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The Effect of Extended Cornell Potential on Heavy and Heavy-Light Meson Masses Using Series Method

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Abstract

The effect of an extended Cornell potential on the mass spectra of heavy and heavy-light mesons is studied. The Cornell potential is extended to include quadratic potential and inverse quadratic potential. The N-radial Schrödinger equation is solved by using series method. The results for charmonium and bottomonium and light-heavy meson masses are obtained. A comparison with other recent works is discussed. The present results are improved in comparison with other recent works and are in a good agreement with experimental data.

Keywords:Schrödinger equation, Cornell potential, Heavy-light mesons

1. Introduction

The quantum chromodynamics (QCD) is the fundamental theory of strong interactions that is one of the four fundamental forces of nature. It describes the interactions of quarks, via their color quantum numbers. The study of quarkonium properties plays a great role in analysis the strong interaction of quarks $[1-4]$.The Schrodinger equation (SE) is a key in different branches of science and its solution plays a great role in studying the properties of quarkonium such as the mass spectra of quarkonium by using potential models $[\ 5-6]$. Most of researchers have calculated the solution of SEby using different methods such as the numerical methods$[7-9]$, the Laplace transform method $[10-11]$, The Nikiforov-Uvarov (UV) method $[12-14]$. The study of quarkonium in the higher dimensional takes much attention in the recent works as $[15-19]$, which give a lot of information about nature of interquark forces. The different types of potential model are used such as a mix between the Cornell potential, harmonic potential oscillator potential or/and inverse harmonic potential $[20-23]$. The Cornell potential includes two terms, the Coulomb and linear terms. In this work, the following potential is employed as in Ref. [5]

$$V(r)= ar^2+ br - \frac{c}{r} +\frac{d}{r^2}, \tag{1}$$

where a, b, c and d are positive potential parameters, which will be fixed by considering experimental data later on.

The aims of this work: The first aim, we treat the difficulty that found in Ref. [6] which used the present method. The second aim, we extend the study to other heavy-light mesons that are not calculated in Ref. [6] and other works. The third aim, the effect of dimensionality number on quarkonium mass is investigated. Thus, the N-dimensional of radial SE is analytically solved to obtain the energy eigenvalues and corresponding wave functions by using power series technique.

The paper is organized as follows: In the section 2, the power series technique and the energy eigenvalues are calculated in the N-dimensional form. In section 3, the masses spectra of heavy and heavy-light quarkonium are calculated. In the section 4, the results are discussed. In the section5, the summary and conclusion are presented.

2. The solution of theSchrödinger equation with extended Cornall Potential

In N-dimensional Hilbert space, the SE for two particles interacting via spherically symmetric potential (1) can be written as in [6]

$$\left[\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}-\frac{l(l+N-2)}{r^2}+2\mu\left(E-ar^2-br+\frac{c}{r}-\frac{d}{r^2}\right)\right]R(r)=0 \tag{2}$$

where l is the angular momentum quantum number, N denotes dimensionality and $\mu = \frac{m_1 m_2}{m_{1+m_2}}$ is thereduced mass of the particles of masses m_1 and m_2, E is the energy eigenvalue corresponding to the radial eigen function. R(r) is the wave function.

Now, we find an approximate solution to equation (2) by making the following choice the wave function as in Ref. [6]

$$R(r) = \exp(-\alpha r^2-\beta r)F(r), \tag{3}$$

where α and β are positive parameters whose values are to be determined in terms of potential parameters a, b, c and d.

Substituting by Eq. (3) intoEq. (4), we obtain

$$\left[\frac{d^2}{dr^2}+\left(\frac{N-1}{r}-4\alpha r-2\beta\right)\frac{d}{dr}+(4\alpha^2-2\mu a)r^2+(4\alpha\beta-2\mu b)r+\left(2\mu c-\beta(N-1)\right)\frac{1}{r}-l(l+N-2))\frac{1}{r^2}+(\beta^2+2\mu E-2\alpha N)\right]F(r)=0. \tag{4}$$

Assume a series solution to the above equation

$$F(r) =\sum_{n=0}^{\infty} a_n\, r^{\frac{3n}{2}+l}, \tag{5}$$

where $a_n$are expansion coefficients to be determined later. It is to be noted that the basic purpose of choosing a power series like $r^{\frac{3n}{2}+l}$ in the above expression is to avoid degeneracy in energy eigenvalues for some states considered. By substituting Eq. (5) into Eq. (4) and collection powers of r, we obtain

$$\sum_{n=0}^{\infty} a_n \Big[\big((3n+2l)(3n+2l+2N-4)-4l(l+N-2)-8\mu d\big)r^{\frac{3n}{2}+l-2}+4\big(\big(2\mu c-\beta(N-1)-\beta(3n+2l)\big)r^{\frac{3n}{2}+l-1}+4\big(\big(\beta^2+2\mu E-2\alpha N-2\alpha(3n+2l)\big)r^{\frac{3n}{2}+l}+8(2\alpha\beta-\mu b)r^{\frac{3n}{2}+l+1}+8(2\alpha^2-\mu a)r^{\frac{3n}{2}+l+2}\Big]=0 \quad (6)$$

This equation, after equating each coefficient of r to zero gives the following relations

$$E = \frac{2\alpha(3n+2l+N)-\beta^2}{2\mu} \quad (7)$$

$$a=\frac{2\alpha^2}{\mu} \quad (8)$$

$$b=\frac{2\alpha\beta}{\mu} \quad (9)$$

$$c=\frac{\beta(N+3n+2l-1)}{2\mu} \quad (10)$$

$$(3n+2l)(3n+2l+2N-4)-4l(l+N-2)-8\mu d=0 \quad (11)$$

The anstaz parameters α and β may be obtained from equations (8) and (9), respectively as

$$\alpha=\sqrt{\frac{a\mu}{2}}, \qquad a>0 \quad (12)$$

$$\beta=b\sqrt{\frac{\mu}{2a}}, \qquad a>0 \quad (13)$$

Thus, the energy eigenvalue becomes

$$E=\sqrt{\frac{a}{2\mu}}(N+3n+2l)-\frac{b^2}{4a} \quad (14)$$

3. Massspectra of heavy and heavy-light mesons

In this section, we calculate spectra of the heavy quarkonium and heavy-light mesons that have quark and antiquark flavor, the mass of quarkonium is calculated in the 3-dimensions (N=3), so we apply the following relation

$$M=m_q+m_{\bar{q}}+E_{nl} \quad (15)$$

where *m* is bare quark mass for quarkonium. By using Eq.(14),we can write Eq.(15) as follows :

$$M=m_q+m_{\bar{q}}+\sqrt{\frac{a}{m_c}}(3+3n+2l)-\frac{b^2}{4a}. \quad (16)$$

In literature, the masses of charm and bottom quarks are taken between 1.2 GeV to 1.8 GeV and 4.8 GeV to 5.3 GeV, respectively $[24,\ 25,\ 2,\ 26-28]$. In this work, we have chosen $m_c = 1.48\ GeV$, $m_b = 4.823\ GeV$, $m_s = 0.419\ GeV$, and $m_d = m_u$=0.220 GeV $[\,29-31]$. The potential parameters a and b for various mesons are determined using Eq.(16). In case of charmonium, the values of a and b are calculated by solving two algebraic equations in *a* and *b*, which are obtained by inserting experimental values of M for 2S,2P in Eq. (16).In case of bottomonium, the values of a and b are calculated by solving two algebraic equations in a and b, which are obtained by inserting experimental values of M for 1S,2S in Eq. (16). In case of $\bar{b}$c, the values of a and b are calculated by solving two algebraic equations in a and b, which are obtained by using experimental values of M for 1S, 2S in Eq. (16). Similarly, for $c\bar{s}$,$b\bar{s}$and $b\bar{q}$, parameters a and b are determined. The value of parameter dis calculated by using Eq. (11).

4. Results and Discussion

Kumar and Chand [6]calculated the mass spectra of $c\bar{c}$ and $b\bar{b}$ within series solution to the N-dimensional radial schr$\ddot{o}$dinger equation for the quark–antiquark interaction potential. The quark-antiquark interactionpotential, which consists of harmonic, linear and coulomb potential terms. There is a defect in this research.

1-The following is derived in Ref. [6]

$$(3n+2l)(3n+2l+2N-4)-4l(l+N-2)=0. \quad (17)$$

This equation was not valid for all n=0,1 … this equation acts as constraints on the given system with l≥0 and n≥ 0. However, the only acceptable solution to this equation is n=0. To overcome on this difficulty, we add the term $\frac{d}{r^2}$ to Cornell plus harmonic potential that displayed in Eq. (1). We obtained the following relation that has not constraints on l≥0 and n≥ 0

$$(3n+2l)(3n+2l+2N-4)-4l(l+N-2)-8\mu d=0. \quad (18)$$

2-Kumar and Chand [6] calculated only the mass spectra of $c\bar{c}$ and $b\bar{b}$.Thus, we extend our calculation to heavy-light meson such as$c\bar{s}$,$\bar{b}$c, $b\bar{s}$ and $b\bar{q}$ and as we see in the following discussion.

In Table (1),we obtained the mass of charmonium by using Eq. (16), where m_c =1.48 GeV. In this table, we note that the present results are improved in comparison with recent Refs.[1, 5, 23, 32]and are agreement in comparison with the experimental data. In this work, we obtained total error for charmonium mass 0.161 as displayed in Table (1). In Ref. [5], the SE is solved by using the asymptotic iteration method and also they used Cornell potential. They obtained the total error for charmonium0.324.In Ref. [23], the SE is solved by using the Nikiforov-Uvarov method with employing the extended Cornell potential which is a particular case from the present potential at d = 0. A similar situation in Ref. [24] that the authors obtained the total error for charmonium equals 0.903. In Ref. [32], the SE is solved by using the Nikiforov-Uvarov method and the authors obtained total error for charmonium 0.223.

In Table (2),we obtained the mass of bottomonium by using Eq. (16), where m_b=4.623GeV. In this table, we note that the present results are improved in comparison with recent Refs. [5, 23]. We obtained the total error 0.098 for bottomonium. In recent Ref.[5], authors obtained the total error for bottomonium0.197. In Ref. [23], authors obtained the total error for bottomonium0.484.

In Table (3),we obtained the mass of $\bar{b}$c by using Eq. (16) where m_b =4.823 GeV and m_c= 1.209 GeV, we note that the present results are improved in comparison with recent Refs.[1, 5, 25, 30, 34].The present result for 1S state of $\bar{b}$c closes with experimental data. In the

Ref. [34], authors calculated the mass spectra of$\bar{b}$c with non-relativistic treatment for $Q\bar{Q}$ systems and they have considered a general power potentialcolor Coulomb term. They obtained to total error for $\bar{b}c$ about 0.011. In Ref.[30],authors calculated the mass spectra of$\bar{b}$c using relativistic quark model based on quasipotential approach. They obtained to total error for$\bar{b}$c 0.0087.

In Table (4), we obtained the mass of$c\bar{s}$by using Eq. (16), where m_c =1.209 GeV and m_s=0.419 GeV. We note that the present results are improved in comparison with recent Refs.[1, 3, 6, 31]. In the Ref. [31], authors obtained the total error for $c\bar{s}$ $equal\ to$ 0.02248. In the present calculations, the obtained results close with experimental data.

In Table (5), we obtained the mass of $b\bar{s}$ by using Eq. (16), where $m_b = 4.823$ GeV and m_s =0.419 GeV, we note that the present results are improved in comparison with recent Ref.[29].The authors obtained total error for $b\bar{s}$0.011. In the present calculation, we obtained total error 0.0001 and we note that 1S and 1P states close with experimental data. In Table (6), we obtained the mass of$b\bar{q}$ by using Eq. (16), where $m_b = 4.823$ GeV and m_d= $m_u = 0.220$ GeV. We note that the present results are improved in comparison with recent Ref.[29]. In Ref. [29], they obtained total error 0.018. In the present work, total error 0.0003 andwe note that 1P state closes with experimental data.

In this work, one interests to study the effect on the dimensional number on quarkonium mass. The motivation for this as a natural consequence of the unification of the two modern theories of quantum mechanics and relativity and the emergence of the string theory. The investigation of the Standard Model particles in extra or higher-dimensional space is a hot topic of interest. From the experimental point of view, the investigation of the existence of extra dimensions is one of the primary goals of the LHC. The search for extra dimensions with the ATLAS and CMS detectors is discussed in Ref. [37]. In Tables (1-6), we note that quarkonium mass increases with increasing dimensionality number when we take N = 5. This means that that binding energy increases with increasing dimensionality number. The investigation of quarkonium in the higher dimensional space has taken attention in the recent works and the dimensionality number plays an important role in changing the binding energy and dissociation temperatures as in Refs. [38-39]. Also, in Ref. [40], the authors showed that the dimensionality number plays an important role for applying of the limitation of non-relativistic models and show also that the quarkonium mass increases with increasing dimensionality number. In the end discussing of the results, we calculate total error of the present work and other research by using

$$\text{Relative error} = \frac{|m-\tilde{m}|}{|m|}, \qquad (19)$$

where, m is the experimental value,$\tilde{m}$is the computed value, and adding all relative error, we get the total error.

Table 1: Mass spectra of charmoniumin (GeV) (a= 0.058GeV^3, b=0.3366 GeV^2)

States	C	Present work	[1]	[5]	[23]	[32]	N = 5	Exp.[33]
1S	1.1486	3.068	3.096	3.096	3.078	3.096	3.461	3.068
1P	2.2972	3.464	3.259	3.214	3.415	3.255	3.857	3.525
1D	3.4459	3.861	3.511	3.412	3.752	3.504	4.253	3.770
2S	2.8716	3.663	3.686	3.686	4.187	3.686	4.055	3.663
2P	4.0202	4.059	3.779	3.773	4.143	3.779	4.451	-
3S	4.5945	4.258	4.037	4.275	5.297	4.040	4.649	4.159
4S	6.3175	4.852	4.263	4.865	6.407	4.269	5.243	4.421
Total error	-	0.161	0.222	0.324	0.903	0.223	-	-

Table 2: Mass spectra of bottomonium in (GeV) (a= 0.1698 GeV^3, b= 0.7131 GeV^2)

States	c	Present work	[5]	[23]	N = 5	Exp. [33]
1S	0.7878	9.460	9.460	9.510	9.835	9.460
1P	1.5757	9.8354	9.492	9.862	10.210	9.900
1D	2.3636	10.210	9.551	10.214	10.586	10.161
2S	1.9697	10.023	10.023	10.627	10.398	10.023
2P	2.7576	10.398	10.038	10.944	10.773	10.260
3S	3.1515	10.585	10.585	11.726	10.961	10.355
4S	4.3334	11.148	11.148	12.834	11.524	10.580
Total error	-	0.098	0.197	0.484	-	-

Table 3: Mass spectra of $\bar{b}$c in (GeV) (a= 0.2281 GeV^3, b=2.410 GeV^2)

States	c	Present work	[5]	[1]	[25]	[34]	[30]	N=5	Exp. [35]
1S	2.594	6.277	6.277	6.277	6.270	6.349	6.332	6.670	6.277
1P	7.258	6.4234	6.340	6.666	6.699	6.715	6.734	7.059	-
1D	10.888	6.569	6.452	-	-	-	7.072	7.448	-
2S	9.073	6.4963	6.814	7.042	6.835	6.821	6.881	7.253	-
2P	12.702	6.6419	6.851	7.207	7.091	7.102	7.126	7.642	-
3S	14.517	6.7148	7.351	7.384	7.193	7.175	7.235	7.837	-
4S	19.961	6.9333	7.889	-	-	-	-	8.420	-
Total error	-	-	-	-	0.0011	0.011	0.0087	-	-

Table 4: Mass spectra of $c\bar{s}$ in (GeV) (a= 0.036 GeV^3, b= 0.4549GeV^2)

States	c	Present work	[31]	[3]	[6]	[1]	N = 5	Exp. [27]
1S	3.038	2.258	2.129	1.968	2.512	1.968	2.562	-
1P	6.077	2.558	2.549	2.565	2.649	2.566	2.859	-
1D	9.115	2.859	2.899	2.857	2.859	2.857	3.156	2.859
2S	7.596	2.709	2.732	2.709	2.709	2.709	3.008	2.709
2P	10.635	3.009	3.018	-	2.860	-	3.305	-
3S	12.154	3.159	3.193	2.932	2.906	2.932	3.454	-
4S	16.712	3.609	3.575	-	3.102	-	3.900	-
Total error	-	-	0.02248	0.00069	-	0.00069	-	-

Table 5: Mass spectra of $b\bar{s}$ in (GeV) (a= 0.255 GeV^3, b=2.26 GeV^2)

States	c	Present work	[29]	N =5	Exp.[$35-36$]
1S	5.098	5.416	5.450	6.579	5.415
1P	10.197	5.830	5.857	7.020	5.830
1D	15.296	6.245	6.182	7.461	-
2S	12.746	6.038	6.012	7.241	-
2P	17.845	6.452	6.279	7.682	-
3S	20.394	6.659	6.429	7.902	-
4S	28.042	7.281	6.773	8.564	-
Total error	-	0.0001	0.011	-	-

Table 6: Mass spectra of $b\bar{q}$ in (GeV)(a= 0.1997GeV^3, b=2.068GeV^2).

States	c	Present work	[29]	N = 5	Exp.[$35-36$]
1S	7.095	5.326	5.371	5.727	5.325
1P	14.190	5.724	5.777	6.125	5.723
1D	21.285	6.122	6.110	6.523	-
2S	17.738	5.923	5.933	6.324	-
2P	24.833	6.321	6.197	6.722	-
3S	28.380	6.520	6.355	6.921	-
4S	39.023	7.117	6.703	7.518	-
Total error	-	0.0003	0.018	-	-

5. Summary and conclusion

In the present work, we have calculated the energy eigenvalues in the N-dimensional form by solving the N- radial Schr$\ddot{o}$dinger equation using power series technique. This method plays an important role in solving SE. In Ref.[6], the series method is used to solve the SE. In this work, we treated the weakness points that found in Ref. [6] by suggested a new form of quark-antiquark interaction that is displayed by Eq. (1). Also, we extended the calculations of Ref. [6] to include the heavy-light meson. The present results are improved in comparison with other recent works such as [5, 6, 23, 25, 29 – 32, 34] and are in a good agreement with experimental data. In addition, we study the effect of dimensionality number on quarkonium masses which is not considered in the previous works. We hope to extent this work to include the effect of medium on quarkonium properties as a future work.

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