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Research Article

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Stability of the inverse boundary value problem for the biharmonic operator: Logarithmic estimates

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Abstract: In this article, we establish *logarithmic* stability estimates for the determination of the perturbation of the biharmonic operator from partial data measurements when the inaccessible part of the domain is flat and homogeneous boundary conditions are assumed on this part. This is an improvement to a log-log type stability estimate proved earlier for the partial data case.

Keywords: Inverse problems, stability estimates, biharmonic equation

MSC 2010: 35J40, 35R30

1 Introduction

Let us consider the boundary-value problem for a perturbation of the biharmonic operator posed in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ with smooth boundary, equipped with the Navier boundary conditions, that is,

$$\begin{aligned} \mathcal{B}_q u := (\Delta^2 + q)u = 0 \quad & \text{in } \Omega, \\ u = f \quad & \text{on } \partial\Omega, \\ \Delta u = g \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $f \in H^{7/2}(\partial\Omega)$ and $g \in H^{3/2}(\partial\Omega)$. Biharmonic operators (with potentials) are widely studied in the context of modelling of hinged elastic beams and suspension bridges. We would like to refer to [7, Chapter 1]) for a discussion of these models and other applications.

If 0 is not an eigenvalue of $\mathcal{B}_q u = 0$ in the domain Ω with the boundary conditions $u|_{\partial\Omega} = 0 = \Delta u|_{\partial\Omega}$, /there exists a unique solution $u \in H^4(\Omega)$ to the problem (1.1) when $(f, g) \in H^{7/2}(\partial\Omega) \times H^{3/2}(\partial\Omega)$ (see [7]).

Let us define the set $\mathcal{Q}_N$ of potentials $q \in H^s(\Omega)$, $s > \frac{n}{2}$, as

$$\mathcal{Q}_N := \{q : \|q\|_{H^s(\Omega)} \leq N \text{ for some } N > 0\} \tag{1.2}$$

and assume that for all $q \in \mathcal{Q}_N$, the value 0 is not an eigenvalue for (1.1) with homogeneous boundary conditions on $\partial\Omega$ and thus the problem (1.1) admits a unique solution for each q.

In this article, we shall consider a bounded domain Ω (with smooth boundary) where $\Omega \subset \{x : x_n < 0\}$ and a part Γ_0 of the boundary $\partial\Omega$ is contained in the plane $\{x : x_n = 0\}$.

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We shall assume that the supports of f and g are contained in $\Gamma := \Omega \setminus \Gamma_0$ and that the boundary measurements $\frac{\partial u}{\partial \nu}$ and $\frac{\partial(\Delta u)}{\partial \nu}$ are available on Γ only. Thus the part Γ_0 is assumed to be an inaccessible part of the boundary.

In order to define the Dirichlet-to-Neumann map that is connected with our boundary measurement we set

$$H_0^t(\Gamma) = \{f \in H^t(\partial\Omega) : \operatorname{supp} f \subset \Gamma\}.$$

The partial Dirichlet-to-Neumann map $\mathcal{N}_q$ can then be defined as

$$\mathcal{N}_q : H_0^{\frac{7}{2}}(\Gamma) \times H_0^{\frac{3}{2}}(\Gamma) \to H^{\frac{5}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma), \quad (f, g) \mapsto \Big(\frac{\partial u}{\partial \nu}|_\Gamma, \frac{\partial(\Delta u)}{\partial \nu}|_\Gamma\Big),$$

where u is the solution to (1.1).

Our aim, in this article, is to address the question of stability in the inverse problem of determination of the potential q from the partial Dirichlet-to-Neumann map $\mathcal{N}_q$. The corresponding question of unique identification of the potential q from the map $\mathcal{N}_q$ was recently studied in [19], wherein the author combined the techniques in [14, 15] with a reflection argument introduced in the work [13] to prove the identification of a first-order perturbation as well. The stability question of recovering the potential q for the operator $\mathcal{B}_q$ was also studied in [6], where by following the methods introduced for the study of the Calderón inverse problem in [1] and [9], logarithmic stability estimates were proved when the boundary measurements are available on the whole boundary. Further log-log type estimates were obtained when the measurements are available only on slightly more than half of the boundary. We shall also like to refer to the works [11, 12] in the context of unique determination of the potential q from $\mathcal{B}_q$.

It will be worthwhile to note that this kind of inverse problem, for the conductivity equation, was first introduced in the work [3]. The uniqueness question for dimensions three or higher was settled in the work [17] based upon the idea of complex geometric optics (CGO) solutions. The method introduced in the proof of the stability estimates in [1] was based on [17]. The work [9] which dealt with the partial data case combined the idea of CGO solutions with the techniques of [2]. For subsequent developments related to the stability issues of the Calderón inverse problem and the inverse problem of the related Schrödinger equation, we refer to the works [4, 5, 8, 10, 18].

In this article, we prove a logarithmic-type stability estimate for the determination of q from the Dirichlet-to-Neumann map $\mathcal{N}_q$. We would like to emphasize that here we deal with a partial data case and thus, for this particular class of domains, we are able to improve the log-log type estimates proved in [6]. The strategy of our proof closely follows that in [10]. We use the reflection argument used in [13, 19] and combine it with a suitable quantitative version of the Riemann–Lebesgue lemma derived in [10].

On the space $H^\alpha(\Gamma) \times H^\beta(\Gamma)$ (which we shall henceforth denote as $H^{\alpha,\beta}(\Gamma)$), we shall consider the norm

$$\|(f, g)\|_{H^{\alpha,\beta}(\Gamma)} := \|f\|_{H^\alpha(\Gamma)} + \|g\|_{H^\beta(\Gamma)}.$$

Let us define

$$\|\mathcal{N}_q\| := \sup\{\|\mathcal{N}_q(f, g)\|_{H^{\frac{5}{2},\frac{1}{2}}(\Gamma)} : \|(f, g)\|_{H^{\frac{7}{2},\frac{3}{2}}(\Gamma)} = 1\}.$$

With the above notations, we now state the main result in this article.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain as described above and let $\mathcal{N}_{q_1}, \mathcal{N}_{q_2}$ be the partial Dirichlet-to-Neumann maps corresponding to the potentials $q_1, q_2 \in \mathcal{Q}_N$. Then there exist constants $C, \alpha, \eta > 0$ such that*

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \le C\big(\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| + |\ln\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\||^{\frac{-2\alpha^2}{n+2}}\big)^{\frac{\eta}{2(1+s)}},$$

where C depends on Ω, n, N, s only and α, η depend on s and n only.

Remark 1.2. We believe that the techniques used in the proof of the stability estimate stated above can be generalized to the study of other geometries (see [19]) like hemispheres and slab-like domains and work in this direction is in progress. In particular, when Γ_0 is part of a hemisphere, similar logarithmic-stability estimates should follow by considering a suitable analogue of the Kelvin transformation.

2 Preliminary results

We begin this section by briefly recollecting the results pertaining to the existence of CGO solutions for the equation $\mathcal{B}_q u = 0$ in a domain Ω. For a detailed exposition and proofs, we refer to the works [14–16].

2.1 Carleman estimates and CGO solutions

The existence of CGO solutions was established using Carleman estimates which we state next. We recall that the standard *semiclassical* Sobolev norm of a function $f \in H^s(\mathbb{R}^n)$ is defined as $\|f\|_{H^s_{\mathrm{scl}}(\mathbb{R}^n)} := \|\langle hD\rangle^s f\|_{L^2(\mathbb{R}^n)}$, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

Proposition 2.1. *Let $q \in \mathcal{Q}_N$ and $\phi = \alpha \cdot x$ for some unit vector α. Then there exist positive constants h_0 ($\ll 1$) and C depending on the dimension n and the constant N in (1.2) only such that for all $0 < h < h_0$ the following estimate holds for any $u \in C_c^\infty(\Omega)$:*

$$\|e^{\frac{\phi}{h}} h^4 \mathcal{B}_q e^{-\frac{\phi}{h}} u\|_{L^2(\Omega)} \geq \frac{h^2}{C} \|u\|_{H^4_{\mathrm{scl}}(\Omega)}.$$

Using this estimate, we can prove the following result guaranteeing the existence of CGO solutions.

Proposition 2.2. *There exist positive constants h_0 ($\ll 1$) and C depending only on the dimension n and the constant N in (1.2) such that for all $0 < h < h_0$ there exist solutions to $\mathcal{B}_q u = 0$ belonging to $H^4(\Omega)$ of the form*

$$u(x, \zeta; h) = e^{\frac{ix\cdot\zeta}{h}} (1 + hr(x, \zeta; h)),$$

where $\zeta \in \mathbb{C}^n$ satisfies $\zeta \cdot \zeta = 0$, $|\mathrm{Re}(\zeta)| = |\mathrm{Im}(\zeta)| = 1$ and $\|r\|_{H^4_{\mathrm{scl}}} \leq Ch$.

2.2 A version of the Riemann–Lebesgue lemma

In order to estimate the terms involving Fourier transforms, we shall use a quantitative version of the Riemann–Lebesgue lemma which we discuss next. The following results were proved in [10] but for the sake of completeness we include the proofs here. In what follows, we shall use the following convention for the definition of the Fourier transform of a function f:

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx.$$

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary and let $f \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Let $\tilde{f}$ denote the extension of f to $\mathbb{R}^n$ by zero. Then there exist $\delta > 0$ and $C > 0$ such that*

$$\|\tilde{f}(\cdot - y) - \tilde{f}(\cdot)\|_{L^1(R)^n} \leq C|y|^\alpha$$

for any $y \in \mathbb{R}^n$ with $|y| < \delta$.

Proof. Given the fact that Ω is a bounded domain with C^1 boundary, we can find a finite number of balls $B_i(x_i)$, $i = 1, \ldots, m$, with centers $x_i \in \partial\Omega$ and C^1 diffeomorphisms

$$\phi_i : B_i(x_i) \to Q := \{x' \in \mathbb{R}^{n-1} : \|x'\| \leq 1\} \times (-1, 1).$$

Let

$$d := \mathrm{dist}\Big(\partial\Omega, \partial\Big(\bigcup_{i=1}^m B_i(x_i)\Big)\Big).$$

Then it follows that $d > 0$.

Let $\tilde{\Omega}_\epsilon = \bigcup_{x\in\partial\Omega} B(x,\epsilon)$, where $B(x,\epsilon)$ is the ball of radius ϵ with center x. Now if $\epsilon < d$, we clearly have that

$$\tilde{\Omega}_\epsilon \subset \bigcup_{i=1}^{m} B_i(x_i).$$

Our next step is to estimate the volume of $\tilde{\Omega}_{|y|}$ when $0 < |y| < \delta \le d$ (where we also assume that $d \le 1$). To do so, we note that for $z_1, z_2 \in B(x,|y|) \cap B_i(x_i)$, we have

$$|\phi_i(z_1) - \phi_i(z_2)| \le \|\nabla\phi_i\|_{L^\infty}|z_1 - z_2| \le C|y|$$

for some positive constant C. This implies

$$\phi_i(\tilde{\Omega}_{|y|} \cap B_i(x_i)) \subset \{x' \in \mathbb{R}^{n-1} : \|x'\| \le 1\} \times (-C|y|, C|y|),$$

and using the transformation formula, we then have the estimate $\operatorname{vol}(\tilde{\Omega}_{|y|}) \le C|y|$.

Therefore,

$$\begin{aligned}\|\tilde{f}(\cdot - y) - \tilde{f}(\cdot)\|_{L^1(\mathbb{R}^n)} &= \int_{\Omega\setminus\tilde{\Omega}_{|y|}} |\tilde{f}(x-y) - \tilde{f}(x)|\,dx + \int_{\tilde{\Omega}_{|y|}} |\tilde{f}(x-y) - \tilde{f}(x)|\,dx \\ &\le C\operatorname{vol}(\Omega)|y|^\alpha + 2\|f\|_{L^\infty}\operatorname{vol}(\tilde{\Omega}_{|y|}) \\ &\le C(|y|^\alpha + |y|) \\ &\le C|y|^\alpha, \quad \text{when } |y| < \delta.\end{aligned}$$

□

The following lemma provides a quantitative version of the Riemann–Lebesgue lemma for functions satisfying the conditions of the previous lemma.

Lemma 2.4. *Let $f \in L^1(\mathbb{R}^n)$ and suppose there exist constants $\delta > 0$, $C_0 > 0$ and $\alpha \in (0,1)$ such that for $|y| < \delta$, we have*

$$\|f(\cdot - y) - f(y)\|_{L^1(\mathbb{R}^n)} \le C_0|y|^\alpha. \tag{2.1}$$

Then there exist constants $C > 0$ and $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, we have the inequality

$$|\mathcal{F}f(\xi)| \le C\big(e^{-\frac{\epsilon^2|\xi|^2}{4\pi}} + \epsilon^\alpha\big),$$

where the constant C depends on C_0, $\|f\|_{L^1}$, n, δ, and α.

Proof. Let us denote $G(x) := e^{-\pi|x|^2}$ and define $G_\epsilon(x) := \epsilon^{-n}G(\frac{x}{\epsilon})$. Let $f_\epsilon := f * G_\epsilon$. Then using the triangle inequality, we write

$$\begin{aligned}|\mathcal{F}f(\xi)| &= |\mathcal{F}f_\epsilon(\xi) + \mathcal{F}(f_\epsilon - f)(\xi)| \\ &\le |\mathcal{F}f_\epsilon(\xi)| + |\mathcal{F}(f_\epsilon - f)(\xi)|.\end{aligned}$$

Now,

$$|\mathcal{F}f_\epsilon(\xi)| = |\mathcal{F}f(\xi)|\cdot|\mathcal{F}G_\epsilon(\xi)| \le \|f\|_{L^1(\mathbb{R}^n)}\epsilon^{-n}\epsilon^n\mathcal{F}G(\epsilon\xi) \le \|f\|_{L^1(\mathbb{R}^n)}e^{-\frac{\epsilon^2|\xi|^2}{4\pi}}. \tag{2.2}$$

Next we estimate the term $|\mathcal{F}(f_\epsilon - f)(\xi)|$. In order to do so, we write it as

$$\begin{aligned}|\mathcal{F}(f_\epsilon - f)(\xi)| &\le \|f_\epsilon - f\|_{L^1(\mathbb{R}^n)} \\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y) - f(x)|G_\epsilon(y)\,dx\,dy \\ &\le \int_{|y|<\delta}\int_{\mathbb{R}^n} |f(x-y) - f(x)|G_\epsilon(y)\,dx\,dy + \int_{|y|\ge\delta}\int_{\mathbb{R}^n} |f(x-y) - f(x)|G_\epsilon(y)\,dx\,dy \\ &\le I_1 + I_2 \quad \text{(say)}.\end{aligned}$$

Now using (2.1), we obtain

$$\begin{aligned} I_1 &= \int_{|y|<\delta} \|f(\cdot - y) - f(y)\|_{L^1(\mathbb{R}^n)} G_\epsilon(y)\,dy \\ &\le C_0 \int_{|y|<\delta} |y|^\alpha G_\epsilon(y)\,dy = C_0 \int_{S^{n-1}} \int_0^\delta r^{n-1} r^\alpha \epsilon^{-n} e^{-\frac{\pi r^2}{\epsilon^2}}\,dr\,d\theta \\ &= C \int_0^\delta \epsilon^{n-1} u^{n-1} \epsilon^\alpha u^\alpha \epsilon^{-n} e^{-\pi u^2} \epsilon\,dr = C\epsilon^\alpha \int_0^\delta u^{n+\alpha-1} e^{-\pi u^2}\,du = C\epsilon^\alpha, \end{aligned} \tag{2.3}$$

where the generic constant C depends on C_0, n, δ, and α.

Also,

$$\begin{aligned} I_2 &\le 2\|f\|_{L^1(\mathbb{R}^n)} \int_{|y|\ge\delta} G_\epsilon(y)\,dy \le C\|f\|_{L^1(\mathbb{R}^n)} \int_\delta^\infty \epsilon^{-n} e^{-\frac{\pi r^2}{\epsilon^2}} r^{n-1}\,dr \le C\|f\|_{L^1(\mathbb{R}^n)} \int_{\frac{\delta}{\epsilon}}^\infty u^{n-1} e^{-\pi u^2}\,du \\ &\le C\|f\|_{L^1(\mathbb{R}^n)} \int_{\frac{\delta}{\epsilon}}^\infty e^{-\pi u}\,du \quad \text{(choosing } \epsilon \text{ sufficiently small, less than some } \epsilon_0 \ll 1) \\ &\le C\|f\|_{L^1(\mathbb{R}^n)} \frac{1}{\pi} e^{-\frac{\pi\delta}{\epsilon}} \\ &\le C\epsilon^\alpha \quad \Big(\text{since } \alpha, \epsilon < 1, \text{ we have } \frac{1}{e^{\frac{\pi\delta}{\epsilon}}} < \frac{\epsilon}{\pi\delta} < \frac{\epsilon^\alpha}{\pi\delta}\Big), \end{aligned} \tag{2.4}$$

where the generic constant C depends on $\|f\|_{L^1(\mathbb{R}^n)}$, n, δ, and α.

From (2.2), (2.3) and (2.4), it therefore follows that

$$|\mathcal{F}f(\xi)| \le C\big(e^{-\frac{\epsilon^2|\xi|^2}{4\pi}} + \epsilon^\alpha\big). \qquad \square$$

Remark 2.5. We would like to note that since, by assumption, the potentials $q \in H^s(\Omega)$ where $s > \frac{n}{2}$, there exists $\alpha > 0$ such that $q \in C^{0,\alpha}(\overline{\Omega})$. Hence the conclusions of Lemma 2.4 hold true for the potentials q.

3 Stability estimates

In this section, we now establish the stability estimate given by Theorem 1.1. As a first step, we derive the following integral identity involving the Dirichlet-to-Neumann map for the operator $\mathcal{B}_q$.

Lemma 3.1. *Let u_1, u_2 be solutions of (1.1) corresponding to $q = q_1, q_2$, respectively. Further let v denote the solution to $\mathcal{B}_{q_1}^* v = 0$ in Ω such that $v = 0 = \Delta v$ on Γ_0. Then the following identity holds true:*

$$\int_\Omega (q_2 - q_1)u_2\bar{v}\,dx = \int_\Gamma \partial_\nu(\Delta(u_1 - u_2))\bar{v}\,dS + \int_\Gamma \partial_\nu(u_1 - u_2)(\overline{\Delta v})\,dS \tag{3.1}$$

Proof. To begin with, let us recall the Green's formula

$$\int_\Omega (\mathcal{B}_q u)\bar{v}\,dx - \int_\Omega u\overline{\mathcal{B}_q^* v}\,dx = \int_{\partial\Omega} \partial_\nu(\Delta u)\bar{v}\,dS + \int_{\partial\Omega} \partial_\nu u(\overline{\Delta v})\,dS - \int_{\partial\Omega} (\Delta u)\overline{\partial_\nu v}\,dS - \int_{\partial\Omega} u(\overline{\partial_\nu(\Delta v)})\,dS$$

for $u, v \in H^4(\Omega)$. Let u_1, u_2 be solutions to (1.1) for q replaced by q_1 and q_2, respectively, and let us define

$$u = u_1 - u_2.$$

Let $v \in H^4(\Omega)$ be the solution to $\mathcal{B}_{q_1}^* v = 0$ in Ω and let us note that on the part of the boundary Γ_0, we have $v = 0 = \Delta v$.

With $q = q_1$ and u, v defined as above, we then apply the Green's formula to get

$$\int_\Omega (\mathcal{B}_{q_1}(u_1 - u_2))\overline{v}\,dx = \int_{\partial\Omega} \partial_\nu(\Delta(u_1 - u_2))\overline{v}\,dS + \int_{\partial\Omega} \partial_\nu(u_1 - u_2)(\overline{\Delta v})\,dS - \int_{\partial\Omega} (\Delta(u_1 - u_2))\overline{\partial_\nu v}\,dS - \int_{\partial\Omega} (u_1 - u_2)(\overline{\partial_\nu(\Delta v)})\,dS. \tag{3.2}$$

The last two terms on the right-hand side of (3.2) vanish since u_1 and u_2 satisfy the same boundary conditions. Also let us note that

$$\mathcal{B}_{q_1}(u_1 - u_2) = \mathcal{B}_{q_1}u_1 - \mathcal{B}_{q_1}u_2 = (\Delta^2 + q_1)u_1 - (\Delta^2 + q_1)u_2 = 0 + q_2u_2 - q_1u_2 = (q_2 - q_1)u_2.$$

The first two integrals on the right-hand side of (3.2) are actually on Γ since the integrands vanish on Γ_0 as v and Δv are 0 on Γ_0. Therefore, the identity (3.2) reduces to

$$\int_\Omega (q_2 - q_1)u_2\overline{v}\,dx = \int_\Gamma \partial_\nu(\Delta(u_1 - u_2))\overline{v}\,dS + \int_\Gamma \partial_\nu(u_1 - u_2)(\overline{\Delta v})\,dS,$$

and thus we have proved the result. □

3.1 A suitable change of coordinates

Given a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and hence we can write $x = (x', x_n)$.

For given $\xi \in \mathbb{R}^n$, $(\xi, \xi' \neq 0)$ we will now choose unit vectors α and β in an appropriate way. These vectors will be used when we construct the CGO solutions. To start with, we define the orthonormal basis $E = \{e_1, \dots, e_n\}$ in $\mathbb{R}^n$ in the following way: Let $e_1 = \xi'/|\xi'|$ and $e_n = (0, \dots, 1)$. Let $e_2, \dots, e_{n-1}$ be such that the n-th component $e_{i,n} = 0$ for $i = 2, \dots, n-1$ and such that E is an orthonormal basis of $\mathbb{R}^n$.

In order to calculate the coordinates of a vector $x \in \mathbb{R}^n$ with respect to the basis E we define the following transformation matrix:

$$T_{ES} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} \xi_1/|\xi'| & \xi_2/|\xi'| & \cdots & \xi_{n-1}/|\xi'| & 0 \\ * & * & \cdots & * & 0 \\ \vdots & \vdots & \cdots & \vdots & 0 \\ * & * & \cdots & * & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Note, that here $*$ describes a matrix entry that we cannot describe more precisely, in general. For the case $n = 3$, for example, we can choose $e_2 = (-\frac{\xi_2}{|\xi'|}, \frac{\xi_1}{|\xi'|}, 0)$. Further, note that T_{ES} is an orthogonal matrix. Hence $T_{ES}^{-1} = T_{ES}^T =: T_{SE}$ and T_{SE} is the matrix that calculates standard coordinates from the coordinates with respect to E.

More precisely, the vector ξ has the following representation with respect to the basis E:

$$\xi_e = T_{ES} \cdot \xi = \begin{pmatrix} \frac{\xi' \cdot \xi'}{|\xi'|} \\ 0 \\ \vdots \\ 0 \\ \xi_n \end{pmatrix}.$$

Since T_{ES} is orthogonal, it is clear that the coordinate transformation defined by T_{ES} preserves the scalar product.

Let $(\beta_{e,1}, \dots, \beta_{e,n})_e$ be the representation of $\beta = (\beta_1, \dots, \beta_n)$ in the new coordinates. Since the coordinate change preserves the n-th coordinate, it follows that $\beta_{e,n} = \beta_n$.

The vector β is perpendicular to ξ and therefore

$$0 = \xi \cdot \beta = (\xi_{e,1}, 0, \dots, 0, \xi_{e,n})_e \cdot (\beta_{e,1}, \dots, \beta_{e,n})_e = \xi_{e,1}\beta_{e,1} + \xi_{e,n}\beta_{e,n} = \xi_{e,1}\beta_{e,1} + \xi_n\beta_n.$$

Also,

$$|\beta| = 1 \implies |(\beta_{e,1}, \dots, \beta_{e,n})_e| = 1.$$

A suitable choice of β is

$$\beta_{e,1} = -\frac{\xi_n}{|\xi|}, \quad \beta_n = \beta_{e,n} = \frac{\xi_{e,1}}{|\xi|}, \quad \beta_{e,j} = 0 \text{ for } j = 2, \dots, n-1.$$

Hence,

$$\beta_n^2 = \frac{\xi_{e,1}^2}{|\xi|^2} = \frac{\xi_1^2 + \dots + \xi_{n-1}^2}{|\xi|^2}.$$

For the choice of the unit vector α perpendicular to both ξ and β, we proceed as follows: We want to choose α such that the n-th coordinate α_n is 0. Since the vectors α and β should be perpendicular to each other, that would mean

$$0 = \alpha \cdot \beta = (\alpha_{e,1}, \dots, \alpha_{e,n})_e \cdot (\beta_{e,1}, \dots, \beta_{e,n})_e = \alpha_{e,1}\beta_{e,1} + \alpha_{e,n}\beta_{e,n} = \alpha_{e,1}\beta_{e,1}$$

since $\alpha_{e,n} = \alpha_n = 0$. In particular, we can therefore choose $\alpha_{e,1} = 0$ and choose $\alpha_{e,2}, \dots, \alpha_{e,n-1}$ such that

$$\alpha_{e,2}^2 + \dots + \alpha_{e,n-1}^2 = 1.$$

Since $\alpha_{e,1} = 0 = \alpha_{e,n}$, the condition $\alpha \cdot \xi = 0$ is also satisfied.

Remark 3.2. It will be important to note that this change of coordinates leading to the above choices of the vectors α and β can be carried out for $\xi, \xi' \neq 0$. In other words, we can carry out this change of coordinates for any ξ which does not lie on the ξ_n-axis.

3.2 The stability estimates

Let $\Omega^* := \{x \in \mathbb{R}^n : (x', -x_n) \in \Omega\}$ denote the reflection of Ω about $x_n = 0$ and we extend a potential $q \in \mathcal{Q}_N$ to Ω^* by reflecting q about $x_n = 0$.

Let us also define

$$\zeta_1 = \frac{h\xi}{2} + \sqrt{1 - h^2\frac{|\xi|^2}{4}}\beta + i\alpha, \quad \zeta_2 = -\frac{h\xi}{2} + \sqrt{1 - h^2\frac{|\xi|^2}{4}}\beta - i\alpha,$$

where α, β are constructed as in Section 3.1, i.e. they are unit vectors in $\mathbb{R}^n$, and α, β and ξ being mutually perpendicular.

Then Proposition 2.2 applied to the domain $\Omega \cup \Omega^*$ guarantees the existence of CGO-solutions to $\mathcal{B}_{q_2}\tilde{u}_2 = 0$ and $\mathcal{B}^*_{q_1}\tilde{v} = 0$ in the domain $\Omega \cup \Omega^*$ of the form

$$\tilde{v}(x, \zeta_1; h) = e^{\frac{ix\cdot\zeta_1}{h}}(1 + hr_1(x, \zeta_1; h)), \quad \tilde{u}_2(x, \zeta_2; h) = e^{\frac{ix\cdot\zeta_2}{h}}(1 + hr_2(x, \zeta_2; h))$$

with $\|r_j\|_{H^4_{\mathrm{scl}}(\Omega\cup\Omega^*)} \leq Ch$, $j = 1, 2$, provided $h \leq h_0$ and $1 - h^2\frac{|\xi|^2}{4}$ is positive.

These CGO-solutions, in turn, provide solutions of $\mathcal{B}_{q_2}u_2 = 0$ and $\mathcal{B}^*_{q_1}v = 0$ in the domain Ω of the form

$$v(x, \zeta_1; h) = e^{\frac{ix\cdot\zeta_1}{h}}(1 + hr_1(x, \zeta_1; h)) - e^{\frac{i(x', -x_n)\cdot\zeta_1}{h}}(1 + hr_1((x', -x_n), \zeta_1; h)),$$
$$u_2(x, \zeta_2; h) = e^{\frac{ix\cdot\zeta_2}{h}}(1 + hr_2(x, \zeta_2; h)) - e^{\frac{i(x', -x_n)\cdot\zeta_2}{h}}(1 + hr_2((x', -x_n), \zeta_2; h)),$$

with $v, u_2 \in H^4(\Omega)$ and satisfying the conditions $v|_{\Gamma_0} = 0 = \Delta v|_{\Gamma_0}$, $u_2|_{\Gamma_0} = 0 = \Delta u_2|_{\Gamma_0}$.

We now estimate the right-hand side of (3.1). To do so, we observe that (see also [6])

$$\begin{aligned}
&\Big|\int_\Gamma \partial_\nu(\Delta(u_1-u_2))\bar v\,dS + \int_\Gamma \partial_\nu(u_1-u_2)\overline{(\Delta v)}\,dS\Big| \\
&\quad\le \|\partial_\nu(\Delta(u_1-u_2))\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)} + \|\partial_\nu(u_1-u_2)\|_{L^2(\Gamma)}\|\Delta v\|_{L^2(\Gamma)} \\
&\quad\le C(\|\partial_\nu(\Delta(u_1-u_2))\|_{L^2(\Gamma)}\|v\|_{H^1(\Omega)} + \|\partial_\nu(u_1-u_2)\|_{L^2(\Gamma)}\|\Delta v\|_{H^1(\Omega)}) \\
&\quad\le C(\|\partial_\nu(\Delta(u_1-u_2))\|_{L^2(\Gamma)} + \|\partial_\nu(u_1-u_2)\|_{L^2(\Gamma)})(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)}) \\
&\quad\le C(\|\partial_\nu(\Delta(u_1-u_2))\|_{H^{\frac12}(\Gamma)} + \|\partial_\nu(u_1-u_2)\|_{H^{\frac52}(\Gamma)})(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)}) \\
&\quad\le C\|(\mathcal{N}_{q_1}-\mathcal{N}_{q_2})(f,g)\|_{H^{\frac52,\frac12}(\Gamma)}(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)}) \\
&\quad\le C\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|\|(f,g)\|_{H^{\frac72,\frac32}(\Gamma)}(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)}) \\
&\quad\le C\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|(\|u_2\|_{H^4(\Omega)} + \|\Delta u_2\|_{H^2(\Omega)})(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)}).
\end{aligned}\tag{3.3}$$

Therefore, we shall now have to estimate the norms of u_2, v and their derivatives that appear in the above expression. Since Ω is a bounded domain, we assume that $\Omega \subset B(0,R)$ for some fixed $R > 0$. Then proceeding as in [6], we can prove

$$\|v\|_{H^1(\Omega)} \le \frac{C}{h}e^{\frac{2R}{h}},\quad \|\Delta v\|_{H^1(\Omega)} \le \frac{C}{h}e^{\frac{2R}{h}},\quad \|\Delta u_2\|_{H^2(\Omega)} \le \frac{C}{h}e^{\frac{2R}{h}},\quad \|u_2\|_{H^4(\Omega)} \le \frac{C}{h^4}e^{\frac{2R}{h}}.$$

Using these estimates in (3.3), we have

$$\begin{aligned}
\Big|\int_\Gamma \partial_\nu(\Delta(u_1-u_2))\bar v\,dS + \int_\Gamma \partial_\nu(u_1-u_2)\overline{(\Delta v)}\,dS\Big| &\le C\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|\Big(\frac{C}{h^4}e^{\frac{2R}{h}} + \frac{C}{h}e^{\frac{2R}{h}}\Big)\Big(\frac{C}{h}e^{\frac{2R}{h}} + \frac{C}{h}e^{\frac{2R}{h}}\Big) \\
&\le C\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|\cdot\frac{C}{h^4}e^{\frac{2R}{h}}\cdot\frac{C}{h}e^{\frac{2R}{h}} \\
&\le \frac{C}{h^5}e^{\frac{4R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|,
\end{aligned}$$

and using the fact that $\frac1h \le e^{\frac{R}{h}}$, we can therefore write

$$\Big|\int_\Gamma \partial_\nu(\Delta(u_1-u_2))\bar v\,dS + \int_\Gamma \partial_\nu(u_1-u_2)\overline{(\Delta v)}\,dS\Big| \le Ce^{\frac{9R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|.$$

We next estimate the left-hand side of (3.1). To do so, we write $q = q_2 - q_1$ and note that

$$\begin{aligned}
\int_\Omega qu_2\bar v dx &= \int_\Omega q\Big[e^{\frac{ix\cdot\zeta_2}{h}}(1+hr_2(x,\zeta_2;h)) - e^{\frac{i(x',-x_n)\cdot\zeta_2}{h}}(1+hr_2((x',-x_n),\zeta_2;h))\Big] \\
&\qquad\times\Big[e^{-\frac{ix\cdot\overline{\zeta_1}}{h}}(1+h\overline{r_1}(x,\zeta_1;h)) - e^{-\frac{i(x',-x_n)\cdot\overline{\zeta_1}}{h}}(1+h\overline{r_1}((x',-x_n),\zeta_1;h))\Big]\,dx \\
&= \int_\Omega q[e^{\frac ih x\cdot(\zeta_2-\overline{\zeta_1})}(1+hr_2(x,\zeta_2;h))(1+h\overline{r_1}(x,\zeta_1;h)) \\
&\qquad + e^{\frac ih (x',-x_n)\cdot(\zeta_2-\overline{\zeta_1})}(1+hr_2((x',-x_n),\zeta_2;h))(1+h\overline{r_1}((x',-x_n),\zeta_1;h)) \\
&\qquad - e^{\frac ih [x\cdot\zeta_2-(x',-x_n)\cdot\overline{\zeta_1}]}(1+hr_2(x,\zeta_2;h))(1+h\overline{r_1}((x',-x_n),\zeta_1;h)) \\
&\qquad - e^{\frac ih [(x',-x_n)\cdot\zeta_2-x\cdot\overline{\zeta_1}]}(1+h\overline{r_1}(x,\zeta_1;h))(1+hr_2((x',-x_n),\zeta_2;h))]\,dx.
\end{aligned}\tag{3.4}$$

Let us introduce the notations

$$(x',-x_n)\cdot\zeta_j \equiv x\cdot\zeta_j^*,\quad r_j((x',-x_n),\zeta_j;h) \equiv r_j^*(x,\zeta_j;h).$$

Then (3.4) can be written as

$$\begin{aligned}\int_\Omega (q_2-q_1)u_2\overline{v}\,dx &= \int_\Omega q\Big[e^{\frac{i}{h}x\cdot(\zeta_2-\overline{\zeta_1})}(1+hr_2)(1+h\overline{r_1}) + e^{\frac{i}{h}x\cdot(\zeta_2^*-\overline{\zeta_1^*})}(1+hr_2^*)(1+h\overline{r_1^*})\\ &\qquad - e^{\frac{i}{h}[x\cdot\zeta_2-x\cdot\overline{\zeta_1^*}]}(1+hr_2)(1+h\overline{r_1^*}) - e^{\frac{i}{h}[x\cdot\zeta_2^*-x\cdot\overline{\zeta_1}]}(1+h\overline{r_1})(1+hr_2^*)\Big]\,dx\\ &= \int_\Omega q\Big[e^{\frac{i}{h}x\cdot(\zeta_2-\overline{\zeta_1})} + e^{\frac{i}{h}x\cdot(\zeta_2^*-\overline{\zeta_1^*})}\Big]\,dx - \int_\Omega q\Big[e^{\frac{i}{h}[x\cdot\zeta_2-x\cdot\overline{\zeta_1^*}]} + e^{\frac{i}{h}[x\cdot\zeta_2^*-x\cdot\overline{\zeta_1}]}\Big]\,dx\\ &\qquad + \int_\Omega qw(x,r_1,r_2,r_1^*,r_2^*)\,dx, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} w &= e^{\frac{i}{h}x\cdot(\zeta_2-\overline{\zeta_1})}(hr_2+h\overline{r_1}+h^2r_2\overline{r_1}) + e^{\frac{i}{h}x\cdot(\zeta_2^*-\overline{\zeta_1^*})}(hr_2^*+h\overline{r_1^*}+h^2r_2^*\overline{r_1^*})\\ &\quad - e^{\frac{i}{h}[x\cdot\zeta_2-x\cdot\overline{\zeta_1^*}]}(hr_2+h\overline{r_1^*}+h^2r_2\overline{r_1^*}) - e^{\frac{i}{h}[x\cdot\zeta_2^*-x\cdot\overline{\zeta_1}]}(h\overline{r_1}+hr_2^*+h^2\overline{r_1}r_2^*). \end{aligned}$$

It can easily be checked that $\frac{i}{h}x\cdot(\zeta_2-\overline{\zeta_1}) = -ix\cdot\xi$ and $\frac{i}{h}x\cdot(\zeta_2^*-\overline{\zeta_1^*}) = -i(x',-x_n)\cdot\xi$, and therefore the first term on the right-hand side of (3.5) is nothing but

$$\int_{\mathbb{R}^n} q(x)e^{-ix\cdot\xi}\,dx = \mathcal{F}q(\xi).$$

Also,

$$\begin{aligned} x\cdot\zeta_2-(x',-x_n)\cdot\overline{\zeta_1} &= (x',x_n)\cdot\zeta_2-(x',-x_n)\cdot\overline{\zeta_1}\\ &= (x',x_n)\cdot\bigg(-\frac{h}{2}\xi'+\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta'-i\alpha',-\frac{h}{2}\xi_n+\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta_n-i\alpha_n\bigg)\\ &\qquad -(x',-x_n)\cdot\bigg(\frac{h}{2}\xi'+\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta'-i\alpha',\frac{h}{2}\xi_n+\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta_n-i\alpha_n\bigg)\\ &= (x',x_n)\cdot\bigg(-h\xi',2\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta_n-2i\alpha_n\bigg)\\ &= -h\xi'x'+2\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta_nx_n-2i\alpha_nx_n\\ &= -h\xi'x'+2\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta_nx_n \end{aligned}$$

since $\alpha_n = 0$. This implies

$$\frac{i}{h}[x\cdot\zeta_2-(x',-x_n)\cdot\overline{\zeta_1}] = -ix'\xi'+\frac{2i}{h}\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta_nx_n = -ix\cdot\bigg(\xi',-\frac{2}{h}\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta_n\bigg).$$

Therefore,

$$\int_\Omega q(x)e^{\frac{i}{h}[x\cdot\zeta_2-x\cdot\overline{\zeta_1^*}]}\,dx = \mathcal{F}q\bigg(\bigg(\xi',-\frac{2}{h}\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta_n\bigg)\bigg). \tag{3.6}$$

Similarly we have,

$$\begin{aligned} (x',-x_n)\cdot\zeta_2-x\cdot\overline{\zeta_1} &= (x',-x_n)\cdot\bigg(-\frac{h}{2}\xi'+\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta'-i\alpha',-\frac{h}{2}\xi_n+\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta_n-i\alpha_n\bigg)\\ &\qquad -(x',x_n)\cdot\bigg(\frac{h}{2}\xi'+\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta'-i\alpha',\frac{h}{2}\xi_n+\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta_n-i\alpha_n\bigg)\\ &= (x',x_n)\cdot\bigg(-h\xi',-2\sqrt{1-h^2\frac{|\xi|^2}{4}}\beta_n+2i\alpha_n\bigg), \end{aligned}$$

which, since $\alpha_n = 0$, implies

$$\frac{i}{h}[(x', -x_n) \cdot \zeta_2 - x \cdot \overline{\zeta_1}] = -ix'\xi' - \frac{2i}{h}\sqrt{1 - h^2\frac{|\xi|^2}{4}}\beta_n x_n = -ix \cdot \left(\xi', \frac{2}{h}\sqrt{1 - h^2\frac{|\xi|^2}{4}}\beta_n\right),$$

and therefore

$$\int_\Omega q(x)e^{\frac{i}{h}[x\cdot\zeta_2^* - x\cdot\overline{\zeta_1}]}\,dx = \mathcal{F}q\left(\left(\xi', \frac{2}{h}\sqrt{1 - h^2\frac{|\xi|^2}{4}}\beta_n\right)\right). \tag{3.7}$$

Using Lemma 2.4, we can estimate terms (3.6) and (3.7) as

$$|\mathcal{F}q((\xi', -\frac{2}{h}\sqrt{1 - h^2\frac{|\xi^2|}{4}}\beta_n))| + |\mathcal{F}q((\xi', \frac{2}{h}\sqrt{1 - h^2\frac{|\xi^2|}{4}}\beta_n))| \le C\Big(e^{-\frac{\epsilon^2}{4\pi}\left[|\xi'|^2 + (\frac{4}{h^2} - |\xi|^2)\beta_n^2\right]} + \epsilon^\alpha\Big).$$

We next estimate the last term on the right-hand side of (3.5) using the bounds on the norms of r_j, $j = 1, 2$. To do so, we observe that

$$\begin{aligned}\left|\iint_\Omega qw(x, r_1, r_2, r_1^*, r_2^*)\,dx\right| &\le \int_\Omega |q|\Big[|hr_2 + h\overline{r_1} + h^2 r_2\overline{r_1}| + |hr_2^* + h\overline{r_1^*} + h^2 r_2^*\overline{r_1^*}| \\ &\quad + |hr_2 + h\overline{r_1^*} + h^2 r_2\overline{r_1^*}| + |h\overline{r_1} + hr_2^* + h^2\overline{r_1}r_2^*|\Big]\,dx \\ &\le C\Big[(h\|r_2\|_{L^2(\Omega)} + h\|\overline{r_1}\|_{L^2(\Omega)} + h^2\|r_2\|_{L^2(\Omega)}\|\overline{r_1}\|_{L^2(\Omega)}) \\ &\quad + (h\|r_2^*\|_{L^2(\Omega)} + h\|\overline{r_1^*}\|_{L^2(\Omega)} + h^2\|r_2^*\|_{L^2(\Omega)}\|\overline{r_1^*}\|_{L^2(\Omega)}) \\ &\quad + (h\|r_2\|_{L^2(\Omega)} + h\|\overline{r_1^*}\|_{L^2(\Omega)} + h^2\|r_2\|_{L^2(\Omega)}\|\overline{r_1^*}\|_{L^2(\Omega)}) \\ &\quad + (h\|\overline{r_1}\|_{L^2(\Omega)} + h\|r_2^*\|_{L^2(\Omega)} + h^2\|\overline{r_1}\|_{L^2(\Omega)}\|r_2^*\|_{L^2(\Omega)})\Big].\end{aligned}$$

Using the bounds on the norms of r_j, $j = 1, 2$, stated in Proposition 2.2 and since $h \ll 1$, we can write

$$\left|\iint_\Omega qw(x, r_1, r_2, r_1^*, r_2^*)\,dx\right| \le Ch.$$

Now for $\xi \ne 0$, $|\xi'| > 0$, we have

$$e^{-\frac{\epsilon^2}{4\pi}\left[|\xi'|^2 + (\frac{4}{h^2} - |\xi|^2)\beta_n^2\right]} = e^{-\frac{\epsilon^2}{4\pi}\left[|\xi'|^2 + (\frac{4}{h^2} - |\xi|^2)\frac{|\xi'|^2}{|\xi|^2}\right]} = e^{-\frac{\epsilon^2}{4\pi}\frac{4}{h^2}\frac{|\xi'|^2}{|\xi|^2}}.$$

Let $\rho > 1$ be a real number to be chosen later. Then for any $\xi \ne 0$ such that $0 < |\xi'| < \rho$, $|\xi_n| < \rho$, the following holds: Since $|\xi|^2 < 2\rho^2$, we have $-\frac{1}{|\xi|^2} < -\frac{1}{2\rho^2}$ and hence

$$-\frac{\epsilon^2}{4\pi}\frac{4}{h^2}\frac{|\xi'|^2}{|\xi|^2} \le -\frac{\epsilon^2}{4\pi}\frac{2}{h^2}\frac{|\xi'|^2}{\rho^2},$$

which then implies that

$$e^{-\frac{\epsilon^2}{4\pi}\frac{4}{h^2}\frac{|\xi'|^2}{|\xi|^2}} \le e^{-\frac{\epsilon^2}{4\pi}\frac{2}{h^2}\frac{|\xi'|^2}{\rho^2}}.$$

Thus for any $\xi \ne 0$ such that $0 < |\xi'| < \rho$, $|\xi_n| < \rho$, we have the estimate

$$|\mathcal{F}q(\xi)| \le C\Big[e^{\frac{9R}{h}}\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| + e^{-\frac{\epsilon^2}{4\pi}\frac{2}{h^2}\frac{|\xi'|^2}{\rho^2}} + \epsilon^\alpha + h\Big].$$

Let $Z_\rho = \{\xi \in \mathbb{R}^n : |\xi'| < \rho, |\xi_n| < \rho\}$. Then using Parseval's identity, we can write

$$\|q\|_{H^{-1}}^2 = \int_{Z_\rho}\frac{|\mathcal{F}q(\xi)|^2}{1 + |\xi|^2}\,d\xi + \int_{Z_\rho^c}\frac{|\mathcal{F}q(\xi)|^2}{1 + |\xi|^2}d\xi \le \int_{Z_\rho}\frac{|\mathcal{F}q(\xi)|^2}{1 + |\xi|^2}\,d\xi + \frac{C}{\rho^2}. \tag{3.8}$$

Now since the set $\{\xi \in \mathbb{R}^n : |\xi'| = 0\}$ is of n-dimensional Lebesgue measure zero, we can ignore it and estimate the integral over Z_ρ as follows:

$$\int_{Z_\rho} \frac{|\mathcal{F}q(\xi)|^2}{1+|\xi|^2}\,d\xi \le C[e^{\frac{18R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2 + \epsilon^{2\alpha} + h^2]\int_{Z_\rho}\frac{d\xi}{1+|\xi|^2} + C\int_{-\rho}^{\rho}\int_{B'(0,\rho)} \frac{e^{-\frac{\epsilon^2}{\pi}\frac{1}{h^2\rho^2}|\xi'|^2}}{1+|\xi|^2}\,d\xi'\,d\xi_n$$

$$\le C\rho^n e^{\frac{18R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2 + C\rho^n\epsilon^{2\alpha} + C\rho^n h^2 + C\int_{-\rho}^{\rho}\int_{B'(0,\rho)} \frac{e^{-\frac{\epsilon^2}{\pi}\frac{1}{h^2\rho^2}|\xi'|^2}}{1+|\xi|^2}\,d\xi'\,d\xi_n.$$

Therefore, from (3.8), we can write

$$\|q\|^2_{H^{-1}} \le C\rho^n e^{\frac{18R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2 + C\rho^n\epsilon^{2\alpha} + C\rho^n h^2 + \frac{C}{\rho^2} + C\int_{-\rho}^{\rho}\int_{B'(0,\rho)} \frac{e^{-\frac{\epsilon^2}{\pi}\frac{1}{h^2\rho^2}|\xi'|^2}}{1+|\xi|^2}\,d\xi'\,d\xi_n. \tag{3.9}$$

In order to estimate the integral

$$\int_{-\rho}^{\rho}\int_{B'(0,\rho)} \frac{e^{-\frac{\epsilon^2}{\pi}\frac{1}{h^2\rho^2}|\xi'|^2}}{1+|\xi|^2}\,d\xi'\,d\xi_n,$$

we choose ϵ such that $h = \epsilon^2$ and proceed as follows:

$$\int_{-\rho}^{\rho}\int_{B'(0,\rho)} \frac{e^{-\frac{\epsilon^2}{\pi}\frac{1}{h^2\rho^2}|\xi'|^2}}{1+|\xi|^2}\,d\xi'\,d\xi_n \le 2\rho\int_{B'(0,\rho)} e^{-\frac{\epsilon^2}{\pi}\frac{1}{h^2\rho^2}|\xi'|^2}\,d\xi' = C\rho\int_0^{\rho} r^{n-2}e^{-\frac{\epsilon^2}{\pi}\frac{1}{h^2\rho^2}r^2}\,dr = C\rho\int_0^{\rho} r^{n-2}e^{-\frac{1}{\pi h\rho^2}r^2}\,dr$$

$$= C\rho^2 h^{\frac12}\rho^{n-2}h^{\frac{n-2}{2}}\int_0^{h^{\frac12}} u^{n-2}e^{-\frac{1}{\pi}u^2}\,du \le C\rho^n h^{\frac{n-1}{2}}\int_0^{\infty} u^{n-2}e^{-\frac{1}{\pi}u^2}\,du \le C\rho^n h^{\frac{n-1}{2}}.$$

Using this in (3.9), we have

$$\|q\|^2_{H^{-1}} \le C\rho^n e^{\frac{18R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2 + C\rho^n h^\alpha + C\rho^n h^2 + C\rho^n h^{\frac{n-1}{2}} + \frac{C}{\rho^2},$$

and since $h \ll 1$, $\frac{n-1}{2} \ge 1$ and $\alpha \in (0,1)$, we can write

$$\|q\|^2_{H^{-1}} \le C\rho^n e^{\frac{18R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2 + C\rho^n h^\alpha + \frac{C}{\rho^2}.$$

Next we choose h such that $\rho^n h^\alpha = \frac{1}{\rho^2}$, that is, $h = 1/\rho^{\frac{n+2}{\alpha}}$. Then we have

$$\begin{aligned}
\|q\|^2_{H^{-1}} &\le C\frac{1}{h^{\frac{n\alpha}{n+2}}}e^{\frac{18R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2 + Ch^{\frac{2\alpha}{n+2}}\\
&\le C\frac{1}{h^{\frac{1}{\alpha}}}e^{\frac{18R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2 + Ch^{\frac{2\alpha}{n+2}} \quad \left(\text{since } \tfrac{n\alpha}{n+2} < \tfrac{1}{\alpha}, \text{ we have } h^{\frac{1}{\alpha}} < h^{\frac{n\alpha}{n+2}}\right)\\
&\le C\frac{1}{h^{\frac{1}{\alpha}}}e^{\frac{18R}{h^{1/\alpha}}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2 + Ch^{\frac{2\alpha}{n+2}}\\
&\le Ce^{\frac{20R}{h^{1/\alpha}}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2 + Ch^{\frac{2\alpha}{n+2}} \quad \left(\text{since } \tfrac{1}{h} < \tfrac{1}{h^{\frac{1}{\alpha}}}\right).
\end{aligned} \tag{3.10}$$

Let $\tilde{h} = \min\{h_0, \epsilon_0^2\}$, $\delta = e^{-\frac{20R}{\tilde{h}^{1/\alpha}}}$ and let us assume $\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\| < \delta$. We then choose

$$\rho = \left\{\frac{1}{20R}|\ln\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\||\right\}^{\frac{\alpha^2}{n+2}}.$$

With this choice of ρ, we have

$$\frac{1}{h^{\frac{1}{\alpha}}} = \rho^{\frac{n+2}{\alpha^2}} = \frac{1}{20R}|\ln\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\||.$$

Now,

$$
\begin{aligned}
\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| < e^{-\frac{20R}{\tilde{h}^{\frac{1}{\alpha}}}} (\ll 1) \implies &\quad \ln\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| < -\frac{20R}{\tilde{h}^{\frac{1}{\alpha}}} \\
\implies &\quad |\ln\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|| > \frac{20R}{\tilde{h}^{\frac{1}{\alpha}}} \\
\implies &\quad \frac{1}{20R}|\ln\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|| > \frac{1}{\tilde{h}^{\frac{1}{\alpha}}} \\
\implies &\quad \frac{1}{h^{\frac{1}{\alpha}}} > \frac{1}{\tilde{h}^{\frac{1}{\alpha}}} \\
\implies &\quad h < \tilde{h} < h_0.
\end{aligned}
$$

For $|\xi'| < \rho$, $|\xi_n| < \rho$, we also have

$$
h^2\frac{|\xi|^2}{4} < h^2\frac{\rho^2}{2} = \frac{1}{2}\rho^2\frac{1}{\rho^{\frac{2(n+2)}{\alpha}}} = \frac{1}{2\rho^{[\frac{2(n+2)}{\alpha}-2]}} < 1,
$$

and thus estimate (3.10) indeed remains valid for our choice of h. Also

$$
\begin{aligned}
\frac{20R}{h^{\frac{1}{\alpha}}} = |\ln\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|| = -\ln\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| \implies &\quad \ln\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| = -\frac{20R}{h^{\frac{1}{\alpha}}} \\
\implies &\quad \|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| = e^{-\frac{20R}{h^{1/\alpha}}} \\
\implies &\quad e^{\frac{20R}{h^{1/\alpha}}}\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| = 1,
\end{aligned}
$$

and therefore from (3.10), it follows that for $\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| < \delta$, we have

$$
\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \le C\big(\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| + |\ln\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\||^{\frac{-2\alpha^2}{n+2}}\big). \tag{3.11}
$$

The case when $\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| \ge \delta$ follows directly keeping in mind the uniform bound N satisfied by the potentials belonging to the set $\mathcal{Q}_N$.

From (3.11), we can now derive an estimate for the L^∞ norm of $q_1 - q_2$ by using the interpolation theorem. We recall that if t_0, t, t_1 are such that $t_0 < t_1$ and $t = (1-\beta)t_0 + \beta t_1$, where $\beta \in (0,1)$, then the H^t-norm of a function f can be estimated, using the interpolation theorem, as

$$
\|f\|_{H^t} \le \|f\|^{1-\beta}_{H^{t_0}} \cdot \|f\|^{\beta}_{H^{t_1}}.
$$

To apply this in our case, we define $\eta > 0$ such that $s = \frac{n}{2} + 2\eta$ and choose $t_0 = -1$, $t_1 = s$ and $t = \frac{n}{2} + \eta = s - \eta$. Then,

$$
t = (1-\beta)t_0 + \beta t_1, \quad \text{where } \beta = \frac{1+s-\eta}{1+s},
$$

and using the Sobolev embedding theorem and the interpolation theorem, we have the estimate

$$
\begin{aligned}
\|q_1 - q_2\|_{L^\infty(\Omega)} \le C\|q_1 - q_2\|_{H^{\frac{n}{2}+\eta}(\Omega)} &\le C\|q_1 - q_2\|^{1-\beta}_{H^{-1}(\Omega)}\|q_1 - q_2\|^{\beta}_{H^s(\Omega)} \le C\|q_1 - q_2\|^{\frac{\eta}{1+s}}_{H^{-1}(\Omega)} \\
&\le C\Big(\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| + |\ln\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\||^{\frac{-2\alpha^2}{n+2}}\Big)^{\frac{\eta}{2(1+s)}},
\end{aligned}
$$

which gives us the stated stability estimate.

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