

New form of the Euler-Bernoulli rod equation applied to robotic systems

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Abstract

This paper presents a theoretical background and an example of extending the Euler-Bernoulli equation from several aspects. Euler-Bernoulli equation (based on the known laws of dynamics) should be supplemented with all the forces that are participating in the formation of the bending moment of the considered mode. The stiffness matrix is a full matrix. Damping is an omnipresent elasticity characteristic of real systems, so that it is naturally included in the Euler-Bernoulli equation. It is shown that Daniel Bernoulli's particular integral is just one component of the total elastic deformation of the tip of any mode to which we have to add a component of the elastic deformation of a stationary regime in accordance with the complexity requirements of motion of an elastic robot system. The elastic line equation mode of link of a complex elastic robot system is defined based on the so-called "*Euler-Bernoulli Approach*" (EBA). It is shown that the equation of equilibrium of all forces present at mode tip point ("*Lumped-mass approach*" (LMA)) follows directly from the elastic line equation for specified boundary conditions. This, in turn, proves the essential relationship between LMA and EBA approaches. In the defined mathematical model of a robotic system with multiple DOF (degree of freedom) in the presence of the second mode, the phenomenon of elasticity of both links and joints are considered simultaneously with the presence of the environment dynamics –

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all based on the previously presented theoretical premises. Simulation results are presented.

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1 Introduction

Modeling and control of elastic robot systems has been a challenge to researchers over the past two decades.

In paper [1], the control of robots with elastic joints in contact with dynamic environment is considered. In [2], the feedback control was formed for the robot with elastic links (two-beam, two-joint systems) with distributed flexibility, robots with elastic links being also dealt with in [3]. In paper [8] a nonlinear control strategy for tip position trajectory tracking of a class of structurally flexible multi-link manipulators is developed. Authors of papers [10] and [11] derived dynamic equations of the joint angle, the vibration of the flexible arm, and the contact force. The paper [12] presents an approach to end point control of elastic manipulators based on the nonlinear predictive control theory. [14-15] presents method for the generation of efficient kinematics and dynamic models of flexible robots. In [17] author discusses the force control problem for flexible joint manipulators. In paper [18] the authors extend the integral manifold approach for the control of flexible joint robot manipulators from the known parameter case to the adaptive case. The author of paper [19] designed a control law for local regulation of contact force and position vectors to desired constant vectors. In paper [20] different from conventional approaches, authors focus on the design of rigid part motion control and the selection of bandwidth of rigid subsystem.

Work [21] presents the derivation of the equations of motion for application mechanical manipulators with elastic links. In [22] the equations are derived using Hamilton's principle, and are nonlinear integro-differential equations.

The formulation is based on expressing the kinetic and potential energies of the manipulator system in terms of generalized coordinates. Method of separation of variables and the Galarkin's approach are suggested in paper [23] for the boundary-value problem with time-dependent

boundary condition.

Our paper presents in detail the procedure of the theoretical implementation of stiffness characteristic to the mathematical model of an elastic robot system. This realistic characteristic of the material of any mode is included through the example of multiple DOF robot mechanism modeling. We also deal with many other phenomena that are part of the nature of elastic robot systems.

We mention details of research done so far (over the preceding 20 years).

The mathematical model of one DOF mechanism with an elastic joint was defined by Spong in 1987 [16]. The same principles are applied in this paper to introduce joint elasticity in the mathematical model.

As far as the introduction of link elasticity in the mathematical model of a robot system is concerned, it is necessary to point out some problems of a physical nature in this field.

In the previous literature [24], [25], [26], [27], [28], [31] the general solution of the motion of an elastic robotic system has been obtained by considering elastic deformations as transversal oscillations that can be determined by the method of particular integrals of D. Bernoulli.

As known, an elastic deformation of a body under consideration may be caused by:

1. *disturbance* forces, which cause the oscillatory nature of motion
2. *stationary* forces, which cause the stationary nature of motion.

Thus, any elastic deformation can be presented by superimposing particular solutions of the oscillatory character of D. Bernoulli and stationary solution of the forced character (See Section III and IV of this paper).

- No relationship between LMA and EBA has been established in the literature. Moreover, an attitude has been formed that these two approaches should be treated in different ways. LMA is taken to be standard and obsolete. Work relying on this approach is seen as having no importance, because in this area everything “has already been done”. On the other hand, EBA is viewed as an approach that should be in the focus of authors’ interest.

- In this paper we have shown that facts differ essentially from what has been accepted: EBA (used in [24] . . .) treats an elastic beam as a system with an infinitely large number of the degrees of freedom, whereas the choice of a ordered shape of an elastic line is equivalent to the introduction of subsequent connections that reduce the number of the degrees of freedom. It permits analyzing the shape of an elastic line of every mode during task performance. When LMA (used in [5], [6] . . .) is applied, the addition of modes increases the number of the degrees of freedom. It permits analyzing the motion of the tip of every mode only. LMA appears to follow directly from EBA, i.e., it is, essentially, just a special case of EBA. Lacking the information on which of these two approaches was developed first, we think that neither of these two methods is superior or inferior – they are “equal” methods treating the same problem from various standpoints. A mathematical method obtained by any method should satisfy the elementary structure of elastic mechanism models known from the literature [32].

Of course, we agree in some points with the approach taken so far in the analysis of elastic robot systems. We agree in *reference trajectory definition*.

- First detailed presentation of the procedure for creating reference trajectory was given in [7]. This especially important result has widely opened up the possibility to the application of different control laws, as well as to the possibility of controlling the position and orientation of elastic robot tip in space. In [26], this has been realized for robotic system with one elastic joint and rigid link, and separately for the system with one rigid joint and one elastic link.
- In our paper the reference trajectory has been synthesized for example which includes elastic joints and links and the presence of environment force. The reference trajectory is calculated from the total dynamic model, when robot tip tracks a desired trajectory in a reference regime in the absence of disturbance. Elastic deformations (of elastic link and elastic joint) exist on a reference level as well in the case when a robot moves along a reference trajectory in a reference regime. With a reference trajectory defined in this way

it is possible to apply a very simple control law through PD local feedback and this, in turn, ensures tracking the robot tip in the Cartesian coordinate space.

As far as a robot working regime is concerned, we think that all of the present forces should participate in the formation of elastic deformations and that it is a rough approximation to assume that elastic effects are generated by gravitation forces only, or by environment dynamics force only [9, 10] or to neglect completely Coriolis' and centrifugal forces and thus make the elastic deviations so small that they do not affect the inertia matrix [13].

Let us emphasize once again that in this paper:

1. we propose a mathematical model solution that includes in its root the possibility for analyzing simultaneously both present phenomena – the elasticity of joints and the elasticity of links, the idea originated from [4] but not on the same principles,
2. we show how the continuously present *environment dynamics force* affects the behavior of an elastic robot system.

See papers [33]-[37].

The analysis of the Euler-Bernoulli equation is given in section 2 while section 3 presents, in general, the ideas for extending the Euler-Bernoulli equation by a damping component as well as an expansion of its solution by the component of a stationary solution of a forced nature. These new ideas are explained in more detail in section 4. In section 5 we analyze a multiple DOF robot mechanism with elastic joints and elastic links in the presence of environment force dynamics. Concluding remarks are presented in section 6.

2 Analysis of the Euler-Bernoulli equation in source shape

A disturbance of the equilibrium state of an elastic body will result in motion, i.e. vibration of particles. This motion is transmitted through the body causing a wave process whose characteristic is that the same

disturbance state prevails in each place of the elastic medium, but with a delayed phase. The equation of the elastic line of a beam bending has the following form:

$$\hat{M}_{1,1} + \beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^2} = 0, \quad \text{or} \quad \hat{M}_{1,1} + \hat{\varepsilon}_{1,1} = 0. \quad (1)$$

$$\hat{\varepsilon}_{1,1} = \beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^2}. \quad (2)$$

Where: $\hat{M}_{1,1} [Nm]$ - load moment, $\hat{\varepsilon}_{1,1}$ - bending moment, $\beta_{1,1} = E_l \cdot I_{mom1,1} [Nm^2]$ - flexural stiffness, where $E_l [N/m^2]$ is the elasticity module and $I_{mom1,1} [m^4]$ the inertia moment of the cross section of the beam under consideration.

The basic dynamic equation of an elastic body has been obtained by applying D'Alembert's principle and adding the inertia force to the Euler-Bernoulli equation. This is how the well known wave equation of transversal oscillations of a flexible beam is obtained:

$$\frac{\partial^2 \hat{y}_{1,1}}{\partial t^2} + c_{1,1}^2 \cdot \frac{\partial^4 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^4} = 0. \quad (3)$$

$c_{1,1}$ is the wave propagation velocity. Equation (3) follows from equation (1), after performing differentiation and assuming that the flexion force is opposed by the inertia force.

The general solution of motion (see Fig. 1), i.e., the form of transversal oscillation of elastic beams may be found using the Daniel Bernoulli's of particular integrals method in the form:

$$\hat{y}_{to1,1}(\hat{x}_{1,1}, t) = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot \hat{T}_{to1,1}(t). \quad (4)$$

$$\begin{aligned} \hat{X}_{1,1}(\hat{x}_{1,1}) = & C_{1,(1,1)} \cos k_{1,1} \hat{x}_{1,1} + C_{2,(1,1)} \sin k_{1,1} \hat{x}_{1,1} + \\ & + C_{3,(1,1)} Ch k_{1,1} \hat{x}_{1,1} + C_{4,(1,1)} Sh k_{1,1} \hat{x}_{1,1}. \end{aligned} \quad (5)$$

$$\hat{T}_{to1,1}(t) = A_{1,1} \cos p_{1,1} t + B_{1,1} \sin p_{1,1} t. \quad (6)$$

$p_{1,1}$ is the circular frequency. $k_{1,1}^4 = \frac{p_{1,1}^2}{c_{1,1}^2}$.

$C_{1,(1,1)}$, $C_{2,(1,1)}$, $C_{3,(1,1)}$ and $C_{4,(1,1)}$, are constants determined from the boundary conditions of mode, and constants $A_{1,1}$ and $B_{1,1}$ are determined

from the initial conditions of motion. By superposing particular solutions (4), any transversal oscillation may be presented in the following form:

$$\hat{y}_{to1}(\hat{x}_{1,j}, t) = \sum_{j=1}^{\infty} \hat{X}_{1,j}(\hat{x}_{1,j}) \cdot \hat{T}_{to1,j}(t). \tag{7}$$

Source equations (1-7) are of special importance for the analysis of elastic bodies.

Remark I: *Equations (1-7) should be expanded by a brief explanation, which we take for granted but is not included in source literature [32]. The authors have written equation (7) in a visionary way, without defining the mathematical model of a link with an infinite number of modes whose solution is equation (7). It seems they have left this task to their followers. The transversal oscillations given by equation (7) describe the motion of an elastic beam to which we have assigned an infinite number of degrees of freedom (modes) and which we may describe by a mathematical model defined with an infinite number of equations of the form:*

$$\hat{M}_{1,j} + \hat{\varepsilon}_{1,j} = 0, \quad j = 1, 2, \dots, j, \dots\infty. \tag{8}$$

The dynamics of each mode is described by one equation. Contrary to today’s interpretations by the authors of numerous papers, equations given in model (8) are not equal in structure. In our opinion, differences between the equations of model (8) arise because of the coupling between the present modes. This explanation is of crucial importance and must be kept in mind in understanding the further presentation in our paper.

Remark II: *The symbol " ^ " generally characterizes the quantities relating to any point on the elastic line of a mode, e.g. $\hat{y}_{1,1}, \hat{x}_{1,1}, \hat{\varepsilon}_{1,1}$. The same quantities without the symbol " ^ " have been defined for the tip of a mode, e.g. $y_{1,1}, x_{1,1}, \varepsilon_{1,1}$.*

Remark III: *Under a mode we understand the presence of coupling between all the modes present in the system. We analyze the system in which the action of coupling forces (inertial, Coriolis’, and elasticity forces) exists between the present modes. To differentiate it from “mode shape” or “assumed mode”, we could call it a coupled mode or, shorter, in the text to follow, a mode. This yields the difference in the structure of Euler-Bernoulli equations for each mode.*

3 Extension of the Euler-Bernoulli equation

A. Damping component

Any elastic body, when exposed to external forces, is deformed – it elongates, shortens, bends, twists, depending on the position and direction of forces acting on it. Only transversal deformations of the prismatic beam will be analyzed in detail in this paper. The point of application of forces displace in time, so these forces perform certain work on this path. Work is opposed by:

- the *potential energy* of the elastic body, which depends on the stiffness characteristic and flexure, and
- the *dissipation energy*, which depends on the damping characteristic and flexure change velocity.

The presence of dissipation energy is especially expressed in an oscillatory regime, while its presence is minimal in a stationary regime, when displacement velocity of the material particles of an elastic body with respect to an equilibrium position is minimal. To include the damping effect into analysis, source equation (1) should be extended as follows:

$$\hat{M}_{1,1} + \beta_{1,1} \cdot \frac{\partial^2(\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})}{\partial \hat{x}_{1,1}^2} = 0. \quad (9)$$

$\eta_{1,1}$ is a factor characterizing the share of damping in the total elasticity characteristic.

B. Whole solution of elastic line

We will consider one more aspect of the expansion of the previously defined source equations (1-7).

Bernoulli's equation (4-6) describes the nature of the motion of real elastic beam only partially. More precisely, it is only one component of motion.

As has already been stated, equations (1-7) have been defined assuming that the elasticity force is opposed by the inertia force only. In addition, it is assumed, by definition, that motion in equation (1) is caused by a suddenly added, and then removed, external force. Daniel Bernoulli's solution (4-6) satisfies these assumed conditions.

To be applicable to a wider analysis of elastic bodies, equations (1-7) should be supplemented by expressions that follow directly from motion dynamics of elastic systems.

As known, an elastic deformation of a body under consideration may be caused by:

- *disturbance* forces, which cause the oscillatory nature of motion
- *stationary* forces, which cause the stationary nature of motion.

By superposing the particular solution of oscillatory nature, and the stationary solution of forced nature, any elastic deformation of a considered mode may be presented in the following general form:

$$\hat{y}_{1,1}(\hat{x}_{1,1}, t) = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot (\hat{T}_{st1,1}(t) + \hat{T}_{to1,1}(t)). \quad (10)$$

$\hat{T}_{st1,1}(t)$ is the stationary part of elastic deformation caused by stationary forces that vary continuously over time.

Remark IV: *The extension of the Euler-Bernoulli equation by a damping component as a realistically present property of elasticity has no direct, causal relationship with the extension of the equation of Daniel Bernoulli's particular integral. These two new phenomena are compatible in the analysis of elastic robot systems, but are not conditioned by each other.*

4 New form of elastic line model and new form of elastic line solution

An elastic beam is made of a certain material characterized by stiffness and damping. The points of working of forces, acting on an elastic beam, displace in time, so these forces perform certain work on this path. A portion of this work transforms into the potential energy $E_{pels1,1}$ and

another into the dissipation energy $\Phi_{pels1,1}$ of elastic body. The energy balance is:

$$A_{d1,1} = E_{pels1,1} + \Phi_{pels1,1}. \quad (11)$$

To emphasize the elasticity effects, in this analysis the kinetic energy of elastic body motion is included in expression $A_{d1,1}$.

Upon a slow change in the load of an elastic body by forces, for example in a stationary state, when the velocity of a change in the position of elastic body material particles, with respect to the equilibrium state of the same body, is slight, dissipation energy $\Phi_{pels1,1}$ can be neglected. There is no such a case in robotics. Because of the possible disturbances during robot task performance, the velocity of a change in the position of elastic body material particles with respect to the equilibrium position can be considerable; therefore $\Phi_{pels1,1}$ can have a considerable share in the total energy balance given by equation (11). What follows from this is that the work of external forces upon a dynamic load of an elastic body is equal to the action of change in the potential and dissipation energy of deformation. This means that the potential and dissipation energy may be expressed and calculated using the work of external forces. A body that has accumulated potential or dissipation energy is in a tension state, so the internal forces tend to return the body to an equilibrium state. The potential and dissipation energy of deformation may also be referred to as *deformation work*. The deformation work caused by the action of all present forces is denoted in this paper by $A_{d1,1}$.

Load force $F_{1,1}$ is consumed on the opposition to stiffness force $F_{1,1}^p$ or on the opposition to damping force $F_{1,1}^\phi$ of elastic mode.

$$F_{1,1} = F_{1,1}^p + F_{1,1}^\phi. \quad (12)$$

$$F_{1,1}^p = C_{s1,1} \cdot r_{1,1}, \quad F_{1,1}^\phi = B_{s1,1} \cdot \dot{r}_{1,1}. \quad (13)$$

$C_{s1,1}$ is the stiffness characteristic and $B_{s1,1}$ is the damping characteristic. The elementary potential energy resulting from the present stiffness of elastic body is:

$$dE_{pels1,1} = C_{s1,1} \cdot r_{1,1} \cdot dr_{1,1}. \quad (14)$$

Integration yields:

$$E_{pels1,1} = \frac{1}{2} C_{s1,1} \cdot r_{1,1}^2. \quad (15)$$

The elementary dissipation energy due to the present damping of elastic body is:

$$d\Phi_{pels1,1} = B_{s1,1} \cdot \dot{r}_{1,1} \cdot d\dot{r}_{1,1}. \tag{16}$$

Integration yields:

$$\Phi_{pels1,1} = \frac{1}{2} B_{s1,1} \cdot \dot{r}_{1,1}^2. \tag{17}$$

From this analysis we may conclude that elastic line equation (1) may be expanded by the damping characteristic of elastic body, as defined by equation (9).

The general solution of the motion of elastic body, described by equation (9), is defined by equation (10). Let us consider this in more detail.

In addition to the Bernoulli's solution defined by equation (5), the geometry of a bent link may also be defined by the following procedure.

A. Simple example with a detailed explanation

It should be noted that the shape of elastic line depends on constant and time-varying factors:

- *constant*: the geometrical characteristics of the link and the characteristics of the material the link is made of
- *time-varying*: the type and level of load during robot task performance.

The load moment for the any point of first mode may also be expressed in the form of equation:

$$\hat{M}_{1,1} = F_{1,1} \cdot (l_{1,1} - \hat{x}_{1,1}). \tag{18}$$

From equation (9) it follows that:

$$\hat{\varepsilon}_{1,1} = \beta_{1,1} \cdot (\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})'' \tag{19}$$

If the coordinate system is set as shown in Fig. 2, the differential equation of elastic line is defined by:

$$\beta_{1,1} \cdot (\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})'' = F_{1,1} \cdot (l_{1,1} - \hat{x}_{1,1}). \tag{20}$$

Doubly integration gives:

$$\beta_{1,1} \cdot (\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})' = -\frac{F_{1,1} \cdot (l_{1,1} - \hat{x}_{1,1})^2}{2} + C_1. \quad (21)$$

$$\beta_{1,1} \cdot (\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1}) = \frac{F_{1,1} \cdot (l_{1,1} - \hat{x}_{1,1})^3}{6} + C_1 \cdot \hat{x}_{1,1} + C_2. \quad (22)$$

Constants C_1 and C_2 are determined from the boundary conditions defined for console.

For $\hat{x}_{1,1} = 0$ and $\hat{y}_{1,1} = 0$, $\dot{\hat{y}}_{1,1} = 0$ it follows that $C_2 = -\frac{F_{1,1} \cdot l_{1,1}^3}{6}$.

For $\hat{x}_{1,1} = 0$ and $(\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})' = 0$ it follows that $C_1 = \frac{F_{1,1} \cdot l_{1,1}^2}{2}$.

By introducing C_1 and C_2 into equations (21) and (22), we obtain:

$$(\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1}) = \frac{F_{1,1} \cdot l_{1,1}^3}{6 \cdot \beta_{1,1}} \cdot \left(\frac{\hat{x}_{1,1}}{l_{1,1}}\right)^2 \cdot \left(3 - \frac{\hat{x}_{1,1}}{l_{1,1}}\right). \quad (23)$$

$$(\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})' = \frac{F_{1,1} \cdot l_{1,1}^2}{2 \cdot \beta_{1,1}} \cdot \frac{\hat{x}_{1,1}}{l_{1,1}} \cdot \left(2 - \frac{\hat{x}_{1,1}}{l_{1,1}}\right). \quad (24)$$

Equations (23) and (24) determine the position and orientation of the elastic line of the mode considered in each its point.

For $x_{1,1} = l_{1,1}$, when $y_{1,1} = y_{\max 1,1}$, hence $\dot{y}_{1,1} = 0$ it follows that from equations (23) and (24):

$$y_{1,1} = r_{1,1} = \frac{F_{1,1} \cdot l_{1,1}^3}{3 \cdot \beta_{1,1}}, \quad (25)$$

$$y_{1,1}' = \alpha_{1,1} = -\frac{F_{1,1} \cdot l_{1,1}^2}{2 \cdot \beta_{1,1}}. \quad (26)$$

$tg\vartheta_{1,1} = \frac{y_{1,1}}{l_{1,1}}$. If we accept $\vartheta_{1,1} \approx tg\vartheta_{1,1}$ for small angles of bending, then

it follows that $\vartheta_{1,1} = \frac{F_{1,1} \cdot l_{1,1}^2}{3 \cdot \beta_{1,1}}$. The rotation angle of the mode considered $\omega_{1,1}$ can also be defined [29]. This angle plays an important role in the dynamics of elastic robot systems.

$$\omega_{1,1} = \alpha_{1,1} - \vartheta_{1,1} = \frac{F_{1,1} \cdot l_{1,1}^2}{6 \cdot \beta_{1,1}} = \frac{\vartheta_{1,1}}{2}. \quad (27)$$

The stiffness characteristic for the tip of first mode is designated from equation (27) as $C_{s1,1} = \frac{3 \cdot \beta_{1,1}}{l_{1,1}^3} [N/m]$, maximal deflection is $r_{1,1}$.

At the moment when the elastic body tip passes through the equilibrium position, $y_{1,1} = 0$ holds (the potential energy of elastic body equals zero), and the tip displacement velocity is $\dot{y}_{1,1}$ (the dissipation energy of elastic body is finite) then:

$$\eta_{1,1} \cdot \dot{y}_{1,1} = \eta_{1,1} \cdot \dot{r}_{1,1} = \frac{F_{1,1} \cdot l_{1,1}^3}{3 \cdot \beta_{1,1}}. \quad (28)$$

From this condition we may calculate the damping characteristic

$$B_{s1,1} = \frac{3 \cdot \beta_{1,1} \cdot \eta_{1,1}}{l_{1,1}^3} [Ns/m].$$

The elasticity moment, which is defined by $\varepsilon_{1,1}$ and which opposes the load moment $M_{1,1}$, is

$$\varepsilon_{1,1} = (C_{s1,1} \cdot r_{1,1} + B_{s1,1} \cdot \dot{r}_{1,1}) \cdot l_{1,1}. \quad (29)$$

Generally speaking, we also know that the tip of the mode of an elastic robot link is affected by:

- *disturbance* forces, which cause the oscillatory nature of motion, and
- *stationary* forces, which cause finite, stationary deformations.

The total force causing elasticity may be expressed, in terms of its components, by:

$$F_{1,1} = F_{st1,1} + \Delta F_{1,1}. \quad (30)$$

The stationary part of elasticity force component is $F_{st1,1}$ which varies continuously over time and, depending on its intensity and direction, causes the stationary component of flexure $r_{st1,1}$.

$$T_{st1,1}(t) = r_{st1,1} = \frac{F_{st1,1} \cdot l_{1,1}^3}{3 \cdot \beta_{1,1}}. \quad (31)$$

$F_{st1,1}$ according to Rayleigh [29], contains two components:

$$F_{st1,1} = W_{u(1,1)} + W_{w(1,1)}. \quad (32)$$

$W_{u(1,1)}$ is the force of the load concentrated on beam tip, and

$$W_{w(1,1)} = \frac{33 \cdot \bar{w}_{1,1} \cdot g \cdot l}{140}. \quad (33)$$

is the force caused by beam eigenweight, where $\bar{w}_{1,1} \cdot g$ is beam weight per unit length.

The assumption that the stationary state of beam is its horizontal position that coincides with axis $x_{1,1}$ is unrealistic. This axis may coincide with a horizontal straight line around which the elastic beam oscillates only if gravitation load is neglected, and this represents a completely idealized case.

$\Delta F_{1,1}$ is a disturbance force, whose action may be momentary or permanent. It causes the oscillatory component of elastic mode motion, which is described by equation (6).

We know that at a moment t_p the action of disturbance force causes an additional flexure $r_{to1,1}(t_p)$, see Fig. 2.

$$T_{to1,1}(t_p) = r_{to1,1}(t_p) = \frac{\Delta F_{1,1} \cdot l_{1,1}^3}{3 \cdot \beta_{1,1}}. \quad (34)$$

Force $\Delta F_{1,1}$ has been removed and the body continues oscillating. Equation (6) describes the oscillatory nature of motion.

If oscillations are caused by an external force $\Delta F_{1,1}$ that has been added and immediately removed, elastic mode oscillations then take place around the position of the stationary state. See Fig. 3.a. Position "B" is defined by a stationary load force $F_{st1,1}$. When beam is at rest, $F_{st1,1}$ is raised only by gravitation force. Position A, B is the stationary position of the elastic line of the first mode. In this case, oscillations take place around position A, B in the range from $A, B \uparrow A, +B'$ to $A, B \downarrow A, -B'$. Position B' is defined by the load force $F_{1,1}$ which represents the sum of stationary load force and disturbance force $F_{1,1} = F_{st1,1} + \Delta F_{1,1}$ at the moment of disturbance.

However, if the disturbance force acts on a robot system in a longer period, oscillations then take place around a new stationary position A, C in the range from $A, C \uparrow A, +C'$ to $A, C \downarrow A, -C'$. See Fig. 3b.

Position C is defined by the total load force $F_{1,1}$.

By analyzing equation (23) we will notice that it contains two components, so its solution may be written as follows:

$$\hat{y}_{1,1} = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot (\hat{T}_{st1,1}(t) + \hat{T}_{to1,1}(t)) = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot \hat{T}_{1,1}(t). \quad (35)$$

Component $\hat{X}_{1,1}(\hat{x}_{1,1})$ describes a possible geometrical relation between $\hat{y}_{1,1}$ and $\hat{x}_{1,1}$. Component $\hat{T}_{1,1}(t)$ describes the dependence of flexure $\hat{y}_{1,1}$ on elasticity force $F_{1,1}$, which is the only time-varying quantity in expression (35).

By combining the particular solution of the oscillatory nature of motion, the stationary solution of the forced nature of motion and the geometry of elastic line of the mode considered, we may obtain the general solution of the motion of the first mode.

By superposing solutions (10), any elastic deformations of an elastic link with an infinite number of degrees of freedom may be presented in the following form:

$$\hat{y}_1(\hat{x}_{1,j}, t) = \sum_{j=1}^{\infty} \hat{X}_{1,j}(\hat{x}_{1,j}) \cdot (\hat{T}_{st1,j}(t) + \hat{T}_{to1,j}(t)). \quad (36)$$

5 Simulation example and results

Robot starts from point "A" (Fig. 4) and moves toward point "B" in the predicted time $T = 2 [s]$.

Dynamics of the environment force is included into the dynamics of system's motion [30]. The adopted velocity profile is trapezoidal, with the period of acceleration/deceleration of $0.2 \cdot T$.

The same example analyzed as in paper [34] only with some what different parameters environment.

Parameters of the environment are: $F_c^o = 30[N]$, $m_e = 1[kg]$, $b_e = 10[N/(m/s)]$, $k_{a1} = 10^4[N/m]$. $\mu = 0.2$.

The characteristics of stiffness and damping of the gear in the real and reference regimes are not the same and neither are the stiffness and damping characteristics of the link.

$$C_{\xi} = 0.99 \cdot C_{\xi}^o, \quad B_{\xi} = 0.99 \cdot B_{\xi}^o, \quad C_{s1,1} = 0.99 \cdot C_{s1,1}^o,$$

$$B_{s1,1} = 0.99 \cdot B_{s1,1}^o, \quad C_{s1,2} = 0.99 \cdot C_{s1,2}^o, \quad B_{s1,2} = 0.99 \cdot B_{s1,2}^o.$$

As can be seen from Fig. 5 in its motion from point “A” to point “B” the robot tip tracks well the reference trajectory in the space of Cartesian coordinates.

As a position control law for controlling local feedback was applied, the tracking of the reference force was directly dependent on the deviation of position from the reference level (see Fig. 6).

The gear deflection angle ξ is given in Fig. 7.

The elastic deformations that are taking place in the vertical plane angle of bending of the lower part of the link (first mode) ϑ_m , as well as the angle of bending of the upper part of the link (second mode) ϑ_e are presented in Fig. 8a., whereas elastic deformations taking place in the horizontal plane: angle of bending of the lower part of the link (first mode) ϑ_q as well as the angle of bending of the upper part of the link (second mode) ϑ_δ are given in Fig. 8b.

As the rigidity of the second mode is about ten times lower compared with that of the first mode, it is then logical that the bending angle for the second mode is about ten times larger compared to that of the first mode.

6 Conclusions

In order to analyze the behavior of an elastic robotic system it is necessary to significantly expand the original form of:

- *Euler-Bernoulli 's equation*, as a direct consequence of the forces involved, and
- *the particular integral of Daniel Bernoulli*.

Based on the known laws of dynamics, the Euler-Bernoulli equation should be supplemented by all forces participating in the formation of bending moment of the considered mode. It is assumed that coupling forces of the present modes (inertial, Coriolis and elasticity forces) are also involved. This yields structural differences between Euler-Bernoulli's equations for each mode. The stiffness matrix is a full matrix. Only a

stiffness characteristic has been attributed to an elastic beam in the literature so far. Realistically, both stiffness and damping characteristics are present in the elastic beam. Apart from stiffness characteristic, damping characteristic has been included in the Euler-Bernoulli equation for the first time. Its presence in the mathematical model of an elastic line of a mode considered is argued theoretically. The shape of elastic line follows directly from motion dynamics of the complete system. Apart from the damping of an elastic link, a variety of other phenomena involved in the motion dynamics of these systems are also analyzed in the paper.

A general form of a transversal elastic deformation is analyzed and defined, which has been obtained by superposing the solution of oscillatory nature (Daniel Bernoulli's solution) and the stationary solution of forced nature. The example of a multiple DOF mechanism is used for defining the elastic line equation of first link based on the so-called EBA. It is shown that the equilibrium equation of all the forces present at mode tip point follows directly from the Euler-Bernoulli equation. This means that LMA is just a special case of EBA.

A dynamic model of an elastic robot system has been defined. A mechanism has been modeled and simulated in the presence of the second mode. Environment force dynamics has been implemented in the dynamics of the behavior of an elastic robot system with elastic link and elastic joint simultaneously. Even when a very simple control law is applied, proper tracking of reference trajectory is achieved.

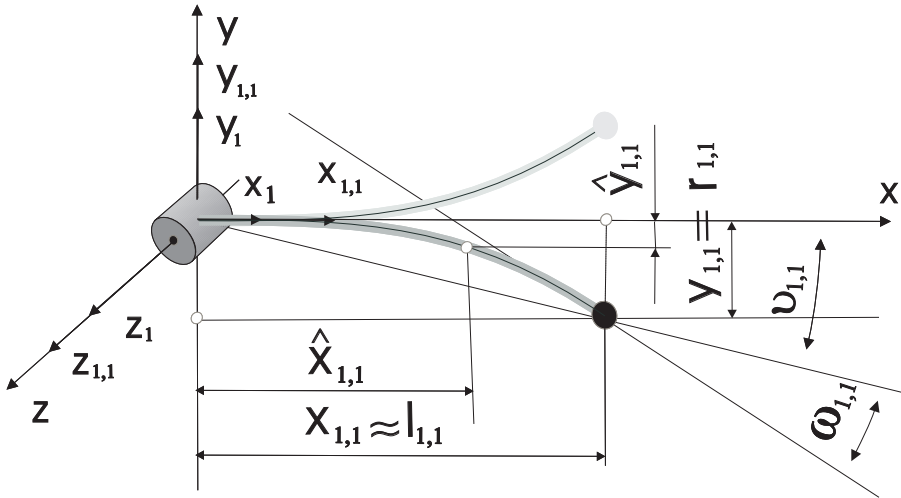


Figure 1: Idealized motion of an elastic body according to Daniel Bernoulli.

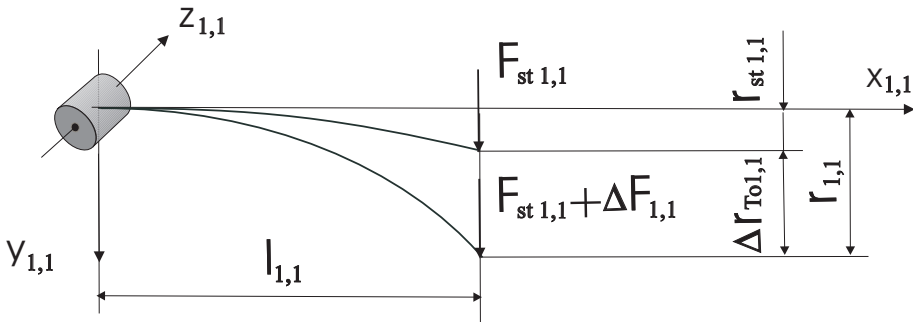


Figure 2: Flexion of elastic body caused by stationary and disturbance force.

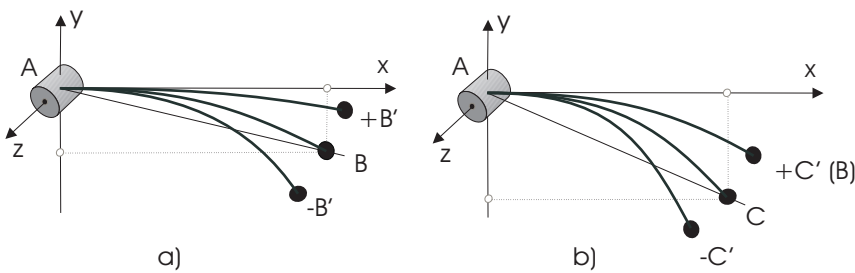


Figure 3: Stationary state position of a bending beam.

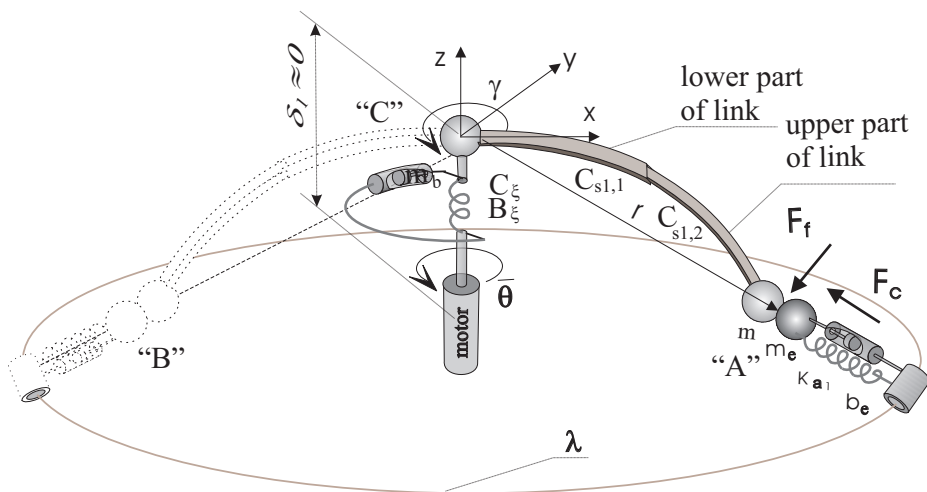


Figure 4: Mechanism robotic.

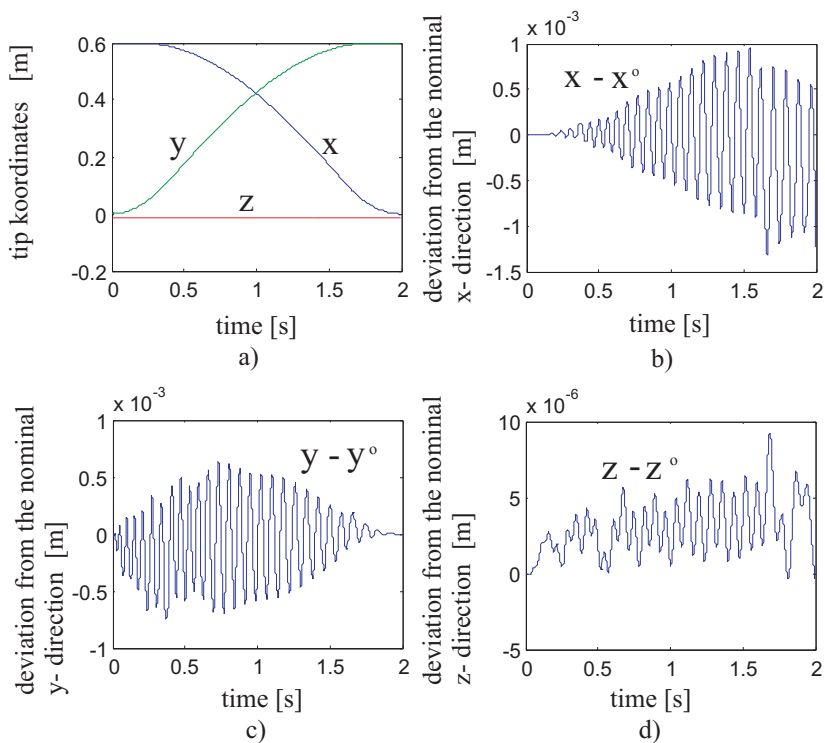


Figure 5: a) Tip coordinates; b), c), and d) deviation of position from the reference.

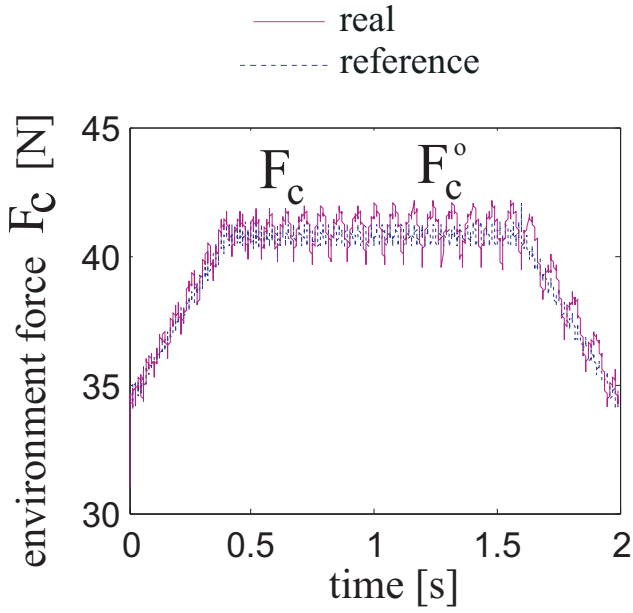


Figure 6: Dynamic force of the environment.

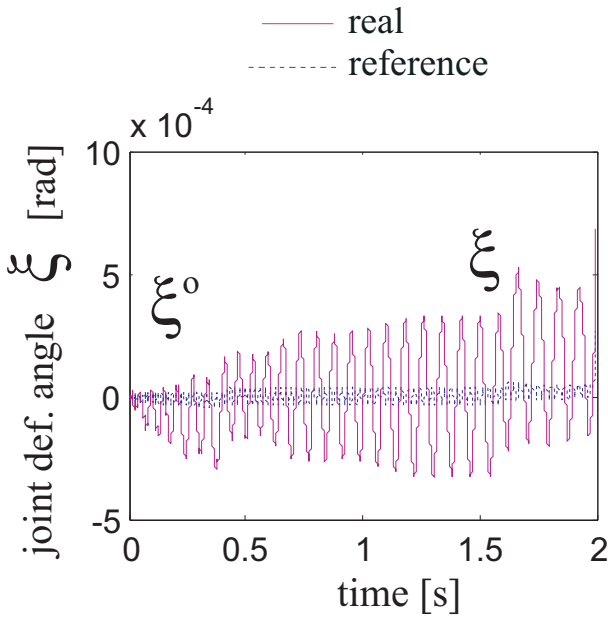


Figure 7: Deflection angle of joint.

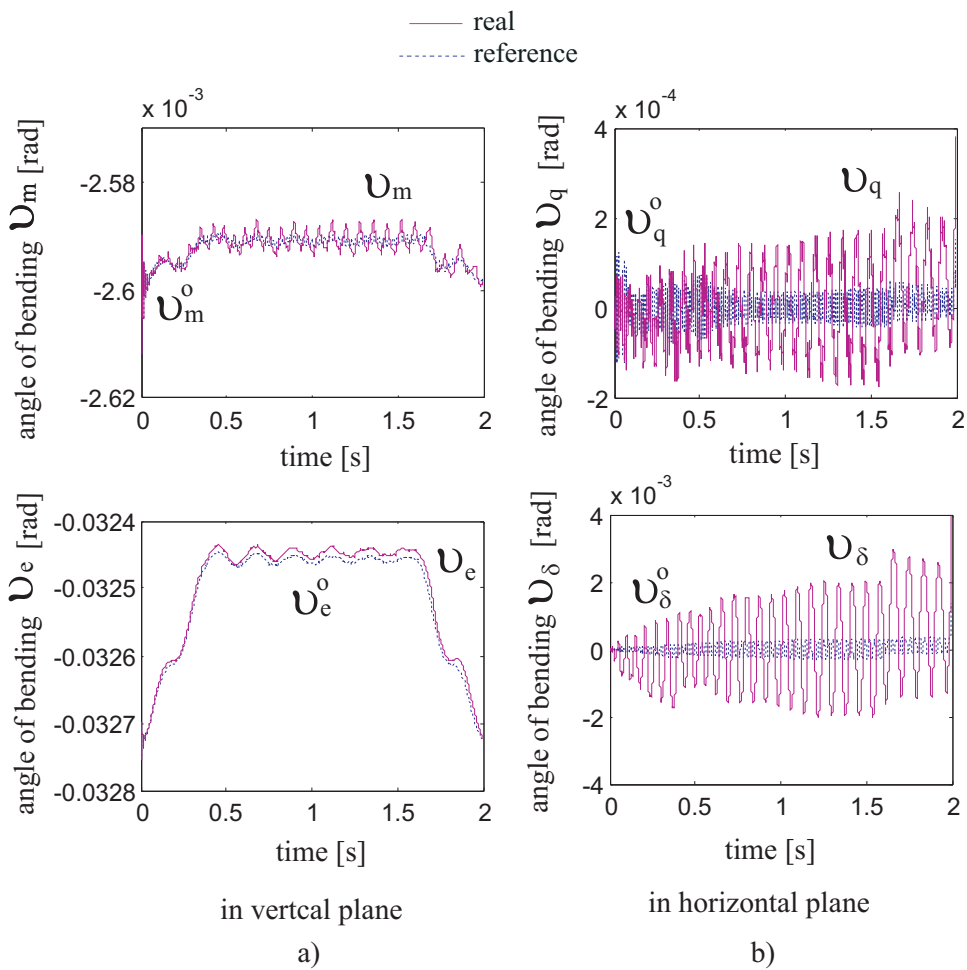


Figure 8: Elastic deformations a) in the vertical plane b) in the horizontal plane.

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Novi oblik Euler-Bernoulli jednačine grede primenjen na robotske sisteme

U ovom radu su date teorijske postavke i primer za proširenje Euler-Bernoulli jednačine sa nekoliko stanovišta. Euler-Bernoulli jednačini, (zasnovano na poznatim zakonima dinamike) treba dodati sve sile koje učestvuju u formiranju momenta savijanja posmatranog moda. Matrica krutosti je puna matrica. Prigušenje je u realnim sistemima uvek prisutna karakteristika elastičnosti tako da je prirodno uključena u Euler-Bernoulli jednačinu. Pokazano je da je partikularni integral Danijela Bernulija samo jedna komponenta ukupne elastine deformacije svakog moda, kojoj je neophodno dodati komponentu elastine deformacije stacionarnog režima prema zahtevima složenosti kretanja elastičnog robotskog sistema. Definisana je jednačina elastične linije moda segmenta složenog elastičnog robotskog sistema zasnovano na takozvanom "*Euler-Bernoulli Approach*" (EBA).

Pokazano je da jednačina ravnoteže svih prisutnih sila u tački vrha moda, ("*Lumped-mass approach*" (LMA)), direktno sledi iz jednačine elastične linije za definisane granične uslove. Time je pokazana suštinska veza između LMA i EBA pristupa.

Definisana je matematički model višestepenog robotskog sistema sa dva moda gde je fenomen elastičnosti i segmenata i zglobova razmatran simultano uz prisustvo dinamike sile okoline a sve na prethodno izloženim teorijskim postavkama. Pokazani su rezultati simulacija.