REDISTRIBUTION OF DAMPING
IN HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

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ABSTRACT. By a change of variables that redistributes damping among the equations of
systems of balance laws in one space dimension, it is demonstrated that dissipation induced
by friction, viscosity or other physical mechanism, manifested in the decay of "entropy"
functionals of quadratic order, also controls the total variation of solutions.

Dedicated, with respect, to Professor Giuseppe Grioli,
on the occasion of his 100th birthday.

1. Introduction

This paper surveys a research program aimed at investigating the Cauchy problem for
nonlinear balance laws, in one space dimension, arising in continuum physics, that either
are, or else exhibit features akin to those encountered in, hyperbolic systems with weakly
dissipative source. The central issue is the interaction and competition between dissipation
and hyperbolicity. Extensive earlier research on this question has established that dissipa-
tion dominates under smooth initial data close to equilibrium, as it thwarts the breaking of
waves, rendering the existence of classical solutions in the large. By contrast, nonlinear
hyperbolicity prevails when the initial data are "large", as waves break and shocks develop.

Building a theory of weak solutions with shocks for such systems rests on mastering
how to exploit the contribution of dissipation. This presents a challenge, because the han-
dling of shocks hinges on controlling the variation of solutions, in a L^1 setting, whereas
damping is typically manifested in the decay of entropy functionals of quadratic order, in
a L^2 setting. Due to this mismatch, the machinery developed for treating hyperbolic sys-
tems of conservation laws deems the terms responsible for damping as impediments rather
than as facilitators. We will attempt to resolve this difficulty by performing a change of
variables that redistributes damping among the equations in such a manner that it becomes
compatible with the L^1 setting.

The approach will be illustrated by means of the following examples: The system gov-
erning gas flow through a porous medium; a system governing the motion of a medium
with damping induced by an internal variable; a system governing heat flow in a medium
with memory; a system governing the motion of viscoelastic media with fading memory;
and finally for general strictly hyperbolic systems of balance laws endowed with a uni-
formly convex entropy incurring positive definite entropy production.

2. Damping under Diagonal Dominance

Our point of departure is the general strictly hyperbolic system of conservation laws

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = 0. \tag{1}$$

The *state vector* U takes values in $\mathbb{R}^n$ and the *flux* F is a given, smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$, whose Jacobian matrix $DF(U)$, for any $U \in \mathbb{R}^n$, possesses real distinct eigenvalues $\lambda_1(U) < \cdots < \lambda_n(U)$ and thereby linearly independent sets $R_1(U), \cdots, R_n(U)$ of eigenvectors.

We prescribe initial data

$$U(x,0) = U_0(x), \qquad -\infty < x < \infty \tag{2}$$

of bounded variation over $(-\infty, \infty)$:

$$TV_{(-\infty,\infty)} U_0(\cdot) = \omega. \tag{3}$$

The fundamental existence theorem for the Cauchy problem, which has been established by the random choice method [13,14], the front tracking algorithm [3] and the vanishing viscosity approach [2], states that when ω is sufficiently small there exists a unique admissible BV solution U of (1), (2) on the upper half-plane $(-\infty, \infty) \times [0, \infty)$. Furthermore,

$$TV_{(-\infty,\infty)} U(\cdot, t) \le a\omega, \qquad 0 \le t < \infty. \tag{4}$$

The requirement that the total variation stay small for all t, which is indispensable under the current state of the art in the theory of hyperbolic conservation laws, detracts from the robustness of the above theorem. Indeed, arbitrarily small, but generic perturbations, allowing the flux F to depend explicitly on (x,t) and/or introducing a *source* term G:

$$\partial_t U(x,t) + \partial_x F(U(x,t), x, t) + G(U(x,t), x, t) = 0 \tag{5}$$

modify the estimate (4) into

$$TV_{(-\infty,\infty)} U(\cdot, t) \le a\omega e^{\kappa t}, \tag{6}$$

with κ positive. As a result, one may only establish the existence of a local BV solution U to the Cauchy problem (5), (2), valid on a time interval on which $a\omega e^{\kappa t}$ remains sufficiently small. Thus, to get a robust theory, one has to consider balance laws with dissipative source.

For a homogeneous, strictly hyperbolic system of balance laws

$$\partial_t U(x,t) + \partial_x F(U(x,t)) + G(U(x,t)) = 0, \tag{7}$$

it has been known [12] that the dissipative features of the source G are encoded in the matrix

$$A = R^{-1}(0) DG(0) R(0), \tag{8}$$

where $R(U)$ is a $n \times n$ matrix-valued function whose column vectors are linearly independent eigenvectors $R_1(U), \cdots, R_n(U)$ of $DF(U)$. Indeed, it has been shown by the random choice method [12], the front tracking algorithm [1] and the vanishing viscosity approach [4] that when $R(U)$ may be selected in such a way that A becomes column diagonally dominant,

$$A_{ii} > \sum_{j \ne i} |A_{ji}|, \quad i = 1, \cdots, n, \tag{9}$$

then the Cauchy problem (7), (2), for ω sufficiently small, admits a unique admissible BV solution U on the upper half-plane $(-\infty, \infty) \times [0, \infty)$. Furthermore,

$$TV_{(-\infty,\infty)}, U(\cdot, t) \leq a\omega e^{-\mu t}, \qquad 0 \leq t < \infty, \tag{10}$$

for some $\mu > 0$.

Whether $R(U)$ rendering A column diagonally dominant exists, may be tested [1] by starting out with the A induced by any $R(U)$ and checking whether the eigenvalues of the matrix $\tilde{A}$, defined by $\tilde{A}_{ii} = A_{ii}$, for $i = 1, \cdots, n$, and $\tilde{A}_{ij} = -|A_{ij}|$, for $i \neq j$, have positive real parts.

Under the diagonal dominance condition (9), the Cauchy problem is robust, as it stays well-posed in the large even after (7) is slightly perturbed into (5); see [12]. However, the difficulty lies in that diagonal dominance fails to hold in the majority of systems of balance laws encountered in continuum physics, with damping induced by friction, viscosity, etc. In the following sections we shall see examples in which this difficulty arises, but it may be remedied by a change of variables that redistribute the damping more equitably among the equations of the system.

3. Frictional Damping

The first example is the simple system

$$\begin{cases} u_t - v_x = 0 \\ \\ v_t - f(u)_x + v = 0 \end{cases} \tag{11}$$

which governs the oscillation of a nonlinear elastic spring immersed in a viscous fluid. In that connection, u is the strain, v denotes the velocity and f is the stress, with $f'(u) > 0$. The damping is induced by the source term v, which accounts for the frictional force exerted on the spring by the viscous fluid. The same system governs the flow of a gas through a porous medium, where now u is the specific volume and $-f$ is the pressure.

We prescribe initial conditions

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \qquad -\infty < x < \infty, \tag{12}$$

with

$$\int_{-\infty}^{\infty} [|u_0(x)| + |v_0(x)|]dx = \delta, \tag{13}$$

$$TV_{(-\infty,\infty)}u_0(\cdot) + TV_{(-\infty,\infty)}v_0(\cdot) = \omega. \tag{14}$$

The following existence theorem is established in [5]:

Theorem 3.1. *When δ and ω are sufficiently small, there exists an admissible BV solution (u, v) to the Cauchy problem (11), (12) on the upper half-plane $(-\infty, \infty) \times [0, \infty)$. Furthermore,*

$$\int_{-\infty}^{\infty} [|u(x, t)| + |v(x, t)|]dx \leq c\delta, \qquad 0 \leq t < \infty, \tag{15}$$

$$TV_{(-\infty,\infty)}u(\cdot, t) + TV_{(-\infty,\infty)}v(\cdot, t) \leq a\omega e^{-\mu t} + b\delta, \quad 0 \leq t < \infty, \tag{16}$$

for some $\mu > 0$.

We outline the steps in the proof that are relevant to the theme of this paper. Despite the dissipative character of the source term, the diagonal dominance condition (9) fails to hold, for any $R(U)$, the reason being that the damping is isolated in the second equation of (11). In order to perform an equitable redistribution of damping between the two equations, we introduce the potential ϕ (which represents the displacement in the case of the elastic spring) by

$$\phi_x = u, \qquad \phi_t = v \tag{17}$$

and change variables in (11) from (u, v) to (u, w) where

$$w = v + \tfrac{1}{2}\phi \tag{18}$$

in which case the system takes the form

$$\begin{cases} u_t - w_x + \tfrac{1}{2}u = 0 \\[2mm] w_t - f(u)_x + \tfrac{1}{2}w = \phi. \end{cases} \tag{19}$$

In contrast to (11), the diagonal dominance condition (9) is satisfied for the system (19). However, in order to complete the construction of the solution, one needs an a priori bound on the total variation of ϕ, which appears on the right-hand side of (19), namely an estimate of the form

$$\int_{-\infty}^{\infty} |\phi_x(x,t)|dx = \int_{-\infty}^{\infty} |u(x,t)|dx \le c\delta, \qquad 0 \le t < \infty. \tag{20}$$

This can be extracted from the inequality

$$\eta_t + q_x + v\eta_v \le 0, \tag{21}$$

for a special entropy-entropy flux pair (η, q), constructed on some neighborhood of the origin and satisfying the following conditions: $\eta(u, v)$ is convex, $v\eta_v(u, v) \ge 0$, and

$$\alpha^{-1}[|u| + |v|] \le \eta(u, v) \le \alpha[|u| + |v|]. \tag{22}$$

The construction of an entropy with the above specifications is feasible because (11) contains only two equations.

The same procedure applies, and a similar existence theorem for the Cauchy problem results, for the more general system

$$\begin{cases} u_t - v_x = 0 \\[2mm] v_t - f(u)_x + v - g(u) = 0, \end{cases} \tag{23}$$

under the assumption that the so-called sub-characteristic condition $|g'(u)| < [f'(u)]^{1/2}$ holds, which renders the source term dissipative; see [6, §13-9].

4. Relaxation with Internal Variable

The next example is the system

$$
\begin{cases}
u_t - v_x = 0 \\[2mm]
v_t - f(u)_x + r_x = 0 \\[2mm]
r_t + \gamma r - u = 0
\end{cases}
\tag{24}
$$

which governs the motion of a viscoelastic medium, with viscosity induced by the internal variable r.

The damping effect of viscosity is manifested in the entropy inequality

$$
\eta_t + q_x + \chi \leq 0,
\tag{25}
$$

where

$$
\eta = \tfrac{1}{2}v^2 + F(u) - ur + \tfrac{1}{2}\gamma r^2,
\tag{26}
$$

$$
q = -v[f(u) - r],
\tag{27}
$$

$$
\chi = (\gamma r - u)^2,
\tag{28}
$$

$$
F(u) = \int_0^u f(\xi)d\xi.
\tag{29}
$$

We normalize f by $f'(0) = 1$ and assume $\gamma > 1$ so that η will be convex.

We prescribe initial conditions

$$
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad r(x,0) = r_0(x), \quad -\infty < x < \infty,
\tag{30}
$$

assuming that

$$
\int_{-\infty}^{\infty} (1 + |x|)^{2s}[u_0^2(x) + v_0^2(x) + r_0^2(x)]dx = \delta^2,
\tag{31}
$$

$$
TV_{(-\infty,\infty)}u_0(\cdot) + TV_{(-\infty,\infty)}v_0(\cdot) + TV_{(-\infty,\infty)}r_0(\cdot) = \omega,
\tag{32}
$$

for some $s > 1$. Furthermore, in order to rule out trivial rigid motions,

$$
\int_{-\infty}^{\infty} u_0(x)dx = 0, \quad \int_{-\infty}^{\infty} v_0(x)dx = 0, \quad \int_{-\infty}^{\infty} r_0(x)dx = 0.
\tag{33}
$$

The following existence theorem is established in [11]:

Theorem 4.1. *When δ and ω are sufficiently small, there exists an admissible BV solution (u, v, r) to the Cauchy problem (24), (4.7) on the upper half-plane $(-\infty, \infty) \times [0, \infty)$. Furthermore,*

$$
\int_{-\infty}^{\infty} [|u(x,t)| + |v(x,t)| + |r(x,t)|]dx \leq c\delta, \qquad 0 \leq t < \infty
\tag{34}
$$

$$
\begin{aligned}
TV_{(-\infty,\infty)}u(\cdot,t) + TV_{(-\infty,\infty)}v(\cdot,t) + \\
+ TV_{(-\infty,\infty)}r(\cdot,t) \leq a\omega e^{-\mu t} + b\delta, \qquad 0 \leq t < \infty
\end{aligned}
\tag{35}
$$

for some $\mu > 0$.

We outline the steps in the proof of the above theorem that are relevant to the theme of this paper. The diagonal dominance condition (9) fails to hold for (24) and thus we have to resort to redistribution of damping. Towards that end, we introduce the potentials φ and ψ by

$$\varphi_x = u, \quad \varphi_t = v, \quad \psi_x = v, \quad \psi_t = f(u) - r \tag{36}$$

and change variables from (u, v, r) to (u, w, z), where

$$w = v + \tfrac{1}{2}\varphi, \qquad z = r - \psi, \tag{37}$$

in which case (24) reduces to

$$
\begin{cases}
u_t - w_x + \tfrac{1}{2}u = 0 \\[2mm]
w_t - f(u)_x + z_x + \tfrac{1}{2}w = \tfrac{1}{4}\varphi \\[2mm]
z_t + (\gamma - 1)z + h(u) = -(\gamma - 1)\psi,
\end{cases}
\tag{38}
$$

with $h(u) = f(u) - u$.

The diagonal dominance condition (9) holds for the system (38) (notice that $h(u) = O(u^2)$ near zero). However, in order to complete the construction of the solution one needs an a priori bound on the total variation of ψ and φ, which appear on the right-hand side of (38). This bound is provided by the estimate (34).

Since (24) consists of three equations, it is no longer possible to establish (34) by constructing L^1-type entropies as we did in Section 3. However, we still have recourse to L^2-type estimates, such as (25). A supplementary L^2 estimate is

$$H_t + Q_x + R = 0, \tag{39}$$

where

$$H = \varphi^2 + \tfrac{1}{\gamma}r^2 + \tfrac{1}{\gamma(\gamma-1)}(\gamma\psi - r)^2, \tag{40}$$

$$Q = -2\varphi\psi, \tag{41}$$

$$R = 2r^2 - \tfrac{2}{\gamma-1}(\gamma\psi - r)h(u). \tag{42}$$

Estimates with the same flavor were derived, in a more general setting, by Ruggeri and Serre [16].

Upon combining (25) with (39), one shows

$$\int_0^\infty \int_{-\infty}^\infty [u^2(x,t) + v^2(x,t) + r^2(x,t)]dxdt \leq c\delta^2, \tag{43}$$

and then

$$t \int_{-\infty}^\infty [u^2(x,t) + v^2(x,t) + r^2(x,t)]dx \leq c\delta^2, \qquad 0 \leq t < \infty, \tag{44}$$

whence the integral of $|u(x,t)| + |v(x,t)| + |r(x,t)|$ over any bounded interval $|x| \leq \lambda t$ is bounded by $c\sqrt{\lambda}\delta$. On the other hand, one shows that the integral of $|u(x,t)| + |v(x,t)| + |r(x,t)|$ over $|x| > \lambda t$ is also bounded by $c\delta$, in virtue of (31) and the finite speed of wave propagation. We thus arrive at (34), concluding the proof of Theorem 4.1.

5. Heat Flow in Media with Memory

The example considered in this section is not a hyperbolic system, in a strict sense, but an evolutionary integrodifferential equation

$$\theta_t(x, t) = \int_0^t a(t - \tau) f(\theta_x(x, \tau))_x d\tau \tag{45}$$

that exhibits similar behavior.

Equation (45) governs the evolution of the temperature θ in a one-dimensional medium with memory. The function f is increasing, normalized by $f'(0) = 1$. For simplicity, the relaxation kernel a is assumed to be of the form

$$a(t) = \sum_{i=1}^n \alpha_i e^{-\lambda_i t}, \tag{46}$$

with $\alpha_i > 0, i = 1, \cdots, n$, and $0 < \lambda_1 < \cdots < \lambda_n$. We scale the space-time variables so that $a(0) = 1, a'(0) = -1$.

Differentiating (45) with respect to t and setting $\theta_x = u, \theta_t = v$ yields the equivalent system

$$\begin{cases} u_t - v_x = 0 \\ \\ v_t - f(u)_x = \int_0^t a'(t - \tau) f(u)_x d\tau. \end{cases} \tag{47}$$

We prescribe initial conditions

$$\theta(x, 0) = \theta_0(x), u(x, 0) = u_0(x) = \theta_{0x}(x), v(x, 0) = 0, \quad -\infty < x < \infty, \tag{48}$$

assuming that, for some $s > 1$ and δ small,

$$\int_{-\infty}^{\infty} \theta_0^2(x) dx \leq \delta^2, \tag{49}$$

$$\int_{-\infty}^{\infty} (1 + |x|)^{2s} u_0^2(x) dx \leq \delta^2, \tag{50}$$

$$TV_{(-\infty,\infty)} u_0(\cdot) \leq \delta. \tag{51}$$

The following existence theorem is established in [9]:

Theorem 5.1. *When δ is sufficiently small, there exists an admissible BV solution (u, v) of the Cauchy problem* (47), (48) *on the upper half-plane* $(-\infty, \infty) \times [0, \infty)$. *Furthermore,*

$$TV_{(-\infty,\infty)} u(\cdot, t) + TV_{(-\infty,\infty)} v(\cdot, t) \leq c\delta, \quad 0 \leq t < \infty. \tag{52}$$

An important step in the proof of the above theorem is to rewrite (47) in the equivalent form

$$\begin{cases} u_t - v_x = 0 \\ \\ v_t - f(u)_x + v = -\int_0^t k'(t - \tau) v d\tau, \end{cases} \tag{53}$$

where k is the resolvent kernel of a'. For a given by (46),

$$k(t) = \kappa_0 + \sum_{i=1}^{n-1} \kappa_i e^{-\mu_i t}, \tag{54}$$

where $\kappa_i > 0$, for $i = 0, 1, \cdots, n - 1$, and $0 < \lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{n-1} < \mu_{n-1} < \lambda_n$. Transforming (47) to (53) was originally suggested by MacCamy [15].

Since (53) does not have the diagonal dominance property (9), we must redistribute the damping. For that purpose, we change variables from (u, v) to (u, w), where

$$w = v + \tfrac{1}{2}\theta, \tag{55}$$

in which case (53) becomes

$$\begin{cases} u_t - w_x + \tfrac{1}{2}u = 0 \\[2mm] w_t - f(u)_x + \tfrac{1}{2}w = \chi, \end{cases} \tag{56}$$

where

$$\chi(x, t) = [\tfrac{1}{2} - k'(0)]\theta(x, t) + k'(t)\theta_0(x) - \int_0^t k''(t - \tau)\theta(x, \tau)d\tau. \tag{57}$$

The diagonal dominance condition (9) holds for (56), but in order to complete the construction of solutions we need a bound on the total variation of $\chi(\cdot, t)$, and for that purpose we seek an estimate

$$\int_{-\infty}^{\infty} [|u(x, t)| + |v(x, t)|]dx \leq c\delta, \qquad 0 \leq t < \infty. \tag{58}$$

As in the previous section, (58) can be established by first deriving the L^2-type estimate

$$t\int_{-\infty}^{\infty} [u^2(x, t) + v^2(x, t)]dx \leq c\delta^2, \qquad 0 \leq t < \infty, \tag{59}$$

and then combining it with (50) and the finite speed of wave propagation. The details are found in [9].

6. Viscoelasticity

The system

$$\begin{cases} u_t - v_x = 0 \\[2mm] v_t - f(u)_x = \int_0^t a'(t - \tau)u_x d\tau \end{cases} \tag{60}$$

governs the motion of a one-dimensional viscoelastic medium with fading memory. The function f, representing the instantaneous elastic response of the medium, is increasing and is normalized by $f(0) = 0, f'(0) = 1$. For simplicity, we assume that the relaxation kernel a is again given by (46) and is normalized by $a'(0) = -1$. However, it is essential here to assume $a(0) < f'(0) = 1$ so that the function

$$g(u) = f(u) - a(0)u, \tag{61}$$

which represents the relaxed elastic response of the medium, is increasing, at least near the origin.

We prescribe initial conditions

$$u(x, 0) = u_0(x), \qquad v(x, 0) = 0, \qquad -\infty < x < \infty, \tag{62}$$

where u_0 satisfies

$$\int_{-\infty}^{\infty} (1 + |x|)^{2s} u_0^2(x) dx < \delta^2, \tag{63}$$

$$\int_{-\infty}^{\infty} u_0(x) dx = 0, \tag{64}$$

$$TV_{(-\infty,\infty)} u_0(\cdot) < \delta, \tag{65}$$

for some $s > 1$ and δ positive small.

The following existence theorem is established in [10].

Theorem 6.1. *When δ is sufficiently small, there exists an admissible BV solution (u, v) to the Cauchy problem (60), (62) on the upper half-plane $(-\infty, \infty) \times [0, \infty)$. Furthermore,*

$$TV_{(-\infty,\infty)} u(\cdot, t) + TV_{(-\infty,\infty)} v(\cdot, t) \le c\delta, \qquad 0 \le t < \infty. \tag{66}$$

Notice that upon setting

$$r(x, t) = -\int_0^t a'(t - \tau) u(x, \tau) d\tau, \tag{67}$$

the system (60) may be written in the equivalent form

$$\begin{cases} u_t - v_x = 0 \\ v_t - f(u)_x + r_x = 0 \\ r_t - u = -\int_0^t a''(t - \tau) u d\tau, \end{cases} \tag{68}$$

which bears close resemblance to (24), and indeed reduces to (24) when $a(t) = \frac{1}{\gamma} e^{-\gamma t}$.

Motivated by the treatment of (47), in the previous section, we write $f(u) = h(u) + u$, where

$$h(u) = f(u) - u, \tag{69}$$

and take the convolution of the second equation in (60) with the resolvent kernel k of a', which yields

$$v_t - f(u)_x + v = -\int_0^t k'(t - \tau) v d\tau + \int_0^t k(t - \tau) h(u)_x d\tau. \tag{70}$$

We now introduce new variables

$$w = v + \tfrac{1}{2}\varphi, \tag{71}$$

$$z = -u - \int_0^t k(t - \tau) h(u) d\tau, \tag{72}$$

where φ is the potential (displacement) with $\varphi_x = u, \varphi_t = v$. By a lengthy but straightforward calculation, (60) is transformed into an equivalent system of three equations in the

variables (u, w, z):

$$\begin{cases} u_t - w_x + \frac{1}{2}u = 0 \\[2mm] w_t - h(u)_x + z_x + \frac{1}{2}w = \chi \\[2mm] z_t + w_x + \frac{1}{2}z = \psi, \end{cases} \tag{73}$$

where

$$\chi = [\tfrac{1}{4} - k'(0)]\varphi + k'(t)\varphi_0 - \int_0^t k''(t-\tau)\varphi d\tau, \tag{74}$$

$$\psi = -h(u) - \tfrac{1}{2}\int_0^t k(t-\tau)h(u)d\tau - \int_0^t k'(t-\tau)h(u)d\tau. \tag{75}$$

The diagonal dominance condition (9) is satisfied by the system (73). However, in order to complete the construction of the solution, we need to control the total variation of $\chi(\cdot, t)$ and $\psi(\cdot, t)$, uniformly on $[0, \infty)$.

With regard to the total variation of $\psi(\cdot, t)$, we first note that both k and k' are integrable over $[0, \infty)$. Furthermore, h, as defined through (69), is $O(u^2)$ near zero, and hence its total variation is majorized by the square of the total variation of u. A term of such size is controlled by the damping action of the source term in (73).

On the other hand, to bound the total variation of $\chi(\cdot, t)$, one needs an estimate of the form

$$\int_{-\infty}^{\infty} |\varphi_x(x,t)|dx = \int_{-\infty}^{\infty} |u(x,t)|dx \le c\delta, \qquad 0 \le t < \infty. \tag{76}$$

Such an estimate results, as in earlier sections, from

$$t\int_{-\infty}^{\infty} [u^2(x,t) + v^2(x,t)]dx \le c\delta, \qquad 0 \le t < \infty, \tag{77}$$

which, in turn, is established with the help of L^2-type estimates. The details are found in [10].

7. General Systems

We conclude with a discussion of the Cauchy problem (7), (2) for general homogeneous strictly hyperbolic systems of balance laws that may fail to satisfy the diagonal dominance condition (9) but still satisfy the weaker hypothesis

$$A_{ii} > 0, \qquad i = 1, \cdots, n, \tag{78}$$

which holds, for instance, for the systems (11) and (23). We follow the references [7] and [8].

Damping is redistributed by the following device. Starting out from any solution $U(x, t)$ of (7), we construct a function $W(x, t)$ by

$$W(x, t) = U(x, t) - P(x, t), \tag{79}$$

where

$$P(x,t) = N \int_{-\infty}^{x} U(y,t)dy, \tag{80}$$

$$N = R(0)MR^{-1}(0), \tag{81}$$

and M is the $n \times n$ matrix with entries $M_{ii} = 0, i = 1, \cdots, n$, and

$$M_{ij} = \frac{A_{ij}}{\lambda_j(0) - \lambda_i(0)}, \qquad \text{for } i \neq j. \tag{82}$$

Then W is a solution of the nonhomogeneous system of balance laws

$$\partial_t W(x,t) + \partial_x \hat{F}(W(x,t), P(x,t)) + \hat{G}(W(x,t), P(x,t), Q(x,t)) = 0, \tag{83}$$

where

$$\hat{F}(W,P) = F(W+P) - F(P), \tag{84}$$

$$\hat{G}(W,P,Q) = G(W+P) - NF(W+P) + DF(P)N[W+P] - Q, \tag{85}$$

$$Q(x,t) = N \int_{-\infty}^{x} G(U(y,t))dy. \tag{86}$$

The gain is that, in contrast to the original system (7), the new system (83) satisfies the diagonal dominance condition, since

$$R^{-1}(0)D_W \hat{G}(0,0,0)R(0) = \text{diag}\{A_{11}, \cdots, A_{nn}\}. \tag{87}$$

We may thus find a BV solution U of the Cauchy problem (7), (2), by solving the Cauchy problem (83), (2), and for that purpose what is needed is an a priori bound on the total variation of $P(\cdot, t)$ and $Q(\cdot, t)$, i.e. an a priori bound on the $L^1(-\infty, \infty)$ norm of $U(\cdot, t)$, uniformly in t on $[0, \infty)$.

In earlier sections, we have seen how one may estimate the $L^1(-\infty, \infty)$ norm of $U(\cdot, t)$, for specific systems. Here we consider the case of a general system (7) endowed with an entropy-entropy flux pair (η, q), where the entropy is convex and it incurs a positive definite entropy production. The result is

Theorem 7.1. *Assume that the strictly hyperbolic system (7) of balance laws is endowed with an entropy-entropy flux pair (η, q) such that the matrices $D^2\eta(0)$ and $D^2\eta(0)DG(0)$ are positive definite. If U_0 satisfies (3), with ω sufficiently small, and*

$$\int_{-\infty}^{\infty} (1 + |x|)^{2s} |U_0(x)|^2 dx = \delta^2 \tag{88}$$

for some $s > 1$ and δ sufficiently small, then there exists an admissible BV solution U of (7), (2) on the upper half-plane $(-\infty, \infty) \times [0, \infty)$. Furthermore,

$$TV_{(-\infty,\infty)}U(\cdot, t) \leq a\omega e^{-\mu t} + b\delta e^{-\nu t}, \qquad 0 \leq t < \infty, \tag{89}$$

for some $\mu > 0, \nu > 0$.

For the proof, one first notes that the positive definiteness of $D^2\eta(0)$ and $D^2\eta(0)DG(0)$ imply (78). The same assumptions together with the entropy inequality

$$\partial_t \eta(U(x,t)) + \partial_x q(U(x,t)) + D\eta(U(x,t))G(U(x,t)) \leq 0 \tag{90}$$

imply that $\int \eta(U(x,t))dx$ decays exponentially, as $t \to \infty$. This in conjunction with (88) and the finite speed of wave propagation yields an estimate

$$\int_{-\infty}^{\infty} |U(x,t)|dx \le ce^{-\nu t}, \qquad 0 \le t < \infty.$$

the assertion of the theorem then follows by working through (83). For the details, see [8].

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