THESIS

Partial hyperbolicity and attracting regions in 3-dimensional manifolds

By: Rafael Potrie Altieri
Advisors: Sylvain Crovisier and Martín Sambarino

Doctorado en Matemática
PEDECIBA
Universidad de la República, Uruguay

Thèse
Pour obtenir le grade de
DOCTEUR DE L'UNIVERSITÉ PARIS 13
Discipline: Mathématiques

A Nati

Acknowledgements

The thesis could not have been done without the financial help by the following institutions: ANR, CNRS, IFUM, ANII, PEDECIBA, CMAT, CSIC, French Embassy in Uruguay (thanks in particular to Graciela Vigo for her constant support and help). I am grateful to all these institutions.

This has been a long trip and many people have of course been there sharing the journey. All of them should know my deep gratitude. However, I wish to express explicit acknowledgement to some of the persons which were specially important for the project itself.

Both Sylvain Crovisier and Martín Sambarino, my advisors, were of course very important and even essential in this project. I want to thank Sylvain for his infinite patience and dedication, he has clearly dedicated more time to this project than it is expected and this has had an important effect on its results, I want to profit to express my great admiration to him as a brilliant mathematician and as a person: He is incredibly generous, humble and kind, and even if sometimes it is hard to get strength to speak to someone which is so good, he makes you feel always comfortable and he is always encouraging, I could never even try to list all what I learned from him. Martín has been my advisor and friend since the very start of my studies, he has been my window to mathematics and he has always been open to pursue with me in the directions which I wanted, as an advisor he has always put priority in my formation over any other subject. As a member of the community of mathematics in Uruguay, he has always been and will always be a role model.

I greatly appreciate Christian Bonatti and Amie Wilkinson for accepting the task of reading this (long) thesis, it is really an honor to me. Christian has been part of this project all along participating in several ways and being always very kind and generous. Amie has accepted to referee this thesis in a difficult moment, I am very happy to thank her also for devoting some of her time to read and listen to me. Thanks also to Julien Barral, Pierre Berger, Álvaro Rovella and Raúl Ures for accepting being part of the jury.

My stay in Paris with Natalia was an important stage for this project. The fact that Andrés Sambarino was there was very important. I thank Andrés and Juliana for their openness and help in every stage of that stay and for sharing many good moments. We met a lot of people in Paris which made our stay great, but I would like to mention specially Freddy and Milena.

Mathematically, when staying in Paris, I was fortunate to meet and discuss with many great mathematicians (some of them frequent visitors of France) which were always open and they were important for the development of this project. Thanks to: Pierre Berger, Jerome Buzzi, Lorenzo Díaz, Tobias Jager, Patrice Le Calvez, Katsutoshi Shinohara, Dawei Yang and specially to Christian Bonatti and Nicolas Gourmelon. Also, I would like to thank the whole Dynamical Systems Group of Villetaneuse for sharing the seminar during the whole stay.

In my visits to Rio, I was fortunate to exchange with other mathematicians which were also important in the development of this project. Jairo Bochi, Pablo Dávalos, Lorenzo Díaz, Andy Hammerlindl and Enrique Pujals. Special thanks to Pablo Guarino, Yuri Ki, Freddy Hernandez and Milena Cortéz not only for discussions but also for their hospitality.

In a visit to Cuernavaca (México) before I started this project (around 2007), I was fortunate to meet Alberto Verjovski who in just an hour made an informal account of more results about topology and foliations that the ones I am able to understand even at this moment. Unexpectedly, those results and my further study of parts of what he taught me resulted very important in this project.

I wish to thank the dynamical systems group of Montevideo. My math discussions with people in Montevideo from the start of undergraduate studies has shaped my mathematical taste as well as it has made a pleasure to study mathematics. I wish to thank specially those which I consider have been essential, though for very different reasons: Diego Armentano, Alfonso Artigue, Joaquín Brum, Matias Carrasco, Marcelo Cerminara, Mariana Haim[1], Jorge Groisman, Pablo Guarino, Nancy Guelman, Pablo Lessa, Jorge Lewowicz, Roberto Markarian, Alejandro Passeggi, Miguel Paternain, Mariana Pereira, Álvaro Rovella, Andrés Sambarino, Martín Sambarino, Armando Treibich, José Vieitez, Mario Wschebor and Juliana Xavier. Among these, I wish to highlight conversations with Andrés Sambarino and Alejandro Passeggi which have always transcended math, discussions with Joaquín Brum and Pablo Lessa whose insight and understanding of different mathematical topics never ceases to surprise me and the courses with Miguel Paternain, Álvaro Rovella and Martín Sambarino which have been by far those which made me grow the most as a mathematician.

I've done my best to write this thesis, but as in other tasks, my best in writing is usually very poor, so, my apologizes go along my gratefulness to anyone who dares to read this (or part of this) thesis.

I wish to thank my friends, both in and outside mathematics, both in Montevideo and in Paris, specially those which have been present all along these years.

I wish to thank my family, specially my parents for their support and initial education which is of course essential for pursuing this kind of goals. I wish also to remember in this section of the thesis my cousin Laura, which passed away during my stay in Paris and has marked me tremendously.

[1] Who in addition devoted a great deal of time in helping me translate the introduction to french.

Finally, I wish to thank Natalia, my wife, my partner and my friend. Nothing would have sense if you were not there.

Abstract

This thesis attempts to contribute to the study of differentiable dynamics both from a semi-local and global point of view. The center of study is differentiable dynamics in manifolds of dimension 3 where we are interested in the understanding of the existence and structure of attractors as well as dynamical and topological implications of the existence of a global partially hyperbolic splitting. The main contributions are new examples of dynamics without attractors where we get a quite complete description of the dynamics around some wild homoclinic classes (see Section 2.2 and subsection 3.3.2) and two results on dynamical coherence of partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ (see Chapter 5).

Resumen

Esta tesis pretende contribuir al estudio de la dinámica diferenciable tanto desde sus aspectos semilocales como globales. El estudio se centra en dinámicas diferenciables en variedades de dimensión 3. Se busca comprender por un lado la existencia y estructura de los atractores así como propiedades topológicas y dinámicas implicadas por la existencia de una descomposición parcialmente hiperbólica global. Las contribuciones principales son la construcción de nuevos ejemplos de dinámicas sin atractores donde se da una descripción bastante completa de la dinámica alrededor de una clase homoclínica salvaje (ver Sección 2.2 y la subsección 3.3.2) y dos resultados sobre la coherencia dinámica de difeomorfismos parcialmente hiperbólicos en $\mathbb{T}^3$ (ver Capítulo 5).

Resumè

Le but de cette thèse est de contribuer à la compréhension des dynamiques différentiables aussi bien dún point de vue semilocal que global. L'etude se concentre sur les diffeomorphismes des variétés de dimension 3. On cherche à comprendre l'existence et la structure de leurs attracteurs, mais aussi à decrire les propriétés topologiques et dynamiques des difféomorphismes partiellement hyperboliques globaux. Les contributions pricipales sont la construction de nouvelle dynamiques sauvages (voir Section 2.2 et subsection 3.3.2) et deux résultats sur la cohérence dynamique des difféomorphismes partiellement hyperboliques dans $\mathbb{T}^3$ (voir Chapitre 5).

Contents

Notations

For a compact metric space X we denote $d(\cdot,\cdot)$ to the metric of X.

M^d will denote a compact connected Riemannian manifold without boundary of dimension $d \in \mathbb{N}$. It is a metric space whose metric is induced by the Riemannian metric $\langle \cdot,\cdot \rangle$.

$\mathrm{Leb}(\cdot)$ will denote the measure induced by any volume form on M of total measure 1. For our purposes it will make no difference which one is it (since we shall not assume the maps to preserve it) and we shall only care about sets having positive, total or zero measure.

For $X \subset M$, we denote $T_X M = \bigcup_{x\in X} T_x M$ with the topology induced by the inclusion $T_X M \subset TM$ into the tangent bundle of M.

$\mathrm{Diff}^r(M)$ $(r \geq 0)$ denotes the set of C^r-diffeomorphisms (homeomorphisms in the case $r = 0$) with the C^r topology (see [Hi]). We shall denote the distance in $\mathrm{Diff}^r(M)$ as $d_{C^r}(\cdot,\cdot)$. It is a Baire space. Similarly, $C^r(M,N)$ denotes the space of C^r-maps from M to N and $\mathrm{Emb}^r(M,N)$ the space of C^r-embeddings.

For $f \in \mathrm{Diff}^1(M)$ we denote as $D_x f : T_x M \to T_{f(x)} M$ the derivative of f over x. Sometimes, we shall not make reference to the point x when it is understood.

For V, W submanifolds of M we say that they intersect *transversally* at $x \in V \cap W$ if we have that $T_x V + T_x W = T_x M$. The set of points in $V \cap W$ where the intersection is transversal is denoted by $V \pitchfork W$. When $V \cap W = V \pitchfork W$ we say that V and W intersect *transversally.*

For V, W compact embedded submanifolds of M (possibly with boundary) which are diffeomorphic to a certain manifold D, we define the C^r-distance between them as the infimum of the C^r-distance between the pairs of embeddings of D in M whose image is respectively V and W.

For Baire spaces (in particular, sets which are metric and complete or open subsets of these) we say that a set $\mathcal{G}$ is *residual* (or $G_\delta-$dense) if it is a countable intersection of open and dense subsets.

We shall say that a property verified by diffeomorphisms in $\text{Diff}^r(M)$ is C^r-*generic* if it is verified by diffeomorphisms in a residual subset of $\text{Diff}^r(M)$. Sometimes, hoping it makes no confusion, we will say that a diffeomorphism f is a C^r-*generic diffeomorphism* to mean that f verifies properties in a residual subset $\text{Diff}^r(M)$ (which will be clear from the context).

$\mathbb{T}^d$ will denote the (flat) d-dimensional torus $\mathbb{R}^d/_{\mathbb{Z}^d}$ with the metric induced by the canonical covering map $p : \mathbb{R}^d \to \mathbb{T}^d$ and the Euclidean metric.

$B_\varepsilon(x)$ denotes the (open) ε-neighborhood of the point x, i.e. the (open) set of points at distance smaller than ε of x.

$B_\varepsilon(K)$ denotes the (open) ε-neighborhood of the set K.

Given a subset A of a metric space X we denote $\text{Int}(A), \overline{A}, \partial A, A^c$ to the interior, closure, frontier and complement of A respectively.

Given a point $x \in A$ we will denote $cc_x(A)$ to the connected component of A containing x.

Given a compact metric space X, we denote $\mathcal{K}(X)$ to be the set of compact subsets of X endowed with the Hausdorff distance:

$$d_H(A,B) = \max\{\sup_{x\in A} \inf_{y\in B}\{d(x,y)\}, \sup_{y\in B} \inf_{x\in A}\{d(x,y)\}\}$$

which is compact.

Given a sequence of sets $A_n \subset X$ a topological space. We define $\limsup A_n = \bigcap_{n\geq 0} \overline{\bigcup_{k\geq n} A_n}$.

We use the symbol $\Box$ to denote the end of a proof of a Theorem, Lemma, Proposition or Corollary. We use the symbol $\diamondsuit$ to denote the end of a Remark, Definition or the proof of some Claim (inside the proof of something else).

Chapter 0

Introduction and presentation of results

0.1 Introduction (English)

0.1.1 Historical account and context

One may[1] claim that the main goal in dynamical systems is to understand the asymptotic behavior of orbits for a given evolution law. Originally, the subject began with the study of ordinary differential equations of the form

$$\dot{x} = X(x) \qquad X : \mathbb{R}^n \to \mathbb{R}^n$$

and the goal was to solve these equations analytically and obtaining, for each initial value $x_0 \in \mathbb{R}^n$ an explicit solution $\varphi_t(x_0)$ to the differential equation.

It was soon realized that even extremely simple equations gave rise to complicated analytic solutions. Moreover, it was realized that the integrated equations did not supply enough understanding of the laws of evolutions.

By studying the famous 3-body problem, Poincaré ([Po]) was probably the first to propose that there should be a qualitative study of evolution rather than a quantitative one and he proposed to study "the behavior of *most* orbits for the *majority* of systems".

At the start, the study focused on stability. Lyapunov studied stable orbits, this means, orbits which contain a neighborhood of points having essentially the same asymptotic behavior. Andronov and Pontryagin, followed by Peixoto, studied stable systems, this means, those whose dynamical properties are robust under perturba-

[1]We warn the reader that the historic context we will present is completely subjective and not necessarily reflects the true historical facts. It must be thought as a plausible context in which the work of this thesis fits.

tions. But it was probably Smale ([Sm_2]) the first to revitalize Poincaré's suggestion by giving to it a more precise formulation:

The goal is to fix a closed manifold M of dimension d and to understand the dynamics of a *large* subset of $\mathrm{Diff}^r(M)$, the space of diffeomorphisms of M endowed with the C^r-topology.

Moreover, he proposed that a subset of diffeomorphisms should be considered large if it was open and dense with this topology, or at least, residual or dense (in a way that by understanding large sets of diffeomorphisms one could not neglect behavior happening in a robust fashion). We will not discuss other possible notions of largeness used in the literature nor the reasons for considering this ones (we refer the reader to [B] or [C_4] for an explanation of this choice).

Structural stability became the center of Smale's program which was strongly based on the hope that even if dynamical systems could not be stable from the point of view of their dynamics (they could be chaotic) they could be, at least the majority of them, stable from the point of view of their orbit structure. This would give that their dynamics and the one of their perturbations could be understood by symbolic or probabilistic methods. Palis and Smale [PaSm] conjectured that structurally stable systems coincide with hyperbolic ones.

Hyperbolicity became the paradigm. Robbin and Robinson ([R, Rob_1]) proved that hyperbolic systems were stable and long afterwards Mañe showed ([M_5]) that C^1-structurally stable dynamics were indeed hyperbolic (this is still unknown in other topologies). Describing the dynamics of hyperbolic diffeomorphisms was the center of attention for dynamicists in the 60's and 70's.

This project was tremendously successful from this point of view and it was not only the semi-local study (through the use of symbolic and ergodic techniques) that was understood but also very deep global aspects as well as some understanding of the topology of basic pieces was achieved.

On semi-local aspects, without being exhaustive, we mention particularly the contributions of Bowen, Newhouse, Palis, Sinai, Ruelle and Smale. We refer the reader to [KH] Part 4 for a panoramic view of a large part of the theory.

On the other hand, the global aspects of the study were mainly associated to the work of Anosov, Bowen, Franks, Shub, Smale, Sullivan and Williams and good part of those can be appreciated in the nice book of Franks [F_4]. It is worth mentioning that, for different reasons, people working in this aspects of differentiable dynamics abandoned the subject and this may be an explanation on why these results are less popular.

However, the program of Smale, as well as the hope that structurally stable systems should be typical among diffeomorphisms of a manifold fell down after some examples of robustly non-hyperbolic dynamics started to appear. The first non hyperbolic examples were those of Abraham-Smale ([AS]) and Newhouse ([New_1]).

This gave rise to the theory of bifurcations, where Newhouse, Palis and Takens (among others) were pioneers and after many work and new examples the initial program was finally adapted in order to contemplate these new examples and to maintain the initial philosophy of Smale. Palis' program [Pa_3], however, has only a semilocal point of view.

After the paradigm of hyperbolicity began to fall, the research started focusing on finding alternative notions, such as non-uniform hyperbolicty (mainly by the Russian school, of which the principal contributors were Pesin and Katok) or the partial hyperbolicity (independently by Hirsch-Pugh-Shub [HPS] and Brin-Pesin [BrPe]). In this thesis, we are mainly interested in the second generalization of hyperbolicity for its condition of geometric structure (in contrast with the measurable structure given by non-uniform hyperbolicity) and its strong relationship with robust dynamical properties. See [BDV] for a panorama on dynamics beyond hyperbolicity.

In his quest for a proof of the stability conjecture, Mañe (independently also Pliss and Liao [Pli, L]) introduced the concept of dominated splitting and showed its close relationship with the dynamics of the tangent map over periodic orbits.

When one studies the space of diffeomorphisms with the C^1-topology the perturbation techniques developed since the 60's by Pugh, Mañe, Hayashi and more recently by Bonatti and Crovisier imply that the periodic orbits capture in a good way (topological and statistical) the dynamics of generic diffeomorphisms. See [C_4] for a survey on this topics.

Recently, Bonatti [B] has proposed a realistic program for the study of the dynamics of C^1-generic diffeomorphisms which extends Palis' program and complements it. It is also the case that this program has a semilocal flavour.

From the global point of view, there is much less work done, and also less proposals on how to proceed (see [PS_2] section 5 for a short survey) although there are some ideas on how to proceed in some cases at least in dimension 3.

In what follows, we will try to present the contributions of this thesis and explain how our results fit in this subjective account of the development of differentiable dynamics.

0.1.2 Attractors in C^1-generic dynamics

It is always possible to decompose the dynamics of a homeomorphism of a compact metric space into its chain-recurrence classes. This is the content of Conley's theory [Co].

This decomposition has proven very useful in the understanding of C^1-generic dynamics[2] thanks to a result by Bonatti and Crovisier ([BC]) which guaranties that it is possible to detect chain recurrence classes of a generic diffeomorphism by its periodic orbits. In a certain sense, the dynamics around periodic orbits has attracted most of the attention in the study of semi-local properties of generic diffeomorphisms and it is hoped that by understanding their behavior one will be able to understand C^1-generic dynamics (see [B]).

If we wish to understand the dynamics of most of the orbits, there are some chain-recurrence classes which stand out from the rest. *Quasi-attractors* are chain-recurrence classes which admit a basis of neighborhoods U_n verifying that $f(\overline{U_n}) \subset U_n$. Such classes always exist, and it was proven in [BC] that, for C^1-generic diffeomorphisms, there is a residual subset of points in the manifold whose forward orbit accumulates in a quasi-attractor.

Sometimes, it is possible to show that these quasi-attractors are isolated from the rest of the chain-recurrence classes and in this case, we say that they are *attractors*. Attractors have the property of being accumulated by the future orbit of nearby points and being dynamically indecomposable. To determine whether attractors exist and the topological and statistical properties of their basins is one of the main problems in non-conservative dynamics. In dimension two, it is possible to show that C^1-generic diffeomorphisms have attractors. This was originally shown by Araujo [Ara] but there was a gap in the proof and this was never published[3]. This result was in a certain way incorporated to the folklore (see for example [BLY]). In this thesis, we present a proof of the following result which appeared in [Pot$_2$] (see Section 3.1).

Theorem. *There exists an open and dense subset $\mathcal{U}$ of the space of C^1-diffeomorphisms of a surface M such that every $f \in \mathcal{U}$ has a hyperbolic attractor. Moreover, if f cannot be perturbed in order to have infinitely many attracting periodic orbits, then every quasi-attractor of f is a hyperbolic attractor and there are finitely many quasi-attractors.*

When a quasi-attractor is hyperbolic, it must be an attractor. To show that when a diffeomorphism has robustly finitely many attracting periodic orbits, all quasi-attractors are hyperbolic it is a key step to show that they admit what is called a *dominated splitting*. This means that there exists a Df-invariant splitting of the tangent bundle over the quasi-attractor into two subbundles which verify a uniform condition of domination (vectors in one subbundle are uniformly less contracted than on the other).

[2] We will use this expression to refer to diffeomorphisms belonging to a residual subset of $\text{Diff}^1(M)$ with the C^1-topology.

[3] See [San] for a very recent correction of the original proof with the use of the results of Pujals and Sambarino [PS$_1$].

Dominated splitting, as well as many other Df-invariant geometric structure, is an important tool for studying the dynamical properties of a chain-recurrence class, and fundamentally, to understand how the class is accumulated by other classes (see Section 1.2).

Aiming at the understanding of the dynamics close to quasi-attractors for C^1-generic diffeomorphisms, we have obtained the following partial result about the structure of those quasi-attractors which are homoclinic classes (see Section 3.2 and [Pot_1]):

Theorem. *Let f be a C^1-generic diffeomorphism and H be a quasi-attractor which contains a periodic orbit p such that the differential of f over p at the period contracts volume. Then, H admits a non-trivial dominated splitting.*

From the point of view of the conclusion of this theorem, it is possible to see by means of examples that the conclusion is in some sense optimal (see [BV]). The same happens with the hypothesis of having a periodic orbit (see [BD_3]). However, the hypothesis on the dissipation of volume along a periodic orbit seems to follow from the fact that H is a quasi-attractor but we were not able to prove this. Proving this seems to require what is known as an *ergodic closing lemma inside a homoclinic class* which is a problem not well understood for the moment (see [B] Conjecture 2).

The main novelty in the proof of this result is the use of a new perturbation result due to Gourmelon [Gou_3] which allows to perturb the differential over a periodic orbit while keeping control on its invariant manifolds. The use of this result combined with Lyapunov stability has allowed us to solve the problem of guaranteeing that a point remains in the class after perturbation. The result responds affirmatively to a question posed in [ABD] (Problem 5.1).

The dream of having C^1-generic dynamics admitting attractors[4] has fallen recently due to a surprising example presented in [BLY] which shows how the recent development of the theory of C^1-generic dynamics has had an important influence in the way we understand dynamics and has simplified questions which seemed unapproachable.

The examples of [BLY] posses what they have called *essential attractors* (see subsection 1.1.5) and it is not yet known whether they posses attractors in the sense of Milnor. In Section 3.3 we review their examples and present new examples from [Pot_3] on which we have a better understanding on how other classes approach their dynamics:

Theorem. *There exists an open set $\mathcal{U}$ of* $\text{Diff}^1(\mathbb{T}^3)$ *of diffeomorphisms such that:*

[4] See the introduction of [BLY] for more historic account on the problem.

- *If $f \in \mathcal{U}$ then f has a unique quasi-attractor and an attractor in the sense of Milnor. Moreover, if f is C^2, then it has a unique SRB measure whose basin has total Lebesgue measure.*

- *For f in a C^r-residual subset of $\mathcal{U}$ we know that f has no attractors.*

- *Chain-recurrence classes different from the quasi-attractor are contained in periodic surfaces.*

The last point of the theorem contrasts with the new results obtained by Bonatti and Shinohara ([BS]). In Section 3.4 we speculate on how both results can fit in the same theory.

0.1.3 Partial hyperbolicity in $\mathbb{T}^3$

In the previous section we have discussed problems which are of semilocal nature (although it is of course possible to ask questions of global nature about attractors and their topology). In this section we treat results of global dynamics.

A well known result in differentiable dynamics, which joins classical results by Mañe ([M_3]) and Franks ([F_1]) can be stated as follows:

Theorem (Mañe-Franks)**.** *Let M be a compact surface. The following three properties for $f \in \mathrm{Diff}^1(M)$ are equivalent:*

(i) f is C^1-robustly transitive.

(ii) f is Anosov.

(iii) f is Anosov and conjugated to a linear Anosov automorphism.

Mañe proved that (i) $\Rightarrow$ (ii) while Franks had proven (ii) $\Rightarrow$ (iii). Robustness of Anosov diffeomorphisms and the fact that linear Anosov automorphisms are transitive gives (iii) $\Rightarrow$ (i).

If we interpret being an Anosov diffeomorphism as having a Df-invariant geometric structure, we can identify the result (i) $\Rightarrow$ (ii) as saying: "an robust dynamical property forces the existence of a Df-invariant geometric structure". In fact, since C^1-perturbations cannot break the dynamical behavior, it is natural to expect that this geometric structure will posses certain rigidity properties.

On the other hand, the direction (ii) $\Rightarrow$ (i) can be thought as a converse statement, showing that Df-invariant geometric structures may imply the existence of certain robust dynamical behavior, in this case, transitivity.

In higher dimensions, the understanding of the relationship between robust dynamical properties and Df-invariant geometric structures is quite less advanced although results in the direction of obtaining a Df-invariant geometric structure from a robust dynamical property do exist. In dimension 3, it follows from a result of [DPU] and this was generalized to higher dimensions by [BDP] (see Section 1.2):

Theorem (Diaz-Pujals-Ures)**.** *Let M be a 3-dimensional manifold and f a C^1-robustly transitive diffeomorphism, then, f is partially hyperbolic.*

The definition of partial hyperbolicity varies throughout the literature and time. We use the definition used in [BDV] which is the one that fits best our approach (see Section 1.2 for precise definitions). A partially hyperbolic diffeomorphism for us will be one which preserves a splitting of the tangent bundle $TM = E \oplus F$ verifying a domination property between the bundles and such that one of them is uniform. For notational purposes, we remove the symmetry of the definition and work with partially hyperbolic diffeomorphisms of the form $TM = E^{cs} \oplus E^u$ where E^u is uniformly expanded.

The first difficulty one encounters when trying to work in the converse direction of the previous theorem is the fact that one has no control on the contraction in the direction E^{cs} other that it is dominated by the expansion in E^u. This forbids us to gain a complete control on the dynamics in that direction as we have in the hyperbolic case. One notable exception is the work of [PS_6] where precise dynamical consequences are obtained from the existence of a dominated splitting in dimension 2.

It is now also time to mention the importance of item (iii) in Mañe-Franks' Theorem which we have neglected so far. In a certain sense, the underlying idea is that in order to obtain a robust dynamical property out of the existence of a Df-invariant geometric structure it can be important to rely on the topological restriction this geometric structure imposes, such as the topology of the manifold or the isotopy class of the diffeomorphism. It is for this reason, and the difficulties that have appeared in the attempt to obtain results in converse direction of Diaz-Pujals-Ures' Theorem that it seems for us a good idea to divide the study in something in the spirit of (ii) $\Rightarrow$ (iii) and (ii)+(iii) $\Rightarrow$ (i) as in Mañe-Franks' Theorem. This means that the study of partially hyperbolic dynamics in a fixed manifold or even in a fixed isotopy class seems to be an important step in the understanding of these relations.

Also in the direction of obtaining results giving topological properties by the existence of a Df-invariant geometric structure the difficulty increases considerably as we raise the dimension. In dimension 2, the sole fact of admitting an continuous line field imposes strong restrictions on the topology of the manifold. In dimension 3, it is well known that every 3-dimensional manifold admits a non-vanishing vector field and moreover, it also admits a codimension one foliation.

This situation may be considered as very bad from the point of view of finding topological properties out of the existence of Df-invariant geometric structures. However, very recently, a beautiful remark by Brin-Burago-Ivanov ([BBI_1, BI]) has renewed the hope:

Remark (Brin-Burago-Ivanov). In a 3-dimensional manifold, if $\mathcal{F}$ is a foliation transverse to the unstable direction of a partially hyperbolic diffeomorphism, then $\mathcal{F}$ has no Reeb components.

♢

Reeb components, and its relationship with partially hyperbolic dynamics had already been studied in [DPU] (Theorem H) and [BWi] (Lemma 3.7) in more restrictive contexts (assuming dynamical coherence and transitivity). This remark is much more general, its strength relies in that the only dependence on the dynamics is in the fact that the unstable foliation cannot have closed curves.

On the one hand, it is known that many 3-manifolds do not admit foliations without Reeb components. On the other, there exists quite a lot of theory regarding its classification (see for example [Pla, Rou]) and therefore, we can expect that progress in the classification of partially hyperbolic diffeomorphisms is within reach.

Unfortunately, another difficulty arises: it is not known if every partially hyperbolic diffeomorphism in a 3-dimensional manifold posses a foliation transverse to the unstable direction. In this thesis, we propose the notion of *almost dynamical coherence* which we show is an open and closed property among partially hyperbolic diffeomorphisms and expect that under this hypothesis more progress can be made.

One of our main results (see Chapter 5 and [Pot_5]) guaranties that in certain isotopy classes of diffeomorphisms of $\mathbb{T}^3$ partial hyperbolicity and almost dynamical coherence are enough to guarantee the existence of a f-invariant foliation tangent to E^{cs}.

Theorem. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an almost dynamically coherent partially hyperbolic diffeomorphism isotopic to a linear Anosov automorphism. Then, f is dynamically coherent.*

Being *dynamically coherent* means that the bundle E^{cs} integrates into an f-invariant foliation.

In the *strong partially hyperbolic* case (i.e. where $TM = E^s \oplus E^c \oplus E^u$ is a Df-invariant splitting with domination properties and E^s and E^u are uniform) we can say more.

As a starting point, in a remarkable paper [BI] it was proved that every strong partially hyperbolic diffeomorphism is almost dynamically coherent. This was used first by [BI] (following a very simple and elegant argument of [BBI_1]) and then in [Par] to give topological conditions these must satisfy. On the other hand, these progress makes expectable that, at least in some special cases, the following conjecture of Pujals may be within reach:

Conjecture (Pujals [BWi])**.** *Let $f : M \to M$ with M a 3-dimensional manifold a strong partially hyperbolic diffeomorphism which is transitive. Then, one of the following possibilities holds (modulo considering finite lifts):*

- *f is leaf conjugate to a linear Anosov automorphism of $\mathbb{T}^3$.*
- *f is leaf conjugate to a skew-product over a linear Anosov automorphism of $\mathbb{T}^2$ (and so $M = \mathbb{T}^3$ or a nilmanifold).*
- *f is leaf conjugate to the time one map of an Anosov flow.*

There has been some progress in the direction of this conjecture lately. Let us mention first the work of [BWi] which makes considerable progress. They work without making any hypothesis on the topology of the manifold but they demand the existence of a closed curve tangent to the center direction and some other technical hypothesis. Then, the work of Hammerlindl [H, H_2] has given a proof of the conjecture when the manifold is $\mathbb{T}^3$ or a nilmanifold by assuming a more restrictive notion of partial hyperbolicity (partial hyperbolicity with absolute domination). Although this notion is verified by many examples, it is in some sense artificial and does not fit well with the results of [DPU].

Of course, to prove Pujals' conjecture, a previous step must be to show dynamical coherence of such diffeomorphisms since leaf conjugacy requires this for a start (see Section 1.4). The work of Hammerlindl relies heavily on previous work by Brin-Burago-Ivanov [BBI_2] who have established dynamical coherence of strong partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ under this more restrictive notion of partial hyperbolicity we mentioned above.

While one could expect the use of this restrictive notion to be mainly technical, a recent example of Rodriguez Hertz-Rodriguez Hertz-Ures ([$RHRHU_3$]) of a non-dynamically coherent strong partially hyperbolic diffeomorphism of $\mathbb{T}^3$ shows that the passage to the general definition should at least use the transitivity hypothesis in a fundamental way and that some difficulties must be addressed. We have completed the panorama ([Pot_5]) by showing:

Theorem. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a strong partially hyperbolic diffeomorphism which does not admit neither a periodic normally attracting torus nor a periodic normally repelling torus, then f is dynamically coherent.*

This theorem responds to a conjecture made by Rodriguez Hertz-Rodriguez Hertz-Ures in $\mathbb{T}^3$ and it also allows one to prove Pujals' conjecture for $M = \mathbb{T}^3$ by further use of the techniques in the proof (see [HP]).

In the end of Chapter 5 we also obtain some results in higher dimensions which are analogous to the classical results of Franks, Newhouse and Manning for Anosov diffeomorphisms in the context of partial hyperbolicity.

0.1.4 Other contributions

In this section we briefly describe other contributions of this thesis.

In Section 2.2 we present a mechanism from [Pot_3] for the localization of chain-recurrence classes which we consider has intrinsic value since it can be applied in many contexts (in this thesis, it is applied in Section 4.A as well as in subsections 3.3.2 and 3.3.3).

In Section 3.3 we present several examples of quasi-attractors and of robustly transitive diffeomorphisms some of which are modifications of well known examples but we consider they may contribute to the understanding of these phenomena.

Then, in Chapter 4 we present results on foliations which we use later in Chapter 5 which we believe may have independent interest. In particular, we mention a quantitative result about the existence of a global product structure of certain transverse foliations which is presented in Section 4.3. Also, in Section 4.A we give a characterization of dynamics of globally partially hyperbolic diffeomorphisms of $\mathbb{T}^2$ to show the techniques we use later in Chapter 5.

We also include in this thesis 4 appendices where some results which we preferred to separate from the main line of the thesis are presented. We make particular emphasis on Appendix C based on [Pot_4] where we prove a result about homeomorphisms of $\mathbb{T}^2$ with a unique rotation vector and Appendix D based on [BCGP] where we present a joint work with Bonatti, Crovisier and Gourmelon. This last work studies the bifurcations of robustly isolated chain-recurrence classes and gives examples of such classes which are not robustly transitive answering to a question posed in [BC].

0.2 Introducción (Español)

0.2.1 Contexto histórico

Se puede[5] decir que el objetivo fundamental de los sistemas dinámicos es comprender el comportamiento asintótico de un estado sujeto a una ley de evolución. Se comenzó por el estudio de ecuaciones diferenciales ordinarias del tipo

[5] Vale aclarar que la introducción histórica que se presentará es subjetiva y no necesariamente refleja con exactitud los hechos históricos. Se puede pensar que lo que sigue es una historia plausible que explica algunas razones por las cuales estudiar los temas aquí presentados.

$$\dot{x} = X(x) \qquad X : \mathbb{R}^n \to \mathbb{R}^n$$

donde se buscaba una solución analítica a la ecuación: buscando para cada valor posible de condición inicial $x_0 \in \mathbb{R}^n$ una solución explícita $\varphi_t(x_0)$.

Rápidamente se vió que ecuaciones extremadamente simples daban lugar a soluciones complicadas que incluso luego de ser integradas tampoco aportaban a la comprensión de la ley de evolución.

Poincaré, interesado en estudiar el famoso problema de los 3-cuerpos ([Po]) fue quizás el primero en proponer que el estudio de la evolución debería ser cualitativo, y de alguna manera propuso estudiar "el comportamiento de la *mayoría* de las órbitas para la *mayoría* de los sistemas".

El comienzo del estudio se centró en la estabilidad. Lyapunov estudió las órbitas estables, órbitas que contienen un entorno de puntos donde el comportamiento es escencialmente el mismo. Por otro lado, Andronov y Pontryagin seguidos por Peixoto, estudiaron sistemas estables, es decir, aquellos cuyas perturbaciones verifican que su estructura dinámica es la misma. Fue quizás Smale ([Sm_2]) el primero en retomar la propuesta de Poincaré dándole una formulación más precisa:

El objetivo es fijar una variedad cerrada M de dimensión d y entender la dinámica de un subconjunto *grande* de $\text{Diff}^r(M)$, el conjunto de difeomorfismos de M munido de la topología de la convergencia uniforme hasta orden r.

Además, propuso que un conjunto fuese considerado grande si era abierto y denso, o en su defecto residual o denso (de esta forma conjunto abierto de sistemas puede ser despreciado). No se hará mención a otras posibles formas de considerar un subconjunto como grande ni se discutirán las razones por las cuales considerar estas nociones (referimos al lector a [B] o [C_4] por más fundamentación).

Smale propuso un programa que se centró en el estudio de la estabilidad estructural y en la esperanza de que si bien las dinámicas típicas podían no ser estables (podían ser caóticas) estas serían estables desde el punto de vista de que sus propiedades dinámicas persistirían frente a perturbaciones. De esta forma, se podría describir la dinámica a través de métodos simbólicos o estadísticos. Palis y Smale [PaSm] conjeturaron que los sistemas estructuralmente estables coincidían con los difeomorfismos hiperbólicos.

La hiperbolicidad fue entonces el paradigma. Robbin y Robinson ([R, Rob_1]) probaron que los sistemas hiperbólicos eran estables. Finalmente Mañe en un célebre resultado ([M_5]) completo la caracterización de la las dinámicas estables en topología C^1 (en otras topologias aún es desconocido). Describir la dinámica de los difeomorfismos hiperbólicos fue la tarea que concentró la mayor atención en esos años 60 y principios de los 70.

Desde el punto de vista de comprender la dinámica de los difeomorfismos hiperbólicos el proyecto fue sumamente exitoso. No sólo se logró comprender cabalmente los aspectos semilocales de su dinámica (a traves del estudio simbólico y ergódico de sus propiedades), sino que se obtuvieron resultados profundos acerca de aspectos globales y de la topología de sus piezas básicas.

Sobre los aspectos semilocales, sin ser exhaustivos, se hace mención a las contribuciones de Bowen, Newhouse, Palis, Sinai, Ruelle y el propio Smale. Se refiere al lector a [KH] Part 4 por una visión panorámica de gran parte de la teoría.

Por otro lado, los aspectos globales fueron fundamentalmente asociados a los trabajos de Anosov, Bowen, Franks, Shub, Smale, Sullivan y Williams y buena parte de estos se pueden apreciar en el libro [F_4] que contiene una muy linda presentación de los trabajos conocidos acerca de la dinámica global de difeomorfismos hiperbólicos. Por diferentes razones estas personas abandonaron estos temas, lo que puede ser una explicación de por qué estos resultados son menos conocidos.

El programa de Smale, así como la esperanza de que los sistemas estructuralmente estables fueran típicos en el espacio de difeomorfismos de una variedad, cayó cuando empezaron a aparecer ejemplos de dinámicas robustamente no hiperbólicas y no estables como los de Abraham-Smale ([AS]) y de Newhouse ([New_1]).

Estos ejemplos dieron lugar a la teoría de bifurcaciones donde Newhouse, Palis y Takens (entre otros) fueron pioneros. Luego de muchos trabajos al respecto los programas iniciales fueron adaptados para contemplar dichas bifurcaciones manteniendo la filosofía inicial de Smale. El programa de Palis [Pa_3] sin embargo, tiene un enfoque principalmente semilocal.

Caído el paradigma de la hiperbolicidad se empezó a buscar nociones alternativas como la hiperbolicidad no uniforme (por parte de la escuela rusa, fundamentalmente Katok y Pesin, ver [KH] Supplement S), o la hiperbolicidad parcial (independientemente por Hirsch-Pugh-Shub [HPS] y Brin-Pesin [BrPe]). Este trabajo se interesa fundamentalmente por esta segunda generalización dada su condición de estructura geométrica (en contraposición a la condición de propiedad medible de la hiperbolicidad no uniforme), y su fuerte vinculación con las propiedades robustas de la dinámica. Véase [BDV] por un panorama general de la dinámica más allá de la hiperbolicidad.

En su búsqueda de la prueba de la conjetura de estabilidad, Mañe (independientemente lo hicieron también Pliss y Liao [Pli, L]) introdujo el concepto de descomposición dominada, y lo que es más importante, mostró su relación con la dinámica de la aplicación tangente sobre las órbitas periódicas.

Cuando se estudia el espacio de difeomorfismos con la topología C^1, gracias a las técnicas de perturbación de órbitas desarrolladas desde los años 60 por Pugh, Mañe, Hayashi y más recientemente por Bonatti y Crovisier, sabemos que de alguna manera las órbitas periódicas de los difeomorfismos genéricos capturan la dinámica de los difeomorfismos (topológica y estadísticamente). Ver [C_4] por un survey sobre estos temas.

Recientemente Bonatti [B] propuso un programa realista para la comprensión de un conjunto grande de difeomorfismos con la topología C^1. Este programa extiende y complementa el programa general de Palis ya mencionado. También en este caso el programa tiene un punto de vista semilocal.

Desde el punto de vista global hay menos trabajo hecho y menos propuestas de trabajo (ver [PS_2] section 5), aunque en dimensión tres hay algunas ideas de cómo proceder en ciertos casos.

En lo que sigue, se presentan las contribuciones de esta tesis pretendiendo mostrar como éstas encajan en este panorama subjetivo del desarrollo de la teoría.

0.2.2 Atractores en dinámica C^1-genérica

Siempre es posible descomponer la dinámica de un homeomorfismo de un espacio métrico compacto en sus clases de recurrencia. Este es el contenido de la teoría de Conley [Co].

Esta descomposición es muy útil en la comprensión de las dinámicas C^1-genéricas[6], debido a un resultado de Bonatti y Crovisier ([BC]) que garantiza que es posible detectar las clases de recurrencia de un difeomorfismo genérico a partir de sus órbitas periódicas. En buena medida, la comprension de la dinámica alrededor de dichas órbitas periódicas se lleva toda la atención y se espera que puedan describir la dinámica de dichos difeomorfismos (ver [B]).

Cuando queremos entender la dinámica de la mayoría de los puntos, existen clases de recurrencia que se destacan sobre otras. Los *quasi-atractores* son clases de recurrencia que admiten una base de entornos U_n que verifican que $f(\overline{U_n}) \subset U_n$. Estas clases siempre existen. Fue probado en [BC] que existe un conjunto residual de puntos de la variedad que convergen a dichas clases cuando se trata de un difeomorfismo genérico.

Algunas veces es posible demostrar que estos quasi-atractores están aislados del resto de las clases de recurrencia, en ese caso, decimos que son *atractores*. Los atractores tienen la propiedad de ser acumulados por la órbita futura de los puntos cercanos y ser dinámicamente indescomponibles. Determinar la existencia de atractores y sus propiedades topológicas y estadísticas es uno de los grandes problemas en sistemas dinámicos. En dimensión dos es posible demostrar que para la mayor parte de los sistemas dinámicos, vistos con la topología C^1, existen atractores. Esto fue demostrado originalmente por Araujo [Ara] aunque la demostración contenía un

[6] Utilizaremos esta expresión para referirnos a difeomorfismos pertenecientes a un conjunto residual de $\text{Diff}^1(M)$ con la topología C^1.

error y nunca fue publicada[7]. El resultado fue de alguna manera incorporado al folklore (ver por ejemplo [BLY]). Esta tesis presenta una prueba del siguiente resultado aparecida por primera vez en [Pot_2] (ver sección 3.1).

Teorema. *Existe un conjunto abierto y denso $\mathcal{U}$ del espacio de difeomorfismos de una superficie con la topología C^1 tal que si $f \in \mathcal{U}$ entonces f tiene un atractor hyperbólico. Más aún, si f no puede ser perturbado para tener infinitos puntos periódicos atractores (pozos), entonces f y sus perturbados verifican que poseen finitos quasi-atractores y estos son atractores hiperbólicos.*

La hiperbolicidad de un quasi-atractor garantiza que este debe ser un atractor. Para mostrar que cuando hay robustamente finitos pozos todos los quasi-atractores son hiperbólicos, es clave demostrar que dichos quasi-atractores poseen lo que se llama una *descomposición dominada*: Existe una descomposición del fibrado tangente sobre el quasi-atractor en dos subfibrados Df-invariantes que verifican una condición uniforme de dominación de uno sobre el otro (la contracción de los vectores en uno de los fibrados es uniformemente menor que en el otro fibrado).

La descomposición dominada, así como otras varias posibles estructuras geométricas Df-invariantes, es una herramienta importante para el estudio de las propiedades dinámicas de una clase de recurrencia, y fundamentalmente, para entender como dicha clase es acumulada por otras clases (ver Sección 1.2).

Buscando comprender la dinámica cerca de los quasi-atractores para las dinámicas C^1-genéricas, se obtuvo el siguiente resultado parcial acerca de la estructura de aquellos quasi-atractores que son clases homoclínicas (ver Sección 3.2 y [Pot_1]):

Teorema. *Para un difeomorfismo C^1 genérico f, si H es un quasi-attractor que contiene un punto periódico p tal que el diferencial de f sobre p en el período contrae volumen, entonces H admite una descomposición dominada no trivial.*

Es posible ver mediante ejemplos ([BV]) que la conclusión del teorema es en cierto sentido óptima. La hipótesis de que existan puntos periódicos en la clase es también necesaria (ver [BD_3]). Sin embargo, la hipótesis acerca de la disipatividad del diferencial sobre la órbita periódica parece ser consecuencia de las otras hipótesis, pero no fue posible eliminarla. Demostrarlo parecería necesitar de algún resultado del tipo *ergodic closing lemma en la clase homoclínica* que es un problema que aún no se logra entender correctamente (ver [B] Conjecture 2).

La mayor novedad en la prueba del teorema es que se utiliza un nuevo resultado perturbativo debido a Gourmelon [Gou_3] que permite perturbar el diferencial de una órbita periódica con cierto control de las variedades invariantes luego de la

[7] Ver [San] por una corrección a la prueba original utilizando los resultados de Pujals y Sambarino [PS_1].

perturbación. El uso de dicho resultado combinado con la estabilidad Lyapunov nos permite resolver el problema de garantizar que un punto pertenece a la clase luego de la perturbación. El resultado responde positivamente a una pregunta realizada en [ABD] (Problem 5.1).

El sueño de que las dinámicas C^1-genéricas tuviesen atractores[8] cayó recientemente debido a un ejemplo sorprendente debido a [BLY] que muestra como el desarrollo reciente de la teoría de la dinámica C^1-genérica ha tenido una gran influencia en la forma de entender la dinámica y ha simplificado preguntas que resultaban a simple vista inabordables.

Los ejemplos de [BLY] poseen lo que llamaron *atractores esenciales* (ver subsección 1.1.5) y no es aún sabido si poseen atractores en el sentido de Milnor. En la Sección 3.3 revemos estos ejemplos y presentamos algunos ejemplos nuevos de [Pot$_3$] sobre los cuales tenemos una mejor comprensión de cómo las otras clases se acumulan a su dinámica.

Teorema. *Existe un abierto $\mathcal{U}$ de $\mathrm{Diff}^1(\mathbb{T}^3)$ de difeomorfismos tales que:*

- *Si $f \in \mathcal{U}$ entonces f tiene un único quasi-atractor y un atractor en el sentido de Milnor. Además, si f es de clase C^2 posee una única medida SRB cuya cuenca es de medida total.*

- *Para f en un residual de $\mathcal{U}$ se verifica que f no tiene atractores.*

- *Las clases de recurrencia diferentes del quasi-atractor están contenidas en superficies periódicas.*

El último punto del teorema entra en contraste con los nuevos resultados obtenidos por Bonatti y Shinohara ([BS]) y en la Sección 3.4 especulamos acerca de cómo ambos resultados podrían llegar a entrar en una misma teoría.

0.2.3 Hiperbolicidad parcial en el toro $\mathbb{T}^3$

Así como la sección anterior trato implícitamente problemas que son de naturaleza semilocal (a pesar de que se pueden hacer preguntas de índole global acerca de la existencia de atractores y de su topología) esta sección trata fundamentalmente acerca de problemas de dinámica global.

Un conocido teorema en dinámica diferenciable, que reune resultados clásicos de Mañe ([M$_3$]) y Franks ([F$_1$]) dice lo siguiente:

Teorema (Mañe-Franks)**.** *Sea M una superficie compacta. Entonces, las tres propiedades siguientes para $f \in \mathrm{Diff}^1(M)$ son equivalentes:*

[8] Ver la introducción de [BLY].

(i) f es C^1-robustamente transitivo.

(ii) f es Anosov.

(iii) f es Anosov y conjugado a un difeomorfismo de Anosov lineal.

Escencialmente, Mañe probó la implicancia (i) $\Rightarrow$ (ii) y Franks la implicancia (ii) $\Rightarrow$ (iii). La robustez de los Anosov y la transitividad de los Anosov lineales da (iii) $\Rightarrow$ (i).

Si se interpreta el ser difeomorfismo de Anosov como que el diferencial de f preserve una estructura geométrica, podemos de alguna manera identificar el resultado (i) $\Rightarrow$ (ii) como diciendo que "una propiedad dinámica robusta fuerza la existencia de una estructura geométrica Df-invariante". De hecho, en vista que las perturbaciones C^1 no pueden romper la propiedad dinámica, no es sorprendente que dicha estructura geométrica tenga propiedades de rigidez frente a perturbaciones.

Por otro lado, la dirección (ii) $\Rightarrow$ (i) se puede ver como diciendo que la existencia de una estructura geométrica invariante está también relacionada a la existencia de una propiedad dinámica robusta, en este caso, la transitividad.

En dimensiones mayores la relación entre las propiedades dinámicas robustas y las estructuras geométricas invariantes está menos desarrollada, aunque existen resultados en la dirección de obtener una estructura geométrica invariante a partir de una propiedad dinámica robusta. En dimensión 3, esto surge de [DPU] y fue generalizado a dimensiones mayores en [BDP] (ver Sección 1.2):

Teorema (Diaz-Pujals-Ures)**.** *Si M es una variedad de dimensión 3 y f es un difeomorfismo C^1-robustamente transitivo, entonces, f es parcialmente hiperbólico.*

Existen diversas definiciones de hiperbolicidad parcial en la literatura, y estas también han variado a lo largo del tiempo. Nosotros seguimos la definición que se utiliza en [BDV] que es la que mejor se ajusta a nuestro enfoque (ver Sección 1.2 por definiciones precisas). Un difeomorfismo parcialmente hiperbólico sera uno que verifica que el fibrado tangente se descompone en una suma Df-invariante $TM = E \oplus F$ que verifica una propiedad de dominación y tal que uno de los dos fibrados es uniforme. Por simplicidad, eliminamos la simetría y consideramos descomposiciones del tipo $TM = E^{cs} \oplus E^u$ con E^u uniformemente expandido.

Una primera dificultad que aparece si nos interesamos en entender las propiedades dinámicas robustas implicadas por la existencia de una descomposición parcialmente hiperbólica, es el no tener control de la contracción en la dirección centro estable E^{cs}. Esto impide que tengamos una comprensión cabal de la dinámica en esa dirección, como si la tenemos en el caso donde los fibrados son uniformes. Una excepción notable a esto es [PS_6] donde se estudian las consecuencias dinámicas de la descomposición dominada en dimensión 2.

Es importante mencionar también, la importancia del ítem (iii) en el Teorema de Mañe y Franks. De alguna manera, la idea que subyace es que para obtener una propiedad dinámica robusta a partir de la existencia de una estructura geométrica invariante, puede ser importante apoyarse en las propiedades topológicas impuestas por dicha estructura, tanto en la topología de la variedad como en la clase de isotopía del difeomorfismo. Es por ello que en vista de ese teorema, y de la dificultad que se ha tenido para obtener resultados en la dirección recíproca al Teorema de Diaz-Pujals-Ures, puede ser importante dividir el estudio en buscar resultados del tipo (ii) $\Rightarrow$ (iii) y del tipo (ii)+(iii) $\Rightarrow$ (i) emulando el Teorema de Mañe y Franks. En particular puede ser importante estudiar propiedades de difeomorfismos parcialmente hiperbólicos en ciertas variedades, o incluso en clases de isotopía fijadas.

Otra dificultad en dimensión 3 es obtener propiedades topológicas a partir de las estructuras invariantes. En dimensión 2, el solo hecho de preservar un campo de vectores impone enormes restricciones en la topología de la variedad. En dimensión 3, es bien sabido que toda variedad admite un campo de vectores no nulo, e incluso, una foliación de codimensión 1.

Esta situación podría ser considerada muy mala desde el punto de vista de obtener resultados en el sentido de encontrar propiedades topológicas de un difeomorfismo que preserva una estructura geométrica. Sin embargo, recientemente, una simple pero poderosa observación de Brin-Burago-Ivanov ([BBI_1, BI]) despertó nuevamente la esperanza:

Observación (Brin-Burago-Ivanov). En una variedad de dimensión 3, si $\mathcal{F}$ es una foliación transversal a la dirección inestable E^u de un difeomorfismo parcialmente hiperbólico f, entonces $\mathcal{F}$ no tiene componentes de Reeb.

$\diamondsuit$

Las componentes de Reeb y su relación con los difeomorfismos parcialmente hiperbólicos ya había sido estudiada, por ejemplo en [DPU] (Theorem H) y [BWi] (Lemma 3.7) en contextos más restrictivos (asumiendo coherencia dinámica y transitividad). Esta observación es más general, su fuerza radica en el hecho que depende de la dinámica sólo en que la foliación inestable no tiene curvas cerradas.

Por un lado, es conocido que diversas variedades no admiten foliaciones sin componentes de Reeb. Por otro lado, existe mucha teoría acerca de su clasificación (ver por ejemplo [Pla, Rou]) y por lo tanto, de alguna manera nos hace esperar que es posible entender los difeomorfismos parcialmente hiperbólicos al menos en cierto grado.

Por otro lado, aparece una nueva dificultad ya que no es conocido si todo difeomorfismo parcialmente hiperbólico posee una foliación transversal a la dirección inestable. En esta tesis se propone la noción de *casi coherencia dinámica* y se prueba que es una propiedad abierta y cerrada en el espacio de difeomorfismos parcialmente hiperbólicos.

Uno de los teoremas principales (ver Capítulo 5 y [Pot_5]) garantiza en ciertas clases de isotopía de difeomorfismos de $\mathbb{T}^3$ que dicha noción es suficiente para que el fibrado E^{cs} sea tangente a una foliación invariante:

Teorema. *Sea $f : \mathbb{T}^3 \to \mathbb{T}^3$ un difeomorfismo parcialmente hiperbólico que es casi dinamicamente coherente y es isotópico a un difeomorfismo de Anosov lineal. Entonces f es dinámicamente coherente.*

Ser *dinámicamente coherente* significa justamente que el fibrado E^{cs} sea tangente a una foliación f-invariante.

En el caso *parcialmente hiperbólico fuerte* (es decir, donde $TM = E^s \oplus E^c \oplus E^u$ es una descomposición Df-invariante con propiedades de dominación y donde E^s y E^u son uniformes) podemos decir más.

En [BI] fue probado que todo difeomorfismo parcialmente hiperbólico fuerte es casi-dinámicamente coherente y esto fue aprovechado por, primero [BI] (siguiendo [BBI_1]) y luego [Par] para dar condiciones topológicas que estos deben satisfacer. Por otro lado, esta condición permite esperar que una conjetura de Pujals sea atacable:

Conjetura (Pujals [BWi]). *Sea $f : M \to M$ con M variedad de dimensión 3 un difeomorfismo parcialmente hiperbólico fuerte y transitivo. Entonces, tenemos las siguientes posibilidades (módulo considerar levantamientos finitos):*

- *f es conjugado por hojas a un difeomorfismo de Anosov en $\mathbb{T}^3$.*
- *f es conjugado por hojas a un skew-product sobre un difeomorfismo de Anosov de $\mathbb{T}^2$ (entonces la variedad es $\mathbb{T}^3$ o una nilvariedad).*
- *f es conjugado por hojas al tiempo 1 de un flujo de Anosov.*

Últimamente ha habido progreso en la dirección de esta conjetura. Para empezar, el trabajo de [BWi] hace un avance considerable sin dar ninguna hipótesis acerca de la topología de la variedad asumiendo la existencia de curvas cerradas tangentes a la dirección central. Luego, los trabajos de Hammerlindl [H, H_2] dan una prueba a la conjetura en caso que la variedad sea $\mathbb{T}^3$ o una nilvariedad pero con una definición más restrictiva de hiperbolicidad parcial. Si bien esta definición es verificada por varios ejemplos, es de alguna manera artificial y no encaja bien con el resultado de [DPU].

Por supuesto, para conseguir probar la conjetura de Pujals, un paso previo es mostrar que los difeomorfismos de ese tipo son dinámicamente coherentes ya que la definición de conjugación por hojas (ver Sección 1.4) requiere la existencia de foliaciones invariantes. El trabajo de Hammerlindl se apoya en trabajos previos de Brin-Burago-Ivanov [BBI_2] que muestran que en $\mathbb{T}^3$, con esta definición restrictiva de hiperbolicidad parcial se tiene coherencia dinámica.

Por otro lado, recientemente apareció un ejemplo debido a Rodriguez Hertz-Rodriguez Hertz-Ures ([$RHRHU_3$]) de un difeomorfismo parcialmente hiperbólico de $\mathbb{T}^3$ que no admite foliaciones invariantes. En esta tesis se completa el panorama ([Pot_5]) mostrando que:

Teorema. *Sea $f : \mathbb{T}^3 \to \mathbb{T}^3$ un difeomorfismo parcialmente hiperbólico fuerte que no admite un toro periódico normalmente atractor ni un toro periódico normalmente repulsor, entonces, f es dinámicamente coherente.*

Esto responde a una conjetura de Rodriguez Hertz-Rodriguez Hertz-Ures en $\mathbb{T}^3$ y también, por lo obtenido en la prueba, permite responder a la conjetura de Pujals en $\mathbb{T}^3$ (ver [HP]).

Sobre el final del Capítulo 5 también se obtienen algunos resultados en dimensiones mayores análogos a los clásicos resultados de Franks, Newhouse y Manning para difeomorfismos de Anosov en el contexto de parcialmente hiperbólicos.

0.2.4 Otras contribuciones

En esta sección se describen otras contribuciones de la tesis.

Por un lado, en la Sección 2.2 se presenta un mecanismo para la localización de clases de recurrencia aparecido en [Pot_3] que se ha considerado tiene valor en si mismo ya que puede ser aplicado en diferentes contextos (en esta tesis se utiliza en la Sección 4.A así como en las subsecciones 3.3.2 y 3.3.3).

En la Sección 3.3 se presentan diversos ejemplos de quasi-atractores y de difeomorfismos robustamente transitivos algunos de los cuales son modificaciones de ejemplos conocidos pero igual consideramos que pueden representar un aporte al entendimiento de estos fenómenos.

Luego, en el Capítulo 4 donde se preparan los resultados sobre foliaciones, que luego serán utilizados en el Capítulo 5, se obtienen resultados que pueden tener interés independiente. En particular, vale mencionar un resultado cuantitativo sobre la existencia de estructura de producto global para foliaciones presentado en la Sección 4.3. También, en la Sección 4.A, se da una clasificación de la dinámica de los difeomorfismos globalmente parcialmente hiperbólicos en $\mathbb{T}^2$ que de alguna manera muestra en un contexto sencillo lo que se hará después en el Capítulo 5.

También se incluyen en la tesis cuatro apéndices donde se presentan resultados que están desviados del cuerpo central de la tesis. Se hace particular énfasis en el Apéndice C, basado en [Pot_4], donde se prueba un resultado acerca de homeomorfismos del toro con un único vector de rotación; y en el Apéndice D, basado en [BCGP], donde se presenta un trabajo conjunto con Bonatti, Crovisier y Gourmelon. En ese trabajo se estudian bifurcaciones de clases de recurrencia robustamente aisladas y se dan ejemplos de ese tipo de clases que no son robustamente transitivas, respondiendo así a una pregunta de [BC].

0.3 Introduction (Français)

0.3.1 Contexte historique

On peut[9] dire que l'objectif principal des systèmes dynamiques est de comprendre le comportement assymptotique d'une loi d'évolution avec certaines conditions initiales dans un espace de configurations. Au début, on étudiait les equations différentielles ordinaires du type

$$\dot{x} = X(x) \qquad X : \mathbb{R}^n \to \mathbb{R}^n$$

et on cherche à résoudre analytiquement l'équation, en essayant de trouver, pour chaque valeur possible de la condition initiale $x_0 \in \mathbb{R}^n$, une solution explicite $\varphi_t(x_0)$.

On a vite remarqué qu'à partir d'équations extrémement simples on obtenait des solutions compliquées qui n'aidaient pas à la compréhension de la loi d'evolution, même après être intégrées.

Poincaré, intéréssé par l'étude du fameux problème des 3 corps ([Po]), a été peût-être le premier à proposer que l'étude de l'évolution se fasse du point de vue qualitatif, et d'une certaine façon, il a suggérer d'étudier le comportement de la "plupart" des orbites pour la "plupart" des systèmes.

Au début, la recherche se centrait autour de la stabilité du système. Lyapunov a étudié les orbites stables, qui contiennent un voisinage de points où le comportement est essentiellement le même. D'autre part, Andronov et Pontryagin, suivis par Peixoto, ont étudié les systèmes stables, c'est à dire, ceux dont la structure dynamique ne change pas sous des perturbations. C'est peût-être Smale le premier qui a repris la suggérence de Poincaré, en présentant une formulation plus précise: l'objectif est de fixer une variété fermée M de dimension d et de comprendre la dynamique d'un sous-ensemble "grand" de $Diff^r(M)$, l'ensemble des difféomorphismes de M muni de la topologie de la convergence uniforme jusqu'à l'ordre r.

Aussi, Smale a proposé qu'un sous-ensemble soit consideré "grand" s'il est soit ouvert et dense, ou bien s'il est à résidual ou même dense (ainsi les ensembles ouverts de systèmes peuvent être négligés). On ne faira pas allusion à d'autres possibles

[9] L'introduction historique que nous présentons est subjective et peut ne pas décrire avec exactitude les faits historiques. On peut penser que ce qui suit est une possible explication historique des raisons pour lesquels les sujets ici présentés ont été abordés.

définitions de "grand" et on ne discutera pas les raisons de notre choix de topologie (voir [B] ou [C_4] pour des explications detailles).

Smale a proposé un programme centré autour de l'étude de la stabilité structurelle. Bien que les dynamiques typiques peuvent ne pas être stables (elles peuvent même être chaotiques), il espérait que leurs propriétes dynamiques soient robustes aux perturbations. Ainsi, Palis et Smale [PaSm] ont conjecturé que les systèmes structurellement stables coïncident avec les difféomorphismes hyperboliques.

L'hyperbolicité est devenue donc le nouveau paradigme. Robbin et Robinson ([R, Rob_1]) ont montré que les systèmes hyperboliques sont stables. Finalement, Mañé ([M_5]) a characterisé les dynamiques stables en topologie C^1 (on ne sait pas encore ce qui arrive dans d'autres topologies). Pendant les années 60 et début des 70, la plupart de l'attention s'est concentrée autour de la description de la dynamique des difféomorphismes hyperboliques.

Du point de vue de la compréhension de la dynamique des difféomorphismes hyperboliques, le programme a été très réussi: il a servi à la compréhension profonde des aspects semi-locaux de la dynamique (à travers l'étude symbolique et ergodique de ses propriétés) mais aussi à l'obtention d'importants résultats sur les aspects globaux et la topologie des pièces basiques.

En ce qui concerne les propriétés semi-locales, nous citons les contributions de Bowen, Newhouse, Palis, Sinai, Ruelle et Smale. Pour une vision panoramique de la théorie, voir [KH] Partie 4.

D'un autre coté, les aspects globaux sont associés fondamentalement aux travaux de Anosov, Bowen, Franks, Shub, Sullivan et Williams, dont la plupart sont présentés dans le très beau livre [F_4], qui rassemble les travaux connus sur la dynamique globale des difféomorphismes hyperboliques. Pour de différentes raisons, ces auteurs ont abandonné l'étude de ces sujets, ce qui peut expliquer qu'ils soient peu connus.

Le programme de Smale a échoué avec l'apparition d'exemples de dynamiques robustement non hyperboliques et non structuralement stables comme ceux de Abraham-Smale ([AS]) et Newhouse ([New_1]).

Ces exemples ont été à la base de la théorie des bifurcations, de laquelle Newhouse, Palis et Takens (entre autres) ont été pionniers. Après plusieurs travails à ce sujet, les premiers programmes ont été adaptés afin de, tout en considérant les bifurcations, maintenir la philosophie initiale de Smale. Malgré cela, l'approche du programme de Palis [Pa_3] est surtout semi-locale.

Après la chute du paradigme de l'hyperbolicité, on a commencé à chercher des notions alternatives comme l'hyperbolicité non uniforme (due à l'école russe, fondamentalement à Katok et Pesin, voir [KH] Supplément S) ou l'hyperbolicité partielle (considérée de façon indépendente par Hirsch, Pugh et Shub [HPS] et par Brin et Pesin [BrPe]). C'est à cette deuxième géneralisation que s'intéresse surtout cette thèse. Nous sommes intéréssés par l'approche géométrique de cette géneralisation (qui contraste avec la condition mesurable de l'hyperbolicité non uniforme) et par son lien étroit avec les propriétés robustes de la dynamique. Le lecteur peut se référer à [BDV] pour une vision générale de la dynamique au delà de l'hyperbolicité.

En cherchant la preuve de la conjecture de stabilité, Mañe (aussi Pliss et Liao [Pli, L], de façon indépendente) a introduit la notion de décomposition dominée et, ce qui est encore plus important, a montré la relation de la dynamique de l'application tangente sur les orbites périodiques.

En ce qui concerne l'étude de l'espace des difféomorphismes sous la topologie C^1, nous savons, grâce aux techniques de perturbation d'orbites, développées depuis les années 60 par Pugh, Mañe, Hayashi et plus récemment par Bonatti et Crovisier, que les orbites périodiques capturent, d'une certaine façon, la dynamique des difféomorphismes génériques (topologique et statistiquement). Voir [C_4] pour une vision générale sur ces sujets.

Récemment Bonatti [B] a proposé un programme réaliste pour la compréhension d'un grand ensemble de difféomorphismes sous la topologie C^1. Ce programme prolonge et complète, aussi d'un point de vue semi-locale, le susmentionné programme général de Palis.

Du point de vue globale, il y a justes quelques idées sur comment procéder en quelques cas de dimension 3.

En ce qui suit, nous présentons les contributions de cette thèse et nous essayons de montrer comment celles-ci s'adaptent à ce panorama subjectif du développement de la théorie.

0.3.2 Attracteurs en dynamique C^1-générique.

Il est toujours possible de décomposer la dynamique d'un homéomorphisme d'un espace métrique compacte en classes de récurrence, comme explique la théorie de Conley [Co].

Cette décomposition est très utile pour la compréhension des dynamiques C^1-génériques[10], grâce à un résultat de Bonatti et Crovisier ([BC]) qui assure que les orbites périodiques sont suffisantes pour détecter les classes de récurrence d'un difféomorphisme générique. Dans une bonne mesure, l'attention se place sur la compréhension de la dynamique autour desdites orbites et on espère qu'elles décrivent la dynamique des difféomorphismes en question (voir [B]).

Lorsque l'on veut comprendre la dynamique de la plupart des points, on trouve des classes de récurrence distinguées: les *quasi-attracteurs* sont des classes de récurrence

[10]Nous utiliserons cette expression pour faire allusion aux difféomorphismes appartenant à un ensemble résiduel de $\text{Diff}^1(M)$ sous la topologie C^1.

qui admitent une base de voisinages U_n qui vérifient $f(\overline{U}_n) \subset U_n$. Les quasi-attracteurs existent toujours. En [BC], les auteurs ont montré que, dans le cas d'un difféomorphisme générique, il existe un ensemble résiduel de points de la variété qui convergent aux dites classes.

Il est possible dans certains cas de prouver que les quasi-attracteurs sont isolés du reste des classes de récurrence. Dans ces cas, on les appelle *attracteurs*. Les attracteurs ont la propriete d'être accumulés par l'orbite future des points proches et d'être dynamiquement indécomposables. Un des grands problèmes des systèmes dynamiques consiste à déterminer leur existence et leur propriétés topologiques et statistiques. En dimension 2, il est possible de prouver que pour la plupart des systèmes dynamiques, sous la topologie C^1, il en existent. Ceci a été prouvé d'abord par Araujo [Ara], mais la preuve contenait une erreur et n'a jamais été publiée[11]. Cette thèse présente une preuve du résultat suivant, apparue pour la première fois en [Pot_2] (voir section 3.1).

Théorème. *Il existe un ensemble ouvert et dense $\mathcal{U}$ de l'espace de difféomorphismes d'une surface sous la topologie C^1 tel que tout $f \in \mathcal{U}$ a un attracteur hyperbolique. Si f ne peut pas être perturbé de façon à obtenir un nombre infini de points périodiques attracteurs (puits), alors pour f et ses perturbés il y a un nombre fini d'attracteurs dont tous sont hyperboliques.*

Tout quasi-attracteur hyperbolique est un attracteur. Pour montrer que tout quasi-attracteur est hyerbolique, dans le cas d'un nombre fini de puits, c'est fondamental de montrer d'abord que les dits attracteurs possèdent ce qu'on appelle une *décomposition dominée*: une décomposition du fibré tangent sur le quasi-attracteur en deux sous-fibrés Df-invariants qui vérifient une condition uniforme de domination de l'un sur l'autre (la contraction des vecteurs sur un des fibrés est uniformément plus petit que la contraction sur l'autre).

La décomposition dominée, ainsi que d'autres possibles structures géométriques Df-invariantes, est un outil important pour l'étude des propriétés dynamiques d'une classe de récurrence et surtout pour comprendre comment ladite classe est accumulée par d'autres classes (voir Section 1.2).

En quête de comprendre la dynamique proche aux quasi-attracteurs pour les systèmes C^1-génériques, le résultat partiel suivant, concernant la structure des quasi-attracteurs qui correspondent à des classes homocliniques, a été obtenu (voir Section 3.2 et [Pot_1]):

Théorème. *Soit f un difféomorphisme C^1-générique. Si H est un quasi-attracteur*

[11]Voir [San] pour une correction de la preuve originelle qui utilise les résultats de Pujals et Sambarino [PS_1].

de f qui contient un point périodique p pour lequel la différentiel à la periode contract le volume, alors H admet une décomposition dominée non triviale.

Il est possible de voir à travers des examples ([BV]) que le théorème est optimal dans un certain sens. Aussi, l'hypothèse de l'existence de points périodiques est nécessaire (voir [BD_3]). Par contre, l'hypothèse sur la dissipativité du différentiel sous l'orbite périodique semble être une conséquence des autres hypothèses, bien qu'elle n'a pas pu être éliminée. Pour le faire, il semble nécessaire d'avoir un résultat du type ergodic closing lemma sur la classe homoclinique, ce qui constitue un problème qui n'est pas encore compris correctement (voir [B] Conjecture 2).

La nouveauté de la preuve de ce théorème se base surtout dans le fat qu'elle utilise un nouveau résultat dû à Gourmelon [Gou_3], qui permet de perturber le différentiel de l'orbite périodique en gardant un certain contrôle des variétés invariantes. L'utilisation de ce résultat et la stabilité Lyapunov du système permettent de garantir que le point reste dans la classe après une perturbation. Ce resultat répond positivement à une question posée en [ABD] (Problème 5.1).

Le rêve des dynamiques C^1-génériques admettant des attracteurs[12] a disparu à cause d'un exemple surprenant dû à [BLY] qui montre que le dévelopement récent de la dynamique C^1-générique a eu une grande influence sur la façon de comprendre la théorie et a simplifié des questions qui résultaient à simple vue intraitables.

Les exemples de [BLY] possèdent ce que l'on a appelé des *attracteurs essentiels* (voir la sous-section 1.1.5); il n'est pas encore connu s'il possèdent ou pas des attracteurs dans les sens Milnor. Dans la Section 3.3 nous reverrons ces exemples et nous présenterons de nouveaux exemples qui apparaissent sur [Pot_2] et sur lesquels nous avons une meilleure compréhension de comment les autres classes accumulent.

Théorème. *Il existe un ouvert $\mathcal{U}$ de $\mathrm{Diff}^1(\mathbb{T}^3)$ tel que:*

- *Chaque $f \in \mathcal{U}$ possède un seul quasi-attracteur et un attracteur au sens Milnor. Aussi, si l'élément est de classe C^2, il possède un seul mesure SRB dont le bassin est de mesure totale.*
- *Pour f dans un ensemble résiduel de $\mathcal{U}$ ne possède pas des attracteurs,*
- *Les classes de récurrence qui ne sont pas un quasi-attracteurs d'un élément f de $\mathcal{U}$ sont contenues dans des surfaces périodiques.*

La dernière conclusion du théorème contraste avec les nouveaux résultats obtenus par Bonatti et Shinohara ([BS]). Dans la Section 3.4, nous discutons comment les deux résultats pourraient s'intégrer dans une même théorie.

[12]Voir l'introduction de [BLY].

0.3.3 Hyperbolicité partielle sur le tore $\mathbb{T}^3$.

La section précédente ayant traité de façon implicite des problèmes de nature semi-locale (bien qu'il est possible de poser des questions sur l'existence et la topologie des attracteurs du point de vue global), nous dédions cette section à des problèmes des dynamiques globales.

À la suite, nous présentons un théorème connu en dynamique différentiable, qui rassemble des résultats classiques de Mañe ([M_3]) et Franks ([F_1]).

Théorème (Mañe-Franks)**.** *Soit M une surface compacte. Pour un difféormorphisme $f \in Diff^1(M)$, les trois conditions suivantes sont équivalentes:*

(i) f est C^1-robustement transitif,

(ii) f f est Anosov,

(iii) f f est Anosov et conjugué à un difféomorphisme Anosov linéaire du tore $\mathbb{T}^2$.

Essentiellement, Mañe a prouvé (i) $\Rightarrow$ (ii) et Franks a prouvé (ii) $\Rightarrow$ (iii). L'autre implication se déduit des faits que les difféomorphismes Anosov sont robustes et que les linéaires sont en plus transitifs.

Si on interprète la condition Anosov comme le fait que le différentiel préserve une certaine structure géométrique, le résultat (i)$\Rightarrow$ (ii) s'identifie, d'une certaine façon, à la condition suivante: "une propriété dynamique robuste implique l'existence d'une structure géométrique Df-invariante". En fait, vu que les perturbations C^1 ne peuvent pas casser la dynamique, il n'est pas surprenant que ladite structure géométrique ait des propriétés de rigidité aux perturbation.

D'autre part, l'implication (ii)$\Rightarrow$ (i) peut s'identifier à la condition suivante: "l'existence d'une propriété géométrique invariante est liée à l'existence d'une propriété dynamique robuste" (dans ce cas, la transitivité).

En dimensions supérieures la relation entre les propriétés dynamiques robustes et les structures géométriques invariantes est moins développée, bien qu'il existe des résultats visant à obtenir une structure géométrique invariante à partir d'une propriété dynamique robuste. En dimension 3, ceci ce déduit de [DPU] et a été généralisé à dimensions supérieures en [BDP] (voir Section 1.2):

Théorème (Díaz-Pujals-Ures)**.** *Soit M un variété de dimension 3. Tout difféomorphisme C^1-robustement transitif de M est partiellement hyperbolique.*

Plusieurs définitions d'hyperbolicité partielle existent dans la littérature, et celles-ci ont changé le long du temps. Nous suivons la définition utilisée en [BDV] qui est celle qui s'adapte le mieux à nôtre approche (voir Section 1.2 pour les définitions précises). Un difféomorphisme partiellement hyperbolique sera tel que le fibré tangent se décompose en une somme Df-invariante $TM = E \oplus F$ qui vérifie la propriété de domination et telle que l'un des fibrés de la décomposition est uniforme. Pour simplifier, nous éliminons la symétrie et considérons les décompositions de la forme $TM = E^{cs} \oplus E^u$, ou E^u est uniformément dilate.

Si on s'intéresse à comprendre les propriétés dynamiques robustes impliquées par l'existence d'une décomposition partiellement hyperbolique, une première difficulté que l'on trouve concerne le manque de contrôle de la contraction sur la direction centre stable E^{cs}. Ceci empêche de comprendre à fond la dynamique sur cette direction, comme on la comprend sur la direction des fibrés uniformes. Une exception notable dans ce sens est [PS_6], travail qui étudie les conséquences dynamiques de la décomposition dominée pour le cas de dimension 2.

Il mérite de mentionner aussi l'importance de (iii) dans le Théorème de Mañe et Franks. D'une certaine façon, il est basé sur l'idée qui suggère que pour obtenir une propriété dynamique robuste à partir de l'existence d'une structure géométrique invariante, il peut être important de s'appuyer sur les propriétés topologiques imposées par ladite structure: la topologie de la variété et la classe d'isotopie du difféomorphisme. Tenant compte de ce théorème et des difficultés à trouver des résultats dans le sens réciproque au Théorème de Diaz-Pujals-Ures , il semble important de diviser le travail en cherchant des résultats tu type (ii)$\Rightarrow$ (iii) et du type (ii)+(iii) $\Rightarrow$ (i), en imitant le Théorème de Mañe et Franks. En particulier, il pourrait être pertinent d'étudier les propriétés des difféomorphismes partiellement hyperboliques dans certaines variétés, ou même dans des classes d'isotopie fixées.

Une autre difficulté en dimension 3 est l'obtention de propriétés topologiques à partir des structures invariantes. En dimension 2, le seul fait de préserver un camp de vecteurs impose des énormes restrictions sur la topologie de la variété. En dimension 3, il est bien connu que toute variété admet un champ de vecteurs non nul, et même un feuilletage de codimension 1.

Cette situation pourrait être considérée mauvaise du point de vue d'obtenir des résultats visant à trouver des propriétés topologiques d'un difféomorphisme qui préserve une structure géométrique. En revanche, une belle remarque faite récemment par Brin-Burago-Ivanov ([BBI, BI]) a redonné vie à l'espoir:

Remarque (Brin-Burago-Ivanov). Dans une variété de dimension 3, si $\mathcal{F}$ est une feuilletage transversale à la direction instable E^u d'un difféomorphisme partiellement hyperbolique, alors $\mathcal{F}$ n'a pas de composantes de Reeb.

$\diamondsuit$

Les composantes de Reeb et leur relation avec les difféomorphismes partiellement hyperboliques avait déjà été étudiée, par exemple dans [DPU] (Theorem H) et [BWi] (Lemma 3.7) en un contexte plus restrictif (la coherence dynamique et la transitivite). La puissance de cette remarque est lie au fait qu'elle dépend de la dynamique seulement dans le fait que le feuilletage instable n'a pas des courbes fermées.

D'un côté, il est connu que plusieures variétés de dimension 3 n'admettent pas des feuilletage sans composantes de Reeb. D'un autre côté, il existe beaucoup de théorie sur leur classification (voir par exemple [Pla, Rou]). Il semble donc possible d'utiliser ces connaisances pour etudier les difféomorphismes partiellement hyperboliques.

Il apparaît une autre difficulté: on ne sait pas si tout difféomorphisme partiellement hyperbolique possède une feuilletage transversale à la direction instable. Dans cette thèse, nous proposons la notion de *presque-cohérence dynamique* et nous prouvons qu'il s'agit d'une propriété ouverte et fermée dans l'espace des difféomorphismes partiellement hyperboliques.

Un des théorèmes principaux (voir Chapitre 5 et [Pot_5]) garantie, dans certaines classes d'isotopie de difféomorphismes de $\mathbb{T}^3$, que cette condition est suffisante pour que le fibré E^{cs} soit tangent à un feuilletage invariante:

Théorème. *Soit f un difféomorphisme partiellement hyperbolique et isotope à un difféomorphisme Anosov linéaire. Si f est presque-dynamiquement cohérent, alors il est dynamiquement cohérent.*

La condition de *cohérence dynamique* signifie justement que le fibré tangent E^{cs} soit tangent à una feuilletage f-invariante.

On peut dire plus pour le cas *partiellement hyperbolique fort* (c'est à dire lorsque il'y a un décomposition $TM = E^s \oplus E^c \oplus E^u$ qui est Df-invariante avec des propriétés de domination et telle que E^s et E^u sont uniformes).

En [BI] les auteurs ont montré que tout difféomorphisme partiellement hyperbolique fort est presque-dynamiquement cohérent et ceci a été repris d'abord par [BI] (suivant [BBI_1]), et après par [Par], pour donner des conditions topologiques nécessaires pour l'hyperbolicité partielle forte.

Avec nos resultats, ca suggère que la suivante conjecture de Pujals pourrait être abordable dans certains varietes:

Conjecture (Pujals [BWi])**.** *Soient M une variété de dimension 3 et f un difféomorphisme partiellement hyperboique fort et transitif de M. Une des trois conditions suivantes doit se vérifier:*

- *f est conjugué par feuilles à un difféomorphisme Anosov de $\mathbb{T}^3$.*

- *f est conjugué par feuilles à un skew-produc sur un difféomorphisme Anosov en $\mathbb{T}^2$ (la variété est donc $\mathbb{T}^3$ ou une nilvariété).*

- *f est conjugué par feuilles au temps 1 d'un flot d'Anosov.*

Il y a eu des progrès dernièrement en ce qui concerne cette conjecture. Tout d'abord, le travail de [BWi] a fait des progrès importants sans imposer d'hypothèse sur la topologie de la variété, en supposant l'existence de courbes fermées tangentes à la direction centrale. Plus tard, les travaux de Hammerlindl [H, H_2] donnent une preuve de la conjecture pour le cas où la variété est $\mathbb{T}^3$ ou une nilvariété mais pour une notion plus restrictive de l'hyperbolicité partielle. Bien que de nombreux exemples vérifient cette définition plus restrictive, elle est dans un sens artificielle et ne s'adapte pas au résultat de [DPU].

Bien sûr, pour obtenir une preuve générale, il faut d'abord montrer que les difféomorphismes de ce genre sont dynamiquement cohérents, puis que la définition de "conjugué par feuilles" (voir Section 1.3) a besoin de l'existence de feuilletages invariantes. Le travail de Hammerlindl s'appui sur des travaux précédents de Brin-Burago-Ivanov ([BBI_2]) qui montrent que cette définition restrictive d'"hyperbolicité partielle, dans le cas de $\mathbb{T}^3$, implique cohérence dynamique.

D'un autre côté, un exemple récent de Rodriguez Hertz-Rodriguez Hertz-Ures ([$RHRHU_3$]) présente un difféomorphisme partiellement hyperbolique de $\mathbb{T}^3$ qui n'admet pas de foliations invariantes. Dans cette thèse nous complétons le paysage avec le suivant résultat ([Pot_5] et Chapitre 5).

Théorème. *Tout difféomorphisme partiellement hyperbolique fort de $\mathbb{T}^3$ qui n'admet pas de tore periodique normallement attracteur ni de tore periodique normallement répulseur est dynamiquement cohérent.*

Ce résultat répond à une conjecture posée par Rodriguez Hertz-Rodriguez Hertz-Ures dans le tore $\mathbb{T}^3$ et sa preuve permet de répondre à la conjecture de Pujals sur $\mathbb{T}^3$ (voir [HP]).

Nous présentons à la fin du Chapitre 5 des résultats en dimension supérieures qui généralisent les résultats obtenus par Franks, Newhouse et Manning pour des difféomorphismes Anosov dans le contexte partiellement hyperbolique.

0.3.4 Autres contributions

Dans cette section nous décrivons des autres contributions de cette thèse.

D'une part, dans la Section 2.2 nous décrivons un mechanisme pour localiser des classes de récurrence qui a été présenté dans [Pot_3] et qu'on considéré important par soi-même. Ce mechanisme peut être appliqué dans des contextes variés (dans cette thèse nous l'utilisons dans la Section 4.A et aussi dans les sous-sections 3.3.2 et 3.3.3).

Dans la Section 3.3, nous présentons divers exemples de quasi-attracteurs et de difféomorphismes robustement transitifs, dont quelques uns sont obtenus à partir de modifications d'exemples connus, mais qui peuvent, d'après nous, représenter une contribution a la compréhension de ces phénomènes.

Le Chapitre 4 est dédié à présenter les idées sur les feuilletages qui seront utilisées après dans le Chapitre 5, nous obtenons des résultats qui peuvent avoir un intérêt indépendent. En particulier, un résultat quantitatif sur l'existence d'une structure de produit global pour les foliations, présenté dans la Section 4.3. De même, dans la Section 4.A, nous donnons une classification de la dynamique des difféomorphismes globalement partiellement hyperbolique en $\mathbb{T}^2$ qui, d'une certaine façon, montre ce qu'on faira dans le Chapitre 5 dans un contexte plus simple.

Nous ajoutons aussi quatre annexes où nous présentons des résultats qui se détachent du corps central de la thèse. Nous soulignons l'annexe C, basé sur [Pot$_4$], où nous prouvons un résultat sur les homéomorphismes du tore qui possèdent un seul vecteur de rotations. Aussi, dans l'annexe D, basé sur [BCGP] nous présentons un travail en collaboration avec Bonatti, Crovisier et Gourmelon, où nous étudions les biffurcations de classes de récurrence robustement isolées et nous donnons des exemples non robustement transitifs, ce qui répond à une question posée en [BC].

0.4 Organization of this thesis

This thesis is organized as follows:

- In Chapter 1 we introduce definitions and known results about differentiable dynamics which will be used along the thesis. This chapter also presents in a systematic way the context in which we will work.

- In Chapter 2 we present background material on semiconjugacies and we also present a mechanism for localization of chain-recurrence classes (see Section 2.2).

- In Chapter 3 we study attractors and quasi-attractors in C^1-generic dynamics.

- In Chapter 4 we give an introduction to the known results in foliations mainly focused in codimension one foliations and particularly in foliations of 3-manifolds. We prove some new results which we will use later in Chapter 5. Also, this chapter contains an appendix which shows similar results as the main results of this thesis in the context of surfaces.

- In Chapter 5 we study global partial hyperbolicity. Most of the chapter is devoted to the study of partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ and in the last section some results in higher dimensions are given.

- Appendix A presents some techniques on perturbations of cocycles over periodic orbits.
- Appendix B gives an example of a decomposition of the plane satisfying some pathological properties.
- Appendix C is devoted to the study of homeomorphisms of $\mathbb{T}^2$ with a unique rotation vector. There we present the results of [Pot_4] and we also give a quite straightforward extension of the results there to certain homeomorphisms homotopic to dehn-twists.
- Appendix D presents the results of [BCGP].

0.5 Reading paths

Being quite long, it seems reasonable to indicate at this stage how to get to certain results without having to read the whole thesis.

First of all, it must be said that Chapter 1, concerning preliminaries, need not be read for those who are acquainted with the subject. In particular, those who are familiar with one or more of the excellent surveys [BDV, C_4, PS_2].

Similar comments go for Chapter 4, in particular the first part covers well known results on the theory of foliations which can be found in [CaCo, Ca] among other nice books. The final part though may be of interest specially for those which are not specialist on the theory of foliations such as the author.

If the reader is interested in the part of this thesis concerned with attractors for C^1-generic dynamics, then, the suggested path (which can be coupled with the suggestions in the previous paragraphs) is first reading Chapters 1 and Chapter 2 and then Chapter 3.

If on the other hand, the reader is interested in the part about global partially hyperbolic dynamics, then some parts of Chapter 1 can be skipped, in particular it is enough with reading Sections 1.2 and 1.4. Then, Chapter 4 is fundamental in those results, but the reader which is familiar with the theory of foliations may skip it in a first read. Finally, the results about global partial hyperbolicity are contained in Chapter 5. If the reader is interested in Section 4.A, then Chapter 2 is suggested.

Chapter 1

Preliminaries

1.1 Recurrence and orbit perturbation tools

1.1.1 Some important dynamically defined sets and transitivity

Let $f : X \to X$ a homeomorphism of X a compact metric space. For a point $x \in X$ we define the following sets:

- The *orbit* of x is the set $\mathcal{O}(x) = \{f^n(x) \ : \ n \in \mathbb{Z}\}$. We can also define the future (resp. past) orbit of x as $\mathcal{O}^+(x) = \{f^n(x) \ : \ n \geq 0\}$ (resp. $\mathcal{O}^-(x) = \{f^n(x) \ : \ n \leq 0\}$).

- The *omega-limit set* (resp.*alpha-limit set*) is the set $\omega(x, f) = \{y \in X \ : \ \exists n_j \to +\infty$ such that $f^{n_j}(x) \to y\}$ (resp. $\alpha(x, f) = \omega(x, f^{-1})$). In general, when f is understood, we shall omit it from the notation.

We can divide the points depending on how their orbit and the nearby orbits behave. We define the following sets:

- $\mathrm{Fix}(f) = \{x \in X \ : \ f(x) = x\}$ is the set of *fixed points*.

- $\mathrm{Per}(f) = \{x \in X \ : \ \#\mathcal{O}(x) < \infty\}$ is the set of *periodic points*. The period of a periodic point x is $\#\mathcal{O}(x)$ which we denote as $\pi(x) = \#\mathcal{O}(x)$.

- We say that a point x is *recurrent* if $x \in \omega(x) \cup \alpha(x)$.

- $\mathrm{Lim}(f) = \overline{\bigcup_x \omega(x) \cup \alpha(x)}$ is the *limit set* of f.

- $\Omega(f) = \{x \in X \ : \ \forall \varepsilon > 0 \ , \ \exists n > 0 \ ; \ f^n(B_\varepsilon(x)) \cap B_\varepsilon(x) \neq \emptyset\}$ it the *nonwandering set* of f.

We refer the reader to [KH, Rob$_2$, Sh] for examples showing the strict inclusions in the following chain of closed sets which is easy to verify:

$$\mathrm{Fix}(f) \subset \mathrm{Per}(f) \subset \mathrm{Lim}(f) \subset \Omega(f)$$

In section 1.1.4 we shall explore another type of recurrence which will play a central role in this text.

For a point $x \in X$ we will define the following sets (as before the reference to the homeomorphism f may be omitted when it is obvious from the context).

- The *stable set* of x is $W^s(x,f) = \{y \in X \ : \ d(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty\}$.
- The *unstable set* of x is $W^u(x,f) = W^s(x, f^{-1})$.

It is clear that $f(W^\sigma(x)) = W^\sigma(f(x))$ for $\sigma = s, u$. The study of these sets and how are they related is one of the main challenges one faces when trying to understand dynamical systems.

Sometimes, it is useful to consider instead the following sets which in some cases (for sufficiently small ε) are related with the stable and unstable sets:

- $S_\varepsilon(x,f) = \{y \in X \ : \ d(f^n(x), f^n(y)) < \varepsilon \ , \ \forall n \geq 0\}$.
- $U_\varepsilon(x,f) = S_\varepsilon(x, f^{-1})$.

Notice the following two properties which are essentially the reason for defining these sets:

$$W^s(x) \subset \bigcup_{n \geq 0} f^{-n}(S_\varepsilon(f^n(x)))$$

$$f(S_\varepsilon(x)) \subset S_\varepsilon(f(x))$$

Similar properties hold for $U_\varepsilon(x)$. It is not hard to make examples where the inclusions are strict. However, the first property is an equality for certain special maps (expansive, or hyperbolic) which will be of importance in this text.

For a set $K \subset X$ we define $W^\sigma(K) = \bigcup_{x \in K} W^\sigma(x)$ with $\sigma = s, u$. Notice that the set is (a priori) smaller than the set of points whose omega-limit is contained in K.

We say that f is *transitive* if there exists $x \in X$ such that $\mathcal{O}(x)$ is dense in X. Sometimes, when f is understood (for example, when X is a compact invariant subset of a homeomorphism of a larger set), we say that X is *transitive.*

It is an easy exercise to show the following equivalences (see for example [KH] Lemma 1.4.2 and its corollaries):

Proposition 1.1.1. *The homeomorphism $f : X \to X$ is transitive if and only if for every U, V open sets there exists $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$, if and only if there is a residual subset of points whose orbit is dense.*

We say that f (or as above that X) is *minimal* if every orbit is dense.

Given an open set U, we define the following compact f-invariant set $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U})$ which we call the *maximal invariant set* in $\overline{U}$.

Many of the dynamical properties one obtains are invariant under what is called *conjugacy*. We say that two homeomorphisms $f : X \to X$ and $g : Y \to Y$ are (topologically) *conjugated* if there exists a homeomorphism $h : X \to Y$ such that:

$$h \circ f = g \circ h$$

When K is an f-invariant set and K' a g-invariant set we say that f and g are *locally conjugated at* K if there exists a neighborhood U of K, a neighborhood V of K' and a homeomorphism $h : U \to V$ such that if a point $x \in U \cap f^{-1}(U)$ then:

$$h \circ f(x) = g \circ h(x)$$

1.1.2 Hyperbolic periodic points

From now on, $f : M^d \to M^d$ will denote a C^1 diffeomorphism.

Given $p \in \mathrm{Per}(f)$ we have the following linear map:

$$D_p f^{\pi(p)} : T_pM \to T_pM$$

We say that $p \in \mathrm{Per}(f)$ is a *hyperbolic periodic point* if $D_p f^{\pi(p)}$ has no eigenvalues of modulus 1. We denote the set of hyperbolic periodic points as $\mathrm{Per}_H(f)$.

It is a direct application of standard linear algebra to show that in fact the set $\mathrm{Per}_H(f)$ is f-invariant. This implies that we can also talk about *hyperbolic periodic orbits*.

For a hyperbolic periodic point p we have that $T_pM = E^s(p) \oplus E^u(p)$ where $E^s(p)$ (resp. $E^u(p)$) corresponds to the eigenspace of $D_p f^{\pi(p)}$ associated to the eigenvalues of modulus smaller than 1 (resp. larger than 1). We have that $D_p f(E^\sigma(p)) = E^\sigma(f(p))$ with $\sigma = s, u$.

We define the (*stable*) *index*[1] of a periodic point p as $\dim E^s(p)$. This also leads to calling *stable eigenvalues* (resp. *unstable eigenvalues*) to those which are of modulus smaller (resp. larger) than 1. We denote the set of index i periodic points as $\mathrm{Per}_i(f)$.

[1] We warn the reader that some authors define the index of a periodic point as the unstable dimension.

The importance of hyperbolic periodic points is related to the fact that their local dynamics is very well understood and it is persistent under C^1-perturbations (for a proof see for example [KH] chapter 6):

Theorem 1.1.2. *Let p be a hyperbolic periodic point of a C^1-diffeomorphism f, then:*

(i) $f^{\pi(p)}$ is locally conjugated to $D_p f^{\pi(p)}$ at p. In particular, there are no periodic points of period smaller or equal to $\pi(p)$ inside a neighborhood U of p.

(ii) There exists a C^1-neighborhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$ there is a unique periodic point p_g of period $\pi(p)$ of g inside U which is also hyperbolic. We say that p_g is the continuation *of p for g.*

(iii) There exists $\varepsilon > 0$ such that S_ε is an embedded C^1 manifold tangent to $E^s(p)$ at p and in particular, one has $S_\varepsilon \subset W^s(p)$.

As a consequence we have that the set of hyperbolic periodic points of period smaller than n is finite. We have that the sets $W^\sigma(p)$ are (injectively) immersed C^1-submanifolds of M diffeomorphic to $\mathbb{R}^{\dim E^\sigma}$. Moreover, one can define $W^\sigma(\mathcal{O}(p))$ which will also be an injectively immersed C^1-submanifold with the same number of connected components as the period of p.

When the stable index s of a hyperbolic periodic point is the same as the dimension of the ambient manifold (resp. $s = 0$), we shall say that it is a *periodic sink* (resp. *periodic source*). In any other case we shall say that it is a *periodic saddle* of index s.

One of the first perturbation results in dynamics was given by Kupka and Smale independently showing:

Theorem 1.1.3 ([Kup, Sm_1])**.** *For every $r \geq 1$, there exists a residual subset $\mathcal{G}_{KS} \subset$ $\mathrm{Diff}^r(M)$ of diffeomorphisms such that if $f \in \mathcal{G}_{KS}$:*

- *All periodic points are hyperbolic (i.e. $\mathrm{Per}(f) = \mathrm{Per}_H(f)$).*
- *Given $p, q \in \mathrm{Per}(f)$ we have that $W^s(p)$ and $W^u(q)$ intersect transversally (recall that this allows the manifolds not to intersect at all).*

Transversal intersections between stable and unstable manifolds yield information on the iterates of those manifolds. A quite useful tool to treat those intersections is given by the celebrated λ-Lemma (or Inclination Lemma) of Palis (see [Pa_1]) which we state as follows (see also [KH] Proposition 6.2.23). The statement we present is for fixed points, but considering an iterate one can of course treat also periodic points as well:

Theorem 1.1.4 (λ-Lemma). *Let p be a hyperbolic fixed point of f a C^1-diffeomorphism of a manifold M and let D be a C^1-embedded disk which intersects $W^s(p)$ transversally. Then, given a compact submanifold B of $W^u(p)$ and $\varepsilon > 0$ there exists $n_0 > 0$ such that for every $n > n_0$ there is a compact submanifold $D_n \subset D$ such that $f^n(D_n)$ is at C^1-distance smaller than ε of B.*

See also Lemma D.1.4.

1.1.3 Homoclinic classes

Given two hyperbolic periodic points p, q we say that they are *homoclinically related* if $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^s(q) \pitchfork W^u(p) \neq \emptyset$.

Given a periodic obit $\mathcal{O}$ we define its *homoclinic class* $H(\mathcal{O})$ as the closure of the set of periodic points homoclinically related to some point in the orbit $\mathcal{O}$.

Notice that by the definition of being homoclinically related, necessarily one has that if two periodic points are homoclinically related, then they have the same stable index. However, this does not exclude the possibility of having periodic points of different index (and even non-hyperbolic periodic points) inside $H(\mathcal{O})$ and this will be "usually" the case outside the "hyperbolic world".

Homoclinic classes were introduced by Newhouse as an attempt to generalize for arbitrary diffeomorphisms the basic pieces previously defined by Smale ([Sm$_2$]) for Axiom A diffeomorphisms. The first and probably the main example of non-trivial homoclinic class is given by the famous *horseshoe* of Smale (see [Sm$_2$] or [KH] 2.5). We have the following properties:

Proposition 1.1.5 ([New$_3$]). *For every hyperbolic periodic orbit $\mathcal{O}$ of a C^1-diffeomorphism f the homoclinic class $H(\mathcal{O})$ is a transitive f-invariant set. Moreover, we have that:*

$$H(\mathcal{O}) = \overline{W^s(\mathcal{O}) \pitchfork W^u(\mathcal{O})}.$$

This proposition essentially follows as an application of the λ-Lemma (see also [Sm$_2$]).

For a periodic point p we denote as $H(p) = H(\mathcal{O}(p))$. Given a hyperbolic periodic point p of a C^1-diffeomorphism, we have by Theorem 1.1.2 that there exists a continuation of p as well as $W^s(p)$ and $W^u(p)$ for close diffeomorphisms. We will sometimes make explicit reference to the diffeomorphism and use the notation $H(p, f)$.

We have the following fact which follows from the continuous variation of stable and unstable manifolds with the diffeomorphism:

Proposition 1.1.6. *Given a hyperbolic periodic point p of a C^1-diffeomorphism f and U an open neighborhood of M such that $H(p) \cap U \neq \emptyset$, there exists $\mathcal{U}$ a C^1-*

neighborhood such that if $H(p_g)$ is the homoclinic class for $g \in \mathcal{U}$ of the continuation p_g of p we have that $H(p_g) \cap U \neq \emptyset$.

Remark 1.1.7 (Semicontinuity). The last statement of the proposition can be stated by saying that homoclinic classes vary *semicontinuously* with respect to the Hausdorff topology. This means that they cannot *implode* (if $f_n \to f$ have hyperbolic periodic points p_n which are the continuation of $p \in \mathrm{Per}_H(f)$ for f_n, we have that if $H(p,f)$ intersects a given open set U, then $H(p_n, f_n)$ also intersects U for large enough n). There exists the possibility that the homoclinic class *explodes* by small perturbations (see for example [Pa$_2$, DS]). However, a classic result in point set topology guaranties that when a map is semicontinuous then it must be continuous in a residual subset (see Proposition 3.9 of [C$_4$] for a precise statement). Other compact sets related to the dynamics which have semicontinuous variation are $\overline{\mathrm{Per}_H(f)}$, $\overline{\mathrm{Per}_i(f)}$ or for a given hyperbolic periodic point p of f the set $\overline{W^\sigma(p_g)}$ ($\sigma = s, u$) for g in a neighborhood of f. All these sets cannot implode but may explode in some situations, from the mentioned result on point set topology, there is a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ where all these sets vary continuously. In the next subsection we shall see a set which also varies semicontinuously but in "the other sense", meaning that it can implode but not explode.

$\diamondsuit$

As it was already mentioned, periodic points in a homoclinic class may not have the same stable index (even if to be homoclinically related they must have the same index). The existence of periodic points of different index in a homoclinic class is one of the main obstructions for hyperbolicity. Given a homoclinic class H we say that its *minimal index* (resp. *maximal index*) is the smallest (resp. largest) stable index of periodic points in H.

1.1.4 Chain recurrence and filtrations

In this section we shall review yet another recurrence property, namely, *chain-recurrence*. From the point of view of recurrence, it can be regarded as the "weakest" form of recurrence for the dynamics. Indeed, it is so weak that it was neglected for much time in differentiable dynamics since its classes seem to have really poor dynamical indecomposability (see for example Appendix C and [Pot$_4$]).

On the other hand, it is by far the best notion when one wishes to decompose the dynamics in pieces, and this is why after being shown to be quite similar to the rest of the notions for C^1-generic dynamics (in [BC]) it became "the" notion of recurrence used in C^1-differentiable dynamics.

We derive the reader to [C$_4$] for a more comprehensive introduction to these concepts (see also [BDV] chapter 10 for another introduction to these topics which is less up to date but still a good introduction).

Definition 1.1.1 (Pseudo-orbits)**.** Given a homeomorphism $f : X \to X$ and points $x, y \in X$ we say that there exists an *ε-pseudo orbit* from x to y and we denote it as $x \dashv_\varepsilon y$ iff there exists points $z_0 = x, \ldots, z_k = y$ such that $k \geq 1$ and

$$d(f(z_i), z_{i+1}) \leq \varepsilon \qquad 0 \leq i \leq k-1$$

We use the notation $x \dashv y$ to express that for every $\varepsilon > 0$ we have that $x \dashv_\varepsilon y$. We also use $x \vdash\dashv y$ to mean $x \dashv y$ and $y \dashv x$.

$\diamondsuit$

We define the *chain-recurrent set* of $f : X \to X$ as

$$\mathcal{R}(f) = \{x \in X \ : \ x \vdash\dashv x\}$$

It is easy to show that inside $\mathcal{R}(f)$ the relation $\vdash\dashv$ is an equivalence relation so we can decompose $\mathcal{R}(f)$ in the equivalence classes which we shall call *chain-recurrence classes*. For a point $x \in \mathcal{R}(f)$ we will denote as $\mathcal{C}(x)$ its chain-recurrence class. Both $\mathcal{R}(f)$ and the chain recurrence classes can be easily seen to be closed (and thus compact).

One can regard chain recurrence classes as maximal *chain transitive sets*. We say that an invariant set $K \subset X$ is chain-transitive if for every $x, y \in K$ we have that $x \dashv y$. We say that a homeomorphism $f : X \to X$ is *chain recurrent* if X is a chain transitive set for f.

For a chain-transitive set K we define its *chain stable set* (resp. *chain unstable set*) as $pW^s(K) = \{y \in X \ : \ \exists z \in K \text{ such that } y \dashv z\}$ (resp. as $pW^u(K)$, the chain stable set for f^{-1}).

Remark 1.1.8. Given a hyperbolic periodic point p of a C^1-diffeomorphism, one has that the homoclinic class of p is a chain-transitive set. In particular, it is always contained in the chain-recurrence class of p. In general, one can have that the inclusion is strict (see for example [DS]). Notice also that if p is a hyperbolic sink (or source) we have that $\mathcal{C}(p) = \mathcal{O}(p)$. Indeed, since p admits neighborhoods whose closure is sent to its interior by f (or f^{-1}), this prevents small pseudo-orbits to leave (or enter) any small neighborhood of p.

$\diamondsuit$

An essential tool for decomposing chain-recurrence classes whose existence is the content of Conley's theory (see [Co] and [Rob$_2$] chapter 9.1) are Lyapunov functions. We remark that the definition we use of Lyapunov function is not the standard one in the literature, we have adapted our definition in order to have the properties of the function given by Conley's Theorem.

Definition 1.1.2 (Lyapunov Functions)**.** Given a homeomorphism $f : X \to X$ we say that $\varphi : X \to [0,1]$ is a *Lyapunov function* if the following conditions are satisfied:

- For every $x \in X$ we have that $\varphi(f(x)) \leq \varphi(x)$ and $\varphi(x) = \varphi(f(x))$ if and only if $x \in \mathcal{R}(f)$.
- Given $x, y \in \mathcal{R}(f)$ then $\varphi(x) = \varphi(y)$ if and only if $\mathcal{C}(x) = \mathcal{C}(y)$.
- The image of $\mathcal{R}(f)$ by φ has empty interior.

$\diamondsuit$

It is remarkable that these functions always exist (see [Rob$_2$] chapter 9.1 for a simple proof of the following theorem also sometimes called *Fundamental theorem of dynamical systems*).

Theorem 1.1.9 (Conley [Co])**.** *For any homeomorphism $f : X \to X$ of a compact metric space X there exists a Lyapunov function $\varphi : X \to \mathbb{R}$.*

Remark 1.1.10 (Filtrations). Lyapunov functions allow to create *filtrations* separating chain recurrence classes. Indeed, consider $\mathcal{C}_1$ and $\mathcal{C}_2$ two distinct chain recurrence classes, and a Lyapunov function φ.

From the definition, we have that without loss of generality, we can assume $\varphi(\mathcal{C}_1) < \varphi(\mathcal{C}_2)$. Since the image by φ of the chain-recurrent set has empty interior, there exists $a \in [0,1] \setminus \varphi(\mathcal{R}(f))$ such that $\varphi(\mathcal{C}_1) < a < \varphi(\mathcal{C}_2)$.

Let $U = \varphi^{-1}((-\infty, a))$ an open set. Since every point such that $\varphi(x) = a$ is not chain-recurrent, we obtain by the definition of Lyapunov function that $f(\overline{U}) \subset U$ and that $\mathcal{C}_1 \subset U$ and $\mathcal{C}_2 \subset \overline{U}^c$.

Moreover, every chain-recurrence class $\mathcal{C}$ admits a basis of neighborhoods U_n such that if Λ_n is the maximal invariant subset of U_n, then $\mathcal{C} = \bigcap_n \Lambda_n$. Moreover, it verifies that if $\mathcal{C}_0$ is a chain recurrence class which intersects U_n then $\mathcal{C}_0$ is contained in U_n (one can consider $\varphi^{-1}((a-\varepsilon_n, a+\varepsilon_n)$ with $\varepsilon_n \to 0$). The sets U_n are sometimes called *filtrating neighborhoods* for $\mathcal{C}$.

$\diamondsuit$

These filtrating neighborhoods (which persist under C^0-small perturbations) allow one to show that the mapping $f \mapsto \mathcal{R}(f)$ is semicontinuous in the sense that it cannot "explode" (see Remark 1.1.7), so it will vary continuously in a residual subset of $\mathrm{Diff}^1(M)$ with respect to the Hausdorff topology on compact sets. See the example of Appendix D.

Pseudo-orbits can be thought of real-orbits of C^0-perturbations of the initial system. We have:

Proposition 1.1.11. *Let $f_n : X \to X$ be a sequence of homeomorphisms such that $f_n \to f$ in C^0-topology and let Λ_n be chain-transitive sets for f_n. Then, in the Hausdorff topology we have that $\Lambda = \limsup \Lambda_n$ is a chain-transitive set.*

PROOF. Consider $x, y \in \Lambda$ and $\varepsilon > 0$. We can consider n large enough so that

- $d_{C^0}(f_n, f) < \frac{\varepsilon}{2}$.
- $d_H(\Lambda_n, \Lambda) < \frac{\varepsilon}{2}$.

Since a $\frac{\varepsilon}{2}$-pseudo-orbit for f_n will be an ε-pseudo-orbit of f and since there are points in Λ_n which are $\frac{\varepsilon}{2}$-close to x and y we conclude.

□

Remark 1.1.12 (Trapping regions). Conley's Theorem implies in particular that a homeomorphism $f : X \to X$ is chain-recurrent if and only if there is no proper (i.e. strictly contained) open set $U \subset X$ such that $f(\overline{U}) \subset U$.

♢

In order to understand the asymptotic behavior of orbits one must then comprehend the dynamics inside chain recurrence classes as well as how the classes are related to each other.

An important concept is then that of *isolation*. We say that a chain-recurrence class $\mathcal{C}$ is *isolated* iff there exists a neighborhood U of $\mathcal{C}$ such that $U \cap \mathcal{R}(f) = \mathcal{C}$. This is equivalent to $\varphi(\mathcal{C})$ being isolated in $\varphi(\mathcal{R}(f))$ for some Lyapunov function (as defined in Definition 1.1.2) of f. Sometimes, non isolated classes will be referred to as *wild chain recurrence classes*.

Remark 1.1.13. In particular, one has that a chain-recurrence class is isolated if and only if it is the maximal invariant set in a neighborhood of itself.

♢

1.1.5 Attracting sets

It seems natural that the goal of understanding the whole orbit structure for general homeomorphisms should be quite difficult. This is why, in general, we content ourselves by trying to understand "almost every" orbit of "almost every" system. This informal statement has various ways to be understood, in particular, it is well known that many different formalizations of "almost every" can be quite different (the paradigma of this is seen in the case of irrational numbers, where Diophantine ones have total Lebesgue measure while Liouville ones form a disjoint residual subset of $\mathbb{R}$).

However, it seems natural in view of the Lyapunov functions to study certain special chain-recurrence classes which are called *quasi-attractors*. In this section we shall define plenty types of attractors which in a certain sense will be the chain-recurrence classes to which we shall pay more attention in view of the discussion above.

Given an open set $U \subset X$ such that $f(\overline{U}) \subset U$ we can consider the set $\Lambda = \bigcap_{n>0} f^n(U)$ which is compact and invariant (it is the maximal invariant subset in U). We call Λ a *topological attractor*[2].

Proposition 1.1.14. *Let U be an open set such that $f(\overline{U}) \subset U$ and let Λ be its maximal invariant set. If $y \in X$ is a point such that for every $\varepsilon > 0$ there exists $z \in \Lambda$ such that $z \dashv_\varepsilon y$, then, $y \in \Lambda$. In particular, for every $z \in \Lambda$ we have that $\overline{W^u(z)} \subset \Lambda$.*

Proof. Let $y \in X$ be such that for every $\varepsilon > 0$ there is some $z \in \Lambda$ such that $z \dashv_\varepsilon y$.

Assume by contradiction that $y \notin \Lambda$. Since Λ is invariant, we can assume (by iterating backwards) that $y \notin \overline{U}$.

Let $\delta > 0$ be such that $d(\partial U, f(\overline{U})) > \delta$. We will show that there cannot be a δ-pseudo orbit from Λ to y.

Indeed, given a point $x \in U$ we have that $f(x)$ is in $f(\overline{U})$ which implies by induction that a δ-pseudo-orbit starting at U must remain in U. This is a contradiction and proves the proposition.

□

The problem with topological attractors is they are not indecomposable in the sense that the dynamics inside Λ may not even be chain-recurrent (and in fact they can admit topological attractors contained inside themselves). On the other hand, they have the virtue of always existing (for example, the whole space is always a topological attractor, and by Remark 1.1.12 there always exist proper topological attractors when the homeomorphism is not chain-recurrent). To obtain in a sense better suited definitions we present now the definition of *attractors* and *quasi-attractors* which will appear throughout this text as one of the main objects of study.

[2]This is the usual definition in the literature related to this subject (see [BDV] Chapter 10). It seems that it could be better to call this sets *attracting sets*, since the word attractor may be sometimes misleading. However, we have chosen to keep this nomenclature.

Definition 1.1.3 (Attractor)**.** We say that a compact invariant set Λ is an *attractor* if it is a topological attractor and it is chain-recurrent. An attractor for f^{-1} is called a *repeller*.

$\diamondsuit$

Remark 1.1.15. We remark that it is usual in the literature also to define attractor by asking the stronger indecomposability hypothesis of being transitive, we use this definition since our context is better suited with the use of chain-recurrence. It is easy to see that if Λ is an attractor, then it is an isolated chain-recurrence class.

$\diamondsuit$

In general, a homeomorphism may not have any attractors, however, it will always have what we call *quasi-attractors*.

Definition 1.1.4. A compact invariant set $\mathcal{Q}$ is a *quasi-attractor* for a homeomorphism $f : X \to X$ if and only if it is a chain-recurrence class and there exists a nested sequence of open neighborhoods $\{U_n\}$ of $\mathcal{Q}$ such that:

- $\mathcal{Q} = \bigcap_n U_n$, and
- $f(\overline{U_n}) \subset U_n$.

A quasi-attractor for f^{-1} is called a *quasi-repeller*.

$\diamondsuit$

Remark 1.1.16.
- If $\varphi : X \to \mathbb{R}$ is a Lyapunov function for a homeomorphism $f : X \to X$, it is clear that $\mathcal{Q}$, the chain-recurrence class for which the value of φ is the minimum must be a Lyapunov stable set. Recall that a compact set Λ is *Lyapunov stable* for f if for every neighborhood U of Λ there exists a neighborhood V of Λ such that $f^n(V) \subset U$ for every $n \geq 0$.
- It is almost direct from the definition that a quasi-attractor must always be a Lyapunov stable set.
- Moreover, although we shall not use it, it is not hard to see that given a quasi-attractor of f one can always construct a Lyapunov function attaining a minimum in the given quasi-attractor. This follows from the proof of Conley's Theorem (see [Rob$_2$] Chapter 9.1).
- A quasi-attractor is a topological attractor if and only if it is an attractor. A quasi-attractor is an attractor if and only if it is isolated (as a chain-recurrence class). This implies that if a homeomorphism has no attractors, then it must have infinitely many distinct chain-recurrence classes.

$\diamondsuit$

We say that a chain-recurrence class $\mathcal{C}$ is a *bi-Lyapunov stable class* iff it is Lyapunov stable for both f and f^{-1}.

We now state a corollary from Proposition 1.1.14:

Corollary 1.1.17. *Let $\mathcal{Q}$ be a quasi-attractor for a homeomorphism $f : X \to X$. If $y \in X$ is a point such that for every $\varepsilon > 0$ there exists $z \in \Lambda$ such that $z \dashv_\varepsilon y$, then, $y \in \mathcal{Q}$. In particular, for every $z \in \mathcal{Q}$ we have that $\overline{W^u(z)} \subset \mathcal{Q}$.*

Proof. In Proposition 1.1.14 it is proved that if y is as in the statement it must belong to U_n for all U_n in the definition of quasi-attractor. This concludes.

□

To finish this section we will define two further notions of attracting sets which will also appear later in the text.

We say that a quasi-attractor $\mathcal{Q}$ is an *essential attractor* (as defined in [BLY]) if it has a neighborhood U which does not intersect any other quasi-attractors. The importance of these classes is given by a conjecture by Hurley [Hur] (known in certain topologies, see Theorem 1.1.22): *For typical dynamics, typical points converge to quasi-attractors.*

Sometimes, the invariant sets which attract important parts of the dynamics need not be chain-recurrence classes, and even not Lyapunov stable. Another important kind of "attracting sets" are *Milnor attractors* (see [Mi]). To define them we first define the *topological basin* of a compact invariant set K as:

$$\mathrm{Bas}(K) = \{y \in X \ : \ \omega(y) \subset K\}$$

We say that a compact f-invariant set K is a *Milnor attractor* if $\mathrm{Leb}(\mathrm{Bas}(K)) > 0$ and for every $K' \subset K$ compact, invariant different from K one has that $\mathrm{Leb}(\mathrm{Bas}(K')) < \mathrm{Leb}(\mathrm{Bas}(K))$.

The definition seems a little stronger than demanding that the basin has positive Lebesgue measure, but a simple Zorn's Lemma argument gives:

Lemma 1.1.18 (Lemma 1 of [Mi])**.** *Let K be a compact invariant set such that $\mathrm{Leb}(\mathrm{Bas}(K)) > 0$ then, there exists $K' \subset K$ a compact invariant set which is a Milnor attractor.*

In some situations, we can have a stronger notion of attractor. We say that a compact f-invariant set K is a *minimal Milnor attractor* if $\mathrm{Leb}(\mathrm{Bas}(K)) > 0$ and $\mathrm{Leb}(\mathrm{Bas}(K')) = 0$ for every $K' \subset K$ compact invariant subset different from K.

1.1.6 Connecting lemmas

In the study of "typical" dynamics in the space of C^1-diffeomorphisms, the main tool is the study of periodic orbits which we hope to describe accurately the recurrent behavior. It is then important to control perturbations of orbits in order to create the desired behavior. This section reviews several orbit perturbation results (in 1.2.4 we shall review perturbations of the derivative which is the other main tool in the study of typical C^1-behavior) such as the closing and connecting lemmas and their consequences. All this results were for many time considered extremely difficult and technical. Nowadays, even if they remain subtle, many proofs have been considerably improved (see in particular [C_4] for simple proofs of some of the results and sketches of the rest). This section intends to be a mere presentation of the results, for an introduction see [C_4] and [BDV] appendix A.

We first introduce the well known *Closing Lemma* of Pugh ([Pu_1, Pu_2]) and a very important consequence which together with Kupka Smale's theorem was one of the first *genericity* results. Its simple and natural statement may hide its intrinsic difficulties, the references above explain why it is not that easy to perform such perturbation.

Theorem 1.1.19 (Closing Lemma [Pu_1])**.** *Given $f \in \mathrm{Diff}^1(M)$, $\mathcal{U}$ a neighborhood of f and $x \in \Omega(f)$, then, there exists $g \in \mathcal{U}$ such that $x \in \mathrm{Per}(g)$.*

The extension of the Closing Lemma to the C^2-topology is far beyond reach of the current techniques except in certain cases where one can control the recurrence and be able perform perturbations in higher topologies (see for example [Pu_3, CP]).

As a consequence of the Closing lemma and the fact that semicontinuous functions are continuous in a residual subset, Pugh obtained the following consequence from his theorem. We shall only sketch the proof to show how to use the techniques (see for example [C_4] Corollary 2.8 for a complete proof).

Corollary 1.1.20. *There exists a C^1-residual subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathcal{G}$ one has that $\overline{\mathrm{Per}(f)} = \Omega(f)$.*

Sketch. Since hyperbolic periodic points persist under C^1-perturbations (see Theorem 1.1.2) we get that the map $f \mapsto \overline{\mathrm{Per}_H(f)}$ which goes from $\mathrm{Diff}^1(M)$ to $\mathcal{K}(M)$ is semicontinuous (see Remark 1.1.7).

We obtain that there is a residual subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ consisting of diffeomorphisms where the map $f \mapsto \overline{\mathrm{Per}_H(f)}$ varies continuously with respect to the Hausdorff metric in $\mathcal{K}(M)$.

We claim that if f is such a continuity point, then $\overline{\mathrm{Per}(f)} = \Omega(f)$. Indeed, if this was not the case, using the Closing Lemma we could make the non-wandering set explode by creating a periodic point far from $\overline{\mathrm{Per}_H(f)}$ by an arbitrarily small perturbation[3] contradicting the continuity and concluding the proof.

$\square$

It may seem that creating a periodic point out of a recurrent (or non-wandering) one is equally as difficult as creating a connection between orbits x and y such that the omega-limit set of one intersects the alpha-limit set of the other. However, the difficulties that arise in this context are considerably larger and it took a long time to handle this case.

The Connecting Lemma was first proven by Hayashi in [Hay]. Then, many improvements appeared (see [Arn$_1$, WX, BC] for example). The statement we present is taken from [C$_1$] Theorem 5 and it is quite stronger.

Theorem 1.1.21 (Connecting Lemma [Hay, C$_1$])**.** *Let $f \in \mathrm{Diff}^1(M)$ and $\mathcal{U}$ a neighborhood of f. Then, there exists $N > 0$ such that every non periodic point $x \in M$ admits two neighborhoods $W \subset \hat{W}$ satisfying that:*

- *The sets $\hat{W}, f(\hat{W}), \dots, f^{N-1}(\hat{W})$ are pairwise disjoint.*
- *For every $p, q \in M \backslash (f(\hat{W}) \cup \dots \cup f^{N-1}(\hat{W}))$ such that p has a forward iterate $f^{n_p}(p) \in W$ and q has a backward iterate $f^{-n_q}(q) \in W$, there exists $g \in \mathcal{U}$ which coincides with f in $M \backslash (f(\hat{W}) \cup \dots \cup f^{N-1}(\hat{W}))$ and such that for some $m > 0$ we have $g^m(p) = q$.*

Moreover, $\{p, g(p), \dots, g^m(p)\}$ is contained in the union of the orbits $\{p, \dots, f^{n_p}(p)\}$, $\{f^{-n_q}(q), \dots, q\}$ and the neighborhoods $\hat{W}, \dots, f^N(\hat{W})$. Also, the neighborhoods $\hat{W}, W$ can be chosen arbitrarily small.

A much harder problem is to create orbits which realize in some sense the ε-pseudo-orbits since this will clearly require making several perturbations. We shall state the consequences of a connecting lemma for pseudo-orbits obtained in [BC] since we shall not use the perturbation result itself (previous partial results can be found in [Ab, BD$_2$, CMP, MP]).

Theorem 1.1.22 ([BC])**.** *There exists a C^1-residual subset $\mathcal{G}_{BC}$ of $\mathrm{Diff}^1(M)$ such that for $f \in \mathcal{G}_{BC}$ one has that:*

- $\overline{\mathrm{Per}(f)} = \mathcal{R}(f)$.
- *For $p \in \mathrm{Per}(f)$ we have that $\mathcal{C}(p) = H(p)$. In particular, homoclinic classes of f are disjoint or equal. Moreover, if two periodic points in $H(p)$ have the same index, then they are homoclinically related.*

[3]Notice that it is not hard to perturb a periodic point in order to make it hyperbolic.

- (Hurley's Conjecture) *There exists a residual subset $R \subset M$ such that for every $x \in R$ we have that $\omega(x)$ is a quasi-attractor.*

- *If $\mathcal{Q}$ is an essential attractor, then there exists U a neighborhood of $\mathcal{Q}$ such that for a residual set of points in U the ω-limit is contained in $\mathcal{Q}$.*

- *If a chain-recurrence class $\mathcal{C}$ is isolated, then, there exists U a neighborhood of $\mathcal{C}$ and $\mathcal{U}$ a neighborhood of f such that for every $g \in \mathcal{U}$ the maximal invariant subset of U is chain-recurrent.*

- ([CMP]) *The closure of the unstable manifold of a periodic orbit is a Lyapunov stable set. Moreover, the homoclinic class of a periodic point $p \in M$ is $H(p) = \overline{W^s(p)} \cap \overline{W^u(p)}$.*

When a chain recurrence class $\mathcal{C}$ has no periodic points we say that $\mathcal{C}$ is an *aperiodic class*.

As a direct consequence we obtain the following properties:

Corollary 1.1.23 ([BC])**.** *For $f \in \mathcal{G}_{BC}$ one has that:*

- *If a chain-recurrence class $\mathcal{C}$ is isolated, then $\mathcal{C}$ is a homoclinic class.*

- *If a chain-recurrence class $\mathcal{C}$ has non-empty interior, then it is a bi-Lyapunov stable homoclinic class.*

PROOF. The statement of those classes being homoclinic classes is a direct consequence of the fact that periodic points are dense in the chain-recurrence set and that chain-recurrence classes containing periodic points coincide with the homoclinic classes of the periodic points.

To prove that a homoclinic class with non-empty interior is bi-Lyapunov stable, notice that since it contains the unstable manifold of any periodic orbit in its interior, from the last statement of Theorem 1.1.22 it follows that it must be a quasi-attractor. The argument is symmetric and it also shows that it must also be a quasi-repeller.

□

The first statement of this corollary poses the following natural question (see [BDV] Problems 10.18 and 10.22):

Question 1.1.24. *Is an isolated homoclinic class of a C^r-generic diffeomorphism C^r-robustly transitive*[4]*?*

[4] We say that a chain recurrence class C of a diffeomorphism f is *C^r-robustly transitive* if there exists a C^r-neighborhood $\mathcal{U}$ of f and a neighborhood U of C such that the maximal invariant of U for $g \in \mathcal{U}$ is transitive.

We have given a negative answer to this question with C. Bonatti, S. Crovisier and N. Gourmelon in [BCGP] (see Appendix D)

In general, another difficult problem is to control that when one perturbs a pseudo-orbit in order to create a real orbit then the new orbit essentially "shadows" the pseudo-orbit (this will be better explained later). In general, this is not possible ([BDT]), however, one can approach pseudo-orbits for C^1-generic diffeomorphisms with a weak form of shadowing (see [Arn_2] for a previous related result).

Theorem 1.1.25 ([C_1] Theorem 1)**.** *There exists a C^1 residual subset $\mathcal{G}_H$ of $\text{Diff}^1(M)$ such that for any $\delta > 0$ there exists $\varepsilon > 0$ such that for any ε-pseudo-orbit $\{z_0, \ldots, z_k\}$ there exist a segment of orbit $\{x, \ldots, f^m(z_k)\}$ which is at Hausdorff distance smaller than δ from the pseudo-orbit. Moreover, if the ε-pseudo-orbit is periodic (i.e. $z_k = z_0$) then one can choose the orbit to be periodic.*

1.1.7 Invariant measures and the ergodic closing lemma

When studying the orbit structure of diffeomorphisms it is sometimes important to understand the recurrence from a more quantitative viewpoint to have better control on how the recurrent points affect the orbits which pass close to them. A main tool for measuring recurrence is ergodic theory which treats dynamics of bi-measurable, measure preserving transformations of measure spaces. When combined with topological dynamics one can obtain lots of information some of which will be used in this text. We will present here some of this theory and refer to the reader to [M_4] for a more complete review of ergodic theory of differentiable dynamics.

Let $f \in \text{Diff}^1(M)$ and μ a regular (probability) measure in the Borel σ-algebra of M. We shall denote the set of regular probability measures of M as $\mathcal{M}(M)$. We say that $\mu \in \mathcal{M}(M)$ is *invariant* if $\mu(f^{-1}(A)) = \mu(A)$ for every measurable set A. We denote the set of invariant measures of f as $\mathcal{M}_f(M) \subset \mathcal{M}(M)$. With the weak-$*$ topology in the space of measures of M we know that $\mathcal{M}(M)$ is convex and compact. It is easy to see that $\mathcal{M}_f(M)$ is also convex and compact.

An invariant measure μ is called *ergodic* if and only if f-invariant measurable sets have μ-measure 0 or 1. We denote the set of ergodic measures as $\mathcal{M}_{erg}(M)$ which can be seen to be the set of extremal points of $\mathcal{M}_f(M)$. We deduce from the usual Krein-Milman theorem (see [Rud]) the following consequence which will allow us in general to concentrate in ergodic measures (see [M_4] II.6 for a more general statement):

Proposition 1.1.26. *Let μ be an invariant measure and $\varphi : M \to \mathbb{R}$ such that*

$$\int_M \varphi d\mu > 0$$

then, there exists an ergodic measure μ' whose support is contained in the support of μ and such that

$$\int_M \varphi d\mu' > 0.$$

The same result holds[5] *for $\geq, <, \leq$.*

The importance of measuring the integral of real-valued functions with invariant measures is given by the well known Birkhoff ergodic theorem which guaranties that for knowing such integrals it is enough to average the values obtained in the orbit of a generic point:

Theorem 1.1.27 (Birkhoff Ergodic Theorem). *Let $f \in \mathrm{Diff}^1(M)$ and $\mu \in \mathcal{M}_f(M)$ an invariant measure. Given $\varphi : M \to \mathbb{R}$ a μ-integrable function we have that for μ-almost every point x there exists*

$$\lim \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^i(x) = \tilde{\varphi}(x).$$

Moreover, $\tilde{\varphi}(x)$ is μ-integrable and f-invariant (i.e. $\tilde{\varphi}(f(x)) = \tilde{\varphi}(x)$) and it satisfies that

$$\int_M \varphi d\mu = \int_M \tilde{\varphi} d\mu$$

Since when μ is ergodic f-invariant functions are (almost) constant we get that for μ-almost every point $x \in M$:

$$\lim \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^i(x) = \int \varphi d\mu$$

One can define the *statistical basin* of an ergodic measure μ as the set of points whose averages with respect to any continuous function φ converge towards $\int \varphi d\mu$:

$$\mathrm{Bas}(\mu) = \left\{ y \in M \ : \ \forall \varphi \in C^0(M, \mathbb{R}) \ , \ \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^i(y) \to \int_M \varphi d\mu \right\}$$

A direct consequence of Birkhoff theorem is that $\mu(\mathrm{Bas}(\mu)) = 1$, however, a C^1-generic diffeomorphism verifies that the support of invariant measures is very small from the topological point of view [ABC$_2$] (as well as from the point of view of Lebesgue measure [AvBo]). This suggest the following definition of measures which are sometimes also called *ergodic attractors* (see [BDV] chapter 11 and references therein for an introduction in the context we are interested in):

[5] In fact, for the non-strict inequalities one needs to use the ergodic decomposition theorem (see [M$_4$] II.6).

Definition 1.1.5 (SRB measures)**.** We say that an ergodic f-invariant measures is an *SRB measure* for f iff:

$$\mathrm{Leb}(\mathrm{Bas}(\mu)) > 0$$

$\diamondsuit$

Notice that in general one has that $\mathrm{supp}(\mu) \subset \overline{\mathrm{Bas}(\mu)}$ and that $\mathrm{Bas}(\mu) \subset \mathrm{Bas}(\mathrm{supp}(\mu))$ so we obtain that the support of an SRB measure always contains a Milnor attractor (see Lemma 1.1.18). In some (quite usual) circumstances, one can in fact guaranty that the support of an SRB measure is indeed a minimal Milnor attractor.

There is a way of generalizing the notion of eigenvalues of the derivative for points which are not periodic:

Definition 1.1.6 (Lyapunov Regular Points)**.** Given a C^1-diffeomorphism f of a d-dimensional manifold M, we say that a point $x \in M$ is *Lyapunov regular* if there are numbers $\lambda_1(x) < \lambda_2(x) < \ldots < \lambda_{m(x)}(x)$ and a decomposition $T_xM = E_1(x) \oplus \ldots \oplus E_{m(x)}(x)$ such that for every $1 \leq j \leq m(x)$ and every $v \in E_j(x) \backslash \{0\}$ we have:

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f^n v\| = \lambda_j(x)$$

The numbers $\lambda_j(x)$ are called *Lyapunov exponents* of the point x and the space $E_j(x)$ is called the *Lyapunov eigenspace* associated to $\lambda_j(x)$. We shall denote as $\mathrm{Reg}(f)$ to the set of Lyapunov regular points of f.

$\diamondsuit$

A remarkable fact is that given an invariant measure, typical points with respect to the measure are Lyapunov regular (see in contrast Theorem 3.14 of [ABC$_2$]).

Theorem 1.1.28 (Oseledet's Theorem [O])**.** *Given $f \in \mathrm{Diff}^1(M)$ and an f-invariant measure μ we have that the set of Lyapunov regular points has μ total measure ($\mu(\mathrm{Reg}(f)) = 1$).*

Remark 1.1.29. Notice that the set of Lyapunov regular points of f is f-invariant. Indeed, $\lambda_i(f(x)) = \lambda_i(x)$ and $E_i(f(x)) = D_xf(E_i(x))$. This implies in particular that given an ergodic measure μ the Lyapunov exponents of μ-generic points is constant. We can thus define $\lambda_i(\mu)$ for an ergodic measure[6] as $\int \lambda_i(x) d\mu$ and $m(\mu) = \int m(x) d\mu$.

$\diamondsuit$

[6]We integrate the quantities only as a way of saying that it equals the value for almost every point.

Given a periodic orbit $\mathcal{O}$ of a diffeomorphism f one can define the following f-invariant ergodic measure:

$$\mu_{\mathcal{O}} = \frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} \delta_x$$

To some extent, the closing lemma and pseudo-orbit connecting lemma give information on the union of the supports of these measures for C^1-generic diffeomorphisms and Theorem 1.1.25 says that in fact every chain-transitive set can be approached in the Hausdorff topology by the supports of such measures for a C^1-generic diffeomorphism. A very important tool yet to be presented is the celebrated Ergodic Closing Lemma of Mañe ([M_3]) which asserts that in fact one can perturb a generic point of a measure in order to shadow it. See [C_4] section 4.1 for a simple and modern proof of this result. An important consequence (with some improvement) is the following:

Theorem 1.1.30 (Ergodic Closing Lemma [M_3], [ABC_2] Theorem 3.8)**.** *There exists a C^1-residual subset $\mathcal{G}_E \subset \text{Diff}^1(M)$ such that if $f \in \mathcal{G}_E$ and μ is an ergodic measure, then, there exists $\mathcal{O}_n$ a sequence of periodic orbits such that:*

- *The measures $\mu_{\mathcal{O}_n}$ converge towards μ in the weak-$*$ topology.*
- *The supports of the measures $\mu_{\mathcal{O}_n}$ converge towards the support of μ in the Hausdorff topology.*
- *$m(\mu_{\mathcal{O}_n}) = m(\mu)$ for every n and the Lyapunov exponents $\lambda_i(\mu_{\mathcal{O}_n})$ converge towards $\lambda_i(\mu)$.*

We say an invariant ergodic measure μ is *hyperbolic* if all its Lyapunov exponents are different from 0.

1.2 Invariant structures under the tangent map

1.2.1 Cocycles over vector bundles

Consider a homeomorphism $f : X \to X$ of a metric space X and a vector bundle $p : \mathcal{X} \to X$. We say that $A : \mathcal{X} \to \mathcal{X}$ is a linear cocycle over f if it is a homeomorphism, we have that $f \circ p(v) = p \circ A(v)$ and $A_x : p^{-1}(\{x\}) \to p^{-1}(\{f(x)\})$ is a linear isomorphism.

This general abstract context particularizes to several applications of which we will only be interested in two:

- The *derivative of a diffeomorphism* $Df : TM \to TM$ is a linear cocycle over f where the vector bundle is given by the trivial projection $TM \to M$.

- $\mathcal{A} = (\Sigma, f, E, A, d)$ is a *large period linear cocycle* of dimension d and bounded by K iff

  $f : \Sigma \to \Sigma$ is a bijection such that every point of Σ is periodic and such that given $n > 0$ there are finitely many points in Σ of period smaller than n (in particular, Σ is at most countable).

  E is a vector bundle over Σ, this is, there exists $p : E \to \Sigma$ such that $p^{-1}(\{x\}) = E_x$ is a d-dimensional vector space endowed with a Euclidean metric $\langle \cdot, \cdot \rangle_x$.

  $A : x \in \Sigma \mapsto A_x \in GL(E_x, E_{f(x)})$ is such that $\|A_x\| < K$ and $\|A_x^{-1}\| < K$.

$\diamondsuit$

In fact, one can think (and it is what it will represent) of large period linear cocycles as the restriction of the derivative of a diffeomorphism to a subset of periodic points (which of course can be a unique periodic orbit). For this reason, in the core of this text we shall restrict ourselves to the study of the derivative of diffeomorphisms (and the restriction of that cocycle to invariant subsets), however, in Appendix A we will use the formalism of large period linear cocycles where some quantitative results can be put in qualitative form.

1.2.2 Dominated splitting

Consider $f \in \text{Diff}^1(M)$ and Λ an f-invariant subset of M. We say that a subbundle $E \subset T_\Lambda M$ is Df-invariant if we have that

$$Df(E(x)) = E(f(x)) \qquad \forall x \in \Lambda$$

Given E and F two Df-invariant subbundles of $T_\Lambda M$ we say that F *ℓ-dominates* E (and we denote it as $E \prec_\ell F$) iff for every $x \in \Lambda$ and any pair of unit vectors $v_E \in E(x)$ and $v_F \in F(x)$ we have that

$$\|D_x f^\ell v_E\| < \frac{1}{2} \|D_x f^\ell v_F\|$$

In general, we say that F *dominates* E (which we denote $E \prec F$) iff there exists ℓ such that $E \prec_\ell F$.

In some examples, there is a stronger form of domination, called *absolute domination* (in contrast to the previous concept sometimes called *pointwise domination*). We say that F *absolutely ℓ-dominates* E (and we denote it as $E \prec_\ell^{ab} F$) iff for any pair of points $x, y \in \Lambda$ and any pair of unit vectors $v_E \in E(x)$ and $v_F \in F(y)$ we have that:

$$\|D_x f^\ell v_E\| < \frac{1}{2}\|D_y f^\ell v_F\|$$

In a similar manner as above, we say that F *absolutely dominates* E (and denote it as $E \prec^{ab} F$) if there exists $\ell > 0$ such that $E \prec^{ab}_\ell F$.

It is possible to define domination in other ways. Notice that the concept of a bundle ℓ-dominating another one (both absolutely as pointwisely) is dependent on the metric chosen in TM. However, if a subbundle E is dominated by other subbundle F then this does not depend on the metric chosen. See [Gou$_1$] for information on possible changes of metric to get domination.

We say that an f-invariant subset Λ admits a *dominated splitting* if $T_\Lambda M = E \oplus F$ where E and F are non-trivial Df-invariant subbundles and $E \prec F$. If one has that $E \prec^{ab} F$ then we say that the dominated splitting is *absolute*.

More generally, a Df-invariant decomposition $T_\Lambda M = E_1 \oplus \ldots \oplus E_k$ over an f-invariant subset Λ is called a *dominated splitting* if for every $1 < i < k$ we have that $(E_1 \oplus \ldots \oplus E_{i-1}) \prec (E_i \oplus \ldots \oplus E_k)$. One can extend to absolute domination in a trivial manner.

A notational parentheses is that we will make a difference in $E \oplus F$ and $F \oplus E$ since in the first case we shall understand that $E \prec F$ and in the second one that $F \prec E$ (this is not always the notation used in the literature).

We shall now give some properties of dominated splittings, the proofs can be found in [BDV] appendix B.

Proposition 1.2.1 (Uniqueness). *Let $T_\Lambda M = E_1 \oplus \ldots \oplus E_k$ and $T_\Lambda M = F_1 \oplus \ldots \oplus F_k$ be two dominated splitting for f with $\dim E_i = \dim F_i$ for every i. Then, $E_i = F_i$ for every i.*

The uniqueness of the decomposition with fixed dimensions of the subbundles allows one to consider the maximal possible decomposition which we shall call the *finest dominated splitting*. When a set does not admit any dominated splitting we will say that its finest dominated splitting is in only one subbundle (we shall explicitly mention the possibility).

Proposition 1.2.2 (Finest dominated splitting). *If an f-invariant set Λ admits a (non-trivial) dominated splitting, then there exists a dominated splitting $T_\Lambda M = E_1 \oplus \ldots \oplus E_k$ ($k \geq 2$) such that every other dominated splitting $T_\Lambda M = F_1 \oplus \ldots \oplus F_l$ verifies that $l \leq k$ and that for every $1 \leq j \leq l$ one has that $F_j = E_i \oplus \ldots \oplus E_{i+t}$ for some $1 \leq i \leq k$ and $0 \leq t \leq k - i$.*

The fact that the invariant set needs not be compact will play an important role since we shall mainly study the behavior over periodic orbits and then use the following proposition to extend that behavior to the closure (it was in fact this property which made Mañe, and probably also Liao and Pliss, consider this notion for attacking the stability conjecture, see [M_3, M_5])

Proposition 1.2.3 (Extension to the closure and to the limit)**.** *Let $f_n \in \mathrm{Diff}^1(M)$ be a sequence of diffeomorphisms converging to a diffeomorphism f and let Λ_n be a sequence of f_n-invariant sets admitting ℓ-dominated splittings $T_{\Lambda_n}M = E_1^n \oplus \ldots \oplus E_k^n$ such that* $\dim E_i^n$ *does not depend on n nor on the point. Then, if*

$$\Lambda = \limsup \Lambda_n = \bigcap_{N>0} \overline{\bigcup_{n>N} \Lambda_n}$$

then Λ is a compact f-invariant set which admits a dominated splitting

$$T_\Lambda M = E_1 \oplus \ldots \oplus E_k$$

such that $E_i(x) = \lim E_k^n(x_k^n)$ for every $x_k^n \in \Lambda_n$ converging to x.

Remark 1.2.4. As a consequence of the previous proposition we obtain that if $T_\Lambda M = E_1 \oplus \ldots \oplus E_k$ is a dominated splitting for an f-invariant subset Λ the following properties are verified:

- The closure $\overline{\Lambda}$ admits a dominated splitting $T_{\overline{\Lambda}}M = E_1' \oplus \ldots \oplus E_k'$ which extends the previous splitting (i.e. restricted to Λ it coincides with $E_1 \oplus \ldots \oplus E_k$). This follows by applying the previous proposition with $f_n = f$, $\Lambda_n = \Lambda$ and $E_i^n = E_i$.
- The bundles E_i vary continuously, this means, if x_n is a sequence in Λ such that $x_n \to x \in \Lambda$ then $E_i(x_n) \to E_i(x)$. This follows by applying the previous proposition with $f_n = f$, $\Lambda_n = \Lambda$, $E_i^n = E_i$ and $x_n^k = x_k$ converging to x.
- There exists $\alpha > 0$ such that for $i \neq j$ the angle[7] between E_i and E_j is larger than α. This follows from the fact that the bundles vary continuously and extend to the closure, so, if the angle is not bounded from below then there must be a point where two bundles have non-trivial intersection contradicting the fact that the sum is direct.

$\diamondsuit$

Given a vector space V of dimension d with an inner product $\langle \cdot, \cdot \rangle$ and a k-dimensional subspace E of V we can express every vector of V in a unique way as $v + v^\perp$ where $v \in E$ and $v^\perp \in E^\perp$. We define $\mathcal{E}$, the *α-cone of E* as:

$$\mathcal{E} = \{v + v^\perp \in V \ : \ \|v^\perp\| \leq \alpha \|v\|\}$$

[7]We can define the angle between two subbundles as arccos of the supremum of the inner product between pairs of unit vectors, one in E_i and the other in E_j.

The interior of $\mathcal{E}$ is denoted as $\mathrm{Int}(\mathcal{E})$ and is the topological interior of $\mathcal{E}$ together with the vector 0. The dimension of $\mathcal{E}$ is the dimension of the largest subspace it contains.

Given a subset K of a manifold M a k-dimensional *cone field* is a continuous association of cones in T_xM to points x in K. It will be given by a continuous k-dimensional subbundle $E \subset T_K M$ together with a continuous function $\alpha : K \to \mathbb{R}$: so, the cone field will associate to $x \in K$ the $\alpha(x)$-cone of $E(x)$ which we shall denote as $\mathcal{E}(x)$.

One can define more general cones and cone fields and this is useful in other contexts ([BoG]), but for us it will suffice to consider this notion.

Proposition 1.2.5 (Cone fields)**.** *If Λ is a f-invariant set admitting a dominated splitting $T_\Lambda M = E \oplus F$. Then, there exists an open neighborhood U of Λ and a $\dim F$-dimensional cone field $\mathcal{E}$ defined in U such that for every $x \in U$ such that $f(x) \in U$ one has that*

$$Df(\mathcal{E}(x)) \subset \mathrm{Int}(\mathcal{E}(f(x)))$$

Conversely, if there exists a k-dimensional cone field $\mathcal{E}$ defined in an open subset U of M verifying that if $x \in U$ and $f(x) \in U$ it satisfies $Df(\mathcal{E}(x)) \subset \mathrm{Int}(\mathcal{E}(f(x)))$ then, if Λ is the maximal invariant subset of U we have a dominated splitting $T_\Lambda M = E \oplus F$ with $\dim F = k$.

We say that a set Λ admits a *dominated splitting of index k* if there is a dominated splitting $T_\Lambda M = E \oplus F$ with $\dim E = k$.

Remark 1.2.6 (Robustness of dominated splitting)*.* The previous proposition allows one to show that if Λ is an f-invariant set which admits a dominated splitting of index k, then there exists a neighborhood U of Λ and a neighborhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$ the maximal invariant set of U for g admits a dominated splitting of index k. Indeed, the first part allows one to construct a k-dimensional cone field in a neighborhood U of Λ which is Df-invariant in the sense explained in the statement of the proposition. This invariance is not hard to see is robust in the C^1-topology and it gives an open neighborhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$ the cone field will verify the converse part of the proposition.

$\diamondsuit$

1.2.3 Uniform subbundles

Given a f-invariant subset Λ and a Df-invariant subbundles $E \subset T_\Lambda M$ we say that E is *uniformly contracted* (resp. *uniformly expanded*) if there exist $N > 0$ (resp. $N < 0$) such that for every $x \in \Lambda$:

$$\|D_x f^N|_{E(x)}\| < \frac{1}{2}$$

Given a C^1-diffeomorphism f on a Riemannian manifold M we define the continuous map $Jf|_E : M \to \mathbb{R}$ such that $Jf|_E(x)$ is the $\dim E$-dimensional (oriented) volume of the parallelepiped generated by the Df image of a orthonormal basis in E.

We say that a Df-invariant subbundle $E \subset T_\Lambda M$ is *uniformly volume contracted* (resp. *uniformly volume expanding*) if there exists $N > 0$ (resp. $N < 0$) such that for every $x \in \Lambda$:

$$|Jf^N|_{E(x)}(x)| < \frac{1}{2}$$

Invariant measures may give a criteria for knowing weather invariant subbundles are uniform (see [C$_3$] Claim 1.7):

Proposition 1.2.7. *Consider a Df-invariant continuous subbundle $E \subset T_\Lambda M$ over a compact f-invariant subset $\Lambda \subset M$. We have that:*

(i) E is uniformly contracted if and only if for every invariant ergodic measure μ such that $\mathrm{supp}(\mu) \subset \Lambda$ *we have that the largest Lyapunov exponent of μ whose Lyapunov eigenspace is contained in E is strictly smaller than* 0.

(ii) E is uniformly volume contracted if and only if for every invariant ergodic measure μ such that $\mathrm{supp}(\mu) \subset \Lambda$ *we have that the sum of all the Lyapunov exponents of μ whose Lyapunov eigenspaces are contained in E is strictly smaller than* 0. *In particular, if E is one dimensional uniform volume contraction implies uniform contraction.*

An analogous statement holds for uniform expansion and uniform volume expansion.

Proof. We first prove (i). If E is uniformly contracted, it is clear that for every vector $v \subset E$ one has that

$$\limsup_n \frac{1}{n} \log \|Df^n v\| < 0$$

Which gives the direct implication. Now, assuming that E is not uniformly contracted, one can prove that there must be points $x_n \in \Lambda$ and $v_n \in E(x_n)$ such that

$$\|D_{x_n} f^j v_n\| \geq \frac{1}{2} \quad 0 \leq j \leq n$$

Consider the invariant measures $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x_n)}$. By compactness of $\mathcal{M}(M)$ we have that (modulo considering a subsequence) $\mu_n \to \mu$ which will be an invariant measure as it is easy to check.

We claim that μ must have a Lyapunov exponent larger or equal to 0 inside E. Indeed, consider (again modulo considering subsequences) the limit $F \subset E$ of the subspaces generated by v_n which will be contained in E by continuity of E. Let x be a generic point in the support of μ and v a vector in $F(x)$. We have that for every $\varepsilon > 0$ there are points x_n arbitrarily close to x such that v_n is arbitrarily close to v. This implies that the derivative along v for x will not be able to contract giving the desired Lyapunov exponent which is larger or equal to 0 (see [C_3] Claim 1.7 for more details, in particular, formalizing this idea requires passing to the unit tangent bundle of the manifold and consider the measures there).

To prove (ii) one proceeds in a similar way by noticing that the change of volume is related to the expansions and contractions in an orthonormal basis and the angles to which they are sent. The fact that the angles between Lyapunov eigenspaces vary subexponentially is a consequence of a stronger version of Oseledet's theorem (see for example [KH] Theorem S.2.9).

$\square$

With some more work one can prove the following result of Pliss [Pli] (see [ABC_2] Lemma 8.4 for a proof):

Lemma 1.2.8 (Pliss). *Let μ be an ergodic measure such that all of its Lyapunov exponents are negative, then μ is supported on a hyperbolic sink.*

Now we are ready to define some notions which will in some sense capture robust dynamical behavior as will be reviewed in subsection 1.2.6.

Definition 1.2.1. Let $f \in \text{Diff}^1(M)$ and Λ a compact f-invariant set such that its finest dominated splitting is of the form $T_\Lambda M = E_1 \oplus \ldots \oplus E_k$ (in this case we allow $k = 1$). We will say that:

- Λ is *hyperbolic* if either $k = 1$ and E_1 is uniformly expanded or contracted or there exists $1 < j \leq k$ such that $E_1 \oplus \ldots \oplus E_{j-1}$ is uniformly contracted and $E_j \oplus \ldots \oplus E_k$ is uniformly expanded.

- Λ is *strongly partially hyperbolic* if E_1 is uniformly contracted and E_k is uniformly expanded.

- Λ is *partially hyperbolic* if either E_1 is uniformly contracted or E_k is uniformly expanded.

- Λ is *volume partially hyperbolic* if both E_1 is uniformly volume contracted and E_k is uniformly volume expanded.

- Λ is *volume hyperbolic* if it is both volume partially hyperbolic and partially hyperbolic.

$\diamondsuit$

The definitions of volume partial hyperbolicity and volume hyperbolicity may vary in the literature as well as those of partial hyperbolicty and strong partial hyperbolicity. We warn the reader for that distinction.

Remark 1.2.9. Using Proposition 1.2.7 and Lemma 1.2.8 we get that if the finest dominated splitting of a compact invariant set is trivial (i.e. $k = 1$) and the set is either hyperbolic or partially hyperbolic then it must be a periodic sink or a source. When the finest dominated splitting is not trivial we have the following implications:

$$\text{Hyperbolic} \Rightarrow \text{Strong Partially Hyperbolic} \Rightarrow$$

$$\Rightarrow \text{Volume hyperbolic} \Rightarrow \text{Partially Hyperbolic}$$

Moreover if one extremal bundle is one-dimensional we have that:

$$\text{Volume Partially Hyperbolic} \Rightarrow \text{Volume Hyperbolic} \Rightarrow \text{Partially Hyperbolic}$$

$\diamondsuit$

Notation (Uniform bundles)**.** Let Λ be a compact f-invariant set admiting a dominated splitting of the form $T_\Lambda M = E_1 \oplus \ldots \oplus E_k$ which is the finest dominated splitting (where k may be equal to 1). Assume that j is the largest value such that E_j uniformly contracted and l the smallest such that E_{j+l} is uniformly expanded. If we denote a Df-invariant subbundle of $T_\Lambda M$ as E^s it will be implicit that $E^s = E_1 \oplus \ldots\ldots E_t$ with $t \leq j$. In a similar way, if we denote a Df-invariant subbundle as E^u it will be implicit that $E^u = E_{j+t} \oplus \ldots \oplus E_k$ with $t \geq l$.

In certain situations we may separate $E^s = E^{ss} \oplus E^{ws}$ (or $E^u = E^{wu} \oplus E^{uu}$) which will denote that the contraction in E^{ss} is stronger than the one in E^{ws}.

$\diamondsuit$

An important part of this thesis will be devoted to study diffeomorphisms such that the whole manifold is a partially hyperbolic (or strong partially hyperbolic) set. We shall say that f is *Anosov* (resp. *partially hyperbolic*, resp. *strong partially hyperbolic*, resp. *volume hyperbolic*) if M is a hyperbolic (resp. partially hyperbolic, resp. strong partially hyperbolic, resp. volume hyperbolic) set for f. We will review these concepts with more detail later.

We remark that there are alternative definitions of these global concepts, for example, it is usual (see [C_4]) to name a diffeomorphism *hyperbolic* if its chain-recurrent set is hyperbolic, in a similar way, it is defined in [CSY] a diffeomorphism to be *partially hyperbolic* if its chain-recurrent set admits a partially hyperbolic splitting.

Notation (Absolute Notions)**.** In many examples one gets a stronger version of these concepts which is given by the fact that the domination provided by the definitions (between the uniform bundles and the "central" or "neutral" ones) may be absolute instead of pointwise as we have been working with. In those cases we will add the word absolute before partial hyperbolicity, strong partial hyperbolicity or volume hyperbolicity depending on the context. Notice that in the hyperbolic case both notions coincide since uniform bundles are naturally absolutely dominated (this is another reason for choosing sometimes the definition of absolute domination).

$\diamondsuit$

We obtain the following robustness property which is quite straightforward from the definitions and Remark 1.2.6.

Proposition 1.2.10 (Robustness)**.** *Assume that Λ is a compact f-invariant set which is hyperbolic, then there exists U a neighborhood of Λ and $\mathcal{U}$ a C^1-neighborhood of f such that for every $g \in \mathcal{U}$ the maximal invariant set of g in U is also hyperbolic. The same holds for the concepts of partial hyperbolicity, strong partial hyperbolicity, volume hyperbolicity, volume partial hyperbolicity and the absolute versions.*

1.2.4 Franks-Gourmelon's Lemma

In this section we shall review some techniques of perturbation which allow to change the derivative of the diffeomorphism over a periodic orbit by a small C^1-perturbation. Notice that this cannot be done in higher topologies (not even in C^2 see [PS_6]).

The classical Franks' Lemma ([F_3]) states the following:

Theorem 1.2.11 (Franks' Lemma [F_3])**.** *Given a C^1-neighborhood $\mathcal{U}$ of a diffeomorphism f, there exists $\varepsilon > 0$ such that:*

- *given any finite set $\{x_1, \ldots, x_k\}$ in M,*

- *any neighborhood U of this finite set*

- *any set of linear transformations $A_i : T_{x_i}M \to T_{f(x_i)}M$ verifying that $\|A_i - D_{x_i}f\| < \varepsilon$ for every $1 \leq i \leq k$*

then there exists $g \in \mathcal{U}$ such that:

- $g = f$ *outside* U.

- $g(x_i) = f(x_i)$ *for every* $1 \leq i \leq k$.

- $D_{x_i}g = A_i$ *for every* $1 \leq i \leq k$.

In sections 3.1 and 3.2 we will use a stronger version of this Lemma which allows to have control on invariant manifolds of periodic points when one perturbs their derivatives. For doing this it is important to have a better understanding of the way one perturbs the cocycle of derivatives in order to make the perturbation step by step and in some sense "follow" the invariant manifolds.

Consider f a C^1-diffeomorphism. We denote as $\mathrm{Per}_j(f)$ the set of (stable) index j hyperbolic periodic points. Let $\mathcal{O}$ be a periodic orbit and E a Df invariant subbundle of $T_{\mathcal{O}}M$. We denote as $D_{\mathcal{O}}f_{/E}$ to the cocycle over the periodic orbit given by its derivative restricted to the invariant subbundle as defined in greater generality in subsection 1.2.1.

Let $\mathcal{O}$ be a periodic orbit and $\mathcal{A}_{\mathcal{O}}$ be a linear cocycle [8] over $\mathcal{O}$. We say that $\mathcal{A}_{\mathcal{O}}$ has a *strong stable manifold of dimension* i if the eigenvalues $|\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_d|$ of $\mathcal{A}_{\mathcal{O}}$ satisfy that $|\lambda_i| < \min\{1, |\lambda_{i+1}|\}$.

If the derivative of $\mathcal{O}$ has strong stable manifold of dimension i then classical results ensure the existence of a local, invariant manifold $W^{s,i}_{\varepsilon}(x)$ tangent to the the subspace generated by the eigenvectors of these i eigenvalues and imitating the behavior of the derivative (see [KH] Theorem 6.2.8 for a precise formulation and recall Theorem 1.1.2). In fact, $W^{s,i}_{\varepsilon}$ is characterized for being the set of points in an ε-neighborhood of $\mathcal{O}$ such that the distance of future iterates of those points and $\mathcal{O}$ goes to zero exponentially at rate faster than $\lambda_i + \varepsilon$ with small ε.

Let $\Gamma_i(\mathcal{O})$ be the set of cocycles over $\mathcal{O}$ which have a strong stable manifold of dimension i.

We endow $\Gamma_i(\mathcal{O})$ with the following distance, $d(\mathcal{A}_{\mathcal{O}}, \mathcal{B}_{\mathcal{O}}) = \max\{\|\mathcal{A}_{\mathcal{O}} - \mathcal{B}_{\mathcal{O}}\|, \|\mathcal{A}_{\mathcal{O}}^{-1} - \mathcal{B}_{\mathcal{O}}^{-1}\|\}$ where the norm is

$$\|\mathcal{A}_{\mathcal{O}}\| = \sup_{p\in\mathcal{O}}\{\frac{\|A_p(v)\|}{\|v\|} \ ; \ v \in T_pM\backslash\{0\}\}.$$

Let g be a perturbation of f such that the cocycles $D_{\mathcal{O}}f$ and $D_{\mathcal{O}}g$ are both in $\Gamma_i(\mathcal{O})$, and let U be a neighborhood of $\mathcal{O}$. We shall say that g *preserves locally the* i*-strong stable manifold of* f *outside* U, if the set of points of the i-strong stable

[8] Recall from subsection 1.2.1 that a linear cocycle $\mathcal{A}$ of dimension n over a transformation $f : \Sigma \to \Sigma$ can in this case be represented by a map $A : \Sigma \to GL(n, \mathbb{R})$. When one point $p \in \Sigma$ is $f-$periodic, the eigenvalues of the cocycle at p are the eigenvalues of the matrix given by $A_{f^{\pi(p)-1}(p)} \ldots A_p$

manifold of $\mathcal{O}$ outside U whose positive iterates do not leave U once they entered it, are the same for f and for g.

We have the following theorem due to Gourmelon which allows to perturb the derivative of periodic orbits while controlling the position of the invariant manifolds of them.

Theorem 1.2.12 ([Gou$_3$]). *Let f be a diffeomorphism, and $\mathcal{O}$ a periodic orbit of f such that $D_{\mathcal{O}}f \in \Gamma_i(\mathcal{O})$ and let $\gamma : [0,1] \to \Gamma_i(\mathcal{O})$ be a path starting at $D_{\mathcal{O}}f$. Then, given a neighborhood U of $\mathcal{O}$, there is a perturbation g of f such that $D_{\mathcal{O}}g = \gamma(1)$, g coincides with f outside U and preserves locally the i-strong stable manifold of f outside U. Moreover, given $\mathcal{U}$ a C^1 neighborhood of f, there exists $\varepsilon > 0$ such that if* $\operatorname{diam}(\gamma) < \varepsilon$ *one can choose $g \in \mathcal{U}$.*

We observe that the Franks' lemma for periodic orbits (Theorem 1.2.11) is the previous theorem with $i = 0$. Also, we remark that Gourmelon's result is more general since it allows to preserve at the same time more than one strong stable and more than one strong unstable manifolds (of different dimensions, see [Gou$_3$]).

1.2.5 Perturbation of periodic cocycles

In view of the techniques of perturbation of the derivative over finite sets of points reviewed in the previous section, it makes sense to try to understand what type of behavior one can create by (small) perturbations of the derivative of periodic orbits.

Of course, eigenvalues depend continuously on the matrices, so a small perturbation has only small effect on the derivative over a periodic orbit. However, the fact that we can perturb a small amount but on many points at a time gives that it is sometimes possible to get large effect by making a small perturbation (of the diffeomorphism) by accumulation of these effects. It turns out that the main obstruction for making such perturbations is the existence of a dominated splitting.

The first results of this kind were obtained by Frank's itself in his paper [F$_3$]. However, the progress made in [M$_3$] started the systematic study of perturbations of cocycles over periodic orbits.

Relevant development was obtained in [BDP] where the concept of *transitions* was introduced. Later, in [BGV] some results were recovered without the need for transitions, and recently, in [BoB], a kind of optimal result was obtained which in turn combines in a very nice way with the recent result of [Gou$_3$] (see Theorem 1.2.12).

In this section we shall present the results we shall use without proofs.

The first result we shall state is the result from [BDP] which uses the notion of *transitions*. It gives a dichotomy between the existence of a dominated splitting and the creation of homotheties by small perturbations along orbits.

Theorem 1.2.13 ([BDP]). *Let H be the homoclinic class of a hyperbolic periodic point p of a C^1-diffeomorphism f. Let Σ_p be the set of periodic points homoclinically related to p and assume that $E \subset T_{\Sigma_p}M$ is a Df-invariant subbundle. We have the following dichotomy:*

- *Either $E = E_1 \oplus E_2$ were E_i are Df-invariant subbundles and $E_1 \prec E_2$.*
- *Or, for every $\varepsilon > 0$ there exists a periodic point $q \subset \Sigma_p$ and a periodic linear cocycle $A : T_{\mathcal{O}(q)}M \to T_{\mathcal{O}(q)}M$ such that $\|A - D_{\mathcal{O}(q)}f\| < \varepsilon$ and we have that $A(f^{\pi(q)-1}) \dots A(q)$ is a linear homothety. Moreover, if $\det(D_p f^{\pi(p)}) \leq 1$ we can consider the homothety to be contracting.*

We shall not present a proof of this fact, the reader can consult [BDV] chapter 7 for a nice sketch of the proof. We will give though a proof of the following result to give the reader a taste on the idea of considering transitions.

Proposition 1.2.14 ([BDP]). *Let H be homoclinic class of a periodic point with $|det(D_p f^{\pi(p)})| > 1$, then there is a dense subset of periodic points in H having the same property.*

PROOF. Let U be an open set in H. There is a periodic point $q \in U$ homoclinically related to p. Consider $x \in W^s(\mathcal{O}(p)) \pitchfork W^u(\mathcal{O}(q))$ and $y \in W^s(\mathcal{O}(q)) \pitchfork W^u(\mathcal{O}(p))$. The set $\mathcal{O}(p) \cup \mathcal{O}(q) \cup \mathcal{O}(x) \cup \mathcal{O}(y)$ is a hyperbolic set. So, using the shadowing lemma (see [KH] Theorem 6.4.15 for example) we can obtain a periodic point $r \in U$, homoclinically related to p such that its orbit spends most of the time near $\mathcal{O}(p)$. Thus, it will satisfy that $|det(D_r f^{\pi(r)})| > 1$.

□

When we wish to use Theorem 1.2.12 we need to not only make small perturbations but also to make them in small paths which do not affect the index of the periodic points during the perturbation. The natural idea of considering the straight line between the initial cocycle and the homothety falls short of providing the desired perturbation and it is quite a difficult problem to really realize the desired perturbation. A recent result of J.Bochi and C.Bonatti ([BoB] which extends previous development in this sense by [BGV]) provides a solution to this problem as well as it investigates which kind of paths of perturbations can be realized in relation to the kind of domination a cocycle admits. We shall state a quite weaker version of their result and avoid the (very natural) mention to large period linear cocycles and work instead with the derivative and paths of perturbations. We refer the interested reader to Appendix A in order to get a more complete account with complete proofs of partial results.

Recall from the previous subsection that given a periodic orbit $\mathcal{O}$ we denote $\Gamma_i(\mathcal{O})$ to be the set of cocycles over $\mathcal{O}$ which have strong stable manifold of dimension i endowed with the distance considered there.

Theorem 1.2.15 ([BGV, BoB]). *Let $f : M \to M$ be a C^1-diffeomorphism and p_n a sequence of periodic points whose periods tend to infinity and their orbits $\mathcal{O}_n$ converge in the Hausdorff topology to a compact set Λ. Let*

$$T_\Lambda M = E_1 \oplus \ldots \oplus E_k$$

be the finest dominated splitting over Λ (where k may be 1). Then, for every $\varepsilon > 0$ there exists $n > 0$ such that an ε-perturbation of the derivative along $\mathcal{O}_n$ makes all the eigenvalues of the orbit in the subspace E_i to be equal. Moreover, if the determinants of $D_{p_n} f^{\pi(p_n)}|_{E_i}$ have modulus smaller than 1 for every n and the periodic orbits have strong stable manifold of dimension j (which must be strictly larger than the dimension of $E_1 \oplus \ldots \oplus E_{i-1}$) then, given $\varepsilon > 0$ there is $n > 0$ and a path $\gamma : [0,1] \to \Gamma_j(\mathcal{O}_n)$ such that:

- $\operatorname{diam}(\gamma) < \varepsilon$.

- $\gamma(0) = D_{\mathcal{O}_n} f$.

- $\gamma(1)$ *has all its eigenvalues of modulus smaller than 1 in E_i.*

We recommend reading Lemma 7.7 of [BDV] whose (simple) argument can be easily adapted to give this result in dimension 2.

1.2.6 Robust properties and domination

In this subsection we shall explain certain results which are consequence of the perturbation results reviewed in the previous subsections.

Possibly, one of the departure points of this study was the study of the *stability conjecture* finally solved in [M$_5$]. We say that a diffeomorphism f is $\mathcal{R}$*-stable* if and only if there exists a C^1-neighborhood of f such that for every $g \in \mathcal{U}$ the diffeomorphism g restricted to $\mathcal{R}(g)$ is conjugated to f restricted to $\mathcal{R}(f)$. See [C$_4$] section 7.7 for a modern proof of the following result (which has been an underlying motivation for the development of differentiable dynamics) and more references for related results.

Theorem 1.2.16 (Stability Conjecture, [PaSm, M$_5$]). *A diffeomorphism f is $\mathcal{R}$-stable if and only if $\mathcal{R}(f)$ is hyperbolic.*

One of the robust properties we will be most interested in is in isolation of classes (and in particular quasi-attractors). For homoclinic classes of generic diffeomorphisms, isolation is enough to guaranty the existence of some invariant geometric structure. The following result of [BDP] was preceded by results in [M_3, DPU].

Theorem 1.2.17. *There exists a residual subset $\mathcal{G}_{BDP} \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathcal{G}_{BDP}$ and $\mathcal{C}$ is an isolated chain-recurrence class, then, $\mathcal{C}$ is volume partially hyperbolic. Moreover, if a homoclinic class H of a diffeomorphism $f \in \mathcal{G}_{BDP}$ does not admit any non-trivial dominated splitting then it is contained in the closure of sinks and sources ($\overline{\mathrm{Per}_0(f) \cup \mathrm{Per}_d(f)}$).*

Sketch. The proof of the first statement goes as follows: First, by a classical Baire argument, one shows that there is a residual subset of $\mathcal{G}_{BDP}$ of $\mathrm{Diff}^1(M)$ such that if $\mathcal{C}$ is an isolated chain-recurrence class of a diffeomorphism $f \in \mathcal{G}_{BDP}$ then $\mathcal{C}$ is a homoclinic class and there exists a neighborhood $\mathcal{U}$ of f and a neighborhood U of $\mathcal{C}$ such that for every $g \in \mathcal{G}_{BDP} \cap \mathcal{U}$ we have that the maximal invariant set of g in U is a homoclinic class (see Theorem 1.1.22 and the discussions after).

Now, consider $\mathcal{C}$ an isolated chain-recurrence class of a diffeomorphism $f \in \mathcal{G}_{BDP}$. Let p be a hyperbolic periodic point of f such that $\mathcal{C} = H(p)$ and consider Σ_p the set of hyperbolic periodic points homoclinically related to p.

Assume that $\mathcal{C}$ does not admit a non-trivial dominated splitting, then, by Remark 1.2.4 we know that Σ_p cannot admit a dominated splitting. Now, by Theorem 1.2.13 we know that we can make an arbitrarily small perturbation of the derivative of some periodic point in order to make it an homothety.

This perturbation can be made dynamically by using Theorem 1.2.11 creating a sink or a source inside U the neighborhood of $\mathcal{C}$. The chain-recurrence class of a sink or a source is the point itself (see Remark 1.1.8), and since sinks and sources persist under perturbations (Theorem 1.1.2), we find a diffeomorphism $g \in \mathcal{G}_{BDP} \cap \mathcal{U}$ having more than one chain-recurrence class in U, a contradiction.

Now, let $T_{\mathcal{C}}M = E_1 \oplus \ldots \oplus E_k$ be the finest dominated splitting over $\mathcal{C} = H(p)$. Assume that E_1 is not uniformly volume contracting (the same argument applied to f^{-1} will show that E_k is uniformly volume expanding).

By Proposition 1.2.7 we know that there exists an ergodic measure μ supported on $H(p)$ such that the sum of the Lyapunov exponents of μ in E_1 is larger or equal to 0. By the Ergodic Closing Lemma (Theorem 1.1.30) there are periodic orbits $\mathcal{O}_n$ converging in the Hausdorff topology towards $\mathrm{supp}(\mu)$ and such that for n large enough, the sum of the eigenvalues of $\mathcal{O}_n$ in the invariant bundle E_1 is larger than or equal to $-\varepsilon$ with small ε. These periodic orbits belong to $H(p)$ since we have assumed that it is an isolated chain-recurrence class.

Now, by the argument of Proposition 1.2.14 we get a dense[9] subset of $H(p)$ of periodic orbits such that the sum of the eigenvalues in E_1 is larger than $-\varepsilon$ with arbitrarily small ε. Since E_1 does not admit a subdominated splitting in $H(p)$ we are able by using again Theorem 1.2.13 to create sources inside U by small perturbations of f.

For the second statement, the proof is very similar. Consider a homoclinic class H of a diffeomorphism f and we can assume that the residual subset $\mathcal{G}_{BDP}$ verifies that for g in a neighborhood of f the continuation H_g of H is close to H in the Hausdorff topology.

Using Proposition 1.2.14 one can make the periodic points which one can turn into sinks or sources ε-dense in H. A Baire argument then allows to show that if the homoclinic class admits no-dominated splitting then it contained in the closure of the set of sinks and sources.

□

We remark that Theorem 1.2.15 together with Franks' Lemma (Theorem 1.2.11) allows also to obtain that if a chain-recurrence class is not accumulated by infinitely many sinks or sources then it admits a non-trivial dominated splitting (see [ABC$_1$]). This results can be used in order to re-obtain the examples presented in [BD$_2$] where homoclinic classes of generic diffeomorphisms accumulated by infinitely many sinks and sources were constructed.

An immediate consequence is that we obtain a criterium for guaranteeing that a homoclinic class is not isolated (this is a key ingredient in constructing examples of dynamics with no attractors).

Corollary 1.2.18. *Let f be a C^1-generic diffeomorphism of M and H a homoclinic class of f which is not volume partially hyperbolic. Then, H is not isolated.*

Remark 1.2.19. Indeed, the proof of Theorem 1.2.17 allows one to show that if $f \in \mathcal{G}_{BDP}$ and H is a homoclinic class such that:

- The finest dominated splitting in H is of the form $T_H M = E_1 \oplus \dots \oplus E_k$.
- H has a periodic point q verifying that $\det(Df^{\pi(q)}|_{E_1}) \geq 1$

then H is contained in the closure of the set of sources of f.

♢

[9] The fact that we are able to create a dense subset of periodic orbits with such behavior is crucial in the proof, since a priori we do not know if the measure for which the volume contraction is not satisfied has total support or not, and whether the finest dominated splitting inside that subbundle is finer or not than the global one.

Another consequence of Theorem 1.2.17 is the global characterization of diffeomorphisms which are robustly transitive (or even, C^1-generic diffeomorphisms which are transitive). The following result is one of the motivations of many of the results we will present in this thesis (the optimality of this result from the point of view of the obtained Df-invariant geometric structure is given by the examples of [BV]):

Corollary 1.2.20. *If $f \in \mathcal{G}_{BDP}$ and $\mathcal{R}(f) = M$, then f is volume partially hyperbolic. Also, if $f \in \mathcal{G}_{BDP}$ and there is a chain-recurrence class $\mathcal{C}$ of f with non-empty interior, then $\mathcal{C}$ admits a non-trivial dominated splitting.*

In dimension 2, this result was shown by Mañe in [M_3]: if a C^1-generic diffeomorphism of a surface is transitive, then, f is Anosov (recall Remark 1.2.9). Indeed, together with a result from Franks ([F_1]) we get the following characterization of robust behaviour in terms of the dynamics of the tangent map:

Theorem 1.2.21 ([F_1, M_3])**.** *A diffeomorphism f of a surface has a C^1-neighborhood $\mathcal{U}$ such that every $g \in \mathcal{U}$ is chain-recurrent if and only if f is an Anosov diffeomorphism of $\mathbb{T}^2$.*

A remarkable feature of this result is that it leaves in evidence the fact that robust dynamical behavior is in relation with the topology of the state space (and even the isotopy class). This relation is given through the appearance of a geometric structure which is invariant under the tangent map of the diffeomorphism. This leads to the following idea whose understanding represents a main challenge:

Robust dynamical behaviour $\Leftrightarrow$ Invariant Structures $\Leftrightarrow$ Topological Properties

Other than in dimension 2, very few is known in this respect other than what it was reviewed in this section (which represents hints on the direction of giving invariant geometric structures by the existence of robust dynamical behavior). In dimension 3, the fact that Corollary 1.2.20 admits a stronger form suggests that it may be possible to search for results with similar taste as Theorem 1.2.21 (we shall review some of the known results later):

Theorem 1.2.22 ([DPU])**.** *Let M be a 3-dimensional manifold and $f \in \mathcal{G}_{BDP}$ be chain-recurrent. Then, f is volume hyperbolic (i.e. partially hyperbolic and volume partially hyperbolic). It can present the following forms of domination:*

$TM = E^{cs} \oplus E^u$,

$TM = E^s \oplus E^c \oplus E^u$ *or*

$TM = E^s \oplus E^{cu}$.

We recommend reading [BDV] chapter 7 for a review on robust transitivity and for a survey of examples which show how these results are optimal from the point of view of the geometric structures obtained.

In dimension 3 the main examples of transitive strong partially hyperbolic diffeomorphisms fall in the following classes: Fiber bundles whose base is Anosov, Time one maps of Anosov flows or Examples derived from Anosov diffeomorphisms in $\mathbb{T}^3$. See [BWi].

Remark 1.2.23. It is important to remark that all this results give pointwise domination and not the absolute one. Indeed, it is not hard to construct examples which verify all these robust properties and fail to admit absolute domination between the invariant subbundles.

$\diamondsuit$

1.2.7 Homoclinic tangencies and domination

In this subsection we shall review certain properties of diffeomorphisms which are far from homoclinic tangencies. We refer the reader to [CSY, LVY] for the latest results on dynamics of diffeomorphisms C^1-far from tangencies.

In this section we shall recall some result whose germ can be traced to [PS_1] where it was proved that a diffeomorphism of a surface which is far away from homoclinic tangencies must admit a dominated splitting on the closure of the saddle hyperbolic periodic points.

First, we define the notion of a homoclinic tangency: Given a hyperbolic periodic saddle p of a C^1-diffeomorphism f, we say that p has a *homoclinic tangency* if there exists a point of non-transverse intersection between $W^s(\mathcal{O}(p))$ and $W^u(\mathcal{O}(p))$.

We denote as $\text{Tang} \subset \text{Diff}^1(M)$ to the set of diffeomorphisms f having a hyperbolic saddle with a homoclinic tangency. As a consequence of results which relate the existence of a dominated splitting with the creation of homoclinic tangencies for periodic orbits [W_1, W_2] as well as some adaptations of the ergodic closing lemma (see [ABC_2]), in [C_3] the following result is proved:

Theorem 1.2.24 ([C_3] Corollary 1.3)**.** *Let f be a diffeomorphism in $\text{Diff}^1(M)\backslash\overline{\text{Tang}}$. Then, there exists a C^1-neighborhood $\mathcal{U}$ of f integers $\ell, N > 0$ and constants $\delta, \rho > 0$ such that for every $g \in \mathcal{U}$ and every ergodic g-invariant measure μ the following holds:*

Let x be a μ-generic point and $T_{\mathcal{O}(x)}M = E_- \oplus E^c \oplus E_+$ be the Oseledet's splitting into the Lyapunov eigenspaces corresponding respectively to Lyapunov exponents in $(-\infty, -\delta)$, $[-\delta, \delta]$ and $(\delta, +\infty)$, so:

- $\dim E^c \leq 1$.

- *The splitting $E_- \oplus E^c \oplus E_+$ is ℓ-dominated (hence it extends to* $\mathrm{supp}(\mu)$*).*

- *For μ-almost every point we have that:*

$$\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} \log \|Df^N|_{E_-(f^{iN}(x))}\| < -\rho \qquad \lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} \log \|Df^{-N}|_{E_+(f^{-iN}(x))}\| > \rho$$

To finish this subsection, we present the following result of [ABCDW] which will be used later.

Theorem 1.2.25 ([ABCDW], [Gou_2])**.** *There exists a residual subset $\mathcal{G}$ of* $\mathrm{Diff}^1(M)$ *such that if $f \in \mathcal{G}$ and H is a homoclinic class of f having periodic points of (stable) index s and s'. Then, for every $s < j < s'$ we have that H contains periodic points of index j. In particular, if there is no perturbation of f which creates a homoclinic tangency for a periodic point in H then H admits a dominated splitting of the form $T_H M = E \oplus G_1 \oplus \dots G_k \oplus F$ with* $\dim E = s$, $\dim F = d - s'$ *and* $\dim G_j = 1$ *for every j.*

We would like to point out that with the new techniques of perturbation given by [BoB] and [Gou_3] one can give a proof of the fact that a homoclinic class of a C^1-generic diffeomorphism is *index complete* which is very direct: First, if a homoclinic class has periodic points of index i and j, then, by perturbing the derivative of some periodic points it is possible to get periodic orbits of index in between (by the results of [BoB]). After, the results of [Gou_3] allows one to make the perturbation in order to keep the necessary homoclinic relations in order guarantee that the point remains in the homoclinic class after perturbation.

1.2.8 Domination and non-isolation in higher regularity

As well as in the case of C^1-topology, we can obtain a similar criterium to obtain non-isolation of a homoclinic class for C^r-generic diffeomorphisms combining the main results of [BD_4] and [PaV] (it is worth also mentioning [Rom]). The only cost will be that we must consider a new open set and that the accumulation by other classes is not as well understood.

We state a consequence of the results in those papers in the following result. We shall only use the result in dimension 3, so we state it in this dimension, it can be modified in order to hold in higher dimension but it would imply defining sectionally dissipative saddles (see [PaV]).

Theorem 1.2.26 ([BD_4] and [PaV])**.** *Consider $f \in \mathrm{Diff}^r(M)$ with* $\dim M = 3$*, a C^1-open set $\mathcal{U}$ of $\mathrm{Diff}^r(M)$ ($r \geq 1$), a hyperbolic periodic point p of f such that its continuation p_g is well defined for every $g \in \mathcal{U}$ and such that:*

- *The homoclinic class $H(p_g)$ admits a partially hyperbolic splitting of the form $T_{H(p)}M = E^{cs} \oplus E^u$ for every $g \in \mathcal{U}$.*

- *The subbundle E^{cs} admits no decomposition in non-trivial $Dg-$invariant subbundles which are dominated.*

- *There is a periodic point $q \in H(p_g)$ such that* $\det(Dg_q^{\pi(q)}|_{E^{cs}(q)}) > 1$.

Then, there exists a C^1-open and C^1-dense subset $\mathcal{U}_1 \subset \mathcal{U}$ and a C^r-residual subset $\mathcal{G}_{PV}$ of $\mathcal{U}_1$ such that for every $g \in \mathcal{G}_{PV}$ one has that $H(p_g)$ intersects the closure of the set of periodic sources of g.

The conditions of the Theorem are used in [BD$_4$] in order to create robust tangencies for a hyperbolic set for diffeomorphisms in an C^1-open and dense subset $\mathcal{U}_1$ of $\mathcal{U}$. Then, using similar arguments as in section 3.7 of [BLY] one creates tangencies associated with periodic orbits which are sectionally dissipative for f^{-1} which allows to use the results in [PaV] to get the conclusion.

1.3 Plaque families and laminations

1.3.1 Stable and unstable lamination

As for periodic orbits, when a compact invariant subset admits a dominated splitting with one uniform extremal subbundle, one can in a sense integrate the subbundle in order to translate the uniformity of the bundle in a dynamical property in the manifold. The following result is classical, the standard proof can be found in [Sh, HPS] (see also [KH] chapter 6).

Theorem 1.3.1 (Strong Unstable Manifold Theorem)**.** *Let Λ be a compact f-invariant set which admits a dominated splitting of the form $T_\Lambda M = E^{cs} \oplus E^u$ where E^u is uniformly expanded. Then, there exists a* lamination *$\mathcal{F}^u$ such that:*

- *For every $x \in \Lambda$ the leaf $\mathcal{F}^u(x)$ through x is an injectively immersed copy of $\mathbb{R}^{\dim E^u}$ tangent at x to $E^u(x)$.*

- *The leaves of $\mathcal{F}^u$ form a partition, this is, for $x, y \in \Lambda$ we have that either $\mathcal{F}^u(x)$ and $\mathcal{F}^u(y)$ are disjoint or coincide.*

- *There exists $\rho > 0$ such that points of $\mathcal{F}^u(x)$ are characterized in the following way:*

$$y \in \mathcal{F}^u(x) \Leftrightarrow \lim_{n\to\infty} \frac{1}{n} \log(d(f^{-n}(x), f^{-n}(y))) < -\rho$$

- *Leaves vary continuously in the C^1-topology: If $x_n \in \Lambda \to x \in \Lambda$ we have that $\mathcal{F}^u(x_n)$ tends uniformly in compact subsets to $\mathcal{F}^u(x)$ in the C^1-topology.*

- *There exists a neighborhood U of Λ such that the leaves also vary continuously in the C^1-topology for points in the maximal invariant subset of U for diffeomorphisms g close to f.*

By *lamination* on a set K we mean a collection of disjoint C^1 injectively immersed manifolds of the same dimension (called *leaves*) such that there exists a compact metric space Γ such that for every point $x \in K$ there exists a neighborhood U and a homeomorphism $\varphi : U \cap K \to \Gamma \times \mathbb{R}^d$ such that if L is a leaf of the lamination and $\tilde{L}$ a connected component of $L \cap U$ then $\varphi|_{\tilde{L}}$ is a C^1-diffeomorphism to $\{s\} \times \mathbb{R}^d$ for some $s \in \Gamma$ (this implies that in K they are tangent to a continuous subbundle of $T_K M$).

For a lamination $\mathcal{F}$ on a compact set $K \subset M$ we shall always denote as $\mathcal{F}(x)$ to the leaf of $\mathcal{F}$ through x. It is worth remarking that in Theorem 1.3.1 the set laminated by $\mathcal{F}^u$ need not coincide with Λ as it may be (and it is in various situations) strictly larger.

For hyperbolic sets, this results gives two transversal laminations which will admit a *local product structure* and dynamical properties. This allows to obtain the well known *shadowing lemma* (see [Sh]). In particular, we obtain the following corollary in quite a direct way:

Proposition 1.3.2. *Let $\mathcal{C}$ be a chain-recurrence class of a diffeomorphism f which is hyperbolic. Then, it is isolated and coincides with the homoclinic class of any of its periodic points. In particular, it is transitive.*

1.3.2 Locally invariant plaque families

In order to search for dynamical or topological consequences of having a geometric structure invariant under the tangent map, it is important to try to "project" into the manifold the information we have on the tangent map.

A model of this kind of projection was given in Theorem 1.3.1 where we saw that the dynamics of the tangent map on uniform bundles project into similar uniform behavior in invariant submanifolds of the same dimension.

When the invariant bundles are not uniform, we are not able to obtain such a description (much in the way it is in general not possible to understand the local behavior of a real-valued function when the derivative is 1) but we are able to obtain certain plaque-families which sometimes help in reducing the ambient dimension and transforming problems in high dimensional dynamics into lower dimensional ones.

Theorem 1.3.3 ([HPS] Theorem 5.5)**.** *Let Λ be a compact f-invariant set endowed with a dominated splitting of the form $T_\Lambda M = E \oplus F$. Then, there exists a continuous map $\mathcal{W} : x \in \Lambda \mapsto \mathcal{W}_x \in \mathrm{Emb}^1(E(x), M)$ such that:*

- *For every $x \in \Lambda$ we have that $\mathcal{W}_x(0) = x$ and the image of $\mathcal{W}_x$ is tangent to $E(x)$ at x.*
- *It is* locally invariant*, i.e. There exists $\rho > 0$ such that $f(\mathcal{W}_x(B_\rho(0))) \subset \mathcal{W}_{f(x)}(E(f(x)))$ for every $x \in \Lambda$.*

Remark 1.3.4. In case one has a dominated splitting of the form $T_\Lambda M = E \oplus F \oplus G$ one can obtain (by applying the previous theorem to $E \oplus F$ and to $F \oplus G$ with f^{-1}) a locally invariant plaque family tangent to F as well.

$\diamondsuit$

We will usually (in case no confusion appears) abuse notation and denote $\mathcal{W}_x$ to $\mathcal{W}_x(E(x))$. Also, $\overline{\mathcal{W}_x}$ will denote the closure of $\mathcal{W}_x(E(x))$ which we can assume is the image of a closed ball of $\mathbb{R}^{\dim E}$. The proof of the theorem allows one to obtain a uniform version of this result, in fact, one obtains that the locally invariant plaque family can also be chosen continuous with respect to the diffeomorphism in a neighborhood of f and defined in the maximal invariant subset by that diffeomorphism in a neighborhood of Λ (see [CP] Lemma 3.7).

Remark 1.3.5. Since these locally invariant manifolds are not dynamically defined they have no uniqueness properties a priori. They may even have wild intersections between them (see [BuW$_2$] for a construction which is slightly more "friendly" which they call *fake foliations*).

$\diamondsuit$

When an invariant plaque family has dynamical properties one can often recover certain uniqueness properties (see Chapter 5 of [HPS] or Lemma 2.4 of [C$_3$]):

Proposition 1.3.6. *Let Λ be a compact set admitting a dominated splitting $T_\Lambda M = E \oplus F$. There exists $\varepsilon > 0$ such that if there exists a plaque family $\{\mathcal{W}_x\}_{x\in\Lambda}$ tangent to E verifying that:*

- *Every plaque $\mathcal{W}_x$ has diameter smaller than ε.*
- *The plaques verify the following* trapping condition*:*

$$\forall x \in \Lambda \quad f(\overline{\mathcal{W}_x}) \subset \mathcal{W}_{f(x)}$$

Then, the following properties are verified:

(Uniqueness) *Any locally invariant plaque family* $\{\mathcal{W}'_x\}_{x\in\Lambda}$ *tangent to* E *verifies that for every* $x \in \Lambda$ *the intersection* $\mathcal{W}'_x \cap \mathcal{W}_x$ *is open relative to both plaques.*

(Coherence) *Given* $x, y \in \Lambda$ *such that* $\mathcal{W}_x \cap \mathcal{W}_y \neq \emptyset$ *then we have that the intersection is open relative to both plaques.*

(Robust Trapping) *There exists* $\delta > 0$ *such that if* $y \in \mathcal{W}_x \cap \Lambda$ *is at distance smaller than* δ *from* x *then we have that* $f(\overline{\mathcal{W}_y}) \subset \mathcal{W}_{f(x)}$*. Moreover, if* Λ *is the maximal invariant set in a neighborhood* U*, then, there exists* $\mathcal{U}$ *a neighborhood of* f *such that for every* $g \in \mathcal{U}$ *the maximal invariant set in* U *has a plaque family which verifies the same trapping condition.*

PROOF. One can choose a neighborhood U of Λ and a neighborhood $\mathcal{U}$ of f such that the maximal invariant set in U will have a dominated splitting for every $g \in \mathcal{U}$. Moreover, there will be a cone field $\mathcal{E}^E$ around E (resp. $\mathcal{E}^F$ of F) which is invariant for every point in U whose backward (resp. forward) iterate is also in U: i.e. verifies that $Df_x^{-1}\mathcal{E}^E(x) \subset \operatorname{Int}\mathcal{E}^E(f^{-1}(x))$ when $x, f^{-1}(x) \in U$ (resp. $Df_x\mathcal{E}^F(x) \subset \operatorname{Int}\mathcal{E}^F(f(x))$ when $x, f(x) \in U$). See Proposition 1.2.5 and Remark 1.2.6.

Choose ε-small enough so that the expansion in F dominates the one in E for every pair of points at distance smaller than ε (this is trivially verified for every ε if the domination is absolute).

Assuming any of the first two properties of the consequence of the proposition does not hold (uniqueness or coherence), one can find points x, z_1, z_2 in a ball of radius smaller than ε such that x can be connected to both z_1 and z_2 by curves γ_1 and γ_2 contained in some plaque of the plaque family and such that z_1 and z_2 can be joined by a curve η (of positive length) which is tangent to $\mathcal{E}^F$. Moreover, for every $n \geq 0$ choose η_n the curve tangent to $\mathcal{E}^F$ joining z_1 and z_2 such that the length of $f^n(\eta_n)$ is minimal among curves joining $f^n(z_1)$ and $f^n(z_2)$ and whose preimage by f^{-n} are tangent to $\mathcal{E}^F$.

Using the trapping condition, one concludes that for every $n > 0$ one has that the points $f^n(x), f^n(z_1), f^n(z_2)$ are contained in a ball of radius ε around $f^n(x)$.

This implies that $f^n(\eta_n)$ remains always of length smaller than ε and since the initial length was at least $\delta > 0$ this implies that the length of γ_1 and γ_2 decreases exponentially fast, in particular:

$$\delta \leq \operatorname{length}(\eta_n) \leq (1+\epsilon)^n \|Df^{-n}|_{F(f^n(x))}\| \operatorname{length}(f^n(\eta_n)) \leq$$
$$\leq (1+\epsilon)^n \|Df^{-n}|_{F(f^n(x))}\| (\operatorname{length}(f^n(\gamma_1) + \operatorname{length}(f^n(\gamma_2)) \leq$$
$$\leq (1+\epsilon)^{2n} \|Df^{-n}|_{F(f^n(x))}\| \|Df^n|_{E(x)}\| (\operatorname{length}(f^n(\gamma_1) + \operatorname{length}(f^n(\gamma_2)) \leq$$

$$\leq const(\lambda(1+\epsilon)^2)^n \to 0$$

which is a contradiction.

To show that trapping holds after perturbation, it is enough to use the fact that the plaque families vary continuously and the trapping condition is C^0-open.

□

This argument uses strongly the fact that plaques are sufficiently small. There are two reasons for this:

- It allows to control the domination between the points involved.
- It allows to control the geometry of the curves joining the points in different plaques and sharing a point in their plaques.

In section 5.1.3 we will review an argument of Brin ([Bri]) which allows to obtain tangent foliations under the existence of a partially hyperbolic splitting. To solve the first problem, he uses absolute domination, and for the second one, he introduces the concept of quasi-isometric foliations which allows him to obtain the desired geometry for comparing distances and lengths.

1.3.3 Holonomy and local manifolds

When there exists an invariant lamination or foliation $\mathcal{F}$ tangent to certain bundle E on some invariant set Λ we will denote the *local leaves* through a point x as $\mathcal{F}_{loc}(x)$. By this we mean that $\mathcal{F}_{loc}(x)$ is the connected component of the leaf $\mathcal{F}(x)$ containing x in a neighborhood of x. We remark that this notion is of course not strictly well defined but when we mention local leaves we will state which are the referred neighborhoods. In some situations, we will use other notations such as $W^\sigma_{loc}(x)$ or $\mathcal{W}^\sigma_{x,loc}$. This notations hold also for locally invariant plaque families which we shall sometimes give similar notation.

When we have two transverse laminations, or even only one and transverse local leaves we can define the *holonomy* between the transversals (which in some sense generalizes the holonomy of foliations, c.f. Chapter 4).

Consider a compact set Λ admitting a lamination $\mathcal{F}$ tangent to a subbundle F of $T_\Lambda M$, denote $\tilde{\Lambda}$ to the union of leaves of $\mathcal{F}$.

Given a plaque family $\{\mathcal{W}_x\}_{x\in\Lambda}$ tangent to a bundle E such that $E \oplus F = T_\Lambda M$ we can define the following set of maps for $x, y \in \Lambda$ in the same leaf of $\mathcal{F}$:

$$\Pi^{\mathcal{F}}_{x,y} : U \cap \tilde{\Lambda} \subset \mathcal{W}_x \to \mathcal{W}_y$$

given by the intersection between the local leaves of $\mathcal{F}$ intersecting $\mathcal{W}_x$ with $\mathcal{W}_y$. Notice that the domain $U \cap \tilde{\Lambda}$ of $\Pi_{x,y}$ can be chosen in order to be open in $\mathcal{W}_x \cap \tilde{\Lambda}$ and contain a neighborhood of x there.

When x, y are sufficiently close in the leaf $\mathcal{F}(x)$, the domains in the transversal can be chosen arbitrarily large, since the transversals are very close.

1.3.4 Control of uniformity of certain bundles

Sometimes one can deduce that certain extremal bundles are uniform. In dimension 2 this follows from a result from [PS$_1$]:

Theorem 1.3.7 ([PS$_1$], [ABCD] Theorem 2)**.** *There exist a residual subset $\mathcal{G}_{PS} \subset \mathrm{Diff}^1(M^2)$ where M^2 is a surface such that if $f \in \mathcal{G}_{PS}$ and Λ is a chain recurrence class admitting a dominated splitting, then, Λ is hyperbolic.*

The proof of this result uses approximation by C^2-diffeomorphisms. At the moment, it is not completely understood the importance of the fact that bundles are extremal and extending this results to higher dimensions as well as for non-extremal bundles represents a main challenge (see [PS$_5$, CP, CSY] for some progress in that direction).

Another important result we will use relates the existence of hyperbolic invariant measures for C^1-diffeomorphisms whose support admits a dominated splitting separating positive and negative Lyapunov exponents. This results extends a well known result of Katok (see [KH] Supplement S) asserting that hyperbolic measures of C^2-diffeomorphisms are contained in the support of a homoclinic class. The cost for doing this is requiring a dominated splitting separating the Lyapunov exponents of the measures (a necessary hypothesis, see [BCS]):

Theorem 1.3.8 ([ABC$_2$] and [C$_3$], Proposition 1.4)**.** *Let μ be an ergodic hyperbolic measure of a C^1-diffeomorphism f (that is, all the Lyapunov exponents are different from zero) such that* $\mathrm{supp}(\mu)$ *admits a dominated splitting $T_{\mathrm{supp}(\mu)}M = E \oplus F$ such that the Lyapunov exponents on E are negative and in F positive. Then, the support of μ is contained in a homoclinic class containing periodic orbits of stable index* $\dim E$.

The proof of this theorem follows from careful application of the existence of locally invariant plaque families as well as ideas in the vein of Lemma 1.2.8 (see also [Pli]).

1.3.5 Central models and Lyapunov exponents

We will present the tool of central models first introduced in [C$_2$] and developed in [C$_3$] which allows to treat the case where there is no knowledge on the Lyapunov exponents along a certain Df-invariant subbundle of dimension 1. The presentation will be incomplete and restricted to the uses we will make of this tool. We strongly recommend the reading of [C_4] Chapter 9 or [C_3] section 2 if the reader is interested in understanding this tool.

Consider a compact f-invariant set Λ which is chain-transitive and we will assume that Λ admits a dominated splitting $T_\Lambda M = E_1 \oplus E^c \oplus E_3$ with $\dim E^c = 1$.

Consider a locally invariant plaque family $\{\mathcal{W}^c_x\}_{x\in\Lambda}$ tangent to E^c. Recall that each $\mathcal{W}^c_x$ is an embedding of $E^c(x)$ in M.

By local invariance, there exists $\rho > 0$ such that $f(\mathcal{W}^c_x([-\rho,\rho])) \subset \mathcal{W}^c_{f(x)}(\mathbb{R})$, where we are identifying $E^c(x)$ with $\mathbb{R}$. Without loss of generality, we can take $\rho = 1$.

When Df-preserves some continuous orientation on E^c (which in particular implies that E^c is orientable) this allows us to define two maps:

$$\hat{f}_1 : \Lambda \times [0,1] \to \Lambda \times [0,+\infty)$$

$$\hat{f}_2 : \Lambda \times [-1,0] \to \Lambda \times (-\infty,0]$$

induced by the way f acts on the locally invariant plaques.

When Df-does not preserve any continuous orientation on E^c, (in particular when E^c is not orientable) one can consider the double covering $\hat{\Lambda}$ of Λ (on which the dynamics will still be chain-transitive) and in a similar way define one map (see [C_3] section 2):

$$\hat{f} : \hat{\Lambda} \times [0,1] \to \hat{\Lambda} \times [0,+\infty)$$

This motivates the study of continuous skew-products of the form (called *central models*):

$$\hat{f} : K \times [0,1] \to K \times [0,+\infty)$$

$$\hat{f}(x,t) = (f_1(x), f_2(x,t))$$

where $f_1 : K \to K$ is chain-transitive, $f_2(x,0) = 0$ and $\hat{f}$ is a local homeomorphism in a neighborhood of $K \times \{0\}$.

For this kind of dynamics, in [C_3] the following classification was proven:

Proposition 1.3.9 (Central Models [C_3] Proposition 2.2)**.** *For a central model $\hat{f}$: $K \times [0,1] \to K \times [0,+\infty)$ one of the following possibilities holds:*

- *The chain-stable and the chain-stable set of $K \times \{0\}$ are non-trivial. In this case there is a segment $\{x\} \times [0,\delta]$ which is chain-recurrent.*

- *The chain-stable set contains a neighborhood of $K \times \{0\}$ and the chain-unstable set is trivial.*

- *The chain-unstable set contains a neighborhood of $K \times \{0\}$ and the chain-stable set is trivial.*

- *Both the chain-stable and the chain-unstable set of $K \times \{0\}$ are trivial.*

As a consequence, for partially hyperbolic dynamics we have (at least one of) the following types of central dynamics:

- **Type (R)** For every neighborhood U of Λ, there exists a curve γ tangent to E^c at a point of Λ such that γ is contained in a compact, invariant, chain-transitive set in U.

- **Type (N)** There are arbitrarily small neighborhoods U_k of the 0 section of E^c such that $f(\mathcal{W}^c_x(\overline{U_k})) \subset \mathcal{W}^c_{f(x)}(U_k)$ (which we call *trapping strips* for f) and there are arbitrarily small trapping strips for f^{-1}.

- **Type (H)** There are arbitrarily small trapping strips for f (case (H_S)) or for f^{-1} (case (H_U)) and the trapping strips belong to the chain-stable set of Λ (case (H_S)) or the chain-unstable set of Λ (case (H_U)).

- **Type (P)** This is only possible in the orientable case and corresponds to the following subtypes: (P_{SN}), (P_{UN}) and (P_{SU}) and corresponds to the case where there is a mixed behavior with respect to the types defined above.

In [C$_3$] it is proved that these types are well defined and more properties are studied.

1.3.6 Blenders

Blenders represent one of the main tools of differentiable dynamics, in particular when searching to prove certain robust properties of diffeomorphisms. They were introduced in [D] and [BD$_1$]. See [BDV] chapter 6 for a nice introduction to these sets, we will only present some properties which we use later. An explicit construction of these sets can be found in Appendix D.

We shall now present *cu*-blenders by its properties: A *cu-blender K* for a diffeomorphism $f : M \to M$ of a 3-dimensional manifold[10] is a compact f-invariant hyperbolic set with splitting $T_K M = E^{ss} \oplus E^s \oplus E^u$ such that the following properties are verified:

[10]Of course they can be defined in more generality, see [BD$_1$, BD$_4$].

- K is the maximal invariant subset in a neighborhood U.
- There exists a cone-field $\mathcal{E}^{ss}$ around E^{ss} defined in all U which is invariant under Df^{-1}.
- There exists a compact region B with non-empty interior (which is called *activating region*) such that every curve contained in U, tangent to $\mathcal{E}^{ss}$ with length larger than δ and intersecting B verifies that it intersects the stable manifold of a point of K.
- There exists an open neighborhood $\mathcal{U}$ of f such that for every g in $\mathcal{U}$ the properties above are verified for the same cone field, the same set B and for K_g the maximal invariant set of U.

For more properties and construction of *cu*-blenders, see [BDV] chapters 6 and [BD$_1$]. There one can see a proof of the following:

Proposition 1.3.10 ([BD$_1$] Lemma 1.9, [BDV] Lemma 6.2)**.** *If the stable manifold of a periodic point $p \in M$ of stable index 1 contains an arc γ tangent to $\mathcal{E}^{ss}$ and intersecting the activating region of a cu-blender K, then, $W^u(p) \subset \overline{W^u(q)}$ for every q periodic point in K.*

1.3.7 Higher regularity and SRB measures

We shall briefly review some of the results from [BV] (see also [VY] for recent advances on this direction) that guaranty the existence of a unique SRB measure in certain partially hyperbolic sets whenever there are some properties verified by the exponents in the center stable direction and f is sufficiently regular (i.e. C^2 is enough).

Consider $f : M \to M$ a C^2-diffeomorphism of a compact manifold such that it contains an open set U such that $f(\overline{U}) \subset U$. We denote:

$$\Lambda = \bigcap_{n \geq 0} f^n(\overline{U})$$

which is a (not-necessarily transitive) topological attractor.

We shall assume that Λ admits a partially hyperbolic splitting of the form $T_\Lambda M = E^{cs} \oplus E^u$ where E^u is uniformly expanding and E^{cs} is dominated by E^u.

Since Λ is a topological attractor, we get that Λ is saturated by unstable manifolds (see Proposition 1.1.14).

To obtain SRB measures for this type of attractors one considers the push-forward by the iterates of the diffeomorphism of Lebesgue measure and by controlling the distortion (here the C^2-hypothesis becomes crucial) one can see that the limiting measure is absolutely continuous with respect to the unstable foliation (for precise definitions see [BDV] chapter 11). After this is done, the fact that for C^2-diffeomorphisms the center-stable leaves are also absolutely continuous, one shows that the limit measures are SRB measures and that their basins covers a full Lebesgue measure of the basin. To obtain this results, further hypothesis are required in [BV] which we pass to review.

We define

$$\lambda^{cs}(x) = \limsup_{n\to+\infty} \frac{1}{n} \log \|D_x f^n|_{E^{cs}(x)}\|$$

which resembles the Lyapunov exponent (only that x needs not be a Lyapunov regular point).

We obtain the following result:

Theorem 1.3.11 (Bonatti-Viana [BV] Theorem A). *Let $f : M \to M$ be a C^2-diffeomorphism such that it admits an open set U verifying $f(\overline{U}) \subset U$ such that $\Lambda = \bigcap_n f^n(U)$, its maximal invariant set is partially hyperbolic with splitting $T_\Lambda M = E^{cs} \oplus E^u$. Assume moreover that for every D disc contained in $\mathcal{F}^u$ there is a positive Lebesgue measure of points x such that $\lambda^{cs}(x) < 0$. Then, there exists finitely many SRB measures $\mu_1, \ldots, \mu_k$ such that $\bigcup_i \mathrm{Bas}(\mu_i)$ has total Lebesgue measure inside $\mathrm{Bas}(\Lambda)$.*

Under certain assumptions, one can see that there is a unique SRB measure. We shall state the following theorem which has slightly more general hypothesis but for which the same proof as in [BV] works (see also [VY] for a further development of these results):

Theorem 1.3.12 (Bonatti-Viana [BV] Theorem B). *Assume that f and Λ satisfy the hypothesis of Theorem 1.3.11 and that moreover there is a unique minimal set of $\mathcal{F}^u$ inside Λ, then, f admits a unique SRB measure in Λ whose statistical basin coincides with the topological one modulo a zero Lebesgue measure set.*

The hypothesis required in [BV] is that $\mathcal{F}^u$ is minimal inside Λ. However, it is not hard to see how the proof of [BV] works for the hypotheis stated above: See the first paragraph of section 5 in [BV], consider the unique minimal set $\tilde{\Lambda}$ of the unstable foliation: we get that there is only one *accessibility class* there as needed for their Theorem B.

1.4 Normal hyperbolicity and dynamical coherence

Consider a lamination $\mathcal{F}$ in a compact set Λ and let $f : M \to M$ be a C^1-diffeomorphism preserving $\mathcal{F}$. We will say that $\mathcal{F}$ is *normally hyperbolic* if there exists a splitting of $T_\Lambda M = E^s \oplus T\mathcal{F} \oplus E^u$ as a Df-invariant sum verifying that the decomposition is partially hyperbolic (in particular, E^s or E^u can be trivial). If the domination is of absolute nature, we say that $\mathcal{F}$ is *absolutely normally hyperbolic.* See [HPS] for the classical reference and [Be] for recent results and some modern proofs of the results (and extensions to general laminations and endomorphisms).

When the lamination $\mathcal{F}$ covers the whole manifold, we say that it is a *foliation* (this corresponds with a C^0-foliation with C^1-leaves and tangent to a continuous distribution in the literature). See Chapter 4.

1.4.1 Leaf conjugacy

Given a lamination $\mathcal{F}$ which is invariant under a diffeomorphism f one wishes to understand which conditions guaranty the fact that for perturbations g of f there will still be a foliation which is g-invariant. As hyperbolicity gives a sufficient condition for structural stability, normal hyperbolicity appears as a natural requirement when one searches for persistence of invariant laminations[11].

In some situations, one obtains something much stronger than persistence of an invariant lamination (notice that for a 0-dimensional foliation-by points- the following notion coincides with the usual conjugacy). For a lamination $\mathcal{F}$ we denote as $K_{\mathcal{F}}$ to the (compact) set which is the union of the leaves of $\mathcal{F}$.

Definition 1.4.1 (Leaf conjugacy)**.** Given $f, g : M \to M$ be C^1-diffeomorphisms such that there are laminations $\mathcal{F}_f$ and $\mathcal{F}_g$ invariant under f and g respectively. We say that $(f, \mathcal{F}_f)$ and $(g, \mathcal{F}_g)$ are *leaf conjugate* if there exists a homeomorphism $h : K_{\mathcal{F}_f} \to K_{\mathcal{F}_g}$ such that:

- For every $x \in M$, $h(\mathcal{F}_f(x)) = \mathcal{F}_g(h(x))$.
- For every $x \in M$ we have that

$$h(\mathcal{F}_f(f(x))) = \mathcal{F}_g(g \circ h(x))$$

If a C^1-diffeomorphism f leaves a lamination $\mathcal{F}$ invariant we say that the foliation $\mathcal{F}$ is *structurally stable* if there exists a neighborhood $\mathcal{U}$ of f such that for $g \in \mathcal{U}$ the

[11] Though in this case, the issue of being a necessary condition is quite more subtle [Be].

diffeomorphism g admits a g-invariant foliation $\mathcal{F}_g$ such that the pairs $(f, \mathcal{F})$ and $(g, \mathcal{F}_g)$ are leaf conjugate.

$\diamondsuit$

The classical result of [HPS] asserts that normal hyperbolicity along with a technical condition called *plaque expansivity* is enough to guarantee structural stability of a lamination:

Theorem 1.4.1 ([HPS] Chapter 7 and [Be] Remark 2.2)**.** *Let f be a C^1-diffeomorphism leaving invariant a foliation $\mathcal{F}$ which is normally hyperbolic and plaque expansive we have that the foliation is structurally stable.*

We shall not give a definition of plaque-expansivity (we refer the reader to [HPS, Be]) but we mention that it is not known if it is a necessary hypothesis and all known normally hyperbolic foliations are either known to be structurally stable or at least suspected.

We do however state the following result which ensures plaque-expansivity and is useful to treat many important examples:

Proposition 1.4.2. *If a normally hyperbolic foliation $\mathcal{F}$ is of class C^1 (this means that the change of charts given by Proposition 4.1.1 are of class C^1) then it is plaque-expansive.*

This extends also to general laminations where the concept of being C^1 is harder to define. We will use this fact later in this thesis.

In general, checking plaque-expansiveness is hard and this makes it an undesirable hypothesis for leaf conjugacy.

1.4.2 Dynamical coherence

One sometimes wishes to consider the inverse problem. We have seen in Theorem 1.3.1 that if a diffeomorphism $f : M \to M$ is partially hyperbolic with splitting $TM = E^{cs} \oplus E^u$ then there exists a (unique) foliation $\mathcal{F}^u$ tangent to E^u which we call the *unstable foliation* of f. In general, it may happen that there is no foliation tangent to E^{cs} (this was remarked by Wilkinson in [Wi] using an example of Smale [Sm$_2$], this is extended in section 3 of [BuW$_1$]).

Definition 1.4.2 (Dynamical coherence)**.** Let $f : M \to M$ be a partially hyperbolic diffeomorphism with splitting $TM = E^{cs} \oplus E^u$. We say that f is *dynamically coherent* if there exists a foliation $\mathcal{F}^{cs}$ everywhere tangent to E^{cs} which is f-invariant in the sense that $f(\mathcal{F}^{cs}(x)) = \mathcal{F}^{cs}(f(x))$. When f is strongly partially hyperbolic with splitting $TM = E^s \oplus E^c \oplus E^u$ we say that it is dynamically coherent if there exists f-invariant foliations tangent to both $E^s \oplus E^c$ and to $E^c \oplus E^u$.

$\diamondsuit$

Remark 1.4.3 (Central Direction). When a strong partially hyperbolic diffeomorphism is dynamically coherent one can intersect the foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively and obtain a foliation $\mathcal{F}^c$ tangent to E^c. Moreover, one can show (see Proposition 2.4 of [BuW$_1$]) that the foliations $\mathcal{F}^c$ and $\mathcal{F}^s$ (resp. $\mathcal{F}^u$) subfoliate the leaves of $\mathcal{F}^{cs}$ (resp. $\mathcal{F}^{cu}$).

$\diamondsuit$

Remark 1.4.4 (Unique integrability). We have not made assumptions in the definition of dynamical coherence about the uniqueness of the f-invariant foliation tangent to E^{cs}. There are many ways to require uniqueness:

- One can ask for $\mathcal{F}^{cs}$ to be the unique f-invariant foliation tangent to E^{cs}. If there exists $n > 0$ such that there exists a unique f^n-invariant foliation, then f is dynamically coherent and with a unique f-invariant foliation. Dynamical coherence in principle does not follow from the existence of an f^n-invariant foliation tangent to E^{cs}.

- One can ask for $\mathcal{F}^{cs}$ to be the unique foliation tangent to E^{cs} which is stronger than the previous requirement.

- One can ask for the following much stronger statement: Any C^1-curve everywhere tangent to E^{cs} is contained in a leaf of $\mathcal{F}^{cs}$.

These (and more) types of uniqueness properties are discussed further in section 2 of [BuW$_1$]. See [PuShWi, BFra] for examples of foliations which satisfy weak forms of uniqueness.

$\diamondsuit$

Notice that if f, a partially hyperbolic diffeomorphism is dynamically coherent, then the f-invariant foliations ($\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$ and $\mathcal{F}^c$) are automatically normally hyperbolic.

Notice that again, the lack of knowledge (in general) of whether the foliations are plaque-expansive does not allow to know if in general the foliation must be structurally stable, in particular, the following is an open question:

Question 1.4.5. *Is dynamical coherence an open property?*

1.4.3 Classification of transitive 3-dimensional strong partially hyperbolic diffeomorphisms

Another main problem in dynamics is to consider a class of systems and try to classify their possible dynamics. For partially hyperbolic diffeomorphisms there is the following conjecture posed by Pujals (see [BWi]):

Conjecture 1.4.6 (Pujals). *Let $f : M^3 \to M^3$ be a transitive strong partially hyperbolic diffeomorphism of a 3-dimensional manifold, then f is leaf conjugate to one of the following models:*

- *Finite lifts of a skew product over an Anosov map of $\mathbb{T}^2$.*

- *Finite lifts of time one maps of Anosov flows.*

- *Anosov diffeomorphisms on $\mathbb{T}^3$.*

In [BWi] this conjecture is treated and some positive results are obtained without any assumptions on the topology of the manifold.

Hammerlindl ([H, H_2]) has made some important partial progress to this conjecture by assuming that the manifold is a nilmanifold and the partial hyperbolicity admits absolute domination. The work done in the present thesis allows to eliminate the need for absolute domination, but it seems that we still lack of tools to attack the complete conjecture. With A. Hammerlindl we plan to use several of the techniques used in this thesis in order to prove Pujals' conjecture for 3-manifolds with fundamental group of polynominal growth (see [HP]).

Notice also that the example of [$RHRHU_3$] which is not transitive does not belong to any of the classes, so that the hypothesis of transitivity cannot be removed.

1.4.4 Accessibility

Let $f : M \to M$ be a strong partially hyperbolic diffeomorphism with splitting $TM = E^s \oplus E^c \oplus E^u$. As stated in Theorem 1.3.1 there exist foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ tangent to the bundles E^s and E^u respectively.

An important notion introduced by Pugh and Shub in the mid 90's is the concept of *accessibility* on which their celebrated conjectures on abundance of ergodicity is based (see [PuSh] and also Chapter 8 of [BDV]).

We define the *accessibility class* of a point $x \in M$ as the set of points $y \in M$ such that there exists an *su-path* from x to y. An *su*-path is a concatenation of finitely many C^1-paths alternatively tangent to E^s of E^u.

We say that a diffeomorphism is *accessible* if there is a unique accessibility class. There has been lots of work devoted to the understanding of accessibility and its relationship with ergodicity, but from the point of view of abundance, we know the following results:

Theorem 1.4.7 ([DoWi]). *There is a C^1-open and dense subset $\mathcal{A}$ of partially hyperbolic diffeomorphisms such that for every $f \in \mathcal{A}$ we have that f is accessible.*

Moreover, it is proved in [BRHRHTU], that if $\dim E^c = 1$ the set of accessible partially hyperbolic diffeomorphisms forms a C^1-open and C^∞-dense set.

We refer the reader to [BuW_2] and [RHRHU_1] for proofs of ergodicity by using accessibility and certain technical conditions we will not discuss.

1.5 Integer 3×3 matrices

We will denote as $GL(d, \mathbb{Z})$ to the group of invertible $d \times d$ matrices with integer coefficients. If one interprets this as having invertibility in the group of integer matrices, it is immediate that the determinant must be of modulus 1, but since there are different uses of this notation in the literature, we will make explicit mention to this when used.

1.5.1 Hyperbolic matrices

Let $A \in GL(3, \mathbb{Z})$ with determinant of modulus 1 and no eigenvalues of modulus 1.

Since A is hyperbolic and the product of eigenvalues is one, we get that A must have one or two eigenvalues with modulus smaller than 1. We say that A has *stable dimension* 1 or 2 depending on how many eigenvalues of modulus smaller than one it has.

We call *stable eigenvalues* (resp. *unstable eigenvalues*) to the eigenvalues of modulus smaller than one (resp. larger than one). The subspace $E^s_A = W^s(0, A)$ (resp $E^u_A = W^u(0, A)$) corresponds to the eigenspace associated to the stable (resp. unstable) eigenvalues.

We shall review some properties of linear Anosov automorphisms on $\mathbb{T}^3$.

We say that a matrix $A \in GL(3, \mathbb{Z})$ (with determinant of modulus 1) is *irreducible* if and only if its characteristic polynomial is irreducible in the field $\mathbb{Q}$. This is equivalent to stating that the characteristic polynomial has no rational roots. It is not hard to prove:

Proposition 1.5.1. *Every hyperbolic matrix $A \in GL(3, \mathbb{Z})$ with determinant of modulus 1 is irreducible. Moreover, it cannot have an invariant linear two-dimensional torus.*

Proof. Assume that a matrix A is not irreducible, this means that A has one eigenvalue in $\mathbb{Q}$.

Notice that the characteristic polynomial of A has the form $-\lambda^3 + a\lambda^2 + b\lambda \pm 1$. By the rational root theorem (see [Hun]), if there is a rational root, it must be ± 1 which is impossible if A is hyperbolic.

Every linear Anosov automorphism is transitive. Let T be a linear two-dimensional torus which is invariant under A. Since the tangent space of T must also be invariant, we get that it must be everywhere tangent to an eigenspace of A. Since we have only 3 eigenvalues, this implies that either T is attracting or repelling, contradicting transitivity.

$\square$

We can obtain further properties of hyperbolic matrices acting in $\mathbb{T}^3$:

Lemma 1.5.2. *Let $A \in SL(3, \mathbb{Z})$ be a hyperbolic matrix. Then, the eigenvalues are simple and irrational. Moreover, if there is a pair of complex conjugate eigenvalues they must be of irrational angle.*

Proof. By the previous proposition, we have that the characteristic polynomial of A is irreducible as a polynomial with rational coefficients.

It is a classic result in Galois´ theory that in a field of characteristic zero, irreducible polynomials have simple roots (see [Hun] Definition V.3.10 and the Remark that follows): In fact, since $\mathbb{Q}[x]$ is a principal ideals domain, if a polynomial has double roots then it can be factorized by its derivative which has strictly smaller degree contradicting irreducibility.

This also implies that if the roots are complex, they must have irrational angle since otherwise, by iterating A we would obtain an irreducible polynomial of degree 3 and non-simple roots (namely, the power of the complex conjugate roots which makes them equal).

$\square$

When A has two different stable eigenvalues $|\lambda_1| < |\lambda_2| < 1$ (resp. unstable eigenvalues $|\lambda_1| > |\lambda_2| > 1$) we call *strong stable manifold of* A (resp. *strong unstable manifold of* A) to the eigenline of λ_1 which we denote as E^{ss}_A (resp. E^{uu}_A).

Remark 1.5.3. For every $A \in SL(3, \mathbb{Z})$ hyperbolic, we know exactly which are the invariant planes of A. If A has complex eigenvalues, then, the only invariant plane is the eigenspace associated to that pair of complex conjugate eigenvalues. If A has 3 different real eigenvalues then there are 3 different invariant planes, one for each pair of eigenvalues. All these planes are totally irrational (i.e. their projection to $\mathbb{T}^3$ is simply connected and dense).

$\diamondsuit$

1.5.2 Non-hyperbolic partially hyperbolic matrices

We prove the following result which plays the role of Lemma 1.5.2 in the non-Anosov partially hyperbolic case.

Lemma 1.5.4. *Let A be a matrix in $GL(3,\mathbb{Z})$ with eigenvalues $\lambda^s, \lambda^c, \lambda^u$ verifying $0 < |\lambda^s| < |\lambda^c = 1| < |\lambda^u| = |\lambda^s|^{-1}$. Let E^s_*, E^c_*, E^u_* be the eigenspaces associated to λ^s, λ^c and λ^u respectively. We have that:*

- *E^c_* projects by p into a closed circle where $p : \mathbb{R}^3 \to \mathbb{T}^3$ is the covering projection.*
- *The eigenlines E^s_* and E^u_* project by p into immersed lines whose closure coincide with a two dimensional linear torus.*

Proof. We can work in the vector field $\mathbb{Q}^3$ over $\mathbb{Q}$ where f_* is well defined since it has integer entries.

Since 1 is an eigenvalue of f_* and is rational, we obtain that there is an eigenvector of 1 in $\mathbb{Q}^3$. Thus, the $\mathbb{R}$-generated subspace (now in $\mathbb{R}^3$) projects under p into a circle.

Since 1 is a simple eigenvalue for f_*, there is a rational canonical form for f_* which implies the existence of two-dimensional $\mathbb{Q}$-subspace of $\mathbb{Q}^3$ which is invariant by f_* and corresponds to the other two eigenvalues (see for example Theorem VII.4.2 of [Hun]).

This plane (as a 2-dimensional $\mathbb{R}$-subspace of $\mathbb{R}^3$) must project by p into a torus since it is generated by two linearly independent rational vectors. This torus is disjoint from the circle corresponding to the eigenvalue 1 and coincides with the subspace generated by E^s_* and E^u_*.

On the other hand, the lines generated by E^s_* and E^u_* cannot project into circles in the torus since that would imply they have rational eigenvalues which is not possible, this implies that the closure of their projection is the whole torus.

□

Chapter 2

Semiconjugacies and localization of chain-recurrence classes

The purpose of this chapter is to present Proposition 2.2.1 which plays an important role in this thesis. It gives conditions under which chain-recurrence classes accumulating a given one to be contained in lower dimensional normally hyperbolic submanifolds. We profit to introduce some notions on semiconjugacies and decompositions of spaces in Section 2.1 and to state and prove a classical result on topological stability of hyperbolic sets which will be useful to then use Proposition 2.2.1.

2.1 Fibers, monotone maps and decompositions of manifolds

Consider two homeomorphisms $f : X \to X$ and $g : Y \to Y$. We say that f is *semiconjugated* to g (or that g is a *factor* of f) if there exists a continuous map $h : X \to Y$ such that

$$h \circ f = g \circ h$$

Semiconjugacies will play an important role in this text, that is why we shall make some effort in understanding certain continuous maps.

Remark 2.1.1. Semiconjugacies preserve some dynamical properties. For example, if $h : X \to Y$ semiconjugates $f : X \to X$ and $g : Y \to Y$ then we have that:

- If $x \in X$ is a periodic point for f, then $h(x)$ is periodic for g.

- If $x \in X$ is recurrent (resp. non-wandering) for f, then $h(x)$ is recurrent (resp. nonwandering) for g.

- If x_2 is in the stable set (resp. unstable set) of x_1 for f, then $h(x_2)$ is in the stable set (resp. unstable set) of $h(x_1)$ for g.

$\diamondsuit$

Let $h : X \to Y$ be a continuous map and $y \in Y$, we call $h^{-1}(\{y\})$ the *fiber* of y by h.

Sometimes, the topology of the fiber gives us information about the map h. We say that h is a *monotone map* if all the fibers are compact and connected. In general we will work with $X = M$ a topological manifold, in that case we will require a stronger property and say that h is a *cellular map* if the fiber of every point is a *cellular set* (i.e. decreasing intersection of topological balls).

Every time we have a continuous and surjective map $h : X \to Y$ we can think Y as $X/_\sim$ where the equivalence classes are given by fibers of h.

Special interest is payed to *cellular decompositions* of manifolds (a partition of a manifold M such that the quotient map is a cellular map) since these quotient spaces are what is known as *generalized manifolds* (see [Da]).

To be more precise, we say that an equivalence relation $\sim$ in a manifold M is a *cellular decomposition* if the following properties are verified:

- If we denote by A_x to the equivalence class of a point x we have that the sets A_x are cellular for every $x \in M$.

- The decomposition is *upper semicontinuous* in the sense that if $x_n \to x$ then we have that $\limsup A_{x_n} \subset A_x$.

When we have a cellular decomposition of a manifold M, we can define a quotient map $\pi : M \to M/_\sim$ and we give to $M/_\sim$ the quotient topology. We have that (see [Da] Proposition I.2.2) that:

Proposition 2.1.2. *The topological space $M/\sim$ is metrizable.*

Also, we can define a function $d : M/_\sim \times M/_\sim \to \mathbb{R}$ by:

$$d(A_x, A_y) = \min\{d(z, w) \ : \ z \in A_x \ , \ w \in A_y\}$$

Notice that this function may not be a metric since the triangle inequality may fail. However, in a certain sense, we have that we can control the topology of $M/_\sim$ using d.

Proposition 2.1.3. *The quotient topology on $M/_\sim$ verifies the following: For every U open set in $M/_\sim$ and $p \in U$ there exists ε such that $B^d_\varepsilon(p) = \{\pi(y) : d(A_y, \pi^{-1}(p)) < \varepsilon\}$ is contained in U. Conversely, for every $\varepsilon > 0$ and $p \in M/_\sim$ we have that $B^d_\varepsilon(p)$ contains a neighborhood of p.*

Proof. First consider an open set $U \in M/_\sim$ with the quotient topology. This means that the preimage $\pi^{-1}(U)$ in M is open.

We must show that for $\pi(x) \in U$ there exists ε such that $B^d_\varepsilon(\pi(x))$ is contained in U (here B^d_ε denotes the ε-ball for the function defined above, that is, $B^d_\varepsilon(\pi(x)) = \{\pi(y) \ : \ d(A_y, A_x) < \varepsilon\}$). For this, we use that the decomposition is upper semicontinuous, thus, given an open set V of $A_x = \pi^{-1}(\pi(x))$ there exists $\varepsilon > 0$ such that for every $y \in B_\varepsilon(A_x)$ we have that $A_y \subset V$. Considering $V = \pi^{-1}(U)$ we have that there is an open set for d contained in U as desired.

Now, let $\varepsilon > 0$ and $\pi(x) \in M/_\sim$ we must show that $B^d_\varepsilon(\pi(x))$ contains an open set for the quotient topology. This is direct since $\pi^{-1}(B^d_\varepsilon(\pi(x)))$ contains the ε-neighborhood of A_x and so it contains an open saturated set again by the upper semicontinuity.

$\square$

2.2 A criterium for localization of chain-recurrence classes

We give a criteria obtained in [Pot$_3$] in order to guaranty that a certain (wild) chain-recurrence class is accumulated by dynamics of lower dimensions. This goes in contraposition with other kind of "wildness" such as universal properties or viral ones though it is not clear at the moment how they are related (see [B]).

Given a homeomorphism $g : \Gamma \to \Gamma$ where Γ is a compact metric space, we say that g is *expansive* if there exists $\alpha > 0$ such that for any pair of distinct points $x \neq y \in \Gamma$ there exists $n \in \mathbb{Z}$ such that $d(g^n(x), g^n(y)) \geq \alpha$.

Given a center-stable plaque family $\{\mathcal{W}^{cs}_x\}_{x\in\Lambda}$ for a partially hyperbolic set Λ and a set C contained in one of the plaques $\mathcal{W}^{cs}_x$ we define the *center-stable frontier* of C which we denote as $\partial^{cs} C$ to the set of points z in $\mathcal{W}^{cs}_x$ such that every ball centered in z intersects both C and $\mathcal{W}^{cs}_x \cap C^c$ (i.e. the frontier with respect to the relative topology).

Proposition 2.2.1. *Let f be a C^1-diffeomorphism and U a filtrating set such that its maximal invariant set Λ admits a partially hyperbolic structure $T_\Lambda M = E^{cs} \oplus E^u$ such that it admits a locally invariant plaque family $\{\mathcal{W}^{cs}_x\}_{x\in\Lambda}$ tangent to E^{cs} whose plaques are contained in U. Assume that there exists a continuous surjective map $h : \Lambda \to \Gamma$, a homeomorphism $g : \Gamma \to \Gamma$ and a chain-recurrence class Q such that:*

- *$h \circ f = g \circ h$.*
- *h is injective in unstable manifolds.*

- *For every $x \in \Lambda$ we have that $h^{-1}(\{h(x)\})$ is contained in $\mathcal{W}^{cs}_x$ and $\partial^{cs} h^{-1}(\{h(x)\}) \subset Q$. In particular $h(Q) = \Gamma$.*

- *The fibers $h^{-1}(\{y\})$ are invariant under unstable holonomy.*

- *g is expansive.*

Then, every chain-recurrence class in U different from Q is contained in the preimage of a periodic orbit by h.

For simplicity, the reader can follow the proof assuming that g is an Anosov diffeomorphism we shall make some footnotes when some differences (which are quite small) appear.

We remark that the hypothesis of having the fibers invariant under unstable holonomy is necessary for proving the result and does not follow from the others in general.

Before starting with the proof we would like to comment on the hypothesis which are quite strong. We are asking the fibers to be invariant under unstable holonomy (which is only defined on Λ) and asking that the center-stable frontier of the fibers to be contained in the chain-recurrence class Q. This implies that in order to have the possibility of Λ being different from Q we must have at least some fibers of h which have interior in some of the center-stable plaques (and by the holonomy invariance in many of the center-stable plaques). In general, the most difficult hypothesis to verify will then be that the center-stable frontier is always contained in the chain-recurrence class, to do this we use different arguments depending on the example.

Proof. Let $R \neq Q$ be a chain recurrence class of f. Then, since $\partial h^{-1}(\{y\}) \subset Q$ for every $y \in \Gamma$, we have that $R \cap int(h^{-1}(\{y\})) \neq \emptyset$ for some $y \in \Gamma$.

Conley's theory gives us an open neighborhood V of R whose closure is disjoint with Q and such that every two points $x, z \in R$ are joined by arbitrarily small pseudo-orbits contained in V.

Since $\overline{V}$ does not intersect Q, using the invariance under unstable holonomy of the fibers, we get that there exists η_0 such that if $d(x,z) < \eta_0$ and $x \in V$, then $h(x)$ and $h(z)$ lie in the same local unstable manifold[1]: In fact, choose $\eta_0 < d(\overline{V}, Q)/2$ and assume that the image of $h(z)$ is not in the unstable manifold of $h(x)$, then, we get that if $\gamma : [0,1] \to U$ is the straight segment joining x and z there is a last point t_0 such that $h(\gamma(t_0)) \subset W^u(h(x), g)$, it is not hard to show that $\gamma(t_0)$ must then belong to $\partial^{cs} h^{-1}(h(\gamma(t_0))) \subset Q$ contradicting that the straight segment cannot intersect Q from the choice of η_0.

[1]The $\zeta-$local unstable set of a point x for an expansive homeomorphism g is the set of points whose orbit remains at distance smaller than ζ for every past iterate. For an expansive homeomorphism, this set is contained in the unstable set.

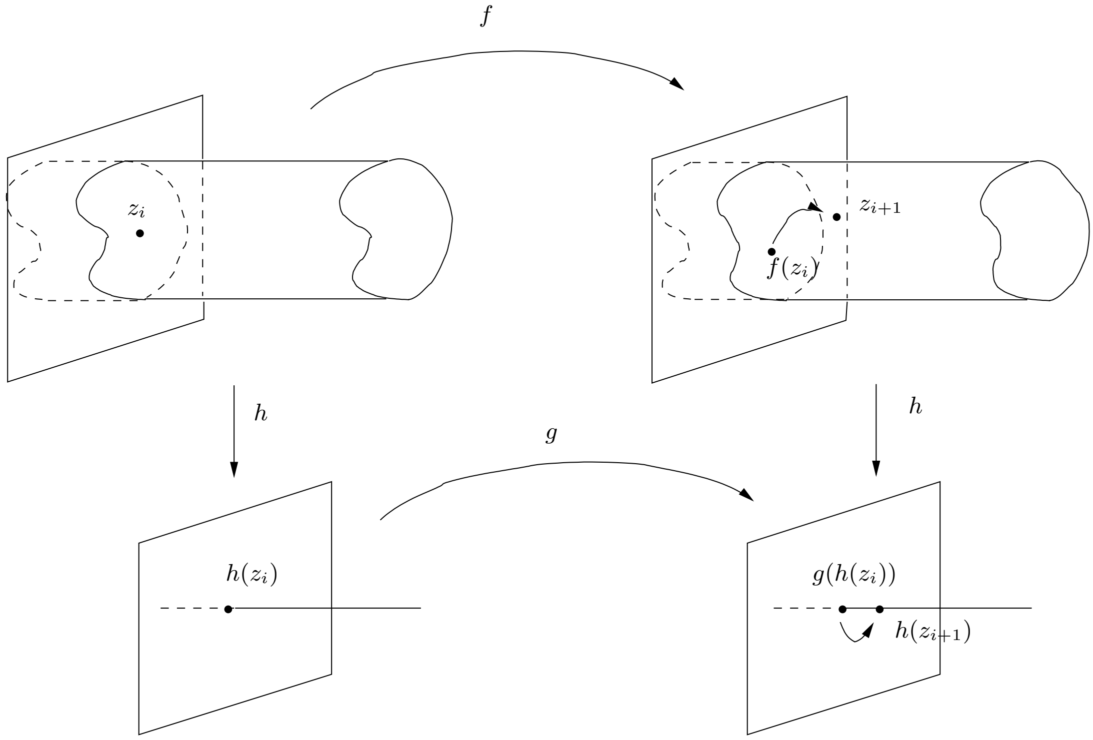

Figure 2.1: Pseudo-orbits for f are sent to pseudo-orbits of g with jumps in the unstable sets.

Given $\zeta > 0$ we choose $\eta > 0$ such that $d(x, z) < \eta$ implies $d(h(x), h(z)) < \zeta$. The semiconjugacy implies then that if $z_0, \dots z_n$ is a $\eta-$pseudo orbit for f, then $h(z_0), \dots, h(z_n)$ is a ζ-pseudo orbit for g (that is, $d(g(h(z_i)), h(z_{i+1})) < \zeta$). Also, if $\eta < \eta_0$ and $z_0, \dots z_n$ is contained in U, then we get that the the pseudo-orbit $h(z_0), \dots, h(z_n)$ has jumps inside local unstable sets (i.e. $h(z_{i+1}) \in W^u_\zeta(g(h(z_i))))$.

Take $x \in R$. Then, for every $\eta < \eta_0$ we take $x = z_0, z_1, \dots, z_n = x$ ($n \geq 1$) a $\eta-$pseudo orbit contained in V joining x to itself. Thus, we have that

$$g^n(W^u(h(x))) = W^u(h(x))$$

so, $W^u(h(x))$ is the unstable manifold for g of a periodic orbit $\mathcal{O}$. Since R is f-invariant and since the semiconjugacy implies that $f^{-n}(x)$ accumulates on $h^{-1}(\mathcal{O})$, we get that R intersects the fiber $h^{-1}(\mathcal{O})$.

We must now prove that $R \subset h^{-1}(\mathcal{O})$ which concludes.

Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $z_0, \dots z_n$ is a $\delta-$pseudo orbit for g with jumps in the unstable manifold, then $z_n \in \mathcal{O}$ implies that $z_0 \in W^u_\varepsilon(\mathcal{O})$ (notice that a pseudo orbit with jumps in the unstable manifold of a periodic orbits can be regarded as a pseudo orbit for a homothety[2] in $\mathbb{R}^k$).

[2]In the general case of g being an expansive homeomorphism, it is very similar since one has

Assume that there is a point $z \in R$ such that $h(z) \in W^u(\mathcal{O})\backslash\mathcal{O}$. Let ε such that $d(h(z), \mathcal{O}) > \varepsilon$. Since R intersects $h^{-1}(\mathcal{O})$ there are δ-pseudoorbits joining z to $h^{-1}(\mathcal{O})$ for every $\delta > 0$. This implies that after sending the pseudo orbit by h we would get δ-pseudo orbits for g, with jumps in the unstable manifold, joining $h(z)$ with $\mathcal{O}$. This contradicts the remark made in the last paragraph.

So, we get that R is contained in $h^{-1}(\mathcal{O})$ where $\mathcal{O}$ is a periodic orbit of g.

□

2.3 Diffeomorphisms homotopic to Anosov ones, C^0 perturbations of hyperbolic sets

Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a diffeomorphism which is isotopic to a linear Anosov automorphism $A : \mathbb{T}^d \to \mathbb{T}^d$ (i.e. the diffeomorphism induced by a hyperbolic matrix in $GL(d, \mathbb{Z})$ with determinant ± 1). Along this text, we assume that $A \in SL(d, \mathbb{Z})$ which does not represent a loss in generality since the results are invariant under considering iterates.

In the context we are working, being isotopic to a linear Anosov automorphism is equivalent to the fact that the induced action f_* of f on the (real) homology (which equals $\mathbb{R}^d$) is hyperbolic (see [F$_1$]).

We shall denote as A to both the diffeomorphism of $\mathbb{T}^d$ and to the hyperbolic matrix $A \in SL(d, \mathbb{Z})$ which acts in $\mathbb{R}^d$ and is the lift of the torus diffeomorphism A to the universal cover.

Let $p : \mathbb{R}^d \to \mathbb{T}^d$ be the covering projection, and $\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ the lift of f to its universal cover. Notice that the fact that $f_* = A$ implies that there exists $K_0 > 0$ such that $d(\tilde{f}(x), Ax) < K_0$ for every $x \in \mathbb{R}^3$.

Classical arguments (see [Wa]) give that there exists K_1 such that for every $x \in \mathbb{R}^d$, there exists a unique $y \in \mathbb{R}^d$ such that

$$d(\tilde{f}^n(x), A^n y) \leq K_1 \qquad \forall n \in \mathbb{Z}$$

We say that the point y *shadows* the point x. Notice that uniqueness implies that the point associated with $x + \gamma$ is $y + \gamma$ where $\gamma \in \mathbb{Z}^d$. We get the following well known result:

Proposition 2.3.1. *There exists $H : \mathbb{R}^d \to \mathbb{R}^d$ continuous and surjective such that $H \circ \tilde{f} = A \circ H$. Also, it is verified that $H(x + \gamma) = H(x) + \gamma$ for every $x \in \mathbb{R}^d$*

that restricted to the unstable set of a periodic orbit, one can obtain a metric inducing the same topology where g^{-1} is an uniform contraction. This follows from [Fa] and can also be deduced using the uniform expansion of f in unstable leaves and the injectivity of the semi-conjugacy along unstable leaves.

and $\gamma \in \mathbb{Z}^d$ so, there exists also $h : \mathbb{T}^d \to \mathbb{T}^d$ homotopic to the identity such that $h \circ f = A \circ h$. Moreover, we have that $d(H(x), x) < K_1$ for every $x \in \mathbb{R}^d$.

PROOF. Any orbit of $\tilde{f}$ is a K_0-pseudo-orbit of the hyperbolic matrix A. This gives that for every x we can associate a unique point y such that

$$d(\tilde{f}^n(x), A^n y) \leq K_1 \qquad \forall n \in \mathbb{Z}$$

We define $H(x) = y$. It is not hard to show that H is continuous. Since it is at distance smaller than K_1 from the identity, we deduce that H is surjective (this follows from a degree argument, see [Hat] Chapter 2). Periodicity follows from the fact that all maps here project to the torus.

□

It is well known and easy to show that $H(W^\sigma(x, \tilde{f})) \subset W^\sigma(H(x), A)$ with $\sigma = s, u$.

The previous result generalizes to general C^0-perturbations of hyperbolic sets. We get the following classical result whose proof is very similar to the previous one:

Proposition 2.3.2. *Let $\Lambda \subset M$ be a hyperbolic set for a diffeomorphism f such that it is maximal invariant in a neighborhood U of Λ. Then, there exists $\varepsilon > 0$ such that for every homeomorphism g which is ε-C^0-close to f in U we have that if Λ_g is the maximal invariant set for g in U then there exists a continuous and surjective map $h : \Lambda_g \to \Lambda$ such that:*

$$f|_\Lambda \circ h = h \circ g|_{\Lambda_g}$$

Chapter 3

Attractors and quasi-attractors

This chapter is devoted to the study of attractors and quasi-attractors for C^1-generic dynamics (see subsection 1.1.5). We will present the results obtained in [Pot_2, Pot_1, Pot_3].

The chapter is organized as follows:

- In Section 3.1 we present a proof of a result by Araujo stating that C^1-generic diffeomorphisms of surfaces have hyperbolic attractors. In addition, we prove a result from [Pot_1] in the context of surfaces which we believe makes its proof more transparent.

- In Section 3.2 we study quasi-attractors of C^1-generic diffeomorphisms in any dimension and present some results regarding their structure. Then, we give as an application some results on bi-Lyapunov stable homoclinic classes.

- In Section 3.3 we present several examples of dynamics without attractors and of robustly transitive attractors. We present the results from [BLY] and then the ones of [Pot_3]. We profit to add some examples which we believe may have some interest in the general theory.

- In Section 3.4 we present a definition which covers a certain class of quasi-attractors in dimension 3 and explain why we believe this class of examples should be studied.

3.1 Existence of hyperbolic attractors in surfaces

In dimension 2, the result of the existence of attractors for C^1-generic diffeomorphisms was announced to be true by Araujo ([Ara]) but the result was never published since there was a gap on its proof. However, it has become a folklore result:

with the techniques of [PS_1] the gap in the proof can be arranged (see [San]). There has also been an announcement of this result in [BLY].

We prove here the following theorem which is similar to the one by Araujo. The proof we give is quite short but based on the recent developments of generic dynamics (mainly [ABC_2],[BC], [MP] and [PS_1]). The proof was made available in [Pot_2].

Theorem 3.1.1. *There is $\mathcal{G} \subset \mathrm{Diff}^1(M^2)$, a residual subset of diffeomorphisms in the surface M^2 such that for every $f \in \mathcal{G}$, there is an hyperbolic attractor. Moreover, if f has finitely many sinks, then f is essentially hyperbolic.*

We say that f is *essentially hyperbolic* if it admits finitely many hyperbolic attractors and such that the union of their basins cover an open and dense subset of M (Araujo proves that the basin of atraction has Lebesgue measure one, his techniques work in this context too, see [San]). This definition comes from [PaT] and is motivated by a new result of [CP] which closes a long standing problem posed by Palis in [PaT] (though a stronger formulation remains open and important).

We will prove the following Theorem in any dimensions in Section 3.2, however, we present it here before since the proofs are easier to follow in the surface case.

Theorem 3.1.2. *There is $\mathcal{G} \subset \mathrm{Diff}^1(M^2)$, a residual subset of diffeomorphisms in the surface M^2 such that for every $f \in \mathcal{G}$ with finitely many sources satisfies that every homoclinic class which is a quasi-attractor is an hyperbolic attractor.*

In particular we get the following using results in [MP] and [BC] (see Theorem 1.1.22):

Corollary 3.1.3. *There is $\mathcal{G} \subset \mathrm{Diff}^1(M^2)$, a residual subset of diffeomorphisms in the surface M^2 such that for every $f \in \mathcal{G}$ with finitely many sources satisfies that generic points converge either to hyperbolic attractors or to aperiodic classes.*

This last Corollary applies for example in a C^1-neighborhood of the well known Henón attractor (see [BDV] Chapter 4). In fact, since hyperbolic attractors which are in a disc which is dissipative are sinks, in the Henón case we get that there are no non-trivial attractors (aperiodic quasi attractors for generic diffeomorphisms cannot be attractors).

3.1.1 Proof of the Theorem 3.1.1

Let $\mathcal{K}$ be the set of all compact subsets of M with the Hausdorff topology.

Let $S : \mathrm{Diff}^1(M^2) \to \mathcal{K}$ be the map such that $S(f) = \overline{\mathrm{Per}_0(f)}$ is the clousure of the set of sinks of f.

Since S is semicontinuous (see Remark 1.1.7), there exists a residual subset $\mathcal{G}_0$ of $\mathrm{Diff}^1(M)$ such that for every $f \in \mathcal{G}_0$, the diffeomorphism f is a continuity point of S.

This implies that we can write $\mathcal{G}_0 = \mathcal{A} \cup \mathcal{I}$ open sets in $\mathcal{G}_0$ such that for every $f \in \mathcal{A}$ the number of sinks is locally constant and finite (that is, there is a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ such that for every $g \in \mathcal{U}$ the number of sinks is the same and they vary continuously), and such that for every $f \in \mathcal{I}$ there are infinitely many sinks.

To prove the Theorem it is enough to work inside $\tilde{\mathcal{A}}$ (an open set in $\mathrm{Diff}^1(M)$ such that $\mathcal{A} = \tilde{\mathcal{A}} \cap \mathcal{G}_0$) since the theorem is trivially satisfied in $\mathcal{I}$.

Let $\mathcal{G} = \mathcal{G}_0 \cap \mathcal{G}_{BC} \cap \mathcal{G}_{BDP} \cap \mathcal{G}_{PS}$ (see Theorems 1.1.22, 1.2.17 and 1.3.7). Let $f \in \tilde{\mathcal{A}} \cap \mathcal{G}$. We must show that f is essentially hyperbolic.

Step 1: We first prove that every quasi-attractor $\mathcal{Q}$ is a hyperbolic attractor.

We have that Λ admits a nested sequence of open neighborhoods U_n such that $\mathcal{Q} = \bigcap_{n\geq 0} U_n$ and such that $f(\overline{U_n}) \subset U_n$. The following lemma holds in every dimension:

Lemma 3.1.4. *Let $\mathcal{Q}$ be a quasi-attractor. Then, there exist an ergodic measure μ supported in $\mathcal{Q}$ such that $\int log(|det(Df)|)d\mu \leq 0$.*

PROOF. Let m_n be the normalized Lebesgue measure in U_n.

Consider μ_n a limit point in the weak-$*$ topology of the sequence of measures given by

$$\nu_k = \frac{1}{k}\sum_{i=1}^{k} f^i_{\#}(m_n)$$

which is an invariant measure supported in $f(\overline{U_n})$. We are here using the following notation: $f_{\#}(\nu)(A) = \nu(f^{-1}(A))$.

Since $f(\overline{U_n}) \subset U_n$, the change of variables theorem and Jensen's inequality (see [Rud]) implies that:

$$\int \log(|\det Df|) dm_n < 0.$$

The same argument, using the fact that $f^k(\overline{U_n}) \subset f^{k-1}(U_n)$ implies that

$$\int \log(|\det Df|) df^k_{\#}(m_n) < 0.$$

We obtain that

$$\int \log(|\det Df|) d\nu_k \leq 0$$

From which we deduce that $\int \log(|\det Df|) d\mu_n \leq 0$. Now, consider a measure μ which is a limit point in the weak-$*$ topology of the measures μ_n.

The measure μ must be an invariant measure, supported on $\mathcal{Q}$ and satisfying that

$$\int \log(|\det Df|) d\mu \leq 0.$$

Using Proposition 1.1.26 one can assume that μ is ergodic.

□

Since the set of sinks varies continuously with f and there are finitely many of them, we can choose n such that there are no sinks in U_n.

Using the Ergodic closing Lemma (Theorem 1.1.30) we get that the support of the measure must admit a dominated splitting: Otherwise we get periodic points converging in the Hausdorff topology to the support of the measure and with $\log(|\det Df^{\pi(p)}|)$ arbitrarily close to zero. If they do not admit a dominated splitting, using a classical argument (see subsection 1.2.5) one can convert them into sinks by applying Franks' Lemma (Theorem 1.2.11), a contradiction.

Also, the measure must be hyperbolic: If no positive exponents exist, since $\int \log(|\det(Df)|) d\mu \leq 0$, one can approach the measure by periodic orbits with both exponents smaller than ε (arbitrarily small) by using the Ergodic closing Lemma (Theorem 1.1.30) and one can create a sink by making a further small perturbation with Franks' Lemma (Theorem 1.2.11).

Using Theorem 1.3.8, we deduce that the support of μ is contained in a homoclinic class. Since $f \in \mathcal{G}_{BC}$ we have that $\mathcal{Q}$ is a homoclinic class.

Also we get periodic points inside the class such that $\log(|\det Df^{\pi(p)}|) < \varepsilon$ for small $\varepsilon > 0$.

Using Proposition 1.2.14 we get that periodic points with this property are dense in the homoclinic class and so we get a dominated splitting $T_{\mathcal{Q}}M = E \oplus F$ in the whole class. In fact, since we are far from sinks and $f \in \mathcal{G}_{BDP}$, we get that F must be uniformly expanding.

Since we are in $\mathcal{G}_{PS}$ we get that $\mathcal{Q}$ is hyperbolic (see Theorem 1.3.7), and thus, $\mathcal{Q}$ is a hyperbolic attractor (see Proposition 1.3.2).

This proves the first assertion of the Theorem.

Step 2: We now prove that in fact f must be essentially hyperbolic.

Suppose first that there are infinitely many non-trivial hyperbolic attractors (recall that we are assuming that there are finitely many sinks). Assume $\mathcal{Q}_n$ is a sequence of distinct hyperbolic attractors such that $\mathcal{Q}_n \to K$ in the Hausdorff topology. From Proposition 1.1.11 we have that K must be a chain-transitive set, this implies that $K \cap S(f) = \emptyset$ (notice that since $S(f)$ is a finite set it will be isolated from the chain-recurrent set).

Notice that there are measures μ_n supported in $\mathcal{Q}_n$ such that

$$\int \log(|det(Df)|)d\mu_n \leq 0.$$

Consider a weak-$*$ limit μ of these measures, so we have that μ is supported in K and verifies that

$$\int \log(|det(Df)|)d\mu \leq 0.$$

So using the same argument as before we deduce that K is contained in a hyperbolic homoclinic class, and thus isolated, a contradiction with the fact that it was accumulated by quasi-attractors.

Since $f \in \mathcal{G}_{BC}$ (see Theorem 1.1.22), generic points in the manifold converge to Lyapunov stable chain recurrence classes and we get that there is an open and dense subset of M in the basin of hyperbolic attractors. This finishes the proof of the Theorem.

□

3.1.2 Proof of the Theorem 3.1.2

Theorem 3.1.2 is implied by the following Theorem from [Pot_1]. This theorem will be extended to higher dimensions in Section 3.2 but we prefer to present a proof in this context since it helps to grasp better the idea involved (which is the use of Theorem 1.2.12):

Theorem 3.1.5. *Let H be a homoclinic class of a C^1-generic surface diffeomorphism f which is a quasi-attractor. Then, if H has a periodic point p such that $det(D_p f^{\pi(p)}) \leq 1$ then H admits a dominated splitting and thus it is a hyperbolic attractor.*

Proof. Consider a generic diffeomorphism f. Consider a periodic point $q \in H$ fixed such that for a neighborhood $\mathcal{U}$ of f the class $H(q_g, g)$ is a quasi-attractor for every g in a residual subset of $\mathcal{U}$ (see subsection 3.2.1).

By genericity, we can assume that every periodic point $q \in H$ verifies that $det(D_p f^{\pi(p)}) \neq 1$ and using Proposition 1.2.14 we deduce there is a dense set of points verifying that the determinant is smaller than 1.

By Theorem 1.2.15 we know that if H does not admit a dominated splitting then for every $\varepsilon > 0$ we can modify the derivative of f along a periodic orbit along a curve with small diameter and which satisfies the hypothesis of Gourmelon's version of Franks' Lemma (Theorem 1.2.12).

We can then apply Theorem 1.2.12 to perturb the periodic point p preserving its strong stable manifold locally.

The idea is to make the perturbation in a small neighborhood of p without breaking the intersection between $W^u(q)$ and $W^s(p)$. This creates a sink in p but such that $W^u(q)$ still intersects its basin. Since $H(q_g, g)$ is still a quasi-attractor, this means that $p_g \in H(q_g, g)$ which is a contradiction (see Lemma 3.2.6 for a more detailed proof).

Now by Theorem 1.3.7 we get that H must be hyperbolic which concludes.

□

The last theorem has some immediate consequences which may have some interest on their own.

We say that an embedding $f : \mathbb{D}^2 \to \mathbb{D}^2$ is *dissipative* if for every $x \in \mathbb{D}^2$ we have that $|det(D_x f)| < b < 1$. Recall that for a dissipative embeddings of the disc, the only hyperbolic attractors are the sinks ([Ply]).

Corollary 3.1.6. *Let $f : \mathbb{D}^2 \to \mathbb{D}^2$ be a generic dissipative embedding. Then, every quasi-attractor which is a homoclinic class is a sink.*

3.2 Structure of quasi-attractors

In this section we explore the existence of invariant structures for quasi-attractors of C^1-generic dynamics in any dimensions. We expect that, for a homoclinic class, being a quasi-attractor imposes sufficiently many structure in the dynamics on the tangent map. We have obtained partial results in this direction.

The main difficulty is that domination is very much related to either isolation of the homoclinic class or with being far from homoclinic tangencies, in this result we manage to deal with homoclinic classes for which we do not know a priori that either of these conditions are satisfied. The main idea is to use the fact that being a quasi-attractor is a somewhat robust property and the perturbation techniques developed by Gourmelon (Theorem 1.2.12). This allows us to prove:

Theorem 3.2.1. *For every f in a residual subset $\mathcal{G}_1$ of $\text{Diff}^1(M)$, if H is a homoclinic class for f which is a quasi-attractor and there is a periodic point $p \in H$ such that $det(D_p f^{\pi(p)}) \leq 1$, then, H admits a non-trivial dominated splitting.*

This theorem is proved in subsection 3.2.3. In view of Lemma 3.1.4 one can ask (see [B] Conjecture 2 for a stronger version of this question):

Question 3.2.2. *Is it true that every quasi-attractor $\mathcal{Q}$ of a C^1-generic diffeomorphism which is a homoclinic class has a periodic point p such that $det(D_p f^{\pi(p)}) \leq 1$?*

Another important task is to determine whether some (extremal) bundles are uniform in order to be able to derive further dynamical properties. In this direction, we have obtained the following result motivated by previous results obtained in [PotS]:

Theorem 3.2.3. *There exists a residual set $\mathcal{G}_2$ of* $\text{Diff}^1(M)$ *such that if f is a diffeomorphism in $\mathcal{G}_2$ and H a homoclinic class which is a quasi-attractor admitting a codimension one dominated splitting $T_H M = E \oplus F$ where* $\dim(F) = 1$*. Then, the bundle F is uniformly expanding for f.*

The proof of this theorem is presented in subsection 3.2.2.

In dimension 3 we have the following corollary about the structure of quasi-attractors:

Corollary 3.2.4. *There exists a residual subset $\mathcal{G}_{QA}$ of* $\text{Diff}^1(M^3)$ *with M a 3-dimensional manifold such that if $f \in \mathcal{G}_{QA}$ and $\mathcal{Q}$ a quasi-attractor of f having a periodic point p such that $det(D_p f^{\pi(p)}) \leq 1$ then, $\mathcal{Q}$ admits a dominated splitting of one of the following forms:*

- *$T_{\mathcal{Q}} M = E^{cs} \oplus E^u$ where E^u is one dimensional and uniformly expanded and E^{cs} may or may not admit a sub-dominated splitting.*

- *$T_{\mathcal{Q}} M = E_1 \oplus E^{cu}$ where the bundle E^{cu} is two-dimensional and verifies that periodic points are volume hyperbolic at the period in E^{cu}.*

Moreover, if $\mathcal{Q}$ is not accumulated by sinks, then in the second case we have that E^{cu} is uniformly volume expanding.

We also explore some properties of quasi-attractors far from homoclinic tangencies and we deduce some consequences for homoclinic classes which are both quasi-attractors and quasi-repellers (bi-Lyapunov stable homoclinic classes) responding to some questions posed in [ABD]. The results about dynamics far from tangencies overlap with [Y].

3.2.1 Persistence of quasi-attractors which are homoclinic classes

The following result will be essential for our proofs of Theorems 3.2.1 and 3.2.3. It states that for C^1-generic diffeomorphisms, the quasi-attractors which are homoclinic classes have a well defined continuation.

Proposition 3.2.5 (Lemma 3.6 of [ABD])**.** *There is a residual subset $\mathcal{G}_{ABD}$ of* $\text{Diff}^1(M)$ *such that if $f \in \mathcal{G}_{ABD}$ and $H(p, f)$ is a homoclinic class of f then, there exists a neighborhood $\mathcal{U}$ of f such that the point p has a continuation in $\mathcal{U}$ and such that:*

- *If $H(p,f)$ is a quasi-attractor, then $H(p_g,g)$ is also a quasi-attractor for every $g \in \mathcal{G}_{ABD} \cap \mathcal{U}$.*

- *If $H(p,f)$ is not a quasi-attractor, then, $H(p_g,g)$ is not a quasi-attractor for any $g \in \mathcal{G}_{ABD} \cap \mathcal{U}$.*

Proof. We reproduce the proof from [ABD].

Consider $\mathcal{G} = \mathcal{G}_{BC} \cap \mathcal{G}_{cont}$ a residual subset of $\text{Diff}^1(M)$ such that for every $f \in \mathcal{G}$ we have that:

- A homoclinic class $H(p,f)$ of f is a quasi-attractor if and only if $H(p,f) = \overline{W^u(p,f)}$ (see Theorem 1.1.22).

- There exists a neighborhood $\mathcal{U}$ of f such that the following maps $g \mapsto H(p_g,g)$ and $g \mapsto \overline{W^u(p_g,g)}$ are well defined and continuous on every $g \in \mathcal{G} \cap \mathcal{U}$ (see Remark 1.1.7).

For a pair $\mathcal{U}$ and p which has a continuation for every $f \in \mathcal{U}$ we let $\mathcal{A}_{(\mathcal{U},p)} \subset \mathcal{U} \cap \mathcal{G}$ be the set of g such that $H(p_g,g) \neq \overline{W^u(p_g,g)}$. Since both sets are compact and vary continuously we deduce that $\mathcal{A}_{(\mathcal{U},p)}$ is open in $\mathcal{U} \cap \mathcal{G}$.

Let $\mathcal{B}_{(\mathcal{U},p)}$ be the complement of the closure of $\mathcal{A}_{(\mathcal{U},p)}$ in $\mathcal{U} \cap \mathcal{G}$ which is also open and verifies that if $g \in \mathcal{B}_{\mathcal{U}}$ then there is a neighborhood of g in $\mathcal{G}$ consisting of diffeomorphisms $\hat{g}$ such that the homoclinic class $H(p_{\hat{g}},\hat{g})$ is a quasi-attractor.

From how we defined $\mathcal{A}_{(\mathcal{U},p)}$ and $\mathcal{B}_{(\mathcal{U},p)}$ we have that their union is open and dense in $\mathcal{G} \cap \mathcal{U}$.

The residual subset $\mathcal{G}_{ABD}$ is then obtained by considering a countable collection of pairs $(\mathcal{U},p)$ where $\mathcal{U}$ varies in a countable basis of the topology of $\text{Diff}^1(M)$ and p is a hyperbolic periodic point of $f \in \mathcal{U}$ which has a continuation to the whole $\mathcal{U}$ (there are clearly at most countably many of them by Kupka-Smale's Theorem 1.1.3). We finally define:

$$\mathcal{G}_{ABD} = \bigcap_{(\mathcal{U},p)} \left(\mathcal{A}_{(\mathcal{U},p)} \cup \mathcal{B}_{(\mathcal{U},p)} \cup \overline{\mathcal{U}}^c \right)$$

□

In general, if X is a Baire topological space and A is a Borel subset of X, then there exists a residual subset G of X such that A is open and closed in G. This usually serves as an heuristic principle, but in general it is not so easy to show that a certain property is Borel. One must prove this kind of result by barehanded arguments.

3.2.2 One dimensional extremal bundle

We will first prove Theorem 3.2.3. The proof strongly resembles the proof of the main theorem of [PotS] and it was indeed motivated by it.

The main difference is that the fact that periodic points must be hyperbolic at the period in the case the homoclinic class has non-empty interior is quite direct by using transitions and the fact that the interior has some persistence properties. This is the content of the following lemma whose proof will serve also as a model for the proof of Theorem 3.2.1. We will make emphasis only in the part of the proof which differs from [PotS] and we refer the reader to that paper for more details in the rest of the proof.

Lemma 3.2.6. *Let H be a homoclinic class which is a quasi-attractor of a C^1-generic diffeomorphism f such that the class has only periodic orbits of stable index smaller or equal to α. So, there exists $K_0 > 0$, $\lambda \in (0,1)$ and $m_0 \in \mathbb{Z}$ such that for every $p \in \mathrm{Per}_\alpha(f|_H)$ of sufficiently large period one has*

$$\prod_{i=0}^{k} \left\| \prod_{j=0}^{m_0-1} Df^{-1}|_{E^u(f^{-im_0-j}(p))} \right\| < K_0\lambda^k \qquad k = \left[\frac{\pi(p)}{m_0}\right].$$

PROOF. Let $\mathcal{G}$ be a residual subset of $\mathrm{Diff}^1(M)$ such that if $f \in \mathcal{G}$ and H is homoclinic class of a periodic point q of index α and a quasi-attractor, there exists a small neighborhood $\mathcal{U}$ of f where the continuation q_g of q is well defined and such that for every $g \in \mathcal{U} \cap \mathcal{G}$ one has that $H(q_g, g)$ is quasi-attractor and g is a continuity point of the map $g \mapsto H(q_g, g)$ (see Proposition 3.2.5 and Remark 1.1.7).

Also, being f generic, we can assume that for every $g \in \mathcal{U} \cap \mathcal{G}$ and every $p \in \mathrm{Per}_\alpha(g) \cap H(q_g, g)$ we have that $H(q_g, g) = H(p, g)$, so, the orbits of p and q_g are homoclinically related (see Theorem 1.1.22).

We can also assume that $\mathcal{U}$ and $\mathcal{G}$ were chosen so that for every $g \in \mathcal{U} \cap \mathcal{G}$ every periodic point in $H(q_g, g)$ has index smaller or equal to α. We can also assume that q_g has index α for every $g \in \mathcal{U}$.

Lemma II.5 of [M_3] asserts that to prove the lemma it is enough to show that there exists $\varepsilon > 0$ such that the set of cocycles

$$\Theta_\alpha = \{D_{\mathcal{O}(p)} f^{-1}|_{E^u} \ : \ p \in \mathrm{Per}_\alpha(f|_H)\}$$

which all have its eigenvalues of modulus bigger than one, verify that every ε-perturbation of them preserves this property. That is, given $p \in \mathrm{Per}_\alpha(f|_H)$ one has that every ε-perturbation $\{A_0, \ldots, A_{\pi(p)-1}\}$ of $D_{\mathcal{O}(p)} f|_{E^u}$ verifies that $A_{\pi(p)-1} \ldots A_0$ has all its eigenvalues of modulus bigger or equal to one.

Therefore, assuming by contradiction that the Lemma is false, we get that $\forall\varepsilon > 0$ there exists a periodic point $p \in \mathrm{Per}_\alpha(f|_H)$ and a linear cocycle $\{A_0, \dots, A_{\pi(p)}\}$ over p satisfying that:

- $\|D_{f^i(p)} f|_{E^u} - A_i\| \leq \varepsilon$,
- $\|D_{f^i(p)} f^{-1}|_{E^u} - A_i^{-1}\| \leq \varepsilon$ and
- $\prod_{i=0}^{\pi(p)-1} A_i$ has some eigenvalue of modulus smaller or equal to 1.

In coordinates $T_{\mathcal{O}(p)}M = E^u \oplus (E^u)^\perp$, since E^u is invariant we have that the form of Df is given by

$$D_{f^i(p)} f = \begin{pmatrix} D_{f^i(p)} f_{/E^u} & K_i^1(f) \\ 0 & K_i^2(f) \end{pmatrix}$$

Let $\gamma : [0,1] \to \Gamma_\alpha$ given in coordinates $T_{\mathcal{O}(p)}M = E^u \oplus (E^u)^\perp$ by

$$\gamma_i(t) = \begin{pmatrix} (1-t) D_{f^i(p)} f|_{E^u} + tA_i & K_i^1(f) \\ 0 & K_i^2(f) \end{pmatrix}$$

whose diameter is bounded by ε (see Lemma 4.1 of [BDP]).

Now([1]), choose a point x of intersection between $W^s(p, f)$ with $W^u(q, f)$ and choose a neighborhood U of the orbit of p such that:

(i) It does not intersect the orbit of q.

(ii) It does not intersect the past orbit of x.

(iii) It verifies that once the orbit of x enters U it stays there for all its future iterates by f.

It is very easy to choose U satisfying (i) since both the orbit of p and the one from q are finite. Since the past orbit of x accumulates in q is not difficult to choose U satisfying (ii). To satisfy (iii) one has only to use the fact that x belongs to the stable manifold of p so, after a finite number of iterates, x will stay in the local stable manifold of p. It is then not difficult to choose a neighborhood U which satisfies (iii) also.

Applying Theorem 1.2.12 we can perturb f to a new diffeomorphism $\hat{g}$ so that the orbit of p has index greater than α and so that it preserves locally its strong stable manifold. This allows to ensure that the intersection between $W^u(q_{\hat{g}}, \hat{g})$ and $W^s(p, \hat{g})$ is non-empty.

[1]The following argument will be referred too by the proof of Theorem 3.2.1.

This intersection is transversal so it persist by small perturbations, the same occurs with the index of p so we can assume that $\hat{g}$ is in $\mathcal{G} \cap \mathcal{U}$.

Using the fact that $H(q_{\hat{g}}, \hat{g})$ is a quasi-attractor we obtain that $p \in H(q_{\hat{g}}, \hat{g})$:

This is because quasi-attractors are saturated by unstable sets, so, since $q_{\hat{g}} \subset H(q_{\hat{g}}, \hat{g})$ we have that $\overline{W^u(q_{\hat{g}}, \hat{g})} \subset H(q_{\hat{g}}, \hat{g})$ and since $W^s(p, \hat{g}) \cap W^u(q_{\hat{g}}, , \hat{g}) \neq \emptyset$, we get by the λ-Lemma (Theorem 1.1.4) and the fact that the quasi-attractor is closed that

$$p \in \overline{W^u(p, \hat{g})} \subset \overline{W^u(q_{\hat{g}}, \hat{g})} \subset H(q_{\hat{g}}, \hat{g})$$

This contradicts the choice of $\mathcal{U}$ since we find a diffeomorphism in $\mathcal{U} \cap \mathcal{G}$ with a periodic point with index bigger than α in the continuation of H, and so the lemma is proved.

$\square$

Remark 3.2.7. One can recover Lemma 2 of [PotS] in this context. In fact, if there is a codimension one dominated splitting of the form $T_H M = E \oplus F$ with $\dim F = 1$ then (using the adapted metric given by [Gou$_1$]) for a periodic point of maximal index one has $\|Df^{-1}|_{F(p)}\| \leq \|Df^{-1}|_{E^u(p)}\|$ so,

$$\prod_{i=0}^{k} \left\| \prod_{j=0}^{m_0-1} Df^{-1}|_{F(f^{-im_0-j}(p))} \right\| < K_0 \lambda^k \qquad k = \left[\frac{\pi(p)}{m_0} \right]$$

And since F is one dimensional one has $\prod_i \|A_i\| = \|\prod_i A_i\|$ so $\|Df^{-\pi(p)}|_{F(p)}\| \leq K_0 \lambda^{\pi(p)}$ (maybe changing the constants K_0 and λ).

In fact, there is $\gamma \in (0,1)$ such that for every periodic point of maximal index and big enough period one has $\|Df^{-\pi(p)}|_{F(p)}\| \leq \gamma^{\pi(p)}$

Also, it is not hard to see, that if the class admits a dominated splitting of index bigger or equal than the index of all the periodic points in the class, then, periodic points should be hyperbolic in the period along F (for a precise definition and discussion on this topics one can read [BGY], [W$_3$]).

$\diamondsuit$

Remark 3.2.8. As a consequence of the proof of the lemma we get that: One can perturb the eigenvalues along an invariant subspace of a cocycle without altering the rest of the eigenvalues. The perturbation will be of similar size to the size of the perturbation in the invariant subspace. See Lemma 4.1 of [BDP]. Notice also that we could have perturbed the cocycle $\{K_i^2(f)\}_i$ without altering the eigenvalues of the cocycle $D_{\mathcal{O}(p)} f|_{E^u}$.

$\diamondsuit$

One can now conclude the proof of Theorem 3.2.3 with the same techniques as in the proof of the main Theorem of [PotS].

Proof of Theorem 3.2.3. We have that $T_H M = E \oplus F$ with $\dim F = 1$. We first prove that the center unstable curves tangent to F should be unstable and with uniform size (this is Lemma 3 of [PotS]). To do this, we first use Lemma 3.2.6 to get this property in the periodic points and then use the results from [PS_4] and [BC] to show that the property extends to the rest of the points. This dynamical properties imply also uniqueness of these central unstable curves.

Assuming the bundle F is not uniformly expanded, one has two cases: one can apply Liao's selecting lemma or not (see [L, W_3]).

In the first case one gets weak periodic points inside the class which contradict the thesis of Lemma 3.2.6.

The second case is similar, if Liao's selecting lemma ([L]) does not apply, one gets a minimal set inside H where the expansion along F is very weak and thus E is uniformly contracting. Using the dynamical properties of the center unstable curves, classical arguments give that we can shadow orbits of this minimal sets by periodic points which are weak in the F direction. Since the stable manifold of this periodic point will be uniform, it will intersect the unstable manifold of a point in H, and then the fact that H is a quasi-attractor implies the point is inside the class and again contradicts Lemma 3.2.6.

For more details see [PotS].

□

3.2.3 Existence of a dominated splitting

We prove here Theorem 3.2.1 which state that a homoclinic class which is a quasi-attractor and has a dissipative periodic orbit admits a dominated splitting.

The idea is the following: in case H does not admit any dominated splitting we can perturb the derivative of some periodic point in order to convert it into a sink with the techniques of [BoB].

We pretend to use Theorem 1.2.12 to ensure that the unstable manifold of a periodic point in the class intersects the stable set of the sink and reach a contradiction as we did in the end of the proof of Lemma 3.2.6.

Proof of Theorem 3.2.1. Let H be a homoclinic class of a C^1-generic diffeomorphism f which is a quasi-attractor. Let us assume that H contains periodic points of index α and we consider $\Delta_\alpha^\eta \subset \mathrm{Per}_\alpha(f|_H)$ the set of index α and η-disippative periodic points in H for some $\eta < 1$.

It is enough to have one periodic point with determinant smaller than one to get that for some $\eta < 1$, the set Δ_α^η will be dense in H (see Proposition 1.2.14). Notice that from hypothesis, and the fact that for generic diffeomorphisms the determinant of the differential at the period is different from one, there is $\eta < 1$ such that Δ_α^η is dense.

Notice that if H admits no dominated splitting, then neither does the cocycle of the derivatives over Δ_α^η. This implies that we can apply Theorem 1.2.15 and there is a periodic point $p \in \Delta_\alpha^\eta$ which can be turned into a sink with a C^1-small perturbation done along a path contained in Γ_α (which maintains or increases the index).

Now we are able to use Theorem 1.2.12 and reach a contradiction. Consider a periodic point $q \in \Delta_\alpha^\eta$ fixed such that for a neighborhood $\mathcal{U}$ of f the class $H(q_g, g)$ is a quasi-attractor for every $g \in \mathcal{U} \cap \mathcal{G}$.

Suppose the class does not admit any dominated splitting, so, as we explained above we have a periodic point $p \in \Delta_\alpha^\eta$ such that f can be perturbed in an arbitrarily small neighborhood of p to a sink for a diffeomorphism $g \in \mathcal{U}$ (which we can assume is in $\mathcal{G} \cap \mathcal{U}$ since sinks are persistent) and preserving locally the strong stable manifold of p. So, we choose a neighborhood of p such that it does not meet the orbit of q nor the past orbit of some intersection of its unstable manifold with the local stable manifold of p with the same argument as in Lemma 3.2.6.

Thus, we get that $W^u(q_g, g) \cap W^s(p, g) \neq \emptyset$ and using Lyapunov stability we reach a contradiction since it implies that $p \in H(q_g)$ which is absurd since p is a sink.

$\square$

One can also deduce some properties on the indices of the possible dominated splitting depending on the indices of the periodic points in the class (see [Pot$_1$]).

Remark 3.2.9.
- Also the same ideas give that periodic points in the class must be volume hyperbolic in the period (not necessarily uniformly, see [BGY] for a discussion on the difference between hyperbolicity in the period and uniform hyperbolicity)

- In fact, we can assume that if a homoclinic class which is a quasi-attractor admits no dominated splitting, then, there exists $\eta > 1$ such that every periodic point p, it has determinant bigger than $\eta^{\pi(p)}$. Otherwise, there would exists a subsequence p_n of periodic points with normalized determinant converging to 1. After composing with a small homothety, we are in the hypothesis of Theorem 3.2.1.

$\diamondsuit$

3.2.4 Quasi-attractors far from tangencies

We present a proof of a result originally proved in [Y]. We believe that having another approach to this result is not entirely devoid of interest (see [CSY] for results that exceed the results here presented on dynamics far from tangencies).

We remark that in the far from tangencies context, J. Yang ([Y]) has proved that quasi-attractors of C^1-generic diffeomorphisms are homoclinic classes (see also [C_4]) and more recently, C.Bonatti, S.Gan, M.Li and D.Yang have proved that for C^1-generic diffeomorphisms far from homoclinic tangencies quasi-attractors are in fact essential attractors (see [BGLY]).

Theorem 3.2.10 ([Y] Theorem 3). *Let $f \in \mathcal{G}$, where $\mathcal{G}$ is a residual subset of $\text{Diff}^1(M) \backslash \overline{\text{Tang}}$, and let H be a Lyapunov stable homoclinic class for f of minimal index α. Let $T_H M = E \oplus F$ be a dominated splitting for H with $\dim E = \alpha$, so, one of the following two options holds:*

1. *E is uniformly contracting.*
2. *E decomposes as $E^s \oplus E^c$ where E^s is uniformly contracting and E^c is one dimensional and H is the Hausdorff limit of periodic orbits of index $\alpha - 1$.*

As in [Y], the proof has 3 stages, the first one is to reduce the problem to the central models introduced by Crovisier, the second one to treat the possible cases and finally, the introduction of some new generic property allowing to conclude in the difficult case.

Our proof resembles that of [Y] in the middle stage (which is the most direct one after the deep results of Crovisier) and has small differences mainly in the other two.

For the first one, we use a recent result of [C_3] (Theorem 3.2.11) and for the last one, we introduce Lemma 3.2.12 which can be compared with the main Lemma of [Y] but the proof and the statement are somewhat different (in particular, ours is slightly stronger). We believe that this Lemma can find some applications (see for example [C_4]).

Before we start the proof of Theorem 3.2.10 we state the following theorem due to Crovisier which will be the starting point for our study:

Theorem 3.2.11 (Theorem 1 of [C_3]). *Let $f \in \mathcal{G}$ where $\mathcal{G} \subset \text{Diff}^1(M) \backslash \overline{\text{Tang}}$ is residual, and K_0 an invariant compact set with dominated splitting $T_{K_0} M = E \oplus F$. If E is not uniformly contracted, then, one of the following cases occurs.*

1. *K_0 intersects a homoclinic class whose minimal index is strictly less than $\dim E$.*

2. *K_0 intersects a homoclinic class whose minimal index is* $\dim E$ *and which contains weak periodic orbits (for every δ there is a sequence of hyperbolic periodic orbits homoclinically related which converge in the Hausdorff topology to a set $K \subset K_0$, whose index is* $\dim E$ *but whose maximal exponent in E is in $(-\delta, 0)$). Also, this implies that every homoclinic class H intersecting K_0 verifies that it admits a dominated splitting of the form $T_H M = E' \oplus E^c \oplus F$ with* $\dim E^c = 1$.

3. *There exists a compact invariant set $K \subset K_0$ with minimal dynamics and which has a partially hyperbolic structure of the form $T_K M = E^s \oplus E^c \oplus E^u$ where* $\dim E^c = 1$ *and* $\dim E^s < \dim E$. *Also, any measure supported on K has zero Lyapunov exponent along E^c.*

Now we are ready to give a proof of Theorem 3.2.10.

Proof of Theorem 3.2.10. Let $\mathcal{G} \subset \mathrm{Diff}^1(M) \backslash \overline{\mathrm{Tang}}$ be a residual subset such that for every $f \in \mathcal{G}$ and every periodic point p of f, there exists a neighborhood $\mathcal{U}$ of f, where the continuation p_g of p is well defined, such that f is a continuity point of the map $g \mapsto H(p_g, g)$ and such that if $H(p, f)$ is a homoclinic class which is a quasi-attractor for f, then $H_g = H(p_g, g)$ is also a quasi-attractor for every $g \in \mathcal{U} \cap \mathcal{G}$. Also, we can assume that for every $g \in \mathcal{U} \cap \mathcal{G}$, the minimal index of H_g is α.

The class admits a dominated splitting of the form $T_H M = E \oplus F$ with $\dim E = \alpha$ (see Theorem 1.2.24). We assume that the subbundle E is not uniformly contracted. This allows us to use Theorem 3.2.11.

Since the minimal index of H is α, option 1) of the theorem cannot occur.

We shall prove that option 3) implies option 2). That is, we shall prove that if E is not uniformly contracted, then we are in option 2) of Theorem 3.2.11.

This is enough to prove the theorem since if we apply Theorem 3.2.11 to E' given by option 2) we get that since $\dim E' = \alpha - 1$ option 1) and 2) cannot happen, and since option 3) implies option 2) we get that E' must be uniformly contracted thus proving Theorem 3.2.10 (observe that the statement on the Hausdorff convergence of periodic orbits to the class can be deduced from option 2) also by using Frank's Lemma).

Claim. *To get option* 2) *in Theorem 3.2.11 is enough to find one periodic orbit of index α in H which is weak (that is, it has one Lyapunov exponent in $(-\delta, 0)$).*

Proof. This follows using the fact that being far from tangencies there is a dominated splitting in the orbit given by Theorem 1.2.24 with a one dimensional central bundle associated with the weak eigenvalue. Let $\mathcal{O}$ be the weak periodic orbit, so we have a dominated splitting of the form $T_{\mathcal{O}} M = E^s \oplus E^c \oplus E^u$.

Using transitions (see Proposition 1.2.14), we can find a dense subset in the class of periodic orbits that spend most of the time near the orbit we found, say, for a small neighborhood U of $\mathcal{O}$, we find a dense subset of periodic points p_n such that the cardinal of the set $\{i \in \mathbb{Z} \cap [0, \pi(p_n) - 1] \ : \ f^i(p_n) \in U\}$ is bigger than $(1 - \varepsilon)\pi(p_n)$.

Since we can choose U to be arbitrarily small, we can choose ε so that the orbits of all p_n admit the same dominated splitting (this can be done using cones for example) and maybe by taking ε smaller to show that p_n are also weak periodic orbits.

$\diamondsuit$

It rests to prove that option 3) implies the existence of weak periodic orbits in the class. To do this, we shall discuss depending on the structure of the partially hyperbolic splitting using the classification given in [C$_3$]. There are 3 different cases according to the possibilities given by Proposition 1.3.9.

We have a compact invariant set $K \subset H$ with minimal dynamics and which has a partially hyperbolic structure of the form $T_K M = E^s \oplus E^c \oplus E^u$ where $\dim E^c = 1$ and $\dim E^s < \dim E$. Also, any measure supported on K has zero Lyapunov exponent along E^c. We shall assume that the dimension of E^s is minimal in the sense that every other compact invariant $\tilde{K}$ satisfying the same properties as K satisfies that $\dim(E^s_{\tilde{K}}) \geq \dim(E^s_K)$ (this will be used only for Case C)).

Case A): There exists a chain recurrent central segment. K has type (R)

Assume that the set $K \subset H$ admits a chain recurrent central segment. That is, there exists a curve γ tangent to E^c in a point of K, which is contained in H and such that γ is contained in a compact, invariant, chain transitive set in U, a small neighborhood of K.

In this case, the results of [C$_2$] (Addendum 3.9) imply that there are periodic orbits in the same chain recurrence class as K (i.e. H) with index $\dim E^s \leq \alpha - 1$, a contradiction.

Case B): K has type (N), (H) or (P_{SN})

If K has type (H), one can apply Proposition 4.4 of [C$_3$] which implies that there is a weak periodic orbit in H giving option 2) of Theorem B.1.

Cases (N) and (P_{SN}) give a family of central curves γ_x $\forall x \in K$ (tangent to E^c, see [C$_3$]) which satisfy that $f(\gamma_x) \subset \gamma_{f(x)}$. It is not difficult to see that there is a neighborhood U of K such that for every invariant set in U the same property will be satisfied (see remark 2.3 of [C$_2$]).

Consider a set $\hat{K} = K \cup \bigcup_n \mathcal{O}_n$ where $\mathcal{O}_n$ are close enough periodic orbits converging in Hausdorff topology to K (these are given, for instance, by Theorem 1.1.25) which we can suppose are contained in U.

So, since for some $x \in K$, we have that $\mathcal{F}^{uu}_{loc}(x)$ will intersect $W^{cs}_{loc}(p_n)$ in a point z (for a point x, the local center stable set, $W^{cs}_{loc}(x)$ is the union of the local strong stable leaves of the points in γ_x).

Since the ω-limit set of z must be a periodic point (see Lemma 3.13 of [C$_2$]) and since H is Lyapunov stable we get that there is a periodic point of index α which is weak, or a periodic point of smaller index in H which gives a contradiction.

Case C): K has type (P_{UN}) or (P_{SU})

One has a minimal set K which is contained in a homoclinic class which is a quasi-attractor and it admits a partially hyperbolic splitting with one dimensional center with zero exponents and type (P_{UN}) or (P_{SU}).

This gives that given a compact neighborhood U of K, there exists a family of C^1-curves $\gamma_x : [0,1] \to U$ ($\gamma_x(0) = x$) tangent to the central bundle such that $f^{-1}(\gamma_x([0,1])) \subset \gamma_{f^{-1}(x)}([0,1))$. This implies that the preimages of these curves remain in U for past iterates and with bounded length.

They also verify that the chain unstable set of K restricted to U (that is, the set of points that can be reached from K by arbitrarily small pseudo orbits contained in U) contains these curves. Since H is a quasi-attractor, this implies that these curves are contained in H.

Assume we could extend the partially hyperbolic splitting from K to a dominated splitting $T_{K'}M = E_1 \oplus E^c \oplus E_3$ in a chain transitive set $K' \subset H$ containing $\gamma_x([0,t))$ for some $x \in K$ and for some $t \in (0,1)$.

Since the orbit of $\gamma_x([0,1])$ remains near K for past iterates, we can assume (by choosing U sufficiently small) that the bundle E_3 is uniformly expanded there. So, there are uniformly large unstable manifolds for every point in $\gamma_x([0,1])$ and are contained in H.

If we prove that E_1 is uniformly contracted in all K', since we can approach K' by weak periodic orbits, we get weak periodic in the class since its strong stable manifold (tangent to E_1) will intersect H.

To prove this, we use that for K the dimension of E^s is minimal. So E_1 must be stable, otherwise, we would get that, using Theorem 3.2.11 again, there is a partially hyperbolic minimal set inside K' with stable bundle of dimension smaller than the one of K, a contradiction.

The fact that we can extend the dominated splitting and approach the point y in $\gamma_x((0,1))$ by weak periodic points is given by Lemma 3.2.12 below.

Lemma 3.2.12. *There exists a residual subset $\mathcal{G}' \subset \mathrm{Diff}^1(M) \backslash \overline{\mathrm{Tang}}$ such that every $f \in \mathcal{G}'$ verifies the following. Given a compact invariant set K such that*

- *K is a chain transitive set.*
- *K admits a partially hyperbolic splitting $T_K M = E^s \oplus E^c \oplus E^u$ where E^s is uniformly contracting, E^u is uniformly expanding and $\dim E^c = 1$.*

- *Any invariant measure supported in K has zero Lyapunov exponents along E^c.*

Then, for every $\delta > 0$, there exists U, a neighborhood of K such that for every $y \in U$ satisfying:

- *y belongs to the local chain unstable set $pW^u(K,U)$ of K (that is, for every $\varepsilon > 0$ there exists an $\varepsilon-$pseudo orbit from K to y contained in U)*

- *y belongs to the chain recurrence class of K*

we have that there exist $p_n \to y$, periodic points, such that:

- *The orbit $\mathcal{O}(p_n)$ of the periodic point p_n has its $\dim E^s + 1$ Lyapunov exponent contained in $(-\delta, \delta)$.*

- *For large enough n_0, if $\tilde{K} = K \cup \overline{\bigcup_{n>n_0} \mathcal{O}(p_n)}$, then we can extend the partially hyperbolic splitting to a dominated splitting of the form $T_{\tilde{K}}M = E_1 \oplus E^c \oplus E_3$.*

The following lemma allows to conclude the proof as we mentioned before. Its proof is postponed to subsection 3.2.5.

This concludes the proof of Theorem 3.2.10

□

3.2.5 Proof of Lemma 3.2.12

We shall first prove a perturbation result and afterwards we shall deduce Lemma 3.2.12 with a standard Baire argument. One can compare this lemma with Lemma 3.2 of [Y] which is a slightly weaker version of this. See [C_4] Chapter 9 for possible applications.

Lemma 3.2.13. *There exists a residual subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that every $f \in \mathcal{G}$ verifies the following. Given:*

- *K a compact chain transitive set.*

- *U a neighborhood of K and $y \in U$ verifying that y is contained in the local chain unstable set $pW^u(K,U)$ of K and in the chain recurrence class of K.*

- *$\mathcal{U}$ a C^1-neighborhood of f.*

Then, there exists $l > 0$ such that, for every $\nu > 0$ and $L > 0$ we have $g \in \mathcal{U}$ with a periodic orbit $\mathcal{O}$ with the following properties:

- *There exists $p_1 \in \mathcal{O}$ such that $d(f^{-k}(y), g^{-k}(p_1)) < \nu$ for every $0 \leq k \leq L$.*

- *There exists $p_2 \in \mathcal{O}$ such that $\mathcal{O} \backslash \{p_2, \ldots, g^l(p_2)\} \subset U$.*

Proof. The argument is similar as the one in section 1.4 of [C$_3$]. We must show that after an arbitrarily small perturbation, we can construct such periodic orbits.

Consider a point y as above. We can assume that y is not chain recurrent in U, otherwise y we would be accumulated by periodic orbits contained in U (see Theorem 1.1.25) and that would conclude without perturbing.

For every $\varepsilon > 0$ we consider an $\varepsilon-$pseudo orbit $Y_\varepsilon = (z_0, z_1, \ldots, z_n)$ with $z_0 \in K$ and $z_n = y$ contained in U. Using that y is not chain recurrent in U, we get that for ε small enough we have that $B_\nu(y) \cap Y_\varepsilon = \{y\}$ where $B_\nu(y)$ is the ball of radius ν and ν is small. So if we consider a Hausdorff limit of the sequence $Y_{1/n}$ we get a compact set Z^- for which y is isolated and such that it is contained in the chain unstable set of K restricted to U. Notice that Z^- is backward invariant.

If we now consider the pair (Δ^-, y) where $\Delta^- = Z^- \backslash \{f^{-n}(y)\}_{n \geq 0}$ we get a pair as the one obtained in Lemma 1.11 of [C$_3$] where y plays the role of x^-. Notice that Δ^- is compact and invariant.

Now, we consider $\tilde{U} \subset U$ a small neighborhood of $K \cup \Delta^-$ such that $y \notin \tilde{U}$. Take $x \in H \cap U^c$ where H is the chain recurrence class of K.

Consider $X_\varepsilon = (z_0, \ldots z_n)$ an $\varepsilon-$pseudo orbit such that $z_0 = x$ and $z_n \in K$. Take z_j the last point of X_ε outside $\tilde{U}$. Since we chose $\tilde{U}$ small we have $z_j \in U \backslash \tilde{U}$. We call $\tilde{X}_\varepsilon$ to $(z_j, \ldots z_n)$.

Consider Z^+ the Hausdorff limit of the sequence $\tilde{X}_{1/n}$ which will be a forward invariant compact set which intersects $U \backslash \tilde{U}$. Since y is not chain recurrent in U we have that $y \notin Z^+$.

We consider a point $x^+ \in Z^+ \cap U \backslash \tilde{U}$. This point satisfies that one can reach K from x^+ by arbitrarily small pseudo orbits. We get that the future orbit of x^+ does not intersect the orbit of y.

Consider $\mathcal{U}$ a neighborhood of f in the C^1 topology. Hayashi's connecting lemma (Theorem 1.1.21) gives us $N > 0$ and neighborhoods $W^+ \subset \hat{W}^+$ of x^+ and $W^- \subset \hat{W}^-$ of y which we can consider arbitrarily small so, we can suppose that

- All the iterates $f^i(\hat{W}^+)$ and $f^j(\hat{W}^-)$ for $0 \leq i, j \leq N$ are pairwise disjoint.

- The iterates $f^i(\hat{W}^+)$ for $0 \leq i \leq N$ are disjoint from the past orbit of y.

Since there are arbitrarily small pseudo orbits going from y to x^+ contained in H and f is generic, Theorem 1.1.25 $\{x_0, \ldots, f^l(x_0)\}$ in a small neighborhood of H and such that $x_0 \in W^-$ and $f^l(x_0) \in W^+$.

The same argument gives us an orbit $\{x_1, \ldots, f^k(x_1)\}$ contained in U such that $x_1 \in W^+$ and $f^k(x_1) \in W^-$. In fact, we can choose it so that $d(f^{-i}(y), f^{k-i}(x_1)) < \nu$

for $0 \leq i \leq L$ (this can be done using uniform continuity of $f^{-1}, \dots, f^{-L}$ and choosing $f^k(x_1)$ close enough to y).

Using Hayashi's connecting lemma (Theorem 1.1.21) we can then create a periodic orbit $\mathcal{O}$ for a diffeomorphism $g \in \mathcal{U}$ which is contained in $\{x_0, \dots, f^l(x_0)\} \cup \hat{W}^+ \cup \dots \cup f^N(\hat{W}^+) \cup \{x_1, \dots, f^k(x_1)\} \cup \hat{W}^- \cup \dots \cup f^N(\hat{W}^-)$ (in the proof of Proposition 1.10 of [C$_3$]) is explained how one can compose two perturbations in order to close the orbit).

Notice that from how we choose $\hat{W}^+$ and $\hat{W}^-$ and the orbit $\{x_1, \dots, f^k(x_1)\}$ we get that the periodic orbit we create with the connecting lemma satisfies that $d(f^{-i}(y), g^{-i}(p_n)) < \nu$ for $0 \leq i \leq L$ and some p_n in the orbit. This is because $\{x_1, \dots, f^{k-1}(x_1)\}$ does not intersect $\hat{W}^- \cup \dots \cup f^N(\hat{W}^-)$ and $\{f^{k-L}(x_1), \dots f^k(x_1)\}$ does not intersect $\hat{W}^+ \cup \dots \cup f^N(\hat{W}^+)$ (in fact, this gives that the orbit of p_n for g contains $\{f^{k-L}(x_1), \dots f^k(x_1)\}$).

Also, since $\hat{W}^+ \cup \dots \cup f^N(\hat{W}^+) \cup \{x_1, \dots, f^k(x_1)\} \subset U$ we get that, except for maybe l consecutive iterates of one point in the resulting orbit, the rest of the orbit is contained in U. This concludes the proof.

□

PROOFOF LEMMA 3.2.12. Take $x \in M$, an let $m, e, t \in \mathbb{N}$. We consider $\mathcal{U}(m, e, t, x)$ the set of C^1 diffeomorphisms $g \in \text{Diff}^1(M)$ with a periodic orbit $\mathcal{O}$ satisfying:

- $\mathcal{O}$ is hyperbolic.
- Its $e + 1$ Lyapunov exponent of $\mathcal{O}$ is contained in $(-1/m, 1/m)$.
- There exists $p \in \mathcal{O}$ such that $d(p, x) < 1/t$.

This set is clearly open in $\text{Diff}^1(M)$. Let $\{x_s\}$ be a countable dense set of M. We define $\mathcal{G}_{m,e,t,s} = \mathcal{U}(m, e, t, x_s) \cup \text{Diff}^1(M) \backslash \overline{\mathcal{U}(m, e, t, x_s)}$ which is open and dense by definition. Consider $\mathcal{G}_1 = \bigcap_{m,e,t,s} \mathcal{G}_{m,e,t,s}$ which is residual. Finally, taking $\mathcal{G}$ as in Lemma 3.2.13, we consider

$$\mathcal{G}' = (\mathcal{G} \cap \mathcal{G}_1) \backslash \overline{\text{Tang}}$$

Consider K compact chain transitive and with a partially hyperbolic splitting $T_K M = E^s \oplus E^c \oplus E^u$ with $\dim E^c = 1$. We assume that any invariant measure supported in K has Lyapunov exponent equal to zero.

Choose $\delta > 0$ small enough. Since f is far from tangencies, Theorem 1.2.24 gives us that every periodic orbit having its $\dim E^s + 1$ Lyapunov exponent in $(-\delta, \delta)$ admits a dominated splitting $E_1 \oplus E^c \oplus E_3$ with uniform strength (that is, if there is a set $\{O_n\}$ of periodic orbits with their $\dim E^s + 1$ Lyapunov exponents in $(-\delta, \delta)$, then the dominated splitting extends to the closure).

We choose U_1, an open neighborhood of K such that every invariant measure supported in U_1 has its Lyapunov exponents in $(-1/2m_0, 1/2m_0)$ where $1/m_0 < \delta$.

We can assume that U_1 verifies that there are Df invariant cones $\mathcal{E}^{uu}$ and $\mathcal{E}^{cu}$ around E^u and $E^c \oplus E^u$ respectively, defined in U_1. Similarly, there are in U_1, Df^{-1} invariant cones $\mathcal{E}^{ss}$, $\mathcal{E}^{cs}$ around E^s and $E^s \oplus E^c$ respectively.

We can assume, by choosing an adapted metric (see [HPS] or [Gou$_1$]), that for every $v \in \mathcal{E}^{ss}$ we have $\|Dfv\| < \lambda\|v\|$ and for every $v \in \mathcal{E}^{uu}$ we have $\|Dfv\| > \lambda^{-1}\|v\|$ for some $\lambda < 1$. There exists $\mathcal{U}_1$, a C^1-neighborhood of f such that for every $g \in \mathcal{U}_1$ the properties above remain true.

Given U neighborhood of K such that $\overline{U} \subset U_1$ we have that any g invariant set contained in U admits a partially hyperbolic splitting.

We now consider $y \in pW^u(K, U)$ which is contained in the chain recurrence class of K.

Claim. *Given t, for any x_s with $d(x_s, y) < 1/2t$ we get that $f \in \mathcal{U}(m_0, \dim E^s, t, s)$.*

PROOF. Since f is in $\mathcal{G}_1'$ it is enough to show that every neighborhood of f intersects $\mathcal{U}(m_0, \dim E^s, t, s)$.

Choose a neighborhood $\mathcal{U}$ of f and consider $\mathcal{U}_0 \subset \mathcal{U}$ given by Franks' Lemma (Theorem 1.2.11) such that we can perturb the derivative of some $g \in \mathcal{U}_0$ in a finite set of points less than ξ and obtain a diffeomorphism in $\mathcal{U}$.

For $\mathcal{U}_0$, Lemma 3.2.13 gives us a value of $l < 0$ such that for any $L > 0$ and there exists $g_L \in \mathcal{U}_0$ and a periodic orbit $\mathcal{O}_L$ of g_L such that there is a point $p_1 \in \mathcal{O}_L$ satisfying that $d(g^{-i}(p_1), f^{-i}(y)) < 1/2t$ $(0 \leq i \leq L)$ and a point $p_2 \in \mathcal{O}_L$ such that $\mathcal{O}_L \backslash \{p_2, \ldots, g^l(p_2)\}$ is contained in U. We can assume that $\mathcal{O}$ is hyperbolic.

We must perturb the derivative of $\mathcal{O}_L$ less than ξ in order to show that the $\dim E^s + 1$ Lyapunov exponent is in $(-1/m_0, 1/m_0)$.

Notice that if we choose L large enough, we can assume that the angle of the cone $Dg^L(\mathcal{C}^\sigma(g^{-L}(p_2))$ is arbitrarily small ($\sigma = uu, cu$). In the same way, we can assume that the angle of the cone $Dg^{-L}(\mathcal{C}^{\tilde{\sigma}}(g^{L+l}(p_2)$ is arbitrarily small ($\tilde{\sigma} = cs, ss$ respectively).

Since l is fixed, we get that for $p \in \mathcal{O}_L \cap U^c$ (if there exists any, we can assume it is p_2), it is enough to perturb less than ξ the derivative in order to get the cones $Dg^L(\mathcal{C}^\sigma(g^{-L}(p_2))$ and $Dg^{-L-l}(\mathcal{C}^{\tilde{\sigma}}(g^{L+l}(p_2)$ transversal (for $\sigma = uu, cu$ and $\tilde{\sigma} = cs, ss$ respectively). This allows us to have a well defined dominated splitting above $\mathcal{O}_L$ (which may be of very small strength) which in turn allow us to define the $\dim E^s + 1$ Lyapunov exponent. Since the orbit $\mathcal{O}_L$ spends most of the time inside U, and any measure supported in U has its center Lyapunov exponent in $(-1/2m_0, 1/2m_0)$ we get the desired property.

$\diamondsuit$

Taking $t = n \to \infty$ and using Theorem 1.2.24, we get a sequence of periodic points $p_n \to y$ such that if $\mathcal{O}(p_n)$ are their orbits, the set $\tilde{K} = K \cup \overline{\bigcup_n \mathcal{O}(p_n)}$ admits a dominated splitting $T_{\tilde{K}}M = E_1 \oplus E^c \oplus E_3$ extending the partially hyperbolic splitting.

This concludes the proof of Lemma 3.2.12.

$\square$

3.2.6 Application: Bi-Lyapunov stable homoclinic classes

In [ABD] the following conjecture was posed (it also appeared as Problem 1 in [BC])

Conjecture 3.2.14 ([ABD])**.** *There exists a residual set $\mathcal{G}$ of* $\mathrm{Diff}^1(M)$ *of diffeomorphisms such that if $f \in \mathcal{G}$ admits a homoclinic class with nonempty interior, then the diffeomorphism is transitive.*

Some progress has been made towards the proof of this conjecture (see [ABD],[ABCD] and [PotS]), in particular, it has been proved in [ABD] that isolated homoclinic classes as well as homoclinic classes admitting a strong partially hyperbolic splitting verify the conjecture. Also, they proved that a homoclinic class with non empty interior must admit a dominated splitting (see Theorem 8 in [ABD]).

In [ABCD] the conjecture was proved for surface diffeomorphisms, other proof for surfaces (which does not use the approximation by C^2 diffeomorphisms) can be found in [PotS] where the codimension one case is studied.

Also, from the work of Yang ([Y], see also subsection 3.2.4 and Proposition 3.2.23 below) one can deduce the conjecture in the case f is C^1-generic and far from homoclinic tangencies.

When studying some facts about this conjecture, in [ABD] it was proved that if a homoclinic class of a C^1-generic diffeomorphism has nonempty interior then this class should be bi-Lyapunov stable. In fact, in [ABD] they proved that isolated and strongly partially hyperbolic bi-Lyapunov stable homoclinic classes for generic diffeomorphisms are the whole manifold.

This concept is a priori weaker than having nonempty interior and it is natural to ask the following question.

Question 3.2.15 (Problem 1 of [BC])**.** *Is a bi-Lyapunov stable homoclinic class of a generic diffeomorphism necessarily the whole manifold?*

It is not difficult to deduce from [BC] that, for generic diffeomorphisms, a chain recurrence class with non empty interior must be a homoclinic class (see Corollary

1.1.23), thus, the answer to Conjecture 3.2.14 must be the same for chain recurrence classes and for homoclinic classes.

However, we know that Question 3.2.15 admits a negative answer if posed for general chain recurrence classes. Bonatti and Diaz constructed (see [BD$_3$]) open sets of diffeomorphisms in every manifold of dimension ≥ 3 admitting, for generic diffeomorphisms there, uncountably many bi-Lyapunov stable chain recurrence classes which in turn have no periodic points.

Although this may suggest a negative answer for Question 3.2.15 we present here some results suggesting an affirmative answer. In particular, we prove that the answer is affirmative for surface diffeomorphisms, and that in three dimensional manifold diffeomorphisms the answer must be the same as for Conjecture 3.2.14.

The main reason for which the techniques in [ABCD] (or in [PotS]) are not able to answer Question 3.2.15 for surfaces, is because differently from the case of homoclinic classes with interior, it is not so easy to prove that bi-Lyapunov stable classes admit a dominated splitting (in fact, the bi-Lyapunov stable chain recurrence classes constructed in [BD$_3$] do not admit any). However, as a consequence of Theorem 3.2.1 we will have this property automatically.

Theorem 3.2.16. *For every f in a residual subset $\mathcal{G}_1$ of $\mathrm{Diff}^1(M)$, if H is a bi-Lyapunov stable homoclinic class for f, then, H admits a dominated splitting. Moreover, it admits at least one dominated splitting with index equal to the index of some periodic point in the class.*

Proof. This follows directly from Theorem 3.2.1 applied either to f or f^{-1}.

$\square$

This theorem solves affirmatively the second part of Problem 5.1 in [ABD]. We remark that Theorem 3.2.16 does not imply that the class is not accumulated by sinks or sources. Also, we must remark that the theorem is optimal in the following sense, in [BV] an example is constructed of a robustly transitive diffeomorphism (thus bi-Lyapunov stable) of $\mathbb{T}^4$ admitting only one dominated splitting (into two two-dimensional bundles) and with periodic points of all possible indexes for saddles.

We recall now that a compact invariant set H is *strongly partially hyperbolic* if it admits a three ways dominated splitting $T_H M = E^s \oplus E^c \oplus E^u$, where E^s is non trivial and uniformly contracting and E^u is non trivial and uniformly expanding.

In the context of Question 3.2.15 it was shown in [ABD] that generic bi-Lyapunov stable homoclinic classes admitting a strongly partially hyperbolic splitting must be the whole manifold. Thus, it is very important to study whether the extremal bundles of a dominated splitting must be uniform.

As a consequence of Theorem 3.2.3 we get the following easy corollaries.

Corollary 3.2.17. *Let H be a bi-Lyapunov stable homoclinic class for a C^1-generic diffeomorphism f such that $T_H M = E^1 \oplus E^2 \oplus E^3$ is a dominated splitting for f and* $\dim(E^1) = \dim(E^3) = 1$*. Then, H is strongly partially hyperbolic and $H = M$.*

PROOF. The class should be strongly partially hyperbolic by applying Theorem 3.2.3 applied to both f and f^{-1}. Corollary 1 of [ABD] (page 185) implies that $H = M$.

□

We say that a hyperbolic periodic point p is *far from tangencies* if there is a neighborhood of f such that there are no homoclinic tangencies associated to the stable and unstable manifolds of the continuation of p. The tangencies are of index i if they are associated to a periodic point of index i, that is, its stable manifold has dimension i.

Corollary 3.2.18. *Let H be a bi-Lyapunov stable homoclinic class for a C^1-generic diffeomorphism f which has a periodic point p of index* 1 *and a periodic point q of index $d-1$ and such that p and q are far from tangencies . Then, $H = M$.*

PROOF. Using Theorem 1.2.25 we are in the hypothesis of Corollary 3.2.17

□

In low dimension, our results have some stronger implications, we obtain:

Theorem 3.2.19. *Let f be a C^1-generic surface diffeomorphism having a bi-Lyapunov stable homoclinic class H. Then, $H = \mathbb{T}^2$ and f is Anosov.*

PROOF. From Theorem 3.2.16 and Theorem 3.2.3 we deduce that H must be hyperbolic. Using the interior and the local product structure, we obtain that $H = M$ (see [ABD]) and thus f is Anosov.

Now, by Franks' theorem ([F_1]) M must be $\mathbb{T}^2$ and f conjugated to a linear Anosov diffeomorphism.

□

Remark 3.2.20. Notice that for proving this Theorem we do not need to use the results of [PS_1] which involve C^2 approximations.

◇

The following proposition gives a complete answer to Problem 5.1 of [ABD] in dimension 3.

Proposition 3.2.21. *Let H be a bi-Lyapunov stable homoclinic class for a C^1-generic diffeomorphism in dimension 3. Then, H has nonempty interior.*

PROOF. Applying Theorem 3.2.16 one can assume that the class H admits a dominated splitting of the form $E \oplus F$, and without loss of generality one can assume that $\dim F = 1$.

Theorem 3.2.3 thus implies that F is uniformly expanded so the splitting is $T_H M = E \oplus E^u$.

Assume first that there exist a periodic point p in H of index 2. Thus, this periodic point has a local stable manifold of dimension 2 which is homeomorphic to a 2 dimensional disc.

Since the class is Lyapunov stable for f^{-1} the stable manifold of the periodic point is completely contained in the class.

Now, using Lyapunov stability for f and the lamination by strong unstable manifolds given by Theorem 1.3.1 one gets (saturating by unstable sets the local stable manifold of p) that the homoclinic class contains an open set. This implies the thesis under this assumption.

So, we must show that if all the periodic points in the class have index 1 then the class is the whole manifold. As we have been doing, using the genericity of f we can assume that there is a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ and an open set $\mathcal{U}$ of f such that for every $g \in \mathcal{U} \cap \mathcal{G}$ all the periodic points in the class have index 1.

We have 2 situations, on the one hand, we consider the case where E admits two invariant subbundles, $E = E^1 \oplus E^2$, with a dominated splitting and thus, we get that E^1 should be uniformly contracting (using Theorem 3.2.3) proving that the homoclinic class is the whole manifold (Corollary 3.2.17).

If E admits no invariant subbundles then, using Theorem 1.2.15, we can perturb the derivative of a periodic point in the class, so that the cocycle over the periodic point restricted to E has all its eigenvalues contracting. So, we can construct a periodic point of index 2 inside the class.

$\square$

Remark 3.2.22. It is very easy to adapt the proof of this proposition to get that: If a bi-Lyapunov stable homoclinic class of a generic diffeomorphism admits a codimension one dominated splitting, $T_H M = E \oplus F$ with $\dim F = 1$, and has a periodic point of index $d-1$, then, the class has nonempty interior.

$\diamondsuit$

Using a Theorems 3.2.10 and 3.2.3 we are able to prove a similar result which is stronger than the previous corollary but which in turn, has hypothesis of a more global nature. We say that a diffeomorphism f is *far from tangencies* if it can not be approximated by diffeomorphisms having homoclinic tangencies for some hyperbolic periodic point. Notice that in the far from tangencies context, it is proved in [Y] that a Lyapunov stable chain recurrence class must be an homoclinic class.

Proposition 3.2.23. *There exists a C^1-residual subset of the open set of diffeomorphisms far from tangencies such that if H is a bi-Lyapunov stable chain recurrence class for such a diffeomorphism, then, $H = M$.*

Proof. First of all, if the class has all its periodic points with index between α and β we know that it admits a 3 ways dominated splitting of the form $T_H M = E \oplus G \oplus F$ where $\dim E = \alpha$ and $\dim F = d - \beta$. This is because we can apply the result of [W_1] (see Theorem 1.2.24) which says that far from homoclinic tangencies there is an index i dominated splitting over the closure of the index i periodic points together with the fact that index α and β periodic points should be dense in the class since the diffeomorphism is generic.

Now, we will show that H admits a strong partially hyperbolic splitting. If E is one dimensional, then it must be uniformly hyperbolic because of Theorem 3.2.3. If not, suppose $\dim E > 1$ then, if it is not uniform, Theorem 3.2.10 implies that it can be decomposed as a uniform bundle together with a one dimensional central bundle, since $\dim E > 1$ we get a uniform bundle of positive dimension.

The same argument applies for F using Lyapunov stability for f^{-1} so we get a strong partially hyperbolic splitting.

Corollary 1 of [ABD] finishes the proof.

□

3.3 Examples

We have seen in section 3.1 that a C^1-generic diffeomorphism of a compact surface admits a hyperbolic attractor. Moreover, we have seen that if a C^1-generic diffeomorphism of a manifold has an attractor, then this attractor must be volume partially hyperbolic (see Theorem 1.2.17, notice that an attractor is an isolated chain-recurrence class).

It seems natural to ask the following question (see [PaPu] Problem 26, [Mi], [BDV] Problems 10.1 and 10.30, [BC]):

Question 3.3.1. *Does a C^r-generic diffeomorphism of a compact manifold have an attractor?*

The question traces back to R. Thom and S. Smale who believed in a positive answer to this question. See [BLY] for a more complete historical account on this problem.

Recently, and surprisingly (notice that even though it was always posed as a question, it had always follow up questions in case the answer was positive), it was shown by [BLY] that this is not the case. We will review their example in subsection 3.3.1.

Their example posses infinitely many sources accumulating on certain quasi-attractors (which as we have seen always exist by Theorem 1.1.22) so it seemed also natural to ask whether a C^r-generic diffeomorphism has either attractors or repellers. A very subtle modification of the example of [BLY] allows one to create examples not having either attractors nor repellers (see [BS], we shall extend this comment in the following section).

It seems natural then, to weaken the notion of attractor in order to continue searching for the chain-recurrence classes which capture "most" of the dynamics of a "typical" diffeomorphism. We have seen in subsection 1.1.5 several notions of attracting sets, and we have payed special attention (specially in this chapter) to *quasi-attractors* (which as we said, always exist by Theorem 1.1.22).

However, the notion of quasi-attractor is not that satisfying since it may even have empty basin (see the examples of [BD_3]). The following natural question was posed in [BLY] and seems the "right" one:

Question 3.3.2. *Does a C^r-generic diffeomorphism admit an essential attractor? and a Milnor attractor?*

As we already mentioned, the first question has been answered in the affirmative for C^1-generic diffeomorphisms far away from tangencies ([BGLY]).

Of course, candidates for such classes will be quasi-attractors, specially those which are homoclinic classes. In view of our Corollary 3.2.4 it seems that there are some tools to attack certain partial questions in dimension 3, and regarding at partially hyperbolic quasi-attractors in dimension 3 which are homoclinic classes seems a reasonable way to proceed. We still lack of examples, but in certain cases, we seem to be acquiring the necessary tools to understand these particular classes and start constructing a theory. In this section, we will review this bunch of examples and we will close the chapter by proposing a direction in order to understand a certain class of quasi-attractors in dimension 3.

3.3.1 The example of Bonatti-Li-Yang

We briefly explain the construction of C.Bonatti, M.Li and D.Yang in [BLY] and the modifications made by C.Bonatti and K.Shinohara in [BS].

They prove the following theorem:

Theorem 3.3.3 ([BLY, BS])**.** *Given a d-dimensional manifold M ($d \geq 3$) we have that for every isotopy class of diffeomorphisms of M there exists an open set $\mathcal{U}$ such that for a diffeomorphisms f in a C^r-residual subset of $\mathcal{U}$ we have that:*

- *There is no attractor nor repeller for f.*

- *Every quasi-attractor $\mathcal{Q}$ of f is an essential attractor.*

- *In every neighborhood of $\mathcal{Q}$ there are aperiodic classes.*

Sketch of the construction of dynamics without attractors

We will outline the construction given in [BLY] of generic dynamics in an open set of diffeomorphisms without attractors. We will make the construction in dimension 3 and in an attracting solid torus, see [BLY] for details on how to extend the dynamics into an attracting ball and other dimensions. Notice that in appendix B of [F_4] it is shown how to construct an Axiom A diffeomorphism C^0-close to a given one having only finitely many sinks as attractors (a simple surgery then allows to obtain that in any isotopy class of diffeomorphisms of a given manifold of dimension ≥ 3 one can construct the desired diffeomorphisms).

We assume from now on some acquaintance with the construction of Smale's solenoid and Plykin's attractor (see for example [KH, Rob_2, Sh]).

One starts with the solid torus $T = \mathbb{D}^2 \times S^1$. As in the solenoid, one "cuts" T by a disk of the form $\mathbb{D}^2 \times \{x\}$ obtaining a solid cylinder of the form $\mathbb{D}^2 \times [-1, 1]$.

Then, one "streches" the resulting filled cylinder in order to be able to "wrap" the torus T more than once. Then one inserts the resulting cylinder in T such that it does not autointersects and "glues" again in the same place one had cut.

One can do this in order that the following conditions are satisfied:

- The resulting map $f : T \to T$ is an injective C^∞ embedding.

- The image of a disk of the form $\mathbb{D}^2 \times \{z\}$ is contained in $\mathbb{D}^2 \times \{z^2\}$ where one thinks of $S^1 \subset \mathbb{C}$ (so that the map $z \mapsto z^2$ is the well known doubling map, see [KH] section 1.7).

- There exists $\alpha > 0$ such that any vector v in the tangent space of a point of the form (p, z) whose angle with respect to to $\mathbb{D}^2 \times \{z\}$ is larger than or equal to α verifies that the image by Df of v makes angle strictly larger than α with $\mathbb{D}^2 \times \{z^2\}$ and norm larger than twice the one of v.

If one also requires that in the f-invariant plaque family given by $\mathbb{D}^2 \times \{z\}$ one has uniform contraction, one obtains the well known Smale's solenoid. C.Bonatti, M.Li and D.Yang have profited from the fact that there is still some freedom to choose the dynamics in this invariant plaque family in order to construct their mentioned example.

Notice that the maximal invariant subset Λ of T admits a partially hyperbolic splitting of the form $T_\Lambda T = E^{cs} \oplus E^u$ where E^{cs} is tangent to $\mathbb{D}^2 \times \{z\}$ in any point of the form (p, z), the subbundle E^u is uniformly expanding and the angle between E^{cs} and E^u is larger than α.

They demand the following two extra properties which are enough to show that in a C^1-neighborhood of f, there will be a residual subset of diffeomorphisms having no attractor at all in T (notice that there must be at least one quasi-attractor since T is a trapping neighborhood, see Theorem 1.1.9):

- Let $1 \in S^1$ be the fixed point of $z \mapsto z^2$. We demand that the dynamics in $\mathbb{D}^2 \times \{1\}$ which is f-invariant has a unique fixed point p which will be hyperbolic and attracting and has complex eigenvalues.
- Let z_0 be a periodic point of $z \mapsto z^2$ of period $k > 0$. We have that $f^k(\mathbb{D}^2 \times \{z_0\}) \subset \mathbb{D}^2 \times \{z_0\}$. We will demand that the dynamics of f^k in that disk is the one of the Plykin attractor on the disk and contains a periodic point q whose determinant restricted to E^{cs} is larger than 1.

The first property allows one to show the following:

Lemma 3.3.4. *There exists a C^1-neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ has a unique quasi-attractor in T which contains the homoclinic class of the continuation of p.*

Proof. Notice that the unstable manifold of every point in the maximal invariant set of g inside T must intersect the stable manifold of the continuation of p. This implies that the closure of the unstable manifold of p (and therefore its homoclinic class) must be contained in every quasi-attractor inside T. See Lemma 3.3.9 for more details.

□

With the second property we can show that generic diffeomorphisms in a neighborhood of f cannot have attractors (recall that an attractor is an isolated quasi-attractor) inside T:

Theorem 3.3.5 ([BLY])**.** *There exists a C^1-neighborhood $\mathcal{U}$ of f such that for every C^1-generic diffeomorphism $g \in \mathcal{U}$ the homoclinic class of the continuation of p is contained in the closure of the set of sources of g, in particular, g has no isolated quasi-attractors.*

Sketch. The fact that p has complex stable eigenvalues implies that E^{cs} admits no sub-dominated splitting.

We now show that the continuation of q for diffeomorphisms in a neighborhood of p belongs to the chain-recurrence class of p. Indeed, the unstable manifold of p intersects the basin of attraction of the Plykin attractor (which is contained in a normally hyperbolic disk) and thus, there are arbitrarilly small pseudo-orbits going from p to the Plykin attractor. Now, since the chain-recurrence class of p is robustly a quasi-attractor by the previous Lemma, we get that the Plykin attractor (and thus q) belongs robustly to the unique quasi-attractor.

By Theorem 1.2.17 we obtain that for C^1-generic diffeomorphisms in a neighborhood of f the homoclinic class of p (which coincides with the unique quasi-attractor for generic diffeomorphisms) is contained in the closure of the set of sources. This concludes.

□

Indeed, in [BLY] it is proved that the same result holds for the C^r-topology (we refer the reader to the next subsection for more details).

Bonatti and Shinohara's result

In the proof of Theorem 3.3.5 the non-isolation of the quasi-attractor follows from the fact that containing a periodic orbit whose determinant at the period is larger than 1, the class cannot be volume hyperbolic, and thus, by Theorem 1.2.17 it cannot be isolated.

One could wonder what happens in the event that the quasi-attractor is indeed volume hyperbolic, so that the creation of sources is not allowed and the criterium given by Corollary 1.2.18 does not apply.

Bonatti and Shinohara, in [BS] have developed a very subtle technique which allows them to eject periodic saddles from the homoclinic class of p even if the class is volume hyperbolic.

Using this technique they are able to construct examples which have no quasi-attractors in T but do not contain sources either [2] thus without attractors nor repellers. A surgery argument allows to prove the second statement of Theorem 3.3.3.

The final item of Theorem 3.3.3 hides some deep consequences of Bonatti and Shinohara's construction. Indeed, they show that such a quasi-attractor (they in fact work in a more general framework which applies to this context) has some *viral* properties as defined in [BCDG] (see also [B]). This allows them to show the existence of quite atypical aperiodic classes (for example, aperiodic classes which are not transitive) as well as to show that there are, for C^1-generic diffeomorphisms in

[2]Notice that they must change the Plykin attractor since it forces the existence of sources even if they are not near the quasi-attractor, see [Ply]. In order to change this, the construction slightly more complicated since it must use Blenders, see subsection 1.3.6.

a neighborhood of the constructed f, uncountably many chain-recurrence classes.

The remarkable feature of their construction is that it relies heavily on the topology of the intersection of the class with the center stable plaques, and indeed, in the next subsection we will show some examples from [Pot$_3$] which have a quite opposed behavior[3].

Both Bonatti-Li-Yang's example and the modifications made by Bonatti and Shinohara yield essential attractors, it would be nice to know if indeed:

Question 3.3.6. *Do these examples admit a Milnor attractor?*

3.3.2 Derived from Anosov examples

After the examples of Bonatti,Li and Yang appeared, it became clear that the use of Theorem 1.2.17 could be a tool yielding examples of dynamics without attractors: It suffices to construct a quasi-attractor which has periodic points which are sectionally dissipative in some sense. Also, the question of understanding ergodic properties of attracting sets and sets whose topological basin is large in some sense becomes an important question.

On the other hand, Bonatti-Li-Yang's example was in a sense, a new kind of *wild* homoclinic class, and the understanding of how the class is accumulated by other classes became a new challenge. Hoping to answer partially to this, I was able to construct some examples whose properties are summarized in the following statement.

Theorem 3.3.7. *There exists a C^1-open set $\mathcal{U}$ of* $\mathrm{Diff}^r(\mathbb{T}^3)$ *such that:*

(a) For every $f \in \mathcal{U}$ we have that f is partially hyperbolic with splitting $TM = E^{cs} \oplus E^u$ and E^{cs} integrates into a f-invariant foliation $\mathcal{F}^{cs}$.

(b) Every $f \in \mathcal{U}$ has a unique quasi-attractor $\mathcal{Q}_f$ which contains a homoclinic class.

(c) Every chain recurrence class $R \neq \mathcal{Q}_f$ is contained in the orbit of a periodic disk in a leaf of the foliation $\mathcal{F}^{cs}$.

(d) There exists a residual subset $\mathcal{G}^r$ of $\mathcal{U}$ such that for every $f \in \mathcal{G}^r$ the diffeomorphism f has no attractors. In particular, f has infinitely many chain-recurrence classes accumulating on $\mathcal{Q}_f$.

(e) For every $f \in \mathcal{U}$ there is a unique Milnor attractor $\tilde{\mathcal{Q}} \subset \mathcal{Q}_f$.

(f) If $r \geq 2$ then every $f \in \mathcal{U}$ has a unique SRB measure whose support coincides with a homoclinic class. Consequently, $\tilde{\mathcal{Q}}$ is a minimal attractor in the sense of

[3]The examples were obtained almost simultaneously.

Milnor. If $r = 1$, then there exists a residual subset $\mathcal{G}_M$ of $\mathcal{U}$ such that for every $f \in \mathcal{G}_M$ we have that $\tilde{\mathcal{Q}}$ coincides with $\mathcal{Q}_f$ and is a minimal Milnor attractor.

The goal of this subsection is to prove this theorem.

By inspection in the proofs, one can easily see that in fact the construction can be made in higher dimensional torus, however, it can only be done in the isotopy classes of Anosov diffeomorphisms. Also, it can be seen that condition (d) can be slightly strengthened in the C^1-topology.

It is clear that this example contrasts with the properties obtained by Bonatti and Shinohara in [BS]. Indeed, for this example, the following question remains unsolved (see [Pot_3] for more discussion on this question):

Question 3.3.8. *Does a C^1-generic diffeomorphism in $\mathcal{U}$ have countably many chain-recurrence classes?*

We remark that the answer to this question for the C^r topology with $r \geq 2$ is false (see [BDV] section 3 and the discussion in [Pot_3]).

In [Pot_3] it is also proved that the example is in the hypothesis of the main theorem of [BuFi] and consequently admits a unique measure of maximal entropy (concept we will not define in this thesis but which is self-explanatory).

Construction of the example

In this section we shall construct an open set $\mathcal{U}$ of $\mathrm{Diff}^r(\mathbb{T}^3)$ for $r \geq 1$ verifying Theorem 3.3.7.

The construction is very similar to the one of Carvalho's example ([Car]) following [BV] with the difference that instead of creating a source, we create an expanding saddle. We do not assume acquaintance of the reader with the referred construction but we will in some stages point the reader to specific parts we will not reproduce.

We start with a linear Anosov diffeomorphism $A : \mathbb{T}^3 \to \mathbb{T}^3$ admitting a splitting $E^s \oplus E^u$ where $\dim E^s = 2$.

We assume that A has complex eigenvalues on the E^s direction so that E^s cannot split as a dominated sum of other two subspaces. For example, the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

which has characteristic polynomial $1 + \lambda^2 - \lambda^3$ works since it has only one real root, and it is larger than one.

Considering an iterate, we may assume that there exists $\lambda < 1/3$ satisfying:

$$\|(DA)_{/E^s}\| < \lambda \;\; ; \;\; \|(DA)^{-1}_{/E^u}\| < \lambda$$

Let q and r be different fixed points of A.

Consider δ small enough such that $B_{6\delta}(q)$ and $B_{6\delta}(r)$ are pairwise disjoint and at distance larger than 400δ (this implies in particular that the diameter of $\mathbb{T}^3$ is larger than 400δ).

Let $\mathcal{E}^u$ be a family of closed cones around the subspace E^u of A which is preserved by DA (that is $D_xA(\mathcal{E}^u(x)) \subset \text{Int}(\mathcal{E}^u(Ax))$). We shall consider the cones are narrow enough so that any curve tangent to $\mathcal{E}^u$ of length bigger than L intersects any stable disk of radius δ. Let $\mathcal{E}^{cs}$ be a family of closed cones around E^s preserved by DA^{-1}.

From now on, δ remains fixed. Given $\varepsilon > 0$ such[4] that $\varepsilon \ll \delta$, we can choose ν sufficiently small such that every diffeomorphism g which is ν-C^0-close to A is semiconjugated to A with a continuous surjection h which is ε-C^0-close to the identity (this is a classical result on topological stability of Anosov diffeomorphisms, see [Wa] and Proposition 2.3.1).

We shall modify A inside $B_\delta(q)$ such that we get a new diffeomorphism $F : \mathbb{T}^3 \to \mathbb{T}^3$ that verifies the following properties:

- F coincides with A outside $B_\delta(q)$ and lies at C^0-distance smaller than ν from A.

- The point q is a hyperbolic saddle fixed point of stable index 1 and such that the product of its two eigenvalues with smaller modulus is larger than 1. We also assume that the length of the stable manifold of q is larger than δ.

- $D_xF(\mathcal{E}^u(x)) \subset \text{Int}(\mathcal{E}^u(F(x)))$. Also, for every $w \in \mathcal{E}^u(x) \setminus \{0\}$ we have $\|DF_x^{-1}w\| < \lambda\|w\|$.

- F preserves the stable foliation of A. Notice that the foliation will no longer be stable.

- For some small $\beta > 0$ we have that $\|D_xFv\| < (1+\beta)\|v\|$ for every v tangent to the stable foliation of A preserved by F and every x.

This construction can be made using classical methods (see [BV] section 6). Indeed, consider a small neighborhood U of q such that $U \subset B_{\nu/2}(q)$ such that U admits a chart $\varphi : U \to \mathbb{D}^2 \times [-1,1]$ which sends q to $(0,0)$ and sends stable manifolds of A in sets of the form $\mathbb{D}^2 \times \{t\}$ and unstable ones into sets of the form $\{s\} \times [-1,1]$. We can modify A by isotopy inside U in such a way that the sets $\mathbb{D}^2 \times \{t\}$ remain an

[4]If K bounds $\|A\|$ and $\|A^{-1}\|^{-1}$ then $\frac{\delta}{10K}$ is enough.

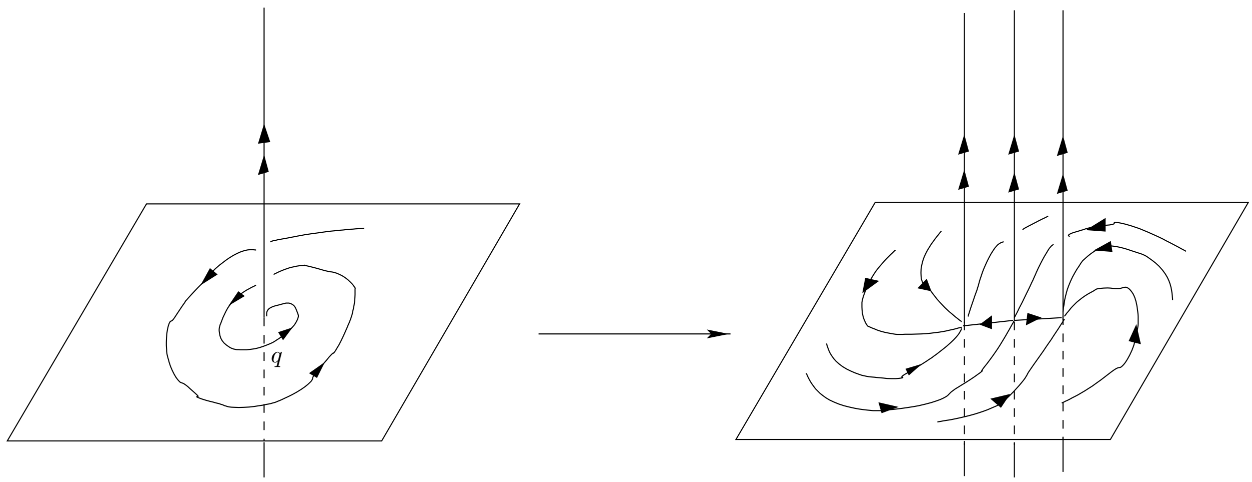

Figure 3.1: Modification of A in a neighborhood of q.

invariant foliation but such that the derivative of q becomes the identity in the tangent space to $\varphi^{-1}(\mathbb{D}^2 \times \{0\})$ which is invariant and such that the dynamics remains conjugated to the initial one. At this point, the norm of the images of unit vectors tangent to the stable foliation of A are not expanded by the derivative.

Now, one can modify slightly the dynamics in $\varphi^{-1}(\mathbb{D}^2 \times \{0\})$ in order to obtain the desired conditions on the eigenvalues of q for F. It is not hard to see that for backward iterates there will be points outside $\varphi^{-1}(\mathbb{D}^2 \times \{0\})$ which will approach q so one can obtain the desired length of the stable manifold of q by maybe performing yet another small modification. All this can be made in order that the vectors tangent to the stable foliation of A are expanded by DF by a factor of at most $(1+\beta)$ with β as small as we desire.

The fact that we can keep narrow cones invariant under DF seems difficult to obtain in view that we made all this modifications. However, the argument of [BV] (page 190) allows to obtain it: This is achieved by conjugating the modification with appropriate homotheties in the stable direction.

The last condition on the norm of DF in the tangent space to the stable foliation of A seems quite restrictive, more indeed in view of the condition on the eigenvalues of q. This condition (as well as property (P7) below) shall be only used (and will be essential) to obtain the ergodic properties of the diffeomorphisms in the open set we shall construct. Nevertheless, one can construct such a diffeomorphism as explained above.

There exists a C^1-open neighborhood $\mathcal{U}_1$ of F such that for every $f \in \mathcal{U}_1$ we have that:

(P1) There exists a continuation q_f of q and r_f of r. The point r_f has stable index 2 and complex eigenvalues. The point q_f is a saddle fixed point of stable index

1, such that the product of its two eigenvalues with smaller modulus is larger than 1 and such that the length of the stable manifold is larger than δ.

(P2) $D_x f(\mathcal{E}^u(x)) \subset \mathrm{Int}(\mathcal{E}^u(f(x)))$. Also, for every $w \in \mathcal{E}^u(x)$ we have

$$\|Df_x w\| \geq \lambda \|w\|.$$

(P3) f preserves a foliation $\mathcal{F}^{cs}$ which is C^0-close to the stable foliation of A. Also, each leaf of $\mathcal{F}^{cs}$ is C^1-close to a leaf of the stable foliation of A.

(P4) For every $x \notin B_\delta(q)$ we have that if $v \in \mathcal{E}^{cs}(x)$ then

$$\|D_x f v\| \leq \lambda \|v\|.$$

This is satisfied for F since $F = A$ outside $B_\delta(q)$.

(P5) There exists a continuous and surjective map $h_f : \mathbb{T}^3 \to \mathbb{T}^3$ such that

$$h_f \circ f = A \circ h_f$$

and $d(h(x), x) < \varepsilon$ for every $x \in \mathbb{T}^3$.

The fact that properties (P1), (P2) and (P4) are C^1-robust is immediate, robustness of (P5) follows from the choice of ν.

Property (P3) holds in a neighborhood of F since F preserves the stable foliation of A which is a C^1-foliation (see [HPS] chapter 7). The foliation $\mathcal{F}^{cs}$ will be tangent to E^{cs} a bidimensional bundle which is f-invariant and contained in $\mathcal{E}^{cs}$. Other way to proceed in order to obtain an invariant foliation is to use Theorem 3.1 of [BuFi] of which all hypothesis are verified here but we shall not state it.

Since the cones $\mathcal{E}^u$ are narrow and from (P3) one has that:

(P6) Every curve of length L tangent to $\mathcal{E}^u$ will intersect any disc of radius 2δ in $\mathcal{F}^{cs}$.

Finally, there exists an open set $\mathcal{U}_2 \subset \mathcal{U}_1$ such that for $f \in \mathcal{U}_2$ we have:

(P7) $\|D_x f v\| \leq (1+\beta)\|v\|$ for every $v \in \mathcal{E}^{cs}(x)$ and every x.

For this examples there exists a unique quasi-attractor for the dynamics.

Lemma 3.3.9. *For every $f \in \mathcal{U}_1$ there exists an unique quasi-attractor $\mathcal{Q}_f$. This quasi attractor contains the homoclinic class of r_f, the continuation of r.*

PROOF. We use the same argument as in [BLY].

There is a center stable disc of radius bigger than 2δ contained in the stable manifold of r_f ((P3) and (P4)). So, every unstable manifold of length bigger than L will intersect the stable manifold of r_f ((P6)).

Let $\mathcal{Q}$ be a quasi attractor, so, there exists a sequence U_n, of neighborhoods of $\mathcal{Q}$ such that $f(\overline{U_n}) \subset U_n$ and $\mathcal{Q} = \bigcap_n \overline{U_n}$.

Since U_n is open, there is a small unstable curve γ contained in U_n. Since Df expands vectors in $\mathcal{E}^u$ we have that the length of $f^k(\gamma)$ tends to $+\infty$ as $n \to +\infty$. So, there exists k_0 such that $f^{k_0}(\gamma) \cap W^s(r_f) \neq \emptyset$. So, since $f(\overline{U_n}) \subset U_n$ we get that $U_n \cap W^s(r_f) \neq \emptyset$, using again the forward invariance of U_n we get that $r_f \in \overline{U_n}$.

This holds for every n so $r_f \in \mathcal{Q}$. Since the homoclinic class of r_f is chain transitive, we also get that $H(r_f) \subset \mathcal{Q}$.

From Conley's theory (subsection 1.1.4), every homeomorphism of a compact metric space there is at least one chain recurrent class which is a quasi attractor. This concludes.

$\square$

The example verifies the mechanism of Proposition 2.2.1

We shall consider $f \in \mathcal{U}_1$ so that it verifies (P1)-(P6).

Let $\mathcal{A}^s$ and $\mathcal{A}^u$ be, respectively, the stable and unstable foliations of A, which are linear foliations. Since A is a linear Anosov diffeomorphism, the distances inside the leaves of the foliations and the distances in the manifold are equal in small neighborhoods of the points if we choose a convenient metric.

Let $\mathcal{A}^s_\eta(x)$ denote the ball of radius η around x inside the leaf of x of $\mathcal{A}^s$. For any $\eta > 0$, it is satisfied that $A(\mathcal{A}^s_\eta(x)) \subset \mathcal{A}^s_{\eta/3}(Ax)$ (an analogous property is satisfied by $\mathcal{A}^u_\eta(x)$ and backward iterates).

The distance inside the leaves of $\mathcal{F}^{cs}$ is similar to the ones in the ambient manifold since each leaf of $\mathcal{F}^{cs}$ is C^1-close to a leaf of $\mathcal{A}^s$. That is, there exists $\rho \approx 1$ such that if x, y belong to a connected component of $\mathcal{F}^{cs}(z) \cap B_{10\delta}(z)$ then $\rho^{-1} d_{cs}(x,y) < d(x,y) < \rho d_{cs}(x,y)$ where $\mathcal{F}^{cs}(z)$ denotes the leaf of the foliation passing through z and d_{cs} the distance restricted to the leaf.

For $z \in \mathbb{T}^3$ we define $W^{cs}_{loc}(z)$ (the *local center stable manifold* of z) as the 2δ-neighborhood of z in $\mathcal{F}^{cs}(z)$ with the distance d_{cs}.

Also, we can assume that for some $\gamma < \min\{\|A\|^{-1}, \|A^{-1}\|^{-1}, \delta/10\}$ the plaque $W^{cs}_{loc}(x)$ is contained in a $\gamma/2$ neighborhood of $\mathcal{A}^s_{2\delta}(x)$, the disc of radius 2δ of the stable foliation of A around x.

Lemma 3.3.10. *We have that $f(\overline{W^{cs}_{loc}(x)}) \subset W^{cs}_{loc}(f(x))$.*

PROOF. Consider around each $x \in \mathbb{T}^3$ a continuous map $b_x : \mathbb{D}^2 \times [-1,1] \to \mathbb{T}^3$ such that $b_x(\{0\} \times [-1,1]) = \mathcal{A}^u_{3\delta}(x)$ and $b_x(\mathbb{D}^2 \times \{t\}) = \mathcal{A}^s_{3\delta}(b_x(\{0\} \times \{t\}))$. For example, one can choose b_x to be affine in each coordinate to the covering of $\mathbb{T}^3$.

Thus, it is not hard to see that one can assume also that $b_x(\frac{1}{3}\mathbb{D}^2 \times \{t\}) = \mathcal{A}^s_\delta(b_x(\{0\} \times \{t\}))$ and that $b_x(\{y\} \times [-1/3, 1/3]) = \mathcal{A}^u_\delta(b_x(\{y\} \times \{0\}))$. Let

$$B_x = b_x(\mathbb{D}^2 \times [-\gamma/2, \gamma/2]).$$

We have that $A(B_x)$ is contained in $b_{Ax}(\frac{1}{3}\mathbb{D}^2 \times [-1/2, 1/2])$. Since f is ε-C^0-near A, we get that $f(B_x) \subset b_{f(x)}(\frac{1}{2}\mathbb{D}^2 \times [-1,1])$.

Let $\pi_1 : \mathbb{D}^2 \times [-1,1] \to \mathbb{D}^2$ such that $\pi_1(x,t) = x$. We have that $\pi_1(b^{-1}_{f(x)}(W^{cs}_{loc}(f(x))))$ contains $\frac{1}{2}\mathbb{D}^2$ from how we chose γ and from how we have defined the local center stable manifolds[5].

Since $f(\mathcal{F}^{cs}(x)) \subset \mathcal{F}^{cs}(f(x))$ and $f(W^{cs}_{loc}(x)) \subset b_{f(x)}(\frac{1}{2}\mathbb{D}^2 \times [-1,1])$ we get the desired property.

$\square$

The fact that $f \in \mathcal{U}_1$ is semiconjugated with A together with the fact that the semiconjugacy is ε-C^0-close to the identity gives us the following easy properties about the fibers (preimages under h_f) of the points.

We denote

$$\Pi^{uu}_{x,z} : U \subset W^{cs}_{loc}(x) \to W^{cs}_{loc}(z)$$

the unstable holonomy where $z \in \mathcal{F}^u(x)$ and U is a neighborhood of x in $W^{cs}_{loc}(x)$ which can be considered large if z is close to x in $\mathcal{F}^u(x)$. In particular, let $\gamma > 0$ be such that if $z \in \mathcal{F}^u_\gamma(x)$ then the holonomy is defined in a neighborhood of radius ε of x.

Proposition 3.3.11. *Consider $y = h_f(x)$ for $x \in \mathbb{T}^3$:*

1. *$h_f^{-1}(\{y\})$ is a compact connected set contained in $W^{cs}_{loc}(x)$.*
2. *If $z \in \mathcal{F}^u_\gamma(x)$, then $h_f(\Pi^{uu}_{x,z}(h_f^{-1}(\{y\})))$ is exactly one point.*

PROOF. (1) Since h_f is ε-C^0-close the identity, we get that for every point $y \in \mathbb{T}^3$, $h_f^{-1}(\{y\})$ has diameter smaller than ε. Since ε is small compared to δ, it is enough to prove that $h_f^{-1}(\{y\}) \subset W^{cs}_{loc}(x)$ for some $x \in h_f^{-1}(\{y\})$.

Assume that for some $y \in \mathbb{T}^3$, $h_f^{-1}(\{y\})$ intersects two different center stable leaves of $\mathcal{F}^{cs}$ in points x_1 and x_2.

[5] In fact, $b^{-1}_{f(x)}(W^{cs}_{loc}(g(x))) \cap \frac{1}{2}\mathbb{D}^2 \times [-1,1]$ is the graph of a C^1 function from $\frac{1}{2}\mathbb{D}^2$ to $[-\gamma/2, \gamma/2]$ if b_x is well chosen.

Since the points are near, we have that $\mathcal{F}^u_\gamma(x_1) \cap W^{cs}_{loc}(x_2) = \{z\}$. Thus, by forward iteration, we get that for some $n_0 > 0$ we have $d(f^{n_0}(x_1), f^{n_0}(z)) > 3\delta$.

Lemma 3.3.10 gives us that $d(f^{n_0}(x_2), f^{n_0}(z)) < 2\delta$. We get that $d(f^{n_0}(x_1), f^{n_0}(x_2)) > \delta$ which is a contradiction since $\{f^{n_0}(x_1), f^{n_0}(x_2)\} \subset h_f^{-1}(\{A^{n_0}(y)\})$ which has diameter smaller than $\varepsilon \ll \delta$.

Also, since the dynamics is trapped in center stable manifolds, we get that the fibers must be connected since one can write them as

$$h^{-1}(\{h(x)\}) = \bigcap_{n\geq 0} f^n(W^{cs}_{loc}(f^{-n}(x))).$$

(2) Since $f^{-n}(h_f^{-1}(\{y\})) = h_f^{-1}(\{A^{-n}(y)\})$ we get that $\text{diam}(f^{-n}(h_f^{-1}(\{y\}))) < \varepsilon$ for every $n > 0$.

This implies that there exists n_0 such that if $n > n_0$ then $f^{-n}(\Pi^{uu}_{x,z}(h_f^{-1}(\{y\})))$ is sufficiently near $f^{-n}(h_f^{-1}(\{y\}))$. So, we have that

$$\text{diam}(f^{-n}(\Pi^{uu}_{x,z}(h_f^{-1}(\{y\})))) < 2\varepsilon \ll \delta.$$

Assume that $h_f(\Pi^{uu}_{x,z}(h_f^{-1}(\{y\})))$ contains more than one point. These points must differ in the stable coordinate of A, so, after backwards iteration we get that they are at distance bigger than 3δ. Since h_f is ε-C^0-close the identity this represents a contradiction.

$\square$

Remark 3.3.12. The second statement of the previous proposition gives that the fibers of h_f are invariant under unstable holonomy.

$\diamondsuit$

The following simple lemma is essential in order to satisfy the properties of Proposition 2.2.1.

Lemma 3.3.13. *For every $f \in \mathcal{U}_1$, given a disc D in $W^{cs}_{loc}(x)$ whose image by h_f has at least two points, then $D \cap \mathcal{F}^u(r_f) \neq \emptyset$ and the intersection is transversal.*

PROOF. Given a subset $K \subset \mathcal{F}^{cs}(x)$ we define its *center stable diameter* as the diameter with the metric d_{cs} defined above induced by the metric in the manifold. We shall first prove that there exists n_0 such that $\text{diam}_{cs}(f^{-n_0}(D)) > 100\delta$:

Since D is arc connected so is $h_f(D)$, so, it is enough to suppose that $\text{diam}(D) < \delta$. We shall first prove that $h_f(D)$ is contained in a stable leaf of the stable foliation of A. Otherwise, there would exist points in $h_f(D)$ whose future iterates separate more than 2δ, this contradicts that the center stable plaques are trapped for f (Lemma 3.3.10).

One now has that, since A is Anosov and that $h_f(D)$ is a connected compact set with more than two points contained in a stable leaf of the stable foliation, there exists $n_0 > 0$ such that $A^{-n_0}(h_f(D))$ has stable diameter bigger than 200δ (recall that $\operatorname{diam} \mathbb{T}^3 > 400\delta$). Now, since h_f is close to the identity, one gets the desired property.

We conclude by proving the following:

Claim. *If there exists n_0 such that $f^{-n_0}(D)$ has diameter larger than 100δ, then D intersects $\mathcal{F}^u(r_f)$.*

Proof. This is proved in detail in section 6.1 of [BV] so we shall only sketch it.

If $f^{-n_0}(D)$ has diameter larger than 100δ, from how we choose δ we have that there is a compact connected subset of $f^{-n_0}(D)$ of diameter larger than 35δ which is outside $B_{6\delta}(q)$.

So, $f^{-n_0-1}(D)$ will have diameter larger than 100δ and the same will happen again. This allows to find a point $x \in D$ such that $\forall n > n_0$ we have that $f^{-n}(x) \notin B_{6\delta}(q)$.

Now, considering a small disc around x we have that by backward iterates it will contain discs of radius each time bigger and this will continue while the disc does not intersect $B_\delta(q)$. If that happens, since $f^{-n}(x) \notin B_{6\delta}(q)$ the disc must have radius at least 3δ.

This proves that there exists m such that $f^{-m}(D)$ contains a center stable disc of radius bigger than 2δ, so, the unstable manifold of r_f intersects it. Since the unstable manifold of r_f is invariant, we deduce that it intersects D and this concludes the proof of the claim.

Transversality of the intersection is immediate from the fact that D is contained in $\mathcal{F}^{cs}$ which is transversal to $\mathcal{F}^u$.

$\diamondsuit$

$\square$

We obtain the following corollary which puts us in the hypothesis of Proposition 2.2.1:

Corollary 3.3.14. *For every $f \in \mathcal{U}_1$, let $x \in \partial h_f^{-1}(\{y\})$ (relative to the local center stable manifold of $h_f^{-1}(\{y\})$), then, x belongs to the homoclinic class of r_f, and in particular, to $\mathcal{Q}_f$.*

Proof. Notice first that the stable manifold of r_f coincides with $\mathcal{F}^{cs}(r_f)$ which is dense in $\mathbb{T}^3$. This follows from the fact that when iterating an unstable curve, it will eventually intersect the stable manifold of r_f, since the stable manifold of r_f is invariant, we obtain the density of $\mathcal{F}^{cs}(r_f)$.

Now, considering $x \in \partial h_f^{-1}(\{y\})$, and $\varepsilon > 0$, we consider a connected component $\tilde{D}$ of $\mathcal{F}^{cs}(r_f) \cap B_\varepsilon(x)$. Clearly, since the fibers are invariant under holonomy and $x \in \partial h_f^{-1}(\{y\})$ we get that $\tilde{D}$ contains a disk D which is sent by h_f to a non trivial connected set. Using the previous lemma we obtain that there is a homoclinic point of r_f inside $B_\varepsilon(x)$ which concludes.

$\square$

The following corollary will allow us to use Theorems 1.2.17 and 1.2.26.

Corollary 3.3.15. *For every $f \in \mathcal{U}_1$ we have that $q_f \in H(r_f)$.*

PROOF. Consider U, a neighborhood of q_f, and D a center stable disc contained in U.

Since the stable manifold of q_f has length bigger than $\delta > \varepsilon$, after backward iteration of D one gets that $f^{-k}(D)$ will eventually have diameter larger than ε, thus $h_f(D)$ will have at least two points, this means that $q_f \in \partial h_f^{-1}(\{h(q_f)\})$. Corollary 3.3.14 concludes.

$\square$

We finish this section by proving the following theorem which is the topological part of Theorem 3.3.7.

Theorem 3.3.16. *(i) For every $f \in \mathcal{U}_1$ there exists a unique quasi-attractor $\mathcal{Q}_f$ which contains the homoclinic class $H(r_g)$ and such that every chain-recurrence class $R \neq \mathcal{Q}_f$ is contained in a periodic disc of $\mathcal{F}^{cs}$.*

(ii) For every $f \in \mathcal{G}_{BC} \cap \mathcal{G}_{BDV} \cap \mathcal{U}_1$ we have that $H(r_f) = \mathcal{Q}_f$ and is contained in the closure of the sources of f.

(iii) For every $r \geq 2$, there exists a C^1-open dense subset $\mathcal{U}_3$ of $\mathcal{U}_1$ and a residual subset $\mathcal{G}^r \subset \mathcal{U}_3 \cap \mathrm{Diff}^r(\mathbb{T}^3)$ such that for every $f \in \mathcal{G}^r$ the homoclinic class $H(r_f)$ intersects the closure of the sources of f.

(iv) For every $f \in \mathcal{U}_1$ there exists a unique Milnor attractor contained in $\mathcal{Q}_f$.

PROOF. Part (i) follows from Proposition 2.2.1 since h_f is the desired semiconjugacy: Indeed, Proposition 3.3.11 and Corollary 3.3.14 show that the hypothesis of the mentioned proposition are verified (notice that A is clearly expansive).

Part (ii) follows from Theorem 1.2.17 using Corollary 3.3.15. Notice that E^{cs} cannot be decomposed in two Df-invariant subbundles since Df has complex eigenvalues in r_f.

Similarly, part (iii) follows from Theorem 1.2.26. The need for considering $\mathcal{U}_3$ comes from [BD$_4$].

To prove (iv) notice that every point which does not belong to the fiber of a periodic orbit belongs to the basin of $\mathcal{Q}_f$: Since there are only countably many periodic orbits and their fibers are contained in two dimensional discs (which have zero Lebesgue measure) this implies directly that the basin of $\mathcal{Q}_f$ has total Lebesgue measure:

Consider a point x whose omega-limit set $\omega(x)$ is contained in a chain recurrence class R different from $\mathcal{Q}_f$. Then, since this chain recurrence class is contained in the fiber $h_f^{-1}(\mathcal{O})$ of a periodic orbit $\mathcal{O}$ of A, which in turn is contained in the local center stable manifold of some point $z \in \mathbb{T}^3$. This implies that some forward iterate of x is contained in $W^{cs}_{loc}(z)$. The fact that the dynamics in W^{cs}_{loc} is trapping (see Lemma 3.3.10) and the fact that $\partial h_f^{-1}(\mathcal{O}) \subset \mathcal{Q}_f$ (see Corollary 3.3.14) gives that x itself is contained in $h_f^{-1}(\mathcal{O})$ as claimed.

Now, Lemma 1.1.18 implies that $\mathcal{Q}_f$ contains an attractor in the sense of Milnor.

□

We have just proved parts (a), (b), (c) and (d) of Theorem 3.3.7 hold in $\mathcal{U}_3$. In fact, for the C^1-topology, we have obtain a slightly stronger property than (d) holds in $\mathcal{U}_1$. Also, we have proved that (e) is satisfied.

Remark 3.3.17. The choice of having complex eigenvalues for A was only used to guaranty that E^{cs} admits no Df-invariant subbundles. One could have started with any linear Anosov map A and modify the derivative of a given fixed or periodic point r to have complex eigenvalues and the construction would be the same.

♢

Ergodic properties

In this section we shall work with $f \in \mathcal{U}_2$ so that properties (P1)-(P7) are verified.

Consider the open set U defined above such that $f(\overline{U}) \subset U$ and consider:

$$\Lambda_f = \bigcap_n f^n(U)$$

We shall show that the hypothesis of Theorem 1.3.11 are satisfied for Λ_f, and thus, we get that there are at most finitely many SRB measures such that the union of their (statistical) basins has full Lebesgue measure in the topological basin of Λ_f. We must show that in every unstable arc there is a positive Lebesgue measure set of points such that $\lambda^{cs}(x) < 0$.

Proposition 3.3.18. *For every $x \in \mathbb{T}^3$ and $D \subset W^{uu}_{loc}(x)$ an unstable arc, we have full measure set of points which have negative Lyapunov exponents in the direction E^{cs}.*

PROOF. The proof is exactly the same as the one in Proposition 6.5 of [BV] so we omit it. Notice that conditions (P2), (P4) and (P7) in our construction imply conditions (i) and (ii) in section 6.3 of [BV].

$\square$

To prove uniqueness of the SRB measure, we must show that there is a unique minimal set of the unstable foliation inside Λ_f to apply Theorem 1.3.12.

However, the fact that the stable manifold of r_f contains $W^{cs}_{loc}(r_f)$, gives that every unstable manifold intersects $W^s(r_f)$ and so we get that every compact subset of Λ_f saturated by unstable sets must contain $\overline{\mathcal{F}^u(r_f)}$. This implies that for every $x \in \overline{\mathcal{F}^u(r_f)}$ we have that $\overline{\mathcal{F}^u(r_f)} = \overline{\mathcal{F}^u(x)}$ and $\overline{\mathcal{F}^u(r_f)}$ is the only compact set with this property (we say that $\overline{\mathcal{F}^u(r_f)}$ is the unique minimal set of the foliation $\mathcal{F}^u$).

We get thus that f admits an unique SRB measure μ and clearly, the support of this SRB measure is $\overline{\mathcal{F}^u(r_f)}$.

We claim that $\overline{\mathcal{F}^u(r_f)} = H(r_f)$: this follows from the fact that the SRB measure μ is hyperbolic (by Proposition 3.3.18) and that the partially hyperbolic splitting separates the positive and negative exponents of μ and so verifies the hypothesis of Theorem 1.3.8.

Finally, since the SRB measure has total support and almost every point converges to the whole support, we get that the attractor is in fact a minimal attractor in the sense of Milnor. We have proved:

Proposition 3.3.19. *If $f \in \mathcal{U}_2$ is of class C^2, then f admits a unique SRB measure whose support coincides with $\overline{\mathcal{F}^u(r_f)} = H(r_f)$. In particular, $\overline{\mathcal{F}^u(r_f)}$ is a minimal attractor in the sense of Milnor for f.*

The importance of considering f of class C^2 comes from the fact that with lower regularity, even if we knew that almost every point in the unstable manifold of r_f has stable manifolds, we cannot assure that these cover a positive measure set due to the lack of absolute continuity in the center stable foliation.

However, the information we gathered for smooth systems in $\mathcal{U}_2$ allows us to extend the result for C^1-generic diffeomorphisms in $\mathcal{U}_2$. Recall that for a C^1-generic diffeomorphisms $f \in \mathcal{U}_2$, the homoclinic class of r_f coincides with $\mathcal{Q}_f$.

Theorem 3.3.20. *There exists a C^1-residual subset $\mathcal{G}_M \subset \mathcal{U}_2$ such that for every $f \in \mathcal{G}_M$ the set $\mathcal{Q}_f = H(r_f)$ is a minimal Milnor attractor.*

PROOF. Notice that since r_f has a well defined continuation in $\mathcal{U}_2$, it makes sense to consider the map $f \mapsto \overline{\mathcal{F}^u(r_f)}$ which is naturally semicontinuous with respect to the Haussdorff topology. Thus, it is continuous in a residual subset $\mathcal{G}_1$ of $\mathcal{U}_2$. Notice that since the semicontinuity is also valid in the C^2-topology, we have that $\mathcal{G}_1 \cap \mathrm{Diff}^2(\mathbb{T}^3)$ is also residual in $\mathcal{U}_2 \cap \mathrm{Diff}^2(\mathbb{T}^3)$.

It suffices to show that the set of diffeomorphisms in $\mathcal{G}_1$ for which $\overline{\mathcal{F}^u(r_f)}$ is a minimal Milnor attractor is a G_δ set (countable intersection of open sets) since we have already shown that C^2 diffeomorphisms (which are dense in $\mathcal{G}_1$) verify this property.

Given an open set U, we define

$$U^+(f) = \bigcap_{n \leq 0} f^n(\overline{U}).$$

Let us define the set $\mathcal{O}_U(\varepsilon)$ as the set of $f \in \mathcal{G}_1$ such that they satisfy one of the following (disjoint) conditions

- $\overline{\mathcal{F}^u(r_f)}$ is contained in U *or*
- $\overline{\mathcal{F}^u(r_f)} \cap \overline{U}^c \neq \emptyset$ and $Leb(U^+(f)) < \varepsilon$

We must show that this sets are open in $\mathcal{G}_1$ (it is not hard to show that if we consider an countable basis of the topology and $\{U_n\}$ are finite unions of open sets in the basis then $\mathcal{G}_M = \bigcap_{n,m} \mathcal{O}_{U_n}(1/m)$).

To prove that these sets are open, we only have to prove the semicontinuity of the measure of $U^+(f)$ (since the other conditions are clearly open from how we chose $\mathcal{G}_1$).

Let us consider the set $\tilde{K} = \overline{U} \backslash U^+(f)$, so, we can write $\tilde{K}$ as an increasing union $\tilde{K} = \bigcup_{n \geq 1} K_n$ where K_n is the set of points which leave $\overline{U}$ in less than n iterates.

So, if $Leb(U^+(f)) < \varepsilon$, we can choose n_0 such that $Leb(\overline{U} \backslash K_{n_0}) < \varepsilon$, and in fact we can consider K'_{n_0} a compact subset of K_{n_0} such that $Leb(\overline{U} \backslash K'_{n_0}) < \varepsilon$.

In a small neighborhood $\mathcal{N}$ of f, we have that if $f' \in \mathcal{N}$, then $K'_{n_0} \subset \overline{U} \backslash U^+(f')$. This concludes.

□

This completes the proof of part (f) of Theorem 3.3.7.

□

3.3.3 Example of Plykin type

The examples in subsection 3.3.2 cannot be embedded in any manifold as the ones of Bonatti-Li-Yang. We were able to adapt the construction in order to get an example with similar properties which can be embedded in any isotopy class of diffeomorphisms of a manifold. However, we were not able to obtain the same strong ergodic properties (see [Pot$_3$] for more discussion and problems).

Theorem 3.3.21. *For every d-dimensional manifold M and every isotopy class of diffeomorphisms of M there exists a C^1-open set $\mathcal{U}$ of* $\mathrm{Diff}^r(M)$ *such that for some open neighborhood U in M:*

(a) Every $f \in \mathcal{U}$ has a unique quasi-attractor $\mathcal{Q}_f$ in U which contains a homoclinic class and has a partially hyperbolic splitting $T_{\mathcal{Q}_f}M = E^{cs} \oplus E^u$ which is coherent.

(b) Every chain recurrence class $R \neq \mathcal{Q}_f$ is contained in the orbit of a periodic leaf of the lamination $\mathcal{F}^{cs}$ tangent to E^{cs} at $\mathcal{Q}_f$.

(c) There exists a residual subset $\mathcal{G}^r$ of $\mathcal{U}$ such that for every $f \in \mathcal{G}^r$ the diffeomorphism f has no attractors. In particular, f has infinitely many chain-recurrence classes.

(d) For every $f \in \mathcal{U}$ there is a unique Milnor attractor $\tilde{\mathcal{Q}} \subset \mathcal{Q}_f$.

The examples here are modifications of the product of a Plykin attractor and the identity on the circle S^1. One can also obtain them in order to provide examples of robustly transitive attractors in dimension 3 with splitting $E^{cs} \oplus E^u$. The author is not aware of other known examples of such attractors other than Carvalho's example which is only possible to be made in certain isotopy classes of diffeomorphisms[6].

In this section we shall show how to construct an example verifying Theorem B. We shall see that we can construct a quasi-attractor with a partially hyperbolic splitting $E^{cs} \oplus E^u$ such that E^{cs} admits no sub-dominated splitting. In case E^{cs} is volume contracting, it will turn out that this quasi-attractor is in fact a robustly transitive attractor (thus providing examples of robustly transitive attractors with splitting $E^{cs} \oplus E^u$ in every 3−dimensional manifold) and when there is a periodic saddle of stable index 1 and such that the product of any two eigenvalues is greater than one and using Theorems 1.2.17 and 1.2.26 we shall obtain that the quasi-attractor will not be isolated for generic diffeomorphisms in a neighborhood.

We shall work only in dimension 3. It will be clear that by multiplying the examples here with a strong contraction, one can obtain examples in any manifold of any dimension.

A main difference between this construction and the one done in section 3.3.2 is the use of *blenders* instead of the argument à la Bonatti-Viana. Blenders were introduced in [BD_1] (see subsection 1.3.6) and constitute a very powerful tool in order to get robust intersections between stable and unstable manifolds of compact

[6]N. Gourmelon communicated me the possibility of constructing examples of this kind by bifurcating other robustly transitive diffeomorphisms such as perturbations of time-one maps of Anosov flows.

sets. We shall only use the facts presented in subsection 1.3.6 and not enter in their definition or construction for which there are many excellent references (we recommend chapter 6 of [BDV] in particular).

Construction of the example

Let us consider $P : \mathbb{D}^2 \hookrightarrow \mathbb{D}^2$ the map given by the Plykin attractor in the disk $\mathbb{D}^2$ (see [Rob_2]).

We have that $P(\mathbb{D}^2) \subset int(\mathbb{D}^2)$, there exist a hyperbolic attractor $\Upsilon \subset \mathbb{D}^2$ and three fixed sources (we can assume this by considering an iterate).

There is a neighborhood N of Υ which is homeomorphic to the disc with 3 holes removed that satisfies that $P(\overline{N}) \subset N$ and

$$\Upsilon = \bigcap_{n\geq 0} P^n(N).$$

It is well known that given $\varepsilon > 0$, one can choose a finite number of periodic points $s_1, \dots, s_N$ and $L > 0$ such that if $A = \bigcup_{i=1}^N W^u_L(s_i)$, then, for every $x \in \Upsilon \setminus A$ one has that A intersects both connected components of $W^s_\varepsilon(x) \setminus \{x\}$.

We now consider the map $F_0 : \mathbb{D}^2 \times S^1 \hookrightarrow \mathbb{D}^2 \times S^1$ given by $F_0(x,t) = (P(x), t)$ whose chain recurrence classes consist of the set $\Upsilon \times S^1$ which is a (non transitive) partially hyperbolic attractor and three repelling circles.

In [BD_1] they make a small C^∞ perturbation F_1 of F_0, for whom the maximal invariant set in $U = N \times S^1$ becomes a C^1-robustly transitive partially hyperbolic attractor Q which remains homeomorphic to $\Upsilon \times S^1$.

This attractor has a partially hyperbolic structure of the type $E^s \oplus E^c \oplus E^u$. One can make this example in order that it fixes the boundary of $\mathbb{D}^2 \times S^1$, this allows to embed this example (and all the modifications we shall make) in any isotopy class of diffeomorphisms of any 3-dimensional manifold (since every diffeomorphism is isotopic to one which fixes a ball, then one can introduce this map by a simple surgery).

In [BD_1] the diffeomorphism F_1 constructed verifies the following properties (see [BD_1] section 4.a page 391, also one can find the indications in [BDV] section 7.1.3):

(F1) F_1 leaves invariant a C^1-lamination $\mathcal{F}^{cs}$ (see [HPS] chapter 7 for a precise definition) tangent to $E^s \oplus E^c$ whose leaves are homeomorphic to $\mathbb{R} \times \mathbb{S}^1$.

(F2) There are periodic points $p_1, \dots, p_N$ of stable index 1 such that for every $x \in Q$ one has that the connected component of $\mathcal{F}^{cs}(x) \setminus \overline{(W^u_L(p_1) \cup \ldots \cup W^u_L(p_N))}$ containing x has finite volume for every $x \in Q \setminus \bigcup_{i=1}^N W^u_L(p_i)$. Here $W^u_L(p_i)$ denotes the L-neighborhood of p_i in its unstable manifold with the metric induced by the ambient.

(F3) There is a periodic point q periodic point of stable index 2 contained in a *cu*-blender K such that for every $1 \leq i \leq N$ the stable manifold of p_i intersects the activating region of K. By Proposition 1.3.10, the unstable manifold of q is dense in the union of the unstable manifolds of p_i.

(F4) The local stable manifold of q intersects every unstable curve of length larger than L.

Before we continue, we shall make some remarks on the properties. The hypothesis (F1) on the differentiability of the lamination $\mathcal{F}^{cs}$ will be used in order to apply the results on normal hyperbolicity of [HPS] (chapter 7, Theorem 7.4, see also Section 1.4).

It can be seen in [BD$_1$] that the construction of F_1 is made by changing the dynamics in finitely many periodic circles and this can be done without altering the lamination $\mathcal{F}^{cs}$ which is C^1 before modification.

This is in fact not necessary; it is possible to apply the barehanded arguments of the proof of Theorem 3.1 of [BuFi] in order to obtain that for the modifications we shall make, there will exist a lamination tangent to the bundle E^{cs}.

Hypothesis (F2) is justified by the fact that the Plykin attractor verifies the same property and the construction of F_1 in [BD$_1$] is made by changing the dynamics in the periodic points by Morse-Smale diffeomorphisms which give rise property (F2) (see section 4.a. of [BD$_1$]). Notice that by continuous variation of stable and unstable sets, this condition is C^1-robust.

Property (F3) is the essence in the construction of [BD$_1$], *cs*-blenders are the main tool for proving the robust transitivity of this examples. As explained in subsection 1.3.6 this is a C^1-open property.

Property (F4) is given by the fact that the local stable manifold of q can be assumed to be $W^s_{loc}(s) \times \mathbb{S}^1$ with a curve removed, where $s \in \Upsilon$ is a periodic point. This is also a C^1-open property.

Let us consider a periodic point $r_1 \in Q$ of stable index 1 and another one r_2 of stable index 2. We can assume they are fixed (modulo considering an iterate of F_1). Consider $\delta > 0$ small enough such that $B_{6\delta}(r_1) \cup B_{6\delta}(r_2)$ is disjoint from:

- the periodic points $p_1, \ldots, p_N, q$ defined above,
- the blender K,
- $\overline{(W^u_L(p_1) \cup \ldots \cup W^u_L(p_N))}$ and
- from $\mathcal{F}^u_{L'}(q)$ (where L' is chosen such that $\mathcal{F}^u_{L'}(q)$ intersects K).

In the same vein as in subsection 3.3.2 we shall first construct a diffeomorphism F_2 modifying F_1 such that:

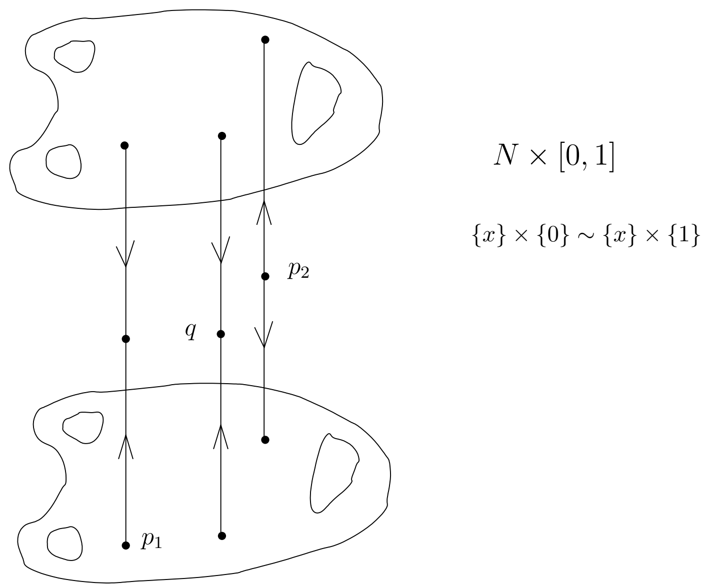

Figure 3.2: How to construct F_1 by small C^∞ perturbations in finitely many circles.

- F_2 coincides with F_1 outside $B_\delta(r_2)$.
- F_2 preserves the center-stable foliation of F_1.
- DF_2 preserves narrow cones $\mathcal{E}^u$ and $\mathcal{E}^{cs}$ around the unstable direction E^u and the center stable direction $E^s \oplus E^c$ of F_1 respectively. Also, vectors in $\mathcal{E}^u$ are expanded uniformly by DF_2 while every plane contained in $\mathcal{E}^{cs}$ verifies that the volume[7] is contracted by DF_2.
- The point r_2 remains fixed for F_2 but now has complex eigenvalues in r_2.

Before we continue with the construction of the example to prove Theorem 3.3.21, we shall make a small detour to sketch the following:

Proposition 3.3.22. *There exists an open C^1-neighborhood $\mathcal{V}$ of F_2 such that for every $f \in \mathcal{V}$ one has that f has a transitive attractor in U.*

SKETCH. Notice that one can choose $\mathcal{V}$ such that for every $f \in \mathcal{V}$ one preserves a center-stable foliation close to the original one. Also, one can assume that properties (F2) and (F3) still hold for the continuations $p_i(f)$ and $q(f)$ since F_2 coincides with F_1 outside $B_\delta(r_2)$ and these are C^1-robust properties.

Also, we demand that for every $f \in \mathcal{V}$, the derivative of f preserves the cones $\mathcal{E}^u$ and $\mathcal{E}^{cs}$, contracts volume in $E^{cs} \subset \mathcal{E}^{cs}$ (the plane tangent to the center-stable foliation) and expands vectors in $E^u \subset \mathcal{E}^u$.

[7]This means with respect to the Riemannian metric which allows to define a notion of 2-dimensional volume in each plane.

Consider now a center stable disk D and an unstable curve γ which intersect the maximal invariant set

$$\mathcal{Q}_f = \bigcap_{n>0} f^n(U).$$

Since by future iterations γ will intersect the stable manifold of $q(f)$ (property (F3)) we obtain that by the $\lambda-$lemma it will accumulate the unstable manifold of $q(f)$. Since the unstable manifold of $q(f)$ is dense in the union of the unstable manifolds $W^u(p_1(f)) \cup \ldots \cup W^u(p_N(f))$ we obtain that the union of the future iterates of γ will also be dense there.

Now, iterating backwards the disk D we obtain, using that Df^{-1} expands volume in the center-stable direction that the diameter of the disk grows exponentially with these iterates.

Condition (F2) will now imply that eventually the backward iterates of D will intersect the future iterates of γ. This implies transitivity.

$\square$

Now, we shall modify F_2 inside $B_\delta(r_1)$ in order to obtain an open set to satisfy Theorem 3.3.21. So we shall obtain F_3 such that:

- F_3 coincides with F_2 outside $B_\delta(r_1)$.
- F_3 preserves the center-stable lamination of F_2.
- DF_3 preserves narrow cones $\mathcal{E}^u$ and $\mathcal{E}^{cs}$ around the unstable direction E^u and the center stable direction E^{cs} of F_2. Also, vectors in $\mathcal{E}^u$ are expanded uniformly by DF_3.
- r_1 is a saddle with stable index 1, the product of any pair of eigenvalues is larger than 1 and the stable manifold of r_1 intersects the complement of $B_{6\delta}(r_1)$.

We obtain a C^1 neighborhood $\mathcal{U}_1$ of F_3 where for $f \in \mathcal{U}$, if we denote

$$\Lambda_f = \bigcap_{n\geq 0} f^n(U):$$

(P1') There exists a continuation of the points $p_1, \ldots, p_N, q, r_1, r_2$ which we shall denote as $p_i(f)$, $q(f)$ and $r_i(f)$. The point $r_1(f)$ is a saddle of stable index 1 and its stable manifold intersects the complement of $B(r_1, 6\delta)$.

(P2') There is a Df-invariant families of cones $\mathcal{E}^u$ in $\mathcal{Q}_f$ and for every $v \in \mathcal{E}^u(x)$ we have that

$$\|D_x f v\| \geq \lambda \|v\|.$$

(P3') f preserves a lamination $\mathcal{F}^{cs}$ which is C^0 close to the one preserved by F_3 and which is trapped in the sense that there exists a family $W^{cs}_{loc}(x) \subset \mathcal{F}^{cs}(x)$ such that for every point $x \in \mathcal{Q}_f$ the plaque $W^{cs}_{loc}(x)$ is homeomorphic to $(0,1) \times \mathbb{S}^1$ and verifies that

$$f(\overline{W^{cs}_{loc}(x)}) \subset W^{cs}_{loc}.$$

Moreover, the stable manifold of $r_1(f)$ intersects the complement of $W^{cs}_{loc}(r_1(f))$.

(P4') Properties (F2),(F3) and (F4) are satisfied for f and every curve γ tangent to $\mathcal{E}^u$ of length larger than L intersects the stable manifold of $q(f)$.

Notice that (P4') implies that there exists a unique quasi-attractor $\mathcal{Q}_f$ in U for every $f \in \mathcal{U}$ which contains the homoclinic class $H(q(f))$ of $q(f)$ (the proof is the same as Lemma 3.3.9).

The example verifies the mechanism of Proposition 2.2.1

We shall show that every $f \in \mathcal{U}$ is in the hypothesis of Proposition 2.2.1 which will conclude the proof of Theorem 3.3.21 as in Theorem 3.3.16. We shall only sketch the proof since it has the same ingredients as the proof of Theorem A, the main difference is that instead of having an a priori semiconjugacy we must construct one.

To construct the semiconjugacy, one uses property (P3'), specifically the fact that $f(\overline{W^{cs}_{loc}(x)}) \subset W^{cs}_{loc}(x)$ (compare with Lemma 3.3.10) to consider for each point $x \in \Lambda_f$ the set:

$$A_x = \bigcap_{n \geq 0} f^n(\overline{W^{cs}_{loc}(f^{-n}(x))})$$

(compare with Proposition 3.3.11 (1)). One easily checks that the sets A_x constitute a partition of Λ_f into compact connected sets contained in local center stable manifolds and that the partition is upper-semicontinuous. It is not hard to prove that if $h_f : \Lambda_f \to \Lambda_f/_\sim$ is the quotient map, then, the map $g : \Lambda_f/_\sim \to \Lambda_f/_\sim$ defined such that

$$h_f \circ f = g \circ h_f$$

is expansive (in fact, $\Lambda_f/_\sim$ can be seen to be homeomorphic to Υ and g conjugated to P). See [Da] for more details on this kind of decompositions and quotients.

Since fibers are contained in center stable sets, we get that h_f is injective on unstable manifolds and one can check that the fibers are invariant under unstable holonomy (see the proof of Proposition 3.3.11 (2)). Stable sets of g are dense in $\Lambda_f/_\sim$.

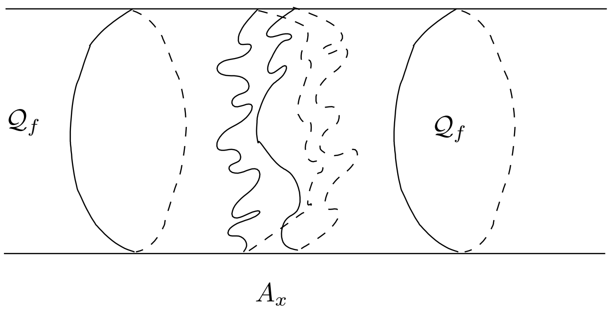

Figure 3.3: The set A_x is surrounded by points in $\overline{W^u(q)} \subset \mathcal{Q}_f$

The point $r_1(f)$ will be in the boundary of $h_f^{-1}(\{h_f(r(f))\})$ since its stable manifold is not contained in $W^{cs}_{loc}(r_1(f))$.

We claim that the boundary of the fibers restricted to center-stable manifolds is contained in the unique quasi-attractor $\mathcal{Q}_f$. This is proven as follows:

Assume that $x \in \partial h_f^{-1}(\{h_f(x)\})$ and consider a small neighborhood V of x. Consider a disk D in $W^{cs}_{loc}(x)$, since x is a boundary point, we get that $h_f(D)$ is a compact connected set containing at least two points in the stable set of $h_f(x)$ for g, so by iterating backwards, and using (F2) (guaranteed for f by (P4')) we get that there is a backward iterate of D which intersects $\mathcal{F}^u(q) \subset \mathcal{Q}_f$ which concludes.

Now, Theorem 3.3.21 follows with the same argument as for Theorem 3.3.16, using Proposition 2.2.1 and the fact that $r_1(f)$ is contained in $\mathcal{Q}_f$.

□

3.3.4 Derived from Anosov revisited

The first examples of non-hyperbolic C^1-robustly transitive diffeomorphisms were given by Shub (see [HPS] chapter 8) in $\mathbb{T}^4$ by considering a skew-product over $\mathbb{T}^2$. Mañe then improved the example to obtain a non-hyperbolic C^1-robustly transitive diffeomorphism of $\mathbb{T}^3$ by deformation of an Anosov diffeomorphism ([M$_1$]). These examples were strongly partially hyperbolic with central dimension 1. Bonatti and Diaz ([BD$_1$]) constructed examples with arbitrary central dimension as well as examples isotopic to the identity but still strongly partially hyperbolic[8].

Finally, Bonatti and Viana ([BV]) constructed examples of robustly transitive diffeomorphisms without any uniform bundle by deforming an Anosov diffeomorphism of $\mathbb{T}^4$ and improving the ideas from Mañe's example. In all the examples constructed by deformation of an Anosov diffeomorphism there is an underlying property which is that the dominated splitting (which must exist by Theorem 1.2.17) has dimensions

[8]Bonatti also constructed an (unpublished) example of robustly transitive partially hyperbolic diffeomorphism of $\mathbb{T}^3$ which was not strongly partially hyperbolic using blenders.

coherent with the splitting the Anosov diffeomorphism has. This simplifies a little the proofs and it is natural to construct the examples this way.

We present here a very similar construction which does not introduce new ideas. However, we believe that such an example is not yet very well understood and may represent an important model for starting to study partially hyperbolic systems with two dimensional center and mixed behavior inside it.

Theorem 3.3.23. *There exists an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^3)$ in the isotopy class of a linear Anosov diffeomorphism A with unstable dimension 2 such that for every $f \in \mathcal{U}$ we have that:*

- *f is partially hyperbolic with splitting $T\mathbb{T}^3 = E^{cs} \oplus E^u$ with $\dim E^u = 1$ and such that E^{cs} admits no subdominated splitting.*
- *f is transitive.*

Proof. Let us consider a linear Anosov automorphism $A \in SL(3, \mathbb{Z})$ such that the eigenvalues of A verify $0 < \lambda_1 < 1/3 < 3 < \lambda_2 < \lambda_3$ and have defined eigenspaces E^s, E^u and E^{uu} respectively.

We denote $p : \mathbb{R}^3 \to \mathbb{T}^3$ the covering projection. Notice that since $\lambda_1\lambda_2\lambda_3 = 1$ we have that $\lambda_1\lambda_2 < 1$.

We choose $\delta > 0$ to be defined later (in the same way as in section 3.3.2) and we make a small C^0-perturbation f_0 of A supported on $B_\delta(p(0))$ such that:

- The tangent map Df_0 preserves narrow cones $\mathcal{C}^u$ and $\mathcal{C}^{cs}$ around E^{uu} and $E^s \oplus E^u$ respectively and uniformly expands vectors in $\mathcal{C}^u$.
- The point $p(0)$ becomes a fixed point with stable index 2 and complex stable eigenvalues.
- The jacobian of f_0 in any $2-$plane inside $\mathcal{C}^{cs}$ is smaller than 1.
- The C^0-distance between f_0 and A is smaller than ε. Here, ε is chosen in a way that every homeomorphism at C^0-distance smaller than 2ε of A is semi-conjugated to A by a continuous map at distance smaller than δ from the identity.

This modification can be made in the same way as we have done for the construction of the example for Theorem 3.3.7.

We shall first consider a small C^1-open neighborhood $\mathcal{U}$ of f_0 such that for every $f \in \mathcal{U}$ we have that:

- The tangent map of Df preserves the cones $\mathcal{C}^u$ and $\mathcal{C}^{cs}$, uniformly expands vectors in $\mathcal{C}^u$ and the jacobian of f in any $2-$plane inside $\mathcal{C}^{cs}$ is smaller than 1.

- The maximal invariant set of f outside $B_\delta(p(0))$ is a hyperbolic set Λ_f with invariant splitting $E^s \oplus E^u \oplus E^{uu}$ which is close to the invariant splitting of A at those points and such that the norm of vectors in an invariant cone around $E^u \oplus E^{uu}$ multiplies by 3 outside $B_\delta(p(0))$.

- The C^0-distance of f and A is smaller than 2ε.

We obtain that for every $f \in \mathcal{U}$ there exists $h_f : \mathbb{T}^3 \to \mathbb{T}^3$ a semiconjugacy with A which is at C^0-distance δ of the identity, that is, we have that $h_f \circ f = A \circ h_f$ and we have that $d(h_f(x), x) < \delta$. Also, we can assume that δ is such that the distance between the connected components of $\tilde{B} = p^{-1}(B_{3\delta}(p(0)))$ is larger than 100δ.

We can choose M such that for every disc D of radius 2δ outside $\tilde{B}$ and every curve of length larger than M and such that its image by H_f (the lift of h_f) to $\mathbb{R}^3$ is close to an arc of stable manifold of A we have that it has an integer translate which intersects D.

Now, consider two open sets U and V in $\mathbb{T}^3$, we must show that there exists $n > 0$ such that $f^n(U) \cap V \neq \emptyset$. We shall work in the universal cover $\mathbb{R}^3$ of $\mathbb{T}^3$. The lift of f shall be denoted as $\tilde{f}$. Let us denote U_0 and V_0 to connected components of $p^{-1}(U)$ and $p^{-1}(V)$ respectively.

Since there is an invariant cone $\mathcal{C}^u$ where vectors are expanded, we get that the diameter of U_0 grows exponentially with future iterates of $\tilde{f}$.

It is not hard to show that there exists a point $x \in U_0$ and $n_0 > 0$ such that for every $n \geq n_0$ we have that $\tilde{f}^n(x) \notin \tilde{B}$. Indeed, once the diameter of $\tilde{f}^{n_1}(U_0)$ is larger than 100δ, there exists a compact connected subset C_1 of U_0 such that $\tilde{f}^{n_1}(C_1)$ does not intersect $\tilde{B}$ and $\text{diam}(\tilde{f}^{-n_1}(C_1)) > 40\delta$. Now, we obtain inductively a decreasing intersection of sets C_k such that $\tilde{f}^{n_1+k}(C_k)$ does not intersect $\tilde{B}$ and has large enough diameter. This implies that in the intersection of all those sets one has the desired point (see the proof the claim inside Lemma 3.3.13).

In a similar fashion, there exists a point $y \in V_0$ such that its backward iterates after some $n_1 < 0$ are disjoint from $\tilde{B}$ (here it is essential the fact that the jacobian of f contracts uniformly the volume in the cone $\mathcal{C}^{cs}$).

Now, we consider a small disk D_1 tangent to a small cone around $E^u \oplus E^{uu}$ centered in $\tilde{f}^{n_0}(x)$ and contained in $\tilde{f}^{n_0}(U_0)$. Iterating forward an using the fact that vectors in that cone are expanded when are outside $p^{-1}(B(p(0), \delta))$ and the point $\tilde{f}^{n_0}(x)$ remains outside $\tilde{B}$ we get that eventually, $\tilde{f}^n(U_0)$ contains a disk $D_2 \subset \tilde{f}^n(U_0)$ whose internal radius is greater than 2δ and whose center is contained in $\tilde{B}^c$.

For past iterates we consider a disk D_3 whose tangent space belongs to a narrow cone around E^{cs} and we know that by volume contraction its backward iterates grow exponentially in diameter[9]. We get that if H_f is the lift of h_f we get that

[9] This could require using dynamical coherence, but we will ignore this issue. In any case, we

$H_f(\tilde{f}^{-n_1}(V_0))$ contains a curve transverse to the cone around $E^u \oplus E^{uu}$ and thus when iterating backwards by A we obtain that its length grows exponentially and it becomes close to the stable direction. Since H_f is at distance smaller than δ of the identity, we get that eventually it has a translate which intersects D_2 by the remark made above which concludes.

□

3.4 Trapping quasi-attractors and further questions

To close this chapter, we will introduce a definition of a kind of quasi-attractors which we believe to be in reach of understanding. The definition is motivated by the examples of Bonatti-Li-Yang as well as the examples presented in subsection 3.3.2. The rest of the examples presented in section 3.3 was essentially introduced in order to show that understanding this kind of quasi-attractors is not the end of the story, even in dimension 3.

Definition 3.4.1 (Trapping quasi-attractors)**.** Let $\mathcal{Q}$ be a quasi-attractor of a diffeomorphism $f : M \to M$ admitting a partially hyperbolic splitting of the form $T_{\mathcal{Q}}M = E^{cs} \oplus E^u$. We will say that $\mathcal{Q}$ is a *trapping quasi-attractor* if it admits a locally f-invariant plaque family $\{\mathcal{W}^{cs}_x\}_{x\in\mathcal{Q}}$ verifying that

$$f(\overline{\mathcal{W}^{cs}_x}) \subset \mathcal{W}^{cs}_x$$

♢

This sets must be compared with *chain-hyperbolic* chain-recurrence classes defined in [CP] which have a similar yet different definition and have played an important role in the proof of the C^1-Palis' conjecture on dynamics far from homoclinic bifurcations.

Hyperbolic attractors are of course examples of trapping quasi-attractors. Both the examples of Bonatti-Li-Yang and the ones presented in section 3.3.2 are non-hyperbolic examples of this type.

As was mentioned, the results of Bonatti-Shinohara and the ones presented in section 3.3.2 present very different properties and it seems a natural to try to understand which are the reasons for this different behaviour.

The author's impression is that the different phenomena is related with the topology of the intersection of the quasi-attractor with center stable plaques: In one case (Bonatti-Li-Yang's example) there is room to eject saddle points, and in the other

can use it because of the results in Chapter 5.

one, the quasi-attractor surrounds the point which one would like to eject prohibiting its removal from the class.

What follows is speculation, and moreover, it is restricted to the 3-dimensional case. We are also assuming that the splitting is of the form $T_Q M = E^{cs} \oplus E^u$ with $\dim E^u = 1$ and such that E^{cs} does not admit a sub-dominated splitting.

It is in principle not obvious how to define the "topology of the intersection of the quasi-attractor with center-stable plaques". We propose that the following should be studied, and we hope pursuing this line in the future:

It should be possible to make a quotient of the dynamics along center-stable leaves in order to obtain an expansive quotient which can be embedded as an expansive attractor of a 3-dimensional manifold (see Section 5.4 for an explicit construction in a particular case). Then, the recent classification by A. Brown [Bro] should allow one to classify these attractors in terms of the topology of the intersection with center-stable plaques. We expect to obtain that if the quotient attractor is one-dimensional then one will be able to perform the techniques of Bonatti and Shinohara in order to get that these quasi-attractors share the same properties as the example of Bonatti-Li-Yang.

In the case the dimension of the attractor has topological dimension 3, we expect that the properties will be similar to the ones obtained for the example in subsection 3.3.2.

It remains to understand also the case where the quotient has topological dimension 2. Although it is not hard to construct examples of this behavior, it seems that a stronger understanding of them needs to be acquired in order to have a more clear picture on the possible dynamics such a quasi-attractor may have.

We finish this section by pointing out that trapping quasi-attractors of course do not cover all the possibilities, even in the case where the decomposition is of the form $T_Q M = E^{cs} \oplus E^u$.

On the one hand, the examples of subsection 3.3.3 show that this may not happen, however, those examples share some property with trapping quasi-attractors since it is possible to find a family of locally invariant plaques (whose topology is not of a disk but a cylinder) which are "trapped". Indeed, this property was the key ingredient in the proof of Theorem 3.3.21.

On the other hand, the example in subsection 3.3.4 seems the real challenge if one wishes to completely understand quasi-attractors in dimension 3 with splitting of the form $T_Q M = E^{cs} \oplus E^u$. The problem is that the center-stable direction contains "unstable" behavior, and this is far less understood. C. Bonatti and Y. Shi ([BSh]) have provided new examples by the study of perturbations the time one map of the Lorenz attractor (see [BDV] chapter 9) which essentially share this problems as well as having several other new interesting properties. It is surely of great interest to have a big list of examples before one attempts to attack the problem of understanding general quasi-attractors in dimension 3.

Chapter 4

Foliations

This chapter has two purposes. One the one hand, it presents general results on foliations and gathers well known material which serves as preliminaries for what we will prove.

On the other hand, we present "almost new" results on foliations: In subsection 4.2.3 we give a classification of Reebless foliations on $\mathbb{T}^3$ (this is "almost new" because the proofs resemble quite nearly those of [BBI$_2$] and similar results exist for C^2-foliations).

In Section 4.3 we prove a result which gives global product structure for certain codimension one foliations on compact manifolds. Our results are slightly more general than the ones which appear, for example, in [HeHi] but with more restrictive hypothesis on the topology of the manifold. Those restrictions on the topology of the manifold have allowed us to prove this result with weaker hypothesis and a essentially different proof.

This chapter contains an appendix which presents similar ideas in the case of surfaces which can be read independently of the rest of the chapter and motivates some of the results of the next chapter.

4.1 Generalities on foliations

4.1.1 Definitions

In section 1.3 we reviewed the concept of lamination, which consists of a partition of a compact subset of a manifold by injectively immersed submanifolds which behave nicely between them. The fact that laminations are only defined in compact subsets suggests that the information they will give about the topology of the manifold is not that strong (although there are many exceptions). In this chapter, we shall work with foliations, that for us will be laminations of the whole manifold and review many results which will give us a lot of information on the relationship between the dynamics and the topology of the phase space.

We will give a partial overview of foliations influenced by the results we use in this thesis. The main sources will be [Ca, CLN, CaCo, HeHi].

Definition 4.1.1 (Foliation). A *foliation* $\mathcal{F}$ of dimension k on a manifold M^d (or codimension $d-k$) is a partition of M on injectively immersed connected C^1-submanifolds tangent to a continuous subbundle E of TM satisfying:

- For every $x \in M$ there exists a neighborhood U and a continuous homeomorphism $\varphi : U \to \mathbb{R}^k \times \mathbb{R}^{d-k}$ such that for every $y \in \mathbb{R}^{d-k}$:

$$L_y = \varphi^{-1}(\mathbb{R}^k \times \{y\})$$

is a connected component of $L \cap U$ where L is an element of the partition $\mathcal{F}$.

$\diamondsuit$

In most of the texts about foliations, this notion refers to a C^0-foliation with C^1-leaves (or foliations of class $C^{1,0+}$ in [CaCo]).

In dynamical systems, particularly in the theory of Anosov diffeomorphisms, flows and or partially hyperbolic systems, this notion is the best suited since it is the one guarantied by these dynamical properties (see Theorem 1.3.1, this notion corresponds to a C^1-lamination of the whole manifold).

Notation. We will denote $\mathcal{F}(x)$ to the *leaf* (i.e. element of the partition) of the foliation $\mathcal{F}$ containing x. Given a foliation $\mathcal{F}$ of a manifold M, we will always denote as $\tilde{\mathcal{F}}$ to the lift of the foliation $\mathcal{F}$ to the universal cover $\tilde{M}$ of M.

$\diamondsuit$

We will say that a foliation is *orientable* if there exists a continuous choice of orientation for the subbundle $E \subset TM$ which is tangent to $\mathcal{F}$. Similarly, we say that the foliation is *transversally orientable* if there exists a continuous choice of orientation for the subbundle $E^{\perp} \subset TM$ consisting of the orthogonal bundle to E. Notice that if M is orientable, then the fact that E is orientable implies that $E^{\perp}$ is also orientable.

Given a foliation $\mathcal{F}$ of a manifold M, one can always consider a finite covering of M and $\mathcal{F}$ in order to get that the lifted foliation is both orientable and transversally orientable.

We remark that sometimes, the definition of a foliation is given in terms of atlases on the manifold, we state the following consequence of our definition:

Proposition 4.1.1. *Let M be a d-dimensional manifold and $\mathcal{F}$ a k-dimensional foliation of M. Then, there exists a C^0-atlas $\{(\varphi_i, U_i)\}$ of M such that:*

- *$\varphi_i : U_i \to \mathbb{R}^k \times \mathbb{R}^{d-k}$ is a homeomorphism.*
- *If $U_i \cap U_j \neq \emptyset$ one has that $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \mathbb{R}^k \times \mathbb{R}^{d-k}$ is of the form $\varphi_i \circ \varphi_j^{-1}(x, y) = (\varphi_{ij}^1(x, y), \varphi_{ij}^2(y))$. Moreover, the maps φ_{ij}^1 are C^1.*
- *The preimage by φ_i of a set of the form $\mathbb{R}^k \times \{y\}$ is contained in a leaf of $\mathcal{F}$.*

An important tool in foliation theory is the concept of *holonomy* (compare with subsection 1.3.3). Given two points x, y in a leaf $\mathcal{F}(x)$ of a foliation $\mathcal{F}$ one can consider transverse disks Σ_x and Σ_y of dimension $d-k$ and a curve $\gamma_{x,y}$ joining these two points and contained in $\mathcal{F}(x)$. It is possible to "lift" this curve to the nearby leaves (by using the atlas given by Proposition 4.1.1) to define a continuous map from a neighborhood of x in Σ_x to a neighborhood of Σ_y. When the curve is understood from the context (for example, when the foliation $\mathcal{F}$ is one dimensional) we denote this map as:

$$\Pi_{x,y}^{\mathcal{F}} : U \subset \Sigma_x \to \Sigma_y$$

These neighborhoods where one can define the maps may depend on the curve γ, however, it can be seen that given two curves $\gamma_{x,y}$ and $\tilde{\gamma}_{x,y}$ which are homotopic inside $\mathcal{F}(x)$ the maps defined coincide in the intersection of their domains, thus, a homotopy class of curves defines a germ of maps from Σ_x to Σ_y.

Considering the curves joining x to itself inside $\mathcal{F}(x)$ one can thus define the following map:

$$\mathrm{Hol} : \pi_1(\mathcal{F}(x)) \to \mathrm{Germ}(\Sigma_x)$$

Which can be seen to be a group morphism. This is useful in some cases in order to see that certain leaves are not simply connected. We call *holonomy group* of a leaf L to the image of the morphism Hol restricted to the fundamental group of L.

An important use of the knowledge of holonomy is given by the following theorem:

Theorem 4.1.2 (Reeb's stability theorem)**.** *Let $\mathcal{F}$ be a foliation of M^d of dimension k and let L be a compact leaf whose holonomy group is trivial. Then, there exists a neighborhood U of L saturated by $\mathcal{F}$ such that every leaf in U is homeomorphic to L. Moreover, the neighborhood U can be chosen arbitrarily small.*

Being an equivalence relation, we can always make a quotient from the foliation and obtain a topological space (which is typically non-Hausdorff) called the *leaf space* endowed with the quotient topology. For a foliation $\mathcal{F}$ on a manifold M we denote the leaf space as $M/_{\mathcal{F}}$.

4.1.2 Generalities on codimension one foliations

Very few is known about foliations in general. However, when the foliation is of codimension 1 there is quite a large general theory (see in particular [HeHi]).

The first important property of codimension one foliations, is the existence of a transverse foliation (which holds in more general contexts, but for our definition of foliation is quite direct):

Proposition 4.1.3. *Given a codimension* 1 *foliation* $\mathcal{F}$ *of a compact manifold* M *there exists a one-dimensional foliation* $\mathcal{F}^\perp$ *transverse to* $\mathcal{F}$*. Moreover, the foliations* $\mathcal{F}$ *and* $\mathcal{F}^\perp$ *admit a* local product structure*, this means that for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that:*

- *Given* $x, y \in M$ *such that* $d(x,y) < \delta$ *one has that* $\mathcal{F}_\varepsilon(x) \cap \mathcal{F}^\perp_\varepsilon(y)$ *consists of a unique point. Here,* $\mathcal{F}_\varepsilon(x)$ *and* $\mathcal{F}^\perp_\varepsilon(y)$ *denote the local leaves*[1] *of the foliations in* $B_\varepsilon(x)$ *and* $B_\varepsilon(y)$*.*

Proof. Assume first that $\mathcal{F}$ is transversally orientable. To prove the existence of a one dimensional foliation transverse to $\mathcal{F}$ consider E the continuous subbundle of TM tangent to $\mathcal{F}$. Now, there exists an arbitrarily narrow cone $\mathcal{E}^\perp$ transverse to E around the one dimensional subbundle $E^\perp$ (the orthogonal subbundle to E).

In $\mathcal{E}^\perp$ there exists a C^1 subbundle F. Since $E^\perp$ is orientable, so is F so we can choose a C^1-vector field without singularities inside F which integrates to a C^1 foliation which will be of course transverse to $\mathcal{F}$.

If $\mathcal{F}$ is not transversally orientable, one can choose a C^1-line field inside the cone field and taking the double cover construct a C^1-vector field invariant under deck transformations. This gives rise to an orientable one dimensional foliation transverse to the lift of $\mathcal{F}$ which projects to a non-orientable one transverse to $\mathcal{F}$.

By compactness of M one checks that the local product structure holds.

$\square$

Remark 4.1.4 (Uniform local product structure)*.* There exists $\varepsilon > 0$ such that for every $x \in M$ there exists $V_x \subset M$ containing $B_\varepsilon(x)$ admitting C^0-coordinates $\psi_x : V_x \to [-1,1]^{d-1} \times [-1,1]$ such that:

- ψ_x is a homeomorphism and $\psi_x(x) = (0,0)$.
- ψ_x sends connected components in V_x of leaves of $\mathcal{F}$ into sets of the form $[-1,1]^{d-1} \times \{t\}$.

[1] More precisely, $\mathcal{F}_\varepsilon(x) = cc_x(\mathcal{F}(x) \cap B_\varepsilon(x))$ and $\mathcal{F}^\perp_\varepsilon(y) = cc_y(\mathcal{F}^\perp(y) \cap B_\varepsilon(y))$. As defined in the Notation section, $cc_x(A)$ denotes the connected component of A containing x.

- ψ_x sends connected components in V_x of leaves of $\mathcal{F}^\perp$ into sets of the form $\{s\} \times [-1,1]$.

In fact, choose $\varepsilon_0 > 0$ and consider δ as in the statement about local product structure of Proposition 4.1.3, we get that if $d(x,y) < \delta$ then $\mathcal{F}_{\varepsilon_0}(x) \cap \mathcal{F}^\perp_{\varepsilon_0}(y)$ consists of exactly one point. Given a point $z \in M$ we can then find a continuous map:

$$\tilde{\psi}_x : \mathcal{F}_{\frac{\delta}{2}}(x) \times \mathcal{F}^\perp_{\frac{\delta}{2}}(x) \to M$$

such that $\tilde{\psi}_x(a,b)$ is the unique point of intersection of $\mathcal{F}_{\varepsilon_0}(x) \cap \mathcal{F}^\perp_{\varepsilon_0}(y)$. By the invariance of domain theorem (see [Hat]) we obtain that $\tilde{\psi}_x$ is a homeomorphism over its image $\tilde{V}_x$. Let ε be the Lebesgue number of the covering of M by the open sets $\tilde{V}_x$. For every point $z \in M$ there exists x such that $B_\varepsilon(z) \subset \tilde{V}_x$. Consider a homeomorphism $\nu_x : [-1,1]^{d-1} \times [-1,1] \to \mathcal{F}_{\frac{\delta}{2}}(x) \times \mathcal{F}^\perp_{\frac{\delta}{2}}(x)$ preserving the coordinates, then it is direct to check that the inverse of $\tilde{\psi}_x$ composed with ν_x is the desired ψ_z.

$\diamondsuit$

In codimension 1 the behaviour of the transversal foliation may detect non-simply connected leafs, this is the content of this well known result of Haefliger which can be thought of a precursor of the celebrated Novikov's theorem:

Proposition 4.1.5 (Haefliger Argument)**.** *Consider a codimension one foliation $\mathcal{F}$ of a compact manifold M. Let $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^\perp$ be the lifts to the universal cover of both $\mathcal{F}$ and the transverse foliation given by Proposition 4.1.3. Assume that there exists a leaf of $\tilde{\mathcal{F}}^\perp$ that intersects a leaf of $\tilde{\mathcal{F}}$ in more than one point, then, $\tilde{\mathcal{F}}$ has a non-simply connected leaf.*

This can be restated in the initial manifold by saying that if there exists a closed curve in M transverse to $\mathcal{F}$ which is nullhomotopic, then there exists a leaf of $\mathcal{F}$ such that its fundamental group does not inject in the fundamental group of M.

This result was first proven by Haefliger for C^2 foliations and then extended to general C^0-foliations by Solodov (see [So]). The idea is to consider a disk bounding a transverse curve to the foliation and making general position arguments (the reason for which Haefliger considered the C^2-case first) in order to have one dimensional foliation with Morse singularities on the disk, classical Poincare-Bendixon type of arguments then give the existence of a leaf of $\mathcal{F}$ with non-trivial holonomy.

Other reason for considering codimension one foliations is that leaves with finite fundamental group do not only give a condition on the local behaviour of the foliation but on the global one:

Theorem 4.1.6 (Reeb's global stability theorem)**.** *Let $\mathcal{F}$ be a codimension one foliation on a compact manifold M and assume that there is a compact leaf L of $\mathcal{F}$*

with finite fundamental group. Then, M is finitely covered by a manifold $\hat{M}$ admitting a fibration $p : \hat{M} \to S^1$ whose fibers are homeomorphic to $\hat{L}$ which finitely covers L and by lifting the foliation $\mathcal{F}$ to $\hat{M}$ we obtain the foliation given by the fibers of the fibration p.

Corollary 4.1.7. *Let $\mathcal{F}$ be a codimension one foliation of a 3-dimensional manifold M having a leaf with finite fundamental group. Then, M is finitely covered by $S^2 \times S^1$ and the foliation lifts to a foliation of $S^2 \times S^1$ by spheres.*

For a codimension one foliation $\mathcal{F}$ of a manifold M, such that the leafs in the universal cover are properly embedded, there is a quite nice description of the leaf space $\tilde{M}/_{\tilde{\mathcal{F}}}$ as a (possibly non-Hausdorff) one-dimensional manifold. When the leaf space is homeomorphic to $\mathbb{R}$ we say that the foliation is $\mathbb{R}$-*covered* (see [Ca]).

4.2 Codimension one foliations in dimension 3

4.2.1 Reeb components and Novikov's Theorem

Consider the foliation of the band $[-1,1] \times \mathbb{R}$ given by the horizontal lines together with the graphs of the functions $x \mapsto \exp\left(\frac{1}{1-x^2}\right) + b$ with $b \in \mathbb{R}$.

Clearly, this foliation is invariant by the translation $(x,t) \mapsto (x, t+1)$ so that it defines a foliation on the annulus $[-1,1] \times S^1$ which we call *Reeb annulus*.

In a similar way, we can define a two-dimensional foliation on $\mathbb{D}^2 \times \mathbb{R}$ given by the cylinder $\partial\mathbb{D}^2 \times \mathbb{R}$ and the graphs of the maps $(x,y) \mapsto \exp\left(\frac{1}{1-x^2-y^2}\right) + b$.

Definition 4.2.1 (Reeb component)**.** Any foliation of $\mathbb{D}^2 \times S^1$ homeomorphic to the foliation obtained by quotienting the foliation defined above by translation by 1 is called a *Reeb component.*

$\diamondsuit$

Another important component of 3-dimensional foliations are dead-end components. They consist of foliations of $\mathbb{T}^2 \times [-1,1]$ such that any transversal which enters the boundary cannot leave the manifold again. An example would be the product of a Reeb annulus with the circle.

Definition 4.2.2 (Dead-end component)**.** A foliation of $\mathbb{T}^2 \times [-1,1]$ such that no transversal can intersect both boundary components is called a *dead-end component.*

$\diamondsuit$

Novikov's theorem (see [No]) was proved for C^2-foliations by the same reason as with Haefliger's argument (see Proposition 4.1.5), with the techniques of Solodov (the techniques of Solodov can be simplified with the existence of a transversal one-dimensional foliation) one can prove it in our context (see [CaCo] Theorems 9.1.3 and 9.1.4 and the Remark on page 286):

Theorem 4.2.1 (Novikov [So, CaCo]). *Let $\mathcal{F}$ be a (transversally oriented) codimension one foliation on a 3-dimensional compact manifold M and assume that one of the following holds:*

- *There exist a positively oriented closed loop transverse to $\mathcal{F}$ which is nullhomotopic, or,*
- *there exist a leaf S of $\mathcal{F}$ such that the fundamental group of S does not inject on the fundamental group of M.*
- *$\pi_2(M) \neq \{0\}$.*

Then, $\mathcal{F}$ has a Reeb component.

4.2.2 Reebless and taut foliations

We will say that a (transversally oriented) codimension one foliation of a 3-dimensional manifold is *Reebless* if it does not contain Reeb components. Similarly, we say that a Reebless foliation is *taut* if it has no dead-end components.

As a consequence of Novikov's theorem we obtain the following corollary on Reebless foliations on 3-manifolds which we state without proof. We say that a surface S embedded in a 3-manifold M is *incompressible* if the inclusion $\imath : S \to M$ induces an injective morphism of fundamental groups.

Corollary 4.2.2. *Let $\mathcal{F}$ be a Reebless foliation on an orientable 3-manifold M and $\mathcal{F}^{\perp}$ a transversal one-dimensional foliation. Then,*

(i) For every $x \in \tilde{M}$ we have that $\tilde{\mathcal{F}}(x) \cap \tilde{\mathcal{F}}^{\perp}(x) = \{x\}$.

(ii) The leafs of $\tilde{\mathcal{F}}$ are properly embedded surfaces in $\tilde{M}$. In fact there exists $\delta > 0$ such that every euclidean ball U of radius δ can be covered by a continuous coordinate chart such that the intersection of every leaf S of $\tilde{\mathcal{F}}$ with U is either empty of represented as the graph of a function $h_S : \mathbb{R}^2 \to \mathbb{R}$ in those coordinates.

(iii) Every leaf of $\mathcal{F}$ is incompressible. In particular, $\tilde{M}$ is either $S^2 \times \mathbb{R}$ and every leaf is homeomorphic to S^2 or $\tilde{M} = \mathbb{R}^3$.

(iv) *For every $\delta > 0$, there exists a constant C_δ such that if J is a segment of $\tilde{\mathcal{F}}^\perp$ then* $\mathrm{Vol}(B_\delta(J)) > C_\delta \,\mathrm{length}(J)$.

Notice that item (iii) implies that every leaf of $\tilde{\mathcal{F}}$ is simply connected, thus, if the manifold M is not finitely covered by $S^2 \times S^1$ then every leaf is homeomorphic to $\mathbb{R}^2$. Also, if M is $\mathbb{T}^3$ one can see that every closed leaf of $\mathcal{F}$ must be a two-dimensional torus (since for every other surface S, the fundamental group $\pi_1(S)$ does not inject in $\mathbb{Z}^3$, see [Ri]).

The last statement of (iii) follows by the fact that the leaves of $\mathcal{F}$ being incompressible they lift to $\tilde{M}$ as simply connected leaves. Applying Reeb's stability Theorem 4.1.6 we see that if one leaf is a sphere, then the first situation occurs, and if there are no leaves homeomorphic to S^2 then all leaves of $\tilde{\mathcal{F}}$ must be planes and by a result of Palmeira ([Pal]) we obtain that $\tilde{M}$ is homeomorphic to $\mathbb{R}^3$.

We give a detailed proof of a similar result in the proof of Corollary 5.1.5 (in particular item (iv)) so we leave this result without proof.

4.2.3 Reebles foliations of $\mathbb{T}^3$

This subsection present results whose proofs are essentially contained in [BBI$_2$]. We mention that there is a paper by Plante [Pla] which is also based on previous developments by Novikov ([No]) and Roussarie ([Rou]) which proves essentially the same results for foliations of class C^2 and extends it to manifolds with almost solvable fundamental group. There exists a result of Gabai ([Ga]) which proves the result of Roussarie for the foliations of lower regularity.

We consider a codimension one foliation $\mathcal{F}$ of $\mathbb{T}^3$ which is transversally oriented and $\mathcal{F}^\perp$ a one dimensional transversal foliation given by Proposition 4.1.3 (the only thing we require to $\mathcal{F}^\perp$ is to be transversal to $\mathcal{F}$ in order to satisfy Remark 4.1.4. We shall assume throughout that $\mathcal{F}$ has no Reeb components.

Let $p : \mathbb{R}^3 \to \mathbb{T}^3$ be the cannonical covering map whose deck transformations are translations by elements of $\mathbb{Z}^3$.

Since $\mathbb{R}^3$ is simply connected, we can consider an orientation on $\tilde{\mathcal{F}}^\perp$ (since $\mathcal{F}^\perp$ is oriented, this orientation is preserved under covering transformations).

Given $x \in \mathbb{R}^3$ we get that $\tilde{\mathcal{F}}^\perp(x) \setminus \{x\}$ has two connected components which we call $\tilde{\mathcal{F}}^\perp_+(x)$ and $\tilde{\mathcal{F}}^\perp_-(x)$ according to the chosen orientation of $\tilde{\mathcal{F}}^\perp$.

By Corollary 4.2.2 (ii) we have that for every $x \in \mathbb{R}^3$ the set $\tilde{\mathcal{F}}(x)$ is an embedded surface in $\mathbb{R}^3$. It is diffeomorphic to $\mathbb{R}^2$ by Corollary 4.2.2 (iii). It is well known that this implies that $\tilde{\mathcal{F}}(x)$ separates $\mathbb{R}^3$ into two connected components[2] whose

[2] One can consider the usual one point compactification of $\mathbb{R}^3$ and apply the well known Jordan-Brower's theorem. See for example [Hat] Proposition 2.B.1. This gives that the complement of $\tilde{\mathcal{F}}(x)$ consists of two connected components. The fact that $\tilde{\mathcal{F}}(x)$ is the boundary of both connected com-

boundary is $\tilde{\mathcal{F}}(x)$. These components will be denoted as $F_+(x)$ and $F_-(x)$ depending on whether they contain $\tilde{\mathcal{F}}^{\perp}_+(x)$ or $\tilde{\mathcal{F}}^{\perp}_-(x)$.

Since covering transformations preserve the orientation and send $\tilde{\mathcal{F}}$ into itself, we have that:

$$F_{\pm}(x) + \gamma = F_{\pm}(x+\gamma) \qquad \forall \gamma \in \mathbb{Z}^3$$

For every $x \in \mathbb{R}^3$, we consider the following subsets of $\mathbb{Z}^3$ seen as deck transformations:

$$\Gamma_+(x) = \{\gamma \in \mathbb{Z}^3 \ : \ F_+(x) + \gamma \subset F_+(x)\}$$

$$\Gamma_-(x) = \{\gamma \in \mathbb{Z}^3 \ : \ F_-(x) + \gamma \subset F_-(x)\}$$

We also consider $\Gamma(x) = \Gamma_+(x) \cup \Gamma_-(x)$.

Remark 4.2.3. There exists a uniform local product structure between $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{\perp}$ (given by Remark 4.1.4). Since leafs of $\tilde{\mathcal{F}}$ do not intersect there exists $\delta > 0$ such that if two points x, y are at distance smaller than δ, then either $F_+(x) \subset F_+(y)$ or $F_+(y) \subset F_+(x)$ (the same for F_-). In particular, if $d(\tilde{\mathcal{F}}(x), \tilde{\mathcal{F}}(x) + \gamma) < \delta$, then $\gamma \in \Gamma(x)$. By Corollary 4.2.2 (i) we also know that if two points are in the same leaf of $\tilde{\mathcal{F}}^{\perp}$ and are at distance smaller than δ, then they are connected by a small arc inside the leaf.

$\diamondsuit$

Lemma 4.2.4. *The following properties hold:*

- *(i) If both $F_+(x) \cap F_+(y) \neq \emptyset$ and $F_-(x) \cap F_-(y) \neq \emptyset$ then, either $F_+(x) \subset F_+(y)$ and $F_-(y) \subset F_-(x)$ or $F_+(y) \subset F_+(x)$ and $F_-(x) \subset F_-(y)$. In both of this cases we shall say that $F_+(x)$ and $F_+(y)$ are* nested *(similar with F_-).*

ponents is proved as follows: First, since $\tilde{\mathcal{F}}(x)$ is differentiable, one can find a normal neighborhood which is an I-bundle (homeomorphic to $\tilde{\mathcal{F}}(x) \times [-1, 1]$ such that the homeomorphism maps $\tilde{\mathcal{F}}(x)$ to $\tilde{\mathcal{F}}(x) \times \{0\}$). This implies that if a point of $\tilde{\mathcal{F}}(x)$ is in the boundary of a connected component of $\mathbb{R}^3 \setminus \tilde{\mathcal{F}}(x)$ then the whole $\tilde{\mathcal{F}}(x)$ must be in its boundary. Now, assume that the boundary of one of the connected components of $\mathbb{R}^3 \setminus \tilde{\mathcal{F}}(x)$ does not coincide with $\tilde{\mathcal{F}}(x)$. This implies that in fact the boundary of the connected component is empty: there cannot be boundary points in the other component since is open and contained in the complement and if one point of $\tilde{\mathcal{F}}(x)$ is in the boundary, then from the argument above, one gets that the boundary coincides with $\tilde{\mathcal{F}}(x)$. This is a contradiction since this connected component would be open and closed, thus the whole $\mathbb{R}^3$ which is not the case.

(ii) If $F_+(x) \cap F_+(y) = \emptyset$ then $F_+(y) \subset F_-(x)$ and $F_+(x) \subset F_-(y)$. A similar property holds if $F_-(x) \cap F_-(y) = \emptyset$.

(iii) In particular, $F_+(x) \subset F_+(y)$ if and only if $F_-(y) \subset F_-(x)$.

Proof. We will only consider the case where $\tilde{\mathcal{F}}(x) \neq \tilde{\mathcal{F}}(y)$ since otherwise the Lemma is trivially satisfied (and case (ii) is not possible).

Assume that both $F_+(x) \cap F_+(y)$ and $F_-(x) \cap F_-(y)$ are non-empty. Since $\tilde{\mathcal{F}}(y)$ is connected and does not intersect $\tilde{\mathcal{F}}(x)$ we have that it is contained in either $F_+(x)$ or $F_-(x)$. We can further assume that $\tilde{\mathcal{F}}(y) \subset F_+(x)$ the other case being symmetric. In this case, we deduce that $F_+(y) \subset F_+(x)$: otherwise, we would have that $F_-(x) \cap F_-(y) = \emptyset$. But this implies that $\tilde{\mathcal{F}}(x) \subset F_-(y)$ and thus that $F_-(x) \subset F_-(y)$ which concludes the proof of (i).

To prove (ii) notice that if $F_+(x) \cap F_+(y) = \emptyset$ then we have that $\tilde{\mathcal{F}}(x) \subset F_-(y)$ and $\tilde{\mathcal{F}}(y) \subset F_-(x)$. This gives that both $F_+(x) \subset F_-(y)$ and $F_+(y) \subset F_-(x)$ as desired.

Finally, if $F_+(x) \subset F_+(y)$ we have that $F_-(x) \cap F_-(y)$ contains at least $F_-(y)$ so that (i) applies to give (iii).

See also [BBI$_2$] Lemma 3.8.

$\square$

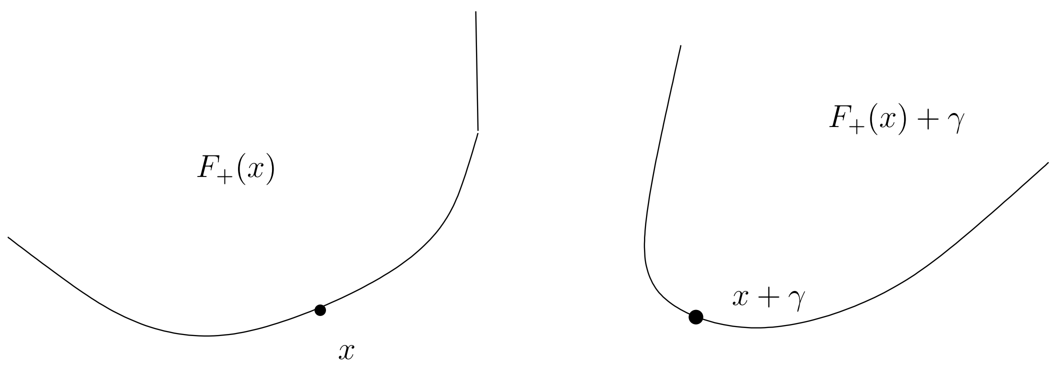

Figure 4.1: When $F_+(x)$ and $F_+(x) + \gamma$ are not nested.

We can prove (see Lemma 3.9 of [BBI$_2$]):

Lemma 4.2.5. *For every $x \in \mathbb{R}^3$ we have that $\Gamma(x)$ is a subgroup of $\mathbb{Z}^3$.*

Proof. Consider $\gamma_1, \gamma_2 \in \Gamma_+(x)$. Since $\gamma_1 \in \Gamma_+(x)$ we have that $F_+(x) + \gamma_1 \subset F_+(x)$. By translating by γ_2 we obtain $F_+(x) + \gamma_1 + \gamma_2 \subset F_+(x) + \gamma_2$, but since $\gamma_2 \in \Gamma_+(x)$ we have $F_+(x) + \gamma_1 + \gamma_2 \subset F_+(x)$, so $\gamma_1 + \gamma_2 \in \Gamma_+(x)$. This shows that $\Gamma_+(x)$ is a semigroup.

Notice also that if $\gamma \in \Gamma_+(x)$ then $F_+(x) + \gamma \subset F_+(x)$, by substracting γ we obtain that $F_+(x) \subset F_+(x) - \gamma$ which implies that $F_-(x) - \gamma \subset F_-(x)$ obtaining that $-\gamma \in \Gamma_-(x)$. We have proved that $-\Gamma_+(x) = \Gamma_-(x)$.

It then remains to prove that if $\gamma_1, \gamma_2 \in \Gamma_+(x)$, then $\gamma_1 - \gamma_2 \in \Gamma(x)$.

Since $F_+(x) + \gamma_1 + \gamma_2$ is contained in both $F_+(x) + \gamma_1$ and $F_+(x) + \gamma_2$ we have that

$$(F_+(x) + \gamma_1) \cap (F_+(x) + \gamma_2) \neq \emptyset.$$

By Lemma 4.2.4 (iii) we have that both $F_-(x) + \gamma_1$ and $F_-(x) + \gamma_2$ contain $F_-(x)$ so they also have non-empty intersection.

Using Lemma 4.2.4 (i), we get that $F_+(x) + \gamma_1$ and $F_+(x) + \gamma_2$ are nested and this implies that either $\gamma_1 - \gamma_2$ or $\gamma_2 - \gamma_1$ is in $\Gamma_+(x)$ which concludes.

□

We close this subsection by proving the following theorem which provides a kind of classification of Reebless foliations in $\mathbb{T}^3$:

Theorem 4.2.6. *Let $\mathcal{F}$ be a Reebless foliation of $\mathbb{T}^3$. Then, there exists a plane $P \subset \mathbb{R}^3$ and $R > 0$ such that every leaf of $\tilde{\mathcal{F}}$ lies in an R-neighborhood of a translate of P. Moreover, one of the following conditions hold:*

(i) Either for every $x \in \mathbb{R}^3$ the R-neighborhood of $\tilde{\mathcal{F}}(x)$ contains $P + x$, or,

(ii) P projects into a two-dimensional torus and (if $\mathcal{F}$ is orientable) there is a dead-end component of $\mathcal{F}$ (in particular, $\mathcal{F}$ has a leaf which is a two-dimensional torus).

Notice that this theorem is mostly concerned with statements on the universal cover so that orientability of the foliation is not necessary. In fact, if one proves the theorem for a finite lift, one obtains the same result since there cannot be embedded incompressible Klein-bottles inside $\mathbb{T}^3$ (see [Ri]). In option (ii), the only thing we need is the fact that transversals remain at bounded distance with the plane P (which does not use orientability since it is a statement on the universal cover).

Proof. By the remark above, we will assume throughout that the foliation is orientable and transversally orientable. This allows us to define as above the sets $F_\pm(x)$ for every x.

We define $G_+(x_0) = \bigcap_{\gamma \in \mathbb{Z}^3} \overline{F_+(x) + \gamma}$ and $G_-(x_0)$ in a similar way.

First, assume that there exists x_0 such that $G_+(x_0) = \bigcap_{\gamma \in \mathbb{Z}^3} \overline{F_+(x) + \gamma} \neq \emptyset$ (see Lemma 3.10 of [BBI$_2$]). The case where $G_-(x_0) \neq \emptyset$ is symmetric. The idea is to prove that in this case we will get option (ii) of the theorem.

There exists $\delta > 0$ such that given a point $z \in G_+(x_0)$ we can consider a neighborhood U_z containing $B_\delta(z)$ given by Corollary 4.2.2 (ii) such that:

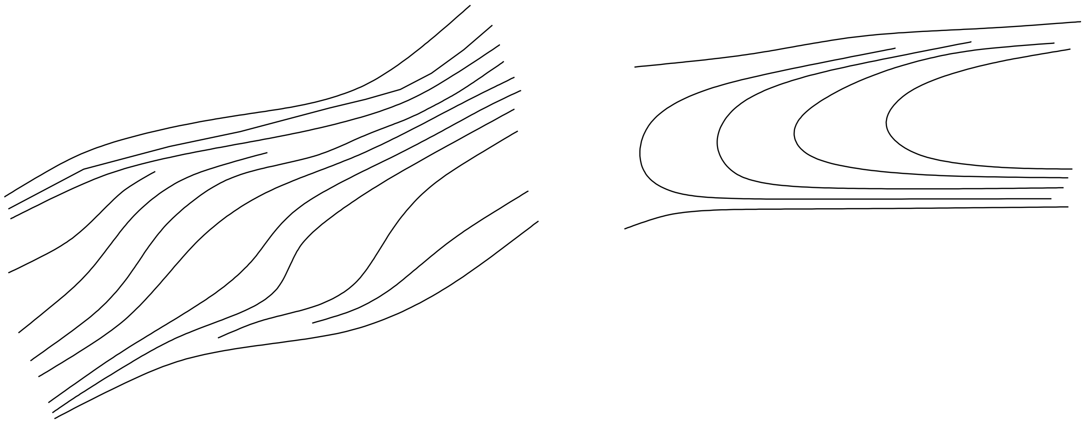

Figure 4.2: How the possibilities on $\tilde{\mathcal{F}}$ look like.

- There is a C^1-coordinate neighborhood $\psi_z : U_z \to \mathbb{R}^2 \to \mathbb{R}$ such that for every $y \in U_z$ we have that $\psi(\tilde{\mathcal{F}}(y) \cap U_z)$ consists of the graph of a function $h_y : \mathbb{R}^2 \to \mathbb{R}$ (in particular, it is connected).

Since $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^\perp$ are orientable, we get that we can choose the coordinates ψ_z in order that for every $y \in U_z$ we have that $\psi_z(F_+(y) \cap U)$ is the set of points (w, t) such that $t \geq h_y(w)$.

Notice that for every $\gamma \in \mathbb{Z}^3$ we have that $(F_+(x) + \gamma) \cap U$ is either the whole U or the upper part of the graph of a function $h_{x+\gamma} : \mathbb{R}^2 \to \mathbb{R}$ in some coordinates in U.

This implies that the intersection $G_+(x_0)$ is a 3-dimensional submanifold of $\mathbb{R}^3$ (modeled in the upper half space) with boundary consisting of leaves of $\tilde{\mathcal{F}}$ (since the boundary components are always locally limits of local leaves).

The boundary is clearly non trivial since $G_+(x_0) \subset F_+(x_0) \neq \mathbb{R}^3$.

Claim. *If $G_+(x_0) \neq \emptyset$ then there exists plane P and $R > 0$ such that every leaf of $\tilde{\mathcal{F}}(x)$ is contained in an R-neighborhood of a translate of P and whose projection to $\mathbb{T}^3$ is a two dimensional torus. Moreover, option (ii) of the proposition holds.*

Proof. Since $G_+(x_0)$ is invariant under every integer translation, we get that the boundary of $G_+(x_0)$ descends to a closed surface in $\mathbb{T}^3$ which is union of leaves of $\mathcal{F}$.

By Corollary 4.2.2 (iii) we get that those leaves are two-dimensional torus whose fundamental group is injected by the inclusion map.

This implies that they are at bounded distance of linear embeddings of $\mathbb{T}^2$ in $\mathbb{T}^3$ and so their lifts lie within bounded distance from a plane P whose projection is a two dimensional torus.

Since leafs of $\tilde{\mathcal{F}}$ do not cross, the plane P does not depend on the boundary component. Moreover, every leaf of $\tilde{\mathcal{F}}(x)$ must lie within bounded distance from a translate of P since every leaf of $\mathcal{F}$ has a lift which lies within two given lifts of some of the torus leafs.

Consider a point x in the boundary of $G_+(x_0)$. We have that $\tilde{\mathcal{F}}(x)$ lies within bounded distance from P from the argument above.

Moreover, each boundary component of $G_+(x_0)$ is positively oriented in the direction which points inward to the interior of $G_+(x_0)$ (recall that it is a compact 3-manifold with boundary).

We claim that η_z lies within bounded distance from P for every $z \in \tilde{\mathcal{F}}(x)$ and η_z positive transversal to $\tilde{\mathcal{F}}$. Indeed, if this is not the case, then $\tilde{\eta}_z$ would intersect other boundary component of $G_+(x_0)$ which is impossible since the boundary leafs of $G_+(x_0)$ point inward to $G_+(x_0)$ (with the orientation of $\tilde{\mathcal{F}}^\perp$).

Now, consider any point $z \in \mathbb{R}^3$, and η_z a positive transversal which we assume does not remain at bounded distance from P. Then it must intersect some translate of $\tilde{\mathcal{F}}(x)$, and the argument above applies. This is a contradiction.

The same argument works for negative transversals since once a leaf enters $(G_+(x_0))^c$ it cannot reenter any of its translates. We have proved that $p(G_+(x_0))$ contains a dead end component. This concludes the proof of the claim.

$\diamondsuit$

Now, assume that (ii) does not hold, in particular $G_\pm(x) = \emptyset$ for every x. Then, for every point x we have that

$$\bigcup_{\gamma \in \mathbb{Z}^3} (F_+(x) + \gamma) = \bigcup_{\gamma \in \mathbb{Z}^3} (F_-(x) + \gamma) = \mathbb{R}^3$$

As in Lemma 3.11 of [BBI$_2$] we can prove:

Claim. *We have that* $\Gamma(x) = \mathbb{Z}^3$ *for every* $x \in \mathbb{R}^3$.

Proof. If for some $\gamma_0 \notin \Gamma(x)$ one has that $F_+(x) \cap (F_+(x) + \gamma_0) = \emptyset$ (the other possibility being that $F_-(x) \cap (F_-(x) + \gamma_0) = \emptyset$)) then, we claim that for every $\gamma \notin \Gamma(x)$ we have that $F_+(x) \cap (F_+(x) + \gamma) = \emptyset$.

Indeed, by Lemma 4.2.4 (i) if the claim does not hold, there would exist $\gamma \notin \Gamma(x)$ such that

$$F_-(x) \cap (F_-(x) + \gamma) = \emptyset \quad \text{and} \quad F_+(x) \cap (F_+(x) + \gamma) \neq \emptyset.$$

By Lemma 4.2.4 (ii) we have:

- $F_-(x) \subset F_+(x) + \gamma$.

- $F_+(x) + \gamma_0 \subset F_-(x)$.

Let $z \in F_+(x) \cap (F_+(x) + \gamma)$ then $z + \gamma_0$ must belong both to $(F_+(x) + \gamma_0) \subset (F_+(x)+\gamma)$ and to $(F_+(x)+\gamma+\gamma_0)$. By substracting γ we get that $z+\gamma_0-\gamma$ belongs to $F_+(x) \cap F_+(x) + \gamma_0$ contradicting our initial assumption. So, for every $\gamma \notin \Gamma(x)$ we have that $F_+(x) \cap (F_+(x) + \gamma) = \emptyset$.

Now, consider the set

$$U_+(x) = \bigcup_{\gamma \in \Gamma(x)} (F_+(x) + \gamma).$$

From the above claim, the sets $U_+(x)+\gamma_1$ and $U_+(x)+\gamma_2$ are disjoint (if $\gamma_1-\gamma_2 \notin \Gamma(x)$) or coincide (if $\gamma_1 - \gamma_2 \in \Gamma(x)$).

Since these sets are open, and its translates by $\mathbb{Z}^3$ should cover the whole $\mathbb{R}^3$ we get by connectedness that there must be only one. This implies that $\Gamma(x) = \mathbb{Z}^3$ and finishes the proof of the claim.

$\diamondsuit$

Consider $\Gamma_0(x) = \Gamma_+(x) \cap \Gamma_-(x)$, the set of translates which fix $\tilde{\mathcal{F}}(x)$.

If $\mathrm{Rank}(\Gamma_0(x)) = 3$, then $p^{-1}(p(\tilde{\mathcal{F}}(x)))$ consists of finitely many translates of $\tilde{\mathcal{F}}(x)$ which implies that $p(\tilde{\mathcal{F}}(x))$ is a closed surface of $\mathcal{F}$. On the other hand, the fundamental group of this closed surface should be isomorphic to $\mathbb{Z}^3$ which is impossible since there are no closed surfaces with such fundamental group ([Ri]). This implies that $\mathrm{Rank}(\Gamma_0(x)) < 3$ for every $x \in \mathbb{R}^3$.

Claim. *For every $x \in \mathbb{R}^3$ there exists a plane $P(x)$ and translates $P_+(x)$ and $P_-(x)$ such that $F_+(x)$ lies in a half space bounded by $P_+(x)$ and $F_-(x)$ lies in a half space bounded by $P_-(x)$.*

Proof. Since $\mathrm{Rank}(\Gamma_0(x)) < 3$ we can prove that $\Gamma_+(x)$ and $\Gamma_-(x)$ are half latices (this means that there exists a plane $P \subset \mathbb{R}^3$ such that each one is contained in a half space bounded by P).

The argument is the same as in Lemma 3.12 of [BBI$_2$] (and the argument after that lemma).

Consider the convex hulls of $\Gamma_+(x)$ and $\Gamma_-(x)$. If their interiors intersect one can consider 3 linearly independent points whose coordinates are rational. These points are both positive rational convex combinations of vectors in $\Gamma_+(x)$ as well as of vectors in $\Gamma_-(x)$. One obtains that $\Gamma_0(x) = \Gamma_+(x) \cap \Gamma_-(x)$ has rank 3 contradicting our assumption.

This implies that there exists a plane $P(x)$ separating these convex hulls.

Consider $z \in \mathbb{R}^3$ and let $\mathcal{O}_+(z) = (z + \mathbb{Z}^3) \cap F_+(x)$. We have that $\mathcal{O}_+(z) \neq \emptyset$ (otherwise $z \in G_-(x)$). Moreover, $\mathcal{O}_+(z) + \Gamma_+(x) \subset \mathcal{O}_+(z)$ because $\Gamma_+(x)$ preserves $F_+(x)$. The symmetric statements hold for $\mathcal{O}_-(z) = (z + \mathbb{Z}^3) \cap F_-(x)$.

We get that $\mathcal{O}_+(z)$ and $\mathcal{O}_-(z)$ are separated by a plane P_z parallel to $P(x)$. The proof is as follows: we consider the convex hull $\mathcal{CO}_+(z)$ of $\mathcal{O}_+(z)$ and the fact that $\mathcal{O}_+(z)+\Gamma_+(x) \subset \mathcal{O}_+(z)$ implies that if v is a vector in the positive half plane bounded by $P(x)$ we have that $\mathcal{CO}_+(z) + v \subset \mathcal{CO}_+(z)$. The same holds for the convex hull of $\mathcal{CO}_-(z)$ and we get that if the interiors of $\mathcal{CO}_+(z)$ and $\mathcal{CO}_-(z)$ intersect, then the interiors of the convex hulls of $\Gamma_+(x)$ and $\Gamma_-(x)$ intersect contradicting that $\mathrm{Rank}(\Gamma_0(x)) < 3$.

Consider δ given by Corollary 4.2.2 (ii) such that every point z has a neighborhood U_z containing $B_\delta(z)$ and such that $\tilde{\mathcal{F}}(y) \cap U_z$ is connected for every $y \in U_z$.

Let $\{z_i\}$ a finite set $\delta/2$-dense in a fundamental domain D_0. We denote as $P^+_{z_i}$ and $P^-_{z_i}$ de half spaces defined by the plane P_{z_i} parallel to $P(x)$ containing $\mathcal{O}_+(z_i)$ and $\mathcal{O}_-(z_i)$ respectively.

We claim that $F_+(x)$ is contained in the δ-neighborhood of $\bigcup_i P^+_{z_i}$ and the symmetric statement holds for $F_-(x)$.

Consider a point $y \in F_+(x)$. We get that $\tilde{\mathcal{F}}(y)$ intersects the neighborhood U_y containing $B_\delta(y)$ in a connected component and thus there exists a $\delta/2$-ball in U_y contained in $F_+(x)$. Thus, there exists z_i and $\gamma \in \mathbb{Z}^3$ such that $z_i + \gamma$ is contained in $F_+(x)$ and thus $z_i + \gamma \in \mathcal{O}_+(z_i) \subset P^+_{z_i}$. We deduce that y is contained in the δ-neighborhood of $P^+_{z_i}$ as desired.

The δ-neighborhood H^+ of $\bigcup_i P^+_{z_i}$ is a half space bounded by a plane parallel to $P(x)$ and the same holds for H^- defined symmetrically. We have proved that $F_+(x) \subset H^+$ and $F_-(x) \subset H^-$. This implies that $\tilde{\mathcal{F}}(x)$ is contained in $H^+ \cap H^-$, a strip bounded by planes $P_+(x)$ and $P_-(x)$ parallel to $P(x)$ concluding the claim.

$\diamondsuit$

We have proved that for every $x \in \mathbb{R}^3$ there exists a plane $P(x)$ and translates $P_+(x)$ and $P_-(x)$ such that $F_\pm(x)$ lies in a half space bounded by $P_\pm(x)$. Let $R(x)$ be the distance between $P_+(x)$ and $P_-(x)$, we have that $\tilde{\mathcal{F}}(x)$ lies at distance smaller than R from $P_+(x)$.

Now, we must prove that the $R(x)$-neighborhood of $\tilde{\mathcal{F}}(x)$ contains $P_+(x)$. To do this, it is enough to show that the projection from $\tilde{\mathcal{F}}(x)$ to $P_+(x)$ by an orthogonal vector to $P(x)$ is surjective. If this is not the case, then there exists a segment joining $P_+(x)$ to $P_-(x)$ which does not intersect $\tilde{\mathcal{F}}(x)$. This contradicts the fact that every curve from $F_-(x)$ to $F_+(x)$ must intersect $\tilde{\mathcal{F}}(x)$.

Since the leaves of $\tilde{\mathcal{F}}$ do not intersect, $P(x)$ cannot depend on x. Since the foliation is invariant under integer translations, we get (by compactness) that $R(x)$ can be chosen uniformly bounded.

$\square$

Remark 4.2.7. It is direct to show that for a given Reebless foliation $\mathcal{F}$ of $\mathbb{T}^3$, the plane

P given by Theorem 4.2.6 is unique. Indeed, the intersection of the R-neighborhoods of two different planes is contained in a $2R$-neighborhood of their intersection line L. If two planes would satisfy the thesis of Theorem 4.2.6 then we would obtain that the complement of every leaf contains a connected component which is contained in the $2R$-neighborhood of L. This is a contradiction since as a consequence of Theorem 4.2.6 we get that there is always a leaf of $\tilde{\mathcal{F}}$ whose complement contains two connected components each of which contains a half space of a plane[3].

$\diamondsuit$

We have used strongly the fact that $\tilde{\mathcal{F}}$ is the lift of a foliation in $\mathbb{T}^3$ so that the foliation is invariant under integer translations, this is why there is more rigidity in the possible foliations of $\mathbb{R}^3$ which are lifts of foliations on $\mathbb{T}^3$. See [Pal] for a classification of foliations by planes of $\mathbb{R}^3$.

4.2.4 Further properties of the foliations

It is not hard to see that:

Proposition 4.2.8. *Let $\mathcal{F}$ be a Reebless foliation of $\mathbb{T}^3$, if option (i) of Theorem 4.2.6 holds, then the leaf space $\mathcal{L} = \mathbb{R}^3/_{\tilde{\mathcal{F}}}$ is homeomorphic to $\mathbb{R}$.*

PROOF. The space of leafs $\mathcal{L}$ with the quotient topology has the structure of a (possibly non-Hausdorff) one-dimensional manifold (see [Ca]). In fact, this follows directly from Corollary 4.2.2 as well as the fact that it is simply connected as a one-dimensional manifold (see Corollary 4.2.2 (i)). To prove the proposition is thus enough to show that it is Hausdorff.

We define an ordering in $\mathcal{L}$ as follows

$$\tilde{\mathcal{F}}(x) \geq \tilde{\mathcal{F}}(y) \quad \text{if} \quad F_+(x) \subset F_+(y).$$

If option (i) of Theorem 4.2.6 holds, given x, y we have that $F_+(x) \cap F_+(y) \neq \emptyset$ and $F_-(x) \cap F_-(y) \neq \emptyset$.

Then, Lemma 4.2.4 (i) implies that $F_+(x)$ and $F_+(y)$ are nested. In conclusion, we obtain that the relationship we have defined is a total order.

Let $\tilde{\mathcal{F}}(x)$ and $\tilde{\mathcal{F}}(y)$ two different leaves of $\tilde{\mathcal{F}}$. We must show that they belong to disjoint open sets.

Without loss of generality, since it is a total order, we can assume that $\tilde{\mathcal{F}}(x) < \tilde{\mathcal{F}}(y)$. This implies that $F_+(y)$ is strictly contained in $F_+(y)$. On the other hand,

[3]Notice that if case (ii) holds this is direct from the existence of a torus leaf and in case (i) this follows from the statement of the last claim in the proof.

this implies that $F_-(y) \cap F_+(x) \neq \emptyset$, in particular, there exists z such that $\tilde{\mathcal{F}}(x) < \tilde{\mathcal{F}}(z) < \tilde{\mathcal{F}}(y)$.

Since the sets $F_+(z)$ and $F_-(z)$ are open and disjoint and we have that $\tilde{\mathcal{F}}(x) \subset F_-(z)$ and $\tilde{\mathcal{F}}(y) \in F_+(z)$ we deduce that $\mathcal{L}$ is Hausdorff as desired.

□

Now, since $\tilde{\mathcal{F}}$ is invariant under deck transformations, we obtain that we can consider the quotient action of $\mathbb{Z}^3 = \pi_1(\mathbb{T}^3)$ in $\mathcal{L}$. For $[x] = \tilde{\mathcal{F}}(x) \in \mathcal{L}$ we get that $\gamma \cdot [x] = [x + \gamma]$ for every $\gamma \in \mathbb{Z}^3$.

Notice that all leaves of $\mathcal{F}$ in $\mathbb{T}^3$ are simply connected if and only if $\pi_1(\mathbb{T}^3)$ acts without fixed point in $\mathcal{L}$. In a similar fashion, existence of fixed points, or common fixed points allows one to determine the topology of leaves of $\mathcal{F}$ in $\mathbb{T}^3$.

In fact, we can prove the following result:

Proposition 4.2.9. *Let $\mathcal{F}$ be a Reebless foliation of $\mathbb{T}^3$. If the plane P given by Theorem 4.2.6 projects into a two dimensional torus by p, then there is a leaf of $\mathcal{F}$ homeomorphic to a two-dimensional torus.*

Proof. Notice first that if option (ii) of Theorem 4.2.6 holds, the existence of a torus leaf is contained in the statement of the theorem.

So, we can assume that option (i) holds. By considering a finite index subgroup, we can further assume that the plane P is invariant under two of the generators of $\pi_1(\mathbb{T}^3) \cong \mathbb{Z}^3$ which we denote as γ_1 and γ_2.

Since leaves of $\tilde{\mathcal{F}}$ remain close in the Hausdorff topology to the plane P we deduce that the orbit of every point $[x] \in \mathcal{L}$ by the action of the elements γ_1 and γ_2 is bounded.

Let γ_3 be the third generator, we get that its orbit cannot be bounded since otherwise it would fix the plane P since it is a translation. So, the quotient of $\mathcal{L}$ by the action of γ_3 is a circle. We can make the group generated by γ_1 and γ_2 act on this circle and we obtain two commuting circle homeomorphisms with zero rotation number. This implies they have a common fixed point which in turn gives us the desired two-torus leaf of $\mathcal{F}$.

□

Also, depending on the topology of the projection of the plane P given by Theorem 4.2.6 we can obtain some properties on the topology of the leaves of $\mathcal{F}$.

Lemma 4.2.10. *Let $\mathcal{F}$ be a Reebless foliation of $\mathbb{T}^3$ and P be the plane given by Theorem 4.2.6.*

(i) Every closed curve in a leaf of $\mathcal{F}$ is homotopic in $\mathbb{T}^3$ to a closed curve contained in $p(P)$. This implies in particular that if $p(P)$ is simply connected, then all the leaves of $\mathcal{F}$ are also simply connected.

(ii) If a leaf of $\mathcal{F}$ is homeomorphic to a two dimensional torus, then, it is homotopic to $p(P)$ (in particular, $p(P)$ is also a two dimensional torus).

PROOF. To see (i), first notice that leafs are incompressible. Given a closed curve γ in a leaf of $\mathcal{F}$ which is not null-homotopic, we know that when lifted to the universal cover it remains at bounded distance from a linear one-dimensional subspace L. Since γ is a circle, we get that $p(L)$ is a circle in $\mathbb{T}^3$. If the subspace L is not contained in P then it must be transverse to it. This contradicts the fact that leaves of $\mathcal{F}$ remain at bounded distance from P.

To prove (ii), notice that a torus leaf T which is incompressible must remain close in the universal cover to a plane P_T which projects to a linear embedding of a 2-dimensional torus. From the proof of Theorem 4.2.6 and the fact that $\mathcal{F}$ is a foliation we get that $P_T = P$. See also the proof of Lemma 3.10 of [BBI$_2$].

□

4.3 Global product structure

4.3.1 Statement of results

We start by defining global product structure:

Definition 4.3.1 (Global Product Structure)**.** Given two transverse foliations (this in particular implies that their dimensions are complementary) $\mathcal{F}_1$ and $\mathcal{F}_2$ of a manifold M we say they admit a *global product structure* if given two points $x, y \in \tilde{M}$ the universal cover of M we have that $\tilde{\mathcal{F}}_1(x)$ and $\tilde{\mathcal{F}}_2(y)$ intersect in a unique point.

♢

Notice that by continuity of the foliations and invariance of domain theorem ([Hat]) we have that if a manifold has two transverse foliations with a global product structure, then, the universal cover of the manifold must be homeomorphic to the product of $\tilde{\mathcal{F}}_1(x) \times \tilde{\mathcal{F}}_2(x)$ for any $x \in \tilde{M}$. Indeed, the map

$$\varphi : \tilde{\mathcal{F}}_1(x) \times \tilde{\mathcal{F}}_2(y) \to \tilde{M} \qquad \varphi(z, w) = \tilde{\mathcal{F}}_1(z) \cap \tilde{\mathcal{F}}_2(w)$$

is well defined, continuous (by the continuity of foliations) and bijective (because of the global product structure), thus a global homeomorphism.

In particular, leaves of $\tilde{\mathcal{F}}_i$ must be simply connected and all homeomorphic between them.

In general, it is a very difficult problem to determine whether two foliations have a global product structure even if there is a local one (this is indeed the main obstruction in the clasification of Anosov diffeomorphisms of manifolds, see [F$_1$]).

However, in the codimension 1 case we again have much more information:

Theorem 4.3.1 (Theorem VIII.2.2.1 of [HeHi]). *Consider a codimension one foliation $\mathcal{F}$ of a compact manifold M such that all the leaves of $\mathcal{F}$ have trivial holonomy. Then, for every $\mathcal{F}^{\perp}$ foliation transverse to $\mathcal{F}$ we have that $\mathcal{F}$ and $\mathcal{F}^{\perp}$ have global product structure.*

This theorem applies for example when every leaf is compact and without holonomy. The other important case (for this thesis) in which this result applies is when every leaf of the foliation is simply connected. Unfortunately, there will be some situations where we will be needing to obtain global product structure but not having neither all leaves of $\mathcal{F}$ simply connected nor that the foliation lacks of holonomy in all its leaves.

We will instead use the following quantitative version of the previous result which does not imply it other than it the situations we will be needing it. The following theorem was proved in [Pot$_5$] and we believe it simplifies certain parts of the previous theorem (at least for the non-expert in the theory of foliations and for the more restrictive hypothesis we include):

Theorem 4.3.2. *Let M be a compact manifold and $\delta > 0$. Consider a set of generators of $\pi_1(M)$ and endow $\pi_1(M)$ with the word length for generators. Then, there exists $K > 0$ such that if $\mathcal{F}$ is a codimension one foliation and $\mathcal{F}^{\perp}$ a transverse foliation such that:*

- *There is a local product structure of size δ between $\mathcal{F}$ and $\mathcal{F}^{\perp}$ (see Remark 4.1.4).*
- *The leaves of $\tilde{\mathcal{F}}$ are simply connected and no element of $\pi_1(M)$ of size less than K fixes a leaf of $\tilde{\mathcal{F}}$.*
- *The leaf space $\mathcal{L} = \tilde{M}/_{\tilde{\mathcal{F}}}$ is homeomorphic to $\mathbb{R}$.*
- *The fundamental group of M is abelian.*

Then, $\mathcal{F}$ and $\mathcal{F}^{\perp}$ admit a global product structure.

4.3.2 Proof of Theorem 4.3.2

Notice that the hypothesis of the Theorem are stable by considering finite lifts and the thesis is in the universal cover so that we can (and we shall) assume that $\mathcal{F}$ is both orientable and transversally orientable.

The first step is to show that leaves of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{\perp}$ intersect in at most one point:

Lemma 4.3.3. *For every $x \in \tilde{M}$ one has that $\tilde{\mathcal{F}}(x) \cap \tilde{\mathcal{F}}^{\perp}(x) = \{x\}$.*

Proof. Assume otherwise, then, by Proposition 4.1.5 (Haefliger argument) one would obtain that there is a non-simply connected leaf of $\tilde{\mathcal{F}}$ a contradiction.

$\square$

In $\mathcal{L} = \tilde{M}/_{\tilde{\mathcal{F}}}$ we can consider an ordering of leafs (by using the ordering from $\mathbb{R}$). We denote as $[x]$ to the equivalence class in $\tilde{M}$ of the point x, which coincides with $\tilde{\mathcal{F}}(x)$.

The following condition will be the main ingredient for obtaining a global product structure:

$(*)$ For every $z_0 \in \tilde{M}$ there exists y^- and $y^+ \in \tilde{M}$ verifying that $[y^-] < [z_0] < [y^+]$ and such that for every $z_1, z_2 \in \tilde{M}$ satisfying $[y^-] \leq [z_i] \leq [y^+]$ $(i = 1, 2)$ we have that $\tilde{\mathcal{F}}^{\perp}(z_1) \cap \tilde{\mathcal{F}}(z_2) \neq \emptyset$.

We get

Lemma 4.3.4. *If property $(*)$ is satisfied, then $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{\perp}$ have a global product structure.*

Proof. Consider any point $x_0 \in \tilde{M}$ and consider the set $G = \{z \in \tilde{M} \ : \ \tilde{\mathcal{F}}^{\perp}(x_0) \cap \tilde{\mathcal{F}}(z) \neq \emptyset\}$. We have that G is open from the local product structure (Remark 4.2.3) and by definition it is saturated by $\tilde{\mathcal{F}}$. We must show that G is closed and since $\tilde{M}$ is connected this would conclude.

Now, consider $z_0 \in \overline{G}$, using assumption $(*)$ we obtain that there exists $[y^-] < [z] < [y^+]$ such that every point z such that $[z^-] < [z] < [z^+]$ verifies that its unstable leaf intersects both $\tilde{\mathcal{F}}(y^-)$ and $\tilde{\mathcal{F}}(y^+)$.

Since $z_0 \in \overline{G}$ we have that there are points $z_n \in G$ such that $z_n \to \tilde{\mathcal{F}}(z_0)$.

We get that eventually, $[y^-] < [z_k] < [y^+]$ and thus we obtain that there is a point $y \in \tilde{\mathcal{F}}^{\perp}(x_0)$ verifying that $[y^-] < [y] < [y^+]$. We get that every leaf between $\tilde{\mathcal{F}}(y^-)$ and $\tilde{\mathcal{F}}(y^+)$ is contained in G from assumption $(*)$. In particular, $z_0 \in G$ as desired.

$\square$

We must now show that property $(*)$ is verified. To this end, we will need the following lemma:

Lemma 4.3.5. *There exists $K > 0$ such that if $\ell > 0$ is large enough, every segment of $\mathcal{F}^{\perp}(x)$ of length ℓ intersects every leaf of $\mathcal{F}$.*

We postpone the proof of this lemma to the next subsection 4.3.3.

Proof of Theorem 4.3.2. We must prove that condition $(*)$ is verified. We consider δ given by the size of local product structure boxes (see Remark 4.2.3) and by Lemma 4.3.5 we get a value of $\ell > 0$ such that every segment of $\mathcal{F}^\perp$ of length ℓ intersects every leaf of $\mathcal{F}$.

There exists $k > 0$ such that every curve of length $k\ell$ will verify that it has a subarc whose endpoints are δ-close and joined by a curve in $\mathcal{F}^\perp$ of length larger than ℓ (so, intersecting every leaf of $\mathcal{F}$).

Consider a point $z_0 \in \tilde{M}$ and a point $z \in \tilde{\mathcal{F}}(z_0)$. Let $\tilde{\eta}_z$ be the segment in $\tilde{\mathcal{F}}^\perp_+(z)$ of length $k\ell$ with one extreme in z. We can project $\tilde{\eta}_z$ to M and we obtain a segment η_z transverse to $\mathcal{F}$ which contains two points z_1 and z_2 at distance smaller than δ and such that the segment from z_1 to z_2 in η_z intersects every leaf of $\mathcal{F}$. We denote $\tilde{z}_1$ and $\tilde{z}_2$ to the lift of those points to $\tilde{\eta}_z$.

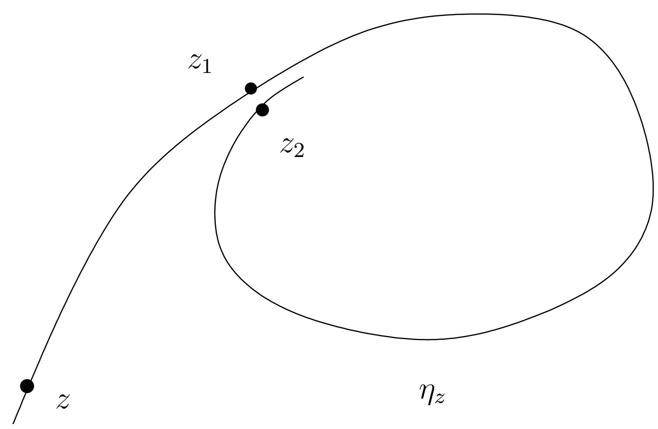

Figure 4.3: The curve η_z.

Using the local product structure, we can modify slightly η_z in order to create a closed curve η'_z through z_1 which is contained in η_z outside $B_\delta(z_1)$, intersects every leaf of $\mathcal{F}$ and has length smaller than $k\ell + \delta$.

We can define Γ_+ as the set of elements in $\pi_1(M)$ which send the half space bounded by $\mathcal{F}(x)$ in the positive orientation into itself.

Since η_z essentially contains a loop of length smaller than $k\ell + \delta$ we have that $\tilde{\eta}_z$ connects $[z_0]$ with $[\tilde{z}_1 + \gamma]$ where γ belongs to Γ_+ and can be represented by a loop of length smaller than $k\ell + \delta$. Moreover, since from z to $\tilde{z}_1$ there is a positively oriented arc of $\tilde{\mathcal{F}}^\perp$ we get that $[z_0] = [z] \leq [\tilde{z}_1]$ (notice that it is possible that $z = \tilde{z}_1$).

This implies that $[\tilde{z}_1 + \gamma] \geq [z_0 + \gamma] > [z_0]$, where the last inequality follows from the fact that the loop is positively oriented and non-trivial (recall that by Lemma 4.3.3 a curve transversal to $\tilde{\mathcal{F}}$ cannot intersect the same leaf twice).

Notice that there are finitely many elements in Γ_+ which are represented by loops of length smaller than $k\ell+\delta$. This is because the fundamental group is abelian so that deck transformations are in one to one correspondence with free homotopy classes of loops.

The fact that there are finitely many such elements in Γ_+ implies the following: There exists $\gamma_0 \in \Gamma_+$ such that for every $\gamma \in \Gamma_+$ which can be represented by a positively oriented loop transverse to $\mathcal{F}$ of length smaller than $k\ell + \delta$, we have

$$[z_0] < [z_0 + \gamma_0] \leq [z_0 + \gamma]$$

We have obtained that for $y^+ = z_0 + \gamma_0$ there exists $L = k\ell > 0$ such that for every point $z \in \tilde{\mathcal{F}}(z_0)$ the segment of $\tilde{\mathcal{F}}^{\perp}_{+}(z)$ of length L intersects $\tilde{\mathcal{F}}(y^+)$.

This defines a continuous injective map from $\tilde{\mathcal{F}}(z_0)$ to $\tilde{\mathcal{F}}(y^+)$ (injectivity follows from Lemma 4.3.3). Since the length of the curves defining the map is uniformly bounded, this map is proper and thus, a homeomorphism. The same argument applies to any leaf $\tilde{\mathcal{F}}(z_1)$ such that $[z_0] \leq [z_1] \leq [y^+]$.

For any z_1 such that $[z_0] \leq [z_1] \leq [y^+]$ we get that $\tilde{\mathcal{F}}^{\perp}(z_1)$ intersects $\tilde{\mathcal{F}}(z_0)$. Since the map defined above is a homeomorphism, we get that also $\tilde{\mathcal{F}}^{\perp}(z_0) \cap \tilde{\mathcal{F}}(z_1) \neq \emptyset$.

A symmetric argument allows us to find y^- with similar characteristics. Using the fact that intersecting with leaves of $\tilde{\mathcal{F}}^{\perp}$ is a homeomorphism between any pair of leafs of $\tilde{\mathcal{F}}$ between $[y^-]$ and $[y^+]$ we obtain $(*)$ as desired.

Lemma 4.3.4 finishes the proof.

□

4.3.3 Proof of Lemma 4.3.5

We first prove the following Lemma which allows us to bound the topology of M in terms of coverings of size δ. Notice that we are implicitly using that $\pi_1(M)$ as before to be able to define a correspondence between (free) homotopy classes of loops with elements of $\pi_1(M)$.

Lemma 4.3.6. *Given a covering $\{V_1, \dots, V_n\}$ of M by contractible open subsets there exists there exists $K > 0$ such that if η is a loop in M such that it intersects each V_i at most once*[4]*, then $[\eta] \in \pi_1(M)$ has norm less than K.*

Proof. We can consider the lift $p^{-1}(V_i)$ to the universal cover of each V_i and we have that each connected component of $p^{-1}(V_i)$ has bounded diameter since they are simply connected in M. Let $C_V > 0$ be a uniform bound on those diameters.

Let K be such that every loop of length smaller than $2nC_V$ has norm less than K in $\pi_1(M)$.

Now, consider a loop η which intersects each of the open sets V_i at most once. Consider η as a function $\eta : [0,1] \to M$ such that $\eta(0) = \eta(1)$. Consider a lift $\tilde{\eta} : [0,1] \to M$ such that $p(\tilde{\eta}(t)) = \eta(t)$ for every t.

[4]More precisely, if η is $\eta : [0,1] \to M$ with $\eta(0) = \eta(1)$ this means that $\eta^{-1}(V_i)$ is connected for every i.

We claim that the diameter of the image of $\tilde{\eta}$ cannot exceed nC_V. Otherwise, this would imply that η intersects some V_i more than once. Now, we can homotope $\tilde{\eta}$ fixing the extremes in order to have length smaller than $2nC_V$. This implies the Lemma.

□

Given δ of the uniform local product structure (see Remark 4.1.4), we say that a loop η is a *δ-loop* if it is transverse to $\mathcal{F}$ and consists of a segment of a leaf of $\mathcal{F}^{\perp}$ together with a curve of length smaller than δ.

Lemma 4.3.7. *There exists $K \geq 0$ such that if $O \subset M$ is an open $\mathcal{F}$-saturated set such that $O \neq M$. Then, there is no δ-loop contained in O.*

Proof. For every point x consider $N_x = B_\delta(x)$ with δ the size of the local product structure boxes. We can consider a finite subcover $\{N_{x_1}, \dots, N_{x_n}\}$ for which Lemma 4.3.6 applies giving $K > 0$.

Consider, an open set $O \neq M$ which is $\mathcal{F}$-saturated. We must prove that O cannot contain a δ-loop.

Let $\tilde{O}_0$ a connected component of the lift $\tilde{O}$ of O to the universal cover $\tilde{M}$. We have that the boundary of $\tilde{O}_0$ consists of leaves of $\tilde{\mathcal{F}}$ and if a translation $\gamma \in \pi_1(M)$ verifies that

$$\tilde{O}_0 \cap \gamma \tilde{O}_0 \neq \emptyset$$

then we must have that $\tilde{O}_0 = \gamma + \tilde{O}_0$. This implies that γ fixes the boundary leafs of $\tilde{O}_0$: This is because the leaf space $\mathcal{L} = \tilde{M}/_{\tilde{\mathcal{F}}}$ is homeomorphic to $\mathbb{R}$ so that $\tilde{O}_0$ being connected and $\tilde{\mathcal{F}}$ saturated is an open interval of $\mathcal{L}$. Since deck transformations preserve orientation, if they fix an open interval then they must fix the boundaries.

The definition of K then guaranties that if an element γ of $\pi_1(M)$ makes $\tilde{O}_0$ intersect with itself, then γ must be larger than K. In particular, any δ-loop contained in O must represent an element of $\pi_1(M)$ of length larger than K.

Now consider a δ-loop η. Corollary 4.2.2 (i) implies that η is in the hypothesis of Lemma 4.3.6. We deduce that η cannot be entirely contained in O since otherwise its lift would be contained in $\tilde{O}_0$ giving a deck transformation γ of norm less than K fixing $\tilde{O}_0$ a contradiction.

□

Corollary 4.3.8. *For the $K \geq 0$ obtained in the previous Lemma, if η is a δ-loop then it intersects every leaf of $\mathcal{F}$.*

Proof. The saturation by $\mathcal{F}$ of η is an open set which is $\mathcal{F}$-saturated by definition. Lemma 4.3.7 implies that it must be the whole M and this implies that every leaf of $\mathcal{F}$ intersects η.

□

PROOF OF LEMMA 4.3.5. Choose K as in Lemma 4.3.7. Considering a covering $\{V_1, \ldots, V_k\}$ of M by neighborhoods with local product structure between $\mathcal{F}$ and $\mathcal{F}^\perp$ and of diameter less than δ.

There exists $\ell_0 > 0$ such that every oriented unstable curve of length larger than ℓ_0 traverses at least one of the $V_i's$. Choose $\ell > (k+1)\ell_0$ and we get that every curve of length larger than ℓ must intersect some V_i twice in points say x_1 and x_2. By changing the curve only in V_i we obtain a δ-loop which will intersect the same leafs as the initial arc joining x_1 and x_2.

Corollary 4.3.8 implies that the mentioned arc must intersect all leafs of $\mathcal{F}$.

□

4.3.4 Consequences of a global product structure

We say that a foliation $\mathcal{F}$ in a Riemannian manifold M is *quasi-isometric* if there exists $a, b \in \mathbb{R}$ such that for every x, y in a same leaf of $\mathcal{F}$ we have:

$$d_{\mathcal{F}}(x, y) \leq a d(x, y) + b$$

where d denotes the distance in M induced by the Riemannian metric and $d_{\mathcal{F}}$ the distance induced in the leaves of $\mathcal{F}$ by restricting the metric of M to the leaves of $\mathcal{F}$. See Section 5.1 for more discussion on quasi-isometry.

Proposition 4.3.9. *Let $\mathcal{F}$ be a codimension one foliation of $\mathbb{T}^3$ and $\mathcal{F}^\perp$ a transverse foliation. Assume the foliations $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^\perp$ lifted to the universal cover have global product structure. Then, the foliation $\tilde{\mathcal{F}}^\perp$ is quasi-isometric. Moreover, if P is the plane given by Theorem 4.2.6, there exists a cone $\mathcal{E}$ transverse to P in $\mathbb{R}^3$ and $K > 0$ such that for every $x \in \mathbb{R}^3$ and $y \in \tilde{\mathcal{F}}^\perp(x)$ at distance larger than K from x we have that $y - x$ is contained in the cone $\mathcal{E}$.*

PROOF. Notice that the global product structure implies that $\mathcal{F}$ is Reebless. Let P be the plane given by Theorem 4.2.6.

Consider v a unit vector perpendicular to P in $\mathbb{R}^3$.

Global product structure implies that for every $N > 0$ there exists L such that every segments of $\tilde{\mathcal{F}}^\perp$ of length L starting at a point x intersect $P + x + Nv$. Indeed, if this was not the case, we could find arbitrarily large segments of leaves of $\tilde{\mathcal{F}}^\perp$ not satisfying this property, by taking a subsequence and translations such that the initial point is in a bounded region, we obtain a leaf of $\tilde{\mathcal{F}}^\perp$ which does not intersect every leaf of $\tilde{\mathcal{F}}$.

This implies quasi-isometry since having length larger than kL implies that the endpoints are at distance at least kN.

Moreover, assuming that the last claim of the proposition does not hold, we get a sequence of points x_n, y_n such that the distance is larger than n and such that the angle between $y_n - x_n$ with P is smaller than $1/\|x_n - y_n\|$.

In the limit (by translating x_n we can assume that it has a convergent subsequence), we get a leaf of $\tilde{\mathcal{F}}^{\perp}$ which cannot intersect every leaf of $\tilde{\mathcal{F}}$ contradicting the global product structure.

□

4.A One dimensional foliations of $\mathbb{T}^2$

This appendix is devoted to characterizing foliations by lines in $\mathbb{T}^2$ where the ideas of the previous sections can be developed in an easier way. The goal is to leave this section independent from the previous ones so that the reader can start by reading this section (even if it will not be used in the remaining of the text).

4.A.1 Classification of foliations

Let $\mathcal{F}$ be a one-dimensional foliation of $\mathbb{T}^2$ and $\mathcal{F}^{\perp}$ any transversal foliation.

We will consider $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{\perp}$ the lifts of these foliations to $\mathbb{R}^2$ with the canonical covering map $p : \mathbb{R}^2 \to \mathbb{T}^2$.

Here, *foliation* will mean a partition of $\mathbb{T}^2$ by continuous flow tangent to a continuous vector field without singularities. This definition implies orientability, the proofs can be easily adapted to cover the non-orientable case.

The first remark is a direct consequence of Poincare-Bendixon's Theorem (see [KH] 14.1.1):

Proposition 4.A.1. *All the leaves of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{\perp}$ are properly embedded copies of $\mathbb{R}$.*

PROOF. By transversality and compactness, there are local product structure boxes of uniform size (see Remark 5.2.8).

Assume there is a leaf $\tilde{\mathcal{F}}(x)$ which intersects a local product structure box in more than one connected component.

This implies that there exists a leaf of $\tilde{\mathcal{F}}$ which is a circle by the argument of the proof of Poincare-Bendixon's theorem. This gives a singularity for the foliation $\tilde{\mathcal{F}}^{\perp}$ a contradiction.

□

This allows us to prove the following:

Proposition 4.A.2. *Given a one dimensional orientable foliation $\mathcal{F}$ of $\mathbb{T}^2$ we have that there exists a subspace $L \subset \mathbb{R}^2$ and $R > 0$ such that every leaf of $\tilde{\mathcal{F}}$ lies in a R-neighborhood of a translate of L. Moreover, one can choose R such that one of the following properties holds:*

(i) Either the R-neighborhood of every leaf of $\tilde{\mathcal{F}}$ contains a translate of L or,

(ii) The line L projects under p to a circle and there is no transversal to $\mathcal{F}$ which intersects every leaf of $\mathcal{F}$.

See figure 4.2, in fact, in option (ii) it can be proved that the foliation has a two-dimensional *Reeb component* (which to avoid confusions we prefer not to define).

Proof. Consider a circle C transverse to $\mathcal{F}$. By Proposition 4.A.1 we know that C is not null-homotopic. The existence of C is not hard to show, it suffices to consider a vector field transverse to $\mathcal{F}$ and perturb it in order to have a periodic orbit.

First, assume that C does not intersect every leaf of $\mathcal{F}$. By saturating C with the leaves of $\mathcal{F}$ we construct O, an open $\mathcal{F}$ saturated set strictly contained in $\mathbb{T}^2$.

We claim that the boundary of the open set consists of leaves of $\mathcal{F}$ homotopic to C: Consider $\tilde{O}_0$ a connected component of the lift of O to the universal cover. Since C is contained in O we have that there is a connected component of the lift of C contained in O. This connected component joins a point $x \in \tilde{O}_0$ with a point $x + \gamma$ where $\gamma \in \mathbb{Z}^2$ represents C in $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$. This implies that γ fixes $\tilde{O}_0$ and in particular its boundary components which must be leafs of $\tilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to the universal cover.

We have that the one-dimensional subspace L generated by the vector γ in $\mathbb{R}^2$ verifies that every leaf of $\tilde{\mathcal{F}}$ lies within bounded distance from a translate of L. Indeed, this holds for the boundary leaves of $\tilde{O}_0$ and by compactness and the fact that leaves do not cross one extends this to every leaf.

Now, assume that there is no circle transverse to $\mathcal{F}$ which intersects every leaf of $\mathcal{F}$. We claim that this means that every transversal to $\mathcal{F}$ must remain at bounded distance from L (which is not hard to prove implies (ii) of the Proposition). Indeed, by the argument above, if this were not the case we would find two closed leaves of $\mathcal{F}$ which are not homotopic, a contradiction with the fact that leaves of $\mathcal{F}$ do not intersect.

So, we can assume that there exists a circle C which is transverse to $\mathcal{F}$ and intersects every leaf of $\mathcal{F}$. By composing with a homeomorphism $H : \mathbb{T}^2 \to \mathbb{T}^2$ isotopic to the identity we can assume that C verifies that its lift is a one-dimensional subspace C_0. If we prove (i) for $H(\mathcal{F})$ we get (i) for $\mathcal{F}$ too since H is at bounded distance from the identity in the universal cover.

By considering the first return map of the flow generated by X to this circle C we obtain a circle homeomorphism $h : C \to C$. By the classical rotation number theory, when lifted to the universal cover $\tilde{C} \cong \mathbb{R}$ we have that the orbit of every point by the lift $\tilde{h}$ has bounded deviation to the translation by some number $\rho \in \mathbb{R}$. We consider the specific lift given orthogonal projecting into C_0 the point of intersection of the flow line with the first integer translate of C_0 it intersects.

We get that the line L we are looking for is generated by $\rho\gamma + \gamma^{\perp}$ where $\gamma \in \mathbb{Z}^2$ is a generator of C and $\gamma^{\perp}$ is the vector orthogonal to γ whose norm equals the distance of C_0 with its closest translate by a vector of $\mathbb{Z}^2$.

□

4.A.2 Global dominated splitting in surfaces

The goal of this section is to show some of the ideas that will appear in Chapter 5) in a simpler context.

Consider $f : \mathbb{T}^2 \to \mathbb{T}^2$ a C^1-diffeomorphism which is partially hyperbolic. Without loss of generality, we will assume that the splitting is of the form $T\mathbb{T}^2 = E \oplus E^u$ where both bundles are one-dimensional and E^u is uniformly expanded. By Theorem 1.3.1 there exists a one-dimensional f-invariant foliation $\mathcal{F}^u$ tangent to E^u. Notice that $\mathcal{F}^u$ cannot have leaves which are circles.

For simplicity, we will assume throughout that the bundles E and E^u are oriented and their orientation is preserved by Df. It is not hard to adapt the results here to the more general case.

We denote as $\tilde{f}$ to a lift of f to the universal cover $\mathbb{R}^2$ and consider the foliation $\tilde{\mathcal{F}}^u$ which is the lift of $\mathcal{F}^u$ to $\mathbb{R}^2$.

Notice that in dimension 2 being partially hyperbolic is equivalent to having a global absolute dominated splitting. The fact that a global dominated splitting implies the existence of a continuous vector field on the manifold readily implies that in an orientable surface, the surface must be $\mathbb{T}^2$.

We will show the following:

Theorem 4.A.3. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a partially hyperbolic diffeomorphism with splitting $T\mathbb{T}^2 = E \oplus E^u$. Then:*

- *f is semiconjugated to an Anosov diffeomorphism of $\mathbb{T}^2$.*
- *There is a unique quasi-attractor $\mathcal{Q}$ of f.*
- *Every chain-recurrence class different from $\mathcal{Q}$ is contained in a periodic interval.*

Since E is a one dimensional bundle uniformly transverse to E^u we can approximate E by a C^1-vector field X which is still transverse to E^u. The vector field X

will be integrable and give rise to a foliation $\mathcal{F}$ which may not be invariant but is transverse to E^u. As usual, we denote as $\tilde{\mathcal{F}}$ to the lift of $\mathcal{F}$ to $\mathbb{R}^2$.

Using this foliation and the things we have proved we will be able to show:

Lemma 4.A.4. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a partially hyperbolic with splitting $T\mathbb{T}^2 = E \oplus E^u$, then $f_* : \mathbb{R}^2 \to \mathbb{R}^2$ is hyperbolic.*

PROOF. Assume that f_* has only eigenvalues of modulus smaller or equal to 1. Then, the diameter of compact sets grows at most polynomially when iterated forward.

Consider an arc γ of $\tilde{\mathcal{F}}^u$ and we iterate it forward. We get that the length of γ grows exponentially while its diameter only polynomially. In $\mathbb{R}^2$ this implies that there will be recurrence of γ to itself and in particular, we will obtain a leaf of $\tilde{\mathcal{F}}^u$ which intersects a leaf of $\tilde{\mathcal{F}}$ twice, a contradiction with Proposition 4.A.1.

Since f_* has determinant of modulus 1 we deduce that f_* must be hyperbolic.

$\square$

We deduce:

Lemma 4.A.5. *There is a global product structure between $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^u$. In particular, $\tilde{\mathcal{F}}^u$ is quasi-isometric.*

PROOF. We apply Proposition 4.A.2 to $\mathcal{F}^u$. We obtain a line L^u which will be f_*-invariant since $\tilde{\mathcal{F}}^u$ is $\tilde{f}$-invariant.

Since L^u is f_*-invariant and f_* is hyperbolic, we can deduce that L^u does not project into a circle so that option (ii) does not hold (recall that hyperbolic matrices have irrational eigenlines).

Moreover, since L^u must project into a dense line in $\mathbb{T}^2$, we get that the foliation $\mathcal{F}^u$ has no holonomy, and this implies by Theorem 4.3.1 that there is a global product structure between $\mathcal{F}^u$ and $\mathcal{F}$.

Quasi-isometry follows exactly as in Proposition 4.3.9.

$\square$

It is possible to give a proof of Theorem 4.3.1 in the lines of the proof of our Theorem 4.3.2. In the case $L \neq L^u$ where L is the line given by Proposition 4.A.2 for $\mathcal{F}$ it is almost direct that there is a global product structure. In the case $L = L^u$ one must reach a contradiction finding a translation which fixes the direction contradicting that L^u is totally irrational.

Remark 4.A.6. With the same argument as in Lemma 4.A.4 we can also deduce that the line L^u given by Proposition 4.A.2 for $\mathcal{F}^u$ must be the eigenline of f_* corresponding to the eigenvalue of modulus larger than 1. Indeed, since $\tilde{\mathcal{F}}^u$ is $\tilde{f}$-invariant, then L^u must be f_*-invariant. Moreover, if L^u corresponds to the stable eigenline of f_* then the diameter of forward iterates of an unstable arc cannot grow more than linearly and the same argument as in Lemma 4.A.4 applies.

$\diamondsuit$

This allows us to show (notice that this also follows from [PS$_4$])

Proposition 4.A.7. *There is a unique f-invariant foliation $\mathcal{F}_E$ tangent to E.*

Proof. The same argument as in Lemma 4.A.5 gives that any foliation tangent to E must have a global product structure with $\mathcal{F}^u$ when lifted to the universal cover.

We first show there exists one f-invariant foliation. To do this, we consider any foliation $\mathcal{F}$ transverse to E^u and we iterate it backwards. Recall that the line L^u close to the foliation $\tilde{\mathcal{F}}^u$ must be the eigenline of the unstable eigenvalue of f_* (see Remark 4.A.6).

Let L be the one dimensional subspace given by Proposition 4.A.2 for $\mathcal{F}$. Since there is a global product structure between $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^u$ we get that $L \neq L^u$: Otherwise by considering points x, y at distance larger than R in the direction orthogonal to L^u we would get that the leaves of $\tilde{F}(x)$ and $\tilde{\mathcal{F}}^u(y)$ cannot intersect due to Proposition 4.A.2.

Iterating backwards, we get that the foliation $\tilde{\mathcal{F}}_m = \tilde{f}^{-m}(\tilde{\mathcal{F}})$ is close to the line $f_*^{-m}(L)$ that as $m \to \infty$ converges to L^s, the eigenline of the stable eigenvalue of f_*.

Moreover, we can prove that there exists a constant R such that for every m we have that every leaf of $\tilde{\mathcal{F}}_m$ lies at distance smaller than R from $f_*^{-m}(L)$. Indeed, consider $R \gg \frac{2\lambda^u K_0}{\cos\alpha}$ where K_0 is the C^0-distance from $\tilde{f}$ and f_*, λ^u the unstable eigenvalue of f_* and α the angle between L and L^s. We get that the R neighborhood of any translate of L is mapped by f_*^{-1} into an $\frac{\cos\alpha'}{\cos\alpha}(\lambda^u)^{-1}R$-neighborhood of $f_*^{-1}(L)$ where $\alpha' < \alpha$ is the angle between $f_*^{-1}(L)$ and L^s. Since $R - \frac{\cos\alpha'}{\cos\alpha}(\lambda^u)^{-1}R > K_0$ from the choice of R we get that every leaf of $\tilde{\mathcal{F}}_1 = \tilde{f}^{-1}(\mathcal{F})$ lies within R-distance from a translate of $f_*^{-1}(L)$. Inductively, we get that each $\tilde{\mathcal{F}}_m$ lies within distance smaller than R from $f_*^{-m}(L)$.

We must show that there exists a unique limit for the backward iterates of any leaf of $\tilde{\mathcal{F}}$. Let us fix R as above.

Let $x \in \mathbb{R}^2$ and we consider $\tilde{\mathcal{F}}_n(x) = \tilde{f}^{-n}(\tilde{\mathcal{F}}(f^n(x)))$. Notice that $\tilde{\mathcal{F}}_n(x)$ is an embedded line which intersects the unstable leaf of each point of $\tilde{\mathcal{F}}(x)$ in exactly one point. Assume there exists $z \in \tilde{\mathcal{F}}(x)$ such that in $\tilde{\mathcal{F}}^u(z)$ there are two different limit points z_1 and z_2 of the sequence $\tilde{\mathcal{F}}_n(x) \cap \tilde{\mathcal{F}}^u(z)$. We have that forward iterates of $\tilde{f}^k(z_i)$ must lie at distance smaller than R from $L^s + \tilde{f}^k(x)$.

Consider $K > 0$ such that if two points lie at distance larger than K inside an unstable leaf then they are at distance larger than R in the direction transverse to L. Then, by choosing k large enough so that the length of the arc of unstable joining

z_1 and z_2 is larger than K we get that $\tilde{f}^k(z_1)$ and $\tilde{f}^k(z_2)$ must be at distance larger than R in the direction transversal to L^s contradicting the previous claim.

A similar argument implies that there cannot be two different f-invariant foliations tangent to E since both should remain close to the stable eigenline of f_*.

$\square$

Since f_* is hyperbolic (Lemma 4.A.4) Proposition 2.3.1 gives that there exists a semiconjugacy $H : \mathbb{R}^2 \to \mathbb{R}^2$ which is C^0-close to the identity, is periodic and verifies that:

$$H \circ \tilde{f} = f_* \circ H$$

We denote as $\tilde{\mathcal{F}}_E$ to the lift of $\mathcal{F}_E$ to the universal cover, we can prove:

Lemma 4.A.8. *The preimage by H of every point is contained in a leaf of $\tilde{\mathcal{F}}_E$.*

Proof. From Proposition 4.A.2 (and the fact that $\tilde{\mathcal{F}}_E$ is $\tilde{f}$-invariant and has a global product structure with $\tilde{\mathcal{F}}^u$) we get that every leaf of $\tilde{\mathcal{F}}_E$ lies within distance smaller than R from a translate of L^s.

Consider points x, y lying in different leaves of $\tilde{\mathcal{F}}_E$. Now, consider $z = \tilde{\mathcal{F}}_E(y) \cap \tilde{\mathcal{F}}^u(x)$. We have that the distance of z and x grows exponentially in the direction of L^u. This implies that by iterating forward, the distance between $\tilde{\mathcal{F}}_E(x)$ and $\tilde{\mathcal{F}}_E(y)$ must grow also exponentially.

We conclude that $d(\tilde{f}^n(x), \tilde{f}^n(y)) \to \infty$ with $n \to +\infty$.

Since H is close to the identity and semiconjugates f with f_* it cannot send x and y to the same point.

$\square$

In order to be able to apply Proposition 2.2.1 we must show the following:

Lemma 4.A.9. *There is a unique quasi-attractor $\mathcal{Q}$ for f. Moreover, every point y which belongs to the boundary of a fiber of H relative to its leaf of $\tilde{\mathcal{F}}_E$ belongs to $\mathcal{Q}$.*

Proof. By Conley's theorem (Theorem 1.1.9), there always exists a quasi-attractor $\mathcal{Q}$ of f. Moreover, we have seen that such quasi-attractors are saturated by unstable sets (see 1.1.16).

Consider any quasi-attractor $\mathcal{Q}$. Let y be a point which is in the boundary of $H^{-1}(\{x\})$ relative to $\tilde{\mathcal{F}}_E(y)$. Given $\varepsilon > 0$, since y is in the boundary of $H^{-1}(\{x\})$ relative to $\tilde{\mathcal{F}}_E(y)$ we obtain that its image by H cannot be contained in the unstable set of x for f_*.

Iterating backwards we obtain a connected set of arbitrarily large diameter in the direction of the stable eigenline of f_*. This implies that for large m we have that

$\tilde{f}^{-m}(B_\varepsilon(y))$ intersects $p^{-1}(\mathcal{Q})$. This holds for every $\varepsilon > 0$ so we get that for every $\varepsilon > 0$ we can construct an ε-pseudo-orbit from y to $\mathcal{Q}$. This implies that $y \in \mathcal{Q}$. Since $\mathcal{Q}$ was arbitrary and quasi-attractors are disjoint it also implies that there is a unique quasi-attractor.

□

We are now able to give a proof of Theorem 4.A.3:

PROOF OF THEOREM 4.A.3. We have proved that f is semiconjugated to a linear Anosov diffeomorphism of $\mathbb{T}^2$ and that there is a unique quasi-attractor.

The last claim of the Theorem follows from the fact that we have proved that the conditions of Proposition 2.2.1 are verified:

- The partially hyperbolic set is the whole $\mathbb{T}^2$ (so that the maximal invariant set in U is also the whole $\mathbb{T}^2$).
- The semiconjugacy is the one given by H. It is injective on unstable manifolds by Lemma 4.A.8.
- Lemma 4.A.9 implies that the frontier of fibers in center stable leaves are all contained in the unique quasi-attractor $\mathcal{Q}$ of f.
- Fibers of H are invariant under unstable holonomy (see the proof of Proposition 3.3.11).

This concludes.

□

Chapter 5

Global partial hyperbolicity

This chapter contains the main contributions of this thesis. In Section 5.1 we present some preliminaries and in particular we introduce the concept of *almost dynamical coherence* which is key in the study we make in this chapter. In particular, this concept allows us to prove the following result in Section 5.2:

Theorem. *Dynamical coherence is an open and closed property among partially hyperbolic diffeomorphisms of* $\mathbb{T}^3$ *isotopic to Anosov.*

We remark that in general it is not known whether dynamical coherence is neither an open nor a closed property. There are not known examples where it is not open but in general, to obtain opennes a technical condition is used (called *plaque-expansiveness*). See [HPS, Be].

Dynamical coherence in the case where the center bundle has dimension larger than one is a widely open subject. It has been remarked by Wilkinson ([Wi]) that one can look at some Anosov diffeomorphisms as partially hyperbolic ones which are not dynamically coherent (see [BuW$_1$] for an overview of dynamical coherence). The proof here presented relies heavily both in the assumption of almost dynamical coherence and in being in the isotopy class of an Anosov automorphism in $\mathbb{T}^3$. Several questions regarding generalizations of these kind of results pop up even in dimension 3. The one which we believe to be more important is the following:

Question 5.0.10. *Is it true that every partially hyperbolic diffeomorphism in dimension* 3 *is almost dynamically coherent?*

Assuming this question admits a positive answer, one could expect to make some progress in the direction of classification of both partially hyperbolic diffeomorphisms of 3-manifolds, and more importantly (due to Theorem 1.2.22) of robustly transitive diffeomorphisms in 3-manifolds.

Another quite natural question to be posed, which is related, is whether some manifolds can admit partially hyperbolic diffeomorphisms but not strong partially hyperbolic ones. We prove in Section 5.2 that there are isotopy classes in $\mathbb{T}^3$ which admit partially hyperbolic diffeomorphisms but not strongly partially hyperbolic ones. This poses the following natural question for which we do not know the answer:

Question 5.0.11. *Let M be a 3-manifold different from $\mathbb{T}^3$. Is every partially hyperbolic diffeomorphism of M isotopic to a strong partially hyperbolic one?*

When we treat strong partially hyperbolic systems we are able to obtain much stronger results concerning integrability. We prove in Section 5.3 the following:

Theorem. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a strong partially hyperbolic diffeomorphism. Then:*

- *Either $E^s \oplus E^c$ is tangent to a unique f-invariant foliation, or,*
- *there exists a f-periodic two-dimensional torus T which is tangent to $E^s \oplus E^c$ and normally expanding.*

This result extends the results of [BBI$_2$] to the pointwise partially hyperbolic case and answers to a conjecture from [RHRHU$_3$] where it is shown that the second possibility of the theorem is non-empty. In the introduction of Section 5.3 we explain the difference between our approach and the one of [BBI$_2$].

In Section 5.1 we present the definition of almost dynamical coherence as well as some properties and we give some preliminaries of results which we will use afterwards.

Finally, in Section 5.4 we comment on some results in higher dimensions as well as to explore some results which allow one to characterize the isotopy class of a partially hyperbolic diffeomorphism.

5.1 Almost dynamical coherence and Quasi-Isometry

5.1.1 Almost dynamical coherence

In general, a partially hyperbolic diffeomorphism may not be dynamically coherent, and even if it is, it is not known in all generality if being dynamically coherent is an open property (see [HPS, Be]). However, all the known examples in dimension 3 verify the following property which is clearly C^1-open:

Definition 5.1.1 (Almost dynamical coherence)**.** We say that $f : M \to M$ partially hyperbolic of the form $TM = E^{cs} \oplus E^u$ is *almost dynamically coherent* if there exists a foliation $\mathcal{F}$ transverse to the direction E^u.

$\diamondsuit$

The introduction of this definition is motivated by the work of [BI] where it was remarked that sometimes it is enough to have a foliation transverse to the unstable direction in order to obtain conclusions.

Almost dynamical coherence is not a very strong requirement, with the basic facts on domination we can show:

Proposition 5.1.1. *Let $\{f_n\}$ a sequence of almost dynamically coherent partially hyperbolic diffeomorphisms converging in the C^1-topology to a partially hyperbolic diffeomorphism f. Then, f is almost dynamically coherent.*

Proof. Let us call $E_n^{cs} \oplus E_n^u$ to the splitting of f_n and $E^{cs} \oplus E^u$ to the splitting of f. We use the following well known facts on domination (see Proposition 1.2.3 and Remark 1.2.4):

- The subspaces E_n^{cs} and E_n^u converge as $n \to \infty$ towards E^{cs} and E^u.
- The angle between E^{cs} and E^u is larger than $\alpha > 0$.

Now, consider f_n such that the angle between E_n^{cs} and E^u is larger than $\alpha/2$. Let $\mathcal{F}_n$ be the foliation transverse to E_n^u.

By iterating backwards by f_n we obtain that $f_n^{-m}(\mathcal{F}_n)$ is, when m is large, tangent to a small cone around E_n^{cs}. From our assumptions, we can thus deduce that $f_n^{-m}(\mathcal{F}_n)$ is transverse also to E^u. This implies that f is almost dynamically coherent as desired.

$\square$

Notice that this proposition implies that if we denote as $\mathcal{PH}^1(M)$ the set of partially hyperbolic diffeomorphisms of M, and $\mathcal{P}$ to a connected component: If $\mathcal{P}$ contains an almost dynamically coherent diffeomorphisms, every diffeomorphism in $\mathcal{P}$ is almost dynamically coherent. In particular, almost dynamical coherent partially hyperbolic diffeomorphisms contain the connected component in $\mathcal{PH}^1(\mathbb{T}^3)$ containing the linear representatives of the isotopy class when these are partially hyperbolic.

As a consequence [BI] (Key Lemma 2.1), every strong partially hyperbolic diffeomorphism of a 3-dimensional manifold is almost dynamically coherent. It is important to remark that it is a mayor problem to determine whether partially hyperbolic diffeomorphisms in the sphere S^3 are almost dynamically coherent (which would solve the question on the existence of robustly transitive diffeomorphisms in the sphere[1]).

The author is not aware of whether the following question is known or still open:

[1]Indeed, by a result of [DPU] a robustly transitive diffeomorphism of a 3-dimensional manifold should be partially hyperbolic. If it were almost dynamically coherent, we would get by Novikov's Theorem that there exists a Reeb component transverse to the strong unstable direction (if it is partially hyperbolic of type $TS^3 = E^s \oplus E^{cu}$ one should consider f^{-1}). This contradicts Proposition 5.1.3.

Question 5.1.2. *Are there any examples of partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ isotopic to a linear Anosov automorphism which are not isotopic to the linear Anosov automorphism through a path of partially hyperbolic diffeomorphisms?*

We end this subsection by stating a property first observed by Brin,Burago and Ivanov ([BBI$_1$, BI]) which makes our definition a good tool for studying partially hyperbolic diffeomorphisms:

Proposition 5.1.3 (Brin-Burago-Ivanov)**.** *Let $f : M \to M$ an almost dynamically coherent partially hyperbolic diffeomorphism with splitting $TM = E^{cs} \oplus E^u$ and let $\mathcal{F}$ be the foliation transverse to E^u. Then, $\mathcal{F}$ has no Reeb components.*

PROOF. If a (transversally oriented) foliation $\mathcal{F}$ on a compact closed 3-dimensional manifold M has a Reeb component, then, every one dimensional foliation transverse to $\mathcal{F}$ has a closed leaf (see [BI] Lemma 2.2).

Since $\mathcal{F}^u$ is one dimensional, transverse to $\mathcal{F}$ and has no closed leafs, we obtain that $\mathcal{F}$ cannot have Reeb components.

$\square$

This has allowed them to prove (see also [Par]):

Theorem 5.1.4 (Brin-Burago-Ivanov [BBI$_1$, BI, Par])**.** *If f is an almost dynamically coherent partially hyperbolic diffeomorphism of a 3-dimensional manifold M with fundamental group of polynomial growth, then, the induced map $f_* : H_1(M, \mathbb{R}) \cong \mathbb{R}^k \to H_1(M, \mathbb{R})$ is partially hyperbolic. This means, it is represented by an invertible matrix $A \in GL(k, \mathbb{Z})$ which has an eigenvalue of modulus larger than 1 and determinant of modulus 1 (in particular, it also has an eigenvalue of modulus smaller than 1).*

SKETCH. We prove the result when $\pi_1(M)$ is abelian, so that it coincides with $H_1(M, \mathbb{Z})$. When the fundamental group is nilpotent, this follows from the fact that the 3-manifolds with this fundamental group are well known (they are circle bundles over the torus) so that one can make other kind of arguments with the same spirit (see [Par] Theorem 1.12).

Assume that every eigenvalue of f_* is smaller or equal to 1. Since the universal cover $\tilde{M}$ is quasi-isometric to $\pi_1(M)$, it is thus quasi-isometric to $H_1(M, \mathbb{R}) \cong \mathbb{R}^k$ (notice that this is trivial if $M = \mathbb{T}^3$).

Now, we have that f_* acting in $H_1(M, \mathbb{R})$ has all of its eigenvalues smaller than one, we obtain that the diameter of a compact set in $\tilde{M}$ grows subexponentially by iterating it with $\tilde{f}$.

Given $R > 0$ the number of fundamental domains needed to cover a ball of radius R in $\tilde{M}$ is polynomial in R.

Consider an unstable arc I. We obtain that $\tilde{f}^n(I)$ has subexponential (in n) diameter but the length grows exponentially (in n). By the previous observation, we obtain that given ε we find points of $\tilde{\mathcal{F}}^u$ which are not in the same local unstable manifold but are at distance smaller than ε, this implies the existence of a Reeb component for $\tilde{\mathcal{F}}$ (Theorem 4.2.1) and contradicts Proposition 5.1.3.

□

Notice that if the growth of the fundamental group is exponential, one can make partially hyperbolic diffeomorphisms which are isotopic to the identity (for example, the time-one map of an Anosov flow). This is because in such a manifold, a sequence K_n of sets with exponentially (in n) many points but polynomial (in n) diameter may not have accumulation points.

As a consequence of combining the Proposition 5.1.3 with Novikov's Theorem 4.2.1 we obtain for $\mathbb{T}^3$ the following consequence (recall Corollary 4.2.2):

Corollary 5.1.5. *Let f be a partially hyperbolic diffeomorphism of $\mathbb{T}^3$ of the form $T\mathbb{T}^3 = E^{cs} \oplus E^u$ $(\dim E^{cs} = 2)$ which is almost dynamically coherent with foliation $\mathcal{F}$. Assume that $\mathcal{F}$ is oriented and transversally oriented and let $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^u$ the lifts of the foliations $\mathcal{F}$ and the unstable foliation $\mathcal{F}^u$ to $\mathbb{R}^3$. Then:*

(i) For every $x \in \mathbb{R}^3$ we have that $\tilde{\mathcal{F}}(x) \cap \tilde{\mathcal{F}}^u(x) = \{x\}$.

(ii) The leafs of $\tilde{\mathcal{F}}$ are properly embedded complete surfaces in $\mathbb{R}^3$. In fact there exists $\delta > 0$ such that every euclidean ball U of radius δ can be covered by a continuous coordinate chart such that the intersection of every leaf S of $\tilde{\mathcal{F}}$ with U is either empty of represented as the graph of a function $h_S : \mathbb{R}^2 \to \mathbb{R}$ in those coordinates.

(iii) Each closed leaf of $\mathcal{F}$ is a two dimensional torus.

(iv) For every $\delta > 0$, there exists a constant C_δ such that if J is a segment of $\tilde{\mathcal{F}}^u$ then $\mathrm{Vol}(B_\delta(J)) > C_\delta \,\mathrm{length}(J)$.

Proof. The proof of (i) is the same as the one of Lemma 2.3 of [BI], indeed, if there were two points of intersection, one can construct a closed loop transverse to $\tilde{\mathcal{F}}$ which descends in $\mathbb{T}^3$ to a nullhomotopic one. By Novikov's theorem (Theorem 4.2.1), this implies the existence of a Reeb component, a contradiction with Proposition 5.1.3.

Once (i) is proved, (ii) follows from the same argument as in Lemma 3.2 in [BBI$_2$]. Notice that the fact that the leafs of $\tilde{\mathcal{F}}$ are properly embedded is trivial after (i), with some work, one can prove the remaining part of (ii) (see also Lemma 5.2.7 for a more general statement).

Part (iii) follows from the fact that if S is an oriented closed surface in $\mathbb{T}^3$ which is not a torus, then it is either a sphere or its fundamental group cannot inject in $\mathbb{T}^3$ (see [Ri] and notice that a group with exponential growth cannot inject in $\mathbb{Z}^3$).

Since $\mathcal{F}$ has no Reeb components, we obtain that if S is a closed leaf of $\mathcal{F}$ then it must be a sphere or a torus. But S cannot be a sphere since in that case, the Reeb's stability theorem (Theorem 4.1.6) would imply that all the leafs of $\mathcal{F}$ are spheres and that the foliated manifold is finitely covered by $S^2 \times S^1$ which is not the case.

The proof of (iv) is as Lemma 3.3 of [BBI$_2$]. Since there cannot be two points in the same leaf of $\tilde{\mathcal{F}}^u$ which are close but in different local unstable leaves, we can find $\epsilon > 0$ and $a > 0$ such that in a curve of length K of $\tilde{\mathcal{F}}^u$ there are at least aK points whose balls of radius ϵ are disjoint (and all have the same volume).

Now, consider $\delta > 0$ and $\tilde{\delta} = \min\{\delta, \epsilon\}$. Let $\{x_1, \dots, x_l\}$ with $l > a\,\text{length}(J)$ be points such that their $\tilde{\delta}$-balls are disjoint. We get that $U = \bigcup_{i=1}^{l} B_{\tilde{\delta}}(x_i) \subset B_\delta(J)$ and we have that $\text{Vol}(U) > l\,\text{Vol}(B_{\tilde{\delta}}(x_i))$. We obtain that $C_\delta = \frac{4\pi}{3} a \delta^3$ works.

□

Notice that most of the previous result can be extended to arbitrary 3-dimensional manifolds. In fact, with a similar proof (see also [Par]) one proves that almost dynamically coherent partially hyperbolic diffeomorphisms can only occur in certain specific 3-manifolds:

Corollary 5.1.6. *Let f be an almost dynamically coherent partially hyperbolic diffeomorphism of a 3-dimensional manifold M with splitting $TM = E^{cs} \oplus E^u$. Then:*

- *The manifold M is irreducible (i.e. $\pi_2(M) = \{0\}$).*

- *The covering space of M is homeomorphic to $\mathbb{R}^3$.*

- *The fundamental group of M is infinite (and different from $\mathbb{Z}$)*

Proof. Let $\mathcal{F}$ be the foliation transverse to E^u.

The first claim follows from the fact that having $\pi_2(M) \neq \{0\}$ implies the existence of a Reeb component for $\mathcal{F}$ by Novikov's Theorem 4.2.1. The last claim follows by the same reason. The fact that the fundamental group cannot be $\mathbb{Z}$ follows from Proposition 5.1.3 since a manifold with $\mathbb{Z}$ as fundamental group (which is of polynomial growth) has $\mathbb{Z}$ as first homology group and admits no automorphisms with eigenvalues of modulus different from 1.

To get the second statement, notice that since the fundamental group of every leaf must inject in the fundamental group of M we have that every leaf of $\tilde{\mathcal{F}}$ must be homeomorphic to $\mathbb{R}^2$ or S^2. By Reeb's stability theorem (Theorem 4.1.6) leafs must be homeomorphic to $\mathbb{R}^2$.

By a result by Palmeira (see [Pal]) we obtain that $\tilde{M}$ must be homeomorphic to $\mathbb{R}^3$.

$\square$

5.1.2 Branched Foliations and Burago-Ivanov's result

We follow [BI] section 4.

We define a *surface* in a 3-manifold M to be a C^1-immersion $\imath : U \to M$ of a connected smooth 2-dimensional manifold (possibly with boundary). The surface is said to be *complete* if it is complete with the metric induced in U by the Riemannian metric of M and the immersion $\imath$. The surface is *open* if it has no boundary.

Given a point x in (the image of) a surface $\imath : U \to M$ we have that there is a neighborhood B of x such that the connected component C containing $\imath^{-1}(x)$ of $\imath^{-1}(B)$ verifies that $\imath(C)$ separates B. We say that two surfaces $\imath_1 : U_1 \to M, \imath_2 : U_2 \to M$ *topologically cross* if there exists a point x in (the image of) $\imath_1$ and $\imath_2$ and a curve γ in U_2 such that $\imath_2(\gamma)$ passes through x and intersects both connected components of a neighborhood of x with the part of the surface defined above removed. This definition is symmetric and does not depend on the choice of B (see [BI]) however we will not use this facts.

Definition 5.1.2. A *branching foliation* on M is a collection of complete open surfaces tangent to a given continuous 2-dimensional distribution on M such that every point belongs to at least one surface and no pair of surfaces of the collection have topological crossings.

$\diamondsuit$

We will abuse notation and denote a branching foliation as $\mathcal{F}_{bran}$ and by $\mathcal{F}_{bran}(x)$ to the set of set of surfaces whose image contains x. We call the (image of) the surfaces, leaves of the branching foliation.

We have the following:

Proposition 5.1.7. *If every point of M belongs to a unique leaf of the branching foliation, then the branching foliation is a true foliation.*

Proof. Let E be the two-dimensional distribution tangent to the branching foliation and we consider $E^{\perp}$ a transverse direction which we can assume is C^1 and almost orthogonal to E.

By uniform continuity we find ε such that for every point p in M the 2ε ball verifies that it admits a C^1-chart to an open set in $\mathbb{R}^3$ which sends E to an almost horizontal xy-plane and $E^{\perp}$ to an almost vertical z-line.

Let D be a small disk in the (unique) surface through p and γ a small arc tangent to $E^\perp$ thorough p. Given a point $q \in D$ and $t \in \gamma$ we have that inside $B_{2\varepsilon}(p)$ there is a unique point of intersection between the curve tangent to $E^\perp$ through q and the connected component of the (unique) surface of $\mathcal{F}_{bran}$ intersected with $B_{2\varepsilon}(p)$ containing t.

We get a well defined continuous and injective map from $D \times \gamma \cong \mathbb{R}^3$ to a neighborhood of p (by the invariance of domain's theorem, see [Hat]) such that it sends sets of the form $D \times \{t\}$ into surfaces of the branching foliation. Since we already know that $\mathcal{F}_{bran}$ is tangent to a continuous distribution, we get that $\mathcal{F}_{bran}$ is a true foliation.

$\square$

Indeed, the result also follows from the following statement we will also use:

Proposition 5.1.8 ([BWi] Proposition 1.6 and Remark 1.10)**.** *Let E be a continuous codimension one distribution on a manifold M and S a (possibly non connected) surface tangent to E which contains a family of disks of fixed radius and whose set of midpoints is dense in M. Then, there exists a foliation $\mathcal{F}$ tangent to E which contains S in its leaves.*

Invariant branching foliations always exist for strong partially hyperbolic diffeomorphisms of 3-dimensional manifolds due to a remarkable result of Burago and Ivanov:

Theorem 5.1.9 ([BI],Theorem 4.1 and Theorem 7.2)**.** *Let $f : M^3 \to M^3$ be a strong partially hyperbolic diffeomorphism with splitting $TM = E^s \oplus E^c \oplus E^u$ into one dimensional subbundles. There exists branching foliations $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$ tangent to $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$ which are f-invariant*[2]*. Moreover, for every $\varepsilon > 0$ there exist foliations $\mathcal{S}_\varepsilon$ and $\mathcal{U}_\varepsilon$ tangent to an ε-cone around E^{cs} and E^{cu} respectively and continuous maps h^{cs}_ε and h^{cu}_ε at C^0-distance smaller than ε from the identity which send the leaves of $\mathcal{S}_\varepsilon$ and $\mathcal{U}_\varepsilon$ to leaves of $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$ respectively.*

We remark that when there exists an f-invariant (branching) foliation, one can assume that every sequence of leaves through points x_k such that $x_k \to x$ verifies that it converges to a leaf through x (see Lemma 7.1 of [BI]).

Convention. We will assume throughout that every branching foliation is *completed* in the sense stated above: For every sequence L_k of leaves in $\mathcal{F}^{cs}_{bran}(x_k)$ such that $x_k \to x$ we have that L_k converges in the C^1-topology to a leaf $L \in \mathcal{F}^{cs}_{bran}(x)$ contained in the branching foliation.

$\diamondsuit$

[2]This means that for every $\mathcal{F}_k \in \mathcal{F}^{\sigma}_{bran}(x)$ there exists $\mathcal{F}_{k'} \in \mathcal{F}^{\sigma}_{bran}(f(x))$ such that $f(\mathcal{F}_k) = \mathcal{F}_{k'}$.

Notice that the existence of the maps h^{cs}_ε and h^{cu}_ε implies that when lifted to the universal cover, the leaves of $\mathcal{S}_\varepsilon$ (resp. $\mathcal{U}_\varepsilon$) remain at distance smaller than ε from lifted leaves of $\mathcal{F}^{cs}_{bran}$ (resp. $\mathcal{F}^{cu}_{bran}$).

We obtain as a corollary the following result we have already announced:

Corollary 5.1.10 (Key Lemma 2.2 of [BI])**.** *A strong partially hyperbolic diffeomorphism on a 3-dimensional manifold is almost dynamically coherent.*

Using the fact that when $x_k \to x$ the leaves through x_k converge to a leaf through x we obtain:

Proposition 5.1.11. *Let $\mathcal{F}_{bran}$ be a branching foliation of $\mathbb{T}^3$ and consider a sequence of points x_k such that there are leaves $\mathcal{F}_k \in \mathcal{F}_{bran}(x_k)$ which are compact, incompressible and homotopic to each other. If $x_k \to x$, then there is a leaf $L \in \mathcal{F}_{bran}(x)$ which is incompressible and homotopic to the leaves $\mathcal{F}_k$.*

Proof. Recall that if $x_k \to x$ and we consider a sequence of leaves through x_k we get that the leaves converge to a leaf through x.

Consider the lifts of the leaves $\mathcal{F}_k$ which are homeomorphic to a plane since they are incompressible. Moreover, the fundamental group of each of the leaves must be $\mathbb{Z}^2$ and the leaves must be homoeomorphic to 2-torus, since it is the only possibly incompressible surface in $\mathbb{T}^3$.

Since all the leaves $\mathcal{F}_k$ are homotopic, their lifts are invariant under the same elements of $\pi_1(\mathbb{T}^3)$. The limit leaf must thus be also invariant under those elements. Notice that it cannot be invariant under further elements of $\pi_1(\mathbb{T}^3)$ since no surface has such fundamental group.

□

The idea of the proof of the previous proposition can be applied to other contexts, however, for simplifying the proof we chose to state it only in this context which is the one of interest for us.

5.1.3 Quasi-isometry and dynamical coherence

We review in this section a simple criterium given by Brin in [Bri] which guaranties dynamical coherence for absolutely dominated partially hyperbolic diffeomorphisms. It involves the concept of quasi-isometry which we will use after in this thesis. We present the sketch of the proof by Brin to show the importance of absolute domination in his argument.

For more information on quasi-isometric foliations we refer the reader to [H_3]. We recall its definition (which already appeared in subsection 4.3.4):

Definition 5.1.3 (Quasi-Isometric Foliation)**.** Consider a Riemannian manifold M (not necessarily compact) and a foliation $\mathcal{F}$ in M. We say that the foliation $\mathcal{F}$ is *quasi-isometric* if distances inside leaves can be compared with distances in the manifold. More precisely, for $x, y \in \mathcal{F}(x)$ we denote as $d_{\mathcal{F}}(x, y)$ as the infimum of the lengths of curves joining x to y, we say that $\mathcal{F}$ is quasi-isometric if there exists $a, b \in \mathbb{R}$ such that for every x, y in the same leaf of $\mathcal{F}$ one has that:

$$d_{\mathcal{F}}(x, y) \leq a d(x, y) + b$$

$\diamondsuit$

In general, this notion makes sense in non-compact manifolds, and it will be used by us mainly in the universal covering space of the manifolds we work with. Notice that if a foliation of a compact manifold is quasi-isometric then all leaves must be compact.

The classic example of a quasi-isometric foliation is a linear foliation in $\mathbb{R}^d$ with the euclidean metric. Indeed, it can be thought that quasi-isometry foliations are in a sense a generalization of these (notice however that even a one dimensional foliation of the plane[3] which is quasi-isometric needs not remain at bounded distance from a one-dimensional "direction").

It is important to remark that the metric in the manifold is quite important, and as in general we work with the universal cover of a compact manifold, this metric is also influenced by the topology of the manifold. See [H$_3$] for more discussion on quasi-isometric foliations and topological and restrictions for their existence.

The argument of the proof of Proposition 1.3.6 can be extended to non-local arguments if one demands that the domination required is absolute and the geometry of leaves is quite special. In fact, Brin has proved in [Bri] the following quite useful criterium (see for example [BBI$_2$], [Par] or [H, H$_2$] for applications of this criterium).

Proposition 5.1.12 ([Bri])**.** *Let $f : M \to M$ be an absolutely partially hyperbolic diffeomorphism with splitting $TM = E^{cs} \oplus E^u$ and such that the foliation $\tilde{\mathcal{F}}^u$ is quasi-isometric in $\tilde{M}$ the universal cover of M. Then, f is dynamically coherent.*

It shows that in fact, the foliation is unique in (almost) the strongest sense, which is that every C^1-embedding of a ball of dimension $\dim E^{cs}$ which is everywhere tangent to E^{cs} is in fact contained in a leaf of the foliation $\mathcal{F}^{cs}$.

We give a sketch of the proof in order to show how the hypothesis are essential to pursue the argument. See [Bri] for a clear exposition of the complete argument.

[3] Consider for example the foliation given by $\{(t, t^3 + b)\}_b$.

Sketch. Assume that there are two embedded balls B_1 and B_2 through a point x which are everywhere tangent to E^{cs} and whose intersection is not relatively open in (at least) one of them.

Then, as in Proposition 1.3.6 it is possible to construct a curve η which has non-zero length, is contained in a leaf of $\mathcal{F}^u$ and joins these two embedded balls.

Let γ_1 and γ_2 two curves contained in B_1 and B_2r respectively joining x to the extremes of η.

Since η is an unstable curve, its length growths exponentially, and by quasi-isometry, we know that the extremal points of the curve are at a distance which grows exponentially with the same rate as the rate the vectors in E^u expand.

On the other hand, the curves γ_1 and γ_2 are forced to grow with at most an exponential rate which is smaller than the one in E^u (by using absolute domination) and so we violate the triangle inequality.

□

5.2 Partially hyperbolic diffeomorphisms isotopic to linear Anosov automorphisms of $\mathbb{T}^3$

In this section we give a proof of the following:

Theorem 5.2.1. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an almost dynamically coherent partially hyperbolic diffeomorphism with splitting of the form $T\mathbb{T}^3 = E^{cs} \oplus E^u$. Assume that f is isotopic to Anosov, then:*

- *f is (robustly) dynamically coherent and has a unique f-invariant foliation $\mathcal{F}^{cs}$ tangent to E^{cs}.*

- *There exists a global product structure between the lift of $\mathcal{F}^{cs}$ to the universal cover and the lift of $\mathcal{F}^u$ to the universal cover.*

- *If f_* has two eigenvalues of modulus larger than 1 then they must be real and different.*

As a consequence of the fact that almost dynamical coherence is an open and closed property (see Proposition 5.1.1 above) we obtain:

Corollary. *Dynamical coherence is an open and closed property among partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ isotopic to Anosov.*

Proof. By Proposition 5.1.1 we know that almost dynamical coherence is an open and closed property. Theorem 5.2.1 then implies that in the isotopy class of Anosov dynamical coherence is open and closed too (since almost dynamical coherence implies dynamical coherence in this context).

□

We shall assume that $f : \mathbb{T}^3 \to \mathbb{T}^3$ is an almost dynamical coherent partially hyperbolic diffeomorphism with splitting of the form $T\mathbb{T}^3 = E^{cs} \oplus E^u$ with $\dim E^u = 1$ and isotopic to a linear Anosov automorphism $A : \mathbb{T}^3 \to \mathbb{T}^3$.

It is important to remark that we are not assuming that the stable dimension of $A = f_*$ coincides with the one of E^{cs}. In fact, many of the arguments below become much easier in the case A has stable dimension 2. The fact that we can treat the case where A has two eigenvalues of modulus larger than one is in the authors' opinion, one of the main contributions of this thesis.

We will denote as $\mathcal{F}$ the foliation given by the definition of almost dynamical coherence which we know is Reebless and it thus verifies the hypothesis of Theorem 4.2.6.

As before, we denote as $p : \mathbb{R}^3 \to \mathbb{T}^3$ the covering projection and we denote as $\tilde{f}$, $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^u$ the lifts of f, $\mathcal{F}$ and $\mathcal{F}^u$ to the universal cover.

We provide an orientation to $\tilde{\mathcal{F}}^u$ and denote as $\tilde{\mathcal{F}}^u_+(x)$ and $\tilde{\mathcal{F}}^u_-(x)$ to the connected components of $\tilde{\mathcal{F}}^u(x) \setminus \{x\}$. Since $\tilde{\mathcal{F}}(x)$ separates $\mathbb{R}^3$ (see subsection 4.2.3) we denote $F_+(x)$ and $F_-(x)$ to the connected components of $\mathbb{R}^3 \setminus \tilde{\mathcal{F}}(x)$ containing respectively $\tilde{\mathcal{F}}^u_+(x)$ and $\tilde{\mathcal{F}}^u_-(x)$.

Proposition 2.3.1 implies the existence of a continuous and surjective function $H : \mathbb{R}^3 \to \mathbb{R}^3$ which verifies

$$H \circ \tilde{f} = A \circ H$$

and such that $d(H(x), x) < K_1$ for every $x \in \mathbb{R}^3$.

5.2.1 Consequences of the semiconjugacy

We can prove:

Lemma 5.2.2. *For every $x \in \mathbb{R}^3$ we have that $H(\tilde{\mathcal{F}}^u_+(x))$ is unbounded.*

PROOF. Otherwise, for some $x \in \mathbb{R}^3$, the unstable leaf $\tilde{\mathcal{F}}^u_+(x)$ would be bounded. Since its length is infinite one can find two points in $\tilde{\mathcal{F}}^u_+(x)$ in different local unstable leafs at arbitrarily small distance. This contradicts Corollary 5.1.5 (i).

□

Remark 5.2.3. Notice that for every $x \in \mathbb{R}^3$ the set $F_+(x)$ is unbounded and contains a half unstable leaf of $\tilde{\mathcal{F}}^u$.

- In the case the automorphism A has stable dimension 2, this implies that $H(F_+(x))$ contains a half-line of irrational slope. Indeed, by Lemma 5.2.2 we have that $H(\tilde{\mathcal{F}}^u_+(x))$ is non bounded and since we know that $H(\tilde{\mathcal{F}}^u(x)) \subset W^u(H(x), A)$ we conclude.

- When A has stable dimension 1, we only obtain that $H(\tilde{\mathcal{F}}^u_+(x))$ contains an unbounded connected set in $W^u(H(x), A)$ which is two dimensional plane parallel to E^u_A.

$\diamondsuit$

One can push forward Lemma 5.2.2 in order to show that H is almost injective in each unstable leaf of $\tilde{\mathcal{F}}^u$, in particular, a similar argument to the one in Lemma 5.2.2 gives that at most finitely many points of an unstable leaf can have the same image under H. Later, we shall obtain that in fact, H is injective on unstable leaves so that we will not give the details of the previous claim (see Remark 5.2.11).

5.2.2 A planar direction for the foliation transverse to E^u

Since $\mathcal{F}$ is transverse to the unstable direction, we get by Corollary 4.2.2 that it is a Reebless foliation so that we can apply Theorem 4.2.6. We intend to prove in this section that option (ii) of Theorem 4.2.6 is not possible when f is isotopic to Anosov (see [RHRHU$_3$] where that possibility occurs). The following simple remark will be essential in what follows:

Remark 5.2.4. Notice that if we apply $\tilde{f}^{-1}$ to the foliation $\tilde{\mathcal{F}}$, then the new foliation $\tilde{f}^{-1}(\tilde{\mathcal{F}})$ is still transverse to E^u so that Theorem 4.2.6 still applies. So, we obtain a plane P' close to $\tilde{f}^{-1}(\tilde{\mathcal{F}})$. We claim that $P' = A^{-1}(P)$ where P is the plane given by Theorem 4.2.6 for $\tilde{\mathcal{F}}$. To prove this, recall that $\tilde{f}$ and A are at bounded distance so, leaves of $\tilde{f}^{-1}(\mathcal{F})$ must remain at bounded distance from $A^{-1}(P)$ and then use the fact that the plane is unique (Remark 4.2.7).

$\diamondsuit$

The result that follows can be deduced more easily if one assumes that A has stable dimension 2.

We say that a subspace P is *almost parallel* to a foliation $\tilde{\mathcal{F}}$ if there exists $R > 0$ such that for every $x \in \mathbb{R}^3$ we have that $P + x$ lies in an R-neighborhood of $\tilde{\mathcal{F}}(x)$ and $\tilde{\mathcal{F}}(x)$ lies in a R-neighborhood of $P + x$.

Proposition 5.2.5. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeomorphism of the form $T\mathbb{T}^3 = E^{cs} \oplus E^u$ (with $\dim E^{cs} = 2$) isotopic to a linear Anosov automorphism and $\mathcal{F}$ a foliation transverse to E^u. Then, there exists a two dimensional subspace $P \subset \mathbb{R}^3$ which is almost parallel to $\tilde{\mathcal{F}}$.*

Proof. It is enough to show that option (i) of Proposition 4.2.6 holds, since it implies the existence of a plane P almost parallel to $\tilde{\mathcal{F}}$.

Assume by contradiction that option (ii) of Proposition 4.2.6 holds. Then, there exists a plane $P \subset \mathbb{R}^3$ whose projection to $\mathbb{T}^3$ is a two dimensional torus and such that every leaf of $\tilde{\mathcal{F}}^u$, being transverse to $\tilde{\mathcal{F}}$, remains at bounded distance[4] from P. Indeed, when there is a dead-end component for $\mathcal{F}$ we get that any transverse foliation must verify that its leaves remain at bounded distance from the boundary torus of the dead-end component which in turn are at bounded distance from the plane P.

Since f is isotopic to a linear Anosov automorphism A we know that P cannot be invariant under A (see Proposition 1.5.1). So, we have that P and $A^{-1}(P)$ intersect in a one dimensional subspace L which projects into a circle in $\mathbb{T}^3$ (notice that a linear curve in $\mathbb{T}^2$ is either dense or a circle, so, if a line belongs to the intersection of two linear two dimensional torus in $\mathbb{T}^3$ which do not coincide, it must be a circle).

We get that for every point x we have that $\tilde{\mathcal{F}}^u(x)$ must lie within bounded distance from P as well as from $A^{-1}(P)$ (since when we apply $\tilde{f}^{-1}$ to $\tilde{\mathcal{F}}$ the leaf close to P becomes close to $A(P)$, see Remark 5.2.4). This implies that in fact $\tilde{\mathcal{F}}^u(x)$ lies within bounded distance from L.

On the other hand, we have that $H(\tilde{\mathcal{F}}^u_+(x))$ is contained in $W^u(H(x), A) = E^u_A + H(x)$ for every $x \in \mathbb{R}^3$. Since H is at bounded distance from the identity, we get that $\tilde{\mathcal{F}}^u(x)$ lies within bounded distance from E^u_A, the eigenspace corresponding to the unstable eigenvalues of A.

Since E^u_A must be totally irrational (see Remark 1.5.3) and L projects into a circle L, we get that $\mathcal{F}^u_+(x)$ remains at bounded distance from $E^u_A \cap L = \{0\}$. This contradicts the fact that $\tilde{\mathcal{F}}^u_+(x)$ is unbounded (Lemma 5.2.2).

$\square$

5.2.3 Global Product Structure in the universal cover

When the plane P almost parallel to $\tilde{\mathcal{F}}$ is totally irrational, one can see that the foliation $\mathcal{F}$ in $\mathbb{T}^3$ is without holonomy, and thus there is a global product structure between $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^u$ which follows directly from Theorem 4.3.1.

This would be the case if we knew that the plane P given by Theorem 4.2.6 is f_*-invariant (see subsection 5.3.6). To obtain the global product structure in our case we will use the fact that iterating the plane P backwards by f_* it will converge to an irrational plane and use instead Theorem 4.3.2.

[4] Notice that if A has stable dimension 2, this already gives us a contradiction since $H(\tilde{\mathcal{F}}^u(x)) = W^u(H(x), A)$ which is totally irrational and cannot acumulate in a plane which projects into a two-torus.

Proposition 5.2.5 implies that the foliation $\tilde{\mathcal{F}}$ is quite well behaved. In this section we shall show that the properties we have showed for the foliations and the fact that $\tilde{\mathcal{F}}^u$ is $\tilde{f}$-invariant while the foliation $\tilde{\mathcal{F}}$ remains with a uniform local product structure with $\tilde{\mathcal{F}}^u$ when iterated backwards (see Lemma 5.2.7) imply that there is a global product structure. Some of the arguments become simpler if one assumes that A has stable dimension 2.

The main result of this section is thus the following:

Proposition 5.2.6. *Given $x, y \in \mathbb{R}^3$ we have that $\tilde{\mathcal{F}}(x) \cap \tilde{\mathcal{F}}^u(y) \neq \emptyset$. This intersection consists of exactly one point.*

Notice that uniqueness of the intersection point follows from Corollary 5.1.5 (i) and will be used to prove the proposition. We must put ourselves in the conditions of Theorem 4.3.2.

We shall proceed with the proof of Proposition 5.2.6.

We start by proving a result which gives that the size of local product structure boxes between $f^{-n}(\mathcal{F})$ and $\mathcal{F}^u$ can be chosen independent of n. We shall denote as $\mathbb{D}^2 = \{z \in \mathbb{C} \ : \ |z| \leq 1\}$.

Lemma 5.2.7. *There exists $\delta > 0$ such that for every $x \in \mathbb{R}^3$ and $n \geq 0$ there exists a closed neighborhood V_x^n containing $B_\delta(x)$ such that it admits C^0-coordinates $\varphi_x^n : \mathbb{D}^2 \times [-1, 1] \to \mathbb{R}^3$ such that:*

- *$\varphi_x^n(\mathbb{D}^2 \times [-1, 1]) = V_x^n$ and $\varphi_x^n(0, 0) = x$.*
- *$\varphi_x^n(\mathbb{D}^2 \times \{t\}) = \tilde{f}^{-n}(\tilde{\mathcal{F}}(\tilde{f}^n(\varphi_x^n(0, t)))) \cap V_x^n$ for every $t \in [-1, 1]$.*
- *$\varphi_x^n(\{s\} \times [-1, 1]) = \tilde{\mathcal{F}}^u(\varphi_x^n(s, 0)) \cap V_x^n$ for every $s \in \mathbb{D}^2$.*

Proof. Notice first that the tangent space to $f^{-n}(\mathcal{F})$ belongs to a cone transverse to E^u and independent of n. Let us call this cone $\mathcal{E}^{cs}$.

Given $\epsilon > 0$ we can choose a neighborhood V_ϵ of x contained in $B_\epsilon(x)$ such that the following is verified:

- There exists a two dimensional disk D containing x such that V_ϵ is the union of segments of $\mathcal{F}^u(x)$ of length 2ϵ centered at points in D. This defines two boundary disks D^+ and D^- contained in the boundary of V_ε.
- By choosing D small enough, we get that there exists $\epsilon_1 > 0$ such that every curve of length ϵ_1 starting at a point $y \in B_{\epsilon_1}(x)$ tangent to $\mathcal{E}^{cs}$ must leave V_ϵ and intersects ∂V_ϵ in $\partial V_\epsilon \setminus (D^+ \cup D^-)$.

Notice that both ϵ and ϵ_1 can be chosen uniformly in $\mathbb{R}^3$ because of compactness of $\mathbb{T}^3$ and uniform transversality of the foliations (see Remark 4.2.3).

This implies that every disk of radius ϵ tangent to $\mathcal{E}^{cs}$ centered at a point $z \in B_{\epsilon_1}(x)$ must intersect the unstable leaf of every point in D, in particular, there is a local product structure of uniform size around each point in $\mathbb{R}^3$.

Now, we can choose a continuous chart (recall that the foliations are with C^1 leaves but only continuous) around each point which sends horizontal disks into disks transverse to E^u and vertical lines into leaves of $\tilde{\mathcal{F}}^u$ containing a fixed ball around each point x independent of $n \geq 0$ giving the desired statement.

$\square$

Remark 5.2.8. We obtain that there exists $\varepsilon > 0$ such that for every $x \in \mathbb{R}^3$ there exists $V_x \subset \bigcap_{n\geq 0} V_x^n$ containing $B_\varepsilon(x)$ admitting C^1-coordinates $\psi_x : \mathbb{D}^2 \times [-1,1] \to \mathbb{R}^3$ such that:

- $\psi_x(\mathbb{D}^2 \times [-1,1]) = V_x$ and $\psi_x(0,0) = x$.
- If we consider $V_x^\varepsilon = \psi_x^{-1}(B_\varepsilon(x))$ then one has that for every $y \in B_\varepsilon(x)$ and $n \geq 0$ we have that:
$$\psi_x^{-1}(\tilde{f}^{-n}(\tilde{\mathcal{F}}(\tilde{f}^n(y))) \cap V_x)$$
is the graph of a function $h_y^n : \mathbb{D}^2 \to [-1,1]$ which has uniformly bounded derivative in y and n.

Indeed, this is given by considering a C^1-chart ψ_x around every point such that its image covers the ε-neighborhood of x and sends the E-direction to an almost horizontal direction and the E^u-direction to an almost vertical direction (see Proposition 5.1.7). See for example [BuW$_2$] section 3 for more details on this kind of constructions.

$\diamondsuit$

This lemma shows that after iterating the foliation backwards, one gets that it becomes nearly irrational so that we can apply Theorem 4.3.2.

Lemma 5.2.9. *Given $K > 0$ there exists $n_0 > 0$ such that for every $x \in \mathbb{R}^3$ and for every $\gamma \in \mathbb{Z}^3$ with norm less than K we have that*

$$\tilde{f}^{-n_0}(\tilde{\mathcal{F}}(x)) + \gamma \neq \tilde{f}^{-n_0}(\tilde{\mathcal{F}}(x)) \qquad \forall x \in \mathbb{R}^3.$$

Proof. Notice that $\tilde{f}^{-n}(\tilde{\mathcal{F}})$ is almost parallel to $A^{-n}(P)$. Notice that $A^{-n}(P)$ has a converging subsequence towards a totally irrational plane $\tilde{P}$ (see Remarks 1.5.3 and 1.5.2).

We can choose n_0 large enough such that no element of $\mathbb{Z}^3$ of norm smaller than K fixes $A^{-n_0}(P)$.

Notice first that $\tilde{f}^{-n_0}(\tilde{\mathcal{F}})$ is almost parallel to $A^{-n_0}(P)$ (see Remark 5.2.4). Now, assuming that there is a translation γ which fixes a leaf of $\tilde{f}^{-n_0}(\tilde{\mathcal{F}})$ we get that the leaf contains a loop homotopic to γ. This implies that it is at bounded distance from the line which is the lift of the canonical (linear) representative of γ (see Lemma 4.2.10). This implies that γ fixes $A^{-n_0}(P)$ and thus has norm larger than K as desired.

$\square$

We can now complete the proof of Proposition 5.2.6.

Proof of Proposition 5.2.6. By Corollary 5.1.5 we know that all the leaves of $\tilde{\mathcal{F}}$ are simply connected. Proposition 4.2.8 implies that the leaf space of $\tilde{\mathcal{F}}$ is homeomorphic to $\mathbb{R}$. All this properties remain true for the foliations $\tilde{f}^{-n}(\tilde{\mathcal{F}})$ since they are diffeomorphisms at bounded distance from linear transformations.

Lemma 5.2.7 gives that the size of the local product structure between $\tilde{f}^{-n}(\tilde{\mathcal{F}})$ and $\tilde{\mathcal{F}}^u$ does not depend on n.

Using Lemma 5.2.9 we get that for some sufficiently large n the foliations $\tilde{f}^{-n}(\tilde{\mathcal{F}})$ and $\tilde{\mathcal{F}}^u$ are in the hypothesis of Theorem 4.3.2 which gives global product structure between $\tilde{f}^{-n}(\tilde{\mathcal{F}})$ and $\tilde{\mathcal{F}}^u$. Since $\tilde{\mathcal{F}}^u$ is $\tilde{f}$-invariant and f is a diffeomorphism we get that there is a global product structure between $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^u$ as desired.

$\square$

Using Proposition 4.3.9 we deduce the following (see figure 5.1) :

Corollary 5.2.10. *The foliation $\tilde{\mathcal{F}}^u$ is quasi-isometric. Moreover, there exist one dimensional subspaces L_1 and L_2 of E^u_A transverse to P and $K > 0$ such that for every $x \in \mathbb{R}^3$ and $y \in \tilde{\mathcal{F}}^u(x)$ at distance larger than K from x we have that $H(y) - H(y)$ is contained in the cone of E^u_A with boundaries L_1 and L_2 and transverse to P.*

Notice that if A has stable dimension 2 then $L_1 = L_2 = E^u_A$.

Proof. This is a direct consequence of Proposition 4.3.9 and the fact that the image of $\tilde{\mathcal{F}}^u(x)$ by H is contained in $E^u_A + H(x)$.

$\square$

Remark 5.2.11. Since points which are sent to the same point by H must have orbits remaining at bounded distance, the quasi-isometry of $\tilde{\mathcal{F}}^u$ implies that H must be injective on leaves of $\tilde{\mathcal{F}}^u$.

$\diamondsuit$

5.2.4 Complex eigenvalues

The following proposition has interest only in the case A has stable dimension 1.

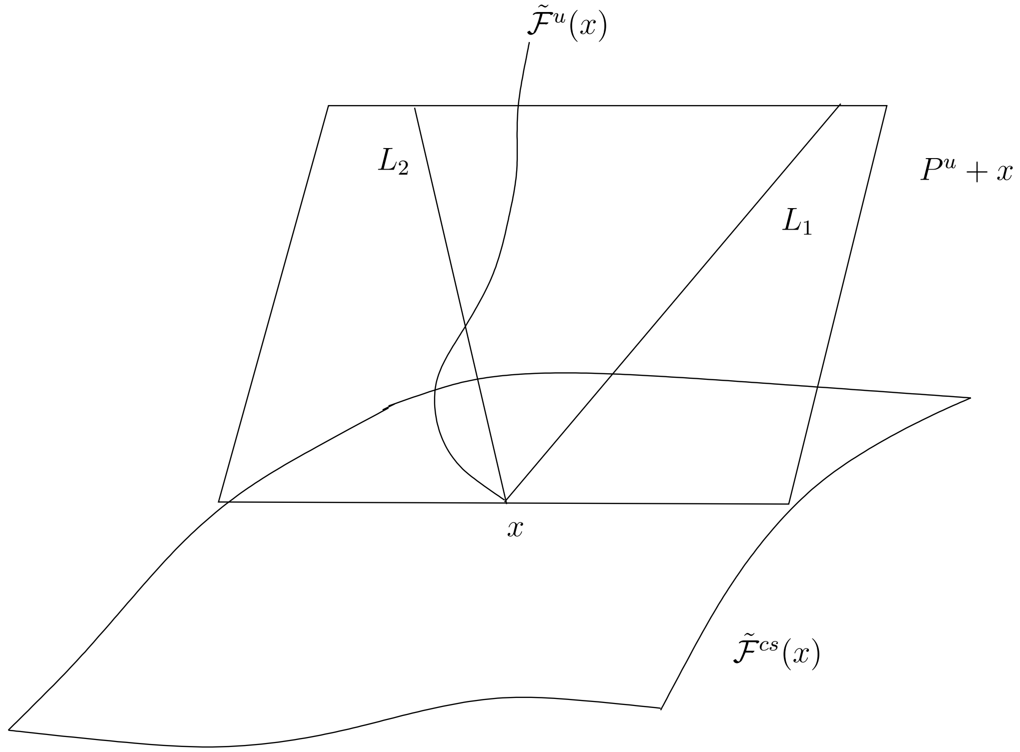

Figure 5.1: The unstable leaf of x remains close to the cone bounded by L_1 and L_2.

Proposition 5.2.12. *The matrix A cannot have complex unstable eigenvalues.*

PROOF. Assume that A has complex unstable eigenvalues, in particular E^u_A is two-dimensional. Consider a fixed point x_0 of $\tilde{f}$.

Recall that by Lemma 5.2.2 the set $\eta = H(\tilde{\mathcal{F}}^u_+(x_0))$ is an unbounded continuous curve in E^u_A. Since x_0 is fixed and since H is a semiconjugacy, we have that η is A-invariant.

On the other hand, by Corollary 5.2.10 we have that η is eventually contained in a cone between two lines L_1 and L_2.

This implies that A cannot have complex unstable eigenvalues (recall that they should have irrational angle by Lemma 1.5.2) since a matrix which preserves an unbounded connected subset of a cone cannot have complex eigenvalues with irrational angle.

$\square$

5.2.5 Dynamical Coherence

In this section we shall show dynamical coherence of almost dynamically coherent partially hyperbolic diffeomorphisms isotopic to linear Anosov automorphisms.

The proof of the following theorem becomes much simpler if one assumes that the plane P almost parallel to $\tilde{\mathcal{F}}$ is A-invariant which as we mentioned before is the most important case (see also subsection 5.3.6)

Theorem 5.2.13. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an almost dynamically coherent partially hyperbolic diffeomorphism of the form $T\mathbb{T}^3 = E^{cs} \oplus E^u$ isotopic to a linear Anosov automorphism. Then, there exists an f-invariant foliation $\mathcal{F}^{cs}$ tangent to E^{cs}. If $\tilde{\mathcal{F}}^{cs}$ denotes the lift to $\mathbb{R}^3$ of this foliation, then $H(\tilde{\mathcal{F}}^{cs}(x)) = P^{cs} + H(x)$ where P^{cs} is an A-invariant subspace and E^u_A is not contained in P^{cs}.*

Proof. Consider the foliation $\tilde{\mathcal{F}}$, by Proposition 5.2.5 we have a plane P which is almost parallel to $\tilde{\mathcal{F}}$.

Let P^{cs} be the limit of $A^{-n}(P)$ which is an A-invariant subspace. Since we have proved that A has no complex unstable eigenvalues (Proposition 5.2.12) and since P is transverse to E^u_A (Proposition 4.3.9), this plane is well defined (see Remark 1.5.3).

Notice that the transversality of P with E^u_A implies that P^{cs} contains E^s_A, the eigenspace associated with stable eigenvalues (in the case where A has stable dimension 2 we thus have $P^{cs} = E^s_A$).

Since P^{cs} is A-invariant, we get that it is totally irrational so that no deck transformation fixes P^{cs}.

Using Remark 5.2.8 we obtain $\varepsilon > 0$ such that for every $x \in \mathbb{R}^3$ there are neighborhoods V_x containing $B_\varepsilon(x)$ admitting C^1-coordinates $\psi_x : \mathbb{D}^2 \times [-1,1] \to V_x$ such that:

- For every $y \in B_\varepsilon(x)$ we have that if we denote as $W^x_n(y)$ to the connected component containing y of $V_x \cap \tilde{f}^{-n}(\tilde{\mathcal{F}}(f^n(y)))$ then the set $\psi_x^{-1}(W^x_n(y))$ is the graph of a C^1-function $h^n_{x,y} : \mathbb{D}^2 \to [-1,1]$ with bounded derivatives.

By a standard graph transform argument (see [HPS] or [BuW$_2$] section 3) using the fact that these graphs have bounded derivative we get that $\{h^{x,y}_n\}$ is pre-compact in the space of functions from $\mathbb{D}^2$ to $[-1,1]$.

For every $y \in B_\varepsilon(x)$ there exists $\mathcal{J}^x_y$ a set of indices such that for every $\alpha \in \mathcal{J}^x_y$ we have a C^1-function $h^{x,y}_{\infty,\alpha} : \mathbb{D}^2 \to [-1,1]$ and $n_j \to +\infty$ such that:

$$h^{x,y}_{\infty,\alpha} = \lim_{j \to +\infty} h^{x,y}_{n_j}$$

Every $h^{x,y}_{\infty,\alpha}$ gives rise to a graph whose image by ψ_x we denote as $W^x_{\infty,\alpha}(y)$. This manifold verifies that it contains y and is everywhere tangent to E^{cs}.

Claim. *We have that $H(W^x_{\infty,\alpha}(z)) \subset P^{cs} + H(z)$ for every $z \in B_\varepsilon(x)$ and every $\alpha \in \mathcal{J}^x_z$.*

Proof. Consider $y \in W^x_{\infty,\alpha}(z)$ for some $\alpha \in \mathcal{J}_z$. One can find $n_j \to \infty$ such that $W^x_{n_j}(z) \to W^x_{\infty,\alpha}(z)$.

In the coordinates ψ_x of V_x, we can find a sequence $z_{n_j} \in W^x_{n_j}(z) \cap \tilde{\mathcal{F}}^u(y)$ such that $z_{n_j} \to y$. Moreover, we have that $\tilde{f}^{n_j}(z_{n_j}) \in \tilde{\mathcal{F}}(f^{n_j}(z))$. Assume that $H(y) \neq H(z)$ (otherwise there is nothing to prove).

We have, by continuity of H that $H(z_{n_j}) \to H(y) \neq H(z)$.

We choose a metric in $\mathbb{R}^3$ so that $(P^{cs})^\perp$ with this metric is A-invariant. We denote as λ to the eigenvalue of A in the direction $(P^{cs})^\perp$.

By Proposition 5.2.5 and the fact that H is at bounded distance from the identity, there exists $R > 0$ such that for every $n_j \geq 0$ we have that $A^{n_j}(H(z_{n_j}))$ is at distance smaller than R from $P + A^{n_j}(H(z))$ since $\tilde{f}^{n_j}(z_{n_j}) \in \tilde{\tilde{\mathcal{F}}}(f^{n_j}(z))$.

Suppose that $H(z_{n_j})$ does not converge to $P^{cs} + H(z)$. We must reach a contradiction.

Consider then $\alpha > 0$ such that the angle between P^{cs} and the vector $H(y) - H(z)$ is larger than $\alpha > 0$. This α can be chosen positive under the assumption that $H(z_{n_j})$ does not converge to $P^{cs} + H(z)$.

Let $n_j > 0$ be large enough such that:

- The angle between $A^{-n_j}(P)$ and P^{cs} is smaller than $\alpha/4$,
- $\|H(z_{n_j}) - H(z)\| > \frac{3}{4}\|H(y) - H(z)\|$,
- $\lambda^{n_j} \gg 2R(\sin(\frac{\alpha}{2})\cos(\beta)\|H(y) - H(z)\|)^{-1}$.

Let v_{n_j} be the vector which realizes $d(H(z_{n_j}) - H(z), A^{-n_j}(P))$ and as $v^\perp_{n_j}$ the projection of v_{n_j} to $(P^{cs})^\perp$. We have that

$$\|v^\perp_{n_j}\| > \frac{1}{2}\sin\left(\frac{\alpha}{2}\right)\|H(y) - H(z)\|$$

Notice that the distance between $A^{n_j}(H(z_{n_j}))$ and $P + A^{n_j}(H(z))$ is larger than $\|A^{n_j} v^\perp_{n_j}\|\cos(\beta)$.

This is a contradiction since this implies that $A^{n_j}(H(z_{n_j}))$ is at distance larger than

$$\lambda^{n_j}\|v^\perp_{n_j}\|\cos(\beta) \gg R$$

from $P + A^{n_j}(H(z))$. This concludes the claim.

$\diamondsuit$

Assuming that P^{cs} does not intersect the cone bounded by L_1 and L_2 this finishes the proof since one sees that each leaf of $\tilde{\mathcal{F}}^u$ can intersect the pre-image by H of $P^{cs} + y$ in a unique point, thus showing that the partition of $\mathbb{R}^3$ by the pre-images of the translates of P^{cs} defines a $\tilde{f}$-invariant foliation (and also invariant under deck transformations). We leave to the interested reader the task of filling the details of the proof in this particular case, since we will continue by giving a proof which works in all cases.

We will prove that H cannot send unstable intervals into the same plane parallel to P^{cs}.

Claim. *Given $\gamma : [0,1] \to \mathbb{R}$ a non-trivial curve contained in $\tilde{\mathcal{F}}^u(x)$ we have that $H(\gamma([0,1]))$ is not contained in $P^{cs} + H(\gamma(0))$.*

Proof. Consider C_ε given by Corollary 5.1.5 (iv) for ε of the size of the uniform local product structure. Moreover, consider L large enough such that $C_\varepsilon L > \mathrm{Vol}(\mathbb{T}^3)$.

Since $\tilde{\mathcal{F}}^u$ is $\tilde{f}$-invariant and P^{cs} is A-invariant we deduce that we can assume that the length of γ is arbitrarily large, in particular larger than $2L$.

We will show that $H(B_\varepsilon(\gamma([a,b]))) \subset P^{cs} + H(\gamma(0))$ where $0 < a < b < 1$ and the length of $\gamma([a,b])$ is larger than L.

Having volume larger than $\mathrm{Vol}(\mathbb{T}^3)$ there must be a deck transformation $\gamma \in \mathbb{Z}^3$ such that $\gamma + B_\varepsilon(\gamma([a,b])) \cap B_\varepsilon(\gamma([a,b])) \neq \emptyset$. This in turn gives that $\gamma + H(B_\varepsilon(\gamma([a,b]))) \cap H(B_\varepsilon(\gamma([a,b]))) \neq \emptyset$ and thus $\gamma + P^{cs} \cap P^{cs} \neq \emptyset$. Since P^{cs} is totally irrational this is a contradiction.

It remains to show that $H(B_\varepsilon(\gamma([a,b]))) \subset P^{cs} + H(\gamma(0))$. By the previous claim, we know that if $z, w \in W^x_{\infty,\alpha}(y)$ for some $\alpha \in \mathcal{J}_y$, then $H(z) - H(w) \in P^{cs}$.

Consider $a, b \in [0,1]$ such that $\tilde{\mathcal{F}}^u(x) \cap B_\varepsilon(\gamma([a,b])) \subset \gamma([0,1])$. By Corollary 5.1.5 we have that such a, b exist and we can choose them in order that the length of $\gamma([a,b])$ is larger than L.

Let $z \in B_\varepsilon(\gamma([a,b]))$ and choose $w \in \gamma([a,b])$ such that $z \in B_\varepsilon(w)$. We get that for every $\alpha \in \mathcal{J}^w_z$ we have that $W^w_{\infty,\alpha}(z) \cap \gamma([0,1]) \neq \emptyset$. Since $H(\gamma([0,1])) \subset P^{cs} + H(\gamma(0))$ and by the previous claim, we deduce that $H(w) \subset P^{cs} + H(\gamma(0))$ finishing the proof.

$\diamondsuit$

Now we are in conditions to show that for every point x and for every point $y \in B_\varepsilon(x)$ there is a unique manifold $W^x_\infty(y)$ tangent to E^{cs} which is a limit of the manifolds $W^x_n(y)$. Using the same argument as in Proposition 5.1.7 we get that the foliations $\tilde{f}^{-n}(\tilde{\mathcal{F}})$ converge to a f-invariant foliation $\tilde{\mathcal{F}}^{cs}$ tangent to E^{cs} concluding the proof of the Theorem.

Indeed, assume that the manifolds $W^x_n(y)$ have a unique limit for every $x \in \mathbb{R}^3$ and $y \in B_\varepsilon(x)$ and that for any pair points $y, z \in B_\varepsilon(x)$ these limits are either disjoint or equal (see the claim below). One has that the set of manifolds $W^x_\infty(y)$ forms an f-invariant plaque family in the following sense:

- $\tilde{f}(W^x_\infty(y)) \cap W^{\tilde{f}(x)}_\infty(\tilde{f}(y))$ is relatively open whenever $\tilde{f}(y) \in B_\varepsilon(\tilde{f}(x))$.

We must thus show that these plaque families form a foliation. For this, we use the same argument as in Proposition 5.1.7. Consider $z, w \in B_\varepsilon(x)$ we have that $W^x_\infty(z) \cap \tilde{\mathcal{F}}^u(w) \neq \emptyset$ and in fact consists of a unique point (see Corollary 5.1.5 (i)). Since the intersection point varies continuously and using that plaques are either disjoint or equal we obtain a continuous map from $\mathbb{D}^2 \times [-1,1]$ to a neighborhood of x sending horizontal disks into plaques. This implies that the plaques form an f-invariant foliation as desired.

It thus remains to show the following:

Claim. *Given $x \in \mathbb{R}^3$ and $y, z \in B_\varepsilon(x)$ we have that there is a unique limit of $W^x_\infty(y)$ and $W^x_\infty(z)$ and they are either disjoint or coincide. More precisely, for every $\alpha \in \mathcal{J}^x_y$ and $\beta \in \mathcal{J}^x_z$ (z could coincide with y) we have that $h^{x,y}_{\infty,\alpha} = h^{x,z}_{\infty,\beta}$ or the graphs are disjoint.*

Proof. Assuming the claim does not hold, one obtains $y, z \in B_\varepsilon(x)$ such that $h^{x,y}_{\infty,\alpha}$ and $h^{x,z}_{\infty,\beta}$ coincide at some point but whose graphs are different for some $\alpha \in \mathcal{J}^x_y$ and $\beta \in \mathcal{J}^x_z$. In particular, there exists a point $t \in \mathbb{D}^2$ which is in the boundary of where both functions coincide. We assume for simplicity[5] that $\psi_x(t)$ belongs to $B_\varepsilon(x)$.

Let $\gamma : [0,1] \to B_\varepsilon(x)$ be a non-trivial arc of $\tilde{\mathcal{F}}^u$ joining the graphs of $h^{x,y}_{\infty,\alpha}$ and $h^{x,z}_{\infty,\beta}$. Since the graphs of both $h^{x,y}_{\infty,\alpha}$ and $h^{x,z}_{\infty,\beta}$ separate V_x we have that every point $w \in \gamma((0,1))$ verifies that for every $\delta \in \mathcal{J}^x_w$ one has that $W^x_{\infty,\delta}(w)$ intersects at least one of $W^x_{\infty,\alpha}(y)$ or $W^x_{\infty,\beta}(z)$. By the first claim we get that $H(w) \in P^{cs} + H(y) = P^{cs} + H(z)$ a contradiction with the second claim.

$\diamondsuit$

$\square$

We can in fact obtain a stronger property since our results allow us to show that in fact E^{cs} is uniquely integrable into a foliation. Notice that there are stronger notions of unique integrability (see [BuW$_1$] and [BFra]).

Proposition 5.2.14. *There is a unique f-invariant foliation $\mathcal{F}^{cs}$ tangent to E^{cs}. Moreover, the plane P^{cs} given by Theorem 4.2.6 for this foliation is A-invariant and contains the stable eigenspace of A.*

Proof. Assume there are two different f-invariant foliations $\mathcal{F}^{cs}_1$ and $\mathcal{F}^{cs}_2$ tangent to E^{cs}.

[5] If it were not the case we would need to change the coordinates and perform the same proof, but not to charge the notation we choose to make this (unnecessary) assumption.

Since they are transverse to E^u they must be Reebless (see Corollary 4.2.2) so that Theorem 4.2.6 applies.

By Remark 5.2.4 we know that since the foliations are f-invariant, the planes P_1^{cs} and P_2^{cs} given by Theorem 4.2.6 are A-invariant. The fact that P^{cs} contains the stable direction of A is given by Remark 1.5.3 and Corollary 5.2.10 since it implies that P^{cs} cannot be contained in E^u_A.

Assume first that the planes P_1^{cs} and P_2^{cs} coincide. The foliations remain at distance R from translates of the planes. By Corollary 5.2.10 we know that two points in the same unstable leaf must separate in a direction transverse to $P_1^{cs} = P_2^{cs}$. If $\mathcal{F}_1^{cs}$ is different from $\mathcal{F}_2^{cs}$ we have a point x such that $\mathcal{F}_1^{cs}(x) \neq \mathcal{F}_2^{cs}(x)$. By the global product structure we get a point $y \in \mathcal{F}_1^{cs}(x)$ such that $\tilde{\mathcal{F}}^u(y) \cap \mathcal{F}_2^{cs}(x) \neq \{y\}$. Iterating forward and using Corollary 5.2.10 we contradict the fact that leaves of $\mathcal{F}_1^{cs}$ and $\mathcal{F}_2^{cs}$ remain at distance R from translates of $P_1^{cs} = P_2^{cs}$.

Now, if $P_1^{cs} \neq P_2^{cs}$ we know that A has stable dimension 1 since we know that E^s_A is contained in both. Using Corollary 5.2.10 and the fact that the unstable foliation is $\tilde{f}$-invariant we see that this cannot happen.

$\square$

Notice also that from the proof of Theorem 5.2.13 we deduce that given a foliation $\mathcal{F}$ transverse to E^{cs} we have that the backward iterates of this foliation must converge to this unique f-invariant foliation. This implies that:

Corollary 5.2.15. *Given a dynamically coherent partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ with splitting $T\mathbb{T}^3 = E^{cs} \oplus E^u$ isotopic to Anosov we know that it is C^1-robustly dynamically coherent and that the f_*-invariant plane P given by Theorem 4.2.6 for the unique f-invariant foliation $\mathcal{F}^{cs}$ tangent to E^{cs} does not change for diffeomorphisms C^1-close to f.*

The robustness of dynamical coherence follows from the fact that being dynamically coherent it is robustly almost dynamically coherent.

We close this Section with a question we were not able to answer in full generality:

Question 5.2.16. *Is it true that P^{cs} corresponds to the eigenspace asociated to the smallest eigenvalues of A?.*

This is true for the case when A has stable index 2 and we show in Proposition 5.3.12 that it is the case in the strong partially hyperbolic case.

5.3 Strong partial hyperbolicity and coherence in $\mathbb{T}^3$

In the strong partially hyperbolic case we are able to give a stronger result independent of the isotopy class of f:

Theorem 5.3.1. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a strong partially hyperbolic diffeomorphism, then:*

- *Either there exists a unique f-invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$ or,*
- *There exists a periodic two-dimensional torus T tangent to $E^s \oplus E^c$ which is (normally) repelling.*

Remark 5.3.2. Indeed, it is not hard to show that in the case there is a repelling torus, it must be an *Anosov tori* as defined in [RHRHU$_2$] (see Proposition 2.1 of [BBI$_1$] or Lemma 4.A.4). In the example of [RHRHU$_3$] it is shown that the second possibility is not empty.

$\diamondsuit$

A diffeomorphism f is *chain-recurrent* if there is no open set U such that $f(\overline{U}) \subset U$ (see [C$_4$] for an introduction to this concept in the context of differentiable dynamics):

Corollary. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ a chain-recurrent strongly partially hyperbolic diffeomorphism. Then, f is dynamically coherent.*

In the strong partially hyperbolic case, when no torus tangent to $E^s \oplus E^c$ nor $E^c \oplus E^u$ exists, we deduce further properties on the existence of planes close to the f-invariant foliations. These results are essential to obtain leaf-conjugacy results (see [H]).

The idea of the proof is to obtain a global product structure between the foliations involved in order to then get dynamical coherence. In a certain sense, this is a similar idea to the one used for the proof of Theorem 5.2.1.

However, the fact that global product structure implies dynamical coherence is much easier in our case due to the existence of f-invariant branching foliations tangent to the center-stable direction (see subsection 5.3.2).

This approach goes in the inverse direction to the one made in [BBI$_2$] (and continued in [H]). In [BBI$_2$] the proof proceeds as follows:

- First they show that the planes close to the two foliations are different. To prove this they use absolute domination.

- They then show (again by using absolute domination) that leaves of $\tilde{\mathcal{F}}^u$ are quasi-isometric. Here absolute domination is essential (since in the examples of [RHRHU$_3$] the lift of the unstable foliation is not quasi-isometric).

- Finally, they use Brin's criterium for absolutely dominated partially hyperbolic systems ([Bri]) to obtain coherence. As it was shown in Proposition 5.1.12 this criterium uses absolute domination in an essential way.

Then, in [H] it is proved that in fact, the planes P^{cs} and P^{cu} close to the f-invariant foliations are the expected ones in order to obtain global product structure and then leaf conjugacy to linear models.

Another difference with the proof there is that in our case it will be important to discuss depending on the isotopy class of f which is not needed in the case of absolute partial hyperbolicity. In a certain sense, the reason why in each case there is a global product structure can be regarded as different: In the isotopic to Anosov case (see subsection 5.3.6) we deduce that the foliations are without holonomy and use Theorem 4.3.1 to get global product structure. In the case which is isotopic to a non-hyperbolic matrix we must first find out which are the planes close to each foliation in order to get the global product structure.

5.3.1 Preliminary discussions

Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a strong partially hyperbolic diffeomorphism with splitting $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$.

We denote as $\mathcal{F}^s$ and $\mathcal{F}^u$ to the stable and unstable foliations given by Theorem 1.3.1 which are one dimensional and f-invariant.

As in the previous sections, we will denote as $p : \mathbb{R}^3 \to \mathbb{T}^3$ to the covering projection and $\tilde{f}$ will denote a lift of f to the universal cover. Recall that $f_* : \mathbb{R}^3 \to \mathbb{R}^3$ which denotes the linear part of f is at bounded distance ($K_0 > 0$) from $\tilde{f}$.

We have already proved:

Theorem 5.3.3. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a strong partially hyperbolic diffeomorphism isotopic to Anosov, then f is dynamically coherent. Moreover, there is a unique f-invariant foliation tangent to $E^{cs} = E^s \oplus E^c$ and a unique f-invariant foliation tangent to $E^{cu} = E^c \oplus E^u$.*

This follows from Theorem 5.2.1 and the fact that strongly partially hyperbolic diffeomorphisms are almost dynamical coherent (Corollary 5.1.10). The uniqueness follows from Proposition 5.2.14. We will give an independent proof in subsection 5.3.6 since in the context of strong partial hyperbolicity the proof becomes simpler.

The starting point of our proof of Theorem 5.3.1 is the existence of f-invariant branching foliations $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$ tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively. By using Theorem 5.1.9 and Theorem 4.2.6 we can deduce the following:

Proposition 5.3.4. *There exist an f_*-invariant plane P^{cs} and $R > 0$ such that every leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$ lies in the R-neighborhood of a plane parallel to P^{cs}.*

Moreover, one can choose R such that one of the following conditions holds:

- *(i) The projection of the plane P^{cs} is dense in $\mathbb{T}^3$ and the R-neighborhood of every leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$ contains a plane parallel to P^{cs}, or,*
- *(ii) The projection of P^{cs} is a linear two-dimensional torus and there is a leaf of $\mathcal{F}^{cs}_{bran}$ which is a two-dimensional torus homotopic to $p(P^{cs})$.*

An analogous dichotomy holds for $\mathcal{F}^{cu}_{bran}$.

Proof. We consider sufficiently small $\varepsilon > 0$ and the foliation $\mathcal{S}_\varepsilon$ given by Theorem 5.1.9.

Let h^{cs}_ε be the continuous and surjective map which is ε-close to the identity sending leaves of $\mathcal{S}_\varepsilon$ into leaves of $\mathcal{F}^{cs}_{bran}$. By taking the lift to the universal cover, we have that there is $\tilde{h}^{cs}_\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$ continuous and surjective which is also at distance smaller than ε from the identity such that it sends leaves of $\tilde{\mathcal{S}}_\varepsilon$ homeomorphically into leaves of $\tilde{\mathcal{F}}^{cs}_{bran}$.

This implies that given a leaf L of $\tilde{\mathcal{F}}^{cs}_{bran}$ there exists a leaf S of $\tilde{\mathcal{S}}_\varepsilon$ such that L is at distance smaller than L from S and viceversa.

Since the foliation $\mathcal{S}_\varepsilon$ is transverse to E^u we can apply Theorem 4.2.6 and we obtain that there exists a plane P^{cs} and $R > 0$ such that every leaf of the lift $\tilde{\mathcal{S}}_\varepsilon$ of $\mathcal{S}_\varepsilon$ to $\mathbb{R}^3$ lies in an R-neighborhood of a translate of P^{cs}. Recall that this plane is unique (see Remark 4.2.7).

From the previous remark, we get that every leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$ lies in an $R + \varepsilon$-neighborhood of a translate of P^{cs} and this is the unique plane with this property.

Since $\tilde{\mathcal{F}}^{cs}_{bran}$ is $\tilde{f}$-invariant, we deduce that the plane P^{cs} is f_*-invariant (see also Remark 5.2.4).

By Proposition 4.2.9 we know that if P^{cs} projects into a two-dimensional torus, we obtain that the foliation $\mathcal{S}_\varepsilon$ must have a torus leaf. The image of this leaf by h^{cs}_ε is a torus leaf of $\mathcal{F}^{cs}_{bran}$. This gives (ii).

Since a plane whose projection is not a two-dimensional torus must be dense we get that if option (ii) does not hold, we have that the image of P^{cs} must be dense. Moreover, option (i) of Theorem 4.2.6 must hold for $\mathcal{S}_\varepsilon$ and this concludes the proof of this proposition.

$\square$

Remark 5.3.5. Assume that $f : \mathbb{T}^3 \to \mathbb{T}^3$ is a strongly partially hyperbolic diffeomorphism which is not isotopic to Anosov. By Theorem 5.1.4 and Corollary 5.1.10 we have that if f is not isotopic to Anosov, then f_* is in the hypothesis of Lemma 1.5.4. Let P be an f_*-invariant plane, then there are the following 3 possibilities:

- P may project into a torus. In this case, $P = E^s_* \oplus E^u_*$ (the eigenplane corresponding to the eigenvalues of modulus different from one).
- If $P = E^s_* \oplus E^c_*$ then P projects into an immersed cylinder which is dense in $\mathbb{T}^3$.
- If $P = E^c_* \oplus E^u_*$ then P projects into an immersed cylinder which is dense in $\mathbb{T}^3$.

$\diamondsuit$

5.3.2 Global product structure implies dynamical coherence

Assume that $f : \mathbb{T}^3 \to \mathbb{T}^3$ is a strong partially hyperbolic diffeomorphism. Let $\mathcal{F}^{cs}_{bran}$ be the f-invariant branching foliation tangent to $E^s \oplus E^c$ given by Theorem 5.1.9 and let $\mathcal{S}_\varepsilon$ be a foliation tangent to an ε-cone around $E^s \oplus E^c$ which remains ε-close to the lift of $\mathcal{F}^{cs}_{bran}$ to the universal cover for small ε.

When the lifts of $\mathcal{S}_\varepsilon$ and $\mathcal{F}^u$ to the universal cover have a global product structure, we deduce from Proposition 4.3.9 the following:

Corollary 5.3.6. *If $\mathcal{S}_\varepsilon$ and $\mathcal{F}^u$ have global product structure, then, the foliation $\tilde{\mathcal{F}}^u$ is quasi-isometric. Indeed, if $v \in (P^{cs})^\perp$ is a unit vector, there exists $\ell > 0$ such that for every $n \geq 0$, every unstable curve starting at a point x of length larger than $n\ell$ intersects $P^{cs} + nv + x$ or $P^{cs} - nv + x$.*

Before we show that global product structure implies coherence, we will show an equivalence to having global product structure between $\tilde{\mathcal{F}}^u$ and $\tilde{\mathcal{S}}_\varepsilon$ which will sometimes be better adapted to our proofs.

Lemma 5.3.7. *There exists $\varepsilon > 0$ such that $\tilde{\mathcal{F}}^u$ and $\tilde{\mathcal{S}}_\varepsilon$ have global product structure if and only if:*

- *For every $x, y \in \mathbb{R}^3$ and for every $L \in \tilde{\mathcal{F}}^{cs}_{bran}(y)$ we have that $\tilde{\mathcal{F}}^u(x) \cap L \neq \emptyset$.*

Proof. First notice that any of the hypothesis implies that $\tilde{\mathcal{S}}_\varepsilon$ cannot have dead-end components. In particular, there exists $R > 0$ and a plane P^{cs} such every leaf of $\tilde{\mathcal{S}}_\varepsilon$ and every leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$ verifies that it is contained in an R-neighborhood of a translate of P^{cs} and the R-neighborhood of the leafs contains a translate of P^{cs} too (see Proposition 5.3.4).

We prove the direct implication first. Consider $x, y \in \mathbb{R}^3$ and L a leaf of $\tilde{\mathcal{F}}^{cs}_{bran}(y)$. Now, we know that L separates in $\mathbb{R}^3$ the planes $P^{cs} + y + 2R$ and $P^{cs} + y - 2R$. One of them must be in the connected component of $\mathbb{R}^3 \setminus L$ which not contains x, without loss of generality we assume that it is $P^{cs} + y + 2R$. Now, we know that there is a leaf S of $\tilde{\mathcal{S}}_\varepsilon$ which is contained in the half space bounded by $P^{cs} + y + R$ not containing L (notice that L does not intersect $P^{cs} + y + R$). Global product product structure implies that $\tilde{\mathcal{F}}^u(x)$ intersects S and thus, it also intersects L.

The converse direction has an analogous proof.

□

We can prove the following result which does not make use of the isotopy class of f.

Proposition 5.3.8. *Assume that there is a global product structure between the lift of $\mathcal{S}_\varepsilon$ and the lift of $\mathcal{F}^u$ to the universal cover. Then there exists an f-invariant foliation $\mathcal{F}^{cs}$ everywhere tangent to $E^s \oplus E^u$.*

Proof. We will show that the branched foliation $\tilde{\mathcal{F}}^{cs}_{bran}$ must be a true foliation (it cannot be branched and use Proposition 5.1.7)).

Assume otherwise, i.e. there exists $x \in \mathbb{R}^3$ such that $\tilde{\mathcal{F}}^{cs}_{bran}(x)$ has more than one complete surface. We call L_1 and L_2 different leaves in $\tilde{\mathcal{F}}^{cs}_{bran}(x)$. There exists y such that $y \in L_1 \setminus L_2$. Using global product structure and Lemma 5.3.7 we get $z \in L_2$ such that:

- $y \in \tilde{\mathcal{F}}^u(z)$.

Consider γ the arc in $\tilde{\mathcal{F}}^u(z)$ whose endpoints are y and z. Let R be the value given by Proposition 5.3.4 and $\ell > 0$ given by Corollary 5.3.6. We consider N large enough so that $\tilde{f}^N(\gamma)$ has length larger than $n\ell$ with $n \gg R$.

By Corollary 5.3.6 we get that the distance between $P^{cs} + \tilde{f}^N(z)$ and $\tilde{f}^N(y)$ is much larger than R. However, we have that, by $\tilde{f}$-invariance of $\tilde{\mathcal{F}}^{cs}_{bran}$ there is a leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$ containing both $\tilde{f}^N(z)$ and $\tilde{f}^N(x)$ and another one containing both $\tilde{f}^N(y)$ and $\tilde{f}^N(x)$. This contradicts Proposition 5.3.4 showing that $\tilde{\mathcal{F}}^{cs}_{bran}$ must be a true foliation.

□

5.3.3 Torus leafs

This subsection is devoted to the proof of the following:

Lemma 5.3.9. *If $\mathcal{F}^{cs}_{bran}$ contains a leaf which is a two-dimensional torus, then there exists a leaf of $\mathcal{F}^{cs}_{bran}$ which is a torus and it is fixed by f^k for some $k \geq 1$. Moreover, this leaf is normally repelling.*

PROOF. Let $T \subset \mathbb{T}^3$ be a leaf of $\mathcal{F}^{cs}_{bran}$ homeomorphic to a two-torus. Since $\mathcal{F}^{cs}_{bran}$ is f-invariant and P^{cs} is invariant under f_* we get that the image of T by f is homotopic to T and a leaf of $\mathcal{F}^{cs}_{bran}$.

Notice that having an f_*-invariant plane which projects into a torus already implies that f_*-cannot be hyperbolic (see Proposition 1.5.1).

By Remark 5.3.5 we have that the plane P^{cs} coincides with $E^s_* \oplus E^u_*$ (the eigenspaces corresponding to the eigenvalues of modulus different from 1 of f_*).

Since the eigenvalue of f_* in E^c_* is of modulus 1, this implies that if we consider two different lifts of T, then they remain at bounded distance when iterated by $\tilde{f}$. Indeed, if we consider two different lifts $\tilde{T}_1$ and $\tilde{T}_2$ of T we have that $\tilde{T}_2 = \tilde{T}_1 + \gamma$ with $\gamma \in E^c_* \cap \mathbb{Z}^3$. Now, we have that $\tilde{f}(\tilde{T}_2) = \tilde{f}(\tilde{T}_1) + f_*(\gamma) = \tilde{f}(\tilde{T}_1) \pm \gamma$.

We shall separate the proof depending on how the orbit of T is.

Case 1: Assume the torus T is fixed by some iterate f^n of f with $n \geq 1$. Then, since it is tangent to the center stable distribution, we obtain that it must be normally repelling as desired.

Case 2: If the orbit of T is dense, we get that $\mathcal{F}^{cs}_{bran}$ is a true foliation by two-dimensional torus which we call $\mathcal{F}^{cs}$ from now on. This is obtained by the fact that one can extend the foliation to the closure using the fact that there are no topological crossings between the torus leaves (see Proposition 5.1.8).

Since all leaves must be two-dimensional torus which are homotopic we get that the foliation $\mathcal{F}^{cs}$ has no holonomy (see Theorem 4.1.6 and Proposition 5.1.11).

Using Theorem 4.3.1, we get that the unstable direction $\tilde{\mathcal{F}}^u$ in the universal cover must have a global product structure with $\tilde{\mathcal{F}}^{cs}$.

Let S be a leaf of $\mathcal{F}^{cs}$ and consider $\tilde{S}_1$ and $\tilde{S}_2$ two different lifts of S to $\mathbb{R}^3$.

Consider an arc J of $\tilde{\mathcal{F}}^u$ joining $\tilde{S}_1$ to $\tilde{S}_2$. Iterating the arc J by $\tilde{f}^n$ we get that its length grows exponentially, while the extremes remain the the forward iterates of $\tilde{S}_1$ and $\tilde{S}_2$ which remain at bounded distance by the argument above.

By considering translations of one end of $\tilde{f}^n(J)$ to a fundamental domain and taking a convergent subsequence we obtain a leaf of $\tilde{\mathcal{F}}^u$ which does not intersect every leaf of $\tilde{\mathcal{F}}^{cs}$. This contradicts global product structure.

Case 3: Let $T_1, T_2 \in \mathcal{F}^{cs}_{bran}$ two different torus leaves. Since there are no topological crossings, we can regard T_2 as embedded in $\mathbb{T}^2 \times [-1, 1]$ where both boundary components are identified with T_1 and such that the embedding is homotopic to the boundary components (recall that any pair of torus leaves must be homotopic). In particular, we get that $\mathbb{T}^3 \setminus (T_1 \cup T_2)$ has at least two different connected components and each of the components has its boundary contained in $T_1 \cup T_2$.

If the orbit of T is not dense, we consider $\mathcal{O} = \overline{\bigcup_n f^n(T)}$ the closure of the orbit of T which is an invariant set.

Recall that we can assume completeness of $\mathcal{F}^{cs}_{bran}$ (i.e. for every $x_n \to x$ and $L_n \in \mathcal{F}^{cs}_{bran}(x_n)$ we have that L_n converges in the C^1-topology to $L_\infty \in \mathcal{F}^{cs}_{bran}(x)$). We get that $\mathcal{O}$ is saturated by leaves of $\mathcal{F}^{cs}_{bran}$ all of which are homotopic torus leaves (see Proposition 5.1.11).

Let U be a connected component of the complement of $\mathcal{O}$. By the previous remarks we know that its boundary ∂U is contained in the union of two torus leaves of $\mathcal{F}^{cs}_{bran}$.

If some component U of $\mathcal{O}^c$ verifies that there exists $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$, by invariance of $\mathcal{O}^c$ we get that f^{2n} fixes both torus leaves whose union contains ∂U. This implies the existence of a periodic normally repelling torus as in Case 1.

We claim that if every connected component of $\mathcal{O}^c$ is wandering, then we can show that every leaf of $\tilde{\mathcal{F}}^u$ intersects every leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$ which allows to conclude exactly as in Case 2.

To prove the claim, consider δ given by the local product structure between these two transverse foliations (one of them branched). This means that given x, y such that $d(x, y) < \delta$ we have that $\tilde{\mathcal{F}}^u(x)$ intersects every leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$ passing through y.

Assume there is a point $x \in \mathbb{R}^3$ such that $\tilde{\mathcal{F}}^u(x)$ does not intersect every leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$. As in subsection 4.2.3 we know that each leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$ separates $\mathbb{R}^3$ into two connected components so we can choose among the lifts of torus leaves, the leaf $\tilde{T}_0$ which is the lowest (or highest depending on the orientation of the semi-unstable leaf of x not intersecting every leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$) not intersecting $\tilde{\mathcal{F}}^u(x)$. We claim that $\tilde{T}_0$ must project by the convering projection into a torus leaf which intersects the boundary of a connected component of $\mathcal{O}^c$. Indeed, there are only finitely many connected components $U_1, \dots, U_N$ of $\mathcal{O}^c$ having volume smaller than the volume of a δ-ball, so if a point is not in U_i for some i, we know that it must be covered by local product structure boxes forcing its unstable leaf to advance until one of those components.

On the other hand, using f-invariance of $\mathcal{F}^u$ and the fact that every connected component of $\mathcal{O}^c$ is wandering, we get that every point in U_i must eventually fall out of $\bigcup_i U_i$ and then its unstable manifold must advance to other component. This concludes the claim, and as we explained, allows to use the same argument as in Case 2 to finish the proof in Case 3.

$\square$

M.A. Rodriguez Hertz and R. Ures were kind to comunicate an alternative proof of this lemma by using an adaptation of an argument due to Haefliger for branched foliations (it should appear in [RHRHU$_3$]).

5.3.4 Obtaining Global Product Structure

In this section we will prove the following result which will allow us to conclude in the case where f_* is not isotopic to Anosov.

Proposition 5.3.10. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a strongly partially hyperbolic diffeomorphism which is not isotopic to Anosov and does not have a periodic two-dimensional torus tangent to $E^s \oplus E^c$. Then, the plane P^{cs} given by Proposition 5.3.4 corresponds to the eigenplane corresponding to the eigenvalues of modulus smaller or equal to 1. Moreover, there is a global product structure between $\tilde{\mathcal{F}}^{cs}_{bran}$ and $\tilde{\mathcal{F}}^u$. A symmetric statement holds for $\tilde{\mathcal{F}}^{cu}_{bran}$ and $\tilde{\mathcal{F}}^s$.*

As noted in Remark 5.3.5 we get that even if a strongly partially hyperbolic diffeomorphism is not isotopic to Anosov, then, f_* still must have one eigenvalue of modulus larger than one and one smaller than one.

The mentioned remark also gives that there are exactly three f_*-invariant lines E^s_*, E^c_* and E^u_* corresponding to the eigenvalues of f_* of modulus smaller, equal and larger than one respectively.

Lemma 5.3.11. *For every $R > 0$ and $x \in \mathbb{R}^3$ we have that $\tilde{\mathcal{F}}^u(x)$ is not contained in an R-neighborhood of $(E^s_* \oplus E^c_*) + x$. Symmetrically, for every $R > 0$ and $x \in \mathbb{R}^3$ the leaf $\tilde{\mathcal{F}}^s(x)$ is not contained in an R-neighborhood of $(E^c_* \oplus E^u_*) + x$.*

Proof. Let C be a connected set contained in an R-neighborhood of a translate of $E^s_* \oplus E^c_*$, we will estimate the diameter of $\tilde{f}(C)$ in terms of the diameter of C.

Claim. *There exists K_R which depends only on $\tilde{f}$, f_* and R such that:*

$$\operatorname{diam}(\tilde{f}(C)) \leq \operatorname{diam}(C) + K_R$$

Proof. Let K_0 be the C^0-distance between $\tilde{f}$ and f_* and consider $x, y \in C$ we get that:

$$d(\tilde{f}(x), \tilde{f}(y)) \leq d(f_*(x), f_*(y)) + d(f_*(x), \tilde{f}(x)) + d(f_*(y), \tilde{f}(y)) \leq$$
$$\leq d(f_*(x), f_*(y)) + 2K_0$$

We have that the difference between x and y in the unstable direction of f_* is bounded by $2R$ given by the distance to the plane $E^s_* \oplus E^u_*$ which is transverse to E^u_*.

Since the eigenvalues of f_* along $E^s_* \oplus E^c_*$ we have that f_* does not increase distances in this direction: we thus have that $d(f_*(x), f_*(y)) \leq d(x, y) + 2|\lambda^u|R$ where λ^u is the eigenvalue of modulus larger than 1. We have obtained:

$$d(\tilde{f}(x), \tilde{f}(y)) \leq d(x, y) + 2K_0 + 2|\lambda^u|R = d(x, y) + K_R$$

which concludes the proof of the claim.

$\diamondsuit$

Now, this implies that if we consider an arc γ of $\tilde{\mathcal{F}}^u$ of length 1 and assume that its future iterates remain in a slice parallel to $E^s_* \oplus E^c_*$ of width $2R$ we have that

$$\operatorname{diam}(\tilde{f}^n(\gamma)) < \operatorname{diam}(\gamma) + nK_R \leq 1 + nK_R$$

So that the diameter grows linearly with n.

The volume of balls in the universal cover of $\mathbb{T}^3$ grows polynomially with the radius (see Step 2 of [BBI$_1$] or page 545 of [BI], notice that the universal) so that we have that $B_\delta(\tilde{f}^{-n}(\gamma))$ has volume which is polynomial $P(n)$ in n.

On the other hand, we know from the partial hyperbolicity that there exists $C > 0$ and $\lambda > 1$ such that the length of $\tilde{f}^n(\gamma)$ is larger than $C\lambda^n$.

Using Corollary 4.2.2 (iv), we obtain that there exists n_0 uniform such that every arc of length 1 verifies that $\tilde{f}^{n_0}(\gamma)$ is not contained in the R-neighborhood of a translate of $E^s_* \oplus E^c_*$. This implies that no unstable leaf can be contained in the R-neighborhood of a translate of $E^s_* \oplus E^c_*$ concluding the proof of the lemma.

$\square$

We are now ready to prove Proposition 5.3.10

Proof of Proposition 5.3.10. Consider the plane P^{cs} given by Proposition 5.3.4 for the branching foliation $\mathcal{F}^{cs}_{bran}$.

If option (ii) of Proposition 5.3.4 holds, we get that there must be a torus leaf in $\mathcal{F}^{cs}_{bran}$ which we assume there is not.

By Lemma 1.5.4 and Remark 5.3.5 the plane P^{cs} must be either $E^s_* \oplus E^c_*$ or $E^c_* \oplus E^u_*$.

Lemma 5.3.11 implies that P^{cs} cannot be $E^c_* \oplus E^u_*$ since $\tilde{\mathcal{F}}^s$ is contained in $\tilde{\mathcal{F}}^{cs}_{bran}$. This implies that $P^{cs} = E^s_* \oplus E^c_*$ as desired.

Now, using Lemma 5.3.11 for $\tilde{\mathcal{F}}^u$ we see that the unstable foliation cannot remain close to a translate of P^{cs}. This implies that $\tilde{\mathcal{F}}^u$ intersects every translate of P^{cs} and since every leaf of $\tilde{\mathcal{S}}_\varepsilon$ is contained in between two translates of P^{cs} which are separated by the leaf, we deduce that every leaf of $\tilde{\mathcal{F}}^u$ intersects every leaf of $\tilde{\mathcal{S}}_\varepsilon$. Now, by Lemma 5.3.7 gives "global prooduct structure" between $\tilde{\mathcal{F}}^u$ and $\tilde{\mathcal{F}}^{cs}_{bran}$ and using Proposition 5.3.8.

□

5.3.5 Proof of Theorem 5.3.1

To prove Theorem 5.3.1, we first assume that f_* is not isotopic to Anosov.

If there is a torus tangent to $E^s \oplus E^c$, then, by Lemma 5.3.9 we obtain a periodic normally repelling torus.

By Proposition 5.3.10 we get that if there is no repelling torus, then there is a global product structure. Now, Proposition 5.3.8 gives the existence of an f-invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$ (see also Lemma 5.3.7).

The proof shows that there must be a unique f-invariant foliation tangent to E^{cs} (and to E^{cu}).

Indeed, we get that every foliation tangent to E^{cs} must verify option (i) of Proposition 5.3.4 when lifted to the universal cover and that the plane which is close to the foliation must correspond to the eigenspace of f_* corresponding to the smallest eigenvalues (Proposition 5.3.10).

Using quasi-isometry of the strong foliations, this implies that if there is another surface tangent to E^{cs} through a point x, then this surface will not extend to an f-invariant foliation since we get that forward iterates will get arbitrarily far from this plane (this is proved exactly as Proposition 5.3.8).

This concludes the proof of Theorem 5.3.1 in case f is not isotopic to Anosov, Theorem 5.3.3 concludes.

□

It may be that there are other (non-invariant) foliations tangent to E^{cs} (see [BFra]) or, even if there are no such foliations there may be complete surfaces tangent to E^{cs} which do not extend to foliations. The techniques here presented do not seem to be enough to discard such situations.

5.3.6 A simpler proof of Theorem 5.3.3. The isotopy class of Anosov.

In Section 5.2 Theorem 5.3.3 is obtained as a consequence of a more general result which is harder to prove. We present here a simpler proof of this result.

Proof of Theorem 5.3.3. Let $\mathcal{F}^{cs}_{bran}$ be the branched foliation tangent to E^{cs} given by Theorem 5.1.9. By Proposition 5.3.4 we get a f_*-invariant plane P^{cs} in $\mathbb{R}^3$ which we know cannot project into a two-dimensional torus since f_* has no invariant planes projecting into a torus (see Remark 1.5.3), this implies that option (i) of Proposition 5.3.4 is verified.

Since for every $\varepsilon > 0$, Theorem 5.1.9 gives us a foliation $\mathcal{S}_\varepsilon$ whose lift is close to $\tilde{\mathcal{F}}^{cs}$, we get that the foliation $\tilde{\mathcal{S}}_\varepsilon$ remains close to P^{cs} which must be totally irrational (see Remark 1.5.3). By Lemma 4.2.10 (i) we get that all leaves of $\mathcal{S}_\varepsilon$ are simply connected, thus, we get that the foliation $\mathcal{S}_\varepsilon$ is without holonomy.

We can apply Theorem 4.3.1 and we obtain that for every $\varepsilon > 0$ there is a global product structure between $\tilde{\mathcal{S}}_\varepsilon$ and $\tilde{\mathcal{F}}^u$ which is transverse to $\mathcal{S}_\varepsilon$ if ε is small enough.

The rest of the proof follows from Proposition. 5.3.8.

□

In fact, using the same argument as in Proposition 5.2.14 we get uniqueness of the foliation tangent to $E^s \oplus E^c$.

We are also able to prove the following proposition which is similar to Proposition 5.3.10 in the context of partially hyperbolic diffeomorphisms isotopic to Anosov, this will be used in [HP] to obtain leaf conjugacy to the linear model.

Notice first that the eigenvalues of f_* verify that they are all different (see Lemma 1.5.2 and Proposition 5.2.12).

We shall name them $\lambda_1, \lambda_2, \lambda_3$ and assume they verify:

$$|\lambda_1| < |\lambda_2| < |\lambda_3| \quad ; \quad |\lambda_1| < 1 \ , \ |\lambda_2| \neq 1 \ , \ |\lambda_3| > 1$$

we shall denote as E^i_* to the eigenline of f_* corresponding to λ_i.

Proposition 5.3.12. *The plane close to the branched foliation $\tilde{\mathcal{F}}^{cs}$ corresponds to the eigenplane corresponding to the eigenvalues of smaller modulus (i.e. the eigenspace $E^1_* \oplus E^2_*$ corresponding to λ_1 and λ_2). Moreover, there is a global product structure between $\tilde{\mathcal{F}}^{cs}$ and $\tilde{\mathcal{F}}^u$. A symmetric statement holds for $\tilde{\mathcal{F}}^{cu}$ and $\tilde{\mathcal{F}}^s$.*

Proof. This proposition follows from the existence of a semiconjugacy H between $\tilde{f}$ and its linear part f_* which is at bounded distance from the identity.

The existence of a global product structure was proven above. Assume first that $|\lambda_2| < 1$, in this case, we know that $\tilde{\mathcal{F}}^u$ is sent by the semiconjugacy into lines parallel to the eigenspace of λ_3 for f_*. This readily implies that P^{cs} must coincide with the eigenspace of f_* corresponding to λ_1 and λ_2 otherwise we would contradict the global product structure.

The case were $|\lambda_2| > 1$ is more difficult. First, it is not hard to show that the eigenspace corresponding to λ_1 must be contained in P^{cs} (otherwise we can repeat the argument in Lemma 5.3.11 to reach a contradiction).

Assume by contradiction that P^{cs} is the eigenspace corresponding to λ_1 and λ_3.

First, notice that by the basic properties of the semiconjugacy H, for every $x \in \mathbb{R}^3$ we have that $\tilde{\mathcal{F}}^u(x)$ is sent by H into $E^u_* + H(x)$ (where $E^u_* = E^2_* \oplus E^3_*$ is the eigenspace corresponding to λ_2 and λ_3 of f_*).

We claim that this implies that in fact $H(\tilde{\mathcal{F}}^u(x)) = E^2_* + H(x)$ for every $x \in \mathbb{R}^3$. In fact, we know from Corollary 5.2.10 that points of $H(\tilde{\mathcal{F}}(x))$ which are sufficiently far apart are contained in a cone of $(E^2_* \oplus E^3_*) + H(x)$ bounded by two lines L_1 and L_2 which are transverse to P^{cs}. If P^{cs} contains E^3_* this implies that if one considers points in the same unstable leaf which are sufficiently far apart, then their image by H makes an angle with E^3_* which is uniformly bounded from below. If there is a point $y \in \tilde{\mathcal{F}}^u(x)$ such that $H(y)$ not contained in E^2_* then we have that $d(\tilde{f}^n(y), \tilde{f}^n(x))$ goes to ∞ with n while the angle of $H(y) - H(x)$ with E^3_* converges to 0 exponentially contradicting Corollary 5.2.10.

Consider now a point $x \in \mathbb{R}^3$ and let y be a point which can be joined to x by a finite set of segments $\gamma_1, \ldots, \gamma_k$ tangent either to E^s or to E^u (an *su*-path, see subsection 1.4.4). We know that each γ_i verifies that $H(\gamma_i)$ is contained either in a translate of E^1_* (when γ_i is tangent to E^s, i.e. it is an arc of the strong stable foliation $\tilde{\mathcal{F}}^s$) or in a translate of E^2_* (when γ_i is tangent to E^u from what we have shown in the previous paragraph). This implies that the *accesibility* class of x verifies that its image by H is contained in $(E^1_* \oplus E^2_*) + H(x)$. The projection of $E^1_* \oplus E^2_*$ to the torus is not the whole $\mathbb{T}^3$ so in particular, we get that f cannot be accesible. From Corollary 5.2.15 this situation should be robust under C^1-perturbations since those perturbations cannot change the direction of P^{cs}.

On the other hand, Theorem 1.4.7 implies that by an arbitrarily small (C^1 or C^r) perturbation of f one can make it accessible. This gives a contradiction and shows that P^{cs} must coincide with $E^1_* \oplus E^2_*$ as desired.

□

5.4 Higher dimensions

In this section we attempt to find conditions that guarantee a partially hyperbolic diffeomorphism to be isotopic to an Anosov diffeomorphism. The progress made so far in this direction is not as strong though we have obtained some partial results. We present here part of what will appear in [Pot$_6$].

We start by giving the property we will require for a partially hyperbolic diffeomorphism and hope it implies being isotopic to an Anosov diffeomorphism.

Definition 5.4.1 (Coherent trapping property)**.** Let $f \in \mathrm{Diff}^1(M)$ be a dynamically coherent partially hyperbolic diffeomorphism of type $TM = E^{cs} \oplus E^u$. We shall say that it admits the *coherent trapping property*[6] if there exists a continuous map

[6]The word coherent (motivated by the existence of an invariant $cs-$foliation) is included to distinguish it from the a priori weaker condition of only having a plaque family trapped by f (it could be that this condition alone implies coherence, see [BuFi] for progress in that direction).

$\mathcal{D}^{cs} : M \to Emb^1(\mathbb{D}^{cs}, M)$ such that $\mathcal{D}^{cs}(x)(0) = x$, the image of $\mathbb{D}^{cs}$ by $\mathcal{D}^{cs}(x)$ is always contained in $\mathcal{F}^{cs}(x)$ and they verify the following trapping property:

$$f(\mathcal{D}^{cs}(x)(\mathbb{D}^{cs})) \subset \mathcal{D}^{cs}(f(x))(int(\mathbb{D}^{cs})) \qquad \forall x \in M.$$

$\diamondsuit$

For notational purposes, and with the risk of abusing notation, we shall denote from now on: $\overline{\mathcal{D}^{cs}_x} = \mathcal{D}^{cs}(x)(\mathbb{D}^{cs})$ and $\mathcal{D}^{cs}_x = \mathcal{D}^{cs}(x)(int(\mathbb{D}^{cs}))$.

We remark the important point that there is no restriction on the size of the plaques $\mathcal{D}^{cs}_x$ so the dynamics can be quite rich in the center stable plaques.

We will prove the following:

Theorem 5.4.1. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism with splitting $TM = E^{cs} \oplus E^u$ having the coherent trapping property and such that one of the following conditions holds:*

- $\dim E^u = 1$ *or*

- $M = \mathbb{T}^d$.

Then, $M = \mathbb{T}^d$ and f is isotopic to a linear Anosov automorphism with stable dimension equal to $\dim E^{cs}$.

In view of Franks-Manning theory [F_1, Man] one can expect that this result also holds for nilmanifolds.

Notice that in general, obtaining a classification result for partially hyperbolic diffeomorphisms with the coherent trapping property should be at least as difficult as having a classification result for Anosov diffeomorphisms since the latter are indeed partially hyperbolic with the coherent trapping property.

The fact that f is dynamically coherent seems to be a strong hypothesis, more in view of the robustness of the conclusion of Theorem 5.4.1. However, we have not been able to remove the hypothesis from our assumptions unless some strong properties are verified.

Along this section we shall assume that $f \in \mathrm{Diff}^1(M)$ is a partially hyperbolic diffeomorphism with the coherent trapping property. Also, we shall call $cs = \dim E^{cs}$ and $u = d - \dim E^{cs} = \dim E^u$.

5.4.1 An expansive quotient of the dynamics

We can define for each $x \in M$

$$A_x = \bigcap_{n \geq 0} f^n(\overline{\mathcal{D}^{cs}_{f^{-n}(x)}})$$

Some obvious properties satisfied by the sets A_x are:

- $f(A_x) = A_{f(x)}$ for every $x \in M$.
- The set A_x is a decreasing union of topological balls (it is a cellular set), so, compact and connected in particular.

We would like to prove that the sets A_x constitute a partition of M and that they vary semicontinuously, so that we can quotient the dynamics. For this, the following lemma is of great use:

Lemma 5.4.2. *For every $y \in \mathcal{F}^{cs}(x)$, there exists n_y such that $f^{n_y}(\overline{\mathcal{D}^{cs}_y}) \subset \mathcal{D}^{cs}_{f^{n_y}(x)}$. The number n_y varies semicontinuously on the point, that is, there exists U a small neighborhood of y such that for every $z \in U$ we have that $n_z \leq n_y$.*

Proof. Consider in $\mathcal{F}^{cs}(x)$ the sets

$$E_n = \{y \in \mathcal{F}^{cs}(x) \ : \ f^n(\overline{\mathcal{D}^{cs}_y}) \subset \mathcal{D}^{cs}_{f^n(x)}\}$$

Notice that there exists $\delta > 0$ (independent of n) such that if $y \in E_n$, then $B_\delta(y) \cap \mathcal{F}^{cs}(x) \subset E_n$. This is given by continuity of f and of the plaque family (using compactness of M) and by the coherent trapping property.

The sets E_n are thus clearly open and verify that $E_n \subset E_{n+1}$ (this is implied by the coherent trapping property).

Now, by the uniform estimate, it is not hard to show that $\bigcup_{n\geq 0} E_n$ is closed, so, since it is not empty, it must be the whole $\mathcal{F}^{cs}(x)$ as claimed.

The fact that the numbers n_y varies semicontinuously is a consequence of the fact that E_n is open (n_y is the first integer such that $y \in E_n$).

$\square$

Corollary 5.4.3. *For $x, y \in M$ we have that $A_x = A_y$ or $A_x \cap A_y = \emptyset$. Moreover, the classes vary semicontinuously, that is, given $x_n \in M$ such that $\lim x_n = x$:*

$$\limsup A_{x_n} = \bigcap_{k>0} \overline{\bigcup_{n>k} A_{x_n}} \subset A_x$$

.

Proof. There exists n_0 fixed such that for every $x \in M$ and $y \in f(\mathcal{D}^{cs}_{f^{-1}(x)})$ we have that $f^{n_0}(\mathcal{D}^{cs}_y) \subset \mathcal{D}^{cs}_{f^{n_0}(x)}$. This is proved first by showing that n_x exists for each $x \in M$ (using the Lemma 5.4.2 and compactness of $f(\overline{\mathcal{D}^{cs}_{f^{-1}(x)}})$) and then, since the numbers n_x vary semicontinuously, the uniform bound n_0 is found.

We know that for every z such that $z \in A_x$ we have that $A_z \in \mathcal{D}^{cs}_x$: Indeed, since $z \in A_x$ we have that $f^{-n_0}(z) \in \mathcal{D}^{cs}_{f^{-n_0}(x)}$ and thus $f^{n_0}(\mathcal{D}^{cs}_{f^{-n_0}(z)}) \subset \mathcal{D}^{cs}_x$ as desired. In fact, this shows that if $z \in A_x$ then $A_z \subset A_x$. In particular, by symmetry, we get that if $A_x \cap A_y \neq \emptyset$, we can find a point z in the intersection and we have that $A_z = A_x$ and $A_z = A_y$ giving the desired statement.

To prove semicontinuity: Consider A_x and an ε neighborhood $A_x(\varepsilon)$ in $\mathcal{F}^{cs}(x)$. Now, fix m such that $f^m(\mathcal{D}^{cs}_{f^{-m}(x)}) \subset A_x(\varepsilon)$. Now, for n large enough, x_n verifies that $d(f^{-m}(x_n), f^{-m}(x))$ is so small that $f^m(\mathcal{D}^{cs}_{f^{-m}(x_n)}) \subset A_x(\varepsilon)$ as wanted. The semicontinuity for points which are not in the same center-stable manifold follows from Lemma 5.4.5 bellow.

□

We get thus a continuous projection by considering the equivalence relation $x \sim y \Leftrightarrow y \in A_x$.

$$\pi : M \to M/_\sim$$

We denote as $g : M/_\sim \to M/_\sim$ the map given by $g([x]) = [f(x)]$ (that is $g \circ \pi = \pi \circ f$). Since π is continuous and surjective (in fact, it is cellular), it is a semiconjugacy. Notice that since f is a diffeomorphism, and g preserves the equivalence classes, one can show that g must be a homeomorphism of $M/_\sim$.

Notice that a priori, we have no knowledge of the topology of $M/_\sim$ except that it is the image by a cellular map of a manifold (see Section 2.1), for example, we do not know a priori if the dimension of $M/_\sim$ is finite. However, in view of Proposition 2.1.2 we know that this quotient is a metric space.

We will prove that it has finite topological dimension dynamically after we prove Theorem 5.4.4 (combined with [M$_2$]).

We say that a homeomorphism has *local product structure* if there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $S_\varepsilon(x) \cap U_\varepsilon(y) \neq \emptyset$ (see Section 1.1).

Recall that a homeomorphism h is *expansive* (with expansivity constant α) if for every $x \in X$ we have that $S_\alpha(x) \cap U_\alpha(x) = \{x\}$.

It is well known that for expansive homeomorphisms we have that $\text{diam}(h^n(S_\varepsilon(x))) \to 0$ uniformly on x for $\varepsilon < \alpha$ (so this coincides with the usual definitions of stable and unstable sets). This implies that $S_\varepsilon(x) \subset W^s(x)$ for an expansive homeomorphism ($\varepsilon < \alpha$).

Theorem 5.4.4. *The homeomorphism g is expansive with local product structure. Moreover, $\pi(\mathcal{F}^{cs}(x)) = W^s(\pi(x))$ and π is injective when restricted to the unstable manifold of any point.*

Proof. The last two claims are direct from Lemma 5.4.2 and the definition of the equivalence classes respectively.

We choose $\varepsilon > 0$ such that:

- $x, y \in M$ and $x \notin f^{-1}(\mathcal{D}^{cs}_{f(y)})$ then there exists $n \geq 0$ such that $d(f^n(x), \overline{\mathcal{D}^{cs}_{f^n(y)}}) > \varepsilon$.

Now, let x, y be two points such that $d(f^n(x), f^n(y)) \leq \varepsilon$ for every $n \in \mathbb{Z}$. From how we choose ε, we have that $f^{-k}(x) \in \mathcal{D}^{cs}_{f^{-k}(y)}$ for every $k \geq 0$ so, $x \in A_y$ as desired.

Since $\pi(\mathcal{F}^{cs}(x)) = W^s(\pi(x))$, and since $d(x, y) < \delta$ implies that $\mathcal{D}^{cs}_x \pitchfork W^{uu}_{loc}(y) \neq \emptyset$ we get that for every two close points, there is a non trivial intersection between the local stable and unstable sets (here, we are using the upper semicontinuity of the sets A_x which imply that if there are two points $\tilde{x}, \tilde{y}$ in $M/_\sim$ which are near, there are points in $\pi^{-1}(\tilde{x})$ and $\pi^{-1}(\tilde{y})$ which are near).

$\square$

Consider two points x, y such that $y \in W^{uu}(x)$. We denote $\Pi^{uu}_{x,y} : \mathcal{D} \subset \mathcal{D}^{cs}_x \to \mathcal{D}^{cs}_y$ as the unstable holonomy from a subset of $\mathcal{D}^{cs}_x$ into a subset of $\mathcal{D}^{cs}_y$. An important useful property is the following:

Lemma 5.4.5. *We have that* $\Pi^{uu}_{x,y}(A_x) = A_y$.

Proof. It is enough to show (by the symmetry of the problem) that $\Pi^{uu}(A_x) \subset A_y$. For n large enough we have that $f^{-n}(\Pi^{uu}(A_x))$ is very close to a compact subset of $\mathcal{D}^{cs}_{f^{-n}(x)}$ and thus, by continuity of $\mathcal{D}^{cs}$ we have that $f^{-n}(\Pi^{uu}(A_x)) \subset \mathcal{D}^{cs}_{f^{-n}(y)}$ which concludes.

$\square$

Some remarks on the topology of the quotient

We shall cite some results from [Da] which help to understand the topology of $M/_\sim$. We refer to the reader to that book for much more information and precise definitions.

Before, we remark that Mañe proved that a compact metric space admitting an expansive homeomorphism must have finite topological dimension ([M_2]).

Corollary IV.20.3A of [Da] implies that, since $M/_\sim$ is finite dimensional, we have that it is a locally compact ANR (i.e. absolute neighborhood retract). In particular, we get that $\dim(M/_\sim) \leq \dim M$ (see Theorem III.17.7). Then, by using Proposition VI.26.1 (or Corollary VI.26.1A) we get that $M/_\sim$ is a $d-$dimensional homology manifold (since it is an ANR, it is a *generalized manifold*). More properties of these spaces can be found in section VI.26 of [Da].

Also, in the cited book, one can find a statement of Moore's theorem (see section IV.25 of [Da]) which states that a cellular decomposition of a surface is approximated by homeomorphisms (this means that the continuous projection is approximated by homeomorphisms in the C^0-topology). In particular, in our case, if $\dim E^{cs} = 2$, we get that $M/_\sim$ is a manifold (see also Theorem VI.31.5 of [Da] and its Corolaries).

Some other results are available, in particular, we notice Edward's cell-like decomposition theorem which asserts that if $\sim$ is a cellular decomposition of a d dimensional manifold ($d \geq 5$) such that $M/_\sim$ has finite topological dimension and such that it has the *disjoint disk property* (see chapter IV.24 of [Da]) then the quotient map is approximated by homeomorphisms. A similar result exists for dimension 3 which is even more technical. Notice than in our case, since we have the decomposition of the center-stable manifold, we can play with the dimensions in order not to be never in dimension 4 by choosing to work with the decomposition on the center stables or the whole manifold.

Also, we remark that it is known that when multiplying a decomposition by $\mathbb{R}^2$ we always get the *disjoint disc property* and in all known decompositions, after multiplying by $\mathbb{R}$ we get a decomposition approximated by homeomorphisms (see section V.26 of [Da]) so in theory, it should be true that always our space $M/_\sim$ is a manifold homeomorphic to M. We show this in the case $M = \mathbb{T}^d$ (the proof should be adaptable for infranilmanifolds).

5.4.2 Transitivity of the expansive homeomorphism

In general, it is not yet known if an Anosov diffeomorphism must be transitive. So, since Anosov diffeomorphisms enter in our hypothesis, there is no hope of knowing if f or g will be transitive without solving this long-standing conjecture. We shall then work with similar hypothesis to the well known facts for Anosov diffeomorphisms, showing that those hypothesis that we know guaranty that Anosov diffeomorphisms are transitive imply transitivity of g.

In particular, we shall prove in this section the following Theorem which implies Theorem 5.4.1:

Theorem 5.4.6. *The following properties hold:*

(T1) *If for every $x, y \in M$ we have that $\mathcal{F}^{uu}(x) \cap \mathcal{D}^{cs}(y) \neq \emptyset$, then g is transitive.*

(T2) *If $\dim E^u = 1$, then g is transitive. Moreover, $M = \mathbb{T}^d$.*

(T3) *If $M = \mathbb{T}^d$, then g is transitive. Moreover, f is homotopic to a linear Anosov diffeomorphism A the topological space $M/_\sim$ is homeomorphic to $\mathbb{T}^d$ and g is conjugated to A.*

Notice that (T1) is trivial, (T2) can be compared to Franks-Newhouse theory ([F_1, New_2]) and (T3) to Franks-Manning theory ([F_2, Man] see also [KH] chapter

18.6 which motivated the proof here presented). It is natural to expect that property (T3) should hold if we consider M an infranilmanifold.

It is important to notice also that it is natural to extend the conjecture about transitivity of Anosov diffeomorphisms to expansive homeomorphisms with local product structure (at least in manifolds). See the results in [Vie, ABP, Hir].

Proof of (T2)

We shall follow the argument of [New$_2$].

From how we defined g, we get that for every $y \in M$ we have that $\pi|_{W^u_{loc}(y)}$ is a homeomorphism over its image and thus, we get that every point in $M/_\sim$ has a one dimensional immersed copy of $\mathbb{R}$ as unstable set.

Also, we have that $g : M/_\sim \to M/_\sim$ is expansive with local product structure. This implies that there is a spectral decomposition for g:

Lemma 5.4.7. *The homeomorphism g has a spectral decomposition.*

PROOF. The proof is exactly as the one for Anosov or Axiom A diffeomorphisms.

It is not hard to show that $Per(g)$ is dense in $\Omega(g)$ with essentially the same proof as in the Anosov case. Let x be a nonwandering point of g, so, every neighborhood of $\pi^{-1}(x)$ has points which return to the neighborhood in arbitrarily large backward iterates. The fact that center stable leaves are invariant and unstable manifolds expand by iterating backwards, gives the existence of a fixed center stable leaf with a point returning near itself. Since the center stable disks are trapped, we obtain a fixed fiber for some iterate, this gives a periodic point for g which is arbitrarily close to x.

The rest of the spectral decomposition, is done by defining homoclinic classes and that needs no more that the local product structure of uniform size (see [New$_3$]). $\square$

By Conley's theory (see Remark 1.1.16), we get a repeller Λ for g which will be saturated by stable sets.

We shall show that $\Lambda = M/_\sim$ which concludes.

To do this, it is enough to show that for every $y \in \Lambda$, we have that y is accumulated by the intersections of both connected components of $W^u(y) \setminus \{y\}$ with Λ.

We can assume that W^u is orientable and g preserves orientation of W^u (otherwise, we take a double cover and g^2, transitivity at this level is even more general than if we do not take the cover nor the iterate). So, for every $y \in \Lambda$ we denote $W^u_+(y)$ and $W^u_-(y)$ the connected components of $W^u(y) \setminus \{y\}$ depending on the orientation.

We define the set

$$A^+ = \{y \in \Lambda \ : \ W^u_+(y) \cap \Lambda \neq \emptyset\}$$

which is an invariant set (we define A^- similarly). It is enough to show that $A^+ = \Lambda$: Indeed, this implies, by compactness that every point intersects Λ in a bounded length of W^u_+. This is enough since being invariant, the length must be zero.

Lemma 5.4.8. *Any point which is not periodic by g belongs to A^+. In fact, there are at most finitely many points not in A^+.*

Proof. The past orbit of every point contains an accumulation point and they are pairwise not in the same local stable set, thus, there is one point in the past orbit such that both components of its unstable set intersect the stable set of the other, and thus Λ, invariance concludes.

The fact that there exists $N > 0$ such that any set with cardinal larger than N has 3 points in a local product structure box, implies that if a point x does not belong to A^+ then its orbit $\mathcal{O}(x)$ must have cardinal smaller than N. This gives that there are at most finitely many points outside A^+ (which must be periodic).

□

To prove that periodic points are in A^+ too, we assume that it is not the case and consider $p \in \Lambda$ such that $p \notin A^+$. We have that $W^s(p)\backslash\{p\}$ is connected [7].

This implies that if $\phi : W^s(p)\backslash\{p\} \to \Lambda$ is the function which sends every point $y \in W^s(p)\backslash\{p\}$ to the first intersection of $W^u_+(y)$ with Λ (the first point of intersection exists since otherwise we would get that $p \in A^+$), then we have that the image of $W^s(p)\backslash\{p\}$ is a unique stable set, say of a point z.

Now, we must show that in fact, we have that $W^u(p)$ must intersect $W^s(z)$ which will be a contradiction and conclude. So, consider in $\pi^{-1}(W^s_{loc}(p)) \subset \mathcal{D}^{cs}_{\tilde{p}}$ (where $\pi(\tilde{p}) = p$) a (small) sphere Σ around $\pi^{-1}(p)$. That is, we assume that $\pi(\Sigma) \subset W^s_{loc}(p)$ (which we can since π is a cellular map).

Now, we consider $y \in \Sigma$ and $I \subset W^u_+(y)$ the interval of the unstable manifold of y from y to the only point in $W^u(y) \cap \pi^{-1}(\phi(\pi(y)))$. This interval can be parametrized in $[0,1]$. We shall call y_t to the point corresponding to $t \in [0,1]$.

We consider the set of points $s \in [0,1]$ such that $W^u_+(p) \cap W^s_{loc}(\pi(y_t)) \neq \emptyset$ which is open by the local product structure. We must show that the supremum of this set, say t_0 belongs to the set and we shall conclude.

Let Υ be the set homeomorphic to $\Sigma \times [0,1]$ given by the map $Y : (x,t) \mapsto I_x \cap \pi^{-1}(\pi(I_x) \cap W^s(\pi(y_t)))$ where I_x is the interval in the unstable manifold of x such that $\pi(I_x) \subset W^u_+(\pi(x))$ and connects $\pi(x)$ with $\phi(\pi(x))$.

[7] We are assuming that $\dim E^{cs} \geq 2$, since if we remove $\pi^{-1}(p)$ to $\mathcal{F}^{cs}(\pi^{-1}(p))$ it remains connected, the claim follows.

We get that $Y(\Sigma \times \{t_0\})$ is homeomorphic to a sphere, and thus it separates $\mathcal{F}^{cs}(y_{t_0})$ in two connected components, one of which is bounded (and thus compact). The image K of this compact component by π is also compact and contained in $W^s(\pi(y_{t_0}))$ and thus we get that there is local product structure well defined around K. This implies that $W^u_+(p) \cap W^s(\pi(y_{t_0})) \neq \emptyset$ and we have shown that t_0 is in the set considered above. This shows that $A^+ = \Lambda$ which as we have already mentioned gives transitivity of f.

Now, following the same proof as in (for example) the appendix of [ABP], we get that $M = \mathbb{T}^d$.

$\square$

Proof of (T3)

We shall follow the proof given in [KH] chapter 18.6.

Before we start with the proof, we shall recall Theorem 18.5.5 of [KH] (the statement is modified in order to fit our needs, notice that for an expansive homeomorphism with local product structure, we have the shadowing property, and thus, specification in each basic piece):

Proposition 5.4.9 (Theorem 18.5.5 of [KH])**.** *Let X a compact metric space and $g : X \to X$ an expansive homeomorphism with local product structure. Then, there exists $h, c_1, c_2 > 0$ such that for $n \in \mathbb{N}$ we have:*

$$c_1 e^{nh} \leq P_n(g) \leq c_2 e^{nh}$$

where $P_n(g)$ is the number of fixed points of g^n.

We shall use several time the very well know Lefschetz formula which relates the homotopy type of a continuous function, with the index of its fixed points (see [F_4] Chapter 5).

Definition 5.4.2. Let $V \subset \mathbb{R}^k$ be an open set, and $F : V \subset \mathbb{R}^k \to \mathbb{R}^k$ a continuous map such that $\Gamma \subset V$ the set of fixed points of F is a compact set, then, $I_\Gamma(F) \in \mathbb{Z}$ (the index of F) is defined to be the image by $(id - F)_* : H_k(V, V - \Gamma) \to H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\})$ of u_Γ where u_Γ is the image of 1 under the composite $H_k(\mathbb{R}^k, \mathbb{R}^k - D) \to H_k(\mathbb{R}^k, \mathbb{R}^k - \Gamma) \cong H_k(V, V - \Gamma)$ where D is a disk containing Γ.

$\diamondsuit$

Remark 5.4.10. In general, if we have a map from a manifold, we can include the manifold in $\mathbb{R}^k$ and extend the map in order to be in the hypothesis of the definition. The value of $I_\Gamma(F)$ does not depend on how we embed the manifold in $\mathbb{R}^k$.

$\diamondsuit$

For hyperbolic fixed points, it is very easy how to compute the index, it is exactly the sign of $\det(Id - D_p f)$. Since the definition is topological, any time we have a set which behaves locally as a hyperbolic fixed point, it is not hard to see that the index is the same.

Lefshetz fixed point formula for the torus can be stated as follows:

Theorem 5.4.11 (Lefshetz fixed point formula ([F_4] p.34-38))**.** *Let $h : \mathbb{T}^d \to \mathbb{T}^d$ be an homeomorphism, so, the sum of the Lefshetz index along a covering of* $\mathrm{Fix}(h)$ *by sets homeomorphic to balls equals $det(Id - h_*)$ where $h_* : H_1(\mathbb{T}^d, \mathbb{Z}) \to H_1(\mathbb{T}^d, \mathbb{Z})$ is the action of h in homology.*

The first thing we must show, is that the *linear* part of f, that is, the action $A = f_* : H_1(\mathbb{T}^d, \mathbb{Z}) \to H_1(\mathbb{T}^d, \mathbb{Z}) \in SL(d, \mathbb{Z})$ is a hyperbolic matrix.

Lemma 5.4.12. *The matrix A is hyperbolic.*

Proof. We can assume (maybe after considering a double covering and f^2) that E^{cs} and E^u are orientable and its orientations preserved by Df. So, it is not hard to show that for every fixed point p of g^n, the index of $\pi^{-1}(p)$ for f is of modulus one and always of the same sign.

So, we know from the Lefshetz formula that

$$|\det(Id - A^n)| = \sum_{g^n(p)=p} |I_{\pi^{-1}(p)}(f)| = \#\,\mathrm{Fix}(g^n).$$

Proposition 5.4.9 and an easy estimate on the growth of $|\det(Id - A^n)| = \prod_{i=1}^d |1 - \lambda_i^n|$ where $\{\lambda_1, \dots, \lambda_d\}$ are the eigenvalues of A gives that A cannot have eigenvalues of modulus 1 and thus A must be hyperbolic (see the argument in Lemma 18.6.2 of [KH]).

$\square$

Proposition 2.3.1 gives the existence of a semiconjugacy $h : \mathbb{T}^d \to \mathbb{T}^d$ isotopic to the identity such that $h \circ f = A \circ h$. Its lift $H : \mathbb{R}^d \to \mathbb{R}^d$ is given by shadowing, in particular, the iterations of the set $H^{-1}(x)$ remain of bounded diameter.

Lemma 5.4.13. *We have that g factors as an intermediate semiconjugacy. More precisely, there exists $\tilde{h} : \mathbb{T}^d/_\sim \to \mathbb{T}^d$ continuous and surjective such that $\tilde{h} \circ \pi = h$.*

Proof. It is enough to show that for every $x \in \mathbb{T}^d/_\sim$ there exists $y \in \mathbb{T}^d$ such that $\pi^{-1}(x) \subset h^{-1}(y)$.

For this, notice that any lifting of $\pi^{-1}(x)$ (that is, a connected component of the preimage under the covering map) to the universal covering $\mathbb{R}^d$ verifies that its iterates remain of bounded size. This concludes by the remark above on H.

□

Now, we shall prove that if $\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ is any lift of f, then there is exactly one fixed fiber of π for $\tilde{f}$.

Lemma 5.4.14. *Let $\tilde{f^n}$ be any lift of f^n to $\mathbb{R}^d$. So, there is exactly one fixed fiber of π.*

PROOF. Since $\tilde{f^n}$ is homotopic to A^n which has exactly one fixed point and each fixed fiber of π contributes the same amount to the index of $\tilde{f^n}$ it must have exactly one fixed fiber.

□

This allows us to show that g is transitive:

Proposition 5.4.15. *The homeomorphism g is transitive.*

PROOF. First, we show that there exists a basic piece of g which projects by $\tilde{h}$ to the whole $\mathbb{T}^d$.

This is easy since otherwise, there would be a periodic point q in $\mathbb{T}^d \setminus \tilde{h}(\Omega(g))$ but clearly, the $g-$orbit of $\tilde{h}^{-1}(q)$ must contain non-wandering points (it is compact and invariant).

This concludes, since considering a point y with dense A-orbit and a point in $\Omega(g) \cap \tilde{h}^{-1}(y)$ we get the desired basic piece.

Now, let Λ be the basic piece of g such that $\tilde{h}(\Lambda) = \mathbb{T}^d$. Assume that there exists $\tilde{\Lambda} \neq \Lambda$ a different basic piece and z a periodic point of $\tilde{\Lambda}$, naturally, we get that $\tilde{h}^{-1}(\tilde{h}(z))$ contains also a periodic point z' in Λ. By considering an iterate, we can assume that z and z' are fixed by g.

So, we get that it is possible to lift $h^{-1}(\tilde{h}(z))$ and chose a lift of f^k which fixes $\pi^{-1}(z)$ and $\pi^{-1}(z')$ contradicting the previous lemma.

□

With this in hand, we will continue to prove that the fibers of h coincide with those of π proving that g is conjugated to A (in particular, $\mathbb{T}^d/_\sim \cong \mathbb{T}^d$).

First, we show a global product structure for the lift of f. Notice that when we lift f to $\mathbb{R}^d$, we can also lift its center-stable and unstable foliation. It is clear that both foliations in $\mathbb{R}^d$ are composed by leaves homeomorphic to $\mathbb{R}^{cs}$ and $\mathbb{R}^u$ respectively (the unstable one is direct, the other is an increasing union of balls, so the same holds).

Lemma 5.4.16. *Given $x, y \in \mathbb{R}^d$, the center stable leaf of x intersects the unstable leaf of y in exactly one point.*

Proof. The fact that they intersect in at most one point is given by the fact that otherwise, we could find a horseshoe for the lift, and thus many periodic points contradicting Lemma 5.4.14 (for more details, see Lemma 18.6.7 in [KH]).

The proof that any two points have intersecting manifolds, is quite classical, and almost topological once we know that both foliations project into minimal foliations (see also Lemma 18.6.7 of [KH]).

$\square$

Now, we can conclude with the proof of Part (T3) of Theorem 5.4.6.

To do this, notice that the map H conjugating $\tilde{f}$ with A is proper, so the preimage of compact sets is compact. Now, assume that A_x, A_y are lifts of fibers of π such that $H(A_x) = H(A_y)$ we shall show they coincide.

Consider K such that if two points have an iterate at distance bigger than K then their image by H is distinct.

We fix $x_0 \in A_x$ and consider a box D_K^n of $\tilde{f}^n(x_0)$ consisting of the points z of $\mathbb{R}^d$ such that $\mathcal{F}^u(z) \cap \mathcal{F}^{cs}_K(x_0) \neq \emptyset$ and $\mathcal{F}^{cs}(z) \cap \mathcal{F}^u_K(x_0) \neq \emptyset$.

It is not hard to show using Lemma 5.4.16 that there exists $\tilde{K}$ independent of n such that every pair of points in D_K^n in the same unstable leaf of $\mathcal{F}^u$ have distance along $\mathcal{F}^u$ smaller than $\tilde{K}$ (this is a compactness argument). An analogous property holds for $\mathcal{F}^{cs}$.

This implies that if $\tilde{f}^n(A_y) \subset D_K^n$ for every $n \in \mathbb{Z}$ then A_y and A_x must be contained in the same leaf of $\mathcal{F}^{cs}$. In fact we get that $\tilde{f}^{-n}(A_y) \subset \mathcal{F}^{cs}_K(\tilde{f}^{-n}(x_0))$ for every $n \geq 0$ and so we conclude that $A_x = A_y$ using Lemma 5.4.2.

$\square$

5.4.3 Some manifolds which do not admit this kind of diffeomorphisms

The arguments used in the previous section also allow to show that certain manifolds (and even some isotopy classes in some manifolds) do not admit partially hyperbolic diffeomorphisms satisfying the coherent trapping property.

To do this, we recall that for general manifolds M^d, and a homeomorphism $h : M \to M$, the Lefschetz number of h, which we denote as $L(h)$ is calculated as $\sum_{i=0}^d trace(h_{*,i})$ where $h_{*,i} : H_i(M, \mathbb{Q}) \to H_i(M, \mathbb{Q})$ is the induced map on (rational) homology[8]. We also have that the sum of index the sum of the Lefshetz index along a covering of $\mathrm{Fix}(h)$ by sets homeomorphic to balls equals $L(h)$.

[8] This is just to avoid torsion elements. Otherwise, one can define the trace with $\mathbb{Z}$ coeficients after making a quotient by the torsion.

A similar argument to the one used in the previous section yields the following result (see also [Shi] for the analog result for Anosov diffeomorphisms[9])

Theorem 5.4.17. *Let f be a partially hyperbolic diffeomorphism of M with the coherent trapping property, then, the action $f_* : H_*(M, \mathbb{Q}) \to H_*(M, \mathbb{Q})$ is strongly partially hyperbolic (it has both eigenvalues of modulus > 1 and < 1).*

As a consequence, several manifolds cannot admit this kind of diffeomorphisms (notably S^d and products of spheres of different dimensions such as $S^1 \times S^2 \times S^3$) and also, for example, there cannot be diffeomorphisms like this acting as the identity on homology. This leads to a natural question: Is every partially hyperbolic diffeomorphism with the coherent trapping property homotopic to an Anosov diffeomorphism?.

Proof. The proof is very similar to the one given in the previous section, so we shall omit some details.

First, we get by counting the fixed points of g^n (the expansive quotient of f^n to $M/_\sim$) and we get an exponential growth.

Now, if there are no eigenvalues of modulus greater than one for f_*, the trace of the map cannot grow exponentially. The same argument applies to f^{-1} so we conclude.

$\square$

[9]Although this result even refines slightly his result even for Anosov diffeomorphisms.

Appendix A

Perturbation of cocycles

A.1 Definitions and statement of results

Before we proceed with the statement and proof of Theorem A.1.4 we shall give some definitions taken from [BGV] and others which we shall adapt to fit our needs.

Recall that $\mathcal{A} = (\Sigma, f, E, A)$ is a *large period linear cocycle*[1] of dimension d bounded by K over an infinite set Σ iff:

- $f : \Sigma \to \Sigma$ is a bijection such that all points in Σ are periodic and such that given $n > 0$ there are only finitely many with period less than n.
- E is a vector bundle over Σ, that is, there is $p : E \to \Sigma$ such that $E_x = p^{-1}(x)$ is a vector space of dimension d endowed with an euclidian metric $\langle , \rangle_x$.
- $A : x \in \Sigma \mapsto A_x \in GL(E_x, E_{f(x)})$ is such that $\|A_x\| \leq K$ and $\|A_x^{-1}\| \leq K$.

In general, we shall denote $A_x^\ell = A_{f^{\ell-1}(x)} \dots A_x$ where juxtaposition denotes the usual composition of linear transformations.

For every $x \in \Sigma$ we denote by $\pi(x)$ its period and $M_x^A = A_x^{\pi(x)}$ which is a linear map in $GL(E_x, E_x)$ (which allows to study eigenvalues and eigenvectors).

For $1 \leq j \leq d$

$$\sigma^j(x, \mathcal{A}) = \frac{log|\lambda_j|}{\pi(x)}$$

Where $\lambda_1, \dots \lambda_d$ are the eigenvalues of M_x^A in increasing order of modulus. As usual, we call $\sigma^j(x, \mathcal{A})$ the j-th Lyapunov exponent of $\mathcal{A}$ at x.

Given an f-invariant subset $\Sigma' \subset \Sigma$, we can always restrict the cocycle to the invariant set defining the cocycle $\mathcal{A}|_{\Sigma'} = (f|_{\Sigma'}, \Sigma', E|_{\Sigma'}, A|_{\Sigma'})$.

[1]Sometimes, we shall abuse notation and call it just *cocycle*.

We shall say that a subbundle $F \subset E$ is *invariant* if $\forall x \in \Sigma$ we have $A_x(F_x) = F_{f(x)}$. When there is an invariant subbundle, we can write the cocycle in coordinates $F \oplus F^{\perp}$ (notice that $F^{\perp}$ may not be invariant) as

$$\begin{pmatrix} \mathcal{A}_F & C_F \\ 0 & \mathcal{A}|F \end{pmatrix}$$

Where C_F is uniformly bounded. This induces two new cocycles: $\mathcal{A}_F = (\Sigma, f, F, A|_F)$ on F (where $A|_F$ is the restriction of A to F) and $\mathcal{A}|F = (\Sigma, f, E|F, A|F)$ on $E|F \simeq F^{\perp}$ where $(A|F)_x \in GL((F_x)^{\perp}, (F_{f(x)})^{\perp})$ is given by $p^2_{f(x)} \circ A_x$ where p^2_x is the projection map from E to $F^{\perp}$. Notice that changing only $\mathcal{A}_F$ affects only the eigenvalues associated to F and changing only $\mathcal{A}|F$ affects only the rest of the eigenvalues, recall Remark 3.2.8. See section 4.1 of [BDP] for more discussions on this decomposition.

As in Section 1.2, if $\mathcal{A}$ has two invariant subbundles F and G, we shall say that F is *ℓ-dominated* by G (and denote it as $F \prec_\ell G$) on an invariant subset $\Sigma' \subset \Sigma$ if for every $x \in \Sigma'$ and for every pair of vectors $v \in F_x \backslash \{0\}$, $w \in G_x \backslash \{0\}$ one has

$$\frac{\|A^\ell_x(v)\|}{\|v\|} \leq \frac{1}{2} \frac{\|A^\ell_x(w)\|}{\|w\|} .$$

We shall denote $F \prec G$ when there exists $\ell > 0$ such that $F \prec_\ell G$.

If there exists complementary invariant subbundles $E = F \oplus G$ such that $F \prec G$ on a subset $\Sigma' \subset \Sigma$, we shall say that $\mathcal{A}$ admits a *dominated splitting* on Σ'.

As in [BGV], we shall say that $\mathcal{A}$ is *strictly without domination* if it is satisfied that whenever $\mathcal{A}$ admits a dominated splitting in a set Σ' it is satisfied that Σ' is finite.

Let Γ_x be the set of cocycles over the orbit of x with the distance

$$d(\mathcal{A}_x, \mathcal{B}_x) = \sup_{0 \leq i < \pi(x), v \in E\backslash\{0\}} \left\{ \frac{\|(A_{f^i(x)} - B_{f^i(x)})v\|}{\|v\|}, \frac{\|(A^{-1}_{f^i(x)} - B^{-1}_{f^i(x)})v\|}{\|v\|} \right\}$$

and let Γ_Σ (or $\Gamma_{\Sigma,0}$) the set of bounded large period linear cocycles over Σ. Given $\mathcal{A} \in \Gamma_\Sigma$ we denote as $\mathcal{A}_x \in \Gamma_x$ to the cocycle $\{A_x, \ldots, A_{f^{\pi(x)-1}(x)}\}$.

We say that the cocycle $\mathcal{A}_x$ has *strong stable manifold of dimension i* if $\sigma^i(x, \mathcal{A}_x) < \min\{0, \sigma^{i+1}(x, \mathcal{A}_x)\}$.

For $0 \leq i \leq d$, let

$$\Gamma_{\Sigma,i} = \{\mathcal{A} \in \Gamma_\Sigma \ : \ \forall x \in \Sigma \ ; \ \mathcal{A}_x \text{ has strong stable manifold of dimension } i\}$$

Following [BGV] we say that $\mathcal{B}$ is a *perturbation* of $\mathcal{A}$ (denoted by $\mathcal{B} \sim \mathcal{A}$) if for every $\varepsilon > 0$ the set of points $x \in \Sigma$ such that $\mathcal{B}_x$ is not $\varepsilon-$close to the cocycle $\mathcal{A}_x$ is finite.

Similarly, we say that $\mathcal{B}$ is a *path perturbation* of $\mathcal{A}$ if for every $\varepsilon > 0$ one has that the set of points $x \in \Sigma$ such that $\mathcal{B}_x$ is not a perturbation of $\mathcal{A}_x$ along a path of diameter $\leq \varepsilon$ is finite. That is, there is a path $\tilde{\gamma} : [0,1] \to \Gamma_\Sigma$ such that $\tilde{\gamma}(0) = \mathcal{A}$ and $\tilde{\gamma}(1) = \mathcal{B}$ such that $\tilde{\gamma}_x : [0,1] \to \Gamma_x$ are continuous paths and given $\varepsilon > 0$ the set of x such that $\tilde{\gamma}_x([0,1])$ has diameter $\geq \varepsilon$ is finite.

In general, we shall be concerned with path perturbations which preserve the dimension of the strong stable manifold, so, we shall say that $\mathcal{B}$ is a *path perturbation of index* i of a cocycle $\mathcal{A} \in \Gamma_{\Sigma,i}$ iff: $\mathcal{B}$ is a path perturbation of $\mathcal{A}$ and the whole path is contained in $\Gamma_{\Sigma,i}$. This induces a relation in $\Gamma_{\Sigma,i}$ which we shall denote as $\sim_i^*$.

We have that $\sim$ and $\sim_i^*$ are equivalence relations in Γ_Σ and $\Gamma_{\Sigma,i}$ respectively, and clearly $\sim_i^*$ is contained in $\sim$.

Remark A.1.1. Notice that if there is an invariant subbundle $F \subset E$ for a cocycle $\mathcal{A}$. Then, a perturbation (resp. path perturbation) of $\mathcal{A}_F$ or $\mathcal{A}|F$ can be completed to an perturbation (resp. path perturbation) of $\mathcal{A}$ which does not alter the eigenvalues associated to $\mathcal{A}|F$ or $\mathcal{A}_F$. Any of this perturbations also preserves the invariance of F, however, one can not control the effect on other invariant subbundles. See section 4.1 of [BDP].

$\diamondsuit$

The *Lyapunov diameter* of the cocycle $\mathcal{A}$ is defined as

$$\delta(\mathcal{A}) = \liminf_{\pi(x)\to\infty} [\sigma^d(x,\mathcal{A}) - \sigma^1(x,\mathcal{A})].$$

If $\mathcal{A} \in \Gamma_{\Sigma,i}$, we define $\delta_{min}(\mathcal{A}) = \inf_{\mathcal{B}\sim\mathcal{A}}\{\delta(\mathcal{B})\}$. Similarly, we define $\delta_{min}^{*,i}(\mathcal{A}) = \inf_{\mathcal{B}\sim_i^*\mathcal{A}}\{\delta(\mathcal{B})\}$. Notice that $\delta_{min}^{*,i}(\mathcal{A}) \geq \delta_{min}(\mathcal{A})$ and a priori it could be strictly bigger.

Remark A.1.2. For any cocycle $\mathcal{A}$, it is easy to see that $\delta_{min}^{*,0}(\mathcal{A}) = \delta_{min}(\mathcal{A})$. It sufficies to consider the path $(1-t)\mathcal{A}+t\mathcal{B}$ where $\mathcal{B}$ is a perturbation of $\mathcal{A}$ having the same determinant over any periodic orbit and verifying $\delta(\mathcal{B}) = \delta_{min}(\mathcal{A})$ (see Lemma 4.3 of [BGV] where it is shown that such a $\mathcal{B}$ exists).

$\diamondsuit$

The following easy Lemma relates the definitions we have just introduced. Its proof is contained in [BDP] and [BGV] except from property (f). We include quick proofs for completeness.

Lemma A.1.3. *Let $\mathcal{A} = (\Sigma, f, E, A)$ and $\mathcal{B} = (\Sigma, f, E, B)$ be two large period cocycles of dimension d and bounded by K.*

(a) If $\mathcal{A}$ is strictly without domination and $\mathcal{B} \sim \mathcal{A}$ (in particular if $\mathcal{B} \sim_i^* \mathcal{A}$) then $\mathcal{B}$ is also strictly without domination.

(b) For every $\ell > 0$ there exists $\nu > 0$ such that if F and G are two invariant subbundles and $F \prec_\ell G$ on Σ', then $\delta((\mathcal{A}_{F\oplus G})|_{\Sigma'}) > \nu$.

(c) If $\delta_{min}(\mathcal{A}) = 0$ there exists an infinite subset $\Sigma' \subset \Sigma$ such that $\mathcal{A}|_{\Sigma'}$ is strictly without domination. Conversely, if $\mathcal{A}$ is strictly without domination, $\delta_{min}(\mathcal{A}) = 0$.

(d) If $\mathcal{A} \sim \mathcal{B}$, then

$$\left| \sum_{j=1}^{d} \sigma^j(x,\mathcal{A}) - \sum_{j=1}^{d} \sigma^j(x,\mathcal{A}) \right| \to 0 \quad as \quad \pi(x) \to \infty$$

(e) Let F, G and H be invariant subbundles of E. So, $F \prec G \oplus H$ if $F \prec G$ and $F|G \prec H|G$. Also, if $F \prec G$ and $G \prec H$ then $F \prec G \oplus H$ and $F \oplus G \prec H$.

(f) Let F, G and H be invariant subbundles of E. So, $F \prec G \oplus H$ implies that $F|G \prec H|G$.

PROOF. Part (a) is Corollary 2.15 of [BGV]. It uses the quite standard fact (see Lemma 2.14 of [BGV]) which asserts that if a cocycle admits certain dominated splitting in a subset Σ', then, there exists ε such that every ε-perturbation of the cocycle remains with dominated splitting in that set.

Assume that $\mathcal{B} \sim \mathcal{A}$ admits dominated splitting in a set Σ', then, the previous argument implies that, modulo removing some finite subset of Σ', the cocycle $\mathcal{A}$ also admits dominated splitting. This implies that Σ' is finite (otherwise, $\mathcal{A}$ would admit a dominated splitting in an infinite set).

Part (b) follows directly from the definition of dominated splitting (ν is going to depend on K, the dimensions of F and G and ℓ).

Part (c) is also standard. The existence of a dominated splitting implies directly the separation of the Lyapunov exponents in each of the invariant subbundles (see part (b)). That is, given a dominated splitting over a set Σ', we get $\varepsilon > 0$ such that for every $x \in \Sigma'$ we have that

$$\sigma^d(x,\mathcal{A}) - \sigma^1(x,\mathcal{A}) > \varepsilon$$

So, if $\delta(\mathcal{A}) = 0$, we get that $\Sigma \backslash \Sigma'$ contains periodic points of arbitrarily large period, so it is infinite as wanted. Notice that $\mathcal{A}$ may have infinite sets admitting a dominated splitting and still verifying $\delta_{min}(\mathcal{A}) = 0$. The converse part is the main result of [BGV] (see Theorem 4.1 of [BGV]).

Part (d) is given by the fact that the determinant is multiplicative, so, an ε-perturbation of a cocycle, can increase at most $(1+\varepsilon)^{\pi(x)}$ the determinant of M_x^A. So, the sum of the exponents can change at most $\log(1+\varepsilon)$ which converges to zero as $\varepsilon \to 0$.

Part (e) is contained in Lemmas 4.4 and 4.6 of [BDP].

To prove property (f) we use that the existence of a dominated splitting admits a change of metric which makes the subbundles ortogonal (see [BDP] section 4.1). So, we can write the cocycle restricted to $F \oplus G \oplus H$ in the form (using coordinates, $G, (G \oplus H) \cap G^{\perp}, F$)

$$\begin{pmatrix} \mathcal{A}_G & \star & 0 \\ 0 & \mathcal{A}(H) & 0 \\ 0 & 0 & \mathcal{A}_F \end{pmatrix}$$

Where $\mathcal{A}(H) = \mathcal{A}|(G\oplus F)$. So the cocycle $\mathcal{A}|G$ is written in coordinates $H|G, F|G$ as

$$\begin{pmatrix} \mathcal{A}(H) & 0 \\ 0 & \mathcal{A}_F \end{pmatrix}$$

Since F is dominated by $G \oplus H$, we get the desired property. Notice that for any $x \in \Sigma$, we have that $\|(A_x|_{G\oplus H})^{-1}\|^{-1} \leq \|(A(H)_x)^{-1}\|^{-1}$ and this guaranties the domination of $F|G \prec H|G$ as desired.

$\square$

We are now ready to state the main result of this appendix. A much stronger version of this result can be found[2] in [BoB]. The proof here presented of this weaker result has some similarities with their proof but I hope that its inclusion in the text is not devoid of interest and introduces some different ways of proving some parts of the result.

Theorem A.1.4. *Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded large period linear cocycle of dimension d. Assume that*

- *$\mathcal{A}$ is strictly without domination.*
- *$\mathcal{A} \in \Gamma_{\Sigma,i}$*
- *For every $x \in \Sigma$, we have $|det(M_x^A)| < 1$ (that is, for all $x \in \Sigma$ we have $\sum_{j=1}^{d} \sigma^j(x, \mathcal{A}) < 0$).*

[2]In fact, the result of [BoB] was announced before I had a proof of this result but I did not notice the overlap and pursued in this direction.

Then, $\delta^{*,i}_{min}(\mathcal{A}) = 0$. In particular, given $\varepsilon > 0$ there exists a point $x \in \Sigma$ and a path γ_x of diameter smaller than ε such that $\gamma_x(0) = \mathcal{A}_x$, the matrix $M_x^{\gamma_x(1)}$ has all its eigenvalues of modulus smaller than one, and such that $\gamma(t) \in \Gamma_i$ for every $t \in [0,1]$.

A.2 Proof of Theorem A.1.4

This section will be devoted to prove this theorem. The proof is by induction.

The following Lemma allows to find several invariant subbundles in order to be able to apply induction. It is proved in [Gou$_3$] Proposition 6.6.

Lemma A.2.1. *For every $\mathcal{A} \in \Gamma_{\Sigma,i}$, there exists $\mathcal{B} \sim_i^* \mathcal{A}$ such that for every $x \in \Sigma$ the eigenvalues of M_x^B have all different modulus and their modulus is arbitrarily near the original one in M_x^A, that is, $|\sigma^i(x,\mathcal{A}) - \sigma^i(x,\mathcal{B})| \to 0$ as $\pi(x) \to \infty$ (in particular, $\delta(\mathcal{B}) = \delta(\mathcal{A})$).*

Sketch We proceed by induction. In dimension 2 the result is the same as in Proposition 3.7 of [BGV] (the only perturbations done there can be made along paths without any difficulty, this was first done in [BC]). Notice that if the eigenvalues are both equal for some $x \in \Sigma$, then necessarily the cocycle belongs to $\Gamma_{\Sigma,2}$ or $\Gamma_{\Sigma,0}$.

We assume the result holds in dimension $< d$. Since there always exists an invariant subspace of dimension 2, you can make independent perturbations and change the eigenvalues as required using the induction hypothesis. For this, one should perturb in the invariant subspace and in the quotient (see Remark A.1.1).

$\square$

If a cocycle $\mathcal{A}$ verifies that for every $x \in \Sigma$, the eigenvalues of M_x^A have all different modulus and different from 1, we shall say that $\mathcal{A}$ is a *diagonal cocycle.*

Remark A.2.2. For a diagonal cocycle $\mathcal{A} \in \Gamma_{\Sigma,i}$ one has well defined invariant one-dimensional subspaces $E_1(x,\mathcal{A}), \dots, E_d(x,\mathcal{A})$ (we shall in general omit the reference to the point and/or the cocycle) associated to the eigenvalues in increasing order of modulus. Also, if $F_l = E_1 \oplus \ldots \oplus E_l$, one gets that $\mathcal{A}_{F_l} \in \Gamma_{\Sigma,j}$ for any $0 \leq j \leq \min\{i,l\}$.

From now on, for diagonal cocycles we shall name $F_l = E_1 \oplus \ldots \oplus E_l$ and $G_l = E_l \oplus \ldots \oplus E_d$.

$\diamondsuit$

As we said, Theorem A.1.4 is easier in dimension 2 (it does not even need the uniformity hypothesis on the determinant). The following Lemma is essentially due to Mañe and will be the base of the induction.

Lemma A.2.3. *Let $\mathcal{A} \in \Gamma_{\Sigma,i}$ be a bounded large period linear cocycle of dimension 2 ($0 \leq i \leq 2$) and strictly without domination such that $|det(M_x^A)| \leq 1$ for every $x \in \Sigma$. Then, there exists $\mathcal{B} \sim_i^* \mathcal{A}$ with the following properties:*

1. $\delta(\mathcal{B}) = 0$.
2. $|det(M_x^A)| = |det(M_x^B)|$ *for every* $x \in \Sigma$.

Proof. This is very standard (see [M$_3$] or section 7.2.1 of [BDV]). With a small perturbation (see Proposition 6.7 of [Gou$_3$]) one can make the angle between the stable and unstable spaces arbitrarily small and not change the determinant.

After that, one can compose with a rotation, of determinant equal to 1 (so, without affecting the product of the modulus of the eigenvalues), since after rotating a small amount one gets complex eigenvalues, there is a moment where the eigenvalues are real and arbitrarily near, there is where we stop.

Notice that if $|det(M_x^A)| > 1$ and $i = 1$ we would be obliged to stop the path longtime before the eigenvalues are nearly equal since the smallest exponent would attain the value 0 which is forbidden for path perturbations of index 1.

$\square$

Before we continue with the induction to prove the general result, we shall make some general perturbative results which loosely state that if two bundles are not dominated, then, after a perturbation which preserves the exponents, we can see the non domination in the bundles associated to the closest bundles.

First we will state a standard linear algebra result we shall use in order to perturb two dimensional cocycles.

Lemma A.2.4. *Given $\varepsilon > 0$ and $K > 0$, there exists $\ell > 0$ such that if $A_1, \dots, A_\ell$ is a sequence in $GL(2,\mathbb{R})$ matrices verifying that* $\max_i\{\|A_i\|, \|A_i^{-1}\|\} \leq K$ *and $v, w \in \mathbb{R}^2$ are vectors with $\|v\| = \|w\| = 1$. Suppose that*

$$\|A_\ell \dots A_1 v\| \geq \frac{1}{2}\|A_\ell \dots A_1 w\|$$

Then, there exists rotations $R_1, \dots, R_\ell$ of angle smaller than ε verifying that

$$R_\ell A_\ell \dots R_1 A_1 \mathbb{R}w = A_\ell \dots A_1 \mathbb{R}v$$

Proof. For simplicity we shall assume that $A_i v = v$ for every i. Since this is made by composing each matrix by a rotation and a homothety, we should change K by K^2 which will be the new bound for the norm of the matrices.

For $\gamma \in P^1(\mathbb{R})$, let $\alpha_i(\gamma) = \frac{\|A_i A_{i-1} \dots A_1 z\|}{\|A_{i-1} \dots A_1 z\|}$ where z is a vector in the direction γ. It is a well known result in linear algebra that if the function $\alpha_i : P^1(\mathbb{R}) \to \mathbb{R}^+$ is not constant then it has a maximum and a minimum in antipodal points and it is monotone in the complement.

We can assume that for every $i \geq 0$, we have that $A_i \dots A_1 \mathbb{R}w$ is at distance larger than ε from $\mathbb{R}v$, otherwise we can perform the perturbation.

Notice also that given $0 < \eta < \nu < 1$ there exists κ such that if for some point $\gamma \in P^1(\mathbb{R})$ we have that $\prod_{i=k}^{j} \alpha_i(\gamma) > \kappa$, then (notice that $\prod_{i=k}^{j} \alpha_i(\mathbb{R}v) = 1$), there is an interval of length ν around γ which does not contain $\mathbb{R}v$ which is mapped by $A_j \dots A_k$ to an interval around $A_j \dots A_k \gamma$ of length η. A similar statement holds for the inverses in the case the product is smaller than κ^{-1}.

If we choose $\nu > 1 - \varepsilon$ and $\eta < \varepsilon$ we get $\kappa = \kappa(\varepsilon)$ which will verify the following: Assume that there exists $\gamma \in P^1(\mathbb{R})$ verifying $\prod_{i=j}^{k} \alpha_i(\gamma) < \kappa^{-1}$ for some j, k, then, since $A_j \dots A_1 \mathbb{R}w$ should be in the interval around γ, we can first rotate it to send it to an extreme point of the interval, and after applying $A_k \dots A_{j+1}$ the vector will be ε close to $\mathbb{R}v$ and so we can finish the perturbation.

This implies that for $\kappa = \kappa(\varepsilon)$ we get that for every $j < k$ we have that $\prod_{i=j}^{k} \alpha_i(\gamma) > \kappa^{-1}$ for any $\gamma \in P^1(\mathbb{R})$.

The hypothesis of the Lemma (and the choice of $\alpha_i(\mathbb{R}v) = 1$ for every i) implies that $\prod_{i=1}^{\ell} \alpha_i(\mathbb{R}w) \leq 2$.

This implies that also (maybe by rechoosing κ) that we can not have a sequence $\alpha_i(\mathbb{R}w)$ verifying $\prod_{i=j}^{k} \alpha_i(\mathbb{R}w) > \kappa$. Which in turn implies that this should also happen for every point $\gamma \in P^1(\mathbb{R})$ (otherwise an iterate of $\mathbb{R}w$ would be ε close to v).

This gives us that, there exists $0 < \rho < 1$ such that the iterates of an interval of length ε remain of length bounded from below by $\rho\varepsilon$. Now, choosing ℓ such that $\ell\rho\varepsilon > 1$ we get that we can take any vector to $\mathbb{R}v$ with rotations of angle less than ε. $\square$

The following proposition is the key step of the proof.

Notice that the Proposition does not use the fact that $\mathcal{A}$ may be chosen strictly without domination, or even with $\delta(\mathcal{A}) = 0$.

Proposition A.2.5. *Given $K > 0$, $k > 0$ and $\varepsilon > 0$, there exists $N > 0$ and ℓ such that if*

- *There exists a diagonal cocycle $\mathcal{A}_x$ of dimension 2 and bounded by K over a periodic orbit of period $\pi(x) > N$.*

- *There exists a unit vector $v \in E_x$ such that*

$$\frac{3}{2} \| A^{\ell}_{f^k(x)}|_{E_1} \| \geq \frac{\| A^{\ell}_{f^k(x)} A^k_x v \|}{\| A^k_x v \|}$$

Then, there exists $\mathcal{B}_x$ a path perturbation of $\mathcal{A}_x$ of diameter smaller than ε verifying that all along the path the cocycle has the same Lyapunov exponents and

$$\|B_x^k|_{E_1(x,\mathcal{B})}\| \leq 2\|A_x^k v\|$$

PROOF. We use coordinates $E_1 \oplus E_1^{\perp}$. With this convention, we have

$$A_x = \begin{pmatrix} \alpha_x & K_x \\ 0 & \beta_x \end{pmatrix} \qquad A_x^n = \begin{pmatrix} \prod_{j=0}^{n-1} \alpha_{f^j(x)} & \star \\ 0 & \prod_{j=0}^{n-1} \beta_{f^j(x)} \end{pmatrix}$$

We also have that $|\alpha_x|$, $|\beta_x|$ and $|K_x|$ are uniformly bounded from above by K and also $|\alpha_x|$ and $|\beta_x|$ are bounded from below by $1/K$ for every $x \in \Sigma$. It is satisfied that $\left|\prod_{j=1}^{\pi(x)} \alpha_{f^j(x)}\right| < \left|\prod_{j=1}^{\pi(x)} \beta_{f^j(x)}\right|$.

We fix $k > 0$ and $\varepsilon > 0$. Let $\ell > 0$ given by Lemma A.2.4 and let $v \in E_x$ a vector in the hypothesis of the Proposition.

We consider the set $\Upsilon \subset E$ of vectors satisfying that if $w_0 \in \Upsilon$ is an unitary vector and if we denote $w_j = \frac{A_x^j w}{\|A_x^j w\|}$ we have

$$\|A^{\ell}_{f^k(x)}|_{E_1(x,\mathcal{A})}\| \geq \frac{1}{2}\|A^{\ell}_{f^k(x)} w_k\|$$

and

$$\|A_x^k w_0\| \leq 2\|A_x^k v\|$$

We remark that Υ is defined just in terms of $\{A_x, \dots A_{f^{k+\ell}(x)}\}$ provided that we maintain the condition on all $A_{f^j(x)}$ being triangular. Also, it is easy to see that Υ is a closed under scalar multiplication, so we shall sometimes consider the unit vectors there and think of it as a subset of the projective line $P^1(\mathbb{R})$.

Notice that if $E_1(x, \mathcal{A}) \in \Upsilon$, the Proposition holds without need to make any perturbation, so we will assume that it is not the case. We shall call θ the distance in $P^1(\mathbb{R})$ between $E_1(x, \mathcal{A})$ and Υ where the distance we consider is the one given by the inner angle between the generated lines.

We shall consider $k + \ell < L < \pi(x) - \ell$ the largest integer (if there exists any) verifying that there exists some $w \in \Upsilon$ satisfying

$$\|A^{\ell}_{f^L(x)} w_L\| \geq \frac{1}{2}\|A^{\ell}_{f^L(x)}|_{E_1(f^L(x),\mathcal{A})}\| \tag{4.1}$$

Claim. *If $\pi(x)$ is large enough, there exists some L with the properties above. Moreover, $L \to \infty$ as $\pi(x) \to \infty$.*

PROOF. Assume that for some $s > 0$ and for arbitrarily large $\pi(x)$ (for simplicity we consider it of the form $\pi(x) = R\ell + s$ with $R \to \infty$), if L exists is smaller than s.

We have that for $w \in \Upsilon$, the nearest vector to $E_1(x,\mathcal{A})$ we have that

$$\|A^{R\ell}_{f^s(x)} w_s\| < \left(\frac{1}{2}\right)^R \|A^{R\ell}_{f^s(x)}|_{E_1(f^s(x),\mathcal{A})}\|$$

A simple calculation gives us, choosing R large enough to satisfy that $K^s\left(\frac{1}{2}\right)^R \leq \tan(\theta/2)$, that the cone of angle $\theta/2$ around $E_1(f^s(x),\mathcal{A})$ is mapped by $A^{\pi(x)}_{f^s(x)}$ inside itself. This contradicts the fact that E_1 is the eigenspace associated to the smallest eigenvalue (i.e. that $\left|\prod_{j=1}^{\pi(x)} \alpha_{f^j(x)}\right| < \left|\prod_{j=1}^{\pi(x)} \beta_{f^j(x)}\right|$).

$\diamondsuit$

We shall need a more quantitative version of this growth:

Claim. *There exists $N > 0$ such that if $\pi(x) > N + 2\ell + k$ then*

$$\left(\prod_{j=k+l}^{L-1} \alpha_{f^j(x)}(1+\varepsilon)^{-1}\right) K^{2\ell+k}\|A^{\pi(x)-L-\ell}_{f^{L+\ell}(x)}\| \leq \prod_{j=1}^{\pi(x)} \alpha_{f^j(x)} \leq$$
$$\leq \left(\prod_{j=k+l}^{L-1} \alpha_{f^j(x)}\right) K^{2\ell+k}\|A^{\pi(x)-L-\ell}_{f^{L+\ell}(x)}\|$$

.

Proof. The second inequality is direct. We prove the claim by contradiction, we assume though that

$$\prod_{j\notin\{k+\ell,\dots,L-1\}} \alpha_{f^j(x)} < \|A^{\pi(x)-L-\ell}_{f^{L+\ell}(x)}\| K^{2\ell+k}(1+\varepsilon)^{-L+k+\ell}$$

Notice that by the previous claim we have that $L \to \infty$ as $\pi(x) \to \infty$ so, as $\pi(x)$ grows, the norm of $A^{\pi(x)-L-\ell}_{f^{L+\ell}(x)}$ grows to infinity compared to the norm of $A^{\pi(x)-L-\ell}_{f^{L+\ell}(x)}|_{E_1}$.

Also, we get that the distance between $A^j_x\Upsilon$ and $E_1(f^j(x),\mathcal{A})$ for $j \geq L$ must be bounded from bellow since the norm of every $A^\ell_{f^i(x)}$ is bounded by K^ℓ so being very close implies that j would satisfy (4.1) contradicting the maximality of L.

So, if $\pi(x)$ is large enough, we get that the vectors far from $E_1(f^{L+\ell}(x),\mathcal{A})$ must be mapped near the direction of maximal expansion of $A^{\pi(x)+k+\ell-L}_{f^{L+\ell}(x)}$ and thus, this allows to find a vector in Υ verifying (4.1) a contradiction with the maximality of L. See Lemma A.2.4 for a similar argument.

$\diamondsuit$

We shall now define the perturbation we will make in order to satisfy the requirements of the Proposition. We shall define a continuous path $\gamma : [0,1] \to \Gamma_x$ of diameter smaller than ε and verifying the required properties. Notice that all along the path, the determinant is never changed in any point, so, the product of the eigenvalues of $M^{\gamma(t)}_x$ remains unchanged for every $t \in [0,1]$.

Using Lemma A.2.4 we can perturb with small rotations the transformations $A_{f^L(x)}, \ldots, A_{f^{L+\ell-1}(x)}$ in order to send $E_1(f^L(x), \mathcal{A})$ into the one dimensional subspace $\mathbb{R}\tilde{w}$ such that $A^{\pi(x)-(L+\ell)}_{f^{L+\ell-1}(x)}\mathbb{R}\tilde{w} = \mathbb{R}w$ where $w \in \Upsilon$ is the vector defining L.

Since this perturbations are made by composing with small rotations, they can be made along small paths. Let $A^t_{f^j(x)} : [0,1] \to GL(E_{f^j(x)}, E_{f^{j+1}(x)})$ where $j \in \{L, \ldots, L+\ell-1\}$ such that

$$\left(A^{\pi(x)-L}_{f^L(x)} A^1_{f^{L+\ell-1}(x)} \ldots A^1_{f^L(x)} A^{L-(k+\ell)}_{f^{k+\ell}(x)}\right) E_1(f^{k+\ell}(x), \mathcal{A}) = \mathbb{R}\tilde{w}$$

The hypothesis we made and the previous Lemma allow us to send $\mathbb{R}w$ to $E_1(f^{k+\ell}(x), \mathcal{A})$ by composing the matrices $A_{f^j(x)}$ with $k \leq j < k+\ell$ with small rotations. Clearly, by choosing properly the rotations, for $k \leq j \leq k+\ell-1$ we can find also paths $A^t_{f^j(x)} : [0,1] \to GL(E_{f^j(x)}, E_{f^{j+1}(x)})$ verifying that

$$\left(A^t_{f^{k+\ell-1}(x)} \ldots A^t_{f^k(x)} A^k_x A^{\pi(x)-L}_{f^L(x)} A^t_{f^{L+\ell-1}(x)} \ldots A^t_{f^L(x)} A^{L-(k+\ell)}_{f^{k+\ell}(x)}\right) E_1(f^{k+\ell}(x), \mathcal{A}) = E_1(f^{k+\ell}(x), \mathcal{A})$$

We shall also perturb the linear transformations $A_{f^j(x)}$ with $j \in \{k+\ell, \ldots, L-1\}$ by multiplying them by matrices of the form

$$\begin{pmatrix} \frac{1}{\alpha(t)} & 0 \\ 0 & \alpha(t) \end{pmatrix}$$

Where $\alpha(t)$ is conveniently chosen in $[1, 1+\varepsilon]$ in order to get that for every $t \in [0,1]$ the two exponents of $M^{\gamma(t)}_{f^{k+\ell}(x)}$ coincide.

To show that the latter can be made, notice that the perturbations we made imply that in our coordinates, the matrix $M^{\gamma(t)}_{f^{k+\ell}(x)}$ is of the same triangular form, so, after applying the first $L-k-\ell$ transformations, the E_1 direction will remain horizontal, so, Claim 2 implies that for every $t \in [0,1]$, there exists $\alpha(t)$ such that the exponents are equal, the fact that this $\alpha(t)$ varies continuously is given by the fact that the eigenvalues of a path of matrices vary continuously. This concludes.

□

We shall extend this two dimensional result to a more general context using this kind of two dimensional perturbations. This will allow us to reduce all the problems to a two dimensional context that we know well how to treat (see Lemma A.2.3).

Proposition A.2.6. *Let $\mathcal{A} \in \Gamma_{\Sigma,i}$ a bounded diagonal cocycle. Assume that for $0 < j < d$ we have that F_j is not dominated by G_{j+1}. Then, there exists $\mathcal{B}$, a path perturbation of $\mathcal{A}$ along a path which does not change any of the Lyapunov exponents, which verifies that $E_j(\mathcal{B})$ is not dominated by $E_{j+1}(\mathcal{B})$.*

PROOF. We shall prove that if F_j is not dominated by G_{j+1} we can perturb as above in order to break the domination between F_j and E_{j+1}. A symmetric argument gives the desired property.

We prove this by induction. So, we will fix j and assume that the proposition holds for every $d < d_0 > j+2$ and prove it in the case $d = d_0$. We assume that F_j is not dominated by G_{j+1} but that $F_j \prec E_{j+1}$ (otherwise there is nothing to prove).

This implies (by property (e) of Lemma A.1.3) that $F_j|E_{j+1}$ is not dominated by $G_{j+2}|E_{j+1}$, so, we can by induction, find a perturbation respecting all the eigenvalues such that $F_j|E_{j+1}$ is not dominated by $E_{j+2}|E_{j+1}$. Now, property (f) of Lemma A.1.3 implies that F_j is not dominated by $E_{j+1} \oplus E_{j+2}$. Using induction again we obtain a perturbation which respects the eigenvalues and such that that F_j is not dominated by E_{j+1} as wanted.

Finally we must prove the Proposition in the case $d = j+2$ in order to conclude. Assume then that F_j is dominated by E_{j+1} (otherwise there is nothing to prove).

We have that there exists s such that $F_j \prec_s E_{j+1}$. For simplicity, we take $s = 1$, that is, for every $x \in \Sigma$ (maybe by considering an infinite subset), and unitary vectors $v_j \in F_j$ and $v_{j+1} \in E_{j+1}$ we have that

$$\|A_x v_j\| \leq \frac{1}{2}\|A_x v_{j+1}\|$$

Since we have assumed that F_j is not dominated by $E_{j+1} \oplus E_{j+2}$ we have that for every $n > 0$ there exists N such that if $\pi(x) > N$ we have that for some point of the orbit of x (which without loss of generality we suppose is x) one has that for some unitary vectors $v_j \in E_j$ and $v \in E_{j+1} \oplus E_{j+2}$ we have that

$$2\|A_x^n v_j\| > \|A_x^n v\|$$

Using standard arguments (see for example Pliss' Lemma [Pli, W_3]) we get that for every k and ℓ there exists N such that if $\pi(x) > N$ we have that (again choosing x conveniently and maybe by changing the vectors by their normalized iterates)

$$\frac{\|A^\ell_{f^k(x)} A^k_x v\|}{\|A^k_x v\|} < \frac{3}{2}\frac{\|A^\ell_{f^k(x)} A^k_x v_j\|}{\|A^k_x v_j\|} < \frac{3}{2}\|A^\ell_{f^k(x)}|_{E_{j+1}}\|$$

and

$$\|A^k_x v\| < \frac{3}{2}\|A^k_x v_j\|$$

This puts us in the hypothesis of Proposition A.2.5 which allow us to make a perturbation of $\mathcal{A}_{E_{j+1}\oplus E_{j+2}}$ without changing the Lyapunov exponents and that breaks the domination between F_j and E_{j+1}.

Notice that given k and ε we get that for periodic points with large period we can perform these perturbations with size smaller than ε, so, we get that we can perturb a sequence of periodic orbits with period going to infinity with arbitrarily small perturbations and break the domination.

$\square$

This proposition allows us to complete the proof of Theorem A.1.4. Before, we shall make some reductions.

Lemma A.2.7. *Let $\mathcal{A} \in \Gamma_{\Sigma,i}$. Then, there exists $\mathcal{B} \sim_i^* \mathcal{A}$ such that $\delta(\mathcal{B}) = \delta_{min}^{*,i}(\mathcal{A})$. Moreover, we can assume that $\mathcal{B}$ is a diagonal cocycle and $\delta(\mathcal{B}_{F_l}) = \delta_{min}^{*,j}(\mathcal{B}_{F_l})$ for every $1 \leq l \leq d$ and $j = \min\{i, l\}$.*

Proof. The existence of $\mathcal{B}$ is proved by following verbatim the proof of Lemma 4.3 in [BGV], since the proof does not introduce new perturbations. The idea is to take a sequence $\mathcal{B}_n$ of path perturbations of index i with $\delta(\mathcal{B}_n)$ converging to $\delta_{min}^{*,i}(\mathcal{A})$ with different eigenvalues (see Lemma A.2.1), and then considering the cocycle $\mathcal{B}$ defined as coinciding with $\mathcal{B}_n$ over the periodic points of period n.

To prove that we can choose $\delta(\mathcal{B}_{F_l}) = \delta_{min}^{*,j}(\mathcal{B}_{F_l})$ we make another diagonal process to first take $\delta(\mathcal{B}_{F_{d-1}})$ to $\delta_{min}^{*,i}(\mathcal{B})$, then $\delta(\mathcal{B}_{F_{d-2}})$ and so on.

$\square$

From now on, we shall use the following notation

$$\overline{\sigma}^j(\mathcal{A}) = \limsup_{\pi(x)\to\infty} \sigma^j(x,\mathcal{A}) \qquad \underline{\sigma}^j(\mathcal{A}) = \liminf_{\pi(x)\to\infty} \sigma^j(x,\mathcal{A})$$

Remark A.2.8. Notice that if $\Sigma' \subset \Sigma$ is an invariant infinite subset, we get that $\delta_{min}^{*,i}(\mathcal{A}|_{\Sigma'}) \geq \delta_{min}^{*,i}(\mathcal{A})$. So, we can always restrict to an infinite invariant subset to prove the Theorem.

This implies that to prove the Theorem, we can assume that the cocycle $\mathcal{A}$ satisfies that for every $1 \leq j \leq d$ we have $\underline{\sigma}^j(\mathcal{A}) = \overline{\sigma}^j(\mathcal{A}) = \sigma^j(\mathcal{A})$. To do this it is enough to make a diagonal process showing that there is an infinite subset $\Sigma_1 \subset \Sigma$ where $\underline{\sigma}^1(\mathcal{A}|_{\Sigma_1}) = \overline{\sigma}^1(\mathcal{A}|_{\Sigma_1})$. Then, inductively, we can construct $\Sigma_k \subset \ldots \subset \Sigma_1 \subset \Sigma$ an infinite subset such that for every $1 \leq j \leq k$ we have $\underline{\sigma}^j(\mathcal{A}|_{\Sigma_k}) = \overline{\sigma}^j(\mathcal{A}|_{\Sigma_k})$. Finally, we restrict $\mathcal{A}$ to Σ_d and use the previous remark.

In this context, we get also that $\delta(\mathcal{A}) = \sigma^d(\mathcal{A}) - \sigma^1(\mathcal{A})$.

$\diamondsuit$

We shall say that a cocycle $\mathcal{A} \in \Gamma_{\Sigma,i}$ of dimension d is *i-incompressible* if it satisfies the following properties (notice that they are quite more restrictive than the ones used in [BGV]):

- $\mathcal{A}$ is a diagonal cocycle.

- $\delta(\mathcal{A}_{F_l}) = \delta^{*,j}_{min}(\mathcal{A}_{F_l})$ where $j = \min\{i, l\}$ for every $1 \leq l \leq d$.

- For every $1 \leq j \leq d$ we have that $\underline{\sigma}^j(\mathcal{A}) = \overline{\sigma}^j(\mathcal{A}) = \sigma^j(\mathcal{A})$.

The previous Lemma and Remark A.2.8 show that to prove Theorem A.1.4 it is enough to work with $i-$incompressible cocycles strictly without domination. Notice that trivially, if $\mathcal{B} \sim_i^* \mathcal{A}$ is a diagonal path perturbation such that for every j we have $\sigma^j(\mathcal{B}) = \sigma^j(\mathcal{A})$, then, $\mathcal{B}$ is also $i-$incompressible.

Proof. of Theorem A.1.4 We shall prove this Theorem by induction. As we said we can work with $i-$incompressible cocycles.

Now we make the standing induction hypothesis, which holds for two dimensional cocycles after Lemma A.2.3. It is easy to see that proving this implies directly the Theorem.

(H) Let $\mathcal{D} \in \Gamma_{\Sigma,i}$ be an $i-$incompressible cocycle of dimension $k < d$ $(0 \leq i \leq k)$ verifying that for every $x \in \Sigma$ we have that $|det(M_x^D)| < 1$. Then, if $j < k$ is the first number such that $\sigma^j(\mathcal{D}) < \sigma^{j+1}(\mathcal{D})$ then, it holds that $F_j \prec G_{j+1}$.

Now, let us consider an $i-$incompressible cocycle $\mathcal{A} \in \Gamma_{\Sigma,i}$ of dimension d. Let j be the smallest number such that $\sigma^j(\mathcal{A}) < \sigma^{j+1}(\mathcal{A})$ (if no such j exists there is nothing to prove).

If $j = d-1$, we shall show that $F_{d-1} \prec E_d$.

We notice first that since the sum of all exponents is ≤ 0, we have that $\sigma^1(\mathcal{A}) = \sigma^{d-1}(\mathcal{A}) < 0$.

Assume that F_{d-1} is not dominated by E_d. So, by Proposition A.2.6 (used for the inverses) we get that for some $\mathcal{B} \sim_i^* \mathcal{A}$ that remains $i-$incompressible (since it does not change the eigenvalues) we have that E_{d-1} is not dominated by E_d. But this is a contradiction since Lemma A.2.3 allows us to decrease the Lyapunov diameter of $\mathcal{B}_{E_{d-1} \oplus E_d}$ contradicting the $i-$incompressibility (notice that this would make the last exponent to decrease).

So it rest to prove the theorem in the case $j < d-1$.

First of all, by induction we get that $F_j \prec E_{j+1} \oplus \ldots \oplus E_{d-1}$.

Now, if F_j is not dominated by G_{j+1} we get that a perturbation which preserves the $i-$incompressibility (given by Proposition A.2.6), allows us to break the domination $F_j \prec E_{j+1} \oplus \ldots \oplus E_{d-1}$ a contradiction.

□

Appendix B

Plane decompositions

We present a construction of a plane diffeomorphism f of bounded C^∞ norm which is semiconjugated to the homothety $x \mapsto x/2$ by a continuous map $h : \mathbb{R}^2 \to \mathbb{R}^2$ whose fibers are all cellular sets of diameter smaller than K and such that it has two disjoint attracting neighborhoods which project by h to the whole plane. We derive some unexpected consequences of the existence of such an example.

We shall denote as $d_2 : \mathbb{R}^2 \to \mathbb{R}^2$ to the map

$$d_2(x) = \frac{x}{2}.$$

The goal of this note is to prove the following theorem (recall subsection 2.1):

Theorem B.0.9. *There exists a C^∞-diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a constants $K > 0$ and $a_K > 0$ such that the following properties are verified:*

- *There exists a (Hölder) continuous cellular map $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $d_{C^0}(h, id) < K$ and $d_2 \circ h = h \circ f$.*
- *There exist open sets V_1 and V_2 such that*
  - $\overline{V_1} \cap \overline{V_2} = \emptyset$
  - $h(V_i) = \mathbb{R}^2$ *for* $i = 1, 2$.
  - $f(\overline{V_i}) \subset V_i$ *for* $i = 1, 2$.
- *The C^∞ norm of f and f^{-1} is smaller than a_K.*

A direct consequence of this Theorem is the existence of $h : \mathbb{R}^2 \to \mathbb{R}^2$ whose fibers are all non trivial and cellular (decreasing intersection of topological disks), the existence of these decompositions of the plane had been shown by Roberts [Ro].

B.1 Construction of f.

For simplicity, we start by considering a curve $\gamma = \{0\} \times [-\frac{1}{4}, \frac{1}{4}]$ (the construction can be made changing γ for any cellular set[1]).

Clearly, $\gamma \subset B_0 = B_1(0)$ the ball of radius one on the origin. We shall also consider the sets $B_n = B_{2^n}(0)$ for every $n \geq 0$. We have that

$$\mathbb{R}^2 = \bigcup_{n \geq 0} B_n$$

We shall define $f : \mathbb{R}^2 \to \mathbb{R}^2$ with the desired properties in an inductive manner, starting by defining it in B_0 and then in the annulus $B_n \setminus B_{n-1}$.

Let us define $f_0 : \overline{B_0} \to B_0$ a C^∞ embedding and disjoint open sets V_1^0 and V_2^0 such that:

(a) f_0 coincides with d_2 in a small neighborhood of ∂B_0.

(b) $\bigcap_{n \geq 0} f_0^n(B_0) = \gamma$.

(c) $f_0(\overline{V_i^0}) \subset V_i^0$ for $i = 1, 2$.

(d) The sets V_i^0 are diffeomorphic to $[-1, 1] \times \mathbb{R}$, separate B_0 in two connected components and intersect $\{0\} \times [-1/4, 1/4]$ in disjoint closed intervals.

Now, we assume that we have defined a C^∞-diffeomorphism $f_n : B_n \to B_{n-1}$ and disjoint open connected sets V_1^n and V_2^n (homeomorphic to a band $\mathbb{R} \times (0, 1)$) such that:

(I1) $f_n|_{B_{n-1}} = f_{n-1}$ and $V_i^{n-1} \subset V_i^n$ for $i = 1, 2$.

(I2) The C^∞-distance between f_n and d_2 in B_n is smaller than a_K.

(I3) $(f_n(\overline{V_i^n}) \setminus \partial B_{n-1}) \subset V_i^{n-1}$ for $i = 1, 2$ and $f_n^n(\overline{V_i^n})$ disconnects B_0.

(I4) V_i^n is $K/2$-dense in B_n.

(I5) f_n coincides with d_2 in a $K/10$-neighborhood of ∂B_n.

We must now construct f_{n+1} assuming we had constructed f_n and this will define a diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ which we shall show has the desired properties.

To construct f_{n+1} and V_i^n we notice that in order to verify (I1), it is enough to define f_{n+1} in $B_n \backslash B_{n-1}$ as well as to add to V_i^n an open set in $B_{n+1} \backslash B_n$ in order to verify the hypothesis.

[1] A cellular set is a decreasing intersection of compact topological disks.

We consider $d_2^{-1}(V_i^n) \cap B_{n+1} \backslash B_n$ which since V_i^n was $K/2$-dense in B_n becomes K-dense in $B_{n+1} \backslash B_n$ for $i = 1, 2$.

We shall use the following lemma whose proof we delay to the end of this section.

Lemma B.1.1. *There exists a_K which only depends on K such that:*

- *Given two open sets A_1, A_2 which are K-dense inside a set of the form $B_n \backslash B_{n-1}$ with sufficiently large n.*
- *The sets A_i verify that for every point in A_i there is a curve going from ∂B_n to ∂B_{n-1} and contained in A_i.*

Then there exists a C^∞ diffeomorphism g of C^∞-norm less than a_K such that coincides with the identity in $K/10$-neighborhood of the boundaries and such that the image by g of the open sets is $K/2$-dense in $B_n \setminus B_{n-1}$.

We consider a diffeomorphism g given by the previous lemma which is $a_K - C^\infty$-close to the identity, coincides with the identity in the $K/10$-neighborhoods of ∂B_{n+1} and ∂B_n and such that $g(V_i^n)$ is $K/2$-dense for $i = 1, 2$.

We define then f_{n+1} in $B_{n+1} \backslash B_n$ as $d_2 \circ g^{-1}$ which clearly glues together with f_n and satisfies properties (I2) and (I5).

To define V_i^{n+1} we consider a very small $\varepsilon > 0$ (in order that $g(V_i^n)$ is also $K/2-\varepsilon$-dense) and for each boundary component C of $g(V_i^n)$ (which is a curve) we consider a curve C' which is at distance less than ε of C inside $g(V_i^n)$ and such that each when it approaches $C \cap \partial B_n$ the distance goes to zero and when it approaches $C \cap \partial B_{n+1}$ the distance goes to ε. This allows to define new V_i^{n+1} as the open set delimited by these curves united with the initial V_i^n. It is not hard to see that it will satisfy (I3) and (I4).

We have then constructed a C^∞-diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is at C^∞ distance a_K of d_2 and such that there are two disjoint open connected sets V_1 and V_2 such that $f(\overline{V_i}) \subset V_i$. and such that both of them are $K/2$-dense in $\mathbb{R}^2$.

We now indicate the proof of the Lemma we have used:

Proof of Lemma B.1.1. The proof follows from the following simple bound:

Claim. *There exists $A > 0$ such that for every pair of curves γ_1, γ_2 in the square $[-2, 2]^2$ which touch both boundaries and intersect $[-1, 1]^2$ we have that there exists a C^∞-diffeomorphism h of C^∞-norm less than A and which coincides with the identity in the boundaries of the cube such that the image of both curves is $1/4$-dense.*

We can assume that n is large enough since we can get a bound by hand on the rest of B_n's.

Now, to prove the Lemma it is enough to subdivide the complement of the $K/10$-neighborhoods of the boundaries of $B_n \setminus B_{n-1}$ into sets S_k such that they contain balls of radius $4K$ and are contained in balls of radius $5K$. Moreover, we can choose these sets S_k in order to verify that for some positive constant B we have:

- S_k is diffeomorphic to $[-2,2]^2$ via a C^∞ diffeomorphism $l_k : [-2,2]^2 \to S_k$ of norm less than B.

- The diffeomorphism l_k sends $1/4$-dense subsets of $[-2,2]^2$ into $K/2$-dense subsets of S_k.

Now, using the claim it is not hard to see that we can construct the desired diffeomorphism g of C^∞-norm less than AB.

$\square$

B.2 Proof of the Theorem

We first show the existence of a continuous function $h : \mathbb{R}^2 \to \mathbb{R}^2$ conjugating f to d_2 which is close to the identity.

This is quite classical, consider a point $x \in \mathbb{R}^2$, so, since $d_{C^0}(f, d_2) < K$ we get that the orbit $\{f^n(x)\}$ is in fact a $K-$pseudo-orbit of d_2. Since d_2 is infinitely expansive, there exists only one orbit $\{d_2^n(y)\}$ which $\alpha(K)$-shadows $\{f^n(x)\}$ and we define $h(x) = y$ (in fact, in this case, it suffices with the past pseudo-orbit to find the shadowing).

We get that h is continuous since when $x_n \to x$ then the pseudo-orbit which shadows must rest near for more and more time, and then, again by expansivity, one concludes. This implies also that h is onto since it is at bounded distance of the identity.

Now, consider any ball B of radius $100\alpha(K)$ in $\mathbb{R}^2$, it is easy to see that $f(B)$ is contained in a ball of radius $50\alpha(K)$ and then, we get a way to identify the preimage of points by h. Consider a point $x \in \mathbb{R}^2$, we get that

$$h^{-1}(h(x)) = \bigcap_{n>0} f^n(B_{100\alpha(K)}(f^{-n}(x)))$$

So, h is also cellular.

It only remains to show that the image under h of both V_1 and V_2 is the whole plane. Since they share equal properties, it will be enough to prove it for one of them.

Lemma B.2.1. *$h(V_i) = \mathbb{R}^2$ for $i = 1, 2$.*

PROOF. We shall show that $h(\overline{V_i})$ is dense. Since h is proper, it is closed: this will imply that it is in fact the whole plane. Using the semiconjugacy and the fact that $f(\overline{V_i}) \subset V_i$ this would prove the lemma.

To prove that $h(\overline{V_i})$ is dense, we consider an arbitrary open set $U \subset \mathbb{R}^2$. Now, choose n_0 such that $d_2^{-n_0}(U)$ contains a ball of radius $10\alpha(K)$. We get that $h^{-1}(d_2^{-n_0}(U))$ contains a ball of radius $9\alpha(K)$ and thus, since $\alpha(K) > K$, we know that since V_i is $K/2$-dense, we get that $V_i \cap h^{-1}(d_2^{-n_0}(U)) \neq \emptyset$. So, since $f(\overline{V_i}) \subset V_i$ we get that $V_i \cap f^{n_0} \circ h^{-1}(d_2^{-n_0}(U)) \neq \emptyset$ which using the semiconjugacy gives us that $h(V_i) \cap U \neq \emptyset$.

This concludes.

□

B.3 Hölder continuity

In this section, we shall prove that in fact h is α-Holder continuous (see also Theorem 19.2.1 of [KH]). Since the boundary of ∂V_i for each i is a space filling curve in arbitrarily small domains and by some easy estimates on the change of Hausdorff dimension by Hölder maps, we see easily that $\alpha \leq \frac{1}{2}$.

To prove the existence of $\alpha > 0$ such that h is α-Hölder, consider C to be a (uniform) bound on $\|Df^{-1}\|$ (recall that f can be choosen "close" to d_2). We choose also $\alpha > 0$ such that $C^\alpha < 2$.

Also, from how we constructed the semiconjugacy h, we see that there exists A_1 and A_2 such that $d(x, y) < A_1$ implies that $d(h(x), h(y)) < A_2$. Now, consider a pair of points $x, y \in \mathbb{R}^2$ such that $\delta = d(x, y)$ is sufficiently small (say, smaller than A_1).

We consider n_0 such that $C^{-n_0}\delta < A_1 \leq C^{-n_0-1}\delta$. We have that

$$2^{-n_0} A_1^\alpha \leq 2^{-n_0} C^{-\alpha n_0} C^{-\alpha} \delta^\alpha \leq C^{-\alpha} \delta^\alpha$$

Now, since $d_2^n \circ h \circ f^{-n}(x) = h(x)$ for every n and x we get

$$d(h(x), h(y)) = d(d_2^{n_0} \circ h \circ f^{-n_0}(x), d_2^{n_0} \circ h \circ f^{-n_0}(y)) = 2^{-n_0} d(h(f^{-n_0}(x)), h(f^{-n_0}(y)))$$

But, from how we choose C, we get that $d(f^{-n_0}(x), f^{-n_0}(y)) < A_1$, so

$$d(h(x), h(y)) \leq 2^{-n_0} A_2$$

From the above, we obtain thus

$$d(h(x), h(y)) \leq 2^{-n_0} A_1^\alpha \frac{A_2}{A_1^\alpha} \leq C^{-\alpha} \frac{A_2}{A_1^\alpha} \delta^\alpha = \left(C^{-\alpha} \frac{A_2}{A_1^\alpha} \right) d(x, y)^\alpha$$

Appendix C

Irrational pseudo-rotations of the torus

This appendix was taken from [Pot_4], but we have added Section C.4.

We consider $\mathrm{Homeo}_0(\mathbb{T}^2)$ to be the set of homeomorphisms homotopic to the identity. We shall say that $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is *non-resonant* if the rotation set of f is a unique vector (α, β) and the values $1, \alpha, \beta$ are irrationally independent (i.e. α, β and α/β are not rational). This ammounts to say that given any lift F of f to $\mathbb{R}^2$, for every $z \in \mathbb{R}^2$ we have that:

$$\lim_{n\to\infty} \frac{F^n(z) - z}{n} = (\alpha, \beta)(\mathrm{mod}\, \mathbb{Z}^2) \tag{C.1}$$

In general, one can define the rotation set of a homeomorphism homotopic to the identity (see [MiZi]). In fact, although we shall not make it explicit, our constructions work in the same way for homeomorphisms of the torus whose rotation set is contained in a segment of slope (α, β) with α, β and α/β irrational and not containing zero.

Non-resonant torus homeomorphisms[1] have been intensively studied in the last years looking for resemblance between them and homeomorphisms of the circle with irrational rotation number (see [Kwap], [LeC], [Jag_1]) and also constructing examples showing some difference between them (see [Fay], [BCL], [BCJL], [Jag_2]).

In [Kwak] the possible topologies of minimal sets these homeomorphisms admit are classified and it is shown that under some conditions, these minimal sets are unique and coincide with the non-wandering set[2]. However, there is one kind of

[1] These are called *irrational pseudo-rotations* by several authors, but since some of them use the term exclusively for conservative ones, we adopt the definition used in [Kwak].

[2] A point x is *wandering* for a homeomorphism f if there exists a neighborhood U of x such that $f^n(U) \cap U = \emptyset$ for every $n \neq 0$. The *non-wandering* set is the closed set of points which are not wandering.

topology of minimal sets where the question of the uniqueness of minimal sets remains unknown. When the topology of a minimal set is of this last kind, [BCJL] constructed an example where the non wandering set does not coincide with the unique minimal set, in fact, they construct a transitive non-resonant torus homeomorphism containing a proper minimal set as a skew product over an irrational rotation.

A natural example of non-resonant torus homeomorphism is the one given by a homeomorphism semiconjugated to an irrational rotation by a continuous map homotopic to the identity. In [Jag_1] it is proved that a non-resonant torus homeomorphism is semiconjugated to an irrational rotation under some quite mild hypothesis.

Under the hypothesis of being semiconjugated by a monotone map[3] which has points whose preimage is a singleton, it is not hard to show the uniqueness of a minimal set (see for example [Kwak] Lemma 14). However, as shown in Appendix B (see also [Ro]), a continuous monotone map may be very degenerate and thus even if there exist such a semiconjugation, it is not clear whether there should exist a unique minimal set nor the kind of recurrence the homeomorphisms should have. Moreover, for general non-resonant torus homeomorphisms, there does not exist a semiconjugacy to the irrational rotation (even when there is "bounded mean motion", see [Jag_2]).

Here, we give a simple and self-contained proof (based on some ideas of [Kwak] but not on the classification of the topologies of the minimal sets) of a result which shows that even if there may be more than one minimal set, the dynamics is in some sense irreducible. Clearly, transitivity of f may not hold for a general non-resonant torus homeomorphism (it may even have wandering points, as in the product of two Denjoy counterexamples; some more elaborate examples may be found in [Kwak]), but we shall show that, in fact, these homeomorphisms are weakly transitive. For a homeomorphism f we shall denote $\Omega(f)$ to the non-wandering set of f (i.e. the set of points x such that for every neighborhood U of x there exists $n > 0$ with $f^n(U) \cap U \neq \emptyset$).

Theorem C.0.1. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a non-resonant torus homeomorphism, then, $f|_{\Omega(f)}$ is weakly transitive.*

Recall that for $h : M \to M$ a homeomorphism, and K an $h-$invariant compact set, we say that $h|_K$ is *weakly transitive* if given two open sets U and V of M intersecting K, there exists $n > 0$ such that $h^n(U) \cap V \neq \emptyset$ (the difference with being transitive is that for transitivity one requires the open sets to be considered relative to K).

This allows to re-obtain Corollary E of [Jag_1]:

[3] A monotone map is a map whose preimages are all compact and connected.

Corollary C.0.2. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a non-resonant torus homeomorphism such that $\Omega(f) = \mathbb{T}^2$. Then, f is transitive.*

In fact, as a consequence of weak-transitivity, we can obtain also the more well known concept of chain-transitivity for non-resonant torus homeomorphisms.

Corollary C.0.3. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a non-resonant torus homeomorphism, then, f is chain-transitive.*

Recall that a homeomorphism h of a compact metric space M is *chain-transitive* if for every pair of points $x, y \in M$ and every $\varepsilon > 0$ there exists an $\varepsilon-$pseudo-orbit $x = z_0, \ldots, z_n = y$ with $n \geq 1$ (i.e. $d(z_{i+1}, h(z_i)) < \varepsilon$).

PROOF. Consider two points $x, y \in M$ and $\varepsilon > 0$.

We first assume that $x \neq y$ are both nonwandering points which shows the idea in a simpler way. From Theorem A we know that there exists a point z and $n > 0$ such that $d(z, f(x)) < \varepsilon$ and $d(f^{n+1}(z), y) < \varepsilon$. We can then consider the $\varepsilon-$pseudo-orbit: $\{x, z, \ldots, f^n(z), y\}$.

Now, for general $x, y \in \mathbb{T}^2$ we consider $n_0 \geq 1$ such that $d(f^{n_0+1}(x), \Omega(f)) < \varepsilon/2$ and $d(f^{-n_0}(y), \Omega(f)) < \varepsilon/2$. Now, by Theorem A there exists $z \in \mathbb{T}^2$ and $n > 0$ such that $d(z, f^{n_0}(x)) < \varepsilon$ and $d(f^{n+1}(z), f^{-n_0}(y)) < \varepsilon$. Considering the following $\varepsilon-$psudo-orbit $\{x, \ldots, f^{n_0-1}(x), z, \ldots, f^n(z), f^{-n_0}(y), \ldots, y\}$ we obtain a pseudo-orbit from x to y and thus proving chain-transitivity.

□

Remark C.0.4. We have proved that in fact, for every $\varepsilon > 0$ the pseudo-orbit can be made with only two "jumps".

As a consequence of our study, we obtain the following result which may be of independent interest:

Proposition C.0.5. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a non-resonant torus homeomorphism and Λ_1 a compact connected set such that $f(\Lambda_1) \subset \Lambda_1$. Then, for every U connected neighborhood of Λ_1, there exists $K > 0$ such that:*

- *If Λ_2 is a compact set which has a connected component in the universal cover of diameter larger than K then*[4]*,*

$U \cap \Lambda_2 \neq \emptyset$.

One could wonder if the stronger property of $\Omega(f)$ being transitive may hold. However, in section C.3 we present an example where $\Omega(f)$ is a Cantor set times $\mathbb{S}^1$, but for which the nonwandering set is not transitive.

[4]This holds if Λ_2 is a connected set such that $f^i(\Lambda_2) \subset \Lambda_2$ for some $i \in \mathbb{Z}$ for example.

C.1 Reduction of the proofs of Theorem C.0.1 and Proposition C.0.5

In this section we shall reduce the proofs of Theorem C.0.1 and Proposition C.0.5 to Proposition C.1.2 and its Addendum C.1.3.

We shall use the word *domain* to refer to an open and connected set. We shall say a domain $U \in \mathbb{T}^2$ is *inessential, simply essential or doubly essential* depending on whether the inclusion of $\pi_1(U)$ in $\pi_1(\mathbb{T}^2)$ is isomorphic to $0, \mathbb{Z}$ or $\mathbb{Z}^2$ respectively[5]. If U is simply essential or doubly essential, we shall say it is *essential.*

Remark C.1.1. Notice that if U and V are two doubly essential domains, then $U \cap V \neq \emptyset$. This is because the intersection number of two closed curves is a homotopy invariant and given two non-homotopic curves in $\mathbb{T}^2$, they have non-zero intersection number, thus, they must intersect. Since clearly, being doubly essential, U and V contain non homotopic curves, we get the desired result.

$\diamondsuit$

We claim that Theorem C.0.1 can be reduced to the following proposition.

Proposition C.1.2. *Given $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ a non-resonant torus homeomorphism and U an open set such that $f(U) \subset U$ and U intersects $\Omega(f)$, then we have that U has a connected component which is doubly essential.*

Almost the same proof also yields the following statement which will imply Proposition B:

Addendum C.1.3. *For f as in Proposition C.1.2, if Λ is a compact connected set such that $f(\Lambda) \subset \Lambda$, then, for every connected open neighborhood U of Λ, we have that U is doubly-essential.*

Notice that the fact that $f(\Lambda) \subset \Lambda$ for Λ compact implies that it contains recurrent points, and in particular, $\Lambda \cap \Omega(f) \neq \emptyset$.

PROOFOF THEOREM A AND PROPOSITION B. Let us consider two open sets U_1 and V_1 intersecting $\Omega(f)$, and let $U = \bigcup_{n>0} f^n(U_1)$ and $V = \bigcup_{n<0} f^n(V_1)$. These sets verify that $f(U) \subset U$ and $f^{-1}(V) \subset V$ and both intersect the nonwandering set.

Proposition C.1.2 (applied to f and f^{-1}) implies that both U and V are doubly essential, so, they must intersect. This implies that for some $n > 0$ and $m < 0$ we have that $f^n(U_1) \cap f^m(V_1) \neq \emptyset$, so, we have that $f^{n-m}(U_1) \cap V_1 \neq \emptyset$ and thus $\Omega(f)$ is weakly transitive.

[5] In [Kwak] these concepts are called *trivial, essential* and *doubly-essential.*

Proposition B follows directly from Addendum C.1.3 since given a doubly-essential domain U in $\mathbb{T}^2$, there exists $K > 0$ such that its lift $p^{-1}(U)$ intersects every connected set of diameter larger than K.

$\square$

Remark C.1.4. Notice that in higher dimensions, Remark C.1.1 does not hold. In fact, it is easy to construct two open connected sets containing closed curves in every homotopy class which do not intersect. So, even if we could show a result similar to Proposition C.1.2, it would not imply the same result.

$\diamondsuit$

C.2 Proof of Proposition C.1.2

Consider a non-resonant torus homeomorphism $f \in \text{Homeo}_0(\mathbb{T}^2)$, and let us assume that U is an open set which verifies $f(U) \subset U$ and $U \cap \Omega(f) \neq \emptyset$.

Since $U \cap \Omega(f) \neq \emptyset$, for some $N > 0$ we have that there is a connected component of U which is f^N-invariant. We may thus assume from the start that U is a domain such that $f(U) \subset U$ and $U \cap \Omega(f) \neq \emptyset$.

Let $p : \mathbb{R}^2 \to \mathbb{T}^2$ be the canonical projection. Consider $U_0 \subset p^{-1}(U)$ a connected component. We can choose F a lift of f such that $F(U_0) \subset U_0$.

We shall denote $T_{p,q}$ to the translation by vector (p, q), that is, the map from the plane such that $T_{p,q}(x) = x + (p, q)$ for every $x \in \mathbb{R}^2$.

Lemma C.2.1. *The domain U is essential.*

Proof. Consider $x \in U_0$ such that $p(x) \in \Omega(f)$. And consider a neighborhood $V \subset U_0$ of x. Assume that there exists $n_0 > 0$ and $(p, q) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $F^{n_0}(V) \cap (V + (p, q)) \neq \emptyset$. Since U_0 is F-invariant, we obtain two points in U_0 which differ by an integer translation, and since U_0 is connected, this implies that U contains a non-trivial curve in $\pi_1(\mathbb{T}^2)$ and thus, it is essential.

To see that there exists such n_0 and (p, q), notice that otherwise, since x is not periodic (because f is a non-resonant torus homeomorphism) we could consider a basis V_n of neighborhoods of $p(x)$ such that $f^k(V_n) \cap V_n = \emptyset$ for every $0 < k \leq n$. Since x is non-wandering, there exists some $k_n > n$ such that $f^{k_n}(V_n) \cap V_n \neq \emptyset$, but since we have that $F^{k_n}(V_n) \cap (V_n + (p, q)) = \emptyset$ for every $(p, q) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, we should have that $F^{k_n}(V_n) \cap V_n \neq \emptyset$ for every n. Since $k_n \to \infty$, we get that f has zero as rotation vector, a contradiction.

$\square$

We conclude the proof of by showing the following lemma which has some resemblance with Lemma 11 in [Kwak].

Lemma C.2.2. *The domain U is doubly-essential.*

PROOF. Assume by contradiction that U is simply-essential.

Since the inclusion of $\pi_1(U)$ in $\pi_1(\mathbb{T}^2)$ is non-trivial by the previous lemma, there exists a closed curve η in U such that when lifted to $\mathbb{R}^2$ joins a point $x \in U_0$ with $x + (p, q)$ (which will also belong to U_0 because η is contained in U and U_0 is a connected component of $p^{-1}(U)$).

We claim that in fact, we can assume that η is a simple closed curve and such that $g.c.d(p, q) = 1$ (the greatest common divisor). In fact, since U is open, we can assume that the curve we first considered is in general position, and by considering a subcurve, we get a simple one (maybe the point x and the vector (p, q) changed, but we shall consider the curve η is the simple and closed curve from the start). Since it is simple, the fact that $g.c.d(p, q) = 1$ is trivial.

If η_0 is the lift of η which joins $x \in U_0$ with $x + (p, q)$, we have that it is compact, so, we get that

$$\tilde{\eta} = \bigcup_{n\in\mathbb{Z}} T_{np,nq}\eta_0$$

is a proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$. Notice that $\tilde{\eta} \subset U_0$.

By extending to the one point compactification of $\mathbb{R}^2$ we get by using Jordan's Theorem (see [Mo] chapter 4) that $\tilde{\eta}$ separates $\mathbb{R}^2$ in two disjoint unbounded connected components which we shall call L and R and such that their closures $L \cup \tilde{\eta}$ and $R \cup \tilde{\eta}$ are topologically a half plane (this holds by Schönflies Theorem, see [Mo] chapter 9).

Consider any pair a, b such that[6] $\frac{a}{b} \neq \frac{p}{q}$, we claim that $T_{a,b}(\tilde{\eta}) \cap U_0 = \emptyset$. Otherwise, the union $T_{a,b}(\tilde{\eta}) \cup U_0$ would be a connected set contained in $p^{-1}(U)$ thus in U_0 and we could find a curve in U_0 joining x to $x + (a, b)$ proving that U is doubly essential (notice that the hypothesis on (a, b) implies that (a, b) and (p, q) generate a subgroup isomorphic to $\mathbb{Z}^2$), a contradiction.

Translations are order preserving, this means that $T_{a,b}(R) \cap R$ and $T_{a,b}(L) \cap L$ are both non-empty and either $T_{a,b}(R) \subset R$ or $T_{a,b}(L) \subset L$ (both can only hold in the case $\frac{a}{b} = \frac{p}{q}$). Also, one can easily see that $T_{a,b}(R) \subset R$ implies that $T_{-a,-b}(L) \subset L$.

Now, we choose (a, b) such that there exists a curve γ from x to $x+(a, b)$ satisfying:

- $T_{a,b}(\tilde{\eta}) \subset L$.
- γ is disjoint from $T_{p,q}(\gamma)$.
- γ is disjoint from $T_{a,b}(\tilde{\eta})$ and $\tilde{\eta}$ except at its boundary points.

[6] We accept division by 0 as being infinity.

We consider $\tilde{\eta}_1 = T_{a,b}(\tilde{\eta})$ and $\tilde{\eta}_2 = T_{-a,-b}(\tilde{\eta})$. Also, we shall denote $\tilde{\gamma} = \gamma \cup T_{-a,-b}(\gamma)$ which joins $x - (a,b)$ with $x + (a,b)$.

We obtain that U_0 is contained in $\Gamma = T_{a,b}(R) \cap T_{-a,-b}(L)$ a band whose boundary is $\tilde{\eta}_1 \cup \tilde{\eta}_2$.

Since U_0 is contained in Γ and is F-invariant, for every point $x \in U_0$ we have that $F^n(x)$ is a sequence in Γ, and since f is a non-resonant torus homeomorphism, we have that $\lim \frac{F^n(x)}{n} = \lim \frac{F^n(x)-x}{n} = (\alpha, \beta)$ is totally irrational.

However, we notice that Γ can be written as:

$$\Gamma = \bigcup_{n \in \mathbb{Z}} T_{np,nq}(\Gamma_0)$$

where Γ_0 is a compact set in $\mathbb{R}^2$. Indeed, if we consider the curve $\tilde{\gamma} \cup T_{a,b}(\eta_0) \cup T_{p,q}(\tilde{\gamma}) \cup T_{-a,-b}(\eta_0)$ we have a Jordan curve. Considering Γ_0 as the closure of the bounded component we have the desired fundamental domain.

So, if we consider a sequence of points $x_n \in \Gamma$ such that $\lim \frac{x_n}{n}$ exists and is equal to v it will verify that the coordinates of v have the same proportion as p/q, thus cannot be totally irrational. This is a contradiction and concludes the proof of the Lemma.

$\square$

We conclude this section by showing how the proof adapts to the case stated in Addendum C.1.3. Consider a compact connected set Λ such that $f(\Lambda) \subset \Lambda$, then, we have that Λ contains points which are recurrent[7].

Let $\tilde{\Lambda}$ be a connected component of $p^{-1}(\Lambda)$ which is F-invariant. Now, if U is an open connected neighborhood of Λ and U_0 is a connected component of $p^{-1}(U)$ containing $\tilde{\Lambda}$. Notice that $d(\partial U, \Lambda) > \delta > 0$ so $d(\partial U_0, \tilde{\Lambda}) > \delta$ also.

Now, the same argument in Lemma C.2.1 can be used in order to show that U must be essential: We can choose a point $x \in \tilde{\Lambda} \subset U_0$ such that $p(x)$ is recurrent and the same argument shows that there will exist $(p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $F^{n_0}(x)$ is δ-close to $x + (p,q)$ and since $F^{n_0}(x)$ must be contained in $\tilde{\Lambda}$ we get that $x + (p,q)$ is contained in U_0 showing that U is essential.

The proof that in fact U is doubly-essential is now the same as in Lemma C.2.2 since one can see that invariance of U was not used in the proof, one only needs that there are points in U_0 such that the orbits by F remain in U_0 and this holds for every point in $\tilde{\Lambda}$.

[7]Since it is a compact invariant set, it contains a minimal set whose points will be all recurrent.

C.3 An example where $f|_{\Omega(f)}$ is not transitive

The example is similar to the one in section 2 of [Jag$_2$], however, we do not know a priori if our specific examples admit or not a semiconjugacy.

Consider $g_1 : S^1 \to S^1$ and $g_2 : S^1 \to S^1$ Denjoy counterexamples with rotation numbers ρ_1 and ρ_2 which are irrationally independent and have minimal invariant sets M_1 and M_2 properly contained in S^1. We shall consider the following skew-product map $f_\beta : \mathbb{T}^2 \to \mathbb{T}^2$ given by:

$$f_\beta(s,t) = (g_1(s), \beta(s)(t))$$

where $\beta : S^1 \to \text{Homeo}_+(S^1)$ is continuous and such that $\beta(s)(t) = g_2(t)$ for every $(s,t) \in M_1 \times S^1$.

The same proof as in Lemma 2.1 of [Jag$_2$] yields:

Lemma C.3.1. *The map f_β is a non-resonant torus homeomorphism and $M_1 \times M_2$ is the unique minimal set.*

Proof. The proof is the same as the one in Lemma 2.1 of [Jag$_2$]. Indeed any invariant measure for f must be supported in $M_1 \times M_2$ and the dynamics there is the product of two Denjoy counterexamples and thus uniquely-ergodic. Since rotation vectors can be computed with ergodic measures, we also get that f_β has a unique rotation vector (ρ_1, ρ_2) which is totally irrational by hypothesis.

□

Clearly, if we restrict the dynamics of f_β to $M_1 \times S^1$ it is not hard to see that the nonwandering set will be $M_1 \times M_2$ (it is a product system there). So, we shall prove that if β is properly chosen, we get that $\Omega(f_\beta) = M_1 \times S^1$. In fact, instead of constructing a specific example, we shall show that for "generic" β in certain space, this is satisfied, this will give the existence of such a β.

First, we define $\mathcal{B}$ to be the set of continuous maps $\beta : S^1 \to \text{Homeo}_+(S^1)$ such that $\beta(s) = g_2$ for every $s \in M_1$. We endow $\mathcal{B}$ with the topology given by restriction from the set of every continuous map from S^1 to $\text{Homeo}_+(S^1)$. With this topology, $\mathcal{B}$ is a closed subset of the set of continuous maps from $S^1 \to \text{Homeo}_+(S^1)$ which is a Baire space, thus, $\mathcal{B}$ is a Baire space.

So, the existence of the desired β is a consequence of:

Lemma C.3.2. *There exists a dense G_δ (residual) subset of $\mathcal{B}$ of maps such that the induced map f_β verifies that $\Omega(f_\beta) = M_1 \times \mathbb{S}^1$.*

Proof. First, we will prove the lemma assuming the following claim:

Claim. *Given $\beta \in \mathcal{B}$, $x \in M_1 \times S^1$, $\varepsilon > 0$ and $\delta > 0$ there exists $\beta' \in \mathcal{B}$ which is $\delta-$close to β such that there exists $k > 0$ with $f^k_{\beta'}(B(x,\varepsilon)) \cap B(x,\varepsilon) \neq \emptyset$.*

Assuming this claim, the proof of the Lemma is a standard Baire argument: Consider $\{x_n\} \subset M_1 \times S^1$ a countable dense set. Using the claim, we get that the sets $\mathcal{B}_{n,N}$ consisting of the functions $\beta \in \mathcal{B}$ such that there exists a point y and a value $k > 0$ such that y and $f^k_\beta(y)$ belong to $B(x_n, 1/N)$ is a dense set. Also, the set $\mathcal{B}_{n,N}$ is open, since the property is clearly robust for C^0 perturbations of f_β. This implies that the set $\mathcal{R} = \bigcap_{n,N} \mathcal{B}_{n,N}$ is a residual set, which implies, by Baire's theorem that it is in fact dense.

For $\beta \in \mathcal{R}$ we get that given a point $x \in M_1 \times S^1$ and $\varepsilon > 0$, we can choose $x_n \in B(x, \varepsilon/2)$ and N such that $1/N < \varepsilon/2$. Since $\beta \in \mathcal{B}_{n,N}$ we have that there exists $k > 0$ such that $f^k_\beta(B(x,\varepsilon)) \cap B(x,\varepsilon) \neq \emptyset$ proving that $M_1 \times S^1$ is nonwandering for f_β as desired.

Proofof the Claim. The point $x \in M_1 \times S^1$ can be written as (s,t) in the canonical coordinates.

Choose an interval $(a,b) \subset (s-\varepsilon, s+\varepsilon)$ contained in a wandering interval of g_1. Then, there exists a sequence of integers $k_n \to +\infty$ such that $g_1^{k_n}((a,b)) \subset (s-\varepsilon, s+\varepsilon)$ for all $n \geq 0$. Further, the orbits of a and b are disjoint and do not belong to M_1. Let $\gamma = (a,b) \times \{t\}$.

We can assume that $f^{k_n}_\beta(\gamma) \cap B(x,\varepsilon) = \emptyset$ for every $n > 0$, otherwise, there is nothing to prove.

We shall thus consider a $\delta-$perturbation of β such that it does not modify the orbit of (a,t) but moves the orbit of (b,t) in one direction making it give a complete turn around $\mathbb{S}^1$ and thus an iterate of γ will intersect $B(x,\varepsilon)$.

Let $s_n = g_1^n(b)$ and $\beta^n(s_0) = \beta(s_{n-1}) \circ \ldots \circ \beta(s_0)$. Note that $\frac{1}{n}\beta^n(s_0)(t) \to \rho(g_2)$ as $n \to \infty$ since $\beta(s_k) \to g_2$ as $k \to \infty$. At the same time, if we let

$$\beta^n_\theta(s_0) = R_\theta \circ \beta(s_{n-1}) \circ R_\theta \circ \beta_{s_{n-2}} \circ \ldots \circ R_\theta \circ \beta(s_0)$$

Then, $\beta^n_\theta(s_0)(t) \to \rho' > \rho(g_2)$ since $R_\theta \circ \beta_{s_k}$ converges to $R_\theta \circ g_2$ which has rotation number strictly greater than g_2 (see for example [KH] Proposition 11.1.9). If we denote by $\tilde{\beta}^n$, respectively $\tilde{\beta}^n_\theta$ the lifts of β^n and β^n_θ to $\mathbb{R}$, then, this implies that there exists n_0 such that for $n > n_0$ one has

$$|\tilde{\beta}^n_\theta(t) - \tilde{\beta}^n(t)| > 1$$

So, if we consider $k_n > n_0$ and we choose β' such that:

- it coincides with β in the g_1-orbit of a,
- it coincides with $R_\theta \circ \beta$ in the points $\{b, g_1(b), \ldots, g_1^{k_n}(b)\}$,

- is at distance smaller than δ from β,

we have that $f_{\beta'}^{k_n}(\gamma) \cap B(x,\varepsilon) \neq \emptyset$ as desired.

$\diamondsuit$

$\square$

C.4 The homotopy class of the dehn-twist

We provide here a simple extension of the results of this appendix to homeomorphisms which are not homotopic to the identity. We call *dehn twist* to the torus homeomorphism whose lift to the universal cover can be written in the form $(x,y) \mapsto (x, x+y)$.

We say that a homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the dehn-twist is *non-resonant* if there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that for every $x \in \mathbb{R}^2$ and for some lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ one has that:

$$\lim_{n\to\infty} \frac{p_1(F^n(x)) - p_1(x)}{n} = \alpha$$

where $p_1 : \mathbb{R}^2 \to \mathbb{R}$ is the projection in the first coordinate. A classical example is given by the projection of the torus of the following plane map:

$$(x,y) \mapsto (x+\alpha, x+y)$$

With essentially the same proof as Theorem C.0.1 we can prove:

Theorem C.4.1. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a non-resonant torus homeomorphism homotopic to a dehn-twist. Then $\Omega(f)$ is weakly transitive.*

The proof is essentially the same as the one of Theorem C.0.1 so we will only give a sketch.

In fact, the result can be reduced to the following statement analogous to Proposition C.1.2.

Proposition C.4.2. *Given $f : \mathbb{T}^2 \to \mathbb{T}^2$ a non-resonant torus homeomorphism homotopic to a dehn-twist and U an open set such that $f(U) \subset U$ and U intersects $\Omega(f)$, then we have that U has a connected component which is doubly essential.*

The proof of the reduction is exactly the same as for Theorem C.0.1.

Let us now prove the proposition.

SKETCH. As in the proof of Proposition C.1.2 we can assume that U is connected and such that $f(U) \subset U$.

Let us call U_0 to a connected component of $p^{-1}(U)$ and we will choose a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ which fixes U_0.

We consider $x \in U_0$ such that $p(x) \in \Omega(f)$. Let $V \subset U_0$ be a connected neighborhood of x.

Assume that there exists $n_0 > 0$ such that $F^{n_0}(V) \cap (V + (p, q)) \neq \emptyset$. Then, as in Lemma C.2.1 we can prove that there is a loop γ contained in U which is homotopic to the loop joining $(0, 0)$ with (p, q).

Now, assume that $p \neq 0$, then, we have that $f(\gamma)$ which is homotopic to $(p, q+1)$ is also contained in U by invariance of U and the fact that f is homotopic to a dehn-twist. Since (p, q) and $(p, q + 1)$ are linearly independent when $p \neq 0$ we get that U should be doubly essential.

To prove that there exists n_0 such that $F^{n_0}(V) \cap (V + (p, q)) \neq \emptyset$ for some (p, q) with $p \neq 0$ we use the fact that the rotation number defined above is irrational and an argument very similar to that of Lemma C.2.1.

$\square$

Appendix D

Tame non-robustly transitive diffeomorphisms

In this appendix, we review the results from [BCGP]. Recall the discussion after Corollary 1.1.23.

Given an $n-$dimensional manifold M (with $n \geq 3$), we consider $\mathrm{Diff}^r(M)$ the set of diffeomorphisms of M endowed with the C^r topology ($r \geq 1$).

Theorem. *There exists a C^1-open set $\mathcal{U} \subset \mathrm{Diff}^r(M)$ $(1 \leq r \leq \infty)$, a C^r-dense subset $\mathcal{D}$ of $\mathcal{U}$ and an open set $U \subset M$ with the following properties:*

(I) Isolation*: For every $f \in \mathcal{U}$, the set $\mathcal{C}_f := U \cap \mathcal{R}(f)$ is a chain-recurrence class.*

(II) Non-robust transitivity*: For every $f \in \mathcal{D}$, the class $\mathcal{C}_f$ is not transitive.*

More precisely:

(1) For any $f \in \mathcal{U}$ there exist a subset $H_f \subset \mathcal{C}_f$ which coincides with the homoclinic class of any hyperbolic periodic $x \in \mathcal{C}_f$. Moreover, each pair of hyperbolic periodic points in $\mathcal{C}_f$ with the same stable dimension is homoclinically related.

(2) For any $f \in \mathcal{U}$ there exist two hyperbolic periodic points $p, q \in \mathcal{C}_f$ satisfying $\dim E^s_p > \dim E^s_q$ and $\mathcal{C}_f$ is the disjoint union of H_f with $W^u(p) \cap W^s(q)$. Moreover the points of $W^u(p) \cap W^s(q)$ are isolated in $\mathcal{C}_f$.

In particular, if $W^u(p) \cap W^s(q) \neq \emptyset$, the class $\mathcal{C}_f$ is not transitive.

(3) One has $\mathcal{D} := \{f \in \mathcal{U} \ : \ W^u(p) \cap W^s(q) \neq \emptyset\}$.

Moreover, this set is a countable union of one-codimensional submanifolds of $\mathcal{U}$.

(4) The chain-recurrent set of any $f \in \mathcal{U}$ is the union of $\mathcal{C}_f$ with a finite number of hyperbolic periodic points (which depend continuously on f).

Remark D.0.3. In the construction, the chain recurrence class $\mathcal{C}_f$ is partially hyperbolic with a one-dimensional central bundle. Thus, it is also far from homoclinic tangencies.

$\diamondsuit$

Remark D.0.4. The isolated points $\mathcal{C}_f \setminus H_f$ are nonwandering for f. However, they do not belong to $\Omega(f|_{\Omega(f)})$ (since they are isolated in $\Omega(f)$ and non-periodic).

$\diamondsuit$

D.1 A mechanism for having isolated points in a chain recurrence class

D.1.1 Preliminaries on invariant bundles

Consider $f \in \mathrm{Diff}^1(M)$ preserving a set Λ.

A Df-invariant subbundle $E \subset T_\Lambda M$ is *uniformly contracted* (resp. *uniformly expanded*) if there exists $N > 0$ such that for every unit vector $v \in E$, we have

$$\|Df^N v\| < \frac{1}{2} \qquad \text{(resp. } > 2).$$

A Df-invariant splitting $T_\Lambda M = E^{ss} \oplus E^c \oplus E^{uu}$ is *partially hyperbolic* if E^{ss} is uniformly contracted, E^{uu} is uniformly expanded, both are non trivial, and if there exists $N > 0$ such that for any $x \in \Lambda$ and any unit vectors $v_s \in E^{ss}_x, v_c \in E^c_x$ and $v_u \in E^{uu}_x$ we have:

$$\|Df^N v_s\| < \frac{1}{2}\|Df^N v_c\| < \frac{1}{4}\|Df^N v_u\|.$$

E^{ss}, E^c and E^{uu} are called the *strong stable*, *center*, and *strong unstable* bundles.

Remark D.1.1. We will sometimes consider a Df-invariant continuous orientation of E^c. When Λ is the union of two different periodic orbits O_p, O_q and of a heteroclinic orbit $\{f^n(x)\} \subset W^u(O_p) \cap W^s(O_q)$, such an orientation exists if and only if above each orbit O_p, O_q, the tangent map Df preserves an orientation of the central bundle.

On a one-dimensional bundle, an orientation corresponds to a unit vector field tangent.

$\diamondsuit$

D.1.2 Cuspidal periodic points

Let p be a hyperbolic periodic point whose orbit is partially hyperbolic with a one-dimensional central bundle. When the central space is stable, there exists a strong stable manifold $W^{ss}(p)$ tangent to E^{ss}_p that is invariant by the iterates f^τ that fix p. It is contained in and separates the stable manifold $W^s(p)$ in two *half stable manifolds* which contain $W^{ss}(p)$ as a boundary.

Let us consider an orientation of E^c_p. The unit vector defining the orientation goes inward on one half stable manifold of p, that we call the *right half stable manifold* $R^s(p)$. The other one is called the *left half stable manifold* $L^s(p)$.

These half stable manifolds are invariant by an iterate f^τ which fixes p if and only if the orientation of E^c_p is preserved by Df^τ_p.

When the central space is unstable, one defines similarly the right and left half unstable manifolds $R^u(p), L^u(p)$.

Definition D.1.1. A hyperbolic periodic point p is *stable-cuspidal* if:

- its orbit is partially hyperbolic, the central bundle is one-dimensional and stable;
- one half stable manifold of p intersects the chain-recurrence class of p only at p.

$\diamondsuit$

When the chain-recurrence class $\mathcal{C}$ containing p is not reduced to the orbit O_p of p, this forces the existence of a Df-invariant orientation on the central bundle of O_p.

In this case, the other half stable manifold intersects $\mathcal{C}$ at points different from p. The choice of the name has to do with the geometry it imposes on $\mathcal{C} \cap W^s(p)$ in a neighborhood of p, see Figure D.1.

This notion appears in [BD$_5$]. It is stronger than the notion of *stable-boundary points* in [CP].

We can define in a similar way the *unstable-cuspidal points*.

Remark D.1.2. If p is a stable-cuspidal point, then the hyperbolic continuation p_g is still stable-cuspidal for every g that is C^1-close to f.

Indeed, there exists a compact set $\Delta \subset L^s(p)$ which meets every orbit of $L^s(p)\setminus\{p\}$ and which is disjoint from $\mathcal{R}(f)$.

By semi-continuity of the chain-recurrent set, a small neighborhood V of Δ is disjoint from $\mathcal{R}(g)$ for any g close to f and meets every orbit of the continuation of $L^s(p) \setminus \{p\}$.

$\diamondsuit$

D.1.3 Description of the mechanism

Let x be a point in a chain-recurrence class $\mathcal{C}$. We introduce the following assumptions (see figure D.2).

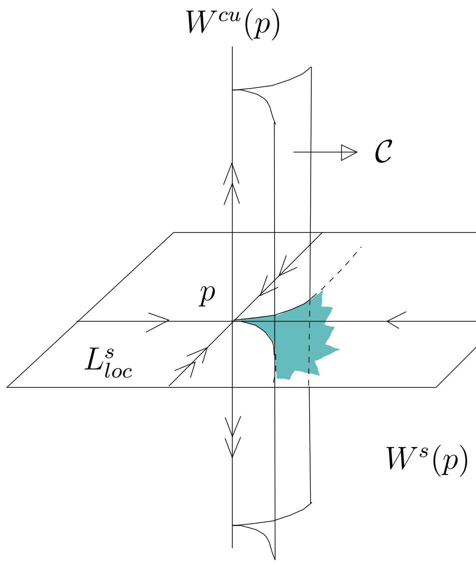

Figure D.1: Geometry of a chain recurrence class $\mathcal{C}$ near a stable-cuspidal fixed point.

(H1) The class $\mathcal{C}$ contains two periodic points p, q such that $\dim(E^s_p) = \dim(E^s_q) + 1$.

(H2) (i) The point p is a stable-cuspidal point.

(ii) The point q is an unstable-cuspidal point.

(H3) The point x belongs to $W^u(p) \cap W^s(q)$. The union Λ of the orbits of x, p, q has a partially hyperbolic decomposition with a one-dimensional central bundle.

Moreover there exists a Df-invariant continuous orientation of the central bundle over Λ such that $\mathcal{C}$ is disjoint from the half manifolds $L^s(p)$ and $R^u(q)$.

Note that from remark D.1.1 and the fact that a central orientation is preserved for cuspidal points, a Df-invariant continuous orientation of the central bundle over Λ always exists.

Proposition D.1.3. *Under (H1)-(H3), the point x is isolated in the chain-recurrence class $\mathcal{C}$. In particular, $\mathcal{C}$ is not transitive.*

D.1.4 Proof of proposition D.1.3

Let q be a periodic point whose orbit is partially hyperbolic and whose central bundle is one-dimensional and unstable. We shall assume that there is an orientation in E^c_q which is preserved by Df.

We fix such an orientation of the central bundle E^c_q, so that the left and right half unstable manifolds of q are defined.

We denote by $d^u + 1$ the unstable dimension of q.

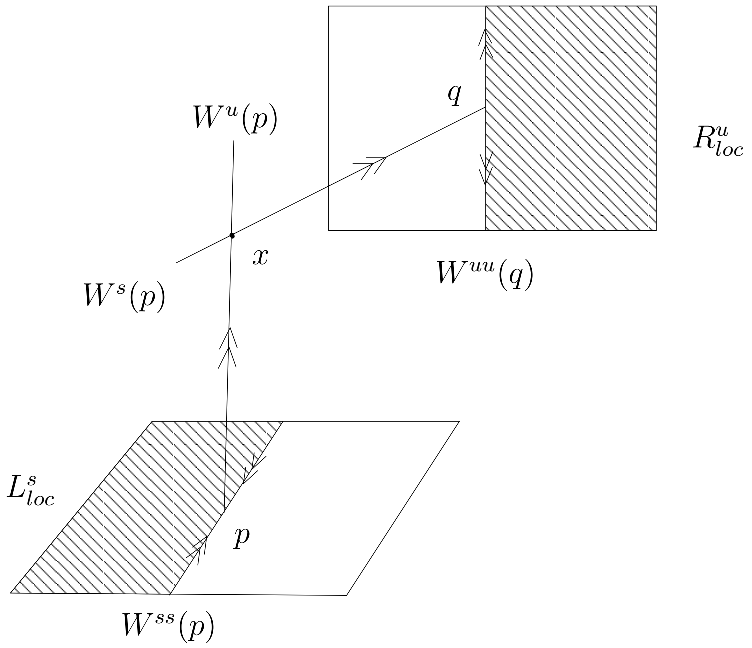

Figure D.2: Hypothesis (H1)-(H3).

Any $x \in W^s(q)$ has uniquely defined stable E^s_x and center stable E^{cs}_x directions: the first one is the tangent space $T_xW^s(q)$; a vector $v \in T_xM \setminus \{0\}$ belongs to the second if the direction of its positive iterates $Df^n(v)$ stays away from the directions of E^{uu}_q.

If $E' \subset E$ are two vector subspaces of T_xM such that E is transverse to E^s_x and E' is transverse to E^{cs}_x (hence E' is one-codimensional in E), then $F = E^{cs}_x \cap E$ is a one-dimensional space whose forward iterates converge to the unstable bundle over the orbit of q. As a consequence, there exists an orientation of F which converges to the orientation of the central bundle by forward iterations.

There is thus a connected component of $E \setminus E'$, such that it intersects F in the orientation of F which converges towards the central orientation, its closure is the *right half plane* of $E \setminus E'$.

The closure of the other component is the *left half plane* of $E \setminus E'$.

Consider a C^1-embedding $\varphi\colon [-1,1]^{d^u} \to M$ such that $x := \varphi(0)$ belongs to $W^s(q)$.

Definition D.1.2. The embedding φ is *coherent with the central orientation at* q if

- $E := D_0\varphi(\mathbb{R}^{d^u+1})$ and $E' := D_0\varphi(\{0\} \times \mathbb{R}^{d^u})$ are transverse to E^s_z, E^{cs}_z respectively;

- the half-plaque $\varphi([0,1] \times [-1,1]^{d^u})$ is tangent to the right half-plane of $E \setminus E'$.

$\diamondsuit$

Let Δ^u be a compact set contained in $R^u(q) \setminus \{q\}$ which meets each orbit of $R^u(q) \setminus \{q\}$.

Lemma D.1.4. *Let $\{\varphi_a\}_{a\in\mathcal{A}}$ be a continuous family of C^1-embeddings that are coherent with the central orientation at q. Consider some $a_0 \in \mathcal{A}$ and a neighborhood V^u of Δ^u. Then, there exist $\delta > 0$ and some neighborhood A of a_0 such that any point $z \in \varphi_a([0,\delta] \times [-\delta,\delta]^{d^u})$ different from $\varphi_a(0)$ has a forward iterate in V^u.*

Proof. Let $\tau \geq 1$ be the period of q and $\chi\colon [-1,1]^d \to M$ be some coordinates such that

- $\chi(0) = q$;
- the image $D^u := \chi((-1,1) \times \{0\}^{d-d^u-1} \times (-1,1)^{d^u})$ is contained in $W^u_{loc}(q)$;
- the image $D^{uu} := \chi(\{0\}^{d-d^u} \times (-1,1)^{d^u})$ is contained in $W^{uu}_{loc}(q)$;
- the image $D^{u,+} := \chi([0,1) \times \{0\}^{d-d^u-1} \times (-1,1)^{d^u})$ is contained in $R^u(q)$;
- $f^{-\tau}(\overline{D^u})$ is contained in D^u.

One deduces that there exists $n_0 \geq 0$ such that:

(i) Any point z close to $\overline{D^{u,+}} \setminus f^{-\tau}(D^u)$ has an iterate $f^k(z)$, $|k| \leq n_0$, in V^u.

The graph transform argument (see for instance [KH, section 6.2]) gives the following generalization of the λ-lemma.

Claim. *There exists $N \geq 0$ and, for all a in a neighborhood A of a_0, there exist some decreasing sequences of disks $(D_{a,n})$ of $[-1,1]^{d^u+1}$ and $(D'_{a,n})$ of $\{0\} \times [-1,1]^{d^u}$ which contain 0 and such that for any $n \geq N$ one has, in the coordinates of χ:*

- *$f^{n\tau}(D_{a,n})$ is the graph of a function $D^u \to \mathbb{R}^{d-d^u-1}$ that is C^1-close to 0;*
- *$f^{n\tau}(D'_{a,n})$ is the graph of a function $D^{uu} \to \mathbb{R}^{d-d^u}$ that is C^1-close to 0.*

Let us consider $a \in A$.

The image by $f^{n\tau}$ of each component of $D_{a,n} \setminus D'_{a,n}$ is contained in a small neighborhood of a component of $D^u \setminus D^{uu}$.

The graph $f^{n\tau}(D'_{a,n})$ which is transverse to a constant cone field around the central direction at q.

Since φ is coherent with the central orientation at q, one deduces that

(ii) $f^{n\tau} \circ \varphi_a\left(([0,1] \times [-1,1]^{d^u}) \cap D_{a,n}\right)$ is contained in a small neighborhood of $D^{u,+}$.

For $\delta > 0$ small, any point $z \in \varphi_a([-\delta,\delta]\times[-\delta,\delta]^{d^u})$ different from $\varphi_a(0)$ belongs to some $D_{a,n}\setminus D_{a,n+1}$, with $n \geq N$. Consequently:

(iii) Any $z \in \varphi_a([-\delta,\delta]\times[-\delta,\delta]^{d^u})\setminus\{\varphi_a(0)\}$ has a forward iterate in $D^u\setminus f^{-1}(D^u)$.

Putting the properties (i-iii) together, one deduces the announced property.

□

Proof of proposition D.1.3. We denote by d^s+1 (resp. d^u+1) the stable dimension of p (resp. the unstable dimension of q)

so that the dimension of M satisfies $d = d^s + d^u + 1$.

Consider a point $x \in W^s(q)\cap W^u(p)$ satisfying (H3) and a C^1-embeddeding $\varphi : [-1,1]^d \to M$ with $\varphi(0) = x$ such that:

- $\varphi(\{0\}\times[-1,1]^{d^s}\times\{0\}^{d^u})$ is contained in $W^s(q)$;
- $\varphi(\{0\}\times\{0\}^{d^s}\times[-1,1]^{d^u})$ is contained in $W^u(p)$;
- $D_0\varphi.(1,0^{d^s},0^{d^u})$ is tangent to E^c_x and has positive orientation.

Note that all the restrictions of φ to $[-1,1]\times\{a^s\}\times[-1,1]^{d^u}$ for $a^s \in \mathbb{R}^{d^s}$ close to 0, are coherent with the central orientation at q.

Consider a compact set $\Delta^u \subset R^u(q)\setminus\{q\}$ that meets each orbit of $R^u(q)\setminus\{q\}$. Since $\mathcal{C}$ is closed and q is unstable-cuspidal, there is a neighborhood V^u of Δ^u in M that is disjoint from $\mathcal{C}$.

The lemma D.1.4 can be applied: the points in $\varphi([0,\delta]\times\{a^s\}\times[-\delta,\delta]^{d^u})$ distinct from $\varphi(0,a^s,0^{d^u})$ have an iterate in V^u, hence do not belong to $\mathcal{C}$.

This shows that

$$\mathcal{C}\cap\varphi\left([0,\delta]\times[-\delta,\delta]^{d-1}\right)\subset\varphi\left(\{0\}\times[-\delta,\delta]^{d^s}\times\{0\}^{d^u}\right).$$

From (H3), if one reverses the central orientation and if one considers the dynamics of f^{-1}, then all the restrictions of φ to $[-1,1]\times[-1,1]^{d^s}\times\{a^u\}$ for $a^u \in \mathbb{R}^{d^u}$ close to 0, are coherent with the central orientation at p.

One can thus argues analogously and gets:

$$\mathcal{C}\cap\varphi\left([-\delta,0]\times[-\delta,\delta]^{d-1}\right)\subset\varphi\left(\{0\}\times\{0\}^{d^s}\times[-\delta,\delta]^{d^u}\right).$$

Both inclusions give that

$$\mathcal{C}\cap\varphi\left([-\delta,\delta]^d\right) = \{\varphi(0)\},$$

which says that $x = \varphi(0)$ is isolated in $\mathcal{C}$.

□

D.2 Construction of the example

In this part we build a collection of diffeomorphisms satisfying the properties (I) and (II) stated in the theorem.

The construction will be made only in dimension 3 for notational purposes. The generalization to higher dimensions is straightforward.

D.2.1 Construction of a diffeomorphism

Let us consider an orientation-preserving C^∞ diffeomorphism H of the plane $\mathbb{R}^2$ and a closed subset $D = D^- \cup C \cup D^+$ such that:

- $H(\overline{D}) \subset \mathrm{Int}(D)$ and $H(\overline{D^- \cup D^+}) \subset \mathrm{Int}(D^-)$;
- the forward orbit of any point in D^- converges towards a sink $S \in D^-$;
- C is the cube $[0,5]^2$ whose maximal invariant set is a hyperbolic horseshoe.

On $C \cap H^{-1}(C)$ the map H is piecewise linear, it preserves and contracts by $1/5$ the horizontal direction and it preserves and expands by 5 the vertical direction (see figure D.3):

- The set $C \cap H(C)$ is the union of 4 disjoint vertical bands I_1, I_2, I_3, I_4 of width 1. We will assume that $I_1 \cup I_2 \subset (0, 2+\frac{1}{3}) \times [0,5]$ and $I_3 \cup I_4 \subset (2+\frac{2}{3}, 5) \times [0,5]$.
- The preimage $H^{-1}(C) \cap C$ is the union of 4 horizontal bands $H^{-1}(I_i)$. We will assume that $H^{-1}(I_1 \cup I_2) \subset [0,5] \times (0, 2+\frac{1}{3})$ and $H^{-1}(I_3 \cup I_4) \subset [0,5] \times (2+\frac{2}{3}, 5)$.

We define a C^∞ diffeomorphism F of $\mathbb{R}^3$ whose restriction to a neighborhood of $D \times [-1,6]$ it is a skew product of the form

$$F\colon (x,t) \mapsto (H(x), g_x(t)),$$

where the diffeomorphisms g_x are orientation-preserving and satisfy (see figure D.4):

(P1) g_x does not depend on x in the sets $H^{-1}(I_i)$ for every $i = 1,2,3,4$.

(P2) For every $(x,t) \in D \times [-1,6]$ one has $4/5 < g'_x(t) < 6/5$.

(P3) g_x has exactly two fixed points inside $[-1,6]$, which are $\{0,4\}$, $\{3,4\}$, $\{1,2\}$ and $\{1,5\}$, when x belongs to $H^{-1}(I_i)$ for i respectively equal to $1,2,3$ and 4. All fixed points are hyperbolic, moreover,

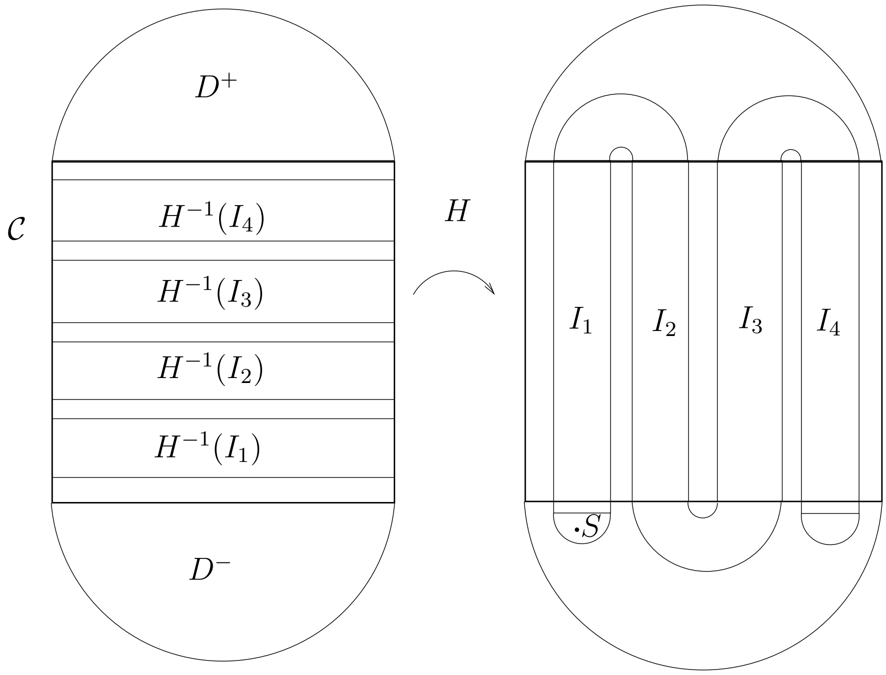

Figure D.3: The map H.

- $g'_x(t) < 1$ for $t \in [-1, 3 + 1/2]$ and $x \in H^{-1}(I_1) \cup H^{-1}(I_2)$.
- $g'_x(t) > 1$ for $t \in [1 + 1/2, 6]$ and $x \in H^{-1}(I_3) \cup H^{-1}(I_4)$.

(P4) For every $(x, t) \in (D^- \cup D^+) \times [-1, 6]$ one has $g_x(t) > t$.

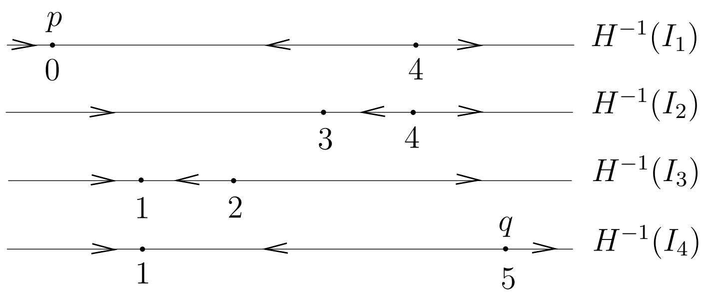

Figure D.4: The map g_x above each rectangle $H^{-1}(I_i)$.

We assume furthermore that the following properties are satisfied:

(P5) $F(D \times [6, 8]) \subset \mathrm{Int}(D \times [6, 8])$;

(P6) there exists a sink which attracts the orbit of any point of $D \times [6, 8]$;

(P7) F coincides with a linear homothety outside a compact domain;

(P8) any forward orbit meets $D \times [-1, 8]$.

One can build a diffeomorphism which coincides with the identity on a neighborhood of the boundary of $D_0 \times (-2, 9)$ and coincides with F in $D \times (-1, 8)$ (D_0 denotes a small neighborhood of D in $\mathbb{R}^2$).

This implies that, on any 3-dimensional manifold, every isotopy class of diffeomorphisms contains an element whose restriction to an invariant set is C^∞-conjugated to F.

On any 3-dimensional manifold, one can consider an orientation-preserving Morse-Smale diffeomorphism and by surgery replace the dynamics on a neighborhood of a sink by the dynamics of F. We denote by f_0 the obtained diffeomorphism.

D.2.2 First robust properties

We list some properties satisfied by f_0, which are also satisfied by any diffeomorphism f in a small C^1-neighborhood $\mathcal{U}_0$ of f_0.

Fixed points By (P3), in each rectangle $\mathrm{Int}(I_i) \times (-1, 6)$, there exists two hyperbolic fixed points p_i, q_i. Their stable dimensions are respectively equal to 2 and 1. Since p_1 and q_4 will play special roles, we shall denote them as $p = p_1$ and $q = q_4$.

Isolation The two open sets $V_0 = \mathrm{Int}(D) \times (-1, 8)$ and $V_1 = V_0 \setminus (C \times [-1, 6])$ are isolating blocks, i.e. satisfy $f(\overline{V_0}) \subset V_0$ and $f(\overline{V_1}) \subset V_1$.

For V_0, the property follows immediatly from the construction.

The closure of the second set V_1 can be decomposed as the union of:

- $D^+ \times [-1, 6]$, which is mapped into $(D^- \times [-1, 6]) \cup (D \times [6, 8])$,
- $D^- \times [-1, 6]$ which is also mapped into $(D^- \times [-1, 6]) \cup (D \times [6, 8])$ and moreover has a foward iterate in $D \times [6, 8]$ by (P4),
- $D \times [6, 8]$ which is mapped into itself and whose limit set is a sink.

Hence, any chain-recurrence class which meets the rectangle $C \times [-1, 6]$ is contained inside. The maximal invariant set in $C \times [-1, 6]$ will be denoted by $\mathcal{C}$.

Any chain-recurrence class which meets V_1 coincides with the sink of $D \times [6, 9]$.

Partial hyperbolicity On $C \times [-1, 6] \subset \mathbb{R}^3$, there exists some narrow cone fields $\mathcal{E}^s, \mathcal{E}^{cs}$ around the coordinate direction $(1, 0, 0)$ and the plane $(x, 0, z)$ which are invariant by Df^{-1}. The vectors tangent to $\mathcal{E}^s$ are uniformly expanded by Df^{-1}.

Similarly there exists some forward invariant cone fields $\mathcal{E}^u, \mathcal{E}^{cu}$ close to the direction $(0,1,0)$ and the plane $(0,y,z)$.

In particular $\mathcal{C}$ is partialy hyperbolic.

Moreover the tangent map Df preserves the orientation of the central direction such that any positive unitary central vector is close to the vector $(0,0,1)$.

Central expansion Property (P2) holds for f when one replaces the derivative $g'_x(t)$ by the tangent map $\|Df|_{E^c}(x,t)\|$ along the central bundle.

Properties (H2) and (H3) The point p is stable-cuspidal and the point q is unstable-cuspidal. More precisely the left half plaque of $W^s(p)$ and the right half plaque of $W^u(q)$ are disjoint from $\mathcal{C}$: since the chain-recurrence classes of p and q are contained in $\mathcal{C}$ this implies property (H2).

Moreover if there exists an intersection point $x \in W^u(p) \cap W^s(q)$ for f, then by the isolating property it is contained in $\mathcal{C}$. By preservation of the central orientation, (H3) holds also.

Let us explain how to prove these properties: it is enough to discuss the case of the left half-plaque of $W^s(p)$ and (arguing as in remark D.1.2) to assume that $f = f_0$.

From (P2) and (P3), we have:

- every point in $C \times [-1,0)$ has a backward iterate outside $C \times [-1,6]$;
- the same holds for every point in $(C \setminus I_1) \times \{0\}$;
- any point in $I_1 \times \{0\}$ has some backward image in $(C \setminus I_1) \times \{0\}$, unless it belongs to $W^u(p)$.

Combining these properties, one deduces that the connected component of $W^s(p) \cap (C \times [-1,0])$ containing p intersects $\mathcal{C}$ only at p.

Note that this is a left half plaque of $W^s(p)$, giving the required property.

Hyperbolic regions By (P3), the maximal invariant set in $Q_p := [0,5] \times [0, 2+\frac{1}{3}] \times [-1, 3+\frac{1}{2}]$ and $Q_q := [0,5] \times [2+\frac{2}{3}, 5] \times [1+\frac{1}{2}, 6]$ are two locally maximal transitive hyperbolic sets, denoted by K_p and K_q.

Their stable dimensions are 2 and 1 respectively. The first one contains p, p_2, the second one contains q, q_3.

Tameness (property (4) of the theorem) since f_0 has been obtained by surgery of a Morse-Smale diffeomorphism, the chain-recurrent set in $M \setminus \mathcal{C}$ is a finite union of hyperbolic periodic orbits.

Any $x \in \mathcal{C}$ has a strong stable manifold $W^{ss}(x)$. Its *local* strong stable manifold $W^{ss}_{loc}(x)$ is the connected component containing x of the intersection $W^{ss}(x) \cap C \times [-1,6]$.

It is a curve bounded by $\{0,5\} \times [0,5] \times [-1,6]$. Symmetrically, we define $W^{uu}(x)$ and $W^{uu}_{loc}(x)$.

D.2.3 Central behaviours of the dynamics

We analyze the local strong stable and strong unstable manifolds of points of $\mathcal{C}$ depending on their central position.

Lemma D.2.1. *There exists an open set $\mathcal{U}_1 \subset \mathcal{U}_0$ such that for every $f \in \mathcal{U}_1$ and $x \in \mathcal{C}$:*

(R1) If $x \in R_1 := C \times [-1, 4+\frac{1}{2}]$, then $W^{uu}_{loc}(x) \cap W^s(p) \neq \emptyset$.

(R2) If $x \in R_2 := C \times [\frac{1}{2}, 6]$, then $W^{ss}_{loc}(x) \cap W^u(q) \neq \emptyset$.

(R3) If $x \in R_3 := C \times [\frac{1}{2}, 2+\frac{1}{2}]$, then $W^{ss}_{loc}(x) \cap W^{uu}_{loc}(y) \neq \emptyset$ for some $y \in K_p$.

(R4) If $x \in R_4 := C \times [2+\frac{1}{2}, 4+\frac{1}{2}]$, then $W^{uu}_{loc}(x) \cap W^{ss}_{loc}(y) \neq \emptyset$ for some $y \in K_q$.

Moreover p_2 belongs to R_2 and q_3 belongs to R_1.

Proof. Properties (R1) and (R2) follow directly from the continuous variation of the stable and unstable manifolds. Similarly $p_2 \in R_2$ and $q_3 \in R_1$ by continuity.

We prove (R3) with classical blender arguments (see [BD$_1$] and [BDV, chapter 6] for more details). The set K_p is a called *blender-horseshoes* in [BD$_4$, section 3.2].

A *cs-strip* $\mathcal{S}$ is the image by a diffeomorphism $\phi : [-1,1]^2 \to Q_p = [0,5] \times [0, 2+\frac{1}{3}] \times [-1, 3+\frac{1}{2}]$ such that:

- The surface $\mathcal{S}$ is tangent to the center-stable cone field and meets $C \times [\frac{1}{2}, 2+\frac{1}{2}]$.
- The curves $\phi(t, [-1,1])$, $t \in [-1,1]$, are tangent to the strong stable cone field and crosses Q_p, i.e. $\phi(t, \{-1,1\}) \subset \{0,5\} \times [0, 2+\frac{1}{3}] \times [-1, 3+\frac{1}{2}]$.
- $\mathcal{S}$ does not intersect $W^u_{loc}(p) \cup W^u_{loc}(p_2)$.

The *width* of $\mathcal{S}$ is the minimal length of the curves contained in $\mathcal{S}$, tangent to the center cone, and that joins $\phi(-1, [-1,1])$ and $\phi(1, [-1,1])$.

Condition (P2) is important to get the following (see [BDV, lemma 6.6] for more details):

Claim. *There exists $\lambda > 1$ such that if $\mathcal{S}$ is a cs-strip of width ε, then, either $f^{-1}(\mathcal{S})$ intersects $W^u_{loc}(p) \cup W^u_{loc}(p_2)$ or it contains at least one cs-strip with width $\lambda\varepsilon$.*

Proof. Using (P2), the set $f^{-1}(\mathcal{S}) \cap C \times [-1, 6]$ is the union of two bands crossing $C \times [-1, 6]$: the first has its two first coordinates near $H^{-1}(I_1)$, the second near $H^{-1}(I_2)$.

Their width is larger than $\lambda\varepsilon$ where $\lambda > 1$ is a lower bound of the expansion of Df^{-1} in the central direction inside Q. We assume by contradiction that none of them intersects $W^u_{loc}(p) \cup W^u_{loc}(p_2)$, nor $C \times [\frac{1}{2}, 2 + \frac{1}{2}]$.

Since $\mathcal{S}$ intersects $C \times [\frac{1}{2}, 2 + \frac{1}{2}]$, from conditions (P2) and (P3) the first band intersects $C \times [\frac{1}{2}, 4]$. By our assumption it is thus contained in $C \times (2 + \frac{1}{2}, 4]$. Using (P2) and (P3) again, this shows that $\mathcal{S}$ is contained in $C \times (2, 4]$.

The same argument with the second band shows that $\mathcal{S}$ is contained in $C \times [-1, 2)$, a contradiction.

$\diamondsuit$

Repeating this procedure, we get an intersection point between $W^u_{loc}(p) \cup W^u_{loc}(p_2)$ and a backward iterate of the $cs-$strip. It gives in turn a transverse intersection point z between the initial $cs-$strip and $W^u(p) \cup W^u(p_2)$. By construction, all the past iterates of z belong to Q_p. Hence z has a well defined local strong unstable manifold. In particular, the intersection y between $W^{uu}_{loc}(z)$ and $W^s_{loc}(p)$ (which exists by (R1)) remains in Q_p both for future and past iterates, thus, it belongs to K_p.

For any point $x \in \mathcal{C} \cap R_3$, one builds a cs-strip by thickening in the central direction the local strong stable manifold. We have proved that this $cs-$strip intersects $W^{uu}_{loc}(y)$ for some $y \in K_p$.

One con consider a sequence of thiner strips. Since K_p is closed and the local strong unstable manifolds vary continuously, we get at the limit an intersection between $W^{ss}_{loc}(x)$ and $W^{uu}_{loc}(y')$ for some $y' \in K_p$ as desired.

This gives (R3). Property (R4) can be obtained similarly.

$\square$

We have controled the local strong unstable manifold of points in $R_1 \cup R_4$ and the local strong stable manifold of points in $R_2 \cup R_3$.

Since neither $R_1 \cup R_4$ nor $R_2 \cup R_3$ cover completely $C \times [-1, 6]$ we shall also make use of the following result:

Lemma D.2.2. *For every diffeomorphism in a small C^1-neighborhood $\mathcal{U}_2 \subset \mathcal{U}_0$ of f_0, the only point whose complete orbit is contained in $C \times [-1, \frac{1}{2}]$ is p; symmetrically, the only point whose complete orbit is contained in $C \times [4 + \frac{1}{2}, 6]$ is q.*

Proof. We argue as for property (H2) in section D.2.2: the set of points whose past iterates stay in $C \times [-1, \frac{1}{2}]$ is the local strong unstable manifold of p. Since p is the only point in its local unstable manifold whose future iterates stay in $C \times [-1, \frac{1}{2}]$ is p we conclude. $\square$

D.2.4 Properties (I) and (II) of the theorem

We now check that (I) and (II) hold for the region $U = \text{Int}(C \times [-1, 6])$ and the neighborhood $\mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2$.

Proposition D.2.3. *For any $f \in \mathcal{U}$, $x \in \mathcal{C}$, there are arbitrarily large $n_q, n_p \geq 0$ such that $W^{uu}_{loc}(f^{n_q}(x)) \cap W^{ss}(y_q) \neq \emptyset$ and $W^{ss}_{loc}(f^{-n_p}(x)) \cap W^{uu}(y_p) \neq \emptyset$ for some $y_q \in K_q$, $y_p \in K_p$.*

Proof. If $\{f^n(x), n \geq n_0\} \subset C \times [4 + \frac{1}{2}, 6]$, for some $n_0 \geq 0$, then $x \in W^{ss}(q)$ by lemma D.2.2.

In the remaining case, there exist some arbitrarily large forward iterates $f^n(x)$ in R_1, so that $W^{uu}_{loc}(f^n(x))$ meets $W^s(p)$ by lemma D.2.1.

Since p is homoclinically related with p_2, by the λ-lemma there exists $k \geq 0$ such that $f^k(W^{uu}_{loc}(f^n(x)))$ contains $W^{uu}_{loc}(x')$ for some $x' \in W^s(p_2) \cap R_4$ because $p_2 \in R_4$.

By lemma D.2.1, $f^k(W^{uu}_{loc}(f^n(x)))$ intersects $W^{ss}_{loc}(y'_q)$ for some $y'_q \in K_q$ showing that $W^{uu}_{loc}(f^n(x)) \cap W^{ss}(y_q) \neq \emptyset$ with $y_q = f^{-k}(y'_q)$ in K_q.

We have obtained the first property in all the cases. The second property is similar.

$\square$

The following corollary (together with the isolation property of section D.2.2) implies that for every $f \in \mathcal{U}$, the properties (I) and (H1) are verified.

Corollary D.2.4. *For every $f \in \mathcal{U}$ the set $\mathcal{C}$ is contained in a chain-transitive class.*

Proof. For any $\varepsilon > 0$ and $x \in \mathcal{C}$, there exists a ε-pseudo-orbit $p = x_0, x_1, \ldots, x_n = p$, $n \geq 1$, which contains x.

Indeed by proposition D.2.3, and using that K_p, K_q are transitive and contain respectively p and q_3, there exists a ε-pseudo-orbit from p to q_3 which contains x.

By lemma D.2.1, the unstable manifold of q_3 intersects the stable manifold of p, hence there exists a ε-pseudo-orbit from q_3 to p.

We take the concatenation of these pseudo-orbits.

$\square$

Now, we show that (H3) holds for a C^r dense set $\mathcal{D}$ of $\mathcal{U}$.

Since (H1) and (H2) are satisfied, proposition D.1.3 implies that the property (II) of the theorem holds with the set $\mathcal{D} \subset \mathcal{U}$.

In fact, as we noticed in section D.2.2 it is enough to get the following.

Corollary D.2.5. *For every $r \geq 1$, the set*

$$\mathcal{D} = \{f \in \mathcal{U},\ W^u(p) \cap W^s(q) \neq \emptyset\}$$

is dense in $\mathcal{U} \cap \mathrm{Diff}^r(M)$. It is a countable union of one-codimensional submanifolds.

In the C^1 topology, this result is direct consequence of the connecting lemma (together with proposition D.2.3).

The additional structure of our specific example allows to make these perturbations in any C^r-topology.

Proof. Fix any $f \in \mathcal{U}$.

By proposition D.2.3, there exists $x \in K_q$ such that $W^u(p)$ intersects $W^{ss}(x)$ at a point y (notice that $y \notin K_q \cup \{p\}$).

Let U be a neighborhood of y such that:

- U is disjoint from the iterates of y, i.e. $\{f^n(y) \ : \ n \in \mathbb{Z}\} \cap U = \{y\}$;
- U is disjoint from $K_q \cup \{p\}$.

Given a C^r neighborhood $\mathcal{V}$ of the identity, there exists a neighborhood $V \subset U$ of y such that, for every $z \in V$, the set $\mathcal{V}$ contains a diffeomorphism g_z which coincides with the identity in the complement of U and maps y at z.

Since K_q is locally maximal, there exists $\bar{x} \in K_q \cap W^s(q)$ near x. In particular $W^{ss}_{loc}(\overline{x})$ intersects V in a point z whose backward orbit is disjoint from U.

For the diffeomorphism $h = g_z \circ f$ (which is C^r-close to f) the manifolds $W^s(q)$ and $W^u(p)$ intersect.

Indeed both f and h satisfy $f^{-1}(y) \in W^u(p)$ and $z \in W^{ss}_{loc}(\overline{x})$.

Since $W^{ss}_{loc}(\overline{x}) \subset W^{ss}(q)$ and $h(f^{-1}(y)) = z$ we get the conclusion.

For each integer $n \geq 1$, the manifolds $f^n(W^{uu}_{loc}(p))$ and $W^{ss}_{loc}(q)$ have disjoint boundary and intersect in at most finitely many points.

One deduces that the set $\mathcal{D}_n$ of diffeomorphisms such that they intersect is a finite union of one-codimensional submanifold of $\mathcal{U}$.

The set $\mathcal{D}$ is the countable union of the $\mathcal{D}_n$.

$\square$

D.2.5 Other properties

We here show properties (1), (2) and (3) of the theorem.

Proposition D.2.6. *For every $f \in \mathcal{U}$ and $x \in \mathcal{C}$ we have:*

- *If $x \notin W^s(q)$, there exist large $n \geq 0$ such that $W^{uu}_{loc}(f^n(x)) \cap W^s(p) \neq \emptyset$.*

- *If $x \notin W^u(p)$, there exists large $n \geq 0$ such that $W^{ss}_{loc}(f^{-n}(x)) \cap W^u(q) \neq \emptyset$.*

Moreover, in the first case x belongs to the homoclinic class of p and in the second it belongs to the homoclinic class of q.

Proof. By lemma D.2.2, any point $x \in \mathcal{C} \setminus W^s(q)$ has arbitrarily large iterates $f^n(x)$ in R_1, proving that $W^{uu}_{loc}(f^n(x)) \cap W^s(p) \neq \emptyset$.

In particular, $W^s(p)$ intersects transversaly $W^{uu}_{loc}(x)$ at points arbitrarily close to x. On the other hand by proposition D.2.3, there exists a sequence z_n converging to x and points $y_n \in K_p$ such that $z_n \in W^u(y_n)$ for each n, proving that $W^{uu}_{loc}(z_n)$ intersects $W^u(p)$ transversaly at a point close to x when n is large.

By the λ-lemma, $W^{uu}_{loc}(y_n)$ is the C^1-limit of a sequence of discs contained in $W^u(p)$. This proves that $W^u(p)$ and $W^s(p)$ have a transverse intersection point close to x, hence x belongs to the homoclinic class of p.

The other properties are obtained analogously. □

Let H_f denotes the homoclinic class of p. The next gives property (1) of the theorem.

Corollary D.2.7. *For every $f \in \mathcal{U}$, the homoclinic class of any hyperbolic periodic point of $\mathcal{C}$ coincides with H_f. Moreover, the periodic points in $\mathcal{C}$ of the same stable index are homoclinically related.*

Proof. Let $z \in \mathcal{C}$ be a hyperbolic periodic point whose stable index is 2.

By proposition D.2.3 $W^{ss}(z)$ intersects $W^{uu}_{loc}(y)$ for some $y \in K_p$, this implies that $W^s(z)$ intersects $W^{uu}_{loc}(y)$ and since $W^{uu}_{loc}(y)$ is accumulated by $W^u(p)$ we get that $W^s(z)$ intersects $W^u(p)$. Now, by proposition D.2.6, $W^u(z)$ intersects $W^s(p)$. Moreover the partial hyperbolicity implies that the intersections are transversal, proving that z and p are homoclinically related.

One shows in the same way that any hyperbolic periodic point whose stable index is 1 is homoclinically related to q.

It remains to prove that the homoclinic classes of p and q coincide.

The homoclinic class of q contains a dense set of points x that are homoclinic to q_3. In particular, x does not belong to $W^u(q)$, hence belongs to the homoclinic class of p by proposition D.2.6.

This gives one inclusion. The other one is similar.

□

Properties (2) and (3) of the theorem follow from corollary D.2.5 and the following.

Corollary D.2.8. *For every $f \in \mathcal{U}$ we have $\mathcal{C} \backslash H_f = W^s(q) \cap W^u(p)$.*

Proof. By corollary D.2.7, a point $x \in \mathcal{C} \backslash H_f$ does not belong to the homoclinic class of q (nor to the homoclinic class of p by definition of H_f).

Proposition D.2.6 gives $\mathcal{C} \backslash H_f \subset W^s(q) \cap W^u(p)$.

Proposition D.1.3 proves that the points of $W^s(q) \cap W^u(p)$ are isolated in $\mathcal{C}$. Since any point in a non-trivial homoclinic class is limit of a sequence of distinct periodic points of the class we conclude that $W^s(q) \cap W^u(p)$ and H_f are disjoint.

□

The proof of the theorem is now complete.

Bibliography

[Ab] F. Abdenur, Generic robustness of spectral decompositions, *Ann. Sci. of the E.N.S* **36** (2003) 213-224.

[ABC$_1$] F. Abdenur, C.Bonatti and S. Crovisier, Global dominated splittings and the C^1-Newhouse phenomenon, *Proceedings of the AMS* **134** (2006) 2229.2237.

[ABC$_2$] F. Abdenur, C.Bonatti and S. Crovisier, Nonuniform hyperbolicity for C^1-generic diffeomorphisms, *Israel Journal of Math.* **183** (2011) pp. 1-60. arXiv:0809.3309.

[ABCD] F.Abdenur , C. Bonatti, S. Crovisier y L.Diaz, Generic diffeomorphisms on compact surfaces. *Fund. Math.* **187** (2005), 127–159.

[ABD] F. Abdenur, C. Bonatti y L.Diaz, Nonwandering sets with non empy interior, *Nonlinearity* **17** (2004), 175–191.

[ABCDW] F.Abdenur, C. Bonatti, S. Crovisier, L.Diaz and L.Wen, Periodic points and homoclinic classes. *Ergodic Theory Dynam. Systems* **27** (2007), 1-22.

[AS] R. Abraham and S. Smale, Non genericity of Ω-stability, *Global analysis I* Proc. Symp. Pure Math AMS **14** (1970) 5–8.

[Ara] A. Araujo, *Existência de atratores hiperbólicos para difeomorfismos de superfície*, Thesis IMPA (1987)

[Arn$_1$] M.C. Arnaud, Création des connexions en topologie C^1, *Ergodic Theory and Dynamical Systems* **21** (2001) 339-381.

[Arn$_2$] M. C. Arnaud, Approximation des ensembles ω-limites des difféomorphismes par des orbites périodiques, *Ann. Sci. Ecole Norm. Sup.* **36**, (2003), 173–190.

[ABP] A. Artigue, J. Brum and R. Potrie, Local product structure for expansive homeomorphisms, *Topology and its Applications* **156** (2009) 674-685. arXiv:0805.1493

[AvBo] A. Avila and J. Bochi, A generic C^1 map has no absolutely continuous invariant probability measure,*Nonlinearity* **19**, 2717-2725, (2006). arXiv:math/0605729

[BCGP] C. Bonatti, S. Crovisier, N. Gourmelon and R. Potrie, Tame dynamics and robust transitivity, *Preprint* arXiv 1112.1002.

[BCJL] F.Beguin, S.Crovisier, T.Jager and F.Le Roux, Denjoy constructions for fibred homeomorphisms of the torus, *Trans. Amer. Math. Soc.* **361** 11 (2009) 5851-5883. arXiv:0707.4373

[BCL] F. Beguin, S.Crovisier and F. Le Roux, Construction of curious minimal uniquely ergodic homeomorphisms on manifolds: the Denjoy-Rees technique. *Ann. Sci. ENS.* **40** (2007), 251-308. arXiv:math/0605438

[Be] P. Berger, Persistence of laminations, *Bulletin of the Brazilian Math. Society* **41** No. 2. (2010), pp. 259-319. arXiv:0807.4952

[BoB] J. Bochi and C. Bonatti, Perturbations of the Lyapunov spectra of periodic orbits, to appear in *Proc. London. Math. Soc.*. arXiv:1004.5029.

[BoG] J. Bochi and N. Gourmelon, Some Characterizations of Domination, *Mathematische Zeitschrift*, **263**, no. 1 (2009), 221-231. arXiv:0808.3811.

[B] C. Bonatti, Towards a global view of dynamical systems, for the C^1-topology, *Ergodic Theory and Dynamical Systems.***31** n.4 (2011) p. 959-993.

[BC] C. Bonatti and S. Crovisier, Récurrence et Généricité, *Inventiones Math.* **158** (2004), 33–104. arXiv:math/0306383

[BCS] C. Bonatti, S. Crovisier and K.Shinohara, *In preparation.*

[BCDG] C. Bonatti, S. Crovisier, L. Díaz and N. Gourmelon, Internal perturbations of homoclinic classes:non-domination, cycles, and self-replication, to appear *Ergodic Theory and Dynamical Systems.* arXiv 1011.2935 (2010).

[BD_1] C. Bonatti y L.J. Díaz, Persistent nonhyperbolic transitive diffeomorphisms, *Annals of Math.*,**143** (1995), 357-396.

[BD_2] C. Bonatti and L.J. Díaz, Connexions hétéroclines et généricité d'une infinité de puits ou de sources, *Ann. Sci. École Norm. Sup.*, **32**, 135–150, (1999).

[BD_3] C. Bonatti and L.J. Díaz, On maximal transitive sets of diffeomorphisms, *Publ. Math. IHES* **96** (2002), 171–197.

[BD_4] C. Bonatti and L. J. Díaz, Abundance of C^1-robust homoclinic tangencies, to appear in *Transactions of the AMS* arXiv 0909.4062.

[BD_5] C. Bonatti and L. J. Díaz, Fragile cycles,*Jour. of Differential Equations* **252** (2012) 4176-4199. arXiv:1103.3255.

[BDP] C. Bonatti, L.J. Díaz y E.R. Pujals, A C^1- generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources, *Annals of Math.* **158** (2003), 355-418. arXiv:math/0610527

[BDPR] C. Bonatti, L. Díaz, E. Pujals and J. Rocha, Robustly transitive sets and heterodimensional cycles,*Geometric methods in dynamics I. Astérisque* **286** (2003), 187–222.

[BDT] C. Bonatti, L.J. Díaz and G. Turcat, Pas de "shadowing lemma" pour des dynamiques partiellement hyperboliques, *C. R. Acad. Sci. Paris*, **330**, (2000), 587–592.

[BDV] C. Bonatti, L.J. Díaz y M. Viana, Dynamics Beyond Uniform Hyperbolicity, *Springer-Verlag* (2005).

[BFra] C. Bonatti and J. Franks, A Hölder continuous vector field tangent to many foliations. *Modern dynamical systems and applications* Cambridge Univ. Press, Cambridge, (2004) 299–306.

[BGV] C. Bonatti, N.Gourmelon and T. Vivier, Perturbations of the derivative along periodic orbits, *Ergodic Theory and Dyn. Syst* **26** (2006), 1307–1337.

[BGLY] C. Bonatti, S. Gan, M. Li and D.Yang, *In preparation.*

[BGY] C. Bonatti, S.Gan and D. Yang, On the hyperbolicity of homoclinic classes, *Disc. and Cont. Dyn. Syst. A* **25** (2009) 1143–1162.

[BLY] C.Bonatti, M.Li and D.Yang, On the existence of attractors, *Preprint* arXiv 0904.4393 (2009) To appear in *Transaction of the AMS.*

[BS] C. Bonatti and K. Shinohara, *In preparation.*

[BSh] C. Bonatti and Y. Shi, *In preparation.*

[BV] C. Bonatti y M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly contracting, *Israel J. Math.* **115** (2000), 157-193.

[BWi] C. Bonatti and A. Wilkinson, Transitive partially hyperbolic diffeomorphisms on 3-manifolds, *Topology* **44** (2005) 475-508.

[Bo] R. Bowen, *Equilibrium states and the ergodic theory of Axiom A diffeomorphisms,* Lect. Notes in Mathematics, **470**, Springer-Verlag (1975).

[Bri] M. Brin, On dynamical coherence, *Ergodic theory and dynamical systems*, **23** (2003), 395-401.

[BBI_1] M. Brin, D. Burago and S. Ivanov, On partially hyperbolic diffeomorphisms of 3-manifolds with commutative fundamental group. *Modern dynamical systems and applications* Cambridge Univ. Press, Cambridge, (2004) 307–312.

[BBI_2] M. Brin, D. Burago and S. Ivanov, Dynamical coherence of partially hyperbolic diffeomorphisms of the 3-torus. *J. Mod. Dyn* **3** (2009) 1-11.

[BrPe] M. Brin and Y. Pesin, Partially hyperbolic dynamical systems, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 170–212.

[Bro] A. Brown, Non-expanding attractors: conjugacy to algebraic models and classification in 3-manifolds, *Journal of Modern Dynamics* **4** (2010) 517 - 548.

[BI] D. Burago and S. Ivanov, Partially hyperbolic diffeomorphisms of 3-manifolds with abelian fundamental groups. *J. Mod. Dyn.* **2** (2008) 541–580.

[BRHRHTU] K.Burns, M.A. Rodriguez Hertz, F. Rodriguez Hertz, A. Talitskaya and R. Ures, Density of accessibility for partially hyperbolic diffeomorphisms with one-dimensional center. *Discrete and Continuous Dynamical Systems A*, (Pesin birthday issue) **22** (2008) 75-88.

[BuW_1] K.Burns and A. Wilkinson, Dynamical coherence and center bunching, *Discrete and Continuous Dynamical Systems A* (Pesin birthday issue) **22** (2008) 89-100.

[BuW_2] K.Burns and A. Wilkinson, On the ergodicity of partially hyperbolic systems, *Ann. of Math.* **171** (2010) 451-489.

[BuFi] J. Buzzi and T. Fisher, Entropic stability beyond partial hyperbolicity, *Preprint* arXiv:1103:2707 (2011).

[BFSV] J. Buzzi, T. Fisher, M. Sambarino and C. Vasquez, Maximal entropy measures for certain partially hyperbolic derived from Anosov systems, *Ergodic Theory and Dynamical Systems* **32** (2012) 63-79. arXiv:0903.3692 and arXiv:0904.1036.

[Ca] D. Calegari, *Foliations and the geometry of 3 manifolds*, Oxford University Press, (2007).

[CLN] C. Camacho and A. Lins Neto, *Teoria geometrica das folheacoes*, Projeto Euclides, IMPA (1979).

[CaCo] A. Candel and L. Conlon, *Foliations I and II*, Graduate studies in Mathematics **60** American Math. Society (2003).

[Car] M. Carvalho, Sinai-Ruelle-Bowen measures for N-dimensional derived from Anosov diffeomorphisms, *Ergodic Theory and Dynamical Systems* **13** (1993) 21–44.

[CMP] C. M. Carballo , C. A. Morales and M. J. Pacifico, Homoclinic classes for generic C^1 vector fields, *Ergodic Theory Dynam. Systems* **23** (2003), 403–415.

[Co] C. Conley, Isolated invariant sets and Morse index, *CBMS Regional Conference Series in Mathematics* **38**, AMS Providence (1978).

[C_1] S. Crovisier, Periodic orbits and chain transitive sets of C^1 diffeomorphisms, *Publ. Math. IHES* **104** (2006), 87–141.

[C_2] S. Crovisier, Birth of homoclinic intersections: a model for the central dynamics of partially hyperbolic systems. *Annals of Math.* **172** (2010)1641-1677. arXiv math/0605387

[C_3] S. Crovisier, Partial hyperbolicity far from homoclinic bifurcations, *Advances in Math.* **226** (2010) 673-726. arXiv-math/0809.4965.

[C_4] S. Crovisier, Perturbation de la dynamique de difféomorphismes en topologie C^1, to appear in *Asterisque*. arXiv:0912.2896

[CP] S. Crovisier and E. Pujals, Essential hyperbolicity versus homoclinic bifurcations, *Preprint* (2011) arXiv:1011.3836.

[CSY] S. Crovisier, M. Sambarino and D.Yang, Partial hyperbolicity and homoclinic tangencies, *Preprint* (2011) arXiv: 1103.0869.

[Da] R. Daverman, *Decompositions of Manifolds* Academic Press 1986

[DPU] L.Diaz, E. Pujals y R.Ures, Partial hyperbolicity and robust transitivity. *Acta Math.* 183 (1999), 1–43.

[D] L. J. Díaz Robust nonhyperbolic dynamics at heterodimensional cycles. *Ergodic Theory and Dynamical Systems*, **15** (1995), 291–315.

[DS] L.J. Díaz and B. Santoro, Collision, explosion and collapse of homoclinic classes, *Nonlinearity* **17** (2004) 1001-1032.

[DoWi] D. Dolgopyat and A. Wilkinson, Stable accesibility is C^1-dense, *Asterisque* **287** (2003), 33-60.

[Fa] A. Fathi, Expansiveness hyperbolicity and Haussdorf dimension, *Comm. Mathematical Physics* **126** (1989) 249-262.

[Fay] B. Fayad. Weak mixing for reparameterized linear flows on the torus. *Ergodic Theory Dyn. Syst.*, **22** 1 (2002), 187–201.

[Fa] A. Fathi, Expansiveness hyperbolicity and Haussdorf dimension, *Comm. Mathematical Physics* **126** (1989) 249-262.

[F$_1$] J. Franks, Anosov diffeomorphisms,*Global Analysis, Proc. Symp. in Pure Math. A.M.S* vol. 14 (1970), 61-94.

[F$_2$] J. Franks, Anosov diffeomophisms on tori, *Trans. Amer. Math. Soc.* **145** (1969) 117–124.

[F$_3$] J. Franks, Nessesary conditions for stability of diffeomorphisms, *Transactions of the A.M.S.* **158**, 301-308 (1971).

[F$_4$] J. Franks, *Homology and dynamical systems*, , CBMS Regional conference series in Mathematics **49** A.M.S. (1982).

[Ga] D. Gabai, Combinatorial volume preserving flows and taut foliations, *Comment. Math. Helv.* **75** (2000), no. 1, 109–124.

[Gou$_1$] N. Gourmelon, Addapted metrics for dominated splitting, *Ergod. Th. and Dyn. Sys.* **27** (2007), 1839-1849.

[Gou$_2$] N. Gourmelon, Generation of homoclinic tangencies by C^1 perturbations. *Discrete and Continuous Dynamical Systems A* **26** (2010) 1–42.

[Gou$_3$] N. Gourmelon, A Franks' lemma that preserves invariant manifolds, *Preprint* arXiv:0912.1121.

[Hae] A. Haefliger, Variétés feuilletées, *Ann. Scuola Norm. Sup. Pisa* **16** (1962) 367-397.

[H] A. Hammerlindl, Leaf conjugacies in the torus, Thesis, and to appear in *Ergodic theory and dynamical systems.*

[H$_2$] A. Hammerlindl, Partial hyperbolicity on 3-dimensional nilmanifolds, *Preprint* arXiv:1103.3724

[H$_3$] A. Hammerlindl, The dynamics of quasi-isometric foliations,*Preprint* arXiv:1110.5662. To appear in *Nonlinearity.*

[HP] A. Hammerlindl and R. Potrie, *In preparation.*

[Hat] A. Hatcher, *Algebraic Topology*,Cambridge University Press, (2002)

[Hay] S. Hayashi, Connecting invariant manifolds and the solution of the C^1-stability and Ω-stability for flows., *Ann. of math.*, **145**, (1999) 81-137,.

[HeHi] G. Hector and U. Hirsch, *Introduction to the Geometry of Foliations, Part B*, Aspects of Math. (1983)

[Hir] K. Hiraide, Expansive homeomorphisms with the pseudo-orbit tracing property of n-tori., *J. Math.Soc.Japan*, **41** (1989), 357–389.

[HPS] M. Hirsch,C. Pugh y M. Shub, Invariant Manifolds, *Springer Lecture Notes in Math.*, **583** (1977).

[Hi] M. Hirsch, *Differential topology*, Springer Graduate Texts in Mathematics **33** (1977).

[Hun] T.W. Hungerford, *Algebra*, Springer Graduate Texts in Mathematics (1974).

[Hur] M. Hurley, *Attractors: persistence, and density of their basins,* Trans. Amer. Math. Soc., **269**, 271–247, (1982).

[Jag_1] T. Jager, Linearisation of conservative toral homeomorphisms, *Inventiones Math.* **176**(3) (2009), 601-616. arXiv:0803.2428

[Jag_2] T. Jager, The concept of bounded mean motion for toral homeomorphisms, *Dynamical Systems. An International Journal* **24**3 (2009), 277-297. arXiv:0802.4187

[KH] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press (1995).

[Kup] I. Kupka, Contribution à la théorie des champs génériques. *Contributions to Differential Equations* **2** (1963) 457–484.

[Kwak] F. Kwakkel, Minimal sets of non-resonant torus homeomorphisms, *Fund. Math.* **211** (2011), 41-76. arXiv:1002.0364

[Kwap] J. Kwapisz. Combinatorics of torus diffeomorphisms. *Ergodic Theory Dynam. Systems* **23** (2003), 559-586.

[LeC] P. Le Calvez, Ensembles invariants non enlaces des diffeomorphismes du tore et de l'anneau. *Invent. Math.* **155** (2004), 561-603.

[LVY] G. Liao, M. Viana and J. Yang, The entropy conjecture for diffeomorphisms away from tangencies,to appear *Journal of the EMS* arXiv: 1012.0514.

[L] S. Liao, On the stability conjecture, *Chinese Ann. Math.* **1** (1980), 9–30.

[Man] A. Manning, There are no new anosov on tori, *Amer. Jour. of Math.* **96** (1974), 422–429.

[M_1] R. Mañe, Contributions to the stability conjecture, *Topology* **17** (1978), 383-396.

[M_2] R. Mañe, Expansive homeomorphisms and topological dimension, *Transactions of the AMS* **252** (1979)

[M_3] R. Mañe, An Ergodic Closing Lemma, *Annals of Math.*, **116** (1982), 503-540.

[M_4] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, (1987).

[M_5] R. Mañe, A proof of the C^1 stability conjecture, *Publ. Math. IHES*,**66** (1987), 161-210.

[Mi] J. Milnor, On the concept of attractor, *Commum. Math. Phys.*, **99** (1985), 177–195.

[MiZi] M. Misiurewicz and K. Ziemian. Rotation sets for maps of tori. *J. Lond. Math. Soc.*, **40**(1989) 490–506.

[Mo] E. Moise, *Geometric topology in dimensions 2 and 3*, Graduate texts in Mathematics **47** Springer (1977).

[MP] C. Morales and M.J. Pacifico, Lyapunov stability of ω-limit sets, *Discrete and continuous dynamical systems* **8** (2002) 671–674.

[New_1] S. Newhouse, Nondensity of axiom A(a) on S^2, *Global analysis I, Proc. Symp. Pure Math. AMS* **14** (1970), 191–202.

[New_2] S. Newhouse, On codimension one Anosov diffeomorphisms, *Amer. Jour. of Math.*, **92** (1970),761–770.

[New_3] S. Newhouse, Hyperbolic Limit Sets, *Transactions of the A.M.S* **167** (1972), 125–150.

[No] S. Novikov, Topology of foliations (Russian), *Trudy Mosk. Mat. Obsh.* **14** (1965) 248-277.

[O] V.I. Oseledec, A multiplicative ergodic theorem, Lyapunov characteristic numbers for dynamical systems, *Trudy Moskov.Mat. Obšč.*, **19**(1968), 119-210.

[Pa_1] J. Palis, On Morse-Smale dynamical systems, *Topology* **8** (1969), 385–405.

[Pa_2] J. Palis , Omega-Explosions, *Bull. Braz. Math. Soc.*, **1**, (1970) 55-57.

[Pa_3] J. Palis, A global perspective for non-conservative dynamics, *Annales de l'Inst. Henri Poincaré, Analyse Non Lineaire*, **22** (2005) 485-507.

[PaPu] J. Palis and C. Pugh, Fifty problems in dynamical systems, *Lecture Notes in Mathematics, Springer-Verlag,***468** (1975) 345-353.

[PaSm] J. Palis and S. Smale, Structural Stability Theorems, *Global Analysis, Proc. Symp. in Pure Math. A.M.S* vol. 14 (1970),223-232.

[PaT] J. Palis and F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge studies in advanced mathematics, **35**, Cambridge University Press (1993).

[PaV] J. Palis and M. Viana, High dimensional diffeomorphisms displaying infinitely many periodic attractors, *Annals of Math* **140** (1994) 207–250.

[Pal] C. Palmeira, Open manifolds foliated by planes, *Annals of Math.* **107** (1978) 109-131.

[Par] K. Parwani, On 3-manifolds supporting partially hyperbolic diffeomorphisms, *Nonlinearity* **23** (2010) 589-604.

[Pla] J.F. Plante, Foliations of 3-manifolds with solvable fundamental group, *Inventiones Math.* **51** (1979) 219-230.

[Pli] V.A. Pliss, On a conjecture of Smale, *Diff. Uravnenija* **8** (1972), 268-282.

[Ply] R.V. Plykin. Sources and currents of A-diffeomorphisms of surfaces. *Math. USSR Sbornik* **94** (1974) 243–264.

[Po] H. Poincare, Sur le problème des trois corps et les équations de la dynamique. *Acta Math.* **13** (1890) 1-270.

[PotS] R. Potrie and M. Sambarino, Codimension one generic homoclinc classes with interior, *Bull. Braz. Math. Soc.* **41** (2010) 125-138. arXiv:0806.3036.

[Pot_1] R. Potrie, Generic bi-Lyapunov stable homoclinc classes, *Nonlinearity* **23** (2010) 1631-1649. arXiv:0903.4090.

[Pot_2] R. Potrie, A proof of existence of attractors for generic surface diffeomorphisms, *Permanent preprint* Available in http://www.cmat.edu.uy/∼rpotrie

[Pot_3] R. Potrie, Wild Milnor attractors accumulated by lower dimensional dynamics, to appear *Ergodic Theory and Dynamical Systems* arXiv:1106.3741.

[Pot_4] R. Potrie, Recurrence for non-resonant torus homeomorphisms, to appear *Proceedings AMS.* arXiv:1103.4576.

[Pot_5] R. Potrie, Partial hyperbolicity and foliations in $\mathbb{T}^3$, *Preprint.*

[Pot_6] R. Potrie, Partially hyperbolic diffeomorphisms with a trapping property, *In Preparation.*

[PS_1] E.R. Pujals and M. Sambarino, Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, *Annals of Math.* **151** (2000), 961-1023. arXiv:math/0005303

[PS_2] E.R. Pujals and M. Sambarino, Homoclinic Bifurcations, Dominated Splitting and Robust Transitivity, *Handbook of Dynamical Systems* Eds: B. Hasselblatt and A.Katok, Vol 1B (2006) Chapter 4, p.327-378.

[PS_3] E.R. Pujals and M. Sambarino, A suficient condition for robustly minimal foliations, *Ergodic Theory and Dynamical Systems* **26** (2006) 281-289.

[PS_4] E.R. Pujals and M. Sambarino, Integrability on codimension one dominated splitting, *Bull. Braz. Math. Soc., N.S.* **38** (2007) 1-19.

[PS_5] E. R. Pujals and M. Sambarino, Density of hyperbolicity for sectionally disipative regions, *Annales de l'Inst. Henri Poincaré, Analyse Non Lineaire* **26** (2009) 1971–2000.

[PS_6] E.R. Pujals and M. Sambarino, On the dynamics of dominated splitting, *Annals of Math.* **169** (2009) 675-740.

[Pu_1] C. Pugh, The closing lemma, *Amer. J. Math* **89** (1967) 956-1009.

[Pu_2] C. Pugh, An improved closing lemma and a general density theorem, *Amer. J. Math* **89** (1967) 1010-1021.

[Pu_3] C. Pugh, A speial C^r-closing lemma. *Geometri dynamics (Rio de Janeiro, 1981), Lecture Notes in Math.* **1007** (1983), 636-650.

[PuShWi] C. Pugh, M. Shub and A. Wilkinson, Hölder foliations, *Duke Math. J.* **86** (1997) 517-546.

[PuSh] C.Pugh and and M. Shub, Stable ergodicity and julienne quasi- conformality, *J. Eur. Math. Soc.* **2** (2000) 1-52.

[Ri] I. Richards, On the classification of non-compact surfaces, *Transactions of the A.M.S* **106** 2 (1963) 259-269.

[Ro] J.H. Roberts, Collections filling a plane, *Duke Math J.* **2** n1 (1939) pp. 10-19.

[R] J. Robbin, A structural stability theorem, *Ann. of Math.* **94** (1971) 447-493.

[Rob$_1$] C. Robinson, Structural stability of C^1-diffeomorphisms. *J. Diff. Equ.* **22** (1976) 28-73.

[Rob$_2$] C. Robinson, *Dynamical Systems, Stability, Symbolic Dynamics, and Chaos.*, CRS Press (1994).

[Rom] N. Romero, Persistence of homoclinic tangencies in higher dimensions, *Ergodic Theory and Dynamical Systems*, **15** (1995), 735-757.

[Rou] R. Roussarie, Plongements dans les variétès feuilletées et classification des feuilletages sans holonomie, *Publ. Math. IHES* **43**(1974) 101-141.

[RHRHU$_1$] M.A. Rodriguez Hertz, F. Rodriguez Hertz and R. Ures, Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1d-center bundle,*Inventiones Math.* **172** (2008) 353-381. arXiv:math/0610864

[RHRHU$_2$] M.A. Rodriguez Hertz, F. Rodriguez Hertz and R. Ures, Tori with hyperbolic dynamics in 3-manifolds, *Journal of Modern Dynamics*, **51**, (2011) 185–202. arXiv:1011.3165.

[RHRHU$_3$] M. A. Rodriguez Hertz, F. Rodriguez Hertz and R. Ures, *A non dynamically coherent example in $\mathbb{T}^3$*, *In preparation.*

[Rud] W. Rudin, *Functional Analysis*, McGraw-Hill (1991).

[San] B. Santiago, *Hyperbolicidade essencial em superficies*, Dissertacao de Mestrado UFRJ (2011).

[Shi] K. Shiraiwa, Manifolds which do not admit Anosov diffeomorphisms, *Nagoya Math J.* **49** (1973), 111-115.

[Sh] M. Shub, Global Stability of Dynamical Systems,*Springer-Verlag* (1987).

[Sm$_1$] S. Smale, Stable manifolds for differential equations and diffeomorphisms.*Ann. Scuola Norm. Sup. Pisa* **17** (1963) 97–116.

[Sm$_2$] S. Smale, Differentiable dynamical systems, *Bulletin A.M.S.* **73** (1967), 747-817.

[So] V. Solodov, Components of topological foliations, *Math. USSR Sbornik*, **47** (1984), 329–343.

[VY] M. Viana and J. Yang, Physical measures and absolute continuity for one-dimensional center direction, *Preprint* IMPA A683 (2010).

[Vie] J.L. Vieitez, Expansive homeomorphisms and hyperbolic diffeomorphisms on three manifolds. *Ergodic Theory and Dynamical Systems*, **16** (1996), 591– 622.

[Wa] P. Walters, Anosov diffeomorphisms are topologically stable, *Topology* **9** (1970) 71–78.

[WX] L. Wen et Z. Xia, C^1-connecting lemmas, *Transactions AMS*, **352**, 5213–5230, (2000).

[W_1] L. Wen, Homoclinic tangencies and dominated splittings, *Nonlinearity* **15** (2002), 1445–1469.

[W_2] L. Wen, Generic diffeomorphisms far away from homoclinic tangencies and heterodimensional cycles, *Bol. Soc. Bras. Mat.* **35** (2004).

[W_3] L. Wen, The selecting lemma of Liao,*Disc.and Cont. Dyn. Sys.* **20** (2008) 159-175.

[Wi] A. Wilkinson, Stable ergodicity of the time-one map of a geodesic flow, *Ergodic Theory and Dynamical Systems* **18** (1998) 1545-1587.

[Y] J.Yang, Lyapunov stable homoclinic classes, *Preprint IMPA* A565 (2007).