Holomorphic Semiflows and Poincaré-Steklov Semigroups

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Contents

Nomenclature

Sets

(a,b)	$:= \{x \in \mathbb{R} \colon a < x < b\}$, an open interval in $\mathbb{R}$
$[a,b]$	$:= \{x \in \mathbb{R} \colon a \leq x \leq b\}$, a closed interval in $\mathbb{R}$
$\mathbb{C}$	complex plane
$\mathbb{C}_+$	$:= \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, the right half-plane
$\mathbb{D}$	$:= \{z \in \mathbb{C} \colon \lvert z\rvert < 1\}$, the open unit disk in $\mathbb{C}$
$\mathbb{N}$	$:= \{0,1,2,\dots\}$, the natural numbers
$\mathbb{R}$	real numbers
$\mathbb{R}_{\geq 0}$	$:= \{x \in \mathbb{R} \colon x \geq 0\}$, the non-negative real numbers

Spaces of functions and operators

$\mathscr{A}^p(\Omega)$	Bergman space on Ω
$\mathscr{H}(\Omega)$	space of holomorphic functions defined on $\Omega \subseteq \mathbb{C}$ with values in $\mathbb{C}$
$\mathscr{H}(\Omega,\tilde{\Omega})$	space of holomorphic functions defined on $\Omega \subseteq \mathbb{C}$ with values in $\tilde{\Omega} \subseteq \mathbb{C}$
$\mathscr{H}^p(\Omega)$	Hardy space on Ω
ℓ^p	space of p-summable sequences
$\mathscr{L}(X)$	space of bounded linear operators $T : X \to X$, where X is a Banach space
$\mathscr{L}(X,Y)$	space of bounded linear operators $T : X \to Y$, where X and Y are Banach spaces
$\mathfrak{h}(\Omega)$	space of harmonic functions defined on $\Omega \subseteq \mathbb{C}$ with values in $\mathbb{R}$
$C(\Omega)$	space of continuous functions on Ω with values in $\mathbb{C}$
$C(\Omega,\tilde{\Omega})$	space of continuous functions on Ω with values in $\tilde{\Omega}$
$C^k(\Omega)$	space of k-times continuously differentiable functions defined on Ω with values in $\mathbb{C}$
$L^p(\Omega)$	space of p-integrable functions on Ω w.r.t. the normalizes Lebesgue measure on Ω

Operations

$\arg z$	argument of a complex number $z \in \mathbb{C}$
$\langle v,A\rangle$	matrix multiplication $v^\perp \cdot A$ if $v \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$
$\langle v,w\rangle$	inner product of $\mathbb{R}^2$ if $v,w \in \mathbb{R}^2$

$\bar{\Omega}$	closure of a set Ω
$\bar{z}$	complex conjugate of a number $z \in \mathbb{C}$
$\dot{u}(t,x)$	$\partial_t u(t,x)$, the partial derivative of u w.r.t. t
$\operatorname{Im} z$	imaginary part of a number $z \in \mathbb{C}$
$\partial\Omega$	boundary of a set Ω
$\partial_x u(t,x)$	partial derivative of u w.r.t. x
$\operatorname{Re} z$	real part of a number $z \in \mathbb{C}$
$C_\varphi f$	$:= f \circ \varphi = f(\varphi)$, the composition operator with symbol φ
$M_a f$	$:= a \cdot f$, the multiplication operator with symbol a

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Introduction

In this thesis, we consider strongly continuous semigroups on Banach spaces of analytic functions. In contrast to the theory of strongly continuous semigroups on general Banach spaces, there is no such comprehensive literature concerning semigroups on Banach spaces of analytic functions, see [72] for an overview. Restricting ourselves to these spaces, we are able to combine results from functional analysis with tools from complex analysis. In general, to take advantage of the specific structure of theses spaces, special operators are studied, for example, shift, integral, or composition operators are examined. In particular, initiated by the paper [20], a large theory on semigroups of composition operators on spaces of analytic functions has grown during the last 40 years ([5, 12, 13, 16, 22, 23, 44, 64, 71, 72]). Such semigroups are defined by the composition of analytic functions with a dynamical system of holomorphic selfmaps of a certain domain Ω. More precisely, let $\Omega \subseteq \mathbb{C}$ be a domain and $\mathscr{H}(\Omega)$ the space of holomorphic functions on Ω. Let $X \hookrightarrow \mathscr{H}(\Omega)$ be a Banach space of analytic functions defined on Ω and $(\varphi_t)_{t\geq 0}$ a dynamical system consisting of holomorphic selfmaps of Ω. Then a semigroup of composition operators on X consists of operators of the form

$$C_{\varphi_t} f = f \circ \varphi_t, \quad t \geq 0, \tag{0.1}$$

with $f \in X$. Composition operators and semigroups of composition operators on Banach spaces of continuous or Lebesgue-integrable functions appear especially in ergodic theory and the theory of positive operator semigroups with numerous approaches in physics and number theory, see [18, 40]. In this setting, composition operators and composition semigroups are called Koopman operators and Koopman semigroups. The idea there is to linearize a given dynamical system of continuous or measure preserving selfmaps so that one can study properties of such a system on another state space. In contrast to this, in the theory of semigroups of composition operators on spaces of analytic functions, operator theoretic properties are deduced from analytic properties of the underlying dynamical system of holomorphic functions, see, e.g., [44, 59, 69]. Furthermore, semigroups of composition operators on spaces of analytic functions are surprisingly connected with the analysis of parabolic equations involving the Dirichlet-to-Neumann operator or similar Poincaré-Steklov operators. The Dirichlet-to-Neumann operator is defined as the linear operator mapping the Dirichlet boundary data $f : \partial\Omega \to \mathbb{C}$ to the Neumann derivative of the solution u of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases} \tag{0.2}$$

provided the solution exists and is sufficiently regular. So, the Dirichlet-to-Neumann operator acts as follows

$$\mathfrak{D}_{\mathcal{N}} : f \mapsto \langle \nu, \nabla u \rangle, \tag{0.3}$$

where ν is the outward pointing normal vector on $\partial\Omega$. In recent years, this operator has been used in the analysis of inverse problems, which apply for instance to image techniques in medicine and also to find defects in materials. Assuming Ω to be the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$, Lax [60] observed that the for Dirichlet boundary data $f \in L^2(\partial\mathbb{B}_n)$ or $f \in C(\partial\mathbb{B}_n)$, the semigroup $(T_t)_{t\geq 0}$ generated by the negative Dirichlet-to-Neumann operator is similar to a semigroup of composition operators $(C_{\varphi_t})_{t\geq 0}$ acting on a Banach space X consisting of harmonic functions. Denoting the space of all (continuous, square-integrable, etc.) Dirichlet boundary values of X by ∂X, we obtain that for $f \in \partial X$ the function $u(t) := T_t f$ is a (mild) solution to the following evolution problem associated with the Dirichlet-to-Neumann operator

$$\begin{cases} \partial_t u + \langle \nu, \nabla u \rangle = 0 & \text{on } (0, \infty) \times \partial\mathbb{B}_n, \\ -\Delta u = 0 & \text{on } (0, \infty) \times \mathbb{B}_n, \\ u(0, \cdot) = f & \text{on } \partial\mathbb{B}_n, \end{cases} \tag{0.4}$$

where ν is the outward pointing normal vector of the unit sphere. The Dirichlet-to-Neumann semigroup, in this case sometimes referred to as Lax semigroup, is similar to a semigroup of composition operators associated with the dynamical system $(\varphi_t)_{t\geq 0}$, given by $\varphi_t(x) = e^{-t}x$ $(x \in \mathbb{B}_n, t \geq 0)$. However, that the negative Dirichlet-to-Neumann operator generates a semigroup on L^2 has been proved by using the method of forms on way more general domains, [9, 10, 11], namely, for a domain $\Omega \subseteq \mathbb{R}^n$ the boundary of which is assumed to have finite $(n-1)$-dimensional Hausdorff measure only. Unfortunately, it has been shown that such a nice representation in terms of a semigroup of composition operators is only possible if the underlying domain is a ball, see [46].

In this thesis, we present a different generalization of the Lax semigroup. Using methods from complex analysis, our purpose is to give new approaches to semigroups generated by operators closely related to the Dirichlet-to-Neumann operator and also to more general operators on boundary spaces of Banach spaces of analytic functions. The Dirichlet-to-Neumann operator is, in fact, a particular example of a more general class of operators, so-called Poincaré-Steklov operators. These operators are defined as linear operators mapping a boundary condition of an elliptic equation to another boundary condition of the same equation. Our aim is to find those Poincaré-Steklov operators that generate a semigroup, similar to a composition semigroup, on Banach spaces that consist of boundary values of functions which belong to a Banach space of analytic functions. It turns out that we can obtain various Dirichlet-to-Neumann and Dirichlet-to-Robin operators which generate semigroups representable as the trace of a semigroup of composition operators and semigroups of weighted composition operators, respectively. By developing this setting, we build a fundament for further investigations of semigroups generated by Poincaré-Steklov operators beyond the content of this thesis. Indeed, our approach allows to apply already established results for semigroups of composition operators (see, e.g., [8, 16, 44]) to investigate, e.g., analyticity, compactness, and long-term behavior of semigroups generated by a Poincaré-Steklov operator.

Let us briefly outline the structure of our thesis. The study of the connection between evolution equations associated with a Poincaré-Steklov operator and semigroups of composition operators can be considered as the first part of our thesis, and it takes place in the first and second chapter. In Chapter 1, we give an overview on the theory of composition semigroups needed for our investigations. In the literature, semigroups of composition operators are usually considered on Hilbert and Banach spaces of analytic functions defined on the unit disk $\mathbb{D}$ or the half-plane $\mathbb{C}_+$. Typical examples are the Bergman spaces $\mathscr{A}^p(\mathbb{D}) := \mathscr{H}(\mathbb{D}) \cap L^p(\mathbb{D})$ and the Hardy spaces

$$\mathscr{H}^p(\mathbb{D}) \colon = \left\{ f \in \mathscr{H}(\mathbb{D})\colon \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\}, \tag{0.5}$$

with $1 \leq p < \infty$. A comprehensive review on the theory of semigroups of composition operators can be found in [72]. Semigroups of composition operators are defined, as mentioned above, by means of a complex dynamical system which we refer to as semiflows of holomorphic selfmaps of a certain domain. An object which is of great importance for us is the generator of a semiflow: Given a semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{D},\mathbb{D})$, its generator $G : \mathbb{D} \to \mathbb{C}$ is defined by

$$G(z) \colon = \lim_{t\to 0^+} \frac{\varphi_t(z) - z}{t}.$$

The semiflow generator is known to be a holomorphic function, see, e.g., [20]. The main purpose of the first chapter is to generalize the existing theory on semigroups of composition operators to more general domains in the complex plane. By the Riemann mapping theorem, the unit disk is conformally equivalent to any simply connected domain strictly contained in the complex plane. However, in some instances we shall need to take into account boundary values of conformal mappings which depend heavily on the regularity of the considered simply connected domain. Fortunately, there is an in-depth study by Pommerenke [66] on the boundary behaviour of conformal mappings. So we rely on the results in loc. cit. in finding appropriate boundary regularity assumptions which allow for a generalization of the results we aim to use in the upcoming chapters. One of our key observations in Chapter 1 on which we build our theory is that the generator of a composition semigroup is a directional derivative, pointing inside the underlying domain due to the structure of the generator of the underlying semiflow. More precisely, given a semiflow $(\varphi_t)_{t\geq 0}$ generated by G and a Banach space $X \hookrightarrow \mathscr{H}(\Omega)$ on which the composition semigroup $(C_{\varphi_t})_{t\geq 0}$ induced by $(\varphi_t)_{t\geq 0}$ is a strongly continuous semigroup of bounded composition operators on X, the generator Γ of $(C_{\varphi_t})_{t\geq 0}$ is given by

$$\Gamma f = G \cdot f' = \langle G, \nabla f \rangle \quad (f \in \operatorname{dom} \Gamma \subseteq X). \tag{0.6}$$

Furthermore, we also consider semigroups of weighted composition operators on a Banach space $X \hookrightarrow \mathscr{H}(\Omega)$ the generator of which admits the form

$$\Gamma f = g \cdot f + G \cdot f' = g \cdot f + \langle G, \nabla f \rangle \quad (f \in \operatorname{dom} \Gamma \subseteq X), \tag{0.7}$$

where $g : \Omega \to \mathbb{C}$ is an appropriate holomorphic function.

In Chapter 2, we first recall from [9] how the Dirichlet-to-Neumann operator is defined on rough domains by using the method of forms, an approach which is based on variational methods and functional analysis. Then we develop our theory by means of the complex analysis and operator theory of composition semigroups to study well-posedness of the evolution problem

$$\begin{cases} \partial_t u - g \cdot u - \langle G, \nabla u \rangle = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ -\Delta u = 0 & \text{on } (0,\infty) \times \Omega, \\ u(0,\cdot) = u_0 & \text{on } \partial\Omega, \end{cases} \tag{0.8}$$

on certain Banach spaces, where $\Omega \subsetneq \mathbb{C}$ is a Jordan domain and G and g are boundary values of appropriate holomorphic functions on Ω. To this end, we establish boundary spaces consisting of distributional boundary values of functions contained in a Banach space $X \hookrightarrow \mathscr{H}(\Omega)$ on which $\Gamma f := g \cdot f + \langle G, \nabla f \rangle$ generates a strongly continuous semigroup of weighted composition operators. We present several examples of spaces on which we obtain well-posedness for the above evolution problem including the scale of $L^p(\partial\Omega)$ spaces. Our results regarding the well-posedness of (0.8) and the connection to semigroups of weighted composition operators are published in [64]. In the last section of Chapter 2, we try to generalize the developed theory to higher dimensional complex domains. For this, we need to overcome several difficulties. Firstly, there is no Riemann mapping theorem in $\mathbb{C}^n$, $n \geq 2$. So complex analysis in higher dimensions depends heavily on the underlying domain. Furthermore, in higher dimensions the composition of a harmonic function with a holomorphic selfmap need not be harmonic in general (which holds for domains in $\mathbb{C}$). Nevertheless, we are able to determine a subspace of $\partial X \subseteq L^p(\partial\Omega)$ for some $\Omega \subseteq \mathbb{C}^n$ such that (0.8) is well-posed on ∂X and the semigroup generated by the Dirichlet-to-Robin operator is the trace of a semigroup of weighted composition operators.

The second part of our thesis builds on our results established in the first and second chapter. We study in Chapter 3 multiplicative perturbations of Dirichlet-to-Neumann and Dirichlet-to-Robin operators by boundary values of holomorphic functions defined on a Jordan domain $\Omega \subseteq \mathbb{C}$. In particular, we are interested in well-posedness of evolution problems of the form

$$\begin{cases} \partial_t u - a(g \cdot u + \langle G, \nabla u \rangle) = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ -\Delta u = 0 & \text{on } (0,\infty) \times \Omega, \\ u(0,\cdot) = u_0 & \text{on } \partial\Omega, \end{cases} \tag{0.9}$$

where a denotes the boundary value of an appropriate holomorphic function on Ω. Our goal is to find multiplicative perturbations in terms of (analytic) properties of the functions G and g such that (0.9) is well-posed. Furthermore, we aim to work out the connection to the semigroup of weighted composition operators which is similar to the semigroup generated by the Dirichlet-to-Robin operator associated to (0.8). To this end, we construct a family of traces of weighted composition operators which approximates the semigroup solving (0.9) by means of Chernoff's formula. For our approximating family we use some results recently found in [26, 42, 44] concerning the analytic extendability of semiflows of holomorphic selfmaps based on the geometric function theory of these dynamical systems.

In Chapter 4, we give some possible applications of our theory. As a first application we use our approximation result from Chapter 3 to contribute to the theory of semigroups of composition operators. More precisely, we determine so-called maximal subspaces of strong continuity. Finding such maximal subspaces is a topic which has been studied in the theory of composition semigroups in the last decade, see, e.g., [5, 22, 23]. As a second application, we try to approximate the semigroup generated by the classical Dirichlet-to-Neumann operator and more general Poincaré-Steklov operators on continuous functions on a Dini-smooth curve in $\mathbb{C}$. We obtain a generalization of the approximation result from Chapter 3 to multiplicative perturbations by positive functions. Lastly, we refine a connection between composition semigroups and stochastic branching processes established in a rather old result from [21].

The last chapter of our thesis contains some open problems which arose from our investigations and we also give some ideas for further research on the developed theory.

People are afraid of new things. You should have taken an existing product and put a clock in it or something.
(H.J.S.)

1. The theory of composition semigroups on spaces of analytic functions

The aim of our investigations is to study a connection between semigroups of composition operators on spaces of analytic functions and evolution equations associated with a Poincaré-Steklov operator. In this chapter, we summarize the theoretical background on semigroups of composition operators needed for our research purposes. In a nutshell, those semigroups provide a very fruitful interaction between complex dynamical systems and operator theory. We aim to apply this theory (and especially the theory of complex analysis) eventually to certain partial differential equations. So, first of all, we recall briefly the basics about operator semigroups and abstract Cauchy problems in Section 1.1. Then we introduce in Section 1.2 the complex dynamical systems we are interested in, namely semiflows of holomorphic selfmaps. We discuss some generation properties and dynamical properties using geometric function theory. In Section 1.3 we collect some facts about semigroups of composition operators on spaces of analytic functions, and in the last section of this chapter we consider a generalization of this concept to semigroups of (cocycle-)weighted composition operators. Most of the result presented here are well-known. We partially contributed to this area by generalizing the ideas which have been developed on the unit disk in the plane only to simply connected domains and Jordan domains. Since we need to present a lot of results for inventing our theory, we decided to omit some proofs of classical results of this area and rather refer to the literature to improve the readability. However, for the reader who is not familiar with semiflows and composition semigroups, we present some crucial ideas and techniques in Appendix A and B.

1.1. Strongly continuous semigroups of bounded operators

This short section is devoted to recalling some basic facts and notions from the theory of operator semigroups. For a comprehensive introduction, we refer to the book by Pazy [63]. However, for our purposes, a very basic understanding of this theory is sufficient.

Let X and Y be Banach spaces. We denote the space of bounded linear operators $T : X \to Y$ by $\mathscr{L}(X,Y)$, and we simply write $\mathscr{L}(X)$ if $X = Y$.

1.1.1 Definition (Strongly continuous semigroup)**.** Let X be a Banach space. A family $(T_t)_{t\geq 0}$ of bounded linear operators in $\mathscr{L}(X)$ is called a strongly continuous semigroup (or C_0-semigroup) if the following properties are satisfied:

1. $T_0 = \mathrm{Id}$,
2. $T_{s+t} = T_s T_t$ for all $s, t > 0$, and
3. $\lim_{t\to 0^+} T_t x = x$ for all $x \in X$.

1.1.2 Definition (Semigroup generator). Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. The infinitesimal generator of $(T_t)_{t\geq 0}$ is the linear operator A : $\mathrm{dom}(A) \subseteq X \to X$ defined by

$$\mathrm{dom}(A) := \left\{ x \in X \mid \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists} \right\} \text{ and } Ax = \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ for } x \in \mathrm{dom}(A).$$

We collect some facts about infinitesimal generators in the following proposition, a proof of which can be found in [63, Ch.1, Thm. 2.4.(c) and Cor. 2.5].

1.1.3 Proposition (Properties of the infinitesimal generator). *Let A be the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on a Banach space X. Then the following are satisfied:*

1. *For all $x \in X$ and $t \geq 0$*
$$AT_t x \in \mathrm{dom}(A) \text{ and } \frac{\mathrm{d}}{\mathrm{d}t} T_t x = AT_t x = T_t Ax.$$
2. *The operator A is a closed and* $\mathrm{dom}(A)$ *is dense in X.*

In particular, this implies for a strongly continuous semigroup $(T_t)_{t\geq 0}$ with infinitesimal generator A that the map $t \mapsto T_t x$, $x \in \mathrm{dom}(A)$, is differentiable for all $t \geq 0$. Therefore, letting $x \in \mathrm{dom}(A)$, the function $u(t) = T_t x$ is a (classical) solution of the following abstract Cauchy problem

$$\begin{cases} \partial_t u(t) - Au(t) = 0, & t \in (0, \infty) \\ u(0) = x. \end{cases} \tag{1.1}$$

In fact, the solution is unique since the semigroup is uniquely determined by its generator, see [48, Thm. 1.4]. A solution in the classical sense is only obtained if the initial value is in the domain of A. Using the concept of mild solution solutions, we even derive a unique solution (in the mild sense) for each initial value in X, see [48, Prop. 6.4]. A mild solution to (1.1) is defined as follows.

1.1.4 Definition. Consider the abstract Cauchy problem (1.1). A continuous function $u : \mathbb{R}_{\geq 0} \to X$ is called a mild solution to (1.1) if $\int_0^t u \in \mathrm{dom} A$ for all $t \geq 0$ and $u(t) = A\int_0^t u(s)\,\mathrm{d}s + x$.

In this thesis, we often prove that the solution in either sense to an abstract Cauchy problem exists by showing that the operator A generates a strongly continuous semigroup. In particular, we use the following definition of well-posedness.

1.1.5 Definition (Well-posedness). The abstract Cauchy problem (1.1) is well-posed in a Banach space X if A is the generator of a strongly continuous semigroup in $\mathscr{L}(X)$.

1.2. Semiflows of holomorphic selfmaps and their generators

In this section, we study dynamical systems of holomorphic selfmaps on a simply connected domain in the complex plane. The study of complex dynamical systems has its roots in particular in investigating discrete dynamical systems obtained from iteration of a holomorphic selfmap. Research on such systems has a long history which started more than a century ago with early works on linearization models (e.g., [55]) and fixed points and asymptotic behavior [36, 76]. Remarkable results concerning the continuous analog of the discrete dynamical system, that is, the fractional iteration of a holomorphic function, have been obtained by Berkson and Porta in the late 70s [20]. These continuous dynamical systems, which we call semiflows, build the main ingredient for the theory of composition semigroups providing an interaction between complex analysis and operator theory. We refer to the books [43] and [70] for a very detailed introduction to the theory of semiflows.

We start this section with a brief summary on the boundary behavior of conformal mappings. Throughout, let $\Omega, \tilde{\Omega} \subseteq \mathbb{C}$ be simply connected domains. We denote the Fréchet space of holomorphic functions $f : \Omega \to \tilde{\Omega}$ by $\mathscr{H}(\Omega, \tilde{\Omega})$ and we write $\mathscr{H}(\Omega)$ if $\tilde{\Omega} = \mathbb{C}$.

Conformal mappings and their boundary behavior. The theory presented in this section (actually, most of the theory presented in this chapter) is usually investigated for holomorphic functions on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ or the open right half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. This is due to the following well-known theorem, see for instance [66, p. 4].

1.2.1 Theorem (Riemann mapping theorem). *Let $\Omega \subsetneq \mathbb{C}$ be a non-empty, simply connected domain. Then there is a conformal map $k : \mathbb{D} \to \Omega$, i.e., k is holomorphic and bijective.*

The boundary behavior of conformal maps on simply connected domains is comprehensively studied in the book of Pommerenke [66]. In general the unrestricted limit to the boundary appears to be a very strong condition, so boundary limits of conformal maps are considered in a slightly weaker sense, namely as nontangential limits. The nontangential limit (or, angular limit) is defined as follows: Given $\alpha \in (0, \frac{\pi}{2})$ and $\beta \in (0, 2\cos\alpha)$, we define the so-called Stolz angle $J_\alpha := \{z \in \mathbb{D} : |\arg(1-\bar{\xi}z)| \le \alpha, |z-\xi| < \beta\}$ at some point $\xi \in \partial\mathbb{D}$. A function $f : \mathbb{D} \to \mathbb{C}$ has a nontangential limit $a \in \mathbb{C} \cup \{\infty\}$ at a point $\xi \in \partial\mathbb{D}$ if $f(z) \to a$ as $z \to \xi$ in each Stolz angle J_α at ξ with $\alpha \in (0, \frac{\pi}{2})$.

Let $\Omega \subsetneq \mathbb{C}$ be a bounded simply connected domain and $\partial\Omega$ its boundary, and let $k : \mathbb{D} \to \Omega$ be conformal. An important result concerning the boundary behavior of conformal maps is the Kellogg-Warschawski theorem from 1932, which states that if $\partial\Omega$ is of class $C^{n,\alpha}$, with $n \in \mathbb{N}$ and $\alpha \in (0,1)$, that is, the nth derivative of the parametrization of $\partial\Omega$ exists and is α-Hölder continuous, then $k^{(n)}$ extends continuously to $\bar{\mathbb{D}}$, see [66, Thm 3.6]. Note that for $n = 1$ and $\alpha = 0$ the result is false in general, and moreover there are even C^1 domains such that no conformal map has a continuous derivative on $\bar{\mathbb{D}}$, furthermore it is even possible that they admit infinite angular derivatives at some point on the boundary, see [50, Exercises and Further Results 14]. For most of our purposes, it suffices to consider Jordan domains which are defined as follows.

1.2.2 Definition (Jordan domain). A Jordan curve C is a curve with an injective continuous parametrization $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) = \gamma(1)$. A simply connected domain bounded by a Jordan curve is called Jordan domain.

A Jordan curve divides the extended complex plane in exactly two connected components (Jordan curve theorem [66, p. 4]), each of which is bounded by the same Jordan curve. We call the unbounded component the *exterior Jordan domain*, and whenever we are writing just *Jordan domain*, we refer to the bounded component.

Caratheodory's theorem states that a conformal map $k : \mathbb{D} \to \Omega$ extends continuously to and one-to-one to $\bar{\mathbb{D}}$ if Ω is a Jordan domain [66, p. 18 & Thm. 2.6], and it can be shown that the angular derivative is finite and non-vanishing a.e. on $\partial\mathbb{D}$ [66, Thm. 6.8]. Sometimes it will be useful to have that k' extends continuously to $\bar{\mathbb{D}}$. This is obtained for simply connected domains bounded by a Dini-smooth Jordan curve [50, Thm 3.5], the definition of which we recall here for convenience.

1.2.3 Definition (Dini-smooth Jordan curve). Let $E \subseteq \mathbb{C}$. The modulus of continuity of a function $f : E \to \mathbb{C}$ is defined by $\omega_f(\delta) := \sup\{|f(z_1) - f(z_2)| : z_1, z_2 \in E, |z_1 - z_2| < \delta\}$. The function f is called Dini-continuous if

$$\int_0^1 \frac{\omega_f(t)}{t}\,\mathrm{d}t < \infty.$$

Let γ be the parametrization of a Jordan curve C. If γ' is Dini-continuous, then C is called Dini-smooth Jordan curve.

We now collect the above mentioned results concerning the boundary behavior of conformal maps in the following theorem for further reference.

1.2.4 Theorem (Boundary behavior of conformal mappings). *Let $\Omega \subsetneq \mathbb{C}$ be a Jordan domain and $k : \mathbb{D} \to \Omega$ conformal. Then*

1. *k extends continuously and one-to-one to $\bar{\mathbb{D}}$,*
2. *k has finite and non-vanishing angular derivative a.e. on $\partial\mathbb{D}$, and*
3. *k' extends continuously to $\bar{\mathbb{D}}$ if $\partial\Omega$ is Dini-smooth.*

Semiflows of holomorphic selfmaps. In what follows, let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain.

1.2.5 Definition (Semiflows of holomorphic selfmaps). A family $(\varphi_t)_{t\geq 0}$ of holomorphic selfmaps in Ω is called a semiflow of holomorphic functions if it satisfies the following properties:

1. $\varphi_0(z) = z$ for all $z \in \Omega$,

2. $\varphi_{s+t}(z) = \varphi_s(\varphi_t(z))$ for all $s, t > 0$ and $z \in \Omega$,

3. $t \mapsto \varphi_t(z)$ is continuous for all $z \in \Omega$.

Since Ω is supposed to be simply connected, by the Riemann mapping theorem there exists a conformal map $k : \Omega \to \mathbb{D}$, and thus every semiflow on Ω is similar to a semiflow on the unit disk.

The following lemma is a well-known result about semiflows of holomorphic selfmaps. The proof presented here stems from [30, Thm. 2].

1.2.6 Lemma. *Let $(\varphi_t)_{t\geq 0}$ be a semiflow of holomorphic functions in $\mathcal{H}(\Omega,\Omega)$. Then φ_t is injective, hence univalent, for every $t \geq 0$.*

Proof. By Definition 1.2.5, a semiflow is assumed to be continuous on $\mathbb{R}_{\geq 0}$, and since $\mathbb{D}$ is bounded, we can apply Vitali's theorem from which we obtain continuity on $\mathbb{R}_{\geq 0} \times \mathbb{D}$. Moreover, $\frac{\mathrm{d}}{\mathrm{d}z}\varphi_t \to 1$ for $t \to 0^+$ locally uniformly. So for small t this yields injectivity on compact subsets.
Now assume that for some $t_0 > 0$

$$\varphi_{t_0}(z_1) = \varphi_{t_0}(z_2) = z_0, \quad z_1 \neq z_2.$$

If $t > t_0, \varphi_t(z_1) = \varphi_{t-t_0}(\varphi_{t_0}(z_1)) = \varphi_{t-t_0}(\varphi_{t_0}(z_2)) = \varphi_t(z_2)$. That means two curves, distinct for $t = 0$, coincide after some t_0. Without loss of generality we assume that this t_0 is the smallest index such that both curves coincide. Choose $r \in (|z_0|, 1)$ and define t_r by $|\varphi'_{t_r}(z) - 1| \leq \frac{1}{2}$ for all $z \in D(0,r)$ and $t < t_r$ which is possible by the first argument of the proof. Let $\varepsilon \in (0, \min(t_r, t_0))$ so that $\varphi_{t_0-\varepsilon}(z_{1,2}) \in D(0,r)$. Then $\varphi_{t_0-\varepsilon}(z_1) \neq \varphi_{t_0-\varepsilon}(z_2)$ implies $\varphi_\varepsilon(\varphi_{t_0-\varepsilon}(z_1)) \neq \varphi_\varepsilon(\varphi_{t_0-\varepsilon}(z_2))$ by injectivity. But from the semigroup property we derive $\varphi_{t_0}(z_1) \neq \varphi_{t_0}(z_2)$, a contradiction. □

We would like to point out that this proof is especially nice because it is elementary and avoids the theory of ordinary differential equations which is the commonly used tool in the literature to prove this result. For all holomorphic selfmaps $\varphi \in \mathcal{H}(\mathbb{D},\mathbb{D})$ that are not elliptic automorphisms, i.e., not an automorphism with one fixed point in $\mathbb{D}$, the asymptotic behavior of the iterates is characterized by the Denjoy-Wolff theorem which states that there is a unique point $b \in \bar{\mathbb{D}}$ such that

$$\varphi_n := \underbrace{\varphi \circ \ldots \varphi}_{n-\text{times}} \stackrel{n\to\infty}{\longrightarrow} b$$

locally uniformly, see, e.g., [70, p. 78]. This theorem can be generalized to semiflows as follows, see [43, Thm. 2.9].

1.2.7 Theorem (Denjoy-Wolff theorem)**.** *Let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathcal{H}(\Omega,\Omega)$ which is not the identity or an elliptic automorphism. Then there is a unique point $b \in \bar{\Omega}$ such that $\varphi_t \to b$ locally uniformly as $t \to \infty$.*

The point b is called Denjoy-Wolff point. If $b \in \Omega$, then b is the unique fixed point of φ_t for all $t \geq 0$, and for $b \in \partial\Omega$ it is a fixed point of φ_t for all $t \geq 0$ in the sense of angular limits such that the angular limit $\lim_{z\to b} \varphi_t'(z) < 1$. We emphasize that there might be other fixed points on the boundary in the angular sense but only one that satisfies $\lim_{z\to b} \varphi_t'(z) < 1$. A detailed proof can be found in [70, Ch. 5].

In the case of a semiflow in $\mathscr{H}(\mathbb{D},\mathbb{D})$, it is common to shift an interior Denjoy-Wolff point to zero and a Denjoy-Wolff point on the boundary to 1, using an appropriate Möbius transform.

Examples

1. The family $(\varphi_t)_{t\geq 0}$ defined by $\varphi_t(z) = ze^{-t}$ $(z \in \mathbb{D})$ is a semiflow in $\mathscr{H}(\mathbb{D},\mathbb{D})$ with Denjoy-Wolff point in 0.

2. The family $(\varphi_t)_{t\geq 0}$ defined by$\varphi_t(z) = ze^{-t}+1-e^{-t}$ $(z \in \mathbb{D})$ is a semiflow in $\mathscr{H}(\mathbb{D},\mathbb{D})$ with Denjoy-Wolff point in 1.

1.2.1. Generators

In 1978, Berkson and Porta [20] established a result concerning the differentiability of semiflows with respect to the time parameter.

1.2.8 Theorem. *Let $\Omega \subsetneq \mathbb{C}$ be a bounded and simply connected domain, and let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathscr{H}(\Omega,\Omega)$. Then there is a holomorphic function $G : \Omega \to \mathbb{C}$ such that for all $t \geq 0$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(z) = G(\varphi_t(z)) \quad (z \in \Omega), \tag{1.2}$$

that is, $(\varphi_t)_{t\geq 0}$ is the solution to the following Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(z) = G(\varphi_t(z)), & (t,z) \in (0,\infty) \times \Omega \\ \varphi_0(z) = z, & z \in \Omega. \end{cases} \tag{1.3}$$

In particular, we derive that the limit

$$G(z) :\, = \lim_{t\to 0^+} \frac{\varphi_t(z) - z}{t} \tag{1.4}$$

exists for every $z \in \Omega$, and so in analogy to the case of operator semigroups, we call the function G the *generator* of the semiflow $(\varphi_t)_{t\geq 0}$. In the literature, a function G such that the Cauchy problem (1.3) has a unique solution for each initial value $z \in \Omega$ is often referred to as *semi-complete vector field*, see, e.g., [43].

One of the main results from [20] is the following:

1.2.9 Theorem (Berkson-Porta representation). *Let $(\varphi_t)_{t\geq 0}$ be a semiflow on the unit disk, which does not consist of elliptic automorphisms, and let $b \in \bar{\mathbb{D}}$ be its Denjoy-Wolff point. Then the generator of $(\varphi_t)_{t\geq 0}$ is given by the so-called Berkson-Porta representation*

$$G(z) = F(z)(\bar{b}z - 1)(z - b), \tag{1.5}$$

where $F : \mathbb{D} \to \mathbb{C}$ is holomorphic and $\mathrm{Re}(F(z)) \geq 0$ $(z \in \mathbb{D})$.

This representation appears to be very useful in proving some results on properties of semiflow generators. In Lemma 1.2.10, we show that the generator of a semiflow which does not consist of elliptic automorphisms has a.e. nontangential limits on $\partial\mathbb{D}$. This relies on the fact that the function F from the Berkson-Porta representation (1.5) is the composition of a bounded holomorphic function and a Möbius transform (this result has already been presented in [16]). Moreover, we derive from (1.5) that the following *angle condition* holds for every semiflow generator G:

$$\operatorname{Re}(G(z)\bar{z}) \leq 0 \text{ for a.e. } z \in \partial\mathbb{D}.$$

Geometrically this means that $G(z)$ is pointing inside $\mathbb{D}$ for a.e. $z \in \partial\mathbb{D}$ or, equivalently, that the angle between $G(z)$ and the outward pointing normal at z is greater than or equal to $\frac{\pi}{2}$ for a.e. $z \in \partial\mathbb{D}$. Conversely, by [4, Thm. 1], if a holomorphic function $G\colon \mathbb{D} \to \mathbb{C}$ extends continuously to $\bar{\mathbb{D}}$ and $\operatorname{Re}(G(z)\bar{z}) \leq 0$ for every $z \in \partial\mathbb{D}$, then G is the generator of a semiflow in $\mathbb{D}$.

1.2.10 Lemma. *Let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathscr{H}(\mathbb{D},\mathbb{D})$, which does not consist of elliptic automorphisms, and let G be its generator. Then G admits nontangential limits a.e. on $\partial\mathbb{D}$ and, furthermore,* $\operatorname{Re}(G(z)\bar{z}) \leq 0$, *for a.e.* $z \in \partial\mathbb{D}$.

Proof. Let $b \in \bar{\mathbb{D}}$ be the Denjoy-Wolff point of $(\varphi_t)_{t\geq 0}$. Then, by [20], the generator is given by (1.5). Without loss of generality, we assume that $F(0) = 1$ (by setting $\tilde{F}(z) := \frac{1}{\operatorname{Re} F(0)}(F(z) - i \operatorname{Im} F(0))$ if necessary). Then F is in the Carathéodory class, and thus it is subordinate to the Möbius transform $l(z) = \frac{1+z}{1-z}$ $(z \in \mathbb{D})$, which maps the unit disk onto the right half plane, in the sense that there is a holomorphic self-map of the unit disk w, with $w(0) = 0$, such that $F = l \circ w$. Such functions are known to be in the Hardy space $\mathscr{H}^p(\mathbb{D})$, $p \in (0,1)$, cf. [37, Thm. 3.2], and so we can apply [37, Thm. 2.2] which states that every function in $\mathscr{H}^p(\mathbb{D})$ has nontangential limits a.e. For a.e. $z \in \partial\mathbb{D}$ we have

$$\begin{aligned}
\operatorname{Re}(F(z)(\bar{b}z - 1)(z - b)\bar{z}) &= \operatorname{Re}(F(z)(\bar{b} - \bar{z})(z - b)) \\
&= \operatorname{Re}(-F(z)|z - b|^2) \\
&\leq 0.
\end{aligned}$$

□

An analogous result holds true for generators of semiflows on Jordan domains.

1.2.11 Lemma. *Let $\Omega \subsetneq \mathbb{C}$ be a Jordan domain. Let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathscr{H}(\Omega,\Omega)$, which does not consist of automorphisms, and let G be its generator. Then* $\operatorname{Re}(G(x)\overline{\nu(x)}) \leq 0$, *for a.e. $x \in \partial\Omega$ for which the nontangential limit of G exists, where $\nu(x)$ is the outer normal vector at x.*

Proof. Let $k\colon \Omega \to \mathbb{D}$ be conformal. Then $(\psi_t)_{t\geq 0} = (k \circ \varphi_t \circ k^{-1})_{t\geq 0}$ is a semiflow in the unit disk. Let $\tilde{G}$ be the generator of $(\psi_t)_{t\geq 0}$. Then $\operatorname{Re}(\tilde{G}(z)\bar{z}) \leq 0$, for a.e. $z \in \partial\mathbb{D}$.

For a.e. $z = k(x) \in \mathbb{D}$, we have

$$\tilde{G}(k(x)) = \lim_{t\to 0} \frac{d}{dt}\psi_t(k(x)) = \lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}t} k \circ \varphi_t(x)$$
$$= \lim_{t\to 0} k'(\varphi_t(x)) \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)$$
$$= k'(x)G(x). \tag{1.6}$$

The function k extends continuously to $\bar{\mathbb{D}}$ (see Theorem 1.2.4(1)) and has non-vanishing angular derivative a.e. (see Theorem 1.2.4(2)), so G admits nontangential limits a.e. Furthermore, for a.e. $x \in \partial\Omega$, we have $\nu(x) = \frac{k(x)}{k'(x)}|k'(x)|^2$. For every $x \in \partial\Omega$ such that the nontangential limit of G exists, there exists a unique $z \in \partial\mathbb{D}$ such that $k(x) = z$, so

$$\mathrm{Re}(G(x)\overline{\nu(x)}) = \mathrm{Re}\left(\frac{\tilde{G}(z)}{k'(x)}\overline{\left(\frac{z}{k'(x)}|k'(x)|^2\right)}\right)$$
$$= Re(\tilde{G}(z)\overline{z})$$
$$\leq 0.$$

This holds for a.e. $x \in \partial\Omega$. □

Next, we transfer the characterization of semiflow generators in the unit disk given above to Jordan domains.

1.2.12 Proposition. *Let $G : \Omega \to \mathbb{C}$ be holomorphic, where $\Omega \subsetneq \mathbb{C}$ is bounded and simply connected. If $\partial\Omega$ is Dini-smooth and G extends continuously to $\bar{\Omega}$ and* $\mathrm{Re}(G(x)\overline{\nu(x)}) \leq 0$ *for all $x \in \partial\Omega$, then G is the generator of a semiflow in $\mathscr{H}(\Omega)$.*

Proof. Let $k : \Omega \to \mathbb{D}$ conformal. Define $\tilde{G}(z) = k'(k^{-1}(z))G(k^{-1}(z))$ for $z \in \mathbb{D}$. Then $\tilde{G}$ is a holomorphic function which admits a uniformly continuous extension to $\bar{\mathbb{D}}$, by Theorem 1.2.4(3). Moreover, for a.e. $z \in \partial\mathbb{D}$,

$$\mathrm{Re}(\tilde{G}(z)\bar{z}) = \mathrm{Re}\left(G(k^{-1}(z))k'(k^{-1}(z))\overline{\left(\frac{k(k^{-1}(z))k'(k^{-1}(z))}{k'(k^{-1}(z))}\right)}\right)$$
$$= \mathrm{Re}(G(k^{-1}(z))\overline{\nu(k^{-1}(z))})$$
$$\leq 0.$$

So we can apply [4, Thm. 1] which shows that $\tilde{G}$ is the generator of a semiflow $(\psi_t)_t$ in $\mathbb{D}$, and by (1.6) G is the generator of the semiflow $(\varphi_t)_t = \left(k^{-1} \circ \psi_t \circ k\right)_t$. □

There are semiflow generators which do not admit a continuous extension to the boundary, so they are not classified by the previous proposition. However a result due to Aharonov et al. [3] gives a simple criterion:

1.2.13 Theorem (Flow invariance condition)**.** *A holomorphic function $G : \mathbb{D} \to \mathbb{C}$ generates a semiflow in $\mathscr{H}(\mathbb{D},\mathbb{D})$ if and only if* $\mathrm{Re}(G(z)\bar{z}) \geq \mathrm{Re}(G(0)\bar{z})(1 - |z^2|)$ *for all $z \in \mathbb{D}$.*

1.2.2. Fractional iteration and associated Koenigs function

For all holomorphic selfmaps φ in the unit disk which are not automorphisms, the embeddability into a semiflow can be characterized in terms of the Denjoy-Wolff point of φ, see for instance [41]. Moreover, in either case the semiflow has a specific representation in terms of a univalent function. These functions and certain subclasses of them play a crucial role in geometric function theory. We collect some well-known facts.

Univalent functions associated to a semiflow. Let $S(\mathbb{D})$ denote the set of functions which are univalent on $\mathbb{D}$, i.e., those functions which are holomorphic and injective on $\mathbb{D}$. Of particular interest for our purposes, especially in the third chapter, are spirallike functions and close-to-convex functions which are subclasses of $S(\mathbb{D})$.
Spirallike functions are generalizations of starlike functions, i.e., functions mapping conformally onto a starlike domain. We denote the set of starlike functions on $\mathbb{D}$ by $S^*(\mathbb{D})$. A domain $B \subseteq \mathbb{C}$ such that $0 \in B$ is said to be λ-spirallike if for each $b_0 \in B$ the logarithmic spiral $b(t) = b_0 \exp(-e^{i\lambda\frac{\pi}{2}} t)(t \geq 0)$ is contained in B. We call a function $h : \mathbb{D} \to \mathbb{C}$, normalized such that $h(0) = 0$, λ-spirallike if $h(\mathbb{D})$ is λ-spirallike and write $h \in S_\lambda^{\circledS}(\mathbb{D})$.
A function $h : \mathbb{D} \to \mathbb{C}$ is called close-to-convex if $h(\mathbb{D})$ is a close-to-convex domain, that is, for each $z \in h(\mathbb{D})$ the ray $\{z + t, t \geq 0\}$ lies in $h(\mathbb{D})$. Close-to-convex domains are sometimes referred to as *convex in one direction* domains. We denote the set of close-to-convex functions by $S^{\text{ctc}}(\mathbb{D})$.
It can be shown that a function in either class is univalent. For a proof of this result as well as a more detailed introduction to spirallike and close-to-convex functions, we refer to [38].
It appears that both classes of univalent functions have a close connection to semiflows of holomorphic functions in $\mathcal{H}(\mathbb{D}, \mathbb{D})$. A function $\varphi \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ that is not an automorphism is called embeddable into a semiflow if there exists a semiflow $(\varphi_t)_{t \geq 0}$ such that $\varphi_1 = \varphi$. Embeddability can be characterized with respect to the position of the Denjoy-Wolff point of φ as follows. Recall that we assume, without loss of generality, that an inner Denjoy-Wolff point is 0 and a boundary Denjoy-Wolff point is 1 by using appropriate Möbius transforms.

1.2.14 Theorem ([41, Thm. 1]). *Let $\varphi \in H(\mathbb{D}, \mathbb{D}) \setminus \text{Aut}(\mathbb{D})$ with Denjoy-Wolff point in 0 and $\varphi'(0) = \gamma \neq 0$. Then φ embeds into a semiflow if and only if*

$$h \circ \varphi = \gamma h \quad \textit{(Schröder's equation)}$$

has a solution $h \in \mathcal{H}(\mathbb{D})$ such that

$$\frac{zh'(z)}{h(z)} = \frac{F(0)}{F(z)}, \tag{1.7}$$

where F is a holomorphic function with positive real-part and $e^{-F(0)} = \gamma$.

1.2.15 Theorem ([41, Thm. 2]). *Let $\varphi \in H(\mathbb{D}, \mathbb{D}) \setminus \text{Aut}(\mathbb{D})$ with Denjoy-Wolff point in 1. Then φ embeds into a semiflow if and only if*

$$h \circ \varphi(z) = h(z) + 1 \quad \textit{(Abel's equation)}$$

has a solution $h \in \mathscr{H}(\mathbb{D})$ *such that*

$$\operatorname{Re}\big((1-z)^2 h'(z)\big) > 0 \tag{1.8}$$

for all $z \in \mathbb{D}$.

1.2.16 Remark. The univalent function h obtained in either theorem is sometimes referred to as *associated univalent function* or *Koenigs function*. It is well known that a univalent function satisfying (1.7) or (1.8) is spirallike or close-to-convex, respectively. In fact, there is a representation for the corresponding semiflow in each case in terms of its Koenigs function: for $\varphi \in H(\mathbb{D},\mathbb{D})\backslash\operatorname{Aut}(\mathbb{D})$ with Denjoy-Wolff point in 0 and $\varphi'(0) = \gamma \neq 0$ embeddable into a semiflow, we obtain

$$\varphi_t(z) = h^{-1}(e^{-\gamma t} h(z)) \quad (z \in \mathbb{D}),$$

and for $\varphi \in H(\mathbb{D},\mathbb{D}) \setminus \operatorname{Aut}(\mathbb{D})$ with Denjoy-Wolff point in 1 embeddable into a semiflow, we have

$$\varphi_t(z) = h^{-1}(h(z) + t) \quad (z \in \mathbb{D}).$$

The set $\mathscr{H}(\Omega,\Omega)$ is rich in semiflows by the construction presented in the remark above. Considering the set of holomorphic selfmaps on doubly or even multiply connected domains is less fruitful. The following results are based on a paper of Heins from 1941 [52] and a recent refinement by Jafari et al. [53]. Doubly connected domains are conformally equivalent to to one of the following domains $\mathbb{C} \setminus \{0\}$, $\mathbb{D} \setminus \{0\}$, or an annulus. The semiflows in $\mathscr{H}(\mathbb{D}\setminus\{0\},\mathbb{D}\setminus\{0\})$ and $\mathscr{H}(\mathbb{C}\setminus\{0\},\mathbb{C}\setminus\{0\})$ are exactly those semiflows of $\mathscr{H}(\mathbb{D},\mathbb{D})$ and $\mathscr{H}(\mathbb{C},\mathbb{C})$, respectively, which fix the origin. Semiflows in $\mathscr{H}(\mathbb{C},\mathbb{C})$ that fix 0 admit the form $\varphi_t(z) = ze^{\alpha t}$ $(z \in \mathbb{C})$ with $\alpha \in \mathbb{C}\setminus\{0\}$, see [53, Thm. 2]. Heins' original theorem [52, Thm. 2.2] is rephrased in the language of semiflows in [53, Thm. 5] as follows: the only semiflows on an annulus A are rotations, i.e., they are of the form $\varphi_t(z) = ze^{i\theta t}$ $(z \in A)$ with $\theta \in \mathbb{R}$. Moreover, all semiflows on n-connected domains $(n > 2)$ are trivial.

1.3. Composition semigroups

Now we study semigroups of operators, defined in terms of a semiflow, on Banach spaces of analytic functions. Throughout this section, let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain.

Admissible spaces

We begin with the definition of a composition operator on the space of holomorphic functions induced by a holomorphic selfmap.

1.3.1 Definition (Composition operator)**.** Let $\varphi \in \mathscr{H}(\Omega,\Omega)$. The operator

$$C_\varphi \colon \mathscr{H}(\Omega) \to \mathscr{H}(\Omega), \quad f \mapsto f \circ \varphi \tag{1.9}$$

is called composition operator and φ its symbol.

There is a rich study on operator theoretic properties of composition operators in terms of their symbols. For a detailed introduction to this topic, we refer the reader to [69]. Also, we give an example how to describe compactness of composition operators using complex analysis in Appendix B.1.

Banach spaces of holomorphic functions. The theory of composition operators and composition semigroups has been invented on several Banach spaces of holomorphic functions in the past decades. In particular, such operators and operator semigroups have been studied on Hardy and Bergman spaces. These Banach spaces, defined below, serve as typical examples for our theory.

1.3.2 Definition (Bergman spaces). Let $p \in [1, \infty)$. The Bergman space $\mathscr{A}^p(\Omega)$ is defined as follows

$$\mathscr{A}^p(\Omega) := \mathscr{H}(\Omega) \cap L^p(\Omega, \mathrm{d}A),$$

where $\mathrm{d}A$ denotes the normalized Lebesgue measure on Ω.

In what follows, we simply write $L^p(\Omega)$ since we consider always the normalized Lebesgue measure on the underlying measurable set Ω. Obviously, the $L^p(\Omega)$-norm, given by

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p \, \mathrm{d}A \right)^{\frac{1}{p}} \quad (f \in L^p(\Omega, \mathrm{d}A))$$

is a norm on $\mathscr{A}^p(\Omega)$, and the Bergman spaces $\mathscr{A}^p(\mathbb{D})$ are known to be Banach spaces in this norm. Note that one can define Bergman spaces also for $p \in (0, 1)$, but in this case these are just quasi-Banach spaces. Bergman spaces and their generalizations to higher dimensions are considered in various modern approaches. For a recent introduction to the theory of Bergman spaces we refer to [39, 57, 78].

Functions in the Bergman space satisfy the following growth estimate: let $p \in [1, \infty)$ and let $f \in \mathscr{A}^p(\Omega)$, then

$$|f(z)| \leq \frac{\|f\|_{\mathscr{A}^p(\Omega)}}{\pi^{1/p}(\mathrm{dist}(z, \Omega))^{2/p}} \quad (z \in \Omega), \tag{1.10}$$

see [39, Thm. 1]. This implies that the Bergman space $\mathscr{A}^p(\Omega)$ is a Banach space for all $p \in [1, \infty)$ (by an application of Montel's theorem) and a Hilbert space for $p \neq 2$ with the inner product induced by the Hilbert space $L^2(\Omega, \mathrm{d}A)$, see [39, p. 8]. The small Bergman spaces are defined similarly by $a^p(\mathbb{D}) := \mathfrak{h}(\Omega) \cap L^p(\Omega)$, where $\mathfrak{h}(\mathbb{D}) := \{u : \Omega \to \mathbb{C} \colon \Delta u = 0\}$ is the space of harmonic functions on Ω.

Next we introduce the so-called Hardy spaces. These spaces are throughout considered in operator theory on spaces of holomorphic functions, and their theory has grown enormously since they have been introduced a century ago. A comprehensive introduction to the theory of Hardy spaces can be found in [37].

1.3.3 Definition (Hardy spaces). Let $p \in [1, \infty)$. The Hardy space $\mathscr{H}^p(\mathbb{D})$ is defined as the space of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{\frac{1}{p}} < \infty. \tag{1.11}$$

Analogously, we define the small Hardy space $h^p(\mathbb{D})$ of all harmonic functions $f \in \mathfrak{h}(\mathbb{D})$ satisfying (1.11).

We can define Hardy spaces of holomorphic functions on simply connected domains. However, there are at least two possible definitions, see [37], either using harmonic majorants or via approximating the boundary of Ω by rectifiable curves. Both definitions are equivalent when domains bounded by analytic Jordan curves are considered. We use the definition in terms of harmonic majorants since it appears to be more convenient in our upcoming investigations.

1.3.4 Definition. Let $\Omega \subsetneq \mathbb{C}$ be bounded and simply connected. For $p \in [1, \infty)$, the Hardy space $\mathscr{H}^p(\Omega)$ consists of those functions $f \in \mathscr{H}(\Omega)$ such that the subharmonic function $|f|^p$ is dominated by a harmonic function $u : \Omega \to \mathbb{R}$.

Equipped with the norm $\|f\|_{\mathscr{H}^p(\Omega)} := (u_0(z_0))^{\frac{1}{p}}$ $(f \in \mathscr{H}^p(\Omega))$, where $z_0 \in \Omega$ is some fixed point and u_0 is the least harmonic majorant for f, the Hardy space over Ω is a Banach space. We can define Hardy spaces also for $p \in (0, 1)$, but these spaces are only quasi-Banach spaces. For more details about Hardy spaces we refer to [37] and especially to [37, Ch. 10] for Hardy spaces over general domains.

Functions in $\mathscr{H}^p(\Omega)$ appear to be particularly fruitful for our theory since they admit nontangential limits a.e. on $\partial\Omega$ and the boundary function is in $L^p(\partial\Omega)$. Moreover, the space of all boundary functions forms a closed subspace of $L^p(\partial\Omega)$.

Semigroups of composition operators. Initiated by the famous paper by Berkson and Porta [20], semigroups of composition operators were studied intensively by many authors on various spaces of holomorphic functions defined on the unit disk, see, for example, [5, 23, 56, 71, 72].

Consider the Fréchet space $\mathscr{H}(\Omega)$ equipped with the topology of uniform convergence on compact subsets of Ω. Let $(K_n)_n$ be an increasing sequence of compact subsets of Ω such that $\bigcup_n K_n = \Omega$. We define a sequence of seminorms on $\mathscr{H}(\Omega)$ as follows

$$p_n(f) := \sup_{z \in K_n} |f(z)| \quad (f \in \mathscr{H}(\Omega)),$$

and a metric induced by these seminorms by

$$d(f, g) := \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(f-g)}{p_n(f-g)+1} \quad (f, g \in \mathscr{H}(\Omega)).$$

For a given semiflow $(\varphi_t)_{t\geq 0}$, we define a family of composition operators $(C_{\varphi_t})_{t\geq 0}$ acting on $\mathscr{H}(\Omega)$ as follows

$$C_{\varphi_t}\colon \mathscr{H}(\Omega)\to\mathscr{H}(\Omega),\quad f\mapsto f\circ\varphi_t. \tag{1.12}$$

By the definition of semiflows, this family is an operator semigroup which is, in particular, strongly continuous since for all compact subsets, we have

$$\sup_{z\in K_n}|f(\varphi_t(z))-f(z)|\overset{t\to 0^+}{\to} 0.$$

This definition makes also sense when the space of harmonic functions $\mathfrak{h}(\Omega):=\{f:\Omega\to\mathbb{C}\colon \Delta f=0\}$ is under consideration as the following lemma shows.

1.3.5 Lemma. *Let $f\in\mathfrak{h}(\Omega)$ and $\varphi\in\mathscr{H}(\Omega,\Omega)$. Then $f\circ\varphi\in\mathfrak{h}(\Omega)$.*

Proof. Denote by φ_1 and φ_2 the real and the imaginary part of φ, respectively. The calculation below relies on the Cauchy-Riemann equations, the harmonicity of f, φ_1 and φ_2, and Schwarz' theorem.

$$\begin{aligned}
\Delta(f\circ\varphi) &= \sum_{k=1}^{2}\partial_k^2 f(\varphi_1,\varphi_2)\\
&= \sum_{k=1}^{2}\partial_k((\partial_1 f)\circ\varphi\cdot\partial_k\varphi_1+(\partial_2 f)\circ\varphi\cdot\partial_k\varphi_2)\\
&= (\partial_1^2 f)\circ\varphi\cdot\partial_1\varphi_1+(\partial_1 f)\circ\varphi\cdot\partial_1^2\varphi_1+(\partial_2 f)\circ\varphi\cdot\partial_1^2\varphi_2+(\partial_{1,2}f)\circ\varphi\cdot\partial_1\varphi_2+\\
&\quad +(\partial_2^2 f)\circ\varphi\cdot\partial_2\varphi_2+(\partial_1 f)\circ\varphi\cdot\partial_2^2\varphi_1+(\partial_2 f)\circ\varphi\cdot\partial_2^2\varphi_2+(\partial_{2,1}f)\circ\varphi\cdot\partial_2\varphi_1\\
&= (\Delta f)\circ\varphi\cdot\partial_1\varphi_1+(\partial_1 f)\circ\varphi\cdot\Delta\varphi_1+(\partial_2 f)\circ\varphi\cdot\Delta\varphi_2+\\
&\quad +(\partial_{1,2}f)\circ\varphi\cdot(\partial_2\varphi_1+\partial_1\varphi_2)\\
&= 0.
\end{aligned}$$

□

1.3.6 Definition. Let $\mathbb{H}(\Omega)$ be either the space of holomorphic functions or the space of harmonic functions defined on Ω with values in $\mathbb{C}$. Let $X\hookrightarrow\mathbb{H}(\Omega)$ be a Banach space and $(\varphi_t)_{t\geq 0}$ a semiflow of holomorphic functions in $\mathscr{H}(\Omega,\Omega)$ generated by G. The space X is called (G)-admissible if the family of operators $(C_{\varphi_t})_{t\geq 0}$ defined by (1.12) satisfies the following two conditions:

(i) X is invariant under C_{φ_t}, i.e., $C_{\varphi_t}X\subseteq X$ for all $t\geq 0$.

(ii) $(C_{\varphi_t})_{t\geq 0}$ is strongly continuous on X.

Note that condition (i), by the closed graph theorem, implies that each C_{φ_t} is bounded.

For a Banach space $X\hookrightarrow\mathbb{H}(\Omega)$ containing polynomials as a dense subset, a general procedure to prove strong continuity of a locally bounded operator semigroup $(C_{\varphi_t})_{t\geq 0}$ is presented by Siskakis [71]: Let $\varepsilon > 0$ and let $f \in X$ and $f_n(z) = \sum_{k=0}^{n} \hat{f}(k) z^n$ and choose $n \in \mathbb{N}$ such that $\|f - f_n\|_X < \varepsilon$, then

$$\begin{aligned}\left\|C_{\varphi_t} f - f\right\|_X &\le \left\|C_{\varphi_t} f - C_{\varphi_t} f_n\right\|_X + \left\|C_{\varphi_t} f_n - f_n\right\|_X + \|f - f_n\|_X \\ &\le \varepsilon\left(\left\|C_{\varphi_t}\right\|_{\mathscr{L}(X)} + 1\right) + \sum_{k=0}^{n} |\hat{f}_k| \left\|\varphi_t^k - z^k\right\|_X .\end{aligned}$$

So it suffices to prove $\varphi_t^k \to z^n$, for $k \in \{0, 1, \ldots, n\}$, in X as $t \to 0^+$. In the Hardy and Bergman spaces it is actually enough to show $\varphi_t \to z$ in X as $t \to 0^+$ which follows by dominated convergence.

Given a semigroup of composition operators $(C_{\varphi_t})_{t\ge 0}$ on a (G)-admissible Banach space $X \hookrightarrow \mathscr{H}(\Omega)$, the generator Γ admits a special form:

$$\Gamma f = \lim_{t\to 0^+} \frac{C_{\varphi_t} f - f}{t} = G \cdot f' \quad (f \in \operatorname{dom}\Gamma). \tag{1.13}$$

Analogously, we calculate the generator of a composition semigroup on a (G)-admissible Banach space $X \hookrightarrow \mathfrak{h}(\Omega)$ as

$$\Gamma f = \lim_{t\to 0^+} \frac{C_{\varphi_t} f - f}{t} = \langle G, \nabla f \rangle \quad (f \in \operatorname{dom}\Gamma). \tag{1.14}$$

Here the product $\langle G, \nabla f \rangle$ is to be understood as the inner product of vectors in $\mathbb{R}^2$. Note that these representations are equal for functions which are complex differentiable by the Cauchy-Riemann equations.

1.3.7 Lemma. *Let $\Omega \subsetneq \mathbb{C}$ and $G : \Omega \to \mathbb{R}^2 \simeq \mathbb{C}$ and $f \in \mathscr{H}(\Omega)$. Then $\langle G, \nabla f \rangle = G \cdot f'$.*

Proof. Let $G(x, y) = (G_1(x, y), G_2(x, y)) \simeq G_1(x, y) + iG_2(x, y)$ and $f(x, y) = f_1(x, y) + if_2(x, y)$. So $f' = \partial_x f_1 + i\partial_x f_2$. Then

$$\begin{aligned}\langle G, \nabla f \rangle &= G_1 \partial_x f + G_2 \partial_y f \\ &= G_1 \partial_x f_1 + G_2 \partial_y f_1 + i(G_1 \partial_x f_2 + G_2 \partial_y f_2) \\ &= G_1 \partial_x f_1 - G_2 \partial_x f_2 + i(G_1 \partial_x f_2 + G_2 \partial_x f_1) \\ &= (G_1 + iG_2)(\partial_x f_1 + i\partial_x f_2) \\ &= G \cdot f'.\end{aligned}$$

□

Next we present a crucial result concerning the boundedness of composition operators on Hardy and Bergman spaces, a proof of which can be found in [69, Sect. 1.3]

1.3.8 Proposition (Littlewood's subordination principle). *Let $\varphi \in \mathscr{H}(\mathbb{D}, \mathbb{D})$. Then*

$$\|C_\varphi\|_{\mathscr{L}(\mathscr{H}^2(\mathbb{D}))} \le \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}. \tag{1.15}$$

The original version of Littlewood's subordination principle is formulated for selfmaps fixing the origin and the Hardy space $\mathscr{H}^2(\mathbb{D})$. Using appropriate Möbius transforms the above generalization for arbitrary holomorphic selfmaps can be obtained. The same result for $\mathscr{H}^p(\mathbb{D})$ is deduced from a decomposition of functions in Hardy spaces into inner and outer functions [37]. The respective result for Bergman spaces follows by integration. For a complete proof we refer to [69, 78].

1.3.9 Corollary. *Let $\varphi \in \mathscr{H}(\mathbb{D},\mathbb{D})$. Then*

1. $\|C_\varphi\|_{\mathscr{L}(\mathscr{H}^p(\mathbb{D}))} \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{1}{p}}$,

2. $\|C_\varphi\|_{\mathscr{L}(\mathscr{A}^p(\mathbb{D}))} \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{2}{p}}$.

Thus, by these results and the discussion above, we obtain (G)-admissibility of the Hardy and Bergman spaces for every possible generator G of a semiflow of holomorphic selfmaps in the unit disk. We extend this result to bounded simply connected domains for the Hardy spaces and to Jordan domains for the Bergman spaces.

1.3.10 Proposition. *Let $\Omega \subsetneq \mathbb{C}$ be bounded and simply connected. Let $(\varphi_t)_{t\geq 0}$ be a semiflow of holomorphic functions in $\mathscr{H}(\Omega,\Omega)$ generated by G. Then the Hardy space $\mathscr{H}^p(\Omega)$ ($p \in [1,\infty)$) is (G)-admissible.*

Proof. Let $k\colon \Omega \to \mathbb{D}$ be conformal. Then there exists a semiflow $(\psi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{D},\mathbb{D})$ such that $(\varphi_t)_{t\geq 0} = (k^{-1}\circ\psi_t\circ k)_t$. By [37, Cor. to Thm. 10.1], $f \in \mathscr{H}^p(\Omega)$ if and only if$f\circ k^{-1} \in \mathscr{H}^p(\mathbb{D})$. This and Littlewood's subordination principle (Corollary 1.3.8) gives invariance since

$$|f\circ\varphi_t|^p = |\underbrace{\underbrace{\underbrace{f\circ k^{-1}}_{\in\mathscr{H}^p(\mathbb{D})}\circ\psi_t}_{\in\mathscr{H}^p(\mathbb{D})}\circ k}_{\in\mathscr{H}^p(\Omega)}|^p.$$

Without loss of generality, we assume that $k^{-1}(0)=z_0$. Then, by [39, p. 168], we have

$$\|f\circ\varphi_t - f\|_{\mathscr{H}^p(\Omega)} = \left\|f\circ\varphi_t\circ k^{-1} - f\circ k^{-1}\right\|_{\mathscr{H}^p(\mathbb{D})} \overset{t\to 0^+}{\to} 0.$$

□

1.3.11 Remark. If we were using the definition of Hardy spaces by approximating level curves (sometimes called Hardy-Smirnov spaces), the preceding proof would involve boundary values of conformal maps. This would have forced us to prescribe conditions concerning the boundary of Ω. Therefore it seems more appropriate to define Hardy spaces via harmonic majorants.

To show admissibility of the Bergman spaces, we assume the underlying domain to be Dini-smooth.

1.3.12 Proposition. *Let $\Omega \subsetneq \mathbb{C}$ be a Dini-smooth domain. Let $(\varphi_t)_{t\geq 0}$ be a semiflow of holomorphic functions in $\mathscr{H}(\Omega,\Omega)$ generated by G. Then the Bergman space $\mathscr{A}^p(\Omega)$ ($p \in [1,\infty)$) is (G)-admissible.*

Proof. Let $k\colon \Omega \to \mathbb{D}$ be conformal. Then there exists a semiflow $(\psi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{D},\mathbb{D})$ such that $(\varphi_t)_{t\geq 0} = (k^{-1} \circ \varphi_t \circ k)_t$. Thus, for $f \in \mathscr{A}^p(\Omega)$,

$$\begin{aligned}\int_\Omega |f \circ \varphi_t|^p &= \int_\mathbb{D} |f \circ \varphi_t \circ k^{-1}|^p |\frac{1}{k'}|^2 \\ &\leq C \int_\mathbb{D} |f \circ k^{-1} \circ \psi_t|^p.\end{aligned}$$

The derivative of k is continuous and non-vanishing on $\bar{\Omega}$, see Theorem 1.2.4(3).

For invariance, we only need to show that $f \circ k^{-1} \in \mathscr{A}^p(\mathbb{D})$. Indeed,

$$\int_\mathbb{D} |f \circ k^{-1}|^p \,\mathrm{d}A = \int_\Omega |f|^p |k'|^2 \,\mathrm{d}A \leq C \|f\|^p_{\mathscr{A}^p(\Omega)} < \infty.$$

Now Littlewood's subordination principle (Corollary 1.3.8) yields invariance.

By the same calculation, we obtain strong continuity of $(T_t)_{t\geq 0}$ on $\mathscr{A}^p(\Omega)$ from strong continuity on $\mathscr{A}^p(\mathbb{D})$. □

1.3.13 Remark. The paper [7] by Arendt and Chalendar also examines (what we call) (G)-admissiblity of certain Banach spaces X of holomorphic functions which are defined on a domain Ω satisfying a weak boundary regularity condition, e.g., a bounded, non-empty, simply connected set. They call such a domain Ω maximal for X if for every $w \in \partial\Omega$ and all $\epsilon > 0$ there exists an $f \in X$ such that f has no holomorphic extension to $\mathscr{H}(\Omega \cup D(w,\epsilon))$, where $D(w,\epsilon) = \{z \in \mathbb{C} \colon |w - z| < \epsilon\}$. In particular, they investigate a condition on a Banach space $X \hookrightarrow \mathscr{H}(\Omega)$, with Ω maximal for X and X containing the identity function, such that a generator of a C_0-semigroup on X of the form $Af = G \cdot f'$ (cf. (1.13)), where G is the generator of a semiflow in $\mathscr{H}(\Omega,\Omega)$, generates a semigroup of composition operators. Namely, the following density condition is developed: *For all $w \in \partial\Omega$ there is an $\epsilon > 0$ such that the space $\{f \in X \colon \exists \tilde{f} \in \mathscr{H}(\Omega \cup D(w,\epsilon)), \tilde{f}|_\Omega = f\}$ is dense in X.* In the language of our setting the space X is then (G)-admissible.

1.4. Semigroups of weighted composition operators

It is also natural to consider semigroups of weighted composition operators. Let $\Omega \subseteq \mathbb{C}$ be bounded and simply connected. Let $\omega : \Omega \to \mathbb{C}$ be holomorphic. For $t \geq 0$ we define a weight as follows

$$m_t = \frac{\omega(\varphi_t)}{\omega}. \tag{1.16}$$

For a family of composition operators $(T_t)_{t\geq 0}$ on $\mathscr{H}(\Omega)$ with semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\Omega,\Omega)$, we define a family of weighted composition operators as follows

$$\begin{aligned} S_t \colon \mathscr{H}(\Omega) &\to \mathscr{H}(\Omega) \\ f &\mapsto m_t \cdot C_{\varphi_t} f. \end{aligned} \tag{1.17}$$

This is again an operator semigroup on $\mathscr{H}(\Omega)$ and also on $\mathfrak{h}(\Omega)$ but the question of strong continuity is more difficult since it depends heavily on the choice of ω.

Special weights that appear to be in particular fruitful for our investigations are so-called cocycles.

1.4.1 Definition (Cocycle). Let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathscr{H}(\Omega, \Omega)$. A family of holomorphic functions $m_t \colon \Omega \to \mathbb{C}$, $t \geq 0$, is called cocycle if

1. $m_0(z) = 1$, $z \in \Omega$,
2. $m_{s+t}(z) = m_t(z) \cdot (m_s(\varphi_t(z))$ for all $t, s \geq 0$ and $z \in \Omega$,
3. $t \mapsto m_t(z)$ is continuous for every $z \in \Omega$.

If there exists a holomorphic function $\omega \colon \Omega \to \mathbb{C}$ such that $m_t(z) = \frac{\omega(\varphi_t(z))}{\omega(z)}$, $z \in \Omega$, then the family $(m_t)_{t\geq 0}$ is called a *coboundary* of $(\varphi_t)_{t\geq 0}$.

It is easy to see that a family of cocycle weighted composition operators is also an operator semigroup on $\mathbb{H}(\Omega)$. Moreover, given an arbitrary holomorphic function g : $\Omega \to \mathbb{C}$, we can easily construct a cocycle to a semiflow $(\varphi_t)_{t\geq 0}$: for $t \geq 0$,

$$m_t(z) = \exp\left(\int_0^t g(\varphi_s(z))\,\mathrm{d}s \right) \quad (z \in \Omega) \tag{1.18}$$

is a cocycle. The following definition is analogous to Definition 1.3.6.

1.4.2 Definition. Let $(S_t)_{t\geq 0}$ be a semigroup of weighted composition operators on $\mathbb{H}(\Omega)$, cf. (1.17), with semiflow generated by the holomorphic function $G : \Omega \to \mathbb{C}$ and cocycle weight in terms of a holomorphic function $g : \Omega \to \mathbb{C}$, see (1.18). A Banach space $X \hookrightarrow \mathbb{H}(\Omega)$ is called (g, G)-admissible if it satisfies the following two conditions:

(i) X is invariant under S_t, i.e., $S_t X \subseteq X$ for all $t \geq 0$.

(ii) $(S_t)_{t\geq 0}$ is strongly continuous on X.

Let $X \subseteq \mathscr{H}(\Omega)$ be (g, G)-admissible. Then the generator Γ of $(S_t)_{t\geq 0}$ is given by

$$\Gamma f = g \cdot f + G \cdot f' \quad (f \in \operatorname{dom}\Gamma). \tag{1.19}$$

In [56, Theorem 2] it has been shown that for certain holomorphic functions $g : \Omega \to \mathbb{C}$ and their associated cocycles $(m_t)_{t\geq 0}$ as in (1.18), and a semiflow $(\varphi_t)_{t\geq 0}$ generated by $G : \Omega \to \mathbb{C}$, the Hardy space $\mathscr{H}^p(\mathbb{D})(p \in [1, \infty))$ is (g, G)-admissible in the sense of Definition 1.4.2. By a slight adjustment of the arguments in Proposition 1.3.10, we obtain the result for Hardy spaces over simply connected sets.

1.4.3 Lemma. *Let $\Omega \subsetneq \mathbb{C}$ be bounded and simply connected. Let $g\colon \Omega \to \mathbb{C}$ be a holomorphic function such that $\sup_{z\in\Omega} \operatorname{Re} g(z) < \infty$, and let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathscr{H}(\Omega,\Omega)$ with generator G. Then $\mathscr{H}^p(\Omega)\,(p \in [1,\infty))$ is (g,G)—admissible.*

Proof. Invariance follows by boundedness of m_t and Proposition 1.3.10. To show strong continuity, we use the same technique as in Proposition 1.3.10, too. Since the real part of $g \circ k^{-1}$ is bounded as well, we obtain the assertion from [71, Theorem 1]. □

Siskakis' proof of [71, Theorem 1], which shows the strong continuity and boundedness of semigroups of weighted composition operators on Hardy spaces, works also for semigroups of m_t-weighted composition operators on the Bergman space $\mathscr{A}^p(\Omega)$, where Ω is a Dini-smooth domain.

1.4.4 Lemma. *Let $\Omega \subsetneq \mathbb{C}$ be a Dini-smooth domain. Let $g\colon \Omega \to \mathbb{C}$ be a holomorphic function such that $\sup_{z\in\Omega} \operatorname{Re} g(z) < \infty$, and let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathscr{H}(\Omega,\Omega)$ with generator G. Then $\mathscr{A}^p(\Omega)\,(p \in [1,\infty))$* is (g,G)-admissible.

Proof. It suffices to prove the statement for $\mathscr{A}^p(\mathbb{D}) = \mathscr{A}^p$ and then to apply the same technique as in Proposition 1.3.12. Due to Siskakis [71, Theorem 1], strong continuity for a semigroup of weighted composition operators on $\mathscr{H}^p(\mathbb{D})$ is achieved if

$$\limsup_{t\to 0} \|m_t\|_\infty \leq 1$$

which is satisfied by our assumptions on g, see [56, Lemma 3.1]. To prove the assertion, we can simply follow the steps in the proof of [71, Theorem 1].
For all $t \geq 0$ we have $m_t \in \mathscr{H}^\infty(\mathbb{D})$. This and the cocycle properties yield that $(S_t)_{t\geq 0}$ defines a family of bounded operators on $\mathscr{A}^p$. Let $f \in \mathscr{A}^p$. By Littlewood's subordination principle

$$\|S_t f\|^p_{\mathscr{A}^p} \leq \|m_t\|_\infty \left(\frac{1+|\varphi_t(0)|}{1-|\varphi_t(0)|}\right)^{\frac{p}{2}} \|f\|^p_{\mathscr{A}^p}, \tag{1.20}$$

thus $\|S_t\|_{\mathscr{L}(\mathscr{A}^p,\mathscr{A}^p)} < \infty$ for all $t \geq 0$.
First, we prove strong continuity if $p > 1$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence such that $t_n \overset{n\to\infty}{\to} 0$. Then we have $\limsup_{n\to\infty} \|S_{t_n} f\|_{\mathscr{A}^p} \leq \|f\|_{\mathscr{A}^p}$. Since $\mathscr{A}^p$ is reflexive and by (1.20), after passing to a subsequence again denoted by $(t_n)_{n\in\mathbb{N}}$, the sequence $(S_{t_n} f)_{t_n}$ is weakly convergent. The weak limit is f because $\lim_{n\to\infty} S_{t_n} f(z) = f(z)$ for all $z \in \mathbb{D}$. By lower-semicontinuity of the $\mathscr{A}^p$ norm, $\|f\|_{\mathscr{A}^p} \leq \liminf_{n\to\infty} \|S_{t_n} f\|_{\mathscr{A}^p}$, and thus

$$\lim_{n\to\infty} \|S_{t_n} f\|_{\mathscr{A}^p} = \|f\|_{\mathscr{A}^p}.$$

Since $\mathscr{A}^p$ is, as a subspace of L^p, a uniformly convex Banach space, we obtain the desired strong continuity by [29, Prop. 3.32].
To show strong continuity in the case $p = 1$, we use that $\mathscr{A}^q$ $(q > 1)$ is dense in $\mathscr{A}^1$. Now, fix $\delta > 0$ and set $C := \sup_{0<t<\delta} \|S_t\|$. Let $\epsilon > 0$. For every $f \in \mathscr{A}^1$ there exists $g \in \mathscr{A}^q$

such that $\|f-g\|_{\mathscr{A}^1} < \frac{\epsilon}{(C+1)2}$. Moreover, for $t < \delta$, we have

$$\begin{aligned}\|S_t f - f\|_{\mathscr{A}^1} &\leq \|S_t f - S_t g\|_{\mathscr{A}^1} + \|S_t g - g\|_{\mathscr{A}^1} + \|f-g\|_{\mathscr{A}^1} \\ &\leq \|S_t f - S_t g\|_{\mathscr{A}^1} + \|S_t g - g\|_{\mathscr{A}^q} + \|f-g\|_{\mathscr{A}^1} \\ &\leq (C+1)\|f-g\|_{\mathscr{A}^1} + \|S_t g - g\|_{\mathscr{A}^q}.\end{aligned}$$

Since $q > 1$, for all $\epsilon > 0$ there exists a sufficiently small $t > 0$ such that $\|S_t g - g\|_{\mathscr{A}^q} < \frac{\epsilon}{2}$. Thus $\|S_t f - f\|_{\mathscr{A}^1} \to 0$ as $t \to 0^+$.

□

1.4.5 Remark. Several authors are especially interested in semigroups of composition operators weighted by the derivative of the semiflow $(\varphi_t)_{t\geq 0}$ with respect to the complex variable, i.e.,

$$S_t f := \varphi_t' \cdot f \circ \varphi_t \quad (f \in X). \tag{1.21}$$

See for example the recent paper [12].

Indeed, this weight is a cocycle given by

$$m_t(z) := \varphi_t'(z) = \exp\left(\int_0^t G'(\varphi_s(z))\,\mathrm{d}s\right), \tag{1.22}$$

see [72, Sect. 7] for details.

The Hardy and the Bergman spaces are by no means the only spaces of analytic functions on which composition operators and composition semigroups have been considered. Other spaces which are also frequently treated in the literature are the disk algebra $\mathscr{A}(\mathbb{D})$, the Dirichlet space $\mathscr{D}$, the Bloch space $\mathscr{B}$ and small Bloch space $\mathscr{B}_0$, and the spaces of analytic functions with bounded or vanishing mean oscillation, BMOA and VMOA, we refer to Section 4.1 for the definition of these spaces. Compared to Hardy and Bergman spaces there are less strongly continuous semigroups of composition operators on the spaces $\mathscr{A}(\mathbb{D})$, $\mathscr{D}$, $\mathscr{B}$, and BMOA, i.e., there are less holomorphic functions G generating a semiflow such that these spaces are (G)-admissible. On these spaces the question of strong continuity is much more delicate, and in fact there is no nontrivial strongly continuous semigroup on $\mathscr{B}$ and BMOA. So in these cases, one is studying so-called maximal subspaces of strong continuity denoted by $[\varphi_t, \mathscr{B}]$ and $[\varphi_t, \mathrm{BMOA}]$ such that a given semiflow $(\varphi_t)_{t\geq 0}$ defines a strongly continuous semigroup of composition operators on $[\varphi_t, \mathscr{B}]$ and $[\varphi_t, \mathrm{BMOA}]$, respectively. In [23] it has been shown that $\mathscr{B}_0 \subseteq [\varphi_t, \mathscr{B}] \subsetneq \mathscr{B}$, and in the recent paper [5] the analogous result for BMOA has been obtained, that is, $\mathrm{VMOA} \subseteq [\varphi_t, \mathrm{BMOA}] \subsetneq \mathrm{BMOA}$. We shall consider subspaces of strong continuity in Chapter 4 (Applications) again where we use the theory developed in the upcoming two chapters.

2. Poincaré-Steklov semigroups on boundary spaces of Banach spaces of analytic functions

This chapter is devoted to elaborating a certain connection between semigroups of composition operators and weighted composition operators presented in the previous chapter and semigroups generated by Poincaré-Steklov operators associated with certain evolution equations. These operators act as follows: given the solution of an elliptic equation with a prescribed boundary condition (like a Dirichlet boundary condition), a Poincaré-Steklov operator maps the boundary values to another boundary condition (e.g., Neumann or Robin boundary conditions) of the same solution. Our observation is based on the Lax semigroup which gives a representation in terms of a composition semigroup for the semigroup generated by the Poincaré-Steklov operator which maps continuous or square integrable Dirichlet boundary values of a solution to the homogeneous Laplace equation on the unit ball to the Neumann derivative of a sufficiently regular solution. This operator, which has been studied intensively during the last decades, is called the Dirichlet-to-Neumann operator. Recently, it has been shown with the method of forms that the Dirichlet-to-Neumann operator generates a semigroup on Lipschitz domains and also on domains with very mild boundary assumptions. Unfortunately, for domains which are not (affine transformations of) the unit ball a representation similar to the Lax semigroup is no longer available. Thus, we try to generalize this approach to more general Poincaré-Steklov operators generating a semigroup which we can represent in terms of (weighted) composition semigroup. With this approach we find certain Dirichlet-to-Robin operators on spaces of distributions and obtain well-posedness for evolution equations associated with these operators.

This chapter is organized as follows. First, in Section 2.1 we recall the definition of the Dirichlet-to-Neumann operator and sketch how the method of forms can be used to show that the Dirichlet-to-Neumann operator is the generator of a semigroup. Then, we present the representation of the Dirichlet-to-Neumann semigroup obtained by Lax emphasizing the connection to composition semigroups. In Section 2.3 we eventually study evolution problems on Jordan domains in the complex plane associated with certain Dirichlet-to-Robin operators generating a semigroup which admits a representation in terms of a weighted composition semigroup. In doing so, we elaborate boundary spaces for Banach spaces, consisting of boundary values of analytic functions in a distributional sense, which are possible domains for our Dirichlet-to-Robin operators. In the last section of this chapter, we discuss a possible generalization of this approach to higher dimensional domains.

2.1. The Dirichlet-to-Neumann operator

Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain, and let $f \in L^2(\partial\Omega)$. We consider the following Dirichlet problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Now, based on this equation, we define the Dirichlet-to-Neumann operator with respect to the Laplacian.

2.1.1 Definition (Dirichlet-to-Neumann operator)**.** The Dirichlet-to-Neumann operator $\mathfrak{D}_{\mathcal{N}}$ on $L^2(\partial\Omega)$ maps the Dirichlet boundary data $f \in L^2(\partial\Omega)$ to the Neumann derivative of the solution u of (2.1), provided the solution exists and is sufficiently regular, i.e.,

$$\begin{aligned} \mathfrak{D}_{\mathcal{N}}\colon \operatorname{dom}(\mathfrak{D}_{\mathcal{N}}) \subseteq L^2(\partial\Omega) &\to L^2(\partial\Omega) \\ f &\mapsto \langle \nu, \nabla u \rangle, \end{aligned} \tag{2.2}$$

where ν is the outward pointing normal on $\partial\Omega$.

In recent years, the Dirichlet-to-Neumann operator has been studied intensively. In the beginning of the 20th century, the Dirichlet-to-Neumann operator was dealt with theoretically, while in the 1980s and 1990s it was used to analyze inverse problems to determine coefficients of a differential operator. These problems apply, e.g., to image techniques in medicine and also to find defects in materials.

A modern approach to this operator is given by the method of forms, which we would like to introduce here very roughly collecting the key ideas.

Let H be a Hilbert space. Let $a : \operatorname{dom} a \times \operatorname{dom} a \to \mathbb{R}$ be a bilinear form with $\operatorname{dom} a$ a real vector space, and let $j : \operatorname{dom}(a) \to H$ be a linear operator with dense range such that, for some $\alpha, \beta \geq 0$,

$$|a(u,w) - a(w,u)| \leq \alpha(a(u,u) - a(w,w)) + \beta(\|j(u)\|_H^2 + \|j(w)\|_H^2)$$

for all $u, w \in \operatorname{dom} a$. The pair (a, j) is then called *sectorial form*. An operator A can be associated to a sectorial form (a, j) as follows: Let $x, y \in H$. Then $x \in \operatorname{dom} A$ and $Ax = y$ if there exists a sequence $(u_n)_n$ in $\operatorname{dom} a$ such that

1. $\lim_{n\to\infty} j(u_n) = x$ in H,
2. $\sup_n a(u_n, u_n) < \infty$, and
3. $\lim_{n\to\infty} a(u_n, w) = \langle y, j(w) \rangle_H$ for all $w \in \operatorname{dom} a$.

The following theorem gives a link between forms and semigroups. For a proof we refer to [11, Thm 6.4].

2.1.2 Theorem. *The operator $-A$, where A is the operator associated with the sectorial form (a, j) defined above, generates a C_0-semigroup on H.*

As it is shown by Arendt and ter Elst in [9], the Dirichlet-to-Neumann operator appears to be an example of an operator associated to a particular sectorial form. We describe this form here briefly. In what follows all derivatives are to be understood in the sense of distributions, and, in particular, a function $u \in H^1(\Omega)$, where $H^1(\Omega)$ is the usual short-hand notation for the Sobolev space $W^{1,2}(\Omega)$, such that $\Delta u \in L^2(\Omega)$ has a weak Neumann derivative if there is a function $\psi \in L^2(\partial\Omega)$ such that

$$\int_\Omega (\Delta u)w + \int_\Omega \nabla u \overline{\nabla w} = \int_{\partial\Omega} \psi w \, \mathrm{d}S$$

for all $w \in H^1(\Omega) \cap C(\bar{\Omega})$. Arendt and ter Elst [11] showed that the Dirichlet-to-Neumann operator described by its graph

$$\mathfrak{D}_{\mathcal{N}} = \{(\phi, \psi) \in L^2(\partial\Omega) \times L^2(\partial\Omega) \colon \exists u \in H^1(\Omega) \text{ s.t.} \Delta u = 0, u|_{\partial\Omega} = \phi, \text{and } \langle \nu, \nabla u \rangle = \psi\}$$

can be obtained as the operator associated with the pair (a, j), where a is the classical Dirichlet form on $V \times V$, with $V = H^1(\Omega) \cap C(\bar{\Omega})$, given by

$$a(u, w) := \int_\Omega \nabla u \overline{\nabla w}$$

and $j : V \to L^2(\partial\Omega)$ the trace operator. The trace operator here is defined as follows: $\phi \in L^2(\partial\Omega)$ is said to be the trace of a function $u \in H^1(\Omega)$ if there exists a sequence $(u_n)_n$ in $H^1(\Omega) \cap C(\bar{\Omega})$ such that

$$\lim_{n\to\infty} u_n = u \text{ in } H^1(\Omega) \text{ and } \lim_{n\to\infty} u_n = \phi \text{ in } L^2(\partial\Omega).$$

Moreover, with their approach they are even able to define the Dirichlet-to-Neumann operator on bounded domains $\Omega \subseteq \mathbb{R}^n$ without any restrictions to the boundary except for assuming finite $(n-1)$ dimensional Hausdorff measure. For more details, we refer to [9, Theorem 3.3].

Together with the generation theorem above, Theorem 2.1.2, we obtain the following.

2.1.3 Theorem. *The operator $-\mathfrak{D}_{\mathcal{N}}$ generates a C_0-semigroup on $L^2(\partial\Omega)$.*

The trajectories of the semigroup generated by $-\mathfrak{D}_{\mathcal{N}}$ can be identified with the traces of the solutions to the following evolution problem

$$\begin{cases} \partial_t u + \langle \nu, \nabla u \rangle = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ -\Delta u = 0 & \text{on } (0, \infty) \times \Omega, \\ u(0, \cdot) = f & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $f \in \operatorname{dom}(-\mathfrak{D}_{\mathcal{N}})$. Note that the approach we presented here is way more general than in the classical theory of forms due to the rough domains on which the authors of [9] aim to define the Dirichlet-to-Neumann operator. Indeed, to associate semigroups to a form, the form is usually assumed to be defined on a Hilbert space instead of an arbitrary vector space, see [10, Sect. 4].

2.2. The Lax semigroup

In the previous section we have discussed a result which allows one to define the Dirichlet-to-Neumann operator on very rough domains $\Omega \subseteq \mathbb{R}^n$, and that the negative Dirichlet-to-Neumann operator defined this way generates an analytic semigroup on $L^2(\partial\Omega)$. However, the property of being a generator on $L^2(\partial\Omega)$ and also on $C(\partial\Omega)$ has been investigated by several authors before, assuming a smoothly bounded domain, see, e.g., [47]. One result which marks the starting point of our investigations appears in Lax' book on functional analysis from 2002. Assuming the underlying domain to be the n-dimensional unit ball $B_n \subseteq \mathbb{R}^n$, he proves that the semigroup $(T_t)_{t\geq 0}$ generated by the negative Dirichlet-to-Neumann operator on $C(\partial\mathbb{B}_n)$ and $L^2(\partial\mathbb{B}_n)$ admits the following representation:

$$T_t f(z) := u(ze^{-t}) \quad (z \in \partial B_n), \tag{2.4}$$

where $f \in C(\partial B_n)$ or $f \in L^2(\partial B_n)$, respectively, and u is the solution to the Dirichlet problem (2.1) with $\Omega = \mathbb{B}_n$. This connects the Dirichlet-to-Neumann operator surprisingly with generators of semigroups of composition operators. Letting $n = 2$, i.e., $B_n = \mathbb{D}$, the semigroup (2.4) is the trace, in the sense of nontangential limits, of a semigroup of composition operators on the small Hardy space $h^2(\mathbb{D})$ if $f \in \operatorname{dom}\mathfrak{D}_{\mathcal{N}} \subseteq L^2(\partial\mathbb{D})$. Note that the small Hardy space $h^2(\mathbb{D})$ is isomorphic to $L^2(\partial\mathbb{D})$, see [67, Thm. 11.30]. The associated semiflow $(\varphi_t)_{t\geq 0}$ is given by $\varphi_t(z) = ze^{-t}$ $(z \in \mathbb{D})$ for all $t \geq 0$, generated by $G(z) = -z = -\nu(z)(z \in \mathbb{D})$. Therefore the infinitesimal generator of the semigroup of composition operators is $\Gamma u = -\langle \nu, \nabla u\rangle$ $(u \in \operatorname{dom}(\Gamma) \subseteq h^2(\mathbb{D}))$. So, for $f \in (\operatorname{dom}(\mathfrak{D}_{\mathcal{N}}) \subseteq)L^2(\partial\mathbb{D})$ and $u \in h^2(\Omega)$ the solution to (2.1), we obtain the following representation of the Dirichlet-to-Neumann operator on $\mathbb{D}$ terms of the generator of a composition semigroup

$$\begin{aligned} -\mathfrak{D}_{\mathcal{N}} f &= -\langle \nu, \nabla u\rangle \\ &= \operatorname{Tr}(\Gamma u). \end{aligned}$$

Unfortunately this representation is optimal in the sense that the semigroup generated by the Dirichlet-to-Neumann operator is only similar to a semigroup of composition operators (in the above sense) if the underlying domain is a ball. This has been shown by Emamirad and Sharifitabar in [46, Thm. 2.2]. It is remarkable that they use different proofs for $n > 2$ and $n = 2$. In the case $n > 2$, their proof is based on an issue with a generalization of Lemma 1.3.5 to higher dimensional domains. More precisely, they prove that composition of a harmonic function with a selfmap φ of class $C^2(\Omega, \Omega)$ is harmonic if and only if $\varphi(x) = Ax + b$, where $B \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is a multiple of an orthogonal matrix, see [46, Lem. 2.1]. For $n = 2$ they show that the outward pointing normal viewed as a complex valued map cannot be extended to a holomorphic function unless the underlying domain is a disk [46, Thm. 3.1]. In particular, there is no semiflow generator which has the outward pointing normal as boundary function.

2.3. Evolution problems associated with a Dirichlet-to-Robin operator

In this section, we extend the idea of the Lax semigroup in two dimensions to a more general class of operators on an appropriately defined boundary space ∂X, a space consisting of boundary distributions of a Banach space $X \hookrightarrow \mathscr{H}(\Omega)$. We are interested in operators which map certain Dirichlet boundary conditions of the elliptic equation (2.1) to very general Robin boundary conditions. Such operators, mapping boundary values of an elliptic equation to another boundary condition of the same equation, are known as Poincaré-Steklov operators. Of course, the Dirichlet-to-Neumann operator is an example of such an operator. Unfortunately, the nice representation of the Dirichlet-to-Neumann semigroup as a composition semigroup appears to be restricted to functions defined on a circle. However, in view of the Lemmas 1.2.11 and 1.3.7, the trace of a generator of a composition semigroup on $h^2(\Omega)$ is an outward pointing derivative for any Jordan domain $\Omega \subseteq \mathbb{C}$. We use this special structure to elaborate a connection between partial differential equations on the boundary of a domain $\Omega \subseteq \mathbb{C}$ associated with a Poincaré-Steklov mapping Dirichlet boundary values to outward pointing derivatives and semigroups of composition operators. Moreover, with the theory of semigroups of weighted composition operators, we extend this approach to operators mapping Dirichlet boundary values to Robin boundary conditions. More precisely, for $f \in \partial X$, we study evolution equations of the following type

$$\begin{cases} \partial_t u - g \cdot u - \langle G, \nabla u \rangle = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ -\Delta u = 0 & \text{on } (0, \infty) \times \Omega, \\ u(0, \cdot) = f & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where $\Omega \subsetneq \mathbb{C}$ is a Jordan domain and G is the generator of a semiflow in $\mathscr{H}(\Omega, \Omega)$ and g denotes the boundary values of an appropriate holomorphic function on Ω. To prove well-posedness of (2.5), we need to elaborate a link between functions in admissible spaces in the sense of Definitions 1.3.6 and 1.4.2 and their (distributional) boundary values.

2.3.1. Preparation: boundary spaces

Finding boundary values of holomorphic functions is a fundamental problem in complex analysis. Strong results concerning the boundary values of functions in Hardy spaces are Fatou's theorem and the theorem by F. and M. Riesz. But, in many spaces of holomorphic functions, convergence to boundary values in a nontangential sense is a rather strong condition. Therefore we consider boundary values in a weaker sense, namely in the sense of distributions.

Let $\Omega \subsetneq \mathbb{C}$ be a Jordan domain. This restriction guarantees existence and nonvanishing of boundary values of derivatives of conformal maps defined on Ω by Theorem 1.2.4. Furthermore, by Carathéodory's theorem, every conformal map from the disk onto a Jordan domain extends continuously and one-to-one to the Jordan curve. We do not know whether the established theory works also for more general simply connected domains with more general boundary.

In what follows, we are exploring boundary distributions of functions in Banach spaces $X \hookrightarrow \mathbb{H}(\Omega)$. Our first aim is to define the boundary space of X consisting of appropriately defined distributional boundary values of elements of X.

2.3.1 Definition. Let $\Omega \subsetneq \mathbb{C}$ be a Jordan domain. Let $X \hookrightarrow \mathbb{H}(\Omega)$ be a Banach space. If for every $f \in X$ there exists a uniquely defined distributional boundary value $f^* \in C^\infty(\partial\Omega)'$ in the following sense

$$\lim_{r\to 1^-} \int_{\partial\Omega} f_r \cdot \phi(x)\,\mathrm{d}x = \langle f^*, \phi\rangle$$

for every $\phi \in C^\infty(\partial\Omega)$, where $f_r(z) := f(k^{-1}(rk(z)))$, and $k : \Omega \to \mathbb{D}$ is any conformal map, then we denote the set consisting of all such boundary values by ∂X. If the map $\mathrm{Tr} : X \to \partial X, f \mapsto f^*$ is injective, then we call ∂X the boundary space corresponding to X. Moreover, we turn ∂X into a Banach space by defining a norm on ∂X by $\|f^*\|_{\partial X} = \|f\|_X$ for every $f^* \in \partial X$.

We collect some examples of boundary spaces so that we can give an idea which type of Dirichlet boundary conditions are possible.

Examples. A first (though artificial) example is the space $X = \mathscr{A} := \mathscr{H}(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ where $\mathscr{A}$ denotes the disk algebra. The restriction to the boundary is an isometric homomorphism from $\mathscr{A}$ into $C(\partial\mathbb{D})$. So $\mathscr{A}$ is a Banach subalgebra of $C(\partial\mathbb{D})$ which is even maximal due to Wermer's maximality theorem, see [67, Thm 18.20]. Thus the boundary space ∂X can be defined as the space of continuous functions on $\partial\mathbb{D}$ which are continuously extendable to holomorphic functions on $\mathbb{D}$. Using the Riemann mapping theorem, we deduce the analogous assertions for the space $\mathscr{A}(\Omega) := \mathscr{H}(\Omega) \cap C(\bar{\Omega})$.
Let $p \in [1, \infty)$ and define X as the Hardy space $\mathscr{H}^p(\Omega)$. Then it is well known that every function in $\mathscr{H}^p(\Omega)$ has nontangential limits a.e. and the boundary function is in $L^p(\partial\Omega)$, see [37, Thm 2.2]. For a comprehensive overview, we refer especially to [37, Chapter 3]. These boundary functions form a closed subspace of $L^p(\partial\Omega)$ which consists of those function in $L^p(\partial\Omega)$ with vanishing negative Fourier coefficients, see [37, Thm. 3.3]. Note that this theory is almost applicable when the analogously defined Hardy space h^p of harmonic functions is considered. However, the case $p = 1$ appears to be different. The boundary space on h^1 consists of finite Borel measures on the unit circle, see [67, Thm. 11.30].

In both examples, the boundary space inherits some properties of the underlying space of holomorphic functions. Moreover, by the Luzin-Privalov theorem, see, e.g., [66, Cor. 6.14], a holomorphic function admitting nontangential boundary values is in either case identically zero if the boundary function vanishes on a set of positive measure. Given a function in one of the boundary spaces from the examples above, we can recover the holomorphic function in X via Cauchy's integral formula and the Poisson integral as well which acts as an isometric isomorphism between X and ∂X.

Boundary distributions of Bergman functions. The theory of boundary values for functions in Hardy spaces on the unit disk is well established. The question of boundary functions is more complicated if one wishes to work on Bergman spaces. In fact, the

Bergman spaces contain functions which do not admit nontangential or radial limits almost everywhere, such as the lacunary series. So it seems more appropriate to define boundary values in the sense of distributions. To establish such distributional boundary values, we emphasize a connection between Hardy and Bergman spaces. For simplicity we use the notation $\mathscr{A}^p := \mathscr{A}^p(\mathbb{D})$ and $\mathscr{H}^p := \mathscr{H}^p(\mathbb{D}), p \geq 1$. The following theorem can be found in [39, Lem. 4].

2.3.2 Theorem. If $f \in \mathscr{A}^1$ and F is an antiderivative of f, then $F \in \mathscr{H}^1$.

For $p \in [1, \infty)$, Theorem 2.3.2 can be generalized to $f \in \mathscr{A}^p$ in the following way.

2.3.3 Theorem. *Let $f \in \mathscr{A}^p$ $(p \geq 1)$ and F an antiderivative of f. Then $F \in \mathscr{H}^p$.*

Proof. Fix $\varepsilon \in (0,1)$. Then we have

$$F(z) = \int_{\varepsilon z}^{z} f(w)\,\mathrm{d}w + F(\varepsilon z) \overset{(w=tz)}{=} \int_{\varepsilon}^{1} f(tz)z\,\mathrm{d}t + F(\varepsilon z)$$

To estimate $M_p(r,F)$, we examine the following two integrals

$$\begin{aligned} M_p(r,F) &= \left(\frac{1}{2\pi} \int_0^{2\pi} |F(re^{it})|^p \,\mathrm{d}t \right)^{\frac{1}{p}} \\ &= \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_{\varepsilon}^{1} f(sre^{it})re^{it} ds + F(\varepsilon re^{it}) \right|^p \mathrm{d}t \right)^{\frac{1}{p}} \\ &\leq \underbrace{\left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_{\varepsilon}^{1} f(sre^{it})re^{it}\,\mathrm{d}s \right|^p \mathrm{d}t \right)^{\frac{1}{p}}}_{=:I_1} + \underbrace{M_p(\varepsilon r, F)}_{=:I_2}. \end{aligned}$$

For the first term we have

$$
\begin{aligned}
I_1^p &= \frac{1}{2\pi}\int_0^{2\pi}\left|\int_\varepsilon^1 f(sre^{it})re^{it}\,\mathrm{d}s\right|^p \mathrm{d}t \\
&\leq r^p \frac{1}{2\pi}\int_0^{2\pi}\left(\int_\varepsilon^1 |f(sre^{it})|\,\mathrm{d}s\right)^p \mathrm{d}t \\
&\leq r^p \frac{1}{2\pi}\int_0^{2\pi}\left(\int_\varepsilon^1 |f(sre^{it})|^p\,\mathrm{d}s\right) \mathrm{d}t \\
&\leq r^{p-1} \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{r\varepsilon}^r |f(ue^{it})|^p \frac{u}{u}\,\mathrm{d}u\right) \mathrm{d}t \\
&\leq \frac{r^{p-2}}{\varepsilon} \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^1 |f(ue^{it})|^p u\,\mathrm{d}u\right) \mathrm{d}t \\
&\leq \frac{r^{p-2}}{\varepsilon}\|f\|_{\mathscr{A}^p}^p.
\end{aligned}
$$

Without loss of generality, we assume $f(0)=0$. Thus we obtain for the second integral

$$
\begin{aligned}
I_2^p &= \frac{1}{2\pi}\int_0^{2\pi} |F(\varepsilon re^{it})|^p\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1 f(s\varepsilon re^{it})\varepsilon re^{it}\,\mathrm{d}s\right|^p \mathrm{d}t \\
&\leq (\varepsilon r)^p \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^1 |f(s\varepsilon re^{it})|^p\,\mathrm{d}s\right) \mathrm{d}t \\
&= (\varepsilon r)^{p-1} \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{\varepsilon r} |f(ue^{it})|^p\,\mathrm{d}u\right) \mathrm{d}t.
\end{aligned}
$$

Since point evaluation on Bergman spaces is (locally uniformly) continuous, see (1.10), we obtain for some $C>0$ independent of r

$$
(\varepsilon r)^{p-1} \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{\varepsilon r} |f(ue^{it})|^p\,\mathrm{d}u\right) \mathrm{d}t \leq (\varepsilon r)^p C\|f\|_{\mathscr{A}^p}^p.
$$

Combining these results, we have

$$M_p(r,F) \le \left(\frac{r^{p-2}}{\varepsilon} + C(\varepsilon r)^p \right) \|f\|_{\mathscr{A}^p}.$$

Letting $r \to 1^-$, the right-hand side is still finite since $\varepsilon \in (0,1)$. □

This theorem remains true if we replace $\mathbb{D}$ by a Jordan domain $\Omega \subsetneq \mathbb{C}$.

2.3.4 Corollary. *Theorem 2.3.3 remains true if $\mathbb{D}$ is replaced by a Dini-smooth domain $\Omega \subsetneq \mathbb{C}$.*

Proof. By [37, Cor. to Thm. 10.1], it is enough to show that $F \circ k^{-1} \in \mathscr{H}^p(\mathbb{D})$ for some conformal mapping $k\colon \Omega \to \mathbb{D}$. Therefore, one can mostly copy the proof of Theorem 2.3.3, noting that for $f \in \mathscr{A}^p(\Omega)$ one has $f \circ k^{-1} \cdot \frac{1}{k'} \in \mathscr{A}^p(\mathbb{D})$.

The derivative of k does not vanish in $\bar{\Omega}$, and so we have

$$\int_{\mathbb{D}} |f \circ k^{-1}|^p \left| \frac{1}{k'} \right|^p \mathrm{d}A \le C \|f \circ k^{-1}\|^p_{\mathscr{A}^p(\mathbb{D})}.$$

It remains to show that $f \circ k^{-1} \in \mathscr{A}^p(\mathbb{D})$:

$$\int_{\mathbb{D}} |f \circ k^{-1}|^p \,\mathrm{d}A = \int_{\Omega} |f|^p |k'|^2 \,\mathrm{d}A \le C \|f\|^p_{\mathscr{A}^p(\Omega)} < \infty.$$

□

Now we can define distributional boundary values for Bergman functions.

2.3.5 Theorem. *Let $p \in [1,\infty)$. Every function $f \in \mathscr{A}^p(\mathbb{D})$ admits a distributional boundary value in $W^{-1,p}(\partial\mathbb{D}) := (W^{1,q}(\partial\mathbb{D}))'$, the dual space of the Sobolev space $W^{1,q}(\partial\mathbb{D})$, where q is the usual conjugate exponent of p.*

Proof. Let $\varphi \in W^{1,q}(\partial\mathbb{D})$. We denote by F an antiderivative of f, so we have

$$\begin{aligned}
\int_{\partial\mathbb{D}} f_r(x)\varphi(x)\,\mathrm{d}x &= \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\varphi(e^{it}) i e^{it} \frac{r}{r}\,\mathrm{d}t \\
&= \underbrace{\frac{1}{2\pi} F(re^{it})\varphi(e^{it}) \frac{1}{r}\Big|_0^{2\pi}}_{=0} - \frac{1}{2\pi}\int_0^{2\pi} F(re^{it})\varphi'(e^{it}) \frac{ie^{it}}{r}\,\mathrm{d}t. \\
&\overset{r\to 1^-}{\longrightarrow} -\frac{1}{2\pi}\int_0^{2\pi} F(e^{it})\varphi'(e^{it}) i e^{it}\,\mathrm{d}t = \langle T_{F'}, \varphi\rangle,
\end{aligned} \tag{2.6}$$

where $T_{F'} =: T_f$ is the distributional derivative of F. By Theorem 2.3.3, $F \in \mathscr{H}^p(\mathbb{D})$. So the convergence in (2.6) is obtained by an application of Hölder's inequality and [37, Thm. 2.6]. □

2.3.6 Corollary. *Let $\Omega \subseteq \mathbb{C}$ be Dini-smooth domain. Then every function $f \in \mathscr{A}^p(\Omega)$ $(p \in [1,\infty))$ admits a distributional boundary value in $W^{-1,p}(\partial\Omega)$.*

Proof. Let $\varphi \in W^{1,q}(\partial\Omega)$, and let $k : \Omega \to \mathbb{D}$ be conformal. For $r \in (0,1)$ we define as usual $f_r : \bar{\Omega} \to \mathbb{C}, x \mapsto f(k^{-1}(rk(x)))$. Thus $f_r \to f$ as $r \to 1^-$. Then

$$\int\limits_{\partial\Omega} f_r(x)\varphi(x)\,\mathrm{d}x = \int\limits_{\partial\mathbb{D}} f_r(k^{-1}(x))\varphi(k^{-1}(x))\left|\frac{1}{k'(x)}\right|^2 \mathrm{d}x$$
$$= \frac{1}{2\pi}\int\limits_0^{2\pi} f(k^{-1}(re^{it}))\varphi(k^{-1}(e^{it}))\left|\frac{1}{k'(e^{it})}\right|^2 ie^{it}\,\mathrm{d}t. \tag{2.7}$$

It is easy to show that $f \circ k^{-1} \in \mathscr{A}^p(\mathbb{D})$ and $\varphi \circ k^{-1} \in L^q(\partial\mathbb{D})$. Since k is conformal, we also have $\varphi \circ k^{-1} \in W^{1,q}(\partial\mathbb{D})$. By [66, Thm 6.8], $\sup_{t\in[0,2\pi)}\left|\frac{1}{k'(e^{it})}\right|^2 < \infty$, and thus by Hölder's inequality and Theorem 2.3.5, we obtain convergence of the integral (2.7) as $r \to 1^-$. □

Distributional boundary values of harmonic and holomorphic functions defined on a simply connected domain with smooth boundary have been studied in [75]. There it has been shown that a holomorphic function admits a distributional boundary value if and only if it lies in the Sobolev space $H^{-k}(\Omega) := W^{-k,2}(\Omega)$ for some $k \in \mathbb{N}$, see [75, Thm. 1.3]. Moreover, by [75, Cor. 1.7], for all $k \in \mathbb{N}$ the map P defined by

$$P : W^{-k-\frac{1}{2},2}(\partial\Omega) \to H^{-k}(\Omega) \cap \mathscr{H}(\Omega,\mathbb{C})$$
$$T_f \mapsto \langle P_z, T_f\rangle = f(z),$$

where P_z is the Poisson kernel for Ω, is an isomorphism. The inverse is given by assigning the distributional boundary value to a given function. Thus, functions in $H^{-k}(\Omega) \cap \mathscr{H}(\Omega,\mathbb{C})$ are uniquely determined by their boundary distributions. Therefore, restricting the map P to the boundary space $\partial\mathscr{A}^p(\Omega)$ for some $p \in [1,\infty)$, we can recover each function in $\mathscr{A}^p(\Omega)$ using the Poisson operator.

2.3.2. Dirichlet-to-Robin operators via composition semigroups

Now, we connect the theory of semigroups of weighted composition operators with evolution equations associated with a Dirichlet-to-Robin operator defined as follows.

2.3.7 Definition (Dirichlet-to-Robin operator)**.** Let $X \hookrightarrow \mathbb{H}(\Omega)$ be a Banach space with boundary space ∂X. Let G be the generator of a semiflow in $\mathscr{H}(\Omega,\Omega)$ and g the boundary value of a holomorphic function on Ω. Then the following operator is called a Dirichlet-to-Robin operator on ∂X

$$\mathfrak{D}_{\mathscr{R}} : \mathrm{dom}(\mathfrak{D}_{\mathscr{R}}) \subseteq \partial X \to \partial X$$
$$f \mapsto (-g \cdot u - G \cdot \nabla u)|_{\partial\Omega},$$

where u is the solution to

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

If $g \equiv 0$, we call the operator

$$\begin{aligned} \mathfrak{D}_G \colon \operatorname{dom}(\mathfrak{D}_G) \subseteq \partial X &\to \partial X \\ f &\mapsto -\langle G, \nabla u\rangle|_{\partial\Omega}, \end{aligned}$$

a Dirichlet-to-Neumann operator.

2.3.8 Theorem (Dirichlet-to-Robin semigroup). *Let $\Omega \subsetneq \mathbb{C}$ be a Jordan domain, and let $G : \Omega \to \mathbb{C}$ be the generator of a semiflow of holomorphic functions in $\mathscr{H}(\Omega, \Omega)$ and $g : \Omega \to \mathbb{C}$ holomorphic such that $X \subseteq \mathbb{H}(\Omega)$ is a (g, G)-admissible space which possesses the boundary space ∂X. Then the evolution problem associated with the Dirichlet-to-Robin operator*

$$\begin{cases} \partial_t u - g \cdot u - \langle G, \nabla u\rangle = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ -\Delta u = 0 & \text{on } (0,\infty) \times \Omega, \\ u(0,\cdot) = f & \text{on } \partial\Omega, \end{cases}$$

is well-posed in ∂X, and the solution is given by the trace of a semigroup of weighted composition operators.

Proof. Let $(S_t)_{t\geq 0}$ be the semigroup of weighted composition operators with semiflow $(\varphi_t)_t$ in $\mathscr{H}(\Omega, \Omega)$ generated by G and weight

$$m_t(z) = \exp\left(\int_0^t g(\varphi_s(z))\,\mathrm{d}s\right) \quad (z \in \Omega).$$

We denote by Γ the generator of $(S_t)_{t\geq 0}$. Then the Dirichlet-to-Robin operator $\mathfrak{D}_{\mathscr{R}} : \operatorname{dom}(\mathfrak{D}_{\mathscr{R}}) \subseteq \partial X \to \partial X, f \mapsto (g \cdot u_0 + G \cdot \nabla u_0)|_{\partial\Omega}$ is given by

$$\begin{aligned} \mathfrak{D}_{\mathscr{R}} f &= \operatorname{Tr}(g \cdot u + \langle G, \nabla u_0\rangle) \\ &= \operatorname{Tr}(\Gamma u_0), \end{aligned}$$

where $f \in \partial X$ and u_0 the solution to the Dirichlet problem

$$\begin{cases} -\Delta u_0 = 0 & \text{on } \Omega, \\ u_0 = f & \text{on } \partial\Omega. \end{cases}$$

So we obtain the Dirichlet-to-Robin semigroup as

$$e^{-t\mathfrak{D}_{\mathscr{R}}} f = \operatorname{Tr}(m_t \cdot u_0 \circ \varphi_t) \quad (f \in \partial X).$$

□

From the first glance, this approach might seem a bit artificial, but we would like to emphasize some crucial points which we find fascinating.

2.3.9 Remark. Compared to the variational approach, we have defined an operator which is closely related to the classical Dirichlet-to-Neumann operator on several spaces of distributions, including in particular the scale of $L^p(\partial\Omega)$ spaces. Indeed, a boundary space in the sense of distributions is not necessary since we can always define boundary values of holomorphic functions by using hyperfunctions. In this case our initial value would be very general. Moreover, our Neumann and Robin boundary conditions appear to be very general too. In particular, the variational approach, since it relies on the theorem of Gauß, seems to lack a possibility to handle coefficients in front of the Neumann derivative. It is worth noting that the function G may degenerate at some point $a \in \partial\Omega$. This is even possible if a is not a fixed point of the generated semiflow $(\varphi_t)_{t\geq 0}$, an example can be found in [43, p. 27]; on the other hand, if a is a non-superrepulsive fixed point of φ (i.e., $\varphi'(a) \neq \infty$), then the angular limit $\lim_{z\to a} G(z) = 0$, see [33, Thm. 1]. We do not see how this can be covered using the method of forms. On the other hand, the method of forms to the Dirichlet-to-Neumann and Dirichlet-to-Robin operators is quite flexible with respect the choice of elliptic operators in the domain Ω, while our approach is restricted to the Laplacian and Jordan domains.

2.3.10 Remark. Let $\varphi \in \mathscr{H}(\Omega, \Omega)$ not an automorphism and denote

$$\varphi_n := \underbrace{\varphi \circ \cdots \circ \varphi}_{n-\text{times}}.$$

If φ has an interior Denjoy-Wolff point and is not an inner function, then for a.e. $z \in \partial\Omega$ and n sufficiently large, $\varphi_n(z)$ lies strictly inside Ω, see [65, Thm. 1.2]. Assume, furthermore, that there exists an n_0 such that $\varphi_n(\partial\Omega) \subseteq K \subseteq \Omega$ for all $n \geq n_0$, where K is some compact subset that lies strictly in Ω, and that φ embeds into a semiflow generated by G and let $(T_t)_{t\geq 0}$ be the semigroup generated by the Dirichlet-to-Robin operator on the boundary space ∂X of a (g, G)-admissible Banach space X. Then, for all $t \geq n_0$ the operator T_t is regularizing in the sense that $T_t\partial X \subseteq L^\infty(\partial\Omega)$, since $\varphi_t(\partial\Omega) \subseteq \Omega$ by the semigroup property of the semiflow. That is, initial values in the space ∂X, which might be a space of distributions, are mapped to bounded functions since their images are restrictions of holomorphic functions in X to a set that lies strictly in Ω.

A weak point of this approach is that we cannot cover the classical Dirichlet-to-Neumann operator on a simply connected domain which is not a disk. However, there is still a connection to semigroups of composition operators which we discuss now.

The Dirichlet-to-Neumann operator on Ω. Let $\Omega \subsetneq \mathbb{C}$ be a Jordan domain with Dini-smooth boundary. Let $k : \Omega \to \mathbb{D}$ be conformal. Then $\nu(z) = \frac{k(z)}{k'(z)}|k'(z)|$ is the unit normal vector at $z \in \partial\Omega$. Since $\partial\Omega$ is Dini-smooth, $k \in C^1(\bar{\Omega})$ by Theorem 1.2.4(3). Thus $G(z) = -\frac{k(z)}{k'(z)}$ $(z \in \Omega)$ is holomorphic in Ω and uniformly continuous on $\bar{\Omega}$, and moreover, $\mathrm{Re}(G\bar{\nu}) \leq 0$ on $\partial\Omega$. In fact, $\mathrm{Re}(G\bar{\nu}) = -1$ on $\partial\Omega$. So, by Proposition 1.2.12, G generates a semiflow in $\mathscr{H}(\Omega,\Omega)$. Therefore, we obtain the following relation between the Dirichlet-to-Neumann operator on $\partial h^2(\Omega) = L^2(\partial\Omega)$ and the semigroup of composition operators on $h^2(\Omega)$. Let $u \in h^2(\Omega)$ be the solution to

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^2(\partial\Omega)$. Then, for $f \in \mathrm{dom}(\mathfrak{D}_{\mathcal{N}})$,

$$\begin{aligned} -\mathfrak{D}_{\mathcal{N}} f &= -\partial_\nu u \\ &= \mathrm{Tr}(\langle G, \nabla u\rangle |k'|), \end{aligned}$$

and $\Gamma u := \langle G, \nabla u\rangle$ is the generator of a semigroup of composition operators on $h^2(\Omega)$ with semiflow generated by G. So the Dirichlet-to-Neumann operator is a multiplicative perturbation of the generator of the semigroup of composition operators. Note that $\arg G = -\arg \nu$ on $\partial\Omega$ and that $h^2(\Omega)$ (in fact, $h^p(\Omega)$ for all $p \in (1,\infty)$) is (G)-admissible. So with Theorem 2.3.8 we obtain a Dirichlet-to-Neumann operator on $L^2(\partial\Omega)$ (or even $L^p(\partial\Omega)$, $p \in (1,\infty)$) mapping Dirichlet boundary values to an outward pointing derivative in normal direction but not normalized to one. We improve this approach in the third chapter using approximation techniques.

Half-planes. Although we discussed solely semiflows of holomorphic selfmaps on bounded domains so far, there exists also a rich theory on semiflows on half-planes. Actually, the famous paper by Berkson and Porta [20] deals with semiflows on the right half-plane. To ensure that Theorems 1.2.8 and 1.2.9 work also on the half-plane, we need to assume that a semiflow of holomorphic selfmaps $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{C}_+,\mathbb{C}_+)$ is a priori continuous on $(0,\infty)\times\mathbb{C}_+$, see [20, Thm 1.1]; this was on bounded domains already satisfied by Vitali's theorem. Also, the Hardy spaces on half-planes are highly considered in the literature, and, in particular, in [13] there is a comprehensive translation of the theory of composition semigroups on Hardy spaces of the disk to Hardy spaces on the half-plane. Roughly speaking, Arvanitidis [13] shows that most of the results which are true for the disk are also true for the plane, except for the fact that there are also semiflows inducing unbounded composition operators on Hardy spaces of the half-plane.

2.4. A remark on generalizations to higher dimensions

Generalizing the ideas we presented from domains in the plane to domains in $\mathbb{C}^n$, we need to overcome several difficulties. Of course, the theory of composition semigroups has to be extended to multidimensional domains. In his comprehensive review on the theory of semigroups of composition operators, Siskakis [72] remarks that there is no theory available for semigroups of composition operators on higher dimensional domains so far. Now, 20 years after his review, it seems that there is still no progress in this direction, at least we are not aware of any paper dealing with such semigroups. On the other hand, there is a rich literature about composition operators on Hardy and Bergman spaces of the unit ball or the polydisk [31, 35, 49] and, moreover, the theory of semiflows is not only generalized to domains in $\mathbb{C}^n$ but also to balls in infinite dimensional Hilbert spaces [1, 2, 24]. Based on the invented theory in the previous section, we give a short straightforward approach to semigroups of composition operators on domains in $\mathbb{C}^n$. Unfortunately, it turns out that a connection to Dirichlet-to-Robin operators, as introduced in the previous sections, is not as flexible and rich as in the plane. One crucial point is boundedness of composition operators. Even for Hardy spaces on the open unit ball $\mathbb{B}^n$ there are symbols that induce unbounded composition operators. Another problem that occurs is that composition of harmonic functions and holomorphic functions does not preserve harmonicity. Moreover, there is no Riemann mapping theorem for domains in $\mathbb{C}^n$ if $n \geq 2$. Even the unit ball and the polydisk $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$ are not biholomorphically equivalent. So complex analysis in higher dimensional domains depends heavily on the choice of the underlying domain. So when we define below the Hardy space on different domains on the ball and the polydisk, they appear not to be conformally equivalent.

We start by fixing some topology in $\mathbb{C}^n$. Every point $z \in \mathbb{C}^n$ is represented as the ordered n-tupel $(z_1, \dots, z_n)$ with $z_i \in \mathbb{C}$ for every $i \in \{1, \dots, n\}$. The inner product of $\mathbb{C}^n$ is given by $\langle z, \zeta \rangle_{\mathbb{C}^n} = \sum_{i=1}^n z_i \cdot \bar{\zeta}_i \,(z, \zeta \in \mathbb{C}^n)$ and the norm by $|z| := \sqrt{\langle z, z \rangle_{\mathbb{C}^n}}\,(z \in \mathbb{C}^n)$.

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $f : \Omega \to \mathbb{C}$ continuous on Ω. The function f is called holomorphic if it is holomorphic in each variable separately which is equivalent to

$$\bar{\partial}_i f := \frac{1}{2}(\partial_{x_i} f + i \partial_{y_i} f) = 0 \quad (z_i = x_i + i y_i)$$

for every $i \in \{1, \dots, n\}$, i.e., f satisfies the Cauchy-Riemann equations in each variable.

For $p \geq 1$ the Hardy space $\mathscr{H}^p(\mathbb{B}^n)$ of the unit ball consists of those holomorphic functions $f : \mathbb{B}^n \to \mathbb{C}$ for which

$$\|f\|^p_{\mathscr{H}^p(\mathbb{B}^n)} :\, = \lim_{r \to 1} \frac{1}{2\pi} \int\limits_{\partial \mathbb{B}^n} |f(r\zeta)|^p \,\mathrm{d}\sigma(\zeta)$$

is bounded. It is well known that every function in $\mathscr{H}^p(\mathbb{B}^n)$ has a boundary function in $L^p(\partial \mathbb{B}^n)$. These boundary functions form a closed subspace of $L^p(\partial \mathbb{B}^n)$ which is isomorphic to $\mathscr{H}^p(\mathbb{B}^n)$, see, e.g., [77, Chapter 4]. Similarly, the Hardy space $\mathscr{H}^p(\mathbb{D}^n)$ on the polydisc $\mathbb{D}^n$ consists of holomorphic functions $f : \mathbb{D}^n \to \mathbb{C}$ such that

$$\|f\|^p_{\mathscr{H}^p(\mathbb{D}^n)} :\, = \lim_{r \to 1} \frac{1}{2\pi} \int\limits_{\mathbb{T}^n} |f(r\zeta)|^p \,\mathrm{d}\sigma(\zeta) < \infty,$$

where $\mathbb{T}^n := \{z \in \mathbb{C} \colon |z_i| = 1 \text{ for } i \in \{1, \dots, n\}\}$. If $n = 1$, these spaces coincide. Unfortunately, a result due to Poincaré states that ball and polydisk are not biholomorphically equivalent, see, e.g., [19].

In several complex variables, one can, analogously to the one-dimensional case, define Hardy-Smirnov spaces using harmonic majorants. But, since composition of a harmonic and a holomorphic function need not be harmonic, these are not conformally equivalent.

Semiflows and their generators The theory of semiflows in the unit disc can be generalized to simply connected domains in $\mathbb{C}^n$, and even to complex manifolds and balls in Hilbert spaces. We collect some facts which can be found in [1]. Consider a simply connected domain $\Omega \subseteq \mathbb{C}^n$, and let $(\varphi_t)_{t\geq 0}$ be a family of holomorphic selfmaps on Ω such that

1. $\varphi_0 = id_\Omega$,
2. $\varphi_{s+t} = \varphi_s \circ \varphi_t \quad (s, t \geq 0)$, and
3. $\varphi_t(z)$ is continuous in t for all $z \in \Omega$.

As in the unit disc, such a family is called semiflow in $\mathscr{H}(\Omega, \Omega)$ and consists of univalent functions. The generator of a semiflow $(\varphi_t)_{t\geq 0}$, defined by

$$G(z) := \lim_{t\to 0^+} \frac{\varphi_t(z) - z}{t} \quad (z \in \Omega),$$

is a holomorphic (semicomplete) vector field that maps Ω to $\mathbb{C}^n$ which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = G(\varphi_t).$$

A result due to Aharonov et al. [3] yields that every generator is a bounded holomorphic function. The following theorem is a generalization to the flow invariance condition Theorem 1.2.13

2.4.1 Theorem. *A holomorphic function $G : \mathbb{B}^n \to \mathbb{C}^n$ is the generator of a semiflow in $\mathbb{B}^n$ if and only if*

$$2(|F(z)|^2 - |\langle F(z), z\rangle_{\mathbb{C}^n}|^2)\operatorname{Re}(F(z), z) + (1 - |z|^2)^2 \operatorname{Re}(\mathrm{d}G \cdot G(z), F(z)) \leq 0,$$

where $F(z) = (1 - |z|^2)G(z) + z\langle G(z), z\rangle_{\mathbb{C}^n}$.

If $n = 1$, the functions F and G coincide, so the theorem reduces to Theorem 1.2.13. As in the one dimensional case, a holomorphic vector field $G : \mathbb{B}^n \to \mathbb{C}^n$ that admits a continuous extension to $\bar{\mathbb{B}}^n$ is the generator of a semiflow in $\mathscr{H}(\mathbb{B}^n, \mathbb{B}^n)$ if and only if $\operatorname{Re}\langle G(z), z\rangle_{\mathbb{C}^n} \leq 0$ for all $z \in \partial\mathbb{B}^n$, see, e.g., [3].

Typical examples of semiflows in the unit ball are linear fractional maps: A holomorphic function $\varphi : \mathbb{B}^n \to \mathbb{C}^n$ is called linear fractional map if there exist a matrix $A \in \mathbb{C}^{n\times n}$, $B, C \in \mathbb{C}^n$, and $D \in \mathbb{C}$ with $D > |C|$ and $DA \neq BC^*$ such that

$$\varphi(z) = \frac{Az + B}{\langle z, C\rangle_{\mathbb{C}^n} + D} \quad (z \in \mathbb{B}^n).$$

Thus a semiflow of linear fractional maps consists of holomorphic selfmaps in $\mathbb{B}^n$ given by

$$\varphi_t(z) = \frac{A_t z + B_t}{\langle z, C_t\rangle_{\mathbb{C}^n} + 1} \quad (z \in \mathbb{B}^n)$$

for appropriate functions $t \mapsto A_t \in \mathbb{C}^{n\times n}$, $t \mapsto B_t \in \mathbb{C}^n$, and $t \mapsto C_t \in \mathbb{C}^n$. A holomorphic vector field $G : \mathbb{B}^n \to \mathbb{C}^n$ is the generator of semiflow of linear fractional maps if and only if there exist $a, b \in \mathbb{C}$ and $A \in \mathbb{C}^{n\times n}$ such that $|\langle b, z\rangle| \leq \langle Az, z\rangle_{\mathbb{C}^n}$ for $z \in \partial\mathbb{B}^n$ and G is given by

$$G(z) = a - \langle z, a\rangle z - (Az + \langle z, b\rangle_{\mathbb{C}^n} z) \quad (z \in \mathbb{B}^n),$$

see [24, Thm. 1.4].

Semigroups of weighted composition operators Let $\Omega \subseteq \mathbb{C}^n$ be a simply connected domain. For a given semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\Omega, \Omega)$, the composition $f \circ \varphi_t$, where $f \in \mathscr{H}(\Omega)$, gives again a holomorphic function in Ω. So we define analogously to the one-dimensional case a semigroup of composition operators $(T_t)_{t\geq 0}$ acting on $\mathscr{H}(\Omega)$ as follows, for all $t \geq 0$

$$\begin{aligned} T_t \colon \mathscr{H}(\Omega) &\to \mathscr{H}(\Omega) \\ f &\mapsto f \circ \varphi_t. \end{aligned} \tag{2.8}$$

Given a holomorphic function $g : \Omega \to \mathbb{C}$ and a semiflow $(\varphi_t)_{t\geq 0}$, we define a cocycle to $(\varphi_t)_{t\geq 0}$ by $m_t = \exp\left(\int_0^t g(\varphi_s)\,\mathrm{d}s\right)$ $(t \geq 0)$. Then

$$\begin{aligned} S_t \colon \mathscr{H}(\Omega) &\to \mathscr{H}(\Omega) \\ f &\mapsto m_t \cdot f \circ \varphi_t. \end{aligned} \tag{2.9}$$

is a semigroup of weighted composition operators on $\mathscr{H}(\Omega)$.

2.4.2 Definition. Let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathscr{H}(\Omega, \Omega)$ generated by G and $g \in \mathscr{H}(\Omega)$. A Banach space $X \hookrightarrow \mathscr{H}(\Omega)$ is called (G)-admissible and (g, G)-admissible if (2.8) and (2.9), respectively, define a strongly continuous semigroup of bounded operators on X.

As in the one-dimensional case, the generator Γ of a semigroup of composition operators acting on a (G)-admissible Banach space $X \hookrightarrow \mathscr{H}(\Omega)$ admits a special form in terms of the semiflow generator $G = (G_1, \dots, G_n)$. Let $f \in \operatorname{dom}(\Gamma)$. Then

$$\begin{aligned} \Gamma f = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t)|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_{1,t}, \dots, \varphi_{n,t})|_{t=0} \\ &= \sum_{i=1}^n \partial_i f(\varphi_t) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \varphi_{i,t}|_{t=0} \\ &= \sum_{i=1}^n \partial_i f \cdot G_i = \langle G, \nabla f\rangle. \end{aligned}$$

In differential geometry such an object is called Lie derivative and denotes a generalized directional derivative in direction of a vector field. Analogously, if X is (g, G)-admissible:

$$\begin{aligned} \Gamma f = \frac{\mathrm{d}}{\mathrm{d}t} m_t \cdot f(\varphi_t)|_{t=0} &= \left(\frac{\mathrm{d}}{\mathrm{d}t} m_t\right) \cdot f(\varphi_t) + m_t \cdot \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi_t)|_{t=0} \\ &= m_t \cdot \left[g(\varphi_t) \cdot f(\varphi_t) + \sum_{i=1}^n \partial_i f(\varphi_t) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \varphi_{i,t} \right]|_{t=0} \\ &= g \cdot f + \langle G, \nabla f\rangle. \end{aligned}$$

Although not to same extent as in the one-dimensional case, the Hardy spaces serve as an example for admissible spaces. In several complex variables, there are semiflows for which a composition operator on the Hardy space is unbounded, see, for instance, [49]. So examining if a given semiflow $(\varphi_t)_{t\geq 0}$ induces a semigroup of bounded operators becomes more delicate. On the other hand, once we have found a semiflow $(\varphi_t)_{t\geq 0}$ generated by G which gives a semigroup of bounded composition operators on $\mathscr{H}^p(\mathbb{B}^n)$, using that the polynomials are dense in $\mathscr{H}^p(\mathbb{B}^n)$, the triangle inequality, and the dominated convergence theorem yield strong continuity. This procedure, suggested by Siskakis, has been explained in Section 1.3 for the one-dimensional case. One particular example is the semiflow $(\varphi_t)_{t\geq 0}$ given by $\varphi_t(z) = e^{-t}z$ which is generated by $G(z) = -z$, the holomorphic extension inward pointing normal of the unit ball $-\nu : \partial\mathbb{B}^n \to \mathbb{C}^n$. Thus $\mathscr{H}^p(\mathbb{B}^n)$ is (G)-admissible. If $f \in \partial\mathscr{H}^p(\mathbb{B}^n) \subseteq L^p(\partial\mathbb{B}^n)$, we obtain well-posedness for the following evolution problem associated with the Dirichlet-to-Neumann operator on a subspace of $L^p(\partial\mathbb{B}^n)$

$$\begin{cases} \partial_t u + \langle \nu, \nabla u\rangle = 0 & \text{on } (0,\infty)\times\partial\mathbb{B}^n, \\ -\Delta u = 0 & \text{on } (0,\infty)\times\mathbb{B}^n, \\ u(0,\cdot) = f & \text{on } \partial\mathbb{B}^n. \end{cases}$$

This is obviously the Lax semigroup for balls in $\mathbb{C}^n$. Theorem 2.3.8 could now be reformulated in the higher dimensional setting but, as a matter of fact, we cannot give such an amount of examples regarding spaces, domains, and coefficents in front of the Neumann derivative as in the one dimensional case. So we omit formulating and proving a well-posedness result whose assumption are rarely satisfied by typical examples. Nevertheless we have found a space, actually a subspace of a subspace of the Hardy space, such that we obtain a well-posed evolution problem associated with a Dirichlet-to-Robin operator for a certain class of domains and arbitrary semiflow generators.

A small space on which our theory works fine. Although there is no Riemann mapping theorem in $\mathbb{C}^n$ when $n > 1$, we can find some domains which are biholomorphically equivalent to the ball $\mathbb{B}^n$ in the class of domains with C^2-boundary. We recall some definitions and results in this direction from [68, Sect. 15.5].

2.4.3 Definition. A bounded domain $\Omega \subseteq \mathbb{C}^n$ has C^2-boundary at a point $\xi \in \partial\Omega$ if there exists a neighborhood W and a function $p \in C^2(W,\mathbb{R})$ such that $\nu(\zeta) \neq 0$ for $\zeta \in W \cap \partial\Omega$ and $\Omega \cap W = \{z \in W : p(z) < 0\}$, where $\nu(\zeta)$ is the outward pointing normal vector at ζ. Such a function p is called local defining function for Ω at ξ. If $\partial\Omega \subseteq W$, then p is called defining function for Ω. If Ω has C^2-boundary at every $\xi \in \partial\Omega$, then Ω is said to have C^2-boundary.

2.4.4 Definition. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with C^2-boundary at $\xi \in \partial\Omega$, and let p be the local defining function for Ω at ξ. If the Hessian of p at ξ, denoted by $H_\xi(p)$, is strictly positive, i.e., there exists a $c > 0$ such that $\langle H_\xi(p)a, a\rangle \geq c|a|^2$ for all $a \in \mathbb{C}^n$, then Ω is said to be strictly pseudoconvex at ξ. We call Ω a strictly pseudoconvex domain if Ω is strictly pseudoconvex at each boundary point.

2.4.5 Theorem ([68, Theorem 15.5.10]). *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain that is strictly pseudoconvex at some point $\xi \in \partial\Omega$. If there is a sequence $(T_n) \in \mathrm{Aut}(\Omega)$ such that*

$$\lim_{n\to\infty} T_n(w) = \xi$$

for some $w \in \Omega$, then Ω is biholomorphically equivalent to the open unit ball $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$.

2.4.6 Corollary. *If $\Omega \subseteq \mathbb{C}^n$ is a bounded domain with C^2 boundary such that* Aut(Ω) *is transitive, then Ω is biholomorphically equivalent to the unit ball $\mathbb{B}_n$.*

A proof of these results can be found in [68, Theorem 15.5.10 and Corollary].

2.4.7 Remark. A generalization of Fatou's theorem to higher dimensional domains due to Stein [74, Theorem 9] asserts that every bounded holomorphic functions defined on a C^2-bounded domain $\Omega \subseteq \mathbb{C}^n$ has admissible limits a.e. (with respect to the $(2n-1)$ dimensional Hausdorff measure σ on $\partial\Omega$) on $\partial\Omega$. The admissible limit is the analogue notion to nontangential limits for $n = 1$. Furthermore, for a biholomorphic mapping $k : \Omega \to \mathbb{B}^n$, we have $k^* : \partial\Omega \to \partial\mathbb{B}^n$, where k^* is the boundary function of k. Krantz [58] has proved that k^* is a.e. bijective as well as $(k^{-1})^*$, the boundary function of the inverse of k. But he emphasizes that he proved that $k^* \circ (k^{-1})^* = (k^{-1})^* \circ k^* = id$ and $(k^{-1})^* = (k^*)^{-1}$ only for the case $n = 1$. On the other hand, sets of positive measure and sets of zero measure are preserved by both boundary functions k^* and $(k^{-1})^*$, see [58, Theorem 4.1].

Now, we try to determine a subspace of holomorphic functions which is (G)-admissible for any possible generator on a simply connected domain $\Omega \subseteq \mathbb{C}^n$ with C^2 boundary biholomorphically equivalent to the ball admitting a boundary space in $L^p(\partial\Omega)$ such that for appropriate functions $g : \Omega \to \mathbb{C}$, the evolution problem associated with a Dirichlet-to-Robin operator

$$\begin{cases} \partial_t u - g\cdot u - \langle G, \nabla u\rangle = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ -\Delta u = 0 & \text{on } (0,\infty)\times\Omega, \\ u(0,\cdot) = f & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

is well-posed in a certain subspace of $L^p(\partial\Omega)$ and the semigroup solving the problem is similar to a semigroup of weighted composition operators.

In the one-dimensional case the problem is well-posed on the boundary space of the Hardy space $\mathscr{H}^p(\Omega)$, where Ω is a simply connected Jordan domain. In several complex variables, there might be a flow for which a composition operator on the Hardy space is unbounded. So the space seems to be too large to be (G)-admissible for every possible semicomplete vector field G. Furthermore, in one dimension it was sufficient to consider the unit disc which depends heavily on the fact that the composition of a harmonic and a holomorphic function is again harmonic. Fortunately there is a (though small) subspace of $\mathscr{H}^p(\Omega)$ that satisfies both boundedness of composition operators for arbitrary symbols and invariance under biholomorphic mappings. This subspace is called Lumer-Hardy space which is defined in terms of pluriharmonic functions which we introduce now.

2.4.8 Definition. Let $\Omega \subseteq \mathbb{C}^n$ be a domain, and let $u \in C^2(\Omega, \mathbb{R})$. Let $a \in \Omega$ and $b \in \mathbb{C}^n$ and set $u_{a,b}(z) := u(a + bz)$ for all $z \in \mathbb{C}$ such that $a + zb \in \Omega$. The function u is said to be pluriharmonic if every $u_{a,b}$ is harmonic.

Pluriharmonic functions are harmonic but the converse is not true. Another important feature is the following lemma.

2.4.9 Lemma. *If f is pluriharmonic in Ω and $\varphi \in \mathscr{H}(\Omega, \Omega)$, then $f \circ \varphi$ is pluriharmonic.*

Using pluriharmonic functions, we can define an analogue of the Hardy-Smirnov space that is invariant under holomorphic mappings.
Let $p \geq 1$. The Lumer-Hardy space $L\mathscr{H}^p(\Omega)$ consists of holomorphic function $f : \Omega \to \mathbb{C}$ such that $|f|^p$ is dominated by a pluriharmonic function. In one complex variable, this is indeed the usual Hardy space, but for $n \geq 2$ it is a proper subspace. As in the Hardy space case for harmonic majorants, we can define a norm in terms of the least pluriharmonic majorant evaluated at 0 (or, if $0 \notin \Omega$, then at some fixed point $z_0 \in \Omega$) . This turns $L\mathscr{H}^p(\mathbb{B}^n)$ into a Banach space. But, in contrast to the usual Hardy spaces, the Lumer-Hardy space is not separable, for $p = 2$ it is not a Hilbert space, and $A(\Omega)$ is not dense in any Lumer-Hardy space. Unfortunately, the Lumer-Hardy space has several pathological properties. We refer to [68, Sect. 7.4 and 9.7] for more details to mentioned properties of the Lumer-Hardy space..
On the other hand, there is one important feature of the Lumer-Hardy space that catched our attention: on $L\mathscr{H}^p(\Omega)$ every composition operator is bounded, see, e.g., [35].
Furthermore, we can prove that $L\mathscr{H}^p(\Omega)$ contains a closed subspace X that is (G)-admissible for every possible generator of semiflows in $\mathscr{H}(\Omega, \Omega)$, and (g, G)-admissible for certain holomorphic functions $g : \Omega \to \mathbb{C}$ with bounded real part. We were not able to figure out if $X = L\mathscr{H}^p(\Omega)$ or not.

2.4.10 Theorem. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. Then for every G generating a semiflow in $\mathscr{H}(\Omega, \Omega)$ there exists a maximal subspace X of the Lumer-Hardy space $L\mathscr{H}^p(\Omega)$ that is (G)-admissible. In particular, $X = \overline{\{f \in L\mathscr{H}^p(\Omega) \colon \langle G, \nabla f \rangle \in L\mathscr{H}^p(\Omega\}}$.*

Proof. It has already been mentioned that every holomorphic function induces a bounded composition operator on $L\mathscr{H}^p(\Omega)$. A proof can be found in [54, Theorem 3.1]. Let $(\varphi_t)_{t\geq 0}$ be the semiflow generated by G. Set $X := \{f \in L\mathscr{H}^p(\Omega) \colon \|f \circ \varphi_t - f\| \to 0 \text{ as } t \to 0^+\}$. Standard arguments (cf. [22]) show that this is indeed a closed subspace of $L\mathscr{H}^p(\Omega)$ which is maximal in the sense that X contains every subspace on which $(\varphi_t)_{t\geq 0}$ induces a strongly continuous semigroup of composition operators. Moreover, we have

$$X \subseteq \overline{\{f \in L\mathscr{H}^p(\Omega) \colon \langle G, \nabla f \rangle \in L\mathscr{H}^p(\Omega\}}$$

by just applying the general theory on generators of C_0-semigroups, cf. Proposition 1.1.3. Conversely, let $f \in L\mathscr{H}^p(\Omega)$ such that $\langle G, \nabla f \rangle \in L\mathscr{H}^p(\Omega)$. So $\langle G(\varphi_t), \nabla f(\varphi_t) \rangle \in L\mathscr{H}^p(\Omega)$

for every $t \geq 0$, and $\|G(\varphi_t) \cdot \nabla f(\varphi_t)\|_{L\mathcal{H}^p(\Omega)} \leq C \|\langle G, \nabla f\rangle\|_{L\mathcal{H}^p(\Omega)}$ for some constant $C > 0$.

$$
\begin{aligned}
|f \circ \varphi_t - f|^p &= \left| \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} f(\varphi_s)\, \mathrm{d}s \right|^p \\
&= \left| \int_0^t \langle G(\varphi_s), \nabla f(\varphi_s)\rangle \, \mathrm{d}s \right|^p \\
&\leq \int_0^t |\langle G(\varphi_s), \nabla f(\varphi_s)\rangle|^p \, \mathrm{d}s \\
&\leq tC \|\langle G, \nabla f\rangle\|_{L\mathcal{H}^p(\Omega)} \to 0 \quad (t \to 0^+).
\end{aligned}
$$

Taking the closure yields the assertion. □

2.4.11 Corollary. *Let G be the generator of a semiflow in $\mathcal{H}(\Omega, \Omega)$ and let*

$$X = \overline{\{f \in L\mathcal{H}^p(\Omega) \colon \langle G, \nabla f\rangle \in L\mathcal{H}^p(\Omega\}}.$$

If $g : \Omega \to \mathbb{C}$ is a holomorphic function with bounded real part in Ω such that $\|m_t - 1\|_\infty \to 0$ as $t \to 0^+$, where $m_t := \exp(\int_0^t g(\varphi_s)\, \mathrm{d}s)$, then X is (g, G)-admissible. Moreover,

$$X = \overline{\{f \in L\mathcal{H}^p(\Omega) \colon g \cdot f + \langle G, \nabla f\rangle \in L\mathcal{H}^p(\Omega\}}.$$

Proof. The semigroup $(S_t)_{t\geq 0}$ defined by $S_t f = m_t \cdot f \circ \varphi_t$ consists of bounded operators on X as a consequences of Theorem 2.4.10 and since $m_t \in \mathcal{H}^\infty(\Omega)$ for all $t \geq 0$ by our assumption on g. The following calculation shows strong continuity.

$$
\begin{aligned}
\|m_t \cdot f \circ \varphi_t - f\|_{L\mathcal{H}^p(\Omega)} &\leq \|m_t \cdot (f \circ \varphi_t - f)\|_{L\mathcal{H}^p(\Omega)} + \|f \cdot (m_t - 1)\|_{L\mathcal{H}^p(\Omega)} \\
&\leq C\underbrace{(\|f \circ \varphi_t - f\|_{L\mathcal{H}^p(\Omega)} + \|f\|_{L\mathcal{H}^p(\Omega)} \|m_t - 1\|_\infty)}_{\to 0,\, t \to 0^+}.
\end{aligned}
$$

It is clear that $X \subseteq \overline{\{f \in L\mathcal{H}^p(\Omega) \colon g \cdot f + \langle G, \nabla f\rangle \in L\mathcal{H}^p(\Omega\}}$. To show the converse, note that since $g \cdot f + \langle G, \nabla f\rangle \in L\mathcal{H}^p(\Omega)$, we have $(g \cdot f + \langle G, \nabla f\rangle)(\varphi_t) \in L\mathcal{H}^p(\Omega))$ for all $t \geq 0$. Thus

$$
\begin{aligned}
|m_t \cdot f(\varphi_t) - f|^p &= \left| \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} m_s \cdot f(\varphi_s)\, \mathrm{d}s \right|^p \\
&\leq \int_0^t |m_s \cdot (g(\varphi_s) \cdot f(\varphi_s) + \langle G(\varphi_s), \nabla f(\varphi_s)\rangle)|^p \, \mathrm{d}s \\
&\leq tC \|g \cdot f + \langle G, \nabla f\rangle\|_{L\mathcal{H}^p(\Omega)} \to 0 \quad (t \to 0^+).
\end{aligned}
$$

□

Using the same technique, we can show that for G the generator of a semiflow in $\mathcal{H}(\mathbb{B}^n, \mathbb{B}^n)$ such that the Hardy space $\mathcal{H}^p(\mathbb{B}^n)$ is (G)-admissible and $g : \mathbb{B}^n \to \mathbb{C}$ with bounded real part such that $\|m_t - 1\|_\infty \to 0$ as $t \to 0^+$, we have that $\mathcal{H}^p(\mathbb{B}^n)$ is (g, G)-admissible.

3. Semiflow generator perturbation and approximation of Poincaré-Steklov semigroups

In this chapter, we study the multiplicative perturbation of the Poincaré-Steklov operators, or, more precisely, Dirichlet-to-Robin operators, constructed in the previous chapter on Jordan domains $\Omega \subsetneq \mathbb{C}$. The geometric function theory of semiflows serves as a driving force in the upcoming investigations. In fact, we search for admissible multiplicative perturbations of semiflow generators such that we obtain well-posedness for the following evolution problem

$$\begin{cases} \partial_t u - a \cdot \langle G, \nabla u \rangle = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ -a\Delta u = 0 & \text{on } (0, \infty) \times \Omega, \\ u(0, \cdot) = f & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where a denotes the boundary value of a holomorphic function in $\mathscr{H}(\Omega)$ (which we still denote by a) and G the boundary value of a semiflow generator. For $m \in \mathscr{H}(\Omega)$, we define by

$$\begin{aligned} M_m : \mathscr{H}(\Omega) &\to \mathscr{H}(\Omega), \\ f &\mapsto m \cdot f \end{aligned}$$

the multiplication operator on $\mathscr{H}(\Omega)$. The above problem is then associated with a multiplicative perturbation of the Dirichlet-to-Neumann operator $\mathfrak{D}_G$ by the multiplication operator M_a, namely,

$$\begin{aligned} M_a \mathfrak{D}_G \colon \operatorname{dom} \mathfrak{D}_G \subseteq \partial X &\to \partial X, \\ f &\mapsto -a \cdot \langle G, \nabla u \rangle. \end{aligned}$$

We prove well-posedness of (3.1) on the boundary space ∂X of a (G)-admissible Banach space $X \hookrightarrow \mathscr{H}(\Omega)$ for certain functions a, and, furthermore, that the semigroup generated by the multiplicative perturbation of the Dirichlet-to-Neumann operator $\mathfrak{D}_G$ can be approximated by the trace of composition operators. Indeed, we find a sufficient condition such that the approximated semigroup is again the trace of a semigroup of composition operators. In contrast to the previous chapter, the underlying semiflow of the approximated semigroup of composition operators need not be given explicitly; it suffices to know the semiflow generated by G. Based on this result we show also well-posedness for evolution problems associated with a multiplicative perturbation of a Dirichlet-to-Robin operator.

Unfortunately, we have no sufficient condition such that the approximated semigroup consists of weighted composition operators although this appears to be true in several examples.

Our observation relies on recent results by Elin et al. [44] and Elin and Jacobzon [42] on analytic extendability of semiflows and the well-known Chernoff formula which we present here for reference, see, e.g., [48, Thm. 5.2].

3.0.1 Theorem (Chernoff's formula). *Let $(T(t))_{t\geq 0}$ be a family of bounded operators on a Banach space X satisfying*

1. $T(0)=I$
2. $\|T(t)^m\| \leq C$ *for some* $C \geq 1$ *and all* $t \geq 0, m \in \mathbb{N}$.
3. $Af := \lim_{t\to 0} \frac{T(t)f-f}{t}$ *exists for all* $f \in D$ *where* $D \subseteq X$ *is a dense subspace, and* $(\lambda - A)D$ *is dense in* X *for some* $\lambda > 0$.

Then the closure of A generates a bounded C_0-semigroup which is given by

$$S_t f = \lim_{n\to\infty} (T(t/n))^n f \text{ for all } f \in X.$$

A crucial tool in this chapter is the special connection between semiflows of holomorphic functions and their associated univalent functions, the so-called Koenigs function, which we already presented in our first chapter, see Remark 1.2.16: a semiflow $(\varphi_t)_{t\geq 0}$ can be represented in terms of its Koenigs function function h. We recall that the representation depends on the position of the Denjoy-Wolff point. For a semiflow $(\varphi_t)_{t\geq 0}$ with Denjoy-Wolff point in 0, we have

$$\varphi_t(z) = h^{-1}(e^{-\gamma t}h(z)) \quad (z \in \mathbb{D}), \tag{3.2}$$

and for a semiflow $(\varphi_t)_{t\geq 0}$ with Denjoy-Wolff point in 1,

$$\varphi_t(z) = h^{-1}(h(z)+t) \quad (z \in \mathbb{D}). \tag{3.3}$$

Using conformal mappings, we easily translate this representation to simply connected domains in $\Omega \subsetneq \mathbb{C}$. Recently, it has been investigated that the associated Koenigs function carries also information about the analytic extendability of the semiflow, see [44] and [42]. We give a brief overview to this topic in Section 3.1. Then, we use these results in Section 3.2 to construct a family of holomorphic selfmaps which do not form a semiflow. But this family is eventually our tool to construct in Section 3.3 a family of operators on certain Banach spaces of analytic functions which satisfy the conditions of Theorem 3.0.1, and the semigroup obtained this way solves the problem posed in the beginning of this Chapter.

3.1. Analytic extendability of semiflows

We recall first the definition of analytic semigroups of bounded operators.

3.1.1 Definition. Let X be a Banach space and $\alpha \in (0,1)$. A strongly continuous semigroup of bounded operators $(T_t)_{t\geq 0}$ on X extends analytically to the sector $\Sigma_{\alpha\frac{\pi}{2}} := \{z \in \mathbb{C} : |\arg z| \leq \alpha\frac{\pi}{2}\}$ if there exists a family of operators $(\tilde{T}_s)_{s\in\Sigma_{\alpha\frac{\pi}{2}}}$ in $\mathscr{L}(X)$ such that $(\tilde{T}_t)_{t\geq 0} = (T_t)_{t\geq 0}$ and

1. $\tilde{T}_{s+t} = \tilde{T}_s\tilde{T}_t$ for all $s,t \in \Sigma_{\alpha\frac{\pi}{2}}$
2. $\lim_{s\in\Sigma_{\alpha\frac{\pi}{2}}\to 0} \tilde{T}_s x = x$ for all $x \in X$ and $s \mapsto T_s x$ is analytic in $\Sigma_{\alpha\frac{\pi}{2}} \setminus \{0\}$.

We usually write $(T_s)_{s\in\Sigma_{\alpha\frac{\pi}{2}}}$ for the analytic extension of a semigroup $(T_t)_{t\geq 0}$ and call $(T_s)_{s\in\Sigma_{\alpha\frac{\pi}{2}}}$ an analytic semigroup. The generator of an analytic operator semigroup (defined analogously to the usual semigroup generator) admits several nice properties and is therefore highly considered in the literature, see, for instance, [48, II 4(a)]. Especially, their stability under multiplicative perturbations will prove to be useful in what follows. We define similarly analytic extensions of semiflows of holomorphic functions.

3.1.2 Definition. Let $\alpha \in (0,1)$. A semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\Omega,\Omega)$ extends analytically to the sector $\Sigma_{\alpha\frac{\pi}{2}}$ if there exists a family of holomorphic selfmaps $(\tilde{\varphi}_s)_{s\in\Sigma_{\alpha\frac{\pi}{2}}}$ in $\mathscr{H}(\Omega,\Omega)$ such that $(\tilde{\varphi}_t)_{t\geq 0} = (\varphi_t)_{t\geq 0}$ and

1. $\tilde{\varphi}_{s+t} = \tilde{\varphi}_s \circ \tilde{\varphi}_t$ for all $s,t \in \Sigma_{\alpha\frac{\pi}{2}}$
2. $s \mapsto \tilde{\varphi}_s(z)$ is analytic for all $z \in \Omega$.

We write $(\varphi_s)_{s\in\Sigma_{\alpha\frac{\pi}{2}}}$ for the analytic extension of $(\varphi_t)_{t\geq 0}$ and call it an analytic semiflow.

One particular reason for studying analytic extendability of semiflows is the following observation, see, e.g., Elin et al. [44]: a strongly continuous semigroup of composition operators extends analytically to some sector Σ if and only if the underlying semiflow extends analytically to Σ. Moreover they prove that there are semiflows in $\mathscr{H}(\mathbb{D},\mathbb{D})$ which do not admit an analytic extension, see [44, Example 2.17], hence there are semigroups of composition operators which are not analytic in any sector.

In 2017, Elin et al. [44] and Elin and Jacobzon [42] published a systematic study of analytic extendability of semiflows in terms of the associated Koenigs function. They show that analytic extendability can be described in terms of geometric properties of the Koenigs function. Their results are summarized below. Since the associated Koenigs function is either spirallike or close-to-convex depending on the position of the Denjoy-Wolff point, analyticity of semiflows is investigated separately for these cases.

Analyticity of semiflows with Denjoy-Wolff point in 0. First we need to recall another generalization of starlike functions: for $0 < \alpha < 1$, a univalent function $h : \mathbb{D} \to \mathbb{C}$ is called strongly α-starlike if $|\arg \frac{zh'(z)}{h(z)}| \leq \alpha\frac{\pi}{2}$ for all $z \in \mathbb{D}$. This class, denoted by $S^*_\alpha(\mathbb{D})$, has been introduced by Brannan and Kirwan [28], and independently by Stankievicz [73]. Elin et al. proved the following correspondence between analyticity of a semiflow with inner Denjoy-Wolff point and the associated Koenigs function, see [44, Thm. 2.13].

3.1.3 Theorem. *Let $0 < \alpha < 1$. A semiflow in the unit disc fixing the origin is analytic in the sector $\Sigma_{\alpha\frac{\pi}{2}} := \{s \in \mathbb{C} : |\arg s| < \alpha\frac{\pi}{2}\}$ if and only if the associated Koenigs function $h : \mathbb{D} \to \mathbb{C}$ is strongly α-starlike.*

3.1.4 Remark. In fact, [44, Thm. 2.13] holds in a more general context for semiflows defined on the unit ball of a complex Banach space. The theorem is a consequence of non-linear analogues of the Lumer-Phillips theorem.

In [62], Lecko investigates several generalizations of strongly starlike domains, and he examines, moreover, that strongly starlike domains and domains of bounded boundary rotation are the same, see [62, Thm. 4.1]. Note that in [62], Lecko names functions of bounded boundary rotation as *spirallike* but not in the sense of our definition, which is misleading since $S^*_\alpha \subseteq S^* \subseteq S^{\circledS}_\lambda$. His results can be summarized as follows.

3.1.5 Theorem ([62, Thm 5.4])**.** *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain. Then the following are equivalent:*

1. *Ω is of bounded boundary rotation of order α.*
2. *Ω is strongly starlike of order α.*
3. *There is a strongly α-starlike function f, such that $f(\mathbb{D}) = \Omega$.*

Combining the results of [44] and [62], we obtain the following.

3.1.6 Corollary. *A semiflow with Denjoy-Wolff point in $\mathbb{D}$ extends analytically to some sector $\Sigma_{\alpha\frac{\pi}{2}}$, for some $\alpha \in (0,1)$, if and only if the associated Koenigs function is of bounded boundary rotation.*

Analyticity of semiflows with Denjoy-Wolff point in 1. In the recent paper [42], analytic extendability of semiflows in the right half-plane with Denjoy-Wolff point at $\{\infty\}$ has been investigated in terms of the associated univalent function. From the geometric point of view, this result appears to be very natural. Considering the representation formula (3.3), a Koenigs function which is *convex in a sector* gives an analytic extendability of the associated semiflow. Let $\theta \in (0,1)$ and denote by $S^{\text{ctc}}_\theta(\mathbb{D})$ the set of univalent functions $h : \mathbb{D} \to \mathbb{C}$ such that $\{h(z) + e^{i\theta\frac{\pi}{2}}t, z \in \mathbb{D}\} \subseteq h(\mathbb{D})$ for all $t \geq 0$ and define $S^{ctc}_{\Sigma_{\alpha\frac{\pi}{2}}}(\mathbb{D}) := \bigcup_{|\theta|<\alpha} S^{\text{ctc}}_\theta(\mathbb{D})$. We call functions in $S^{ctc}_{\Sigma_{\alpha\frac{\pi}{2}}}(\mathbb{D})$ *convex in the sector* $\Sigma_{\alpha\frac{\pi}{2}}$. Shifting the right half-plane conformally to the unit disk and using the Berkson-Porta representation for semiflow generators, we rewrite here a crucial result from [42].

3.1.7 Theorem ([42, Thm. 4.3])**.** *Let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathcal{H}(\mathbb{D},\mathbb{D})$ with Denjoy-Wolff point 1, generated by G, and with Koenigs function h.Then the following are equivalent:*

(i) The semiflow $(\varphi_t)_{t\geq 0}$ extends analytically to the sector $\Sigma_{\alpha\frac{\pi}{2}}$.

(ii) $h \in S^{ctc}_{\Sigma_{\alpha\frac{\pi}{2}}}(\mathbb{D})$.

(iii) $|\arg F| \le (1-\alpha)\frac{\pi}{2}$, *where F is the holomorphic function with positive real part from the Berkson-Porta representation* $G(z) = (1-z)^2 F(z)\,(z \in \mathbb{D})$.

Note that in [42] the sector need not be symmetric with respect to the real axis, but we restrict to this case for simplicity.

3.2. Admissible semiflow generator perturbation

This section is devoted to the study of the behavior of semiflow generators under multiplicative perturbations. More precisely, we investigate which multiplicative perturbations are still semifow generators; such perturbations will be called admissible. Then, we define a family of holomorphic functions from a semiflow and an admissible multiplicative perturbation of its generator. This family does not form a semiflow in general but we prove that it is still consisting of holomorphic selfmaps. This leads eventually to a family of operators which satisfies the assumptions of Chernoff's formula, Theorem 3.0.1.

We introduce the following convenient notation for semiflows: let $\Omega \subsetneq \mathbb{C}$, and consider a semiflow $(\varphi_t)_{t\ge 0}$ in $\mathscr{H}(\Omega,\Omega)$, and set $\varphi(t,z) := \varphi_t(z)$ for all $z \in \Omega$. In what follows, we denote the set of generators of semiflows in $\mathscr{H}(\Omega,\Omega)$ by $\mathscr{G}(\Omega)$ and briefly write $\mathscr{G}$ if the underlying simply connected domain is clear. If we want to refer to functions $G \in \mathscr{G}$ that generate a semiflow with Denjoy-Wolff point $b \in \bar{\Omega}$, we write $G \in \mathscr{G}_b$, and we drop the b if we do not refer to a specific Denjoy-Wolff point.

3.2.1 Remark. Although we heavily rely on the systematic study of analytic extendability of semiflows invented in [42, 44], there are also several results on analytic extendability of semigroups of composition operators on $\mathscr{H}^2(\mathbb{D})$ not based on the geometric function theory of the associated Koenigs function. For the theory of semigroups with underlying semiflow admitting an inner Denjoy-Wolff we refer to [16] where the authors obtain the following result for semigroups on $\mathscr{H}^2(\mathbb{D})$ the generator of which admits a representation of the form (1.13), see [16, Thm. 2.8]:

3.2.2 Theorem. *Let* $G \in \mathscr{H}(\mathbb{D})$ *such that A given by* $Af = G \cdot f'$ *has dense maximal domain in* $\mathscr{H}^2(\mathbb{D})$. *Then the following are equivalent:*

1. *A generates an analytic semigroup of composition operators on* $\mathscr{H}^2(\mathbb{D})$.
2. *There is* $\alpha \in (0,1)$ *such that A,* $e^{i\alpha\frac{\pi}{2}}A$, *and* $e^{-i\alpha\frac{\pi}{2}}A$ *generate a strongly continuous semigroup of composition operators.*
3. *There is* $\alpha \in (0,1)$ *such that* $\sup\{\operatorname{Re} e^{\pm i\alpha\frac{\pi}{2}} Af \cdot \bar{f} : f \in \operatorname{dom} A, \|f\|_{\mathscr{H}^2(\mathbb{D})} = 1\} < \infty$, *and there exists* $\lambda > 0$ *such that* $(\lambda - A)\operatorname{dom} A = \mathscr{H}^2(\mathbb{D})$.

Also, there is a paper from 1983 [34] where the author presents an example of a semiflow with Denjoy-Wolff point on the boundary which induces an analytic semigroup of composition operators on $\mathscr{H}^2(\mathbb{D})$.

Multiplicative perturbation of semiflow generators. We define admissible perturbations as follows:

3.2.3 Definition (admissible perturbation). Let $G \in \mathscr{G}(\mathbb{D})$ generate the semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{D},\mathbb{D})$ and let $a : \mathbb{D} \to \mathbb{C}$ be a holomorphic function with positive real part. We call a an admissible perturbation if $a \cdot G \in \mathscr{G}$

Clearly $a \cdot G \in \mathscr{G}_b(\mathbb{D})$ is not satisfied for arbitrary choices of a and G since the following result leads to counterexamples.

3.2.4 Lemma. *Let G be the generator of a semiflow in in $\mathscr{H}(\mathbb{D},\mathbb{D})$ fixing 0 and $a : \mathbb{D} \to \mathbb{C}$ a holomorphic function with positive real part. Then $a \cdot G \in \mathscr{G}_0(\mathbb{D})$ if and only if* $\operatorname{Re} a(z)G(z)\bar{z} \leq 0$ *for all $z \in \mathbb{D}$.*

Proof. This is an easy consequence of the flow invariance condition, Theorem 1.2.13. □

One possible counterexample is thus $G(z) = -z(1-z)$ and $a(z) = \frac{1}{2} - 2i$ where $z \in \mathbb{D}$, since, for $z = x + iy$, we have

$$-\|z\|^2 \operatorname{Re}(1-z)\left(\frac{1}{2} - 2i\right) = \|z\|^2\left(\frac{1}{2}(x-1) + 2y\right),$$

and thus $\operatorname{Re} a(z)G(z)\bar{z}$ attains positive and negative values in $\mathbb{D}$.

Assuming the analytic extendability of a semiflow generated by G and the image of a lying in the sector of analyticity, $a \cdot G$ generates a semiflow. This is caused by the following (known) fact:

3.2.5 Lemma. *Let $G \in \mathscr{G}_b(\mathbb{D})$ generate a semiflow which extends analytically to the sector $\Sigma_{\alpha\frac{\pi}{2}}$ for some $\alpha \in (0,1)$. Then $|\arg F| \leq (1-\alpha)\frac{\pi}{2}$ $(z \in \mathbb{D})$, where F is the holomorphic function with positive real part from the Berkson-Porta representation.*

Proof. Let $b \in \bar{\mathbb{D}}$ be the Denjoy-Wolff point of the semiflow generated by G. For every $|\zeta| < \alpha$ there exists a holomorphic function F_ζ with positive real part such that generator of $(\varphi_{te^{i\zeta\frac{\pi}{2}}})_{t\geq 0}$ is given by $G_\zeta(z) = (\bar{b} - z)(1 - bz)F_\zeta(z)$. Since $F_\zeta(z) = e^{i\zeta}F(z)$, where $F = F_0$, and F_ζ and F have positive real part, $|\arg F| \leq (1-\alpha)\frac{\pi}{2}$. □

Note that the assertion of Lemma 3.2.5 appears also in [26, Thm. 5.2] by Bracci et al. in terms of filtrations of infinitesimal generators in $\mathscr{G}_0(\mathbb{D})$. For the case of a Denjoy-Wolff point on the boundary see Theorem 3.1.7 above which is a conformally equivalent version of [42, Thm. 4.3]. Furthermore in both cases the converse holds also true. We use this for the following proposition.

3.2.6 Proposition. *Let $a : \mathbb{D} \to \mathbb{C}$ such that $a(\mathbb{D}) \subseteq \Sigma_{\alpha\frac{\pi}{2}}$, $\alpha \in (0,1)$, where $\Sigma_{\alpha\frac{\pi}{2}}$ is maximal in the sense that there is no smaller sector that contains $a(\mathbb{D})$. Let $G \in \mathscr{G}_b(\mathbb{D})$. Then $a \cdot G \in \mathscr{G}$ if and only if the semiflow $(\varphi_t)_{t\geq 0}$ generated by G extends analytically to $\Sigma_{\alpha\frac{\pi}{2}}$.*

Proof. We can represent G as usual by $G(z) = (\bar{b} - z)(1 - bz)F(z)$ with F being a holomorphic function with positive real part and $b \in \bar{\mathbb{D}}$ the Denjoy-Wolff point of the semiflow generated by G.

If G generates a semiflow which extends analytically to $\Sigma_{\alpha\frac{\pi}{2}}$, then, by Lemma 3.2.5, $|\arg a \cdot F| \leq |(1-\alpha) + \alpha|\frac{\pi}{2} = \frac{\pi}{2}$. Thus $a \cdot G$ admits a Berkson-Porta representation, and hence generates a semiflow in $\mathscr{H}(\mathbb{D}, \mathbb{D})$.

Conversely, we have $|\arg a \cdot F| \leq \frac{\pi}{2}$ and thus $|\arg F| \leq (1-\alpha)\frac{\pi}{2}$. Then, by [26, Thm. 5.2] and [42, Thm 4.3], G generates a semiflow which is analytic in $\Sigma_{\alpha\frac{\pi}{2}}$. □

Now, we define a family of functions which serves as a substitute of a semiflow in our approximation result. Let $a \in \mathscr{H}(\mathbb{D})$ be an admissible perturbation of G generating the semiflow $(\varphi_t)_{t\geq 0}$. Subsequently, we show that the famliy $\varphi_{ta}(z) := \varphi(t \cdot a(z), z)$, $z \in \mathbb{D}$, consists of holomorphic selfmaps of $\mathbb{D}$ for all $t > 0$. As indicated in the beginning of this chapter, the class of semiflows in $\mathscr{H}(\mathbb{D}, \mathbb{D})$ divides in two subclasses according to the position of the Denjoy-Wolff point. Using the representation formulas (3.2) and (3.3), we show that for every admissible perturbation a of a generator $G \in \mathscr{G}$, we always obtain a family of holomorphic selfmaps $(\varphi_{ta})_{t\geq 0}$. We begin with the case of an inner Denjoy-Wolff point.

Perturbation of generators in $\mathscr{G}_0$. Let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathscr{H}(\mathbb{D}, \mathbb{D})$ with inner Denjoy-Wolff point in 0. Then we denote by $h : \mathbb{D} \to \mathbb{C}$ the associated Koenigs function, that is, the (univalent) λ-spirallike function such that $h(\varphi_t(z)) = e^{-tc}h(z)$, where $c = e^{i\lambda\frac{\pi}{2}}$ for some $0 < \lambda < 1$, see (3.2). Let $a : \mathbb{D} \to \mathbb{C}$ be holomorphic with $\operatorname{Re} a > 0$. To obtain a family of holomorphic selfmaps, we need to check if $e^{-cta(z)}h(z) \in h(\mathbb{D})$ for all $t \geq 0$ and all $z \in \mathbb{D}$.

Geometrically this perturbation acts as follows. Let $z \in \mathbb{D}$. A point on the logarithmic spiral $\gamma(t) = e^{-ct}h(z)$ moves along the spiral in the positive sense if $\operatorname{Re} a(z) \geq 1$ or in the negative sense if $\operatorname{Re} a(z) < 1$. Then it rotates around the origin by $t(\operatorname{Re} c \operatorname{Im} a(z) + \operatorname{Im} c \operatorname{Re} a(z)) \mod 2\pi$ and it is translated toward or away from the origin depending on the sign of $\operatorname{Im} c \cdot \operatorname{Im} a(z)$. So we need to figure out if after these operations the point lies still in the image of h. From the geometric interpretation we obtain the following example.

Example. In general, we can barely hope that for a general function a the above geometric condition is satisfied. However a special property of (normalized) univalent functions, Koebe's one-quarter theorem [50, Cor. 4.8], states that there is always a disk of radius less than $\frac{1}{4}$ contained in $h(\mathbb{D})$. Thus, for c and a satisfying one of the conditions

1. $c \in \mathbb{R}_{>0}$,
2. $\operatorname{Im} c < 0$ and $a(\mathbb{D}) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z > 0 \wedge \operatorname{Im} z \geq 0\}$, or
3. $\operatorname{Im} c > 0$ and $a(\mathbb{D}) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z > 0 \wedge \operatorname{Im} z \leq 0\}$

and t large enough, $e^{-cta(z)}h(z) \in h(\mathbb{D})$ for all $z \in \mathbb{D}$. If $h(\mathbb{D})$ is a disk centered at the origin, then $e^{-cta(z)}h(z) \in h(\mathbb{D})$ for all $t \geq 0$ and all $z \in \mathbb{D}$. This is caused by the fact that since h is here actually starlike, $c \in \mathbb{R}_{\geq 0}$, see [70, Prop. 5.2.4].

Fortunately, using admissible perturbations, we obtain holomorphic selfmaps as desired.

3.2.7 Lemma. *Let $\alpha \in (0,1)$ and $a \in \mathcal{H}(\mathbb{D})$ such that $a(\mathbb{D}) \subseteq \Sigma_{\alpha\frac{\pi}{2}}$. Let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathcal{H}(\mathbb{D},\mathbb{D})$ generated by $G \in \mathcal{G}_0$ which extends analytically to $\Sigma_{\alpha\frac{\pi}{2}}$. Then $(\varphi_{ta})_{t\geq 0}$ is a family of holomorphic selfmaps.*

Proof. Denote by h the Koenigs function of $(\varphi_t)_{t\geq 0}$. Then, there exists a $c \in \mathbb{C}$ with $\operatorname{Re} c \geq 0$ such that $(\varphi_t)_{t\geq 0}$ is given by $\varphi_t(z) := h^{-1}(e^{-ct}h(z))\,(z \in \mathbb{D})$. Since the Denjoy-Wolff point of this semiflow is 0, by the Berkson-Porta representation, the generator of $(\varphi_t)_{t\geq 0}$ is given by $G(z) = -zF(z)$ where F is a holomorphic function with positive real part and $F(0) = c$. There is the following representation of h, see the proof of [41, Theorem 1]:

$$h(z) = z\exp\left(\int_0^z \frac{F(0)-F(\xi)}{\xi F(\xi)}\,\mathrm{d}\xi\right). \tag{3.4}$$

For $|\zeta| < \alpha$, the family $(\varphi_t^\zeta)_{t\geq 0}$ given by $\varphi_t^\zeta := \varphi_{e^{i\zeta\frac{\pi}{2}}t}$ is a semiflow in $\mathcal{H}(\mathbb{D},\mathbb{D})$ with Denjoy-Wolff point in 0 since $(\varphi_t)_{t\geq 0}$ extends analytically to $\Sigma_{\alpha\frac{\pi}{2}}$. Analogously as above we find a holomorphic function F_ζ with positive real part such that the generator of $(\varphi_t^\zeta)_{t\geq 0}$ is given by $G_\zeta(z) = -zF_\zeta(z)$. For an analytic semiflow $(\varphi_s)_{s\in\Sigma_{\alpha\frac{\pi}{2}}}$ the generator of the semiflow restricted to a ray $\gamma(t) = e^{i\zeta\frac{\pi}{2}}t$, $t \geq 0$, is obviously just the rotation of the generator of the semiflow $(\varphi_t)_{t\geq 0}$ by $\zeta\frac{\pi}{2}$. Thus, $F_\zeta(z) = e^{i\zeta\frac{\pi}{2}}F(z)$. Moreover, there exists a univalent function h_ζ such that

$$\varphi_t^\zeta(z) = h_\zeta^{-1}(e^{-te^{i\zeta\frac{\pi}{2}}c}h_\zeta(z)) \quad (z \in \mathbb{D}). \tag{3.5}$$

Using the representation formula (3.4), we obtain $h_\zeta = h$ for all $|\zeta| < \alpha$. Thus, for all $s \in \Sigma_{\alpha\frac{\pi}{2}}$,

$$\varphi_s(z) = h^{-1}(e^{-cs}h(z))\,(z \in \mathbb{D}).$$

Since $a(\mathbb{D}) \subseteq \Sigma_{\alpha\frac{\pi}{2}}$, we have $a(z)\cdot t \in \Sigma_{\alpha\frac{\pi}{2}}$ for all $t \geq 0$ and $z \in \mathbb{D}$, and thus $e^{-a(z)t}h(z) \in h(\mathbb{D})$ for all $t \geq 0$ and $z \in \mathbb{D}$. □

In view of Theorem 3.1.3, we introduce the following subset of univalent functions

$$S^{\circledcirc}_{\lambda,\Sigma_{\alpha\frac{\pi}{2}}}(\mathbb{D}) := \{h \in S^{\circledcirc}_\lambda(\mathbb{D}) \,|\, \exp\left(-e^{i(\theta+\lambda)\frac{\pi}{2}}t\right)h(\mathbb{D}) \subseteq h(\mathbb{D}),\ -\alpha < \theta < \alpha,\ t > 0\}.$$

A semiflow $(\varphi_t)_{t\geq 0}$ in $\mathcal{H}(\mathbb{D},\mathbb{D})$ fixing the origin extends analytically to $\Sigma_{\alpha\frac{\pi}{2}}$ if and only if $h \in S^{\circledcirc}_{\lambda,\Sigma_{\alpha\frac{\pi}{2}}}$, where h is the associated Koenigs function. So for a generator G of a semiflow fixing the origin and with Koenigs function $h \in S^{\circledcirc}_{\lambda,\Sigma_{\alpha\frac{\pi}{2}}}(\mathbb{D})$ any holomorphic function $a : \mathbb{D} \to \mathbb{C}$ with $a(\mathbb{D}) \subseteq \Sigma_{\alpha\frac{\pi}{2}}$ is an admissible perturbation for G.

Perturbation of semiflows with Denjoy-Wolff point in 1. Let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathscr{H}(\mathbb{D},\mathbb{D})$ with Denjoy-Wolff point 1. Then we denote by $h:\mathbb{D}\to\mathbb{C}$ the associated Koenigs function, that is, the (univalent) close-to-convex function such that $h(\varphi_t(z)) = h(z)+t$. Then, to prove that $(\varphi_{ta})_{t\geq 0}$ forms a family of holomorphic selfmaps for $a:\mathbb{D}\to\mathbb{C}$ holomorphic with $\operatorname{Re} a > 0$, we need to check if $h(z)+a(z)\cdot t\in h(\mathbb{D})$ for all $t\geq 0$ and $z\in\mathbb{D}$. For an admissible perturbation of the generator of $(\varphi_t)_{t\geq 0}$, the Koenigs function h is convex in the sector $\Sigma_{\alpha\frac{\pi}{2}}$ which is, by Theorem 3.1.7, equivalent to the analytic extendability of the semiflow to the sector $\Sigma_{\alpha\frac{\pi}{2}}$.

3.2.8 Lemma. *Consider $G\in\mathscr{G}_1$ generating a semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{D},\mathbb{D})$ with Denjoy-Wolff point in 1 and Koenigs function h. Suppose $(\varphi_t)_{t\geq 0}$ extends analytically to some sector $\Sigma_{\alpha\frac{\pi}{2}}$ for $\alpha\in(0,1)$. Then $(\varphi_{ta})_{t\geq 0}$ is a family of holomorphic selfmaps in $\mathscr{H}(\mathbb{D},\mathbb{D})$ for any admissible perturbation $a\in\mathscr{H}(\mathbb{D})$ such that $a(\mathbb{D})\subseteq\Sigma_{\alpha\frac{\pi}{2}}$.*

Proof. The claim follows directly from [42, Thm. 3.1] which shows that $\varphi_s = h^{-1}(h+s)$ for all $s\in\Sigma_{\alpha\frac{\pi}{2}}$. □

We emphasize that for an admissible perturbation a of $G\in\mathscr{G}$ the family $(\varphi_{ta})_{t\geq 0}$ is not a semiflow since the semigroup property fails, but we still derive differentiability in $t=0$.

3.2.9 Lemma. *Let a be an admissible perturbation of $G\in\mathscr{G}$ generating the semiflow $(\varphi_t)_{t\geq 0}$. Then $(\varphi_{ta})_{t\geq 0}$ is differentiable in $t=0$ for all $z\in\mathbb{D}$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{at}(z)\Big|_{t=0} = a(z)G(z).$$

Proof. The proof follows from the representation formulas (3.2) and (3.3). Differentiating these equations in $t=0$, we derive

$$G = -c\frac{h}{h'} \text{ and } G = \frac{1}{h'}$$

for a Denjoy-Wolff point of $(\varphi_t)_{t\geq 0}$ in 0 or 1, respectively. Let $G\in\mathscr{G}_0$ and h the associated Koenigs function (which is λ-spirallike, say, and denote $c := e^{i\lambda\frac{\pi}{2}}$) of $(\varphi_t)_{t\geq 0}$, then for $z\in\mathbb{D}$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{ta}(z)|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}h^{-1}\left(e^{-cta(z)}h(z)\right)|_{t=0}\\ &= -c\frac{h(z)}{h'(z)}a(z) = a(z)\cdot G(z),\end{aligned}$$

by the chain rule. Analogously for $G\in\mathscr{G}_0$ and h the close-to-convex Koenigs function of $(\varphi_t)_{t\geq 0}$, we derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{ta}(z)|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}h^{-1}\left(h(z)+ta(z)\right)|_{t=0}\\ &= \frac{1}{h'}a(z) = a(z)\cdot G(z).\end{aligned}$$

□

3.2.10 Remark. Assume $a \cdot G$ generates a semiflow. Using the Berkson-Porta representation of a generator G, i.e., $G = (\bar{b} - z)(1 - bz)F(z)$, we calculate the Koenigs function $\tilde{h}$ associated to the semiflow generated by $a \cdot G$. For a Denjoy-Wolff point in 0, we use the representation of the Koenigs function given in the proof of [41, Thm. 1] which already appeared above (3.4) :

$$\begin{aligned}\tilde{h}(z) &= z \exp\left(\int_0^z \frac{a(0)F(0) - F(\xi)a(\xi)}{\xi a(\xi)F(\xi)} \,\mathrm{d}\xi \right) \\ &= z \exp\left(\int_0^z \frac{1 - F(\xi)}{\xi F(\xi)} \,\mathrm{d}\xi + \int_0^z \frac{a(0) - a(\xi)}{\xi a(\xi)F(\xi)} \,\mathrm{d}\xi \right) \\ &= h(z) \exp\left(\int_0^z \frac{a(0) - a(\xi)}{\xi a(\xi)F(\xi)} \,\mathrm{d}\xi \right).\end{aligned}$$

Analogously, using now the representation formula from the proof of [41, Thm. 2], we obtain for a Denjoy-Wolff point in 1:

$$\tilde{h}(z) = \int_0^z \frac{a(0)F(0)}{(1-\zeta)^2 a(\zeta)F(\zeta)} \,\mathrm{d}\zeta.$$

3.2.11 Remark. As usual, by the Riemann mapping theorem, a semiflow in $\mathscr{H}(\Omega, \Omega)$ is similar to a semiflow in $\mathscr{H}(\mathbb{D}, \mathbb{D})$ and we can write the Koenigs function and the generator in terms of the Koenigs function and the generator of the similar semiflow in $\mathscr{H}(\mathbb{D}, \mathbb{D})$ and a conformal map. It easily follows that the results of the previous sections are also true for any simply connected domain $\Omega \subsetneq \mathbb{C}$.

In what follows we denote by $\mathscr{G}_{b, \Sigma_{\alpha \frac{\pi}{2}}}(\Omega)$ the set of holomorphic functions on Ω which generate a semiflow in $\mathscr{H}(\Omega, \Omega)$ with Denjoy-Wolff point $b \in \bar{\Omega}$ that extends analytically to $\Sigma_{\alpha \frac{\pi}{2}}$.

In our approximation results Theorem 3.3.2 and 3.3.6 we use such families as a substitute for a semiflow.

3.2.12 Definition. Let $a \in \mathscr{H}(\Omega)$ be an admissible perturbation of G, generating the semiflow $(\varphi_t)_{t \geq 0}$. Then we call the family $(\varphi_{ta})_{t \geq 0}$ defined by $\varphi_{ta}(z) := \varphi(t \cdot a(z), z)$, $z \in \Omega$, $t \geq 0$, a semiflow approximating family.

3.3. Perturbation result

The idea of approximating the semigroup generated by a multiplicative perturbation of a Dirichlet-to-Neumann operator by (the trace of) composition operators is based on the following simple observation: suppose that $(\varphi_t)_{t \geq 0}$ is a semiflow in $\mathscr{H}(\Omega, \Omega)$ generated by G and let $X \hookrightarrow \mathscr{H}(\Omega)$ be a (G)-admissible Banach space. Then for all $f \in X$ such that $G \cdot f' \in X$ we obtain the following multiplicative perturbation of the generator Γ of a semigroup of composition operators

$$\begin{aligned}\lim_{t\to 0} \frac{f\circ\varphi_{ta} - f}{t} &= \lim_{t\to 0} a\frac{f\circ\varphi_{ta} - f}{ta} \\ &= a\cdot G\cdot f' =: M_a\Gamma f.\end{aligned}$$

For our main result, we need to show that $M_a\Gamma$ is the generator of a C_0-semigroup on X. To this end, we recall the following theorem on perturbations of generators of analytic semigroups due to Jung [54].

3.3.1 Theorem ([54, Theorem 2.2]). *Let Γ be the generator of a C_0-semigroup $(T_t)_{t\geq 0}$ analytic in some sector $\Sigma_{\alpha\frac{\pi}{2}}$. Let M be a bounded operator such that $\sigma(M) \subseteq \Sigma_{\alpha\frac{\pi}{2}}$ and γ a curve contained in $\Sigma_{\alpha\frac{\pi}{2}}$ that surrounds $\sigma(M)$. If all powers of*

$$\mathscr{F}(t) = \frac{1}{2\pi i}\int_\gamma (\lambda - M)^{-1} T(\lambda t)\,\mathrm{d}\lambda \tag{3.6}$$

are bounded by $C \geq 1$ in the operator norm for all $t \geq 0$, then $M\Gamma$ generates a bounded semigroup.

3.3.1. Approximation of a multiplicatively perturbed Dirichlet-to-Neumann semigroup

As mentioned in the introduction to this chapter, we aim to apply Chernoff's formula to get an approximation for the semigroup generated by a multiplicative perturbation of a Dirichlet-to-Neumann operator. Indeed, one can approximate the semigroup obtained by Jung's theorem using Chernoff's formula. With these tools in hand we are ready to state and prove our main result of this chapter.

3.3.2 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a Jordan domain and let $\alpha \in (0,1)$. Let $X \hookrightarrow \mathscr{H}(\Omega)$ be a (G)-admissible Banach space with $G \in \mathscr{G}_{b,\Sigma_{\alpha\frac{\pi}{2}}}(\Omega)$, such that every composition operator induced by a symbol fixing $b \in \bar{\Omega}$ is a contraction on X and X possesses a boundary space ∂X. Let $a : \Omega \to \mathbb{C}$ be a holomorphic function with $a(\Omega) \subseteq \Sigma_{\alpha\frac{\pi}{2}}$, and assume $M_a \in \mathscr{L}(X)$. Then*

$$\begin{cases} \partial_t u - a\cdot\langle G, \nabla u\rangle = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ -a\Delta u = 0 & \text{on } (0,\infty)\times\Omega, \\ u(0,\cdot) = f & \text{on } \partial\Omega, \end{cases}$$

is well-posed in ∂X and the semigroup generated by the multiplicative perturbation of the Dirichlet-to-Neumann operator $M_a\mathfrak{D}_G$ can be approximated by composition operators.

Proof. The following diagram shows that $M_a\mathfrak{D}_G$ is similar to the operator $M_a\Gamma$, where Γ is the generator of the semigroup of composition operators associated with the semiflow $(\varphi_t)_{t\geq 0}$ generated by G on X,

$$\begin{array}{ccc} \operatorname{dom} A \subseteq \partial X & \xrightarrow{M_a \mathfrak{D}_G} & \partial X \\ \uparrow \mathrm{Tr} & & \uparrow \mathrm{Tr} \\ \operatorname{dom} \Gamma \subseteq X & \xrightarrow{M_a \Gamma} & X. \end{array}$$

Using Theorem 3.3.1, we prove well-posedness by showing that $M_a\Gamma$ is the generator of a semigroup. It is immediate that the family of operators defined by (3.6) satisfies the assumptions of Chernoff's formula, Theorem 3.0.1.
The semigroup $(C_{\varphi_t})_{t\geq 0}$ generated by Γ extends analytically to $\Sigma_{\alpha\frac{\pi}{2}}$ by our assumption on the analytic extendability of $(\varphi_t)_{t\geq 0}$, and since M_a is a bounded multiplication operator on X, we have $\sigma(M_a) \subseteq a(\Omega) \subseteq \Sigma_{\alpha\frac{\pi}{2}}$. Let γ be any closed curve in $\Sigma_{\alpha\frac{\pi}{2}}$ surrounding $\sigma(M_a)$. Then, for $f \in X$,

$$\begin{aligned} \mathscr{F}(t)f &= \frac{1}{2\pi i} \int_\gamma (\lambda - M_a)^{-1} T(\lambda t) f \, \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f \circ \varphi_{\lambda t}}{\lambda - a} \, \mathrm{d}\lambda \\ &= f \circ \varphi_{ta}(\cdot) \\ &= C_{\varphi_{ta}} f =: T_a(t) f. \end{aligned}$$

The family $(\varphi_{ta})_{t\geq 0}$ has b as common fixed point. Thus, by assumption, all powers of $(T_a(t))_{t\geq 0}$ are uniformly bounded by 1 for all $t \geq 0$. So we obtain that $M_a\Gamma$ is the generator of a bounded C_0-semigroup $(C_t)_{t\geq 0}$ which can be approximated by Chernoff's formula and the trace of which is the semigroup generated by the multiplicatively perturbed Dirichlet-to-Neumann operator $M_a\mathfrak{D}_G$. □

We give some examples of Banach spaces which fit into the setting of our main theorem: Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain with $0 \in \Omega$. Let $p \in [1, \infty)$. If $\Omega = \mathbb{D}$, then the Hardy space $\mathscr{H}^p(\mathbb{D})$ and the Bergman space $\mathscr{A}^p(\mathbb{D})$ are spaces on which every composition operator induced by a symbol fixing the origin is a contraction. This is due to Littlewood's subordination principle, Proposition 1.3.8 and Corollary 1.3.9. When the norm is appropriately defined, the Hardy spaces serve also as an example if Ω is a Jordan domain, see [37, Ch. 10]. So we obtain well-posedness for (3.1) on $\partial\mathscr{H}^p(\Omega) \subseteq L^p(\partial\Omega)$ (for any Jordan domain $\Omega \subsetneq \mathbb{C}$) and $\partial\mathscr{A}^p(\mathbb{D}) \subseteq (W^{1,q}(\partial\mathbb{D}))'$, where q is the conjugate exponent to p. However, this is obvious since by Propositions 1.3.10 and 1.3.12 any admissible perturbation induces a semigroup generator and Hardy and Bergman spaces are admissible for any possible generator, and then the claim follows from our Theorem 2.3.8. However, we derive a nice connection to the notion of infinite divisibility which also gives an idea of the approximating property of such a family of holomorphic functions introduced in Definition 3.2.12.

3.3.2. An application to infinite divisibility of holomorphic selfmaps

Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain. Holomorphic selfmaps $\varphi \in \mathscr{H}(\Omega,\Omega)$ embeddable into a semiflow are known to be infinitely divisible, i.e., there exists a function $\mu_n \in \mathscr{H}(\Omega,\Omega)$ for any $n \in \mathbb{N}$ such that

$$\varphi = \underbrace{\mu_n \circ \cdots \circ \mu_n}_{n\text{-fold}}.$$

Let $\mathbb{C}^+$ denote the upper half plane. A result due to Anshelevich and Williams [6, Cor. 1.2] states that given an infinitely divisible function $\varphi \in \mathscr{H}(\mathbb{C}^+,\mathbb{C}^+)$ fixing $i\infty$, i.e., the conformally equivalent selfmap in the unit disk fixes 1, a fixed set $\{\mu_n : n \in \mathbb{N}\}$, and an increasing sequence $(k_n)_{n\in\mathbb{N}}$, then

$$\underbrace{\mu_n \circ \cdots \circ \mu_n}_{k_n\text{-fold}} \to \varphi \iff k_n(\mu_n(z) - z) \to G \text{ locally uniformly,} \tag{3.7}$$

where G is the generator of the semiflow in which φ embeds. This result occurs as corollary to a limit theorem in non-commutative probability, namely a limit theorem concerning monotone independence. Anshelevich and Williams use a sophisticated method for their proof involving a converse of Chernoff's formula.
Let $G \in \mathscr{G}_0$ be the generator of a semiflow $(\varphi_t)_{t\geq 0}$ with Denjoy-Wolff point in 0, which extends analytically to some sector, and let a be an admissible perturbation of G. With our approach, we can easily construct a set $\{\mu_n : n \in \mathbb{N}\}$ using approximating semiflows to find a divisibility set in the above sense (3.7) for a function ψ embeddable into the semiflow generated by $a \cdot G$.

3.3.3 Corollary. *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain, and let $p \in [1,\infty)$. Suppose $a \in \mathscr{H}(\Omega)$ is such that $M_a \in \mathscr{L}(\mathscr{H}^p(\Omega))$ and an admissible perturbation of $G \in \mathscr{G}_0$ which generates the semiflow $(\varphi_t)_{t\geq 0}$. Then for all $t \geq 0$*

$$\underbrace{\varphi_{\frac{t}{n}a} \circ \cdots \circ \varphi_{\frac{t}{n}a}}_{n\text{-fold}} \to \psi_t \quad (n \to \infty) \tag{3.8}$$

where $(\psi_t)_{t\geq 0}$ is the semiflow generated by $a \cdot G$.

Proof. The space $\mathscr{H}^p(\Omega)$ is admissible for every possible generator in $\mathscr{G}$ (see Proposition 1.3.10), so in particular for G and $a \cdot G$. Furthermore, since $G \in \mathscr{G}_0$, the semiflow $(\varphi_t)_{t\geq 0}$ fixes 0, hence $\varphi_{ta}(0) = 0$ for all $t \geq 0$. Thus, by Littlewood's subordination principle (Proposition 1.3.8 and Corollary 1.3.9), the family of operators defined by $(T_a(t))_{t\geq 0} := (C_{\varphi_{ta}})_{t\geq 0}$ consists of contractions on $\mathscr{H}^p(\Omega)$. From Theorem 3.3.2 we know that $(T_a(t))_{t\geq 0}$ approximates via Chernoff's formula the solution to (3.1). But, in fact, by $(a \cdot G)$-admissibility of X, the (unique) solution to (3.1) is given by $(C_{\psi_t})_{t\geq 0}$ by Theorem 2.3.8. Therefore, we derive for all $f \in \mathscr{H}^p(\Omega)$

$$\left\| \left[C_{\varphi_{\frac{t}{n}a}} \right]^n f - C_{\psi_t} f \right\|_{\mathscr{H}^p(\Omega)} \to 0 \quad (n \to \infty).$$

In particular this holds true for the identity function. Furthermore, point-evaluation is a bounded linear functional on X and so we have for all $z \in \Omega$ and $t \geq 0$

$$\underbrace{\varphi_{\frac{t}{n}a} \circ \cdots \circ \varphi_{\frac{t}{n}a}}_{n\text{-fold}}(z) \to \psi_t(z) \quad (n \to \infty).$$

□

This corollary holds also true if $\mathcal{H}^p(\Omega)$ is replaced by $\mathcal{A}^p(\mathbb{D})$. As an example, we deduce that $[e^{-\frac{t}{n}(1-z)}z]^n \to \frac{e^{-t}z}{(e^{-t}-1)z+1}$ as $n \to \infty$ since $a(z) := (1-z)$ is and admissible perturbation for $G(z) := -z$ and satisfies the conditions of the former corollary.

Moreover, infinite divisibility gives for appropriate functions a a sufficient condition such that the semigroup solving (3.1) is actually similar to a semigroup of composition operators.

3.3.4 Theorem. *Suppose in addition to the assumptions in Theorem 3.3.2 that the space X is a uniformly convex Banach space. Then X is $(a \cdot G)$-admissible if*

$$\left[\varphi_{\frac{t}{n}a}(z)\right]^n : = \underbrace{\varphi_{\frac{t}{n}a}(z) \circ \cdots \circ \varphi_{\frac{t}{n}a}}_{n\text{-fold}} \to \psi_t(z)\,(n \to \infty)\ \textit{for all } z \in \Omega$$

and

$$\limsup_{n\to\infty} \left\| \left[C_{\varphi_{\frac{t}{n}a}} \right]^n f \right\| \leq \left\| C_{\psi_t} f \right\| \textit{ for all } f \in X$$

where $(\psi_t)_{t\geq 0}$ denotes the semiflow generated by $a \cdot G$.

Proof. Let $f \in X$ and $t \geq 0$. There exists a weakly convergent subsequence of $(f \circ [\varphi_{\frac{t}{n}a}]^n)_n$ the limit of which is $f \circ \psi_t$ since

$$f \circ [\varphi_{\frac{t}{n}a}]^n(z) \to f \circ \psi_t(z)$$

for all $z \in \Omega$. Therefore $\|f \circ \psi_t\| \leq \liminf_{n\to\infty} \left\| f \circ [\varphi_{\frac{t}{n}a}]^n \right\|$. Thus we derive

$$\left\| \left[C_{\varphi_{\frac{t}{n}a}} \right]^n f - C_{\psi_t} f \right\| \to 0 \quad (n \to \infty)$$

for all $f \in X$ and $t \geq 0$. But the strong limit of $\left(\left[C_{\varphi_{\frac{t}{n}a}} \right]^n \right)_n$ is the (unique) solution to (3.1), so the solution is (the trace of) a semigroup of composition operators. In particular

$$\|f \circ \psi_t - f\| \to 0 \quad (t \to 0^+)$$

for all $f \in X$ by [29, Prop. 3.32], which proves $(a \cdot G)$-admissibility. □

This result appears to be useful in determining maximal subspaces of strong continuity. We present these applications in the fourth chapter. Note also that uniform convexity is not a necessary condition since the spaces $\mathcal{H}^1(\Omega)$ and $\mathcal{A}^1(\Omega)$ are $(a \cdot G)$-admissible. Next, we generalize our approximation approach to Dirichlet-to-Robin operators.

3.3.3. Approximation of a multiplicatively perturbed Dirichlet-to-Robin semigroup

To extend our perturbation result to evolution problems associated with a multiplicatively perturbed Dirichlet-to-Robin operator, we need to consider analytic semigroups of cocycle weighted composition operators, and thus the notion of cocycles has to be extended. Let $(\varphi_s)_{s\in\Sigma_{\alpha\frac{\pi}{2}}}$ be an analytic semiflow. We construct a cocycle associated with $(\varphi_s)_{s\in\Sigma_{\alpha\frac{\pi}{2}}}$ as follows: Let $s \in \Sigma_{\alpha\frac{\pi}{2}}$, and let $\gamma_s : [0, w_s] \to \mathbb{C}$ be any rectifiable curve in $\Sigma_{\alpha\frac{\pi}{2}}$ such that $\gamma_s(0) = 0$ and $\gamma_s(w_s) = s$. We define for $z \in \Omega$

$$m_s = \exp\left(\int_0^{w_s} g(\varphi_{\gamma_s})\,\mathrm{d}\gamma_s\right). \tag{3.9}$$

We can show that (3.9) is an analytic cocycle of the analytic semiflow $(\varphi_s)_{s\in\Sigma_{\alpha\frac{\pi}{2}}}$, i.e., it satisfies the cocycle properties from Definition 1.4.1 where the continuity condition (3) is replaced by $m_\cdot(\cdot)$ is analytic in $\Sigma_{\alpha\frac{\pi}{2}} \times \Omega$. First, we easily see that $m_0 = 1$. Moreover, the function $g \circ \varphi(\cdot, z) : \Sigma_{\alpha\frac{\pi}{2}} \to \mathbb{C}$ is holomorphic for all $z \in \Omega$ and thus $s \mapsto m_s(z)$ is analytic in $\Sigma_{\alpha\frac{\pi}{2}}$ for all $z \in \Omega$. For the cocycle property we consider the following curve: Let $s, t \in \Sigma_{\alpha\frac{\pi}{2}}$, then we define

$$\gamma_{s+t}(r) = \begin{cases} \gamma_t(r), & 0 \le r \le w_t, \\ \tilde{\gamma}_s(r) := \gamma_s(r - w_t) + t, & w_t \le r \le w_s + w_t =: w_{s+t}. \end{cases}$$

If we can show the cocycle property for this particular curve, then it holds for any curve joining 0 and $s + t$. Indeed,

$$\begin{aligned} m_{t+s} &= \exp\left(\int_0^{w_{t+s}} g(\varphi_{\gamma_{s+t}})\,\mathrm{d}\gamma_{s+t}\right) \\ &= \exp\left(\int_0^{w_t} g(\varphi_{\gamma_t})\,\mathrm{d}\gamma_t\right) \exp\left(\int_{w_t}^{w_s+w_t} g(\varphi_{\tilde{\gamma}_s})\,\mathrm{d}\tilde{\gamma}_s\right) \\ &= m_t \cdot \exp\left(\int_0^{w_s} g \circ \varphi_{\gamma_s}(\varphi_t)\,\mathrm{d}\tilde{\gamma}_s\right) \\ &= m_t \cdot (m_s(\varphi_t)). \end{aligned}$$

In analogy to Definition 3.2.12 we define a family of approximating cocycles:

3.3.5 Definition. Let $a \in \mathscr{H}(\Omega)$ with $a(\Omega) \subseteq \Sigma_{\alpha\frac{\pi}{2}}$ be an admissible perturbation of the function G which generates the semiflow $(\varphi_t)_{t\ge 0}$. We set $\gamma_{ta} = a \cdot t$ for $t \ge 0$, and define the following family of holomorphic functions

$$m_{ta} \colon = \exp\left(\int_0^{w_t} g(\varphi_{\gamma_{sa}})\,\mathrm{d}\gamma_{sa} \right). \tag{3.10}$$

We call the family $(m_{ta})_{t\geq 0}$ an approximating cocycle to $(\varphi_{ta})_{t\geq 0}$.

Note that $(m_{ta})_{t\geq 0}$ is not a cocycle since $(\varphi_{ta})_{t\geq 0}$ is not a semiflow. We obtain the following generalization of Theorem 3.3.2.

3.3.6 Theorem. *Assume in addition to the assumptions in Theorem 3.3.2 that there is a holomorphic function $g : \Omega \to \mathbb{C}$ with $\sup \operatorname{Re} g \leq 0$ such that X is (g, G)-admissible and $\|f \cdot h\|_X \leq \|h\|_\infty \|f\|_X$ for all $h \in \mathscr{H}^\infty$ and for all $f \in X$. Then the evolution problem*

$$\begin{cases} \partial_t u - a \cdot (g \cdot u + \langle G, \nabla u \rangle) = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ -a\Delta u = 0 & \text{on } (0,\infty) \times \Omega, \\ u(0,\cdot) = f & \text{on } \partial\Omega, \end{cases} \tag{3.11}$$

is well-posed and the semigroup generated by the multiplicative perturbation of the Dirichlet-to-Robin operator $\mathfrak{D}_{\mathscr{R}}$ can be approximated by a family of operators which are similar to weighted composition operators.

Proof. We note that the semigroup of weighted composition operators, generated by $\Gamma f = g \cdot f + \langle G, \nabla f \rangle$ $(f \in \operatorname{dom} \Gamma)$, extends to an analytic semigroup in $\Sigma_{\alpha \frac{\pi}{2}}$. Set $(T_a(t))_{t\geq 0} := (m_{ta} \cdot C_{\varphi_{ta}})_{t\geq 0}$, where m_{ta} is the approximating cocycle to the approximating semiflow $(\varphi_{ta})_{t\geq 0}$ as defined in (3.10). In particular, m_{ta} is a bounded holomorphic function in Ω for all $t \geq 0$, and furthermore $\sup_t \|m_{ta}\|_\infty \leq 1$ since $\sup \operatorname{Re} g \leq 0$ by assumption. Similar to the proof of Theorem 3.3.2, we calculate

$$\begin{aligned} \mathscr{F}(t)f &= \frac{1}{2\pi i} \int_\gamma \frac{m_{t\lambda} \cdot f \circ \varphi_{t\lambda}}{\lambda - a}\,\mathrm{d}t \\ &= m_{ta} \cdot f \circ \varphi_{ta}(\cdot) = T_a(t)f. \end{aligned}$$

Hence we obtain for all $f \in X$

$$\begin{aligned} \|\mathscr{F}(t)f\|_X \, &\left\| m_{ta} \cdot C_{\varphi_{ta}} f \right\|_X \\ &\leq \|m_{ta}\|_\infty \|f\|_X \leq \|f\|_X\,, \end{aligned}$$

and thus also all powers of $\mathscr{F}$ are bounded by 1 in the operator norm $X \to X$. Now the assertion follows analogously as in the proof of Theorem 3.3.2. □

3.3.7 Remark. Unfortunately, here we have no sufficient condition that the approximated semigroup is (the trace of) a semigroups of weighted composition operators. In the above theorem, we approximate by a sequence of powers the composition of a multiplication and a composition operator. So, when the space X is not $(a \cdot g, a \cdot G)$-admissible a priori, then we can barely hope that the limit from the Chernoff approximation formula is again a semigroup of weighted composition operators. Consider a generator G of a semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\Omega,\Omega)$ that extends analytically to some sector $\Sigma_{\alpha\frac{\pi}{2}}$. Let a be an admissible perturbation of G and let $g:\Omega\to\mathbb{C}$ be holomorphic such that $m_t = \exp\left(\int_0^t g(\varphi_s)\,\mathrm{d}s\right)$ is a cocycle to $(\varphi_t)_{t\geq 0}$. Then by definition $a\cdot G$ generates a semiflow $(\psi_t)_{t\geq 0}$ in $\mathscr{H}(\Omega,\Omega)$. Furthermore, the family of functions $(n_t)_{t\geq 0}$ defined by $n_t = \exp\left(\int_0^t (a\cdot g)(\psi_s)\,\mathrm{d}s\right)$ is a cocycle to $(\psi_t)_{t\geq 0}$. So, the semigroup $S_t := n_t C_{\psi_t}$ is our candidate for the semigroup generated by the multiplicative perturbation of the Dirichlet-to-Robin operator in the above problem (3.11), and it would be the solution to (3.11) if X was $(a\cdot g, a\cdot G)$-admissible, for instance, if X was a Hardy or Bergman space. It would be interesting to find conditions for analogue versions of Corollary 3.3.3 and Theorem 3.3.4.

4. Applications

In this chapter, we collect some possible applications of our theory presented in the last two chapters. These approaches are independent of each other and cover different areas to which our results contribute. First, we discuss an application of our approximation result from the previous chapter to maximal subspaces of strong continuity, which we already mentioned in this thesis. In a nutshell, these spaces are admissible subspaces of larger spaces which are not admissible themselves for a specific generator. In the last years, such subspaces have been studied on several spaces of holomorphic function. With our results we find a condition of when maximal subspaces are equal. In our second application, we look for an approximation of the Dirichlet-to-Neumann semigroup and more general Poincaré-Steklov on spaces of continuous functions on a Dini-smooth curve. This is based on our construction of an approximating family of operators from our perturbation results and a similar approximation formula presented in [45]. In the last section, we give a connection to probability theory, more precisely stochastic branching processes, using our construction of Dirichlet-to-Neumann and Dirichlet-to-Robin operators as traces of (weighted) composition semigroups. Actually we generalize a rather old result from [21] using the modern theory of composition semigroups.

4.1. Maximal subspaces of strong continuity

The notion of maximal subspaces of strong continuity has already been mentioned at the end of Chapter 1 where we emphasized that there are some spaces of holomorphic functions that are not admissible for every possible generator but which admit subspaces which are admissible for certain or even all possible semiflow generators. Typical examples in this direction are the Bloch space and BMOA on which no non-trivial semiflow induces a semigroup of composition operators. On the other hand, for every semiflow generator the small Bloch space and VMOA are admissible. (We refer to [22] for a definition of these spaces.) Moreover, one can find semiflows such that there are spaces strictly included between VMOA and BMOA (or small Bloch and Bloch) which are admissible.

In this section we recall some well-known general facts on the theory of maximal subspaces of strong continuity and recall some recent results, and we give some thoughts on how our approximation result from the previous chapter could make a contribution to this area.

4.1.1 Definition (Maximal subspaces of strong continuity). Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain, and let $X \hookrightarrow \mathscr{H}(\Omega)$ be a Banach space and G be the generator of the semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\Omega,\Omega)$. A subspace $[\varphi_t, X] \subseteq X$ is called maximal subspace of strong continuity if $[\varphi_t, X]$ is (G)-admissible and for any subspace $Y \subseteq X$ such that Y is (G)-admissible we have $Y \subseteq [\varphi_t, X]$.

The following theorem can be found in [23, Thm. 1] and gives a nice characterization of maximal subspaces of strong continuity. In fact, we already referred to this result in Section 2.4 in determining a subspace of the Lumer-Hardy space on the boundary space of which we constructed a Dirichlet-to-Robin operator on domains in $\mathbb{C}^n$.

4.1.2 Theorem. *Let $X \hookrightarrow \mathscr{H}(\mathbb{D})$ be a Banach space containing the constant functions, and let G be the generator of a semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{D},\mathbb{D})$. Suppose that $\sup_{t\geq 0} \left\|C_{\varphi_t}\right\|_X < \infty$. Then*

$$[\varphi_t, X] = \overline{\{f \in X : \langle G, \nabla f\rangle \in X\}}.$$

Maximal subspaces of strong continuity are studied mainly on BMOA and the Bloch space $\mathscr{B}$, and also on the disk algebra $\mathscr{A}(\mathbb{D})$ and $\mathscr{H}^\infty(\mathbb{D})$, see [5, 22, 23]. The papers [22, 23] can be considered as a starting point for the study of maximal subspaces of strong continuity of BMOA and $\mathscr{B}$. We recall that the Bloch space $\mathscr{B}$ is defined as the space of functions $f \in \mathscr{H}(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty.$$

In particular, the authors in loc. cit. observe that $[\varphi_t, \mathscr{B}] \subsetneq \mathscr{B}$ for every nontrivial semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{D},\mathbb{D})$, see [23, Thm. 3]. This result has been generalized to a more general class of Banach spaces of holomorphic functions including the space BMOA, see [5, Thm. 1].

4.1.3 Theorem. *Let $\mathscr{H}^\infty(\mathbb{D}) \subseteq X \subseteq \mathscr{B}$. Then $[\varphi_t, X] \subsetneq X$ for every nontrivial semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{D},\mathbb{D})$.*

Moreover, in the same paper, the following result is obtained, see [5, Cor. 1.3]:

4.1.4 Theorem. *Let $\mathscr{H}^\infty(\mathbb{D}) \subseteq X \subseteq \mathscr{H}(\mathbb{D})$ and denote by X_π the closure of the polynomials in X. Then $X_\pi \subseteq [\varphi_t, X]$ for any semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{D},\mathbb{D})$.*

This theorem gives in particular a unified proof for (G)-admissibility of the Hardy and Bergman spaces for any possible semiflow generator since the polynomials are dense in these spaces.

Now, it would be interesting to apply our result deduced from the approximation result Theorem 3.3.2 in the context of maximal subspaces of strong continuity. Translating to the language of subspaces of strong continuity and neglecting the connection to an evolution problem on the boundary space, our Theorem 3.3.4 can be rephrased as follows.

4.1.5 Theorem. *Consider a simply connected domain $\Omega \subsetneq \mathbb{C}$. Let $\alpha \in (0,1)$ and let $G \in \mathscr{G}_{b,\Sigma_{\frac{\pi}{2}}}$ generate the semiflow $(\varphi_t)_{t\geq 0}$. Assume $[\varphi_t, X] \subseteq X \hookrightarrow \mathscr{H}(\Omega)$ is a uniformly convex Banach space such that every composition operator induced by a symbol fixing $b \in \bar{\Omega}$ is a contraction. Let $a \in \mathscr{H}(\Omega)$ be an admissible perturbation of G with $M_a \in \mathscr{L}(X)$ such that*

$$\underbrace{\varphi_{\frac{t}{n}a} \circ \cdots \circ \varphi_{\frac{t}{n}a}}_{n\text{-fold}}(z) \to \psi_t(z)\,(n\to\infty) \text{ for all } z \in \Omega$$

and

$$\limsup_{n\to\infty} \left\| \left[C_{\varphi_{\frac{t}{n}a}} \right]^n f \right\|_X \leq \left\| C_{\psi_t} f \right\|_X \text{ for all } f \in [\varphi_t, X]$$

where $(\psi_t)_{t\geq 0}$ *denotes the semiflow generated by* $a \cdot G$*. Then* $[\varphi_t, X] = [\psi_t, X]$*.*

Unfortunately, the theorem above does not apply to maximal subspaces of strong continuity in BMOA and in the Bloch space since these spaces appear to be non-reflexive. We leave it as an open question if maximal subspaces of strong continuity of BMOA and the Bloch space are stable under admissible perturbations.

4.2. A generalization of an approximation formula

In this section, we extend our approximation approach from the third chapter to multiplicative perturbations by positive functions. Combined with our observation done in the end of Subsection 2.3.2 on the representation of the classical Dirichlet-to-Neumann operator $\mathfrak{D}_{\mathcal{N}}$ as the trace of a multiplicatively perturbed semigroup of composition operators on Dini-smooth domains, we deduce an approximation of the semigroup generated by $-\mathfrak{D}_{\mathcal{N}}$ by composition operators. Inspired by an approximation formula obtained by Emamirad and Laadnani [45], we prove a generalized approximation result for Dirichlet-to-Neumann and Dirichlet-to-Robin operators which are multiplicatively perturbed by a positive continuous function. The idea of the proof, which is mainly based on [45] and our approximation theorems from the third chapter, appears to be restricted to Dirichlet boundary values in $C(\partial\Omega)$. This space contains, as a closed subspace, the boundary space of the disk algebra. So we recall some facts about semigroups of composition operators on the disk algebra.

Semigroups of composition operators on the disk algebra. We recall that the disk algebra $\mathscr{A}(\mathbb{D})$ is defined as the space of holomorphic functions on $\mathbb{D}$ which admit a continuous extension to $\bar{\mathbb{D}}$. Semigroups of composition operators on $\mathscr{A}(\mathbb{D})$ have been studied in [32] and the main result [32, Thm. 1.2] can be rephrased as follows:

4.2.1 Theorem. *Let* $(\varphi_t)_{t\geq 0}$ *be a semiflow in* $\mathscr{H}(\mathbb{D},\mathbb{D})$ *generated by* G*. Then* $\mathscr{A}(\mathbb{D})$ *is* (G)*-admissible if and only if* $(\varphi_t)_{t\geq 0}$ *is a semiflow in* $\mathscr{A}(\mathbb{D}) \cap \mathscr{H}(\mathbb{D},\mathbb{D})$*.*

This result has been reproved in [5] in the context of maximal subspaces of strong continuity:

4.2.2 Theorem. *Let* $(\varphi_t)_{t\geq 0}$ *be a semiflow in* $\mathscr{H}(\mathbb{D},\mathbb{D})$ *generated by* G*. Then* $[\varphi_t, \mathscr{A}(\mathbb{D})] = \mathscr{A}(\mathbb{D})$ *if and only if* $(\varphi_t)_{t\geq 0}$ *is a semiflow in* $\mathscr{A}(\mathbb{D}) \cap \mathscr{H}(\mathbb{D},\mathbb{D})$*.*

We shall use this characterization to give a refined approach to our investigations in the end of Subsection 2.3.2 which we summarize here briefly for convenience. Let Ω be a Dini-smooth domain. We showed by using conformal mappings that the 'classical' Dirichlet-to-Neumann operator on a boundary space $L^p(\partial\Omega)$ is given by the trace of a multiplicative perturbation of a generator of a semigroup of composition operators. Analogously, this construction applies to the Dirichlet-to-Neumann operator on $C(\partial\mathbb{D})$. More precisely, let $u \in \mathfrak{h}(\Omega) \cap C(\bar{\Omega})$ be the solution to the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f \in C(\partial\Omega)$, and let $k : \Omega \to \mathbb{D}$ be a conformal map. Then, for $f \in \mathrm{dom}(\mathfrak{D}_{\mathcal{N}})$,

$$\begin{aligned} -\mathfrak{D}_{\mathcal{N}} f &= -\langle \nu, \nabla u \rangle \\ &= \mathrm{Tr}(\langle G, \nabla u \rangle |k'|), \end{aligned}$$

where G is the generator of a semiflow given by $G = -\frac{k}{k'}$. Indeed, $\mathscr{A}(\Omega)$ is admissible for G since the Dini-smoothness of Ω gives that k and k' extend continuously to $\bar{\Omega}$ where k' is also non-vanishing on $\bar{\Omega}$ by Theorem 1.2.4(3). From the representation formula (3.2), we obtain that the Koenigs function associated to the semiflow generated by G is given by k. We define for $t \geq 0$ the following family of functions:

$$\varphi_{t|k'|}(z) := k^{-1}\left(e^{-t|k'(z)|}k(z)\right) \quad (z \in \Omega). \tag{4.2}$$

This family is obviously not holomorphic and $|k'|$ is not an admissible perturbation of G in the sense of Definition 3.2.3. So, unfortunately, our approach illustrated in the third chapter does not cover this multiplicative perturbation of a Dirichlet-to-Neumann operator which would yield an approximation by composition operators for the (classical) Dirichlet-to-Neumann operator on a Dini-smooth domain which is not a disk. This is also in line with the optimality result concerning the representation of the Lax semigroup from [46]. Fortunately, a similar problem has already been considered in [45, Sect. 4], where the author also aims to construct an approximating family similar to the Lax semigroup for the Dirichlet-to-Neumann operator on $C(\partial\Omega)$ associated with the following evolution problem

$$\begin{cases} \partial_t u + \langle \nu, \gamma \nabla u \rangle = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ \mathrm{div}\, \gamma \nabla u = 0 & \text{on } (0,\infty) \times \Omega, \\ u(0,\cdot) = f & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where Ω is a domain in $\mathbb{R}^n$ which satisfies the property of the interior ball and $\gamma \in C^\infty(\bar{\Omega}, \mathbb{R}^{n\times n})$ represents electrical conductivity on Ω. The property of the interior ball means that for all $y \in \partial\Omega$ there is a tangent surface $\mathscr{T}_y$ and a ball tangent to $\mathscr{T}_y$ which is totally contained in Ω. So to each $y \in \partial\Omega$ we find $x_y \in \Omega$ which is the center of the biggest ball tangent to $\mathscr{T}_y$ of radius, say, $r_y > 0$. For each $0 < r \leq r_y$ the authors of [45] construct an approximating family for the ball centered at $x_{r,y} = \frac{r}{r_y} x_y + (1 - \frac{r}{r_y}) y$ with radius r using the following family of operators

$$V_r(t) f = \mathrm{Tr}\, u(x_{r,y} + e^{-\frac{t}{r}\gamma(y)} r\, \nu(y)), \tag{4.4}$$

where $f \in C(\partial\Omega)$ and u is the γ-harmonic lifting of f. Assuming that r_y is neither 0 nor ∞ for every $y \in \partial\Omega$, he showed by applying Chernoff's formula that (4.4) approximates the Dirichlet-to-Neumann semigroup solving (4.3), see [45, Thm 4.5]. Note that the author in [45] not only aims for a Lax semigroup on domains which are not disks but also for a generalization of the Lax semigroup to more general elliptic operators. So far, we avoided considering other elliptic operators than the Laplacian. Indeed, there is also a theory of composition operators related to so-called pseudoholomorphic functions but we were not able to fit them in our setting. We comment on this in the fifth chapter. So, inspired by the approximation formula obtained in [45], we can construct analogously an approximating family for evolution problems associated with a positive multiplicative perturbation of a Dirichlet-to-Neumann on a Dini-smooth domain with $\gamma = I$, the identity matrix.

4.2.3 Theorem. *Let $a : \bar{\Omega} \to \mathbb{R}$ be a bounded, positive, continuous function, and let $G \in \mathscr{G}_{b,\Sigma_{\alpha\frac{\pi}{2}}}(\Omega)$ for some $\alpha \in (0,1)$ and $b \in \bar{\Omega}$. Let $X \hookrightarrow \mathscr{H}(\Omega)$ be (G)-admissible with boundary space in $\partial X \subseteq C(\partial\Omega)$. Then*

$$\begin{cases} \partial_t u - a\langle G, \nabla u\rangle = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ -\Delta u = 0 & \text{on } (0,\infty)\times\Omega, \\ u(0,\cdot) = f & \text{on } \partial\Omega, \end{cases}$$

is well-posed in ∂X and the semigroup generated by the multiplicatively perturbed Dirichlet-to-Neumann operator can be approximated by traces of composition operators.

Proof. Let $(\varphi_t)_{t\geq 0}$ be the semiflow generated by G and denote by h the associated Koenigs-function. Without loss of generality, we can assume $b \in \{0,1\}$. Set $\varphi_{ta}(z) := \varphi(ta(z), z)$ for $z \in \Omega$ and $t \geq 0$. Here the perturbation a affects just the time parameter, and thus

$$e^{-tca(z)}h(z) \in h(\Omega) \quad \text{or} \quad h(z) + ta(z) \in h(\Omega)$$

for all $z \in \Omega$ and $t \geq 0$ if $(\varphi_t)_{t\geq 0}$ has Denjoy-Wolff point in 0 or 1, respectively. So φ_{ta} is a family of continuous selfmaps on Ω. Furthermore $(\varphi_{ta})_{t\geq 0}$ consists of continuous selfmaps extending continuously to the boundary $\partial\Omega$ by Theorem 4.2.2. The composition operator $C_{\varphi_{ta}}$ maps functions in $\mathscr{H}(\Omega) \cap C(\bar{\Omega})$ to functions in $C(\bar{\Omega})$. We define

$$V(t)\colon \partial X \to C(\partial\Omega),\, f \mapsto \operatorname{Tr} C_{\varphi_{ta}} u, \tag{4.5}$$

where u is the solution to Dirichlet problem (4.1) with $f \in \partial X$. More precisely, u is the Poisson integral of f, i.e., $u = Pf$. We only need to show that the family $V(t)$ satisfies Chernoff's formula for all $f \in C(\partial\Omega)$. First, we easily derive that $V(0) = I$. By the maximum principle $\|V(t)f\|_\infty \leq \|f\|_\infty$ for all $f \in \partial X$. This is the same argument as the one used in [45, Thm 4.5]. We are now left to show that $\lim_{t\to 0}\frac{\mathrm{d}}{\mathrm{d}t}V(t)f$ exists on a dense domain in $\partial X \subseteq C(\partial\Omega)$. We do this as in the proof of Theorem 3.3.2, and obtain by Jung's theorem, Theorem 3.3.1, that the limit $\lim_{t\to 0}\frac{\mathrm{d}}{\mathrm{d}t}V(t)f$ exists and is generator of a C_0-semigroup. In particular, $\lim_{t\to 0}\frac{\mathrm{d}}{\mathrm{d}t}V(t)f = \operatorname{Tr}(a\langle G, \nabla u\rangle) := M_a\mathfrak{D}_G$ for all $f \in \operatorname{dom} M_a\mathfrak{D}_G$, as desired. □

Now, we can not only approximate the classical Dirichlet-to-Neumann operator on a Dini-smooth domain by the trace of composition operators but also positive multiplicative perturbations of Poincaré-Steklov operators mapping continuous Dirichlet boundary conditions to outward pointing derivatives. Note also that the assumption in Theorem 3.3.2 on the contractivity of the composition operators is already satisfied by the maximum principle. Furthermore, just as in Theorem 3.3.6 we extend this observation to Dirichlet-to-Robin operators. Before we state and prove this result, we give a condition for Banach spaces of holomorphic functions to be (g,G)-admissible which applies in particular to the disk algebra:

4.2.4 Lemma. *Let $\Omega \subsetneq \mathbb{C}$ be simply connected and $(\varphi_t)_{t\geq 0}$ a semiflow in $\mathscr{H}(\Omega,\Omega)$ generated by G. Let $\mathscr{H}^\infty \subseteq X \hookrightarrow \mathscr{H}(\Omega)$ be a Banach $\mathscr{H}^\infty$-module and $g : \Omega \to \mathbb{C}$ a bounded holomorphic function such that $\|m_t - 1\|_\infty \to 0$ as $t \to 0^+$, where $m_t := \exp(\int_0^t g(\varphi_s))\,\mathrm{d}s$. Then X is (g,G)-admissible if and only if X is (G)-admissible.*

Proof. Suppose X is (G)-admissible. Then $\|m_t \cdot (f \circ \varphi_t)\|_X \leq C\,\|m_t\|_\infty \|f \circ \varphi_t\|_X$ for some $C > 0$ and

$$\begin{aligned}\|m_t \cdot (f \circ \varphi_t) - f\|_X &\leq \|m_t \cdot (f \circ \varphi_t - f)\|_X + \|f \cdot (m_t - 1)\|_X \\ &\leq C(\underbrace{\|f \circ \varphi_t - f\|_X + \|m_t - 1\|_\infty \|f\|_X}_{\to 0,\, t \to 0^+}).\end{aligned}$$

Conversely, $\|f \circ \varphi_t\|_X \leq C \left\|\frac{1}{m_t}\right\|_\infty \|m_t \cdot f \circ \varphi_t\|_X < \infty$ for some $C > 0$ and

$$\begin{aligned}\|f \circ \varphi_t - f\|_X &\leq C \left\|\frac{1}{m_t}\right\|_\infty \|m_t \cdot f \circ \varphi_t - m_t \cdot f\|_X \\ &\leq C(\underbrace{\|m_t \cdot (f \circ \varphi_t) - f\|_X + \|m_t - 1\|_\infty \|f\|_X}_{\to 0,\, t \to 0^+}).\end{aligned}$$

□

We finish this section with the following generalization of Theorem 4.2.3.

4.2.5 Theorem. *Let $a : \bar{\Omega} \to \mathbb{R}$ be a bounded, positive, continuous function, let $G \in \mathscr{G}_{b,\Sigma_{\alpha\frac{\pi}{2}}}(\Omega)$ for some $\alpha \in (0,1)$ and $b \in \bar{\Omega}$, and let $g : \Omega \to \mathbb{C}$ be holomorphic such that $\sup \operatorname{Re} g \leq 0$. Let $X \hookrightarrow \mathscr{H}(\Omega)$ be a (g,G)-admissible Banach space with boundary space in $\partial X \subseteq C(\partial\Omega)$. Then*

$$\begin{cases} \partial_t u - a(g \cdot u + \langle G, \nabla u\rangle) = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ -\Delta u = 0 & \text{on } (0,\infty) \times \Omega, \\ u(0,\cdot) = f & \text{on } \partial\Omega, \end{cases}$$

is well-posed in ∂X and the semigroup generated by the multiplicatively perturbed Dirichlet-to-Robin operator can be approximated by traces of weighted composition operators.

Proof. Again, we construct a family of operators (in this case weighted composition operators) which satisfy Chernoff's formula. We define φ_{ta} as in the proof of Theorem 4.2.3.

Then, let

$$m_{ta}(z) := \exp\left(\int_0^{ta(z)} g(\varphi_{sa}(z))\,\mathrm{d}s \right) \tag{4.6}$$

with $z \in \Omega$ and $t \geq 0$. So we deduce that $\|m_{ta}\|_\infty \leq 1$ since by assumption $\sup \operatorname{Re} g \leq 0$. We define the following family of traces of weighted composition operators

$$W(t)f :\, = \operatorname{Tr}\left(M_{m_{ta}} C_{\varphi_{ta}} u\right), \tag{4.7}$$

where u is the solution to Dirichlet problem (4.1) with $f \in \partial X$. Obviously, $W(0) = I$ and by means of the maximum principle and the upper bound of m_{ta}, we deduce $W(t)$ is a contraction on ∂X for every $t \geq 0$. Jung's theorem, Theorem 3.3.1, yields that the limit $\lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}t} W(t)f$ exists on a dense subspace of ∂X and is generator of a C_0-semigroup, and, as desired, $\lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}t} W(t)f = \operatorname{Tr}(a(g + \langle G, \nabla u \rangle))$. □

4.3. Branching processes

So far, we focussed our study on the interaction between semiflows, composition semigroups, and partial differential equations involving a Poincaré-Steklov operator. In this last section of our applications another object comes into play: stochastic branching processes. We give a representation of the semigroup generated by certain Dirichlet-to-Neumann and Dirichlet-to-Robin operators in terms of a semigroup deduced from a stochastic process with state space in $\mathbb{N}$. Our results are inspired by a note on the connection between semiflows of holomorphic functions and stochastic branching processes in the book [43, Section 5.2] by Elin and Shoikhet and a paper by Bharucha-Reid [21] published in 1965. We begin with a short introduction to stochastic branching processes. We note that a very elementary understanding of probability theory suffices to establish the results we aim to present.

Branching processes. Let $(A, \mathscr{F}, \mathbb{P})$ be a probability space and consider a discrete random variable Z with values in $\mathbb{N} := \{0, 1, 2, \dots\}$ with distribution $p_k := \mathbb{P}[Z = k]$, $k \in \mathbb{N}$. The generating function of Z is defined by

$$\varphi(z) := \mathbb{E}[z^Z] = \sum_{k\in\mathbb{N}} p_k z^k,\ z \in \mathbb{D}. \tag{4.8}$$

We note that the generating function is a holomorphic function on $\mathbb{D}$. Moreover, $\varphi \in \mathscr{H}(\mathbb{D}, \mathbb{D})$ since $\sum_k p_k = 1$.

A time homogeneous branching process, also known as Galton-Watson process, is a Markov chain $(Z(n))_n$ with state space $\mathbb{N}$ to a given transition probability law $p_{ij}(n) := \mathbb{P}[Z(n) = j | Z(0) = i]$, $i, j, n \in \mathbb{N}$, satisfying the Chapman-Kolmogorov equation, i.e.,

$$p_{ij}(n+1) = \sum_{k\in\mathbb{N}} p_{ik}(n) p_{kj}(1) \tag{4.9}$$

for all $i, j, n \in \mathbb{N}$, and such that $\sum_j p_{ij}(n) \le 1$ for all $i, n \in \mathbb{N}$. Such processes appear in various applications, for instance, in the context of extinction probabilities for a population. Actually the theory of branching processes was initiated by the question if aristocratic surnames will become extinct. We explain this model in more detail: let $(Z(n))_n$ be a homogeneous branching process with transition probability law $p_{ij},\ i, j \in \mathbb{N}$, and assume $Z(0) = s_0 \in \mathbb{N}$, that is, the initial generation consists of s_0 individuals. Each of these individuals produces independently offsprings in the next generation (i.e., after one time step). Suppose that the nth generation consists of s_n individuals, and denote by $Z^i(n)$ the random variable which corresponds to the ith individual of the nth generation in our model, $i \in \{1, \dots, s_n\}$. Then the random variable $Z(n+1)$, which models the number of individuals in the $(n+1)$th generation, is given by

$$Z(n+1) = \sum_{i=1}^{Z(n)} Z^i(n). \tag{4.10}$$

Next, we show that the generating functions of the individuals form a discrete semigroup in $\mathscr{H}(\mathbb{D}, \mathbb{D})$. We give some details on this observation based on [14, 43] here and refer the interested reader especially to [14, Ch. 1, Sect. 1] for a more involved introduction to this whole theory.

Let $(Z(n))_{n\in\mathbb{N}}$ be a homogeneous branching process with transition probabilities $p_{ij}(n)$, $i, j, n \in \mathbb{N}$. First we note that the generating function of the ith individual after the first time step $Z^i(1)$ can be written in terms of transition probabilities as follows

$$\varphi(z) := \mathbb{E}[z^{Z^i(1)}] = \sum_{k\in\mathbb{N}} p_{1k}(1) z^k.$$

Thus each individual in this generation has the same generating function. Then, by (4.10) and since the offsprings are i.i.d., we calculate the generating function of $Z(1)$ as follows

$$\mathbb{E}[z^{Z(1)} | Z(0) = s_0] = \mathbb{E}[z^{\sum_{i=1}^{s_0} Z^i(1)}] = \prod_{i=1}^{s_0} \mathbb{E}[z^{Z^i(1)}] = (\varphi(z))^{s_0}.$$

This yields, for a process starting with s_0 individuals,

$$\mathbb{E}[z^{Z(1)}] = \sum_{k\in\mathbb{N}} p_{s_0 k}(1) z^k = (\varphi(z))^{s_0}.$$

Using (4.9), we obtain

$$\begin{aligned}\varphi_{n+1}(z) = \mathbb{E}[z^{Z^1(n+1)}] &= \sum_j p_{1j}(n+1)z^j \\ &= \sum_j \sum_k p_{1k}(n)p_{kj}(1)z^j \\ &= \sum_k p_{1k}(n) \sum_j p_{kj}(n)z^j \\ &= \sum_k p_{1k}(n)(\varphi(z))^k \\ &= \varphi_n(\varphi(z)).\end{aligned}$$

Thus, by induction, φ_n is the n-fold iterate of φ.

Now we consider the time-continuous analogue to $(Z(n))_{n\in\mathbb{N}}$, namely a Markov branching process $(Z(t))_{t\geq 0}$.

4.3.1 Definition. A time continuous Markov branching process is a stochastic process $(Z(t))_{t\geq 0}$ with transition probabilities $p_{ij}(t) = \mathbb{P}[Z(t) = j|Z(0) = i]$ on a probability space $(A, \mathscr{F}, \mathbb{P})$ such that

1. $Z(t)$ is an $\mathbb{N}$-valued random variable for all $t \geq 0$,
2. $(Z(t))_{t\geq 0}$ is a stationary Markov chain with respect to $\mathscr{F}_t = \sigma\{Z(s)|s \leq t\}$, and, in particular, the transition probabilities satisfy the time-continuous Chapman-Kolmogorov equation
$$p_{ij}(t_1 + t_2) = \sum_{k\in\mathbb{N}} p_{ik}(t_1)p_{kj}(t_2) \quad (t_1, t_2 > 0; i, j \in \mathbb{N}), \tag{4.11}$$
3. $$\left(\sum_j p_{1j}(t)z^j\right)^k = \sum_j p_{kj}(t)z^j, \text{ and} \tag{4.12}$$
4. $\lim_{t\to 0^+} p_{ij}(t) = \delta_{ij}$.

Now we show that this definition guarantees that as in the time-discrete case, the generating function of an individual at time t is nothing but the tth (so the fractional) iterate of the generating function φ of an individual at time 1. In other words, φ embeds into a semiflow. Let $(Z(t))_{t\geq 0}$ be a time continuous Markov branching process. For $t \geq 0$, the generating function of each individual of $Z(t)$ is given by

$$\varphi_t(z) = \sum_{k\in\mathbb{N}} p_{1k}(t)z^k. \tag{4.13}$$

Using (4.12), we derive

$$(\varphi_t(z))^k = \underbrace{\varphi_t(z)\cdot\varphi_t(z)\cdot\cdots\cdot\varphi_t(z)}_{k-\text{fold}} = \sum_j p_{kj}(t)z^j. \tag{4.14}$$

With (4.11), we deduce for all $t, s > 0$

$$\begin{aligned}
\varphi_{t+s}(z) = \mathbb{E}[z^{Z^1(t+s)}] &= \sum_j p_{1j}(t+s)z^j \\
&= \sum_j \sum_k p_{1k}(t)p_{kj}(s)z^j \\
&= \sum_k p_{1k}(t)\sum_j p_{kj}(s)z^j \\
&= \sum_k p_{1k}(t)(\varphi_s(z))^k \\
&= \varphi_t(\varphi_s(z)).
\end{aligned}$$

Thus the family $(\varphi_t)_{t\geq 0}$ satisfies the semigroup property, and $\varphi_0(z) = z$ for all $z \in \mathbb{D}$ since $p_{1k} = \delta_{1k}$ $(k \in \mathbb{N})$. The continuity in the time variable is satisfied by definition. Thus the family of generating functions $(\varphi_t)_{t\geq 0}$ forms a semiflow in $\mathscr{H}(\mathbb{D},\mathbb{D})$.

A time-continuous Markov process $(Z(t))_{t\geq 0}$ is called regular if $\sum_{k\in\mathbb{N}} \mathbb{P}[Z(t) = k|Z(0) = 1] = 1$. In this case, $\varphi_t(1) = 1$ for all $t \geq 0$. We note that the point 1 is not necessarily the Denjoy-Wolff point of $(\varphi_t)_{t\geq 0}$. It can be shown that the generator G of $(\varphi_t)_{t\geq 0}$ admits a special form:

$$G(z) = a\left(z - \sum_{n=0}^{\infty} \tilde{p}_n z^n\right), \tag{4.15}$$

where $a > 0$ and $\sum_{n\in\mathbb{N}} \tilde{p}_n = 1$, and the $\tilde{p}_n \geq 0$ are so-called infinitesimal probabilities, see [14, Ch. 3, Sect. 2].

A stochastic process associated to a Dirichlet-to-Neumann operator. Now we use the special properties of semiflows coming from a time-continuous Markov process to show that such processes are in a certain way connected to the Poincaré-Steklov semigroups we constructed in the previous chapters from composition operators. For convenience, we assume that a Markov branching process $(Z(t))_{t\geq 0}$ satisfies $Z(0) = 1$ a.s., so the generating function of $Z(1)$ embeds into a semiflow by the construction above.

4.3.2 Definition. Given a time continuous Markov branching process $(Z(t))_{t\geq 0}$ with the transition probabilities $p_{ij}(t)$, we call the corresponding family $(\varphi_t)_{t\geq 0}$, obtained by fractional iteration of the generating function of $Z(1)$, a Markov semiflow in $\mathscr{H}(\mathbb{D},\mathbb{D})$.

Let $G : \mathbb{D} \to \mathbb{C}$ be the analytic function given by (4.15) generating a semiflow in $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{D},\mathbb{D})$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = a\left(\varphi_t - \sum_k \tilde{p}_n(\varphi_t)^k\right).$$

Note that in this case $\lim_{z\to 1} G(z) = 0$ implies that 1 is a fixed point (but not necessarily the Denjoy-Wolff point) of $(\varphi_t)_{t\geq 0}$.

4.3.3 Remark. The class of semiflow generators which are zero in the angular sense at one point at the boundary divides actually in three subclasses. Each subclass gives information on the position of the Denjoy-Wolff point of the generated semiflow. It is also interesting that these classes correspond to a certain classification for the extinction probability of a branching process. We refer to [43, Section 5.2] for a nice illustration in tabular form.

Given the generator of a Markov semiflow, we can construct φ_t for all $t \geq 0$ from the infinitesimal probabilities. The infinitesimal probabilities $\tilde{p}_n$ uniquely determine the transition probabilities $p_{ij}(t)$ by Kolmogorov's backward equation, see [14, Ch. 3, Sect. 2], i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{1n}(t) = a\left(p_{1n}(t) - \sum_k \tilde{p}_k p_{kj}(t)\right),$$

from which we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_n p_{1n}(t)z^n &= a\left(\sum_n p_{1n}(t)z^n - \sum_k \tilde{p}_n \sum_n p_{kn}(t)z^n\right)\\ &= a\left(\sum_n p_{1n}(t)z^n - \sum_k \tilde{p}_n \left(\sum_n p_{1n}(t)z^n\right)^k\right).\end{aligned}$$

Thus $\varphi_t(z) = \sum_n p_{1n}(t)z^n$.

Consider the following evolution problem associated with the Dirichlet-to-Neumann operator $\mathfrak{D}_G$ which degenerates (at least) in one point:

$$\begin{cases} \partial_t u - \langle G, \nabla u\rangle = 0 & \text{on } (0,\infty)\times\partial\mathbb{D},\\ -\Delta u = 0 & \text{on } (0,\infty)\times\mathbb{D},\\ u(0,\cdot) = f & \text{on } \partial\mathbb{D}.\end{cases} \tag{4.16}$$

As the main result of our second chapter, we have proven that this problem is well-posed on the boundary space of any (G)-admissible Banach space $X \hookrightarrow \mathbb{H}(\mathbb{D})$, Theorem 2.3.8. The following theorem is inspired by an observation in [21] and gives a representation of the semigroup generated by a Dirichlet-to-Neumann operator as in (4.16) in terms of a Markov semigroup associated with a Markov process with state space $\mathbb{N}$.

4.3.4 Theorem. *Given a time continuous Markov process $(Z(t))_{t\geq 0}$ with transition probabilities $(p_{ij}(t))_{t\geq 0}$, there is a Banach space on which the Markov semigroup associated to $(Z(t))_{t\geq 0}$ is similar to the semigroup generated by the Dirichlet-to-Neumann operator (4.16) on the boundary space of a (G)-admissible Banach space $X \hookrightarrow \mathscr{H}(\mathbb{D})$.*

Proof. First, we identify the Fréchet space of holomorphic functions $\mathscr{H}(\mathbb{D})$ with the subspace $\ell_{\mathscr{H}(\mathbb{D})} \subseteq \mathbb{C}^{\mathbb{N}}$ consisting of sequences $(c_n)_n$ such that $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence greater or equal to 1. Let $\ell_X := \{(c_n)_n \in \mathbb{C}^{\mathbb{N}} : \sum_n c_n z^n \in X\} \subseteq \ell_{\mathscr{H}(\mathbb{D})}$. Then, we define an operator H which maps a sequence in ℓ_X to a function in X, i.e.,

$$H : \ell_X \to X, c_n \mapsto \sum_n c_n z^n.$$

Note that H is an isometric isomorphism if we define $\|x\|_{\ell_X} := \|Hx\|_X$ for all $x \in \ell_X$, and for $f \in X$, we have $H^{-1}f = (\hat{f}(n))_n$, where $\hat{f}(n)$ denotes the nth Fourier coefficient of f. Given a denumerable Markov process $(Z(t))_{t\geq 0}$ with transition probabilities $p_{ij}(t), t > 0$, we define a family of operators $(S(t))_{t\geq 0}$ on ℓ_X given by $(S(t)x)_j := (\sum_k x_k p_{kj}(t))_j$. From the transition probabilities, we construct a semiflow in $\mathscr{H}(\mathbb{D}, \mathbb{D})$ fixing 1 generated by G given in terms of infinitesimal probabilities, see (4.15). Now we show that $(S(t))_{t\geq 0}$ is a bounded semigroup on ℓ_X. Let $x = (x_n)_n \in \ell_X$ for $x \in \ell_X$. Then

$$\begin{aligned} H(S(t)x) &= \sum_j \sum_n x_n p_{nj}(t) z^j \\ &= \sum_n x_n \sum_j p_{nj}(t) z^j \\ &= \sum_n x_n (\varphi_t(z))^n. \end{aligned}$$

Thus $S(t)x = H^{-1}(C_{\varphi_t}(Hx))$ and by (G)-admissibility of X, composition by φ_t is a bounded operator on X, and since H is an isometric isomorphism, $S(t) \in \mathscr{L}(\ell_X)$ for all t. The semigroup property follows from the semigroup property of $(\varphi_t)_{t\geq 0}$ induced by the Chapman-Kolmogorov equation (4.11). Now, denoting $L = \operatorname{Tr} H$, we deduce that $T(t) = LS(t)L^{-1}$, where $(T_t)_{t\geq 0}$ is the semigroup generated by the Dirichlet-to-Neumann operator associated with the evolution problem (4.16). □

Examples. The semigroup associated with a Markov process $(Z_t)_{t\geq 0}$ with state space $\mathbb{N}$ and transition probabilities $p_{ij}(t)$ is similar to the semigroup generated by the Dirichlet-to-Neumann operator (4.16) on $\partial\mathscr{H}^2(\mathbb{D})$ and $\partial\mathscr{A}^2(\mathbb{D})$: the spaces $\mathscr{H}^2(\mathbb{D})$ and $\mathscr{A}^2(\mathbb{D})$ are both (G)-admissible for every semiflow generator, and they are isomorphic to $\ell^2(\mathbb{N})$ and $\ell^2((\sqrt{1+n})^{-1}) := \{(x_n)_n | \sum_n \left(\frac{x_n}{\sqrt{1+n}}\right)^2 < \infty\}$, respectively.

4.3.5 Remark. We would like to emphasize the following observation concerning the history of composition semigroups on spaces of analytic functions. In [21], Theorem 4.3.4 has been proved for the Hardy space $\mathscr{H}^2(\mathbb{D})$ only. Although this paper is highly using the theory of composition semigroups, like the representation of the generator of a composition semigroup, it has not been cited yet in this context. This is remarkable since it has been published more than 10 years before the paper by Berkson and Porta [20], which is often considered as the starting point for the theory of semigroups of composition operators on spaces of analytic functions.

4.3.6 Remark. If we consider the evolution equation associated with the 'classical' Dirichlet-to-Neumann operator on $L^2(\partial\mathbb{D})$, we can analogously derive a process with which we associate a semigroup that is similar to the semigroup generated by the classical Dirichlet-to-Neumann operators, namely, the dishonest denumerable process $(p_{ij}(t))_{t\geq 0} = (\delta_{ij} e^{-ti})_{t\geq 0}$.

The process associated to a Dirichlet-to-Robin operator. Now we generalize this approach to evolution problems associated with the Dirichlet-to-Robin operator, namely,

$$\begin{cases} \partial_t u - \beta \cdot u - \langle G, \nabla u \rangle = 0 & \text{on } (0,\infty) \times \partial\mathbb{D}, \\ -\Delta u = 0 & \text{on } (0,\infty) \times \mathbb{D}, \\ u(0,\cdot) = f & \text{on } \partial\mathbb{D}, \end{cases} \tag{4.17}$$

where G generates a Markov semiflow and $\beta > 0$.

4.3.7 Theorem. *Given a time continuous Markov process $(Z(t))_{t\geq 0}$ with transition probabilities $p_{ij}(t)$, there is a Banach space on which the sub-Markovian semigroup associated with the (dishonest) process corresponding to $e^{-t\beta} p_{ij}(t)$ is similar to the semigroup generated by the Dirichlet-to-Robin operator in (4.17) on ∂X, the boundary space of a (β, G)-admissible Banach space $X \hookrightarrow \mathscr{H}(\mathbb{D})$.*

Proof. Let $\mathfrak{D}_{\mathscr{R}}$ be the Dirichlet-to-Robin operator associated with (4.17). Using our approach for the correspondence between weighted composition semigroups and the semigroup $(T_t)_{t\geq 0}$ generated by $\mathfrak{D}_{\mathscr{R}}$, we obtain $\mathrm{Tr}(w_t C_{\varphi_t})\mathrm{Tr}^{-1} = T_t$, where $w_t C_{\varphi_t}$ is the weighted composition semigroup associated with the semiflow $(\varphi_t)_{t\geq 0}$ generated by G and weight function $w_t = e^{-t\beta}$. Note that the weight does not depend on the space variable.
Let ℓ_X and let $H : \ell_X \to X, c_n \mapsto \sum_n c_n z^n$ as above. Let $(Y(t))_{t\geq 0}$ the weighted process corresponding to $e^{-t\beta} p_{ij}(t)$. Then the corresponding semigroup $Q(t) : \ell_X \to \ell_X$ is given by $Q(t) = e^{-t\beta} S(t)$, where $S(t)$ is the Markov semigroup on ℓ_X associated with $Z(t)$. Then, we derive for all $(x_n)_n = x \in \ell_X$

$$\begin{aligned} (w_t C_{\varphi_t}) H x &= e^{-t\beta} \sum_{n=0}^{\infty} c_n (\varphi_t(z))^n \\ &= e^{-t\beta} \sum_{n=0}^{\infty} c_n \sum_{k=0}^{\infty} p_{nk}(t) z^k \\ &= e^{-t\beta} \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} c_n p_{nk}(t) \right) z^k \\ &= \sum_{k=0}^{\infty} (S(t)x)_k \cdot e^{-t\beta} z^k. \end{aligned}$$

So, $w_t C_{\varphi_t} = H Q(t) H^{-1}$ and therefore, denoting $L = \mathrm{Tr} H$ we obtain the claimed equivalence $T_t = L Q(t) L^{-1}$. □

Unfortunately, this approach seems to be restricted to the unit disk. Transforming Markov semiflows to other simply connected domains changes the coefficients of the corresponding power series, and we cannot expect those coefficients to be positive after transformation (or even real).

5. Summary, open problems, and outlook

The main impetus for this thesis was the observation by Lax [60] that the semigroup generated by the classical Dirichlet-to-Neumann operator on $C(\partial\mathbb{B}^n)$ or $L^2(\partial\mathbb{B}^n)$ can be represented as the trace of a semigroup of composition operators. Although this appears to be a very special case -such a representation is only available if the underlying domain is a ball- we were inspired to build on this observation a new theory which relates operators which act like a Dirichlet-to-Neumann operator, i.e., sending Dirichlet boundary values to an outward pointing derivative, to semigroups of composition operators. We summarize below our main results and open questions which arose during our investigations.

We found that a large class of Poincaré-Steklov operators, namely Dirichlet-to-Neumann and Dirichlet-to-Robin operators, are infinitesimal generators of semigroups which are similar to semigroups of composition operators and semigroups of weighted composition operators acting on certain Banach spaces of analytic functions, which we called here admissible spaces. By associating a (possibly very general) boundary space to such an admissible space, we proved well-posedness for certain evolution problems associated with Dirichlet-to-Neumann and Dirichlet-to-Robin operators on boundary spaces of admissible spaces. Our method allows one to construct Dirichlet-to-Robin operators with very general coefficients in front of the Neumann derivative and the Dirichlet boundary values, namely boundary values of appropriate holomorphic functions. Regarding this our method stands out against the method of forms and the variational approach covering only the classical Dirichlet-to-Neumann operator which appears in our context only as a special example. On the other hand, the variational approach is quite flexible concerning the underlying domain. Our results are obtained mostly on planar domains bounded by a Jordan curve since we had to take into account boundary values of conformal maps. However, there might be some room for improvements, e.g., to simply connected domains with less smooth boundary. Also, we did not discuss our approach on unbounded domains in detail, which would be another interesting refinement.

A generalization of our approach to higher dimensional domains seems possible, but we could not achieve the same results as in the planar case since some theory we used in the planar case does not carry over to higher dimensions. In particular, there is no Riemann mapping theorem for higher dimensional domains. Function theory in higher dimensional domains lacks also some results which appear to be very fruitful in investigating operator theoretic properties of infinitesimal generators of composition semigroups. For example, in the one-dimensional case we can use the argument principle to determine the point spectrum of the generator of a composition semigroup (see Appendix B.2), and by similarity we determine the point spectrum of a Dirichlet-to-Neumann operator. Unfortunately, we have not found any similar result for analysis of several complex variables, so that we could not give a straightforward generalization to this approach. Nevertheless, it would be interesting to give a more involved study of semigroups of composition operators for domains in $\mathbb{C}^n$. Actually, one could also think about a generalization to infinite dimensional domains since the theory of semiflows on balls in Hilbert spaces is already available.

Based on our theory invented for Dirichlet-to-Neumann and Dirichlet-to-Robin operators in the second chapter on Jordan domains, in the third chapter we constructed by means of Chernoff's formula an approximating family consisting of (weighted) composition operators to obtain well-posedness of a multiplicatively perturbed version of the evolution equations considered in the second chapter. In this approach, we only discussed multiplicative perturbations by boundary values of holomorphic functions mapping into some sector in the right half-plane. Our method relies on the representation of semiflows in terms of a univalent function (the Koenigs function). We proved that given a Banach space X which is admissible for some semiflow generator $G : \Omega \to \mathbb{C}$ and $a : \Omega \to \mathbb{C}$ an admissible multiplicative perturbation of G, the operator $A : \operatorname{dom} A \subseteq X \to X$, $f \mapsto a\langle G, \nabla f\rangle$, generates a semigroup which appears to be the limit of a sequence of composition operators which stem from the semigroup generated by $\Gamma : \operatorname{dom}\Gamma \subseteq X \to X$, $f \mapsto \langle G, \nabla f\rangle$. Moreover, we found a condition such that the approximated semigroup is actually a semigroup of composition operators, hence we proved that then X is (aG)-admissible. We also proved a generalization of our approximation result to multiplicatively perturbed Dirichlet-to-Robin operators. However, the question whether the approximated semigroup consists of cocycle-weighted composition operators seems highly non-trivial and remains open. We also conjecture that our approximation results might also work under some milder assumptions. For instance, we assumed all composition operators on the underlying domain to be contractions for symbols fixing the Denjoy-Wolff point of the semiflow which induces a strongly continuous semigroup of composition operators on X. However, the approximating family of operators in Chernoff's formula need not be contracting.

In the fourth chapter we presented some possible applications of our theory developed in this thesis. First we applied our approximation results from the third chapter to maximal subspaces of strong continuity. We found that, under certain circumstances, maximal subspaces of strong continuity do not change under admissible perturbations. It would be nice to figure out a generalization of this result which does not rely on the reflexivity of the space. As a second application we derived a generalization of our approximation approach to multiplicative perturbations by positive continuous functions, by using an idea from [45]. Actually, our aim was to approximate Poincaré-Steklov operators which are multiplicative perturbations by positive functions of Dirichlet-to-Neumann operators on $L^p(\partial\Omega)$ where $\Omega \subseteq \mathbb{C}$ is a Dini-smooth domain. In particular, this would yield an approximation of the classical Dirichlet-to-Neumann operator on a simply connected domain by composition operators as we have shown that this operator is representable as the trace of a multiplicative perturbation by a positive function of a generator of a semigroup of composition operators. Unfortunately, our approach works only on $C(\partial\Omega)$, and we have not been able to find a generalization to $L^p(\partial\Omega)$. Lastly, we presented a nice connection to stochastic branching processes which was based on a result in [21] published more than 50 years ago.

Outlook. Finally, let us collect some ideas for further research on our approach to Poincaré-Steklov semigroups which we have not touched upon in our thesis but which could give interesting and natural generalizations of our results. As we established our theory exclusively on spaces of holomorphic and harmonic functions, the elliptic equation associated to our Poincaré-Steklov operator is always the Laplace equation. However, the classical Dirichlet-to-Neumann operator is known to be the infinitesimal generator of semigroup also when the considered Dirichlet problem is the following conductivity equation

$$\begin{cases} -\operatorname{div}\sigma\nabla u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where σ is some appropriate function on Ω. We recall that real and imaginary part of a holomorphic function are harmonic functions by the Cauchy-Riemann equations. Roughly speaking, one can define pseudo-holomorphic functions such that the real part satisfies the conductivity equation (5.1) and the imaginary part satisfy the conductivity equation (5.1) with σ replaced by $\frac{1}{\sigma}$. For such pseudo-holomorphic functions one similarly defines Hardy spaces, and, surprisingly, a lot of results from the holomorphic Hardy space carry over to the pseudo-holomorphic case. Unfortunately, composition operators on a Hardy space of pseudo-holomorphic functions map into a Hardy space of different pseudo-holomorphic functions (i.e., real part and imaginary part satisfy different conductivity equations). For an overview on this topic, we refer to [17, 61]. We conjecture that for pseudo-holomorphic functions which are continuous up to the boundary one can establish an approximation formula as in Section 4.2 for a semigroup solving

$$\begin{cases} \partial_t(u+iv) - \langle G, \sigma\nabla u + i\frac{1}{\sigma}\nabla v\rangle = 0 & (0,\infty)\times\partial\Omega, \\ \begin{pmatrix} \operatorname{div}\sigma & 0 \\ 0 & \operatorname{div}\frac{1}{\sigma} \end{pmatrix}\begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix} = 0 & (0,\infty)\times\Omega, \\ u+iv = f & \partial\Omega, \end{cases} \tag{5.2}$$

where $f \in C(\partial\Omega)$. Another interesting generalization of our theory might be to consider evolution families instead of semiflows. An evolution family is defined as a family of holomorphic selfmaps $(\varphi_{t,s})_{t\geq s\geq 0}$ of the unit disk such that

1. $\varphi_{s,s} = \mathrm{id}$.
2. $\varphi_{t,s} = \varphi_{u,t} \circ \varphi_{t,s}$ for all $0 \leq s \leq t \leq u < \infty$.

An evolution family is of order $d \in [1,\infty)$ if for all $z \in \mathbb{D}$ and $t_0 > 0$ there is a function $k_{z,t_0} \in L^d([0,t_0),\mathbb{R}_{\geq 0})$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,t_0}(x)\,\mathrm{d}x$$

for all $0 \leq s \leq u \leq t \leq t_0$. Obviously, given a semiflow $(\varphi_t)_{t\geq 0}$ we can define an evolution family by $\tilde{\varphi}_{t,s} := \varphi_{t-s}$ for all $0 \leq s \leq t < \infty$. The paper [27] describes in detail when an evolution family stems from a semiflow.

There is a lot of results for evolution families which are quite similar to those obtained in the theory of semiflows of holomorphic selfmaps, we refer to [25] for a review. For example, for an evolution family $(\varphi_{t,s})_{t\geq s\geq 0}$ of order $d \in [1,\infty)$ there exists a so-called Herglotz vector field $G : \mathbb{D} \times \mathbb{R}_{\geq 0} \to \mathbb{C}$ of order d, i.e.,

1. $G(\cdot,t)$ is holomorphic for all $t \geq 0$ and $G(z,\cdot)$ is measurable for all $z \in \mathbb{D}$,
2. $G(\cdot,t) \in \mathcal{G}$ for all $t \geq 0$, and
3. for all compact sets $K \subseteq \mathbb{D}$ and $t_0 > 0$, there exists $k_{K,t_0} \in L^d([0,t_0],\mathbb{R}_{\geq 0})$ such that $|G(z,t)| \leq k_{K,t_0}(t)$ for all $z \in K$ and a.e. $t \in [0,t_0]$

such that for all $z \in \mathbb{D}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{t,s}(z) = G(\varphi_{t,s}(z),t) \text{ for a.e. } t,s \in [0,\infty),\ t \geq s,$$

see [25, Thm. 1.1]. Furthermore, this Herglotz vector field has a Berkson-Porta representation [25, Thm. 4.8]: for any Herglotz vector field G, there is a measurable function $\tau : \mathbb{R}_{\geq 0} \to \bar{\mathbb{D}}$ and a function $F : \mathbb{D} \times \mathbb{R}_{\geq 0} \to \mathbb{C}$ such that $F(\cdot,t)$ is a holomorphic function with positive real part for all $t \in \mathbb{R}_{\geq 0}$ such that

$$G(z,t) = (z - \tau(t))(z\bar{\tau}(t) - 1)F(z,t).$$

There is a rich theory on such evolution families which provides in particular a nice interaction between complex dynamical systems and probability (stochastic Loewner equations). It might be interesting to consider families of composition operators associated with an evolution family, and to study if there is also a connection to partial differential equations associated with a Poincaré-Steklov operator on a boundary space of a Banach space of analytic functions.

A. Generation theory of semiflows

The theory of semiflows is used throughout in this thesis. We present in this appendix chapter some proofs of classical results in this area to give an idea of the geometric function theory behind semiflows of holomorphic functions.

A.1. Semiflow generators

This section is devoted to prove the main results concerning semiflow generators from the famous paper by Berkson and Porta [20]. First, the existence of a generator for a given semiflow, and secondly the representation formula.

The proof of the following theorem stems from [51] and gives a different (in our opinion nicer) proof for [20, Thm. 1].

A.1.1 Theorem. *Let $(\varphi_t)_{t\geq 0}$ be a semiflow in $\mathscr{H}(\mathbb{D},\mathbb{D})$. Then there is a holomorphic function G such that*

$$\lim_{t\to 0^+} \frac{\varphi_t(z)-z}{t} = G(z)$$

for all $z\in\mathbb{D}$. Moreover, $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(z) = G(\varphi_t(z))$ for all $z\in\mathbb{D}$ and $t\geq 0$.

Proof. Let $(\varphi_t)_{t\geq 0}$ be a semiflow in $H(\mathbb{D},\mathbb{D})$ and fix $r\in(0,1)$. We define by $K_r := \{z\in\mathbb{C}\colon |z|\leq r\}$ the closed disk of radius r centered in the origin. Consider a test function $\eta\in C_c^\infty(\mathbb{R})$ such that $\eta(t)=0$ for $t\notin(0,1)$, $\eta(t)>0$ for $t\in(0,1)$, and $\int_{\mathbb{R}}\eta = 1$. For $\delta>0$ we set $\eta_\delta(t) := \eta(\frac{t}{\delta})$.
Let $\rho\in(r,1)$ and choose $\delta>0$ such that for all $t\in(0,\delta)$

$$|\varphi_t(z)|\leq\rho \quad (z\in K_r)$$

and

$$\mathrm{Re}(\varphi_t'(z))\geq\frac{1}{2} \quad (z\in K_\rho).$$

Note that this is possible since a semiflow (and its complex derivative) converges locally uniformly. Then, define the function

$$F(z) := \int_{\mathbb{R}} \eta_\delta(t)\varphi_t(z)\,\mathrm{d}t = \int_0^\delta \eta_\delta(t)\varphi_t(z)\,\mathrm{d}t$$

and note that this is a holomorphic function in $\mathbb{D}$. Furthermore, we derive

$$\begin{aligned}\operatorname{Re}(F'(z)) &= \int_0^\delta \eta_\delta(t)\operatorname{Re}(\varphi_t'(z))\,\mathrm{d}t \\ &\geq \frac{1}{2}\int_0^\delta \eta_\delta(t)\,\mathrm{d}t = \frac{\delta}{2} > 0.\end{aligned}$$

Thus F is univalent for all $z \in \operatorname{int} K_\rho$. Moreover, by the semigroup property, we obtain

$$\begin{aligned}F(\varphi_s(z)) &= \int_{\mathbb{R}} \eta_\delta(t)\varphi_t(\varphi_s(z))\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \eta_\delta(t)\varphi_{s+t}(z)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \eta_\delta(\tau - s)\varphi_\tau(z)\,\mathrm{d}\tau = \eta_\delta * \varphi_s(z)\end{aligned}$$

This yields that the map $t \mapsto F(\varphi_t)$ is infinitely differentiable. Since $\varphi_t(z) \in K_\rho$ for all $z \in K_r$ and $t \in (0, \delta)$, we derive

$$\varphi_t(z) = F^{-1}(F(\varphi_t(z))) \quad \text{for } z \in \operatorname{int} K_r \text{ and } t \in (0, \delta).$$

Let $z \in \operatorname{int} K_r$ and note that the inverse function F^{-1} is holomorphic, therefore we get that $t \mapsto \varphi_t(z)$ is infinitely differentiable for all $t \in (0, \delta)$.
Moreover, given $z \in \operatorname{int} K_r$, we calculate the derivative of φ_t for $t \in (0, \delta)$ as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(z) &= \frac{\mathrm{d}}{\mathrm{d}t}F^{-1}(F(\varphi_t(z))) \\ &= (F^{-1})'(F(\varphi_t(z))) \cdot \frac{\mathrm{d}}{\mathrm{d}t}F(\varphi_t(z)) \\ &= -(F^{-1})'(F(\varphi_t(z))) \cdot \int_{\mathbb{R}} \dot{\eta}_\delta(\tau - t)\varphi_\tau(z)\,\mathrm{d}\tau \\ &= -\frac{1}{F'(\varphi_t(z))} \cdot \int_{\mathbb{R}} \dot{\eta}_\delta(s)\varphi_s(\varphi_t((z)))\,\mathrm{d}s\end{aligned}$$

Set

$$G(z) := -\frac{1}{F'(z)} \cdot \int_{\mathbb{R}} \eta_\delta(s)\varphi_s(z)\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(z)|_{t=0}. \tag{A.1}$$

Then G is a holomorphic function and $G(\varphi_t) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t$. Letting $r \to 1$ shows that G is holomorphic in $\mathbb{D}$. In particular, G is the generator of $(\varphi_t)_{t\geq 0}$. □

Given a semiflow $(\varphi_t)_{t\geq 0}$ and its generator as derived in the above theorem, we obtain that the ordinary differential equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} w = G(w) \\ w(0) = z \end{cases}$$

has a unique solution for all $z \in \mathbb{D}$ given by $w(t) = \varphi_t(z)$, since G is, as it is holomorphic, locally Lipschitz continuous.

Let G be the generator of a semiflow $(\varphi_t)_{t\geq 0}$ in $\mathscr{H}(\mathbb{D}, \mathbb{D})$. Then we obtain $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(z) = G(z)\frac{\partial}{\partial z}\varphi_t(z)$ since, for $t \geq 0$ and $z \in \mathbb{D}$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t,z) &= G(\varphi(t,z)) \\ &= \lim_{s\to 0} \frac{\mathrm{d}}{\mathrm{d}s}\varphi_s(\varphi_t(z)) \\ &= \lim_{s\to 0} \frac{\mathrm{d}}{\mathrm{d}s}\varphi_t(\varphi_s(z)) \\ &= \lim_{s\to 0} \left(\frac{\partial}{\partial z}\varphi_t\right)(\varphi_s(z)) \cdot \frac{\mathrm{d}}{\mathrm{d}s}\varphi_s(z) \\ &= \lim_{s\to 0} \left(\frac{\partial}{\partial z}\varphi_{t+s}\right)(z) \cdot \frac{\mathrm{d}}{\mathrm{d}s}\varphi(s,z) \\ &= G(z)\frac{\partial}{\partial z}\varphi_t(z). \end{aligned}$$

Lastly, we would like to prove at least one direction of the Berkson-Porta representation, namely that each semiflow generator is representable as the product of a particular quadratic polynomial and a holomorphic function with positive real part. This result has been used frequently in our thesis. The proof we present here is the an adaption of the original proof by Berkson and Porta [20] found in Avicou's Ph.D. thesis [15].

A.1.2 Theorem (Berkson-Porta representation). *Let G be the generator of a semiflow $(\varphi_t)_{t\geq 0}$ with Denjoy-Wolff point $b \in \mathbb{D}$. Then there exists a holomorphic function F with positive real part such that $G(z) = (z - b)(\bar{b}z - 1)F(z)$.*

Proof. Let G be the generator of a semiflow $(\varphi_t)_{t\geq 0}$ and $b \in \mathbb{D}$ its Denjoy-Wolff point. Let $z \in \mathbb{D}$ and define the following function:

$$f(t) = (1 - |b|^2)\left(\frac{1}{1 - \left|\frac{\varphi_t(z) - b}{\bar{b}\varphi_t(z) - 1}\right|^2}\right) \quad (t \in [0, \infty)).$$

The Schwarz-Pick lemma [70, Prop. 1.1.3] and the semigroup property yield that f is decreasing, so $\frac{\mathrm{d}}{\mathrm{d}t}f(t) \leq 0$. In particular, we calculate

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{\varphi_t(z)-b}{\bar{b}\varphi_t(z)-1}\right|^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varphi_t(z)-b}{\bar{b}\varphi_t(z)-1}\cdot\overline{\left(\frac{\varphi_t(z)-b}{\bar{b}-1}\right)}\right)\\
&= \frac{\dot{\varphi}_t(z)(\bar{b}\varphi_t(z)-1)-\dot{\varphi}_t(z)(b\bar{\varphi}_t(z)-|b|^2)}{(\bar{b}\varphi_t(z)-1)^2}\cdot\overline{\left(\frac{\varphi_t(z)-b}{\bar{b}\varphi_t(z)-1}\right)}+\\
&\quad+\frac{\overline{\dot{\varphi}_t}(z)(b\bar{\varphi}_t(z)-1)-\overline{\dot{\varphi}_t}(z)(b\bar{\varphi}_t(z)-|b|^2)}{\left(\overline{\bar{b}\varphi_t(z)-1}\right)^2}\cdot\left(\frac{\varphi_t(z)-b}{\bar{b}\varphi_t(z)-1}\right)\\
&= \dot{\varphi}_t(z)\frac{|b|^2-1}{(\bar{b}\varphi_t(z)-1)^2}\cdot\overline{\left(\frac{\varphi_t(z)-b}{\bar{b}\varphi_t(z)-1}\right)}+\overline{\dot{\varphi}_t}(z)\frac{|b|^2-1}{\left(\overline{\bar{b}\varphi_t(z)-1}\right)^2}\cdot\\
&\quad\cdot\left(\frac{\varphi_t(z)-b}{\bar{b}\varphi_t(z)-1}\right).
\end{aligned}
$$

Letting $t \to 0^+$, we derive

$$
\begin{aligned}
\lim_{t\to 0^+}\frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{\varphi_t(z)-b}{\bar{b}-1}\right|^2 &= (|b|^2-1)\left(G(z)\frac{\bar{z}-\bar{b}}{(\bar{z}-1)^2\overline{\overline{(\bar{b}z-1)}}}+\bar{G}(z)\frac{z-b}{(\bar{z}-1)\overline{\overline{(\bar{b}z-1)^2}}}\right)\\
&= 2(|b|^2-1)\operatorname{Re}G(z)(\bar{z}-\bar{b})(1-b\bar{z}).
\end{aligned}
$$

Thus

$$
\lim_{t\to 0^+}\frac{\mathrm{d}}{\mathrm{d}t}f(t) = -\left(\frac{2\operatorname{Re}G(z)(\bar{z}-\bar{b})(1-b\bar{z})}{\left(1-\left|\frac{\varphi_t(z)-b}{\bar{b}-1}\right|^2\right)^2}\right) \leq 0,
$$

which in particular implies $\operatorname{Re}G(z)(\bar{z}-\bar{b})(1-b\bar{z}) \geq 0$. Since b is a zero of G, we can define the following function

$$
F(z) := G(z)(z-b)^{-1}(1-\bar{b}z)^{-1}
$$

which admits positive real part by our calculations. Furthermore $G(z) = (z-b)(\bar{b}z-1)F(z)$ as desired. □

A.2. Embeddability

In this chapter, we discuss the property of embeddability into a semiflow by means of a holomorphic selfmap of the unit disk with inner Denjoy-Wolff point. The following two Lemmata can be found in [38, §2.7].

A.2.1 Lemma. *Let* $\phi \in \mathscr{H}(\mathbb{D})$ *such that* $\operatorname{Re}\phi(z) > 0$ *for all* $z \in \mathbb{D}$ *and* $w \colon \mathbb{R}_{\geq 0} \to \mathbb{C}$ *be a solution to*

$$\begin{cases} \dot{w}(t) = -w(t)\phi(w(t)), \\ w(0) = \xi \in \mathbb{D}. \end{cases} \tag{A.2}$$

Then $|w|$ *is decreasing and* $|w(t)| \to 0$ *as* $t \to \infty$.

Proof. Let $w \neq 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \log|w(t)| = \operatorname{Re}(\dot{w}(t)\frac{1}{w(t)}) = -\operatorname{Re}(\phi(w(t)) < 0$$

This implies that $|w|$ is decreasing in $(0, \infty)$. As $\operatorname{Re}\phi$ is harmonic, the maximum principle yields that $\sup_{B_{|\xi|}} \operatorname{Re}\phi = \sup_{\partial B_{|\xi|}} \operatorname{Re}\phi$. Thus, we find $\delta > 0$ such that $-\operatorname{Re}\phi(w(t)) < -\delta$ for all $t > 0$. So,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \log|w(t)| &< -\delta \\ \Rightarrow \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \log|w(s)|\,\mathrm{d}s &< \int_0^t \delta\,\mathrm{d}s \\ \Rightarrow |w(t)| &< e^{-\delta t}|\xi| \end{aligned}$$

which implies $|w(t)| \to 0$ for $t \to \infty$. □

A.2.2 Lemma. *Let* $h \in \mathscr{H}(\mathbb{D})$, *such that* $\varphi(0) = 0, \varphi'(0) \neq 0$, *and* $\varphi(z) \neq 0\,(0 < |z| < 1)$, *and assume there is* $\lambda \in \mathbb{C}$ *such that* $\operatorname{Re}(\lambda \frac{h(z)}{zh'(z)}) > 0$ *for all* $z \in \mathbb{D}$. *Then* h *is spirallike.*

Proof. Let $\xi \in \mathbb{D}$. Set $\phi(z) := \lambda \frac{h(z)}{zh'(z)}$ and let $w(t, \xi)$ a solution to (A.2) and define $v(t, \xi) := h(w(t, \xi))$. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} v(t, \xi) = h'(w(t, \xi)) \frac{\mathrm{d}}{\mathrm{d}t} w(t, \xi) &= -w(t, \xi) h'(w(t, \xi))\phi(z) \\ &= -\lambda h(w(t, \xi)) = -\lambda v(t, \xi). \end{aligned}$$

That is, v is the (unique) solution to

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} v = -\lambda v, \\ v(0) = h(\xi). \end{cases}$$

Hence $v(t) = h(\xi)e^{-\lambda t}$ and thus $h(\xi)e^{-\lambda t} \subseteq h(\mathbb{D})$ for all $\xi \in \mathbb{D}$. □

A.2.3 Theorem ([41, Thm. 1])**.** *Let* $\varphi \in H(\mathbb{D}, \mathbb{D}) \setminus \operatorname{Aut}(\mathbb{D})$ *with Denjoy-Wolff point in 0 and* $\varphi'(0) = \gamma \neq 0$. *Then* φ *embeds into a semiflow if and only if*

$$h \circ \varphi = \gamma h \quad \textit{(Schröder's equation)}$$

has a solution $h \in \mathscr{H}(\mathbb{D})$ such that

$$\frac{zh'(z)}{h(z)} = \frac{F(0)}{F(z)}, \tag{A.3}$$

where F is a holomorphic function with positive real-part and $e^{-F(0)} = \gamma$.

Proof. Let $\varphi \in H(\mathbb{D},\mathbb{D}) \setminus \mathrm{Aut}(\mathbb{D})$ with Denjoy-Wolff point in 0 and $\varphi'(0) = \gamma \neq 0$.

Assume we are given a solution h to (A.3) such that F is a holomorphic function with positive real-part and $e^{-F(0)} = \gamma$. Then

$$0 < \mathrm{Re}\left(\frac{1}{F(z)}\right) = \mathrm{Re}\left(F(0)\frac{zh'(z)}{h(z)}\right), \quad \text{for all } z \in \mathbb{D}.$$

Lemma A.2.2 yields that h is spirallike. We set

$$\varphi_t(z) :\ = h^{-1}(e^{-tp(0)}h(z)).$$

Note that $\varphi_1 = \varphi$ and $\varphi_0 = \mathrm{id}$. Then $(\varphi_t)_{t\geq 0}$ is a semiflow since for all $z \in \mathbb{D}$

$$\begin{aligned}
\varphi_{s+t}(z) &= h^{-1}(e^{-(s+t)F(0)}h(z)) \\
&= h^{-1}(e^{-sF(0)}e^{-tF(0)}h(z)) \\
&= h^{-1}(e^{-sF(0)}h(h^{-1}(e^{-tF(0)}h(z)))) \\
&= \varphi_s \circ \varphi_t(z).
\end{aligned}$$

Continuity in t is obvious.

Conversely, let φ embed into the semiflow $(\varphi_t)_{t\geq 0}$. Then the generator is given by the Berkson-Porta representation

$$G(z) = -zF(z),$$

where F has positive real part. Let

$$h(z) :\ = z\exp\left(\int_0^z \frac{F(0) - F(\xi)}{\xi F(\xi)}d\xi\right).$$

Then for all $z \in \mathbb{D}$

$$\frac{zh'(z)}{h(z)} = \frac{F(0)}{F(z)}.$$

Furthermore h is the unique solution to Schröder's equation: First observe that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}h(\varphi(t,z)) &= h'(\varphi_t(z))\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(z) \\
&= h'(\varphi_t(z))G(\varphi_t(z)) \\
&= -h'(\varphi_t(z))F(\varphi_t(z))\varphi_t(z) \\
&= -F(0)h(\varphi_t(z)).
\end{aligned}$$

So $\zeta(t) = h(\varphi_t(z))$ is a solution to

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\zeta = -F(0)\zeta, \\ \zeta(0) = h(z). \end{cases}$$

In particular,

$$\zeta(t) = h(z)e^{-F(0)t} = h(\varphi_t),$$

which is for $t = 1$ Schröder's equation. □

We only illustrate the case of an inner Denjoy-Wolff point since the key ideas are the same for a boundary Denjoy-Wolff point. For a boundary Denjoy-Wolff point, one needs a close-to-convex function instead of a spirallike function and Abel's equation instead of Schröder's equation.

B. Results on composition operators and composition semigroups on Hardy spaces

In this appendix chapter, we present some classical results on composition operators and composition semigroups on Hardy spaces to emphasize how complex analysis comes into play in investigating operator theoretic properties. We begin with a short discussion the Hilbert space $\mathscr{H}^2(\mathbb{D})$. Based on this we present a proof of Littlewood's subordination principle which shows boundedness of composition operators on Hardy spaces, and we give also an easy criterion for compactness of a composition operator in terms of the associated symbol. Lastly, we show how the point spectrum of the generator of a composition semigroup can be determined by using results from complex analysis and in particular the associated Koenigs function.

We recall the definition of the Hardy spaces.

B.0.1 Definition (Hardy spaces)**.** Let $p \in [1, \infty)$. The Hardy space $\mathscr{H}^p(\mathbb{D})$ is defined as the space of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{\mathscr{H}^p(\mathbb{D})} := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \,\mathrm{d}t \right)^{\frac{1}{p}} < \infty. \tag{B.1}$$

We recall that $\mathscr{H}^p(\mathbb{D})$ is a Banach space and $\mathscr{H}^2(\mathbb{D})$ is a Hilbert space. We show that for $f \in \mathscr{H}^2(\mathbb{D})$

$$\|f\|^2_{\mathscr{H}^2(\mathbb{D})} = \sum_{n \in \mathbb{N}} |\hat{f}(n)|^2,$$

where $\hat{f}(n)$ denotes the nth Fourier coefficient of f. Let $f \in \mathscr{H}^2(\mathbb{D})$. First note that for $r \in (0, 1)$

$$|f(re^{it})|^2 = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \hat{f}(n)\overline{\hat{f}(m)} r^{n+m} e^{i(n-m)t}.$$

Integrating with respect to t yields

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \,\mathrm{d}t = \sum_{n \in \mathbb{N}} |\hat{f}(n)|^2 r^{2n}$$

by the L^2-orthogonality of the system $\{e^{int}\}_{n\in\mathbb{N}}$. Thus

$$\sum_{n\in\mathbb{N}} |\hat{f}(n)|^2 = \sup_{0<r<1} \sum_{n\in\mathbb{N}} |\hat{f}(n)|^2 r^{2n} = \|f\|^2_{\mathscr{H}^2(\mathbb{D})}.$$

The following calculation shows that point evaluation is a bounded linear functional on $\mathscr{H}^2$:

$$\begin{aligned} |f(z)| &\le \sum_{k=0}^{\infty} |\hat{f}(k)||z|^k \\ &\le (\sum |\hat{f}(k)|^2)^{1/2} (\sum |z|^2 k)^{1/2} \\ &= \frac{\|f\|}{\sqrt{1-z^2}}. \end{aligned} \tag{B.2}$$

Functions in $\mathscr{H}^p(\mathbb{D})$ admit nontangential limits a.e. on $\partial\mathbb{D}$ and, moreover, the boundary function is in $L^p(\partial\mathbb{D})$, [37, Thm 2.2]. One easily derives that $\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it}) - f(e^{it})|^2 \,\mathrm{d}t \to 0$ as $r \to 1^-$ for all $f \in \mathscr{H}^2(\mathbb{D})$ (see also [37, Thm. 2.6]):

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it}) - f(e^{it})|^2 \,\mathrm{d}t &\le \liminf_{s\to 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it}) - f(se^{it})|^2 \,\mathrm{d}t \\ &\le \sum_{n=1}^{\infty} |\hat{f}(n)|^2 (1-r^n)^2 \to 0 \quad (r \to 1^-). \end{aligned}$$

B.1. Composition operators: boundedness and compactness

For convenience, we set $\mathscr{H}^2 := \mathscr{H}^2(\mathbb{D})$ and $\|\cdot\|$ instead of $\|\cdot\|_{\mathscr{H}^2}$. In what follows, we present the proof of Littlewood's subordination principle as it is given in [69].

B.1.1 Proposition (Littlewood's subordination principle). *Let* $\varphi \in \mathscr{H}(\mathbb{D},\mathbb{D})$ *with* $\varphi(0) = 0$. *Then*

$$\|C_\varphi\|_{\mathscr{L}(\mathscr{H}^2(\mathbb{D}))} \le 1. \tag{B.3}$$

Proof. Let φ be a holomorphic selfmap of the unit disk fixing the origin. We first note that the multiplication operator $M_\varphi : \mathscr{H}^2 \to \mathscr{H}^2, f \mapsto f \cdot \varphi$ is contractive.
Next we define the backward-shift operator on $\mathscr{H}^2$ as follows

$$B : \mathscr{H}^2 \to \mathscr{H}^2 \quad f(z) \mapsto \sum_{n=0}^{\infty} \hat{f}(n+1) z^n.$$

Then

$$f(z) = f(0) + zBf(z) \text{ and} \tag{B.4}$$

$$(B^n f)(0) = \hat{f}(n) \tag{B.5}$$

We prove the assertion first for polynomials. Let $f \in \mathscr{H}^2$ be a polynomial. Then by (B.4) and $\varphi(0) = 0$, we derive

$$C_\varphi f(z) = f(\varphi(z)) = f(0) + \varphi(z)Bf(z) = f(0) + M_\varphi C_\varphi(Bf(z)) \tag{B.6}$$

Thus

$$\|C_\varphi f\|^2 = |f(0)|^2 + \|M_\varphi C_\varphi Bf\|^2 \leq |f(0)|^2 + \|C_\varphi Bf\|^2. \tag{B.7}$$

For all $n \in \mathbb{N}$, we get

$$\|C_\varphi B^n f\|^2 \leq |B^n f(0)|^2 + \|C_\varphi B^{n+1} f\|^2. \tag{B.8}$$

Then (B.6)-(B.8) together yield

$$\|C_\varphi f\|^2 \leq \sum_{k=0}^{n} |B^k f(0)|^2 + \|C_\varphi B^{n+1} f\|^2.$$

Assume f has degree n. For $m > n$, we derive

$$\|C_\varphi f\|^2 \leq \sum_{k=0}^{m} |B^k f(0)|^2 \leq \sum_{k=0}^{\infty} |B^k f(0)|^2 = \|f\|^2.$$

Now assume that $f \in \mathscr{H}^2$ is not a polynomial. First note that by (B.2) if there is a sequence $(f_n)_n$ such that $f_n \to f$ in $\mathscr{H}^2$ as $n \to \infty$, then $f_n \to f$ locally uniformly as $n \to \infty$. More precisely, for $r < 1$ and a sequence $(f_n)_n$ with $f_n \to f$ in $\mathscr{H}^2$, we derive

$$\sup_{|z|<r} |f_n(z) - f(z)| \leq \frac{\|f_n - f\|}{\sqrt{1 - r^2}}.$$

Now, let $r \in (0,1)$. Defining the sequence $(f_n)_{n\in\mathbb{N}}$ by $f_n = \sum_{k=0}^{n} \hat{f}(n)$ we derive

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} |f \circ \varphi(re^{it})|^2 \, \mathrm{d}t &= \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} |f_n \circ \varphi(re^{it})|^2 dt \\
&\leq \limsup_{n\to\infty} \|f_n \circ \varphi\| \\
&\leq \limsup_{n\to\infty} \|f_n\|. \\
&\leq \|f\|.
\end{aligned}$$

Letting $r \to 1^-$ gives the desired assertion of the proposition. □

Using Möbius transforms, we shift a fixed point in the origin of a holomorphic selfmap of the unit disk to any point in $\bar{\mathbb{D}}$. By a change of variables we obtain

$$\|f \circ \varphi\|_{\mathscr{H}^2(\mathbb{D})} \leq \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}} \|f\|_{\mathscr{H}^2(\mathbb{D})}$$

for any holomorphic selfmap $\varphi \in \mathscr{H}(\mathbb{D},\mathbb{D})$. It is well-known that functions in Hardy spaces can be decomposed into a product of an inner and an outer function. Actually the inner function is the product of a Blaschke product and a singular inner function; for more details on the factorization of functions in Hardy spaces, we refer to [37, Thm. 2.8]. For every $f \in \mathscr{H}^p(\mathbb{D})$ there is a function $f_i \in \mathscr{H}(\mathbb{D})$, with $|f_i| \leq 1$ in $\mathbb{D}$ and $|f| = 1$ on $\partial\mathbb{D}$, and a $c \in \partial\mathbb{D}$ such that the function $f_o := c\exp(\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log f(e^{it})\,\mathrm{d}t) \in \mathscr{H}^p(\mathbb{D})$ and $\|f_o\|_{\mathscr{H}^p(\mathbb{D})} = \|f\|_{\mathscr{H}^p(\mathbb{D})}$, such that $f = f_i \cdot f_o$. Using this we generalize Littlewood's subordination principle to functions in $\mathscr{H}^p(\mathbb{D})$ for $p \neq 2$: Let $\varphi \in \mathscr{H}(\mathbb{D},\mathbb{D})$ and $f \in \mathscr{H}^p(\mathbb{D})$. Then

$$\begin{aligned}\left\|C_\varphi f\right\|_p &\leq \left\|C_\varphi f_o\right\|_p \\ &= \left\|C_\varphi f_o^{p/2}\right\|_2^{2/p} \\ &\leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{p}{2}} \underbrace{\left\|f_o^{p/2}\right\|_2^{2/p}}_{=\|f_o\|_p=\|f\|_p}.\end{aligned}$$

As an example, we would like to present how certain symbols induce compactness of a composition operator.

B.1.2 Theorem ([70, Sect. 2.2]). *Let $\varphi \in \mathscr{H}(\mathbb{D},\mathbb{D})$ with $\|\varphi\|_\infty < 1$. Then C_φ is compact on $\mathscr{H}^2$.*

Proof. We define for $f \in \mathscr{H}^2$ the following family of operators $T_n f := \sum_{k=0}^n \hat{f}(k)\varphi^k$, and we note that $T_n\colon \mathscr{H}^2 \to \operatorname{span}\{\varphi^k | k \leq n\}$ is a finite rank operator for all $n \in \mathbb{N}$. We show that C_φ is compact by proving $\|T_n - C_\varphi\| \to 0$. Let $f \in \mathscr{H}^2$. Then

$$\begin{aligned}\|(C_\varphi - T_n)f\| &= \left\|f \circ \varphi - \sum_{k=0}^n \hat{f}(k)\varphi^k\right\| \\ &= \left\|\sum_{k=n+1}^\infty \hat{f}(k)\varphi^k\right\| \\ &\leq \sum_{k=n+1}^\infty |\hat{f}(k)|\|\varphi^k\|_\infty \\ &\leq \left(\sum_{k=n+1}^\infty |\hat{f}(k)|^2\right)^{1/2} \left(\sum_{k=n+1}^\infty \|\varphi\|_\infty^{2k}\right)^{1/2} \\ &\leq \|f\| \frac{1}{\sqrt{1-\|\varphi\|_\infty^2}} \|\varphi\|_\infty^{n+1}.\end{aligned}$$

Thus

$$\|C_\varphi - T_n\| \to 0, n \to \infty.$$

□

There is a rich theory on compactness of composition operators in terms of their symbols which can be found in [70].

B.2. Point spectrum of the generator of a composition semigroup

The point spectrum of the generator of a semigroup of composition operators on the Hardy space has already been determined by Berkson and Porta in [20]. A generalization of this result to weighted composition operators has been obtained by Siskakis in [71], and for (unweighted) composition semigroups on Hardy spaces of the half plane by Arvanitidis [13]. In any of these, the point spectrum is calculated by means of the argument principle from one-dimensional complex analysis. We present here, as an example, how the point spectrum can be calculated for a generator of a composition semigroup induced by a semiflow fixing the origin. Let $(\varphi_t)_{t\geq 0}$ be a semiflow generated by G, with Denjoy-Wolff point in 0. Let Γ be the generator of the composition semigroup $(C_{\varphi_t})_{t\geq 0}$ on $\mathscr{H}^p(\mathbb{D})$. We show how the point spectrum of Γ can be determined with tools from complex analysis. To calculate the point spectrum, we need to solve the following equation

$$\Gamma f(z) = G(z)f'(z) = \lambda f(z) \tag{B.9}$$

By the Berkson-Porta representation, there is a holomorphic function $F : \mathbb{D} \to \mathbb{C}$ with positive real part such that $G(z) = -zF(z)$, $z \in \mathbb{D}$.

Assume we are given a solution $f \neq 0$ to (B.9) for some $\lambda \in \mathbb{C}$. Then f admits at most countably many zeros on compact subset of $\mathbb{D}$. We rewrite (B.9) as follows:

$$-\frac{f'(z)}{f(z)} = \lambda \frac{1}{zF(z)}.$$

Now choose $r > 0$ such that f admits no zero on $|z| = r$ and, say, $k \in \mathbb{N}$ zeros in $\{z \in \mathbb{C} : |z| < r\}$. Then, we derive candidates for the eigenvalues of Γ by the argument principle:

$$\begin{aligned} -\frac{1}{2\pi i} \int\limits_{|z|=r} \frac{f'(\xi)}{f(\xi)}\, \mathrm{d}\xi &= \lambda \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{1}{F(\xi)\xi}\, \mathrm{d}\xi \\ \Rightarrow -kF(0) &= \lambda. \end{aligned}$$

Indeed, we can use the Koenigs function of $(\varphi_t)_{t\geq 0}$ to tell if these candidates are eigenvalues of Γ.

B.2.1 Lemma. *Consider Γ as above and let h be the Koenigs function associated to $(\varphi_t)_{t\geq 0}$. Then $-kF(0) \in \sigma_p(\Gamma)$ if and only if $h^k \in \mathscr{H}^p$. In particular $\sigma_p(\Gamma) = \{-kF(0) \colon h^k \in \mathscr{H}^p, k \in \mathbb{N}\}$.*

Proof. Let $k \in \mathbb{N}$ such that

$$h^k(z) = z^k \exp\left(\int\limits_0^z -k \frac{F(\xi) - F(0)}{\xi F(xi)}\, \mathrm{d}\xi \right) =: f(z) \quad (z \in \mathbb{D})$$

belongs to $\mathscr{H}^p$. Then

$$
\begin{aligned}
G(z)f'(z) = -zF(z)\Bigg(&kz^{k-1}\exp\left(\int_0^z -k\frac{F(\xi)-F(0)}{\xi F(\xi)}\,\mathrm{d}\xi\right)+\\
&-kz^k\frac{F(z)-F(0)}{zF(z)}\exp\left(\int_0^z -k\frac{F(\xi)-F(0)}{\xi F(\xi)}\,\mathrm{d}\xi\right)\Bigg)\\
&=-kF(z)z^k\exp\left(\int_0^z -k\frac{F(\xi)-F(0)}{\xi F(\xi)}\,\mathrm{d}\xi\right)\left(1-\frac{F(z)-F(0)}{F(z)}\right)\\
&=-kF(0)f(z).
\end{aligned}
$$

Hence $f\in\operatorname{dom}\Gamma\setminus\{0\}$ and f solves the equation $G(z)f'(z)=-kF(0)f(z)$. So $-kF(0)\in\sigma_p$.

Conversely, let $-kF(0)\in\sigma_p(\Gamma)$ and $f\in\operatorname{dom}\Gamma\setminus\{0\}$ admitting k zeros in 0. Then there is a holomorphic function g with $g(0)\neq 0$ such that $f(z)=z^k g(z)$. Thus f satisfies (B.9) by our calculations above. Then

$$
\begin{aligned}
&G(z)f'(z)=-kF(0)f(z)\\
\Leftrightarrow\ &-zF(z)z^kg(z)=-kF(0)z^kg(z)\\
\Leftrightarrow\ &zF(z)(kz^{k-1}g(z)+z^kg'(z))-kF(0)z^kg(z)=0\\
\Leftrightarrow\ &z^kF(z)g(z)+z^{k+1}F(z)g'(z)-kF(0)z^kg(z)=0\\
\Leftrightarrow\ &kz^k(F(z)-F(0))g(z)+g'(z)z^{k+1}F(z)=0.
\end{aligned}
$$

We divide the last equation for $z\neq 0$ by $z^{k+1}F(z)$:

$$
\begin{aligned}
&-kg(z)\frac{F(z)-F(0)}{zF(z)}=g'(z)\\
\Leftrightarrow\ &-k\frac{F(z)-F(0)}{zF(z)}=\frac{g'(z)}{g(z)}\\
\Rightarrow\ &\int_0^z -k\frac{F(\xi)-F(0)}{\xi F(\xi)}\,\mathrm{d}\xi=\int_0^z\frac{g'(\xi)}{g(\xi)}\,\mathrm{d}\xi\\
\Leftrightarrow\ &\int_0^z -k\frac{F(\xi)-F(0)}{\xi F(\xi)}\,\mathrm{d}\xi=\log\frac{g(z)}{g(0)}\\
\Leftrightarrow\ &\exp\left(\int_0^z -k\frac{F(\xi)-F(0)}{\xi F(\xi)}\,\mathrm{d}\xi\right)=\frac{g(z)}{g(0)}.
\end{aligned}
$$

Since $z^kg(z)\in\mathscr{H}^p$ by assumption, we deduce $h^k(z)=z^k\frac{g(z)}{g(0)}\in\mathscr{H}^p$. □

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation zum Thema *Holomorphic Semiflows and Poincaré-Steklov Semigroups* am Institut für Analysis der TU Dresden unter Betreuung von Prof. Dr. Ralph Chill ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt. Ich erkenne die Promotionsordnung in der Version vom 23.02.2011 an.

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