

Equation (4.5) shows that $Q_\sigma^{(0)}$, $\sigma\in gl(M|N)$, generates a $gl(M|N)$ superalgebra, and that $Q_\sigma^{(n)}$ (n fixed) form a representation of it. The second term in (4.6) reflects the nonlinear commutation relation of the super-Yangian.

Note that we have

$$Q_{\mathbb{I}}^{(0)}=N\quad\text{and}\quad Q_{\mathbb{I}}^{(1)}=P, \tag{4.7}$$

so that Eq. (4.5) shows that $Q_\sigma^{(n)}$ commutes with N and P. Moreover, we have the supersymmetrylike relations:

$$\begin{matrix}\{Q_\sigma^{(0)},Q_\sigma^{(0)}\}=2iN,\\ \{Q_\sigma^{(0)},Q_\sigma^{(1)}\}=2iP,\end{matrix}\quad\text{as soon as}\quad\sigma^2=\mathbb{I}\quad\text{and}\quad[\sigma]=1. \tag{4.8}$$

However, let us remark that $Q_{\mathbb{I}}^{(2)}$ is not the NLSS Hamiltonian:

$$Q_{\mathbb{I}}^{(2)}=H+\frac{g^2}{4}\int dxdydz\,sg(x-y)sg(y-z)\Phi^\dagger(y)\Phi(x)\cdot\Phi^\dagger(x)\sigma\Phi(z)\cdot\Phi^\dagger(z)\Phi(y).$$

$Q_{\mathbb{I}}^{(2)}$ corresponds to a central generator which, if it were the Hamiltonian, would lead to nonlocal equation of motion for Φ. On the contrary, H commutes with the generators $Q_\sigma^{(n)}$ and provides local equation of motion.

B. Super-Yangian generators in terms of ZF generators

We have obtained the ZF-algebra (2.15) and (2.17) from the commutation relations of the quantum monodromy matrix. This shows the central importance of this algebra and one is naturally led to take it as a starting point. This is the very idea developed in Ref. 29 and we use it to construct a realization of the generators of the super-Yangian symmetry in terms of the ZF oscillators.

First of all, we need to generalize all the basic results of Ref. 29 to our graded formalism. It is actually readily obtained since the fundamental idea of the properties given in Refs. 29 and 4 is the possibility of relabelling the auxiliary spaces which holds for our global formalism as the reader can check. Thus, we are in position to apply any result from Ref. 29 in our context. Here is our strategy: we start from the ZF algebra (corresponding to the algebra $\mathcal{A}_R$ in Ref. 29), introduce the associated well-bred vertex operator $T(\lambda)$ and use the explicit expression of our R-matrix to derive the first two terms of the expansion of $T(\lambda)$ in power series of λ^{-1}. Then we show that this approach actually coincides with the previous Lax pair formulation so that we have a realization of the generators of the super-Yangian symmetry for the hierarchy associated to the nonlinear super-Schrödinger equation in terms of the ZF oscillators. This completes and confirms the deep relationships between the quantum canonical field description (cf. Sec. IV A) and the ZF algebra approach.

Definition 4.1: The vertex operators $T_{ij}(\lambda)$ $(i,j=1,...,K)$ *associated to the ZF algebra* $\mathcal{A}_R$ *are defined by* $T(\lambda)=T^{ij}(\lambda)E_{ij}\in\mathcal{A}_R\otimes\mathbb{C}^{K^2}$ *with*

$$T_\infty(\lambda)=\mathbb{1}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}a^\dagger_{n\ldots 1}T^{(n)}_{\infty 1\cdots n}a_{1\cdots n}, \tag{4.9}$$

where

$$a^\dagger_{n\ldots 1}=(a_{1\cdots n})^\dagger=a^\dagger_n(k_n)\cdots a^\dagger_1(k_1),$$

$$T^{(n)}_{\infty 1\cdots n}=T^{(n)}_{\infty 1\cdots n}(\lambda,k_1,\ldots,k_n)\in(\mathbb{C}^{\otimes K^2})^{\otimes(n+1)}(\lambda,k_1,\ldots,k_n),$$

and integration is implied over the spectral parameters $k_1,\ldots,k_n$ *(the summation over the auxiliary spaces being understood as in the Appendix).*

$T_\infty(\lambda)$ *is said to be well-bred (on* $\mathcal{A}_R$*) if*

$$T_\infty(\lambda)a_1(\mu)=R_{1\infty}(\mu-\lambda)a_1(\mu)T_\infty(\lambda)\quad\text{and}\quad T_\infty(\lambda)a^\dagger_1(\mu)=a^\dagger_1(\mu)R_{\infty 1}(\lambda-\mu)T_\infty(\lambda) \tag{4.10}$$

with R given by (2.18).

Then, from Ref. 29 we can directly assert the following.

Property 4.2: The well-bred vertex operators $T_\infty(\lambda)$ *obey Faddeev–Reshetikhin–Takhtajan (FRT) relations*

$$R_{\infty\infty'}(\lambda-\mu)T_\infty(\lambda)T_{\infty'}(\mu)=T_{\infty'}(\mu)T_\infty(\lambda)R_{\infty\infty'}(\lambda-\mu), \tag{4.11}$$

so that they generate the super-Yangian algebra $Y(gl(M|N))$*. In addition, they form a symmetry super-algebra for the hierarchy* $H^{(n)}$ *defined by*

$$H^{(n)}=\int_{-\infty}^{\infty}dk\,k^n a^\dagger(k)a(k),\quad n\in\mathbb{Z}_+, \tag{4.12}$$

forming an Abelian algebra of Hermitian operators and governing the flows of the scattering operators a, $a^\dagger$ *as follows:*

$$e^{iH^{(n)}t}a(k)e^{-iH^{(n)}t}=e^{-ik^nt}a(k),$$

$$e^{iH^{(n)}t}a^\dagger(k)e^{-iH^{(n)}t}=e^{ik^nt}a^\dagger(k).$$

Now, recalling the results obtained in Sec. II C 4, Property 3.6 and Eqs. (3.73)–(3.75), we see that both descriptions of our integrable system (in terms of canonical fields or ZF scattering operators) are equivalent. But in this operation, we have gained an explicit realization of the super-Yangian generators.

To do this, we use the inductive relations obtained in Theorem 3.3 of Ref. 29 order by order in the spectral parameter λ. Let us rewrite

$$T_\infty(\lambda)=\mathbb{1}+\frac{ig}{\lambda}\sum_{p=0}^{\infty} T_\infty^{\{p\}}\lambda^{-p}, \tag{4.13}$$

where, accordingly,

$$T_\infty^{\{p\}}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}a_{n\cdots 1}^{\dagger}T_{\infty 1\cdots n}^{(n)\{p\}}a_{1\cdots n}$$

for some $T_{\infty 1\cdots n}^{(n)\{p\}}\in(\mathbb{C}^{\otimes K^2})^{\otimes(n)}(k_1,\ldots,k_n)$.

Our goal is to determine $T_\infty^{\{0\}}$ and $T_\infty^{\{1\}}$, that is the first two ''levels'' of the super-Yangian generators. To do this we note that the inductive relations of Theorem 3.3 in Ref. 29 at first order in λ take the form

$$T_{\infty 0\cdots n}^{(n+1)}=T_{\infty 1\cdots n}^{(n)}-T_{\infty 0\cdots n-1}^{(n)}+O(\lambda^{-2}), \tag{4.14}$$

which, under the knowledge of

$$T_{\infty 0}^{(1)\{0\}}=\mathbb{1}+P_{\infty 0},$$

yields

$$T_{\infty 0\cdots n}^{(n+1)\{0\}}=(-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}P_{\infty k},$$

where P_{ij} is the super-permutation of auxiliary spaces i and j, so that

$$T_\infty^{\{0\}}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}a_{n\cdots 0}^{\dagger}P_{\infty k}a_{0\cdots n}. \tag{4.15}$$

Now that we have the explicit form of $T_\infty^{\{0\}}$ we can use it to evaluate the commutator $[T_{\infty'}^{\{1\}},T_\infty^{\{0\}}]$ directly and compare the result to that obtained from the FRT relations (4.11) at order λ^{-2}. The latter calculation yields

$$[T_{\infty'}^{\{1\}},T_\infty^{\{0\}}]=[P_{\infty'\infty},T_\infty^{\{1\}}]. \tag{4.16}$$

As for the former, the well-bred relations (4.10) at order λ^{-2} read

$$[T_\infty^{\{0\}},a_0(\mu)]=(\mathbb{1}+P_{0\infty})a_0(\mu),$$

$$[T_\infty^{\{1\}},a_0(\mu)]=\mu(\mathbb{1}+P_{0\infty})a_0(\mu)+ig(\mathbb{1}+P_{0\infty})a_0(\mu)(\mathbb{1}+T_\infty^{\{0\}}),$$

$$[T_\infty^{\{0\}},a_0^{\dagger}(\mu)]=-a_0^{\dagger}(\mu)(\mathbb{1}+P_{\infty 0}),$$

$$[T_\infty^{\{1\}},a_0^{\dagger}(\mu)]=-\mu a_0^{\dagger}(\mu)(\mathbb{1}+P_{\infty 0})+iga_0^{\dagger}(\mu)(\mathbb{1}+P_{\infty 0})(\mathbb{1}-T_\infty^{\{0\}}),$$

which will be useful in calculating

$$[T_{\infty'}^{\{1\}},T_\infty^{\{0\}}]=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}[T_{\infty'}^{\{1\}},a_{n\cdots 0}^{\dagger}P_{\infty k}a_{0\cdots n}].$$

Note that this procedure can be iterated to evaluate $T_{\infty'}^{\{n\}}$ for an arbitrary n through $[T_{\infty'}^{\{n\}},T_\infty^{\{0\}}]$. Now,

$$
\begin{aligned}
[T^{\{1\}}_{\infty'},a^{\dagger}_{n\cdots 0}P_{\infty k}a_{0\cdots n}]&=\sum_{i=0}^{n} a^{\dagger}_{n\cdots 0}P_{\infty k}a_0\cdots[T^{\{1\}}_{\infty'},a_i]\cdots a_n+\sum_{i=0}^{n} a^{\dagger}_{n}\cdots[T^{\{1\}}_{\infty'},a^{\dagger}_i]\cdots a^{\dagger}_0P_{\infty k}a_{0\cdots n}\\
&=[P_{\infty\infty'},(\mu_k-n-1)a^{\dagger}_{n\cdots 0}P_{\infty k}a_{0\cdots n}]+a^{\dagger}_{n\cdots 0}[P_{\infty\infty'},P_{\infty k}]T^{\{0\}}_{\infty'}a_{0\cdots n}\\
&\quad+\sum_{i=0}^{k-1} a^{\dagger}_{n\cdots 0}[P_{\infty k},P_{\infty\infty'}]P_{\infty' i}a_{0\cdots n}+\sum_{i=k+1}^{n} a^{\dagger}_{n\cdots 0}P_{\infty' i}[P_{\infty k},P_{\infty\infty'}]a_{0\cdots n}.
\end{aligned}
$$

This expression can be considerably simplified in $[T^{\{1\}}_{\infty'},T^{\{0\}}_{\infty}]$ using the properties of the binomial coefficients to combine the last three terms. Inserting (4.15) and using the property

$$
\sum_{n=k}^{i-1}\binom{N}{n}\alpha_k^n\alpha_{i-n}^{N-n}=\alpha_k^N-\alpha_i^N,\quad\text{where}\quad \alpha_k^n=(-1)^{k-1}\binom{n-1}{k-1},
$$

proved in Ref. 4, we get (after a convenient relabeling of the auxiliary spaces)

$$
[T^{\{1\}}_{\infty'},T^{\{0\}}_{\infty}]=\left[P_{\infty\infty'},\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{k=1}^{n}\alpha_k^n a^{\dagger}_{1\cdots n}\left\{(\mu_k-ign)P_{\infty k}-ig\sum_{i=1}^{k-1}P_{\infty i}P_{\infty k}\right\}a_{n\cdots 1}\right].
$$

Comparing this last expression with (4.16), we get the explicit form for $T^{\{1\}}_{\infty}$ (up to a term proportional to $\mathbb{I}_{\infty}$).

To conclude, we can recast this expression as

$$
T^{\{1\}}_{\infty}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{k=1}^{n}\alpha_k^n a^{\dagger}_{1\cdots n}\left(\mu_kP_{\infty k}-ig\sum_{i=1}^{k-1}P_{\infty k}P_{\infty i}\right)a_{n\cdots 1}+igT^{\{0\}}_{\infty}T^{\{0\}}_{\infty}. \tag{4.17}
$$

In the case of $gl(N)$, we recover the results of Ref. 4, although in a different basis:

$$
T^{\{0\}}_{ij}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}\sum_{k=0}^{n}\alpha_k^n a^{\dagger}_{n\cdots 0}E^{(k)}_{ji}a_{0\cdots n},
$$

$$
T^{\{1\}}_{ij}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{k=1}^{n}\alpha_k^n a^{\dagger}_{1\cdots n}\left(\mu_kE^{(k)}_{ji}-ig\sum_{\ell=1}^{k-1}\sum_{m=1}^{N}E^{(\ell)}_{jm}E^{(k)}_{mi}\right)a_{n\cdots 1}+ig(T^{\{0\}})^2_{ji},
$$

where $E^{(\ell)}_{ij}$ denotes the E_{ij} matrix in the ℓth auxiliary space.

For $gl(M|N)$, similar formulas may also be obtained, taking care of the $\mathbb{Z}_2$-graded tensor products.

V. CONCLUSION

We solved a vectorial version of the nonlinear Schrödinger equation which contains fermions and bosons at the same time. We first introduced it classically using a $\mathbb{Z}_2$-graded formalism. At the quantum level, special attention was paid to the resolution using a super ZF algebra associated to the R-matrix of the super-Yangian $Y(gl(M|N))$. The integrability and symmetry of our system was studied through a Lax pair formalism and it is worth stressing the deep interplay between canonical and (ZF) algebraic formalisms. The ZF algebra allowed us to compute the correlation functions. Further investigations can be performed in this direction to study super-versions of known integrable systems. One can also study these super-versions when a boundary is introduced, using generalizations of the ZF algebra (boundary algebras).

APPENDIX: AUXILIARY SPACES

1. Graded spaces

We define in the auxiliary space, a K-column vector e_j with 1 at row j and 0 elsewhere, its transpose, the row vector $e_i^\dagger=(0,...,1,...,0)$ and the matrices E_{ij}, with 1 at position (i,j).

Here and below, the vectors e_i, $e_i^\dagger$, and the matrices E_{ij} will be $\mathbb{Z}_2$-graded:

$$[e_i]=[e_i^\dagger]=[i]; \quad [E_{ij}]=[i]+[j] \quad \text{with } [i]=\begin{cases} 0 & \text{for } i=1,...,M, \\ 1 & \text{for } i=M+1,...,N. \end{cases}$$

Accordingly, the tensor product of auxiliary spaces will be also $\mathbb{Z}_2$-graded, e.g.,

$$(\mathbb{I}\otimes e_i)(E_{jk}\otimes\mathbb{I})=(-1)^{[i]([j]+[k])}E_{jk}\otimes e_i .$$

We will consider even objects in the following sense: $v=v_ie_i$ and $U=U_{ij}E_{ij}$ (summation on repeated indices is understood) are even iff $[v_i]=[i]$ and $[U_{ij}]=[i]+[j]$. For example, the field $\Phi(x)$ is even.

Note that, when dealing with the tensor product of auxiliary spaces, one has to be careful not to confuse (even) objects like $\boldsymbol{\lambda}_1=\boldsymbol{\lambda}\otimes\mathbb{I}=\Sigma_{i=1}^K\lambda_ie_i\otimes\mathbb{I}$ with their ($\mathbb{Z}_2$-graded) components λ_i, $i=1,...,K$. As a (tentative) clarifying notation, we will use boldface letters for the even objects, and ordinary letters for their components.

Finally, in order to apply our formalism to derive the classical NLSS equation, we will use the global Kronecker symbol,

$$\delta_{12}=\delta^{ij}(e_i\otimes e_j^\dagger)=(e_i\otimes e_i^\dagger), \tag{A1}$$

and, accordingly,

$$\delta_{21}=(-1)^{[i]}(e_i^\dagger\otimes e_i). \tag{A2}$$

2. Poisson brackets

For F and G two $(\Phi,\Phi^\dagger)$-functionals, their Poisson bracket is defined by

$$\{F,G\}=i\sum_{\ell=1}^{K}\int_{-\infty}^{\infty}dx(-1)^{[F][\ell]}\left((-1)^{[\ell]}\frac{\delta F}{\delta\phi_\ell(x)}\frac{\delta G}{\delta\phi_\ell^\dagger(x)}-\frac{\delta F}{\delta\phi_\ell^\dagger(x)}\frac{\delta G}{\delta\phi_\ell(x)}\right). \tag{A3}$$

This bracket is a graded Poisson bracket, i.e., it is bilinear, graded antisymmetric, and obeys the graded Leibniz rule and graded Jacobi identity.

To any graded PB, one can associate a "global" Poisson bracket, defined for the even functionals $\mathbf{F}$ and $\mathbf{G}$. We introduce the notation u_α to denote either e_i ($\alpha=(0,i)$ and $[\alpha]=[i]$), $e_i^\dagger$ ($\alpha=(i,0)$ and $[\alpha]=[i]$), or E_{ij} ($\alpha=(i,j)$ and $[\alpha]=[i]+[j]$), so that any even object $\mathbf{F}$ can be written $\mathbf{F}=\Sigma_\alpha F_\alpha u_\alpha$ with $[F_\alpha]=[\alpha]$.

On any even object, one defines the global PB

$$\{\mathbf{F}_1,\mathbf{G}_2\}=\sum_{\alpha,\beta}\{\mathbf{F}_\alpha,\mathbf{G}_\beta\}u_\alpha\otimes u_\beta . \tag{A4}$$

It is bilinear, antisymmetric, and obeys Leibniz rule and Jacobi identity. Let us stress that this global PB is not graded (because of the use of auxiliary spaces), but its "component" version indeed is graded.

Lemma A.1: The global PB (A4) corresponding to the graded PB (A3) can be rewritten as

$$\{\mathbf{F}_1,\mathbf{G}_2\}=i\int_{\mathbb{R}}dx\left(\frac{\delta\mathbf{F}_1}{\delta\Phi_{3/2}(x)}\frac{\delta\mathbf{G}_2}{\delta\Phi^\dagger_{3/2}(x)}-\frac{\delta\mathbf{G}_2}{\delta\Phi_{3/2}(x)}\frac{\delta\mathbf{F}_1}{\delta\Phi^\dagger_{3/2}(x)}\right), \tag{A5}$$

where we have introduced a third auxiliary space (labeled $\frac{3}{2}$) which is "inserted" between the space 1 and the space 2. We have also defined

$$\frac{\delta}{\delta\Phi(x)}=\sum_{j=1}^{K}e_j^\dagger\frac{\delta}{\delta\phi_j(x)}\quad\text{and}\quad\frac{\delta}{\delta\Phi^\dagger(x)}=\sum_{j=1}^{K}(-1)^{[j]}e_j\frac{\delta}{\delta\phi_j^\dagger(x)} \tag{A6}$$

Proof: Direct calculation. ■

[1] E. Gutkin, Phys. Rep. **167**, 1 (1988).
[2] V. E. Zakharov and A. B. Shabat, Sov. Phys. JHEP **34**, 62 (1972); V. E. Zakharov and S. V. Manakov, Theor. Math. Phys. **19**, 551 (1975); P. P. Kulish, S. V. Manakov, and L. D. Faddeev, *ibid.* **28**, 615 (1976); E. K. Sklyanin, L. D. Faddeev, and L. A. Takhtajan, *ibid.* **40**, 688 (1980); P. P. Kulish and E. K. Sklyanin, Lect. Notes Phys. **151**, 61 (1982).
[3] S. Murakami and M. Wadati, J. Phys. A **29**, 7903 (1996).
[4] M. Mintchev, E. Ragoucy, P. Sorba, and P. Zaugg, J. Phys. A **32**, 5885 (1999).
[5] G. H. Roelofs and P. H. M. Kersten, J. Math. Phys. **33**, 2185 (1992).
[6] F. B. Estabrook and H. D. Wahlquist, J. Math. Phys. **17**, 1293 (1976).
[7] Z. Popowicz, Phys. Lett. A **194**, 375 (1994).
[8] L. Bonora, S. Krivonos, and A. Sorin, Nucl. Phys. B **477**, 835 (1996); E. Ivanov, *$N=4$ supersymmetric integrable models*, talk given at the international seminar "Supersymmetry and quantum field theory," Kharkov, Ukraine, January 1997.
[9] J. C. Brunelli and A. Das, Rev. Math. Phys. **7**, 1181 (1995).
[10] F. Delduc and L. Gallot, J. Math. Phys. **39**, 4729 (1998).
[11] J. C. Brunelli and A. Das, J. Math. Phys. **36**, 268 (1995).
[12] P. P. Kulish, Sov. Phys. Dokl. **25**, 912 (1980).
[13] H. Fan, F. Pu, and B. Zhao, Nucl. Phys. B **299**, 52 (1988).
[14] R. R. Rosales, Stud. Appl. Math. **59**, 117 (1978).
[15] B. Davies, J. Phys. A **14**, 2631 (1981).
[16] E. Sklyanin, Sov. Phys. Dokl. **24**, 107 (1979).
[17] H. B. Thacker, Rev. Mod. Phys. **53**, 253 (1981).
[18] F. Pu and B. Zhao, Phys. Rev. D **30**, 2253 (1984).
[19] F. Pu, Y. Wu, and B. Zhao, J. Phys. A **20**, 1173 (1987).
[20] A. B. Zamolodchikov and A. B. Zamolodchikov, Ann. Phys. **120**, 253 (1979).
[21] L. D. Faddeev, Sov. Sci. Rev., Sect. C, Math. Phys. Rev. **1**, 107 (1980).
[22] A. Liguori and M. Mintchev, Commun. Math. Phys. **169**, 635 (1995).
[23] M. Gattobigio, A. Liguori, and M. Mintchev, J. Math. Phys. **40**, 2949 (1999).
[24] V. Caudrelier and E. Ragoucy, Int. J. Mod. Phys. A to appear, math-ph/0306010.
[25] E. Sklyanin and L. D. Faddeev, Sov. Phys. Dokl. **23**, 902 (1978); J. Honerkamp, P. Weber, and A. Wiesler, Nucl. Phys. B **152**, 266 (1979).
[26] H. B. Thacker and D. Wilkinson, Phys. Rev. D **19**, 3660 (1979); D. B. Creamer, H. B. Thacker, and D. Wilkinson, *ibid.* **21**, 1523 (1980).
[27] B. Davies and E. Gutkin, Physica A **151**, 167 (1988).
[28] E. K. Sklyanin, J. Sov. Math **19**, 1546 (1982) (Ph.D. thesis).
[29] E. Ragoucy, J. Phys. A **35**, 7929 (2002).