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A note on a two-temperature model in linear thermoelasticity

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Abstract
We discuss the so-called two-temperature model in linear thermoelasticity and provide a Hilbert space framework for proving well-posedness of the equations under consideration. With the abstract perspective of evolutionary equations, the two-temperature model turns out to be a coupled system of the elastic equations and an abstract ordinary differential equation (ODE). Following this line of reasoning, we propose another model which is entirely an abstract ODE. We also highlight an alternative method for a two-temperature model, which might be of independent interest.

Keywords
Evolutionary equations, thermoelasticity, two-temperature model, coupled systems.

1. Introduction

Chen and Gurtin [1] and Chen et al. [2, 3] have given the formulation of the theory of heat conduction related to a deformable body which is based on two different temperatures. Here the first one is the conductive temperature, ϕ, and the other one is the thermodynamic temperature, θ. Chen et al. [2] discussed that these two temperatures are equal in the absence of a heat supply in the case of time-independent situations and the difference between these two temperatures is proportional to the heat supply, where, in the time-dependent case, these two temperatures are different, in general. Before these studies, by doing the study of the transient coupled thermoelastic boundary value problem in half space, Boley and Tolins [4] gave the conclusion that the strain and two temperatures are found to have an explanation in the form of a wave plus a response taking place immediately through the body. The uniqueness and reciprocity theorems for the two-temperature thermoelasticity theory in the case of a homogeneous and isotropic solid were reported by Iesan [5]. Subsequently, investigations were carried out on the basis of this theory by several researchers like Warren and Chen [6], Warren [7], Amos [8], Chakrabarti [9], and so on. This theory (2TT) has drawn the attention of researchers in recent years and some specific features of this theory have been reported (see [10–18] and the references there-in).

A structural formulation for linear material laws in classical mathematical physics was introduced by Picard [19] who considered a class of evolutionary problems which covers a number of initial boundary value problems of classical mathematical physics. The corresponding solution theory is also established in [19]. Prior to this,

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Picard [20] also reported the structural formulation for linear thermoelasticity in nonsmooth media. Recently, Mukhopadhyay et al. [21] have studied various models of thermoelasticity theory and have shown that these models can be treated within the common structural framework of evolutionary equations, and considering the flexibility of the structural perspective they obtained well-posedness results for a large class of generalized models allowing for more general material properties such as anisotropies, inhomogeneities, and so on. It should be noted that evolutionary equations in the form just discussed have also been studied with regards to homogenization theory; see for example [22–24]. The aim of this article is to analyze the two-temperature thermoelastic model given by Chen and Gurtin [1] as a first-order system within the framework of evolutionary equations; see for example [25]. The model of thermoelasticity we shall discuss was originally conceived as a constant coefficient model. There is little harm in this assumption at this point, since we shall dispose of this simplification completely when we discuss more general models in the last two sections. An alternative two-temperature thermoelastic model is proposed in which we can avoid involving roots of an unbounded operator. It is believed that the general perspective on two-temperature thermoelasticity to be presented may shed some new light on the theory of homogenization of such models.

In Section 2, we discuss the functional analytic background needed for discussing the two-temperature model. Section 3 discusses the two-temperature model in detail. In this section, we will also give a suitable Hilbert space framework allowing for well-posedness of the respective equation. An observation in Section 3 is that the heat equation part is replaced by an abstract ordinary differential equation (ODE) with an infinite-dimensional state space. More precisely, in the heat equation part the *only* unbounded operator involved is the time derivative. Having realized this property of the two-temperature model, we propose in the two concluding sections, Sections 4 and 5, alternative systems of thermoelasticity, the first one being entirely an abstract ODE in the sense just discussed. The second one describes a possible alternative model, which does not involve square roots of operators.

2. Functional analytic preliminaries

In this section, we shall elaborate on some standard concepts in functional analysis needed in the following. Most frequently, we will have occasion to use the square root and the modulus of an operator.

Definition 1. Let H_0, H_1 be Hilbert spaces. Let $C\colon D(C) \subseteq H_0 \to H_0$ be a non-negative-definite, selfadjoint operator, that is, for all $\phi \in D(C)$ we have $\langle \phi, C\phi \rangle \geq 0$ and $C = C^*$. Then $\sqrt{C}$ is defined as the unique non-negative-definite operator satisfying $\sqrt{C}\sqrt{C} = C$. For $A\colon D(A) \subseteq H_0 \to H_1$, a closed and densely defined linear operator, we define the modulus of A, $|A|$, by

$$|A| := \sqrt{A^*A}.$$

Recall that $D(A) = D(|A|)$ and that $\|A\phi\| = \|\,|A|\,\phi\|$ for all $\phi \in D(A)$. We record the following standard fact.

Proposition 2. *Let H_0, H_1 be Hilbert spaces, and $A\colon D(A) \subseteq H_0 \to H_1$ be densely defined, closed, linear. Then*

$$\overline{\left(\sqrt{1+|A^*|^2}\right)^{-1} A} = A\left(\sqrt{1+|A|^2}\right)^{-1} \in L(H_0, H_1)$$

with $\left\| A\left(\sqrt{1+|A|^2}\right)^{-1} \right\| \leq 1.$

Proof. Let $A = U\,|A|$ with a partial isometry U, being in particular a contraction i.e. $\|U\| \leq 1$. We have by the spectral theorem $\left(\sqrt{1+|A|^2}\right)^{-1} \phi \in D(|A|)$ for all $\phi \in H_0$, and thus

$$\begin{aligned}\left\| A\left(\sqrt{1+|A|^2}\right)^{-1} \phi \right\| &= \left\| |A|\left(\sqrt{1+|A|^2}\right)^{-1} \phi \right\| \\ &\leq \|\phi\|, \qquad (\phi \in H_0),\end{aligned}$$

establishing the boundedness and the norm estimate of the operator $A\left(\sqrt{1+|A|^2}\right)^{-1}$. As the operator $\left(\sqrt{1+|A^*|^2}\right)^{-1}A$ is densely defined, for the asserted equality in the proposition, it suffices to establish the inclusion

$$\left(\sqrt{1+|A^*|^2}\right)^{-1}A \subseteq A\left(\sqrt{1+|A|^2}\right)^{-1}. \tag{2.1}$$

Next, we prove (2.1): for this, by induction, we show the inclusion

$$(1+AA^*)^{-n}A \subseteq A(1+A^*A)^{-n} \quad (n \in \mathbb{N}). \tag{2.2}$$

For proving the latter inclusion for $n = 1$, observe that for $\phi \in D(AA^*A)$ we have

$$(1+AA^*)A\phi = A(1+A^*A)\phi.$$

Hence, substituting $\psi := (1+A^*A)\phi$, we get

$$A(1+A^*A)^{-1}\psi = (1+AA^*)^{-1}A\psi.$$

So, for every $n \in \mathbb{N}$ the inductive step can be shown as follows:

$$\begin{aligned}(1+AA^*)^{-(n+1)}A &= (1+AA^*)^{-n}(1+AA^*)^{-1}A\\ &\subseteq (1+AA^*)^{-n}A(1+A^*A)^{-1}\\ &\subseteq A(1+A^*A)^{-n}(1+A^*A)^{-1}\\ &= A(1+A^*A)^{-(n+1)}.\end{aligned}$$

For the proof of (2.1), we recall that for every real number $x > 0$ with $|x| < 1$ the binomial series gives

$$\sqrt{1+x} = \sum_{n=0}^{\infty}\binom{1/2}{n}x^n. \tag{2.3}$$

Putting $x_\varepsilon := -\varepsilon y(1+y)^{-1}$ for some $y \geq 0$ and $\varepsilon \in]0,1[$, we have $|x_\varepsilon| \leq \varepsilon, 1 + x_\varepsilon = (1+(1-\varepsilon)y)(1+y)^{-1}$, which also leads to

$$\sqrt{1+x_\varepsilon} = \sqrt{(1+(1-\varepsilon)y)(1+y)^{-1}} \to \sqrt{(1+y)^{-1}} \quad (\varepsilon \to 1). \tag{2.4}$$

Moreover, plugging x_ε into the series (2.3), we arrive at

$$\begin{aligned}\sqrt{1+x_\varepsilon} &= \sum_{n=0}^{\infty}\binom{1/2}{n}x_\varepsilon^n = \sum_{n=0}^{\infty}\binom{1/2}{n}\left(-\varepsilon y(1+y)^{-1}\right)^n\\ &= \sum_{n=0}^{\infty}\binom{1/2}{n}\left((-\varepsilon)^n y^n(1+y)^{-n}\right).\end{aligned}$$

By the functional calculus for selfadjoint operators, we may replace y in the latter expression by A^*A and AA^*, respectively. Thus, for $\varepsilon \in]0,1[$, we set

$$\begin{aligned}B_{1,\varepsilon} &:= \sum_{n=0}^{\infty}\binom{1/2}{n}\left((-\varepsilon)^n (AA^*)^n (1+(AA^*))^{-n}\right),\\ B_{2,\varepsilon} &:= \sum_{n=0}^{\infty}\binom{1/2}{n}\left((-\varepsilon)^n (A^*A)^n (1+(A^*A))^{-n}\right).\end{aligned} \tag{2.5}$$

Note that $B_{1,\varepsilon}$ and $B_{2,\varepsilon}$ define bounded linear operators. Moreover, by the spectral theorem (write AA^* and A^*A as multiplication operators in a suitable L^2-space), we get, invoking (2.4),

$$B_{1,\varepsilon} \to \sqrt{(1+AA^*)^{-1}} \text{ and } B_{2,\varepsilon} \to \sqrt{(1+A^*A)^{-1}} \tag{2.6}$$

as $\varepsilon \to 1$ in the strong operator topology. Thus, for $\varepsilon \in]0,1[$, with the help of (2.2) and (2.5) we get

$$\begin{aligned}
B_{1,\varepsilon}A &= \sum_{n=0}^{\infty} \binom{1/2}{n} \left((-\varepsilon)^n (AA^*)^n (1+(AA^*))^{-n}\right) A \\
&\subseteq \sum_{n=0}^{\infty} \binom{1/2}{n} \left((-\varepsilon)^n (AA^*)^n A (1+(A^*A))^{-n}\right) \\
&= \sum_{n=0}^{\infty} \binom{1/2}{n} \left((-\varepsilon)^n A (A^*A)^n (1+(A^*A))^{-n}\right) \\
&= A \sum_{n=0}^{\infty} \binom{1/2}{n} \left((-\varepsilon)^n (A^*A)^n (1+(A^*A))^{-n}\right) \\
&= AB_{2,\varepsilon}.
\end{aligned}$$

Thus, the closedness of A together with (2.6) yields the asserted inclusion (2.1). □

Another fact used in the following is mentioned in the next proposition.

Proposition 3. *Let H_0, H_1 be Hilbert spaces, $A\colon D(A) \subseteq H_0 \to H_1$ be densely defined, closed, linear, and $\kappa \in L(H_1)$ with $0 \in \varrho(\kappa)$. Then κA is densely defined and closed and we have*

$$(\kappa A)^* = A^*\kappa^*.$$

Proof. The operator κA is clearly densely defined. Moreover, if $(\phi_n)_n$ is a sequence in $D(A)$ such that $(\phi_n)_n$ and $(\kappa A\phi_n)_n$ are convergent to $\psi \in H_0$ and $\eta \in H_1$, we infer, by the continuous invertibility of κ and the closedness of A, $\psi \in D(A)$ and $A\psi = \kappa^{-1}\psi$. Hence, κA is closed. The equality $(\kappa A)^* = A^*\kappa^*$ is also easy. □

Next, we briefly recall the functional analytic setting in which we are going to discuss the two-temperature model later on. A more detailed discussion can be found in [19, 25] or (particularly concerning the time derivative) in [26]. See also [20].

Definition 4. Let $\nu > 0$, and H be a Hilbert space. Define $L^2_\nu(\mathbb{R}, H)$ to be the space of (equivalence classes of) square integrable functions $f\colon \mathbb{R} \to H$ with respect to the measure with Lebesgue density $x \mapsto e^{-2\nu x}$. Denote the space of L^2_ν-functions f with distributional derivative f' representable as $L^2_\nu(\mathbb{R}, H)$-function by $H_{\nu,1}(\mathbb{R}, H)$. Define

$$\partial_0\colon H_{\nu,1}(\mathbb{R}, H) \subseteq L^2_\nu(\mathbb{R}, H) \to L^2_\nu(\mathbb{R}, H), f \mapsto f'.$$

Note that we will not notationally distinguish between the time derivative realized as an operator in $L^2_\nu(\mathbb{R}, H_1)$ and $L^2_\nu(\mathbb{R}, H_2)$ for possibly different Hilbert spaces H_1 and H_2. The reason for introducing this particularly weighted L^2-space is the fact that ∂_0 becomes a *continuously invertible* operator. In fact, one has $\|\partial_0^{-1}\| \leq 1/\nu$; see [26].

For a closed and densely defined linear operator $C\colon D(C) \subseteq H_0 \to H_1$ between the Hilbert spaces H_0 and H_1, the lifted operator as an abstract multiplication operator from $L^2_\nu(\mathbb{R}, H_0)$ to $L^2_\nu(\mathbb{R}, H_1)$ will be denoted by the same notation. With these conventions, we can come to (a special case of) the solution theory first established in [19]. We mention here possible generalizations to non-autonomous [27, 28] or non-linear frameworks [29, 30]. Denoting the range of an operator M_0 by $R(M_0)$ and its kernel by $N(M_0)$ we recall the following general solution theory result from [19, 25].

Theorem 5. *Let H be a Hilbert space, $M_0 = M_0^*$, $M_1 \in L(H)$, $A\colon D(A) \subseteq H \to H$ skew-selfadjoint. Assume there exists $c > 0$ such that $\langle M_0\phi, \phi\rangle \geq c\langle \phi, \phi\rangle$ and $\mathfrak{Re}\,\langle M_1\psi, \psi\rangle \geq c\langle \psi, \psi\rangle$ for all $\phi \in \overline{R(M_0)}$, $\psi \in N(M_0)$. Then there exists $\nu_0 \geq 0$ such that for all $\nu > \nu_0$ the operator sum*

$$\mathcal{B} := \partial_0 M_0 + M_1 + A$$

is closable as an operator in $L^2_\nu(\mathbb{R}, H)$ and the closure $\overline{\mathcal{B}}$ is continuously invertible in $L^2_\nu(\mathbb{R}, H)$. Moreover, $\overline{\mathcal{B}}^{-1}$ is causal *in the sense that given $f \in L^2_\nu(\mathbb{R}, H)$ with the property that $f = 0$ on $(-\infty, a]$ for some $a \in \mathbb{R}$, then $\overline{\mathcal{B}}^{-1} f = 0$ on $(-\infty, a]$.*

The latter theorem tells us that the non-homogeneous problem $\overline{\mathcal{B}}u = f$ admits a unique solution for all $f \in L^2_\nu(\mathbb{R}, H)$ given ν sufficiently large. In [25] how to invoke initial value problems in this context has been shown. Note that it is also possible to show that the solution u does *not* depend on the parameter ν. That is, let $\mu, \nu > 0$ be sufficiently large: then the solution operators $\overline{\mathcal{B}_\nu}^{-1}$ and $\overline{\mathcal{B}_\mu}^{-1}$ established in $L^2_\nu(\mathbb{R}, H)$ and $L^2_\mu(\mathbb{R}, H)$, respectively, coincide on the intersection of the respective domain, that is, on $L^2_\nu(\mathbb{R}, H) \cap L^2_\mu(\mathbb{R}, H)$.

Later on, we will also need the operations skew : $\mathbb{C}^{3\times 3} \to \mathbb{C}^{3\times 3}, A \mapsto \frac{1}{2}\left(A - A^{\mathrm{T}}\right)$ and sym : $\mathbb{C}^{3\times 3} \to \mathbb{C}^{3\times 3}, A \mapsto \frac{1}{2}\left(A + A^{\mathrm{T}}\right)$.

3. The two-temperature model

In this section, we shall have a deeper look into the two-temperature model found in [1]. For this, however, we have to introduce several vector analytical operators. In the whole section, we assume we are given an open set $\Omega \subseteq \mathbb{R}^n$.

Definition 6. We denote by $\overset{\circ}{C}_\infty(\Omega)$ the set of smooth functions with compact support. Then, we define, as usual, $\operatorname{Grad}\Phi$ to be the symmetric part of the 3×3-matrix-valued derivative of a smooth vector field Φ, $\operatorname{grad}\phi$ to be the gradient of a smooth function ϕ and $\operatorname{Div}\Psi$ and $\operatorname{div}\psi$ to be the row-wise and the usual divergence for a smooth matrix-valued function Ψ and a smooth vector-valued function ψ, respectively. Reusing the notation Grad, grad, Div and div for the respective $L^2(\Omega)$-realizations, we further define

$$\overset{\circ}{\operatorname{Grad}} := \overline{\operatorname{Grad}\big|_{\overset{\circ}{C}_\infty(\Omega)^3}}$$

$$\overset{\circ}{\operatorname{Div}} := \overline{\operatorname{Div}\big|_{\operatorname{sym}\left[\overset{\circ}{C}_\infty(\Omega)^{3\times 3}\right]}}$$

$$\overset{\circ}{\operatorname{grad}} := \overline{\operatorname{grad}\big|_{\overset{\circ}{C}_\infty(\Omega)}}$$

$$\overset{\circ}{\operatorname{div}} := \overline{\operatorname{div}\big|_{\overset{\circ}{C}_\infty(\Omega)^3}}$$

and their respective $L^2(\Omega)$-type adjoints

$$-\operatorname{Div} := \left(\operatorname{Grad}\big|_{\overset{\circ}{C}_\infty(\Omega)^3}\right)^*$$

$$-\operatorname{Grad} := \left(\operatorname{Div}\big|_{\operatorname{sym}\left[\overset{\circ}{C}_\infty(\Omega)^{3\times 3}\right]}\right)^*$$

$$-\operatorname{div} := \left(\operatorname{grad}\big|_{\overset{\circ}{C}_\infty(\Omega)}\right)^*$$

$$-\operatorname{grad} := \left(\operatorname{div}\big|_{\overset{\circ}{C}_\infty(\Omega)^3}\right)^*.$$

Note that here Div maps from and Grad maps into the Hilbert space $L^2_{\mathrm{sym}}(\Omega) := L^2\left(\Omega, \mathrm{sym}\left[\mathbb{C}^{3\times 3}\right]\right)$ of 3×3-symmetric-matrix-valued L^2-type mappings.

In the so-called two-temperature models of Chen and Gurtin [1], apart from the temperature θ another temperature ϕ, the conductive temperature, is introduced (together with a reference temperature $T_0 \in]0, \infty[$) such that

$$\theta - (\phi - T_0) = \alpha \operatorname{div} q. \tag{3.1}$$

Here $\alpha \in]0, \infty[$ is a parameter, called the *two-temperature parameter*. Assuming homogeneous Dirichlet boundary conditions, Fourier's law is then formulated in terms of the conductive temperature as

$$q = -\kappa \,\mathring{\operatorname{grad}}\, (\phi - T_0), \tag{3.2}$$

where $\kappa \in L(L^2(\Omega)^3)$ is a selfadjoint operator with $\kappa \geq c > 0$. In addition, the two-temperature system consists of the heat equation with mass density $\varrho_0 \in L^\infty(\Omega)$, $\varrho_0 \geq c_0 > 0$, that is,

$$\partial_0 (\varrho_0 T_0 \eta) + \operatorname{div} q = \varrho_0 Q$$

or, for our purposes, more conveniently,

$$\partial_0 (\varrho_0 \eta) + \operatorname{div} (q/T_0) = \varrho_0 Q/T_0, \tag{3.3}$$

where q is the heat flux as in (3.2), η is the entropy and Q is the heat source. For the entropy η we have the following material law relating the entropy to the temperature θ and the strain tensor $\mathcal{E} = \mathring{\operatorname{Grad}} u$, u being the displacement,

$$\varrho_0 T_0 \eta = \varrho_0 \lambda \theta + T_0 \gamma^* \mathcal{E} \tag{3.4}$$

for some scalar $\lambda > 0$, and an operator $\gamma \in L(L^2(\Omega), L^2_{\mathrm{sym}}(\Omega))$. Next, the strain tensor $\mathcal{E} = \mathring{\operatorname{Grad}} u$ is related to the stress tensor σ and the temperature via the elasticity tensor $C = C^* \in L\left(L^2_{\mathrm{sym}}(\Omega)\right)$ which is strictly positive definite and γ in the following way:

$$\mathcal{E} = C^{-1}\sigma + C^{-1}\gamma\theta. \tag{3.5}$$

The two-temperature model is completed by the balance of momentum

$$\varrho_0 \partial_0^2 u - \operatorname{Div} \sigma = \varrho_0 F \tag{3.6}$$

for some given external force F.

In the following, we will show that Theorem 5 is applicable to the equations (3.1) to (3.6). Hence, the Hilbert space setting introduced in the previous section provides a functional analytic framework such that for all right-hand sides F and Q there exists a unique solution to the two-temperature model depending continuously on F and Q. So, the task to be solved in the next lines is to find the right unknowns and, hence, the right operators M_0, M_1 and A, making Theorem 5 applicable.

It should be noted that our reformulation of the two-temperature model reveals that the introduction of the second temperature transforms the heat equation into an ODE with an infinite-dimensional state space.

A first step towards our main goal in this section is the following observation yielded by (3.1) and (3.2).

Proposition 7. *Let $\kappa = \kappa^* \in L(L^2(\Omega)^3)$ be strictly positive definite. Assume that $T_0, \alpha \in]0, \infty[$ and $\theta \in L^2(\Omega)$ and $q \in D(\operatorname{div})$, $\phi \in D(\mathring{\operatorname{grad}})$ satisfy (3.1) and (3.2). Then with[1] $\kappa_\alpha := \sqrt{\alpha}\kappa\sqrt{\alpha}$ we have*

$$\sqrt{1 - \sqrt{\kappa_\alpha}\,\mathring{\operatorname{grad}}\operatorname{div}\sqrt{\kappa_\alpha}}\sqrt{\kappa}^{-1} q = -\sqrt{\kappa}\,\mathring{\operatorname{grad}}\sqrt{1 - \operatorname{div}\kappa_\alpha \mathring{\operatorname{grad}}}^{-1}\theta. \tag{3.7}$$

Proof. Plugging in Fourier's law we can rewrite (3.1) as

$$\theta = \left(1 - \operatorname{div}\kappa_\alpha \mathring{\operatorname{grad}}\right)(\phi - T_0). \tag{3.8}$$

The operator $\sqrt{\kappa_\alpha}\mathring{\mathrm{grad}} : D(\mathring{\mathrm{grad}}) \subseteq L^2(\Omega) \to L^2(\Omega)^3$ is a closed densely defined linear operator, since κ and hence $\sqrt{\kappa_\alpha}$ are boundedly invertible: see Proposition 3. Moreover, its adjoint is given by $\left(\mathring{\mathrm{grad}}\right)^* \sqrt{\kappa_\alpha} = -\operatorname{div}\sqrt{\kappa_\alpha}$ (Proposition 3) and thus $-\operatorname{div}\kappa_\alpha\mathring{\mathrm{grad}}$ is a selfadjoint, non-negative operator. In particular, $1-\operatorname{div}\kappa_\alpha\mathring{\mathrm{grad}}$ is boundedly invertible. Hence, rephrasing (3.2) in terms of the temperature θ, we are led to

$$
\begin{aligned}
q &= -\kappa\mathring{\mathrm{grad}}\left(1-\operatorname{div}\kappa_\alpha\mathring{\mathrm{grad}}\right)^{-1}\left(1-\operatorname{div}\kappa_\alpha\mathring{\mathrm{grad}}\right)(\phi - T_0)\\
&= -\kappa\mathring{\mathrm{grad}}\left(1-\operatorname{div}\kappa_\alpha\mathring{\mathrm{grad}}\right)^{-1}\theta.
\end{aligned}
$$

Next, applying Proposition 2 to $A := \sqrt{\kappa_\alpha}\mathring{\mathrm{grad}}$ we infer

$$
\sqrt{1-\sqrt{\kappa_\alpha}\mathring{\mathrm{grad}}\operatorname{div}\sqrt{\kappa_\alpha}}^{\,-1}\sqrt{\kappa_\alpha}\mathring{\mathrm{grad}} \subseteq \sqrt{\kappa_\alpha}\mathring{\mathrm{grad}}\sqrt{1-\operatorname{div}\kappa_\alpha\mathring{\mathrm{grad}}}^{\,-1}
$$

and

$$
\sqrt{1-\operatorname{div}\kappa_\alpha\mathring{\mathrm{grad}}}^{\,-1}\operatorname{div}\sqrt{\kappa_\alpha} \subseteq \operatorname{div}\sqrt{\kappa_\alpha}\sqrt{1-\sqrt{\kappa_\alpha}\mathring{\mathrm{grad}}\operatorname{div}\sqrt{\kappa_\alpha}}^{\,-1},
$$

which leads us to rewrite Fourier's law as

$$
\begin{aligned}
\sqrt{\sqrt{\alpha}^{-1}\kappa\sqrt{\alpha}^{-1}}^{\,-1} q &= -\sqrt{\kappa_\alpha}\mathring{\mathrm{grad}}\sqrt{1-\operatorname{div}\kappa_\alpha\mathring{\mathrm{grad}}}^{\,-1}\sqrt{1-\operatorname{div}\kappa_\alpha\mathring{\mathrm{grad}}}^{\,-1}\theta\\
&= -\sqrt{1-\sqrt{\kappa_\alpha}\mathring{\mathrm{grad}}\operatorname{div}\sqrt{\kappa_\alpha}}^{\,-1}\sqrt{\kappa_\alpha}\mathring{\mathrm{grad}}\sqrt{1-\operatorname{div}\kappa_\alpha\mathring{\mathrm{grad}}}^{\,-1}\theta,
\end{aligned}
$$

yielding the assertion. □

With the latter observation, we are in the position to rewrite the two-temperature model as a system in the spirit of Theorem 5.

Theorem 8. *Let $\kappa = \kappa^* \in L(L^2(\Omega)^3)$, $C = C^* \in L(L^2_{\mathrm{sym}}(\Omega))$, $\gamma \in L(L^2(\Omega), L^2_{\mathrm{sym}}(\Omega))$, $\varrho_0 = \varrho_0^* \in L(L^2(\Omega))$, $\lambda, \alpha, T_0 \in]0,\infty[$. Moreover, we assume that κ, C and ϱ_0 are strictly positive definite. Then the system* (3.1) *to* (3.6) *may be rewritten into*

$$
(\partial_0 M_0 + M_1 + A)\,U = J \tag{3.9}
$$

with $\partial_0 u = v$ and

$$
U = \begin{pmatrix} v\\ \sigma\\ \theta\\ \sqrt{1-\sqrt{\kappa_\alpha}\mathring{\mathrm{grad}}\operatorname{div}\sqrt{\kappa_\alpha}}\sqrt{\kappa}^{-1} q/T_0 \end{pmatrix}, \quad J = \begin{pmatrix} \varrho_0 F\\ 0\\ \varrho_0 Q/T_0\\ 0\end{pmatrix},
$$

where $\sqrt{\alpha}\kappa\sqrt{\alpha} = \kappa_\alpha$,

$$
M_0 = \begin{pmatrix} \varrho_0 & 0 & 0 & 0\\ 0 & C^{-1} & C^{-1}\gamma & 0\\ 0 & \gamma^* C^{-1} & \left(\varrho_0 T_0^{-1}\lambda + \gamma^* C^{-1}\gamma\right) & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad A = \begin{pmatrix} 0 & -\operatorname{Div} & 0 & 0\\ -\mathring{\mathrm{Grad}} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},
$$

and

$$
\begin{aligned}
M_1 &= \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -M_{1,32}^*\\ 0 & 0 & M_{1,32} & T_0\end{pmatrix}\\
M_{1,32} &= \sqrt{1-\sqrt{\kappa_\alpha}\mathring{\mathrm{grad}}\operatorname{div}\sqrt{\kappa_\alpha}}^{\,-1}\sqrt{\kappa}\mathring{\mathrm{grad}}\\
&= \sqrt{\kappa}\mathring{\mathrm{grad}}\sqrt{1-\operatorname{div}\kappa_\alpha\mathring{\mathrm{grad}}}^{\,-1}.
\end{aligned}
$$

In particular, there exists $\nu_0 \geq 0$ *such that for all* $\nu > \nu_0$ *the equation in* (3.9) *admits for every* $J \in L^2_\nu\left(\mathbb{R}, L^2(\Omega)^3 \oplus L^2_{\mathrm{sym}}(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega)^3\right)$ *a unique solution* $U \in D\left(\overline{\partial_0 M_0 + M_1 + A}\right) \subseteq L^2_\nu\left(\mathbb{R}, L^2(\Omega)^3 \oplus L^2_{\mathrm{sym}}(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega)^3\right)$. *The solution operator is continuous and causal.*

Proof. Before computing that the equation $(\partial_0 M_0 + M_1 + A)\, U = J$ is a reformulation of the two-temperature model, we establish the well-posedness issue first. For this, note that $M_0 = M_0^*$ and $A = -A^*$. Next, we check that M_0 is strictly positive definite on its range. For the purpose of symmetric Gauss elimination, we define the transformation matrix

$$S := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \gamma & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence,

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\gamma & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \gamma^* & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \left(S^{-1}\right)^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\gamma^* & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We compute that

$$\begin{aligned}
\left(S^{-1}\right)^* M_0 S^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\gamma^* & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varrho_0 & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}\gamma & 0 \\ 0 & \gamma^* C^{-1} & \left(\varrho_0 T_0^{-1}\lambda + \gamma^* C^{-1}\gamma\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\gamma & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \varrho_0 & 0 & 0 & 0 \\ 0 & C^{-1} & 0 & 0 \\ 0 & 0 & \varrho_0 T_0^{-1}\lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Next, as bijective transformation S reduces the space

$$R := L^2(\Omega)^3 \oplus L^2_{\mathrm{sym}}(\Omega) \oplus L^2(\Omega) \oplus \{0\},$$

we infer $R(M_0) = R$. Moreover, for $\phi \in R$ we compute

$$\begin{aligned}
\langle M_0\phi, \phi\rangle &= \langle M_0 S^{-1} S\phi, S^{-1} S\phi\rangle \\
&= \langle \left(S^{-1}\right)^* M_0 S^{-1} S\phi, S\phi\rangle \\
&\geq \widetilde{c}\langle\phi, \phi\rangle
\end{aligned}$$

for some $\widetilde{c} > 0$. On $N(M_0)$, the operator $\mathfrak{Re}\, M_1$, the real part of M_1, is given by multiplication by $T_0 > 0$. Hence, the assertion concerning well-posedness follows, once we have established that M_1 defines a bounded linear operator. This, however, is a direct consequence of Proposition 2. Indeed,

$$\begin{aligned}
\left|\sqrt{\kappa}\,\mathrm{gr\mathring{a}d}\sqrt{1 - \operatorname{div}\kappa_\alpha \mathrm{gr\mathring{a}d}}^{\,-1}\phi\right|_0 &= \frac{1}{\sqrt{\alpha}}\left|\left|\sqrt{\kappa_\alpha}\,\mathrm{gr\mathring{a}d}\right|\sqrt{1 + \left|\sqrt{\kappa_\alpha}\,\mathrm{gr\mathring{a}d}\right|^2}^{\,-1}\phi\right|_0 \\
&\leq \frac{1}{\sqrt{\alpha}}\,|\phi|_0 \quad \left(\phi \in L^2(\Omega)\right).
\end{aligned}$$

As a next step we proceed to show that the two-temperature model admits the asserted reformulation. For this, in turn, it suffices to observe the following consequence of equations (3.4) and (3.5):

$$\begin{aligned}
\varrho_0 T_0 \eta &= \varrho_0 \lambda\theta + T_0 \gamma^* \mathcal{E} \\
&= \varrho_0 \lambda\theta + T_0 \gamma^* \left(C^{-1}\sigma + C^{-1}\gamma\theta\right).
\end{aligned}$$

Hence,

$$\varrho_0\eta = \left(\varrho_0 T_0^{-1}\lambda + \gamma^* C^{-1}\gamma\right)\theta + \gamma^* C^{-1}\sigma.$$

Moreover, from $\mathcal{E} = \mathring{\mathrm{Grad}}u$ and $\partial_0 u = v$ it follows that

$$\partial_0\mathcal{E} - \mathring{\mathrm{Grad}}v = 0$$

and the balance of momentum (3.6) reads as

$$\varrho_0\partial_0 v - \mathrm{Div}\,\sigma = \varrho_0 F.$$

Recalling (3.7) from Proposition 7, we note that

$$\mathrm{div}\,(q/T_0) = \mathrm{div}\,\sqrt{\kappa}\sqrt{1-\sqrt{\kappa_\alpha}\mathring{\mathrm{grad}}\,\mathrm{div}\,\sqrt{\kappa_\alpha}}^{-1}\left(\sqrt{1-\sqrt{\kappa_\alpha}\mathring{\mathrm{grad}}\,\mathrm{div}\,\sqrt{\kappa_\alpha}}\sqrt{\kappa}^{-1}q/T_0\right),$$

which eventually establishes the assertion. □

Note that $M_{1,32}$ has moved from its place in A for the limit case $\alpha = 0$ to the material law.

Remark 9. Symbolizing non-vanishing entries in the block operator matrices under consideration by ★, clearly, the pattern of M_0 is

$$M_0 = \begin{pmatrix} \bigstar & 0 & 0 & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

But the pattern of M_1 is

$$\mathfrak{Re}\,M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bigstar \end{pmatrix}, \ \mathfrak{Im}\,M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bigstar \\ 0 & 0 & \bigstar & 0 \end{pmatrix}.$$

Moreover,

$$A = \begin{pmatrix} 0 & -\mathrm{Div} & 0 & 0 \\ -\mathring{\mathrm{Grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that the system has partly been turned into an ODE in an infinite-dimensional state space.

4. A two-temperature, two-strain model

In this section, we shall elaborate briefly on the possibility of developing an alternative model, such that the whole partial differential equation (PDE) part in the two-temperature model discussed in the previous section vanishes. We start with basically the same model as in Theorem 8. As a preparation for deriving the two-temperature, two-strain model, we consider first the following system, which is unitarily congruent to the one in Theorem 8:

$$\partial_0 \left(\left(\begin{pmatrix} \varrho_0 & 0 & 0 & 0 \\ 0 & 1 & C^{-1/2}\gamma & 0 \\ 0 & \gamma^* C^{-1/2} & \left(\varrho_0 T_0^{-1}\lambda + \gamma^* C^{-1}\gamma\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right. \right.$$

$$\left. \left. \times \begin{pmatrix} v \\ C^{-1/2}\sigma \\ \theta \\ \left(1 - \sqrt{\kappa_\alpha}\,\mathring{\operatorname{grad}}\,\operatorname{div}\sqrt{\kappa_\alpha}\right)^{1/2} \sqrt{\kappa}^{-1} q/T_0 \end{pmatrix} \right) \right)$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -M_{1,32}^* \\ 0 & 0 & M_{1,32} & T_0 \end{pmatrix} \begin{pmatrix} v \\ C^{-1/2}\sigma \\ \theta \\ \left(1 - \sqrt{\kappa_\alpha}\,\mathring{\operatorname{grad}}\,\operatorname{div}\sqrt{\kappa_\alpha}\right)^{1/2} \sqrt{\kappa}^{-1} q/T_0 \end{pmatrix}$$

$$+ A \begin{pmatrix} v \\ C^{-1/2}\sigma \\ \theta \\ \left(1 - \sqrt{\kappa_\alpha}\,\mathring{\operatorname{grad}}\,\operatorname{div}\sqrt{\kappa_\alpha}\right)^{1/2} \sqrt{\kappa}^{-1} q/T_0 \end{pmatrix}$$

$$= \begin{pmatrix} f \\ 0 \\ \varrho_0 Q/T_0 \\ 0 \end{pmatrix}$$

with

$$A = \begin{pmatrix} 0 & -\operatorname{Div} C^{1/2} & 0 & 0 \\ -C^{1/2}\mathring{\operatorname{Grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$M_{1,32} = \sqrt{\kappa}\,\mathring{\operatorname{grad}}\sqrt{1 - \operatorname{div}\kappa_\alpha \mathring{\operatorname{grad}}}^{-1}.$$

Taking this as a starting point and substituting $C_\beta := \sqrt{\beta}C\sqrt{\beta}$ for some $\beta > 0$, we may propose analogously a similar modification of the elastic part yielding

$$\partial_0 \left(\left(\begin{pmatrix} \varrho_0 & 0 & 0 & 0 \\ 0 & 1 & C^{-1/2}\gamma & 0 \\ 0 & \gamma^* C^{-1/2} & \left(\varrho_0 T_0^{-1}\lambda + \gamma^* C^{-1}\gamma\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} v \\ \sqrt{1 - \sqrt{C_\beta}\,\mathring{\operatorname{Grad}}\,\operatorname{Div}\sqrt{C_\beta}}\,\sqrt{C}^{-1}\sigma \\ \theta \\ \sqrt{1 - \sqrt{\kappa_\alpha}\,\mathring{\operatorname{grad}}\,\operatorname{div}\sqrt{\kappa_\alpha}}\sqrt{\kappa}^{-1} q/T_0 \end{pmatrix} \right) \right)$$

$$+ \begin{pmatrix} 0 & -M_{1,10}^* & 0 & 0 & 0 \\ M_{1,10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -M_{1,32}^* \\ 0 & 0 & 0 & M_{1,32} & T_0 \end{pmatrix} \begin{pmatrix} v \\ \sqrt{1 - \sqrt{C_\beta}\,\mathring{\operatorname{Grad}}\,\operatorname{Div}\sqrt{C_\beta}}\,\sqrt{C}^{-1}\sigma \\ \theta \\ \sqrt{1 - \sqrt{\kappa_\alpha}\,\mathring{\operatorname{grad}}\,\operatorname{div}\sqrt{\kappa_\alpha}}\sqrt{\kappa}^{-1} q/T_0 \end{pmatrix}$$

$$= \begin{pmatrix} f \\ 0 \\ \varrho_0 Q/T_0 \\ 0 \end{pmatrix}$$

where now

$$M_{1,10} = -\sqrt{C}\,\mathring{\operatorname{Grad}}\sqrt{1 - \operatorname{Div} C_\beta \mathring{\operatorname{Grad}}}^{-1}.$$

and

$$M_{1,32} = \sqrt{\kappa}\,\mathring{\mathrm{grad}}\sqrt{1 - \operatorname{div}\kappa_\alpha \mathring{\mathrm{grad}}}^{-1}.$$

Clearly, the pattern of M_0 is still

$$M_0 = \begin{pmatrix} \bigstar & 0 & 0 & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

But the pattern of M_1 is now

$$\mathrm{sym} M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bigstar \end{pmatrix}, \ \mathrm{skew} M_1 = \begin{pmatrix} 0 & \bigstar & 0 & 0 \\ \bigstar & 0 & 0 & 0 \\ 0 & 0 & 0 & \bigstar \\ 0 & 0 & \bigstar & 0 \end{pmatrix}.$$

Moreover,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that the system has completely been turned into an abstract ODE.

Remark 10.

- Taking the general perspective used here into account for more complex materials, the Maxwell-Cattaneo-Vernotte (MCV) model of heat conduction [31–33] can also be easily applied to include the generalized model as introduced in [12]. For implementing the MCV model, we merely have to take M_0 with the pattern

$$M_0 = \begin{pmatrix} \bigstar & 0 & 0 & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & 0 & 0 & \bigstar \end{pmatrix}$$

  as strictly positive definite.
- Moreover, if we change the parameter α (and β) to be a bounded, selfadjoint, strictly positive-definite operator in an appropriate Hilbert space, we gain further flexibility for material modelling within the framework of the first-order system.
- Given the intricate rationale used in deriving the model in the first place it is somewhat disappointing to see that it merely serves to approximate a PDE by an ODE, which of course is always possible: compare this with for example the Yosida approximation or the above strategy, which amounts to replacing an unbounded skew-selfadjoint operator A by the bounded skew-selfadjoint operator $A\sqrt{1-\alpha A^2}^{-1} = \sqrt{1-\alpha A^2}^{-1}A$, $\alpha \in]0,\infty[$.

5. An alternative two-temperature model

In this section, we will make an attempt to establish an alternative two-temperature model from a purely structural point of view. For this, we proceed as follows.

Note that a transition to the ODE setting can also be achieved for example by approximating A with $A(1+\varepsilon A)^{-1}$ (Yosida approximation). Indeed,

$$A(1+\varepsilon A)^{-1} \overset{\varepsilon\to 0}{\to} A \tag{5.1}$$

point-wise on $D(A)$.

This way the occurrence of a square root (of inverses) of unbounded operators can be avoided. We assume the conditions of Theorem 8. For notational convenience we set $D := \sqrt{\kappa}\,\mathring{\mathrm{grad}}$. Applying the idea of using (5.1) to our initial two-temperature model yields

$$
\begin{aligned}
&\partial_0 \left(\begin{pmatrix} \varrho_0 & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}\gamma & 0 \\ 0 & \gamma^* C^{-1} & \left(\varrho_0 T_0^{-1}\lambda + \gamma^* C^{-1}\gamma\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right. \\
&\left. \times \begin{pmatrix} v \\ \sigma \\ \theta \\ \left(1+\varepsilon^2 DD^*\right)\sqrt{\kappa}^{-1} q/T_0 + \varepsilon D\theta \end{pmatrix} \right) \\
&+ \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon D^* D\left(1+\varepsilon^2 D^* D\right)^{-1} & -D^*\left(1+\varepsilon^2 DD^*\right)^{-1} \\ 0 & 0 & D\left(1+\varepsilon^2 D^* D\right)^{-1} & \varepsilon DD^*\left(1+\varepsilon^2 DD^*\right)^{-1} + T_0 \end{pmatrix} \right. \\
&\left. \times \begin{pmatrix} v \\ \sigma \\ \theta \\ \left(1+\varepsilon^2 DD^*\right)\sqrt{\kappa}^{-1} q/T_0 + \varepsilon D\theta \end{pmatrix} \right) \\
&+ A \begin{pmatrix} v \\ \sigma \\ \theta \\ \left(1+\varepsilon^2 DD^*\right)\sqrt{\kappa}^{-1} q/T_0 + \varepsilon D\theta \end{pmatrix} U \\
&= \begin{pmatrix} \varrho_0 F \\ 0 \\ \varrho_0 Q/T_0 \\ 0 \end{pmatrix}
\end{aligned}
$$

with

$$
A = \begin{pmatrix} 0 & -\operatorname{Div} & 0 & 0 \\ -\mathring{\mathrm{Grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

This reduces to

$$
\begin{aligned}
\partial_0 \varrho_0 v - \operatorname{Div}\sigma &= \varrho_0 F \\
\partial_0 (\sigma + \gamma\theta) - C\mathring{\mathrm{Grad}} v &= 0 \\
\partial_0 \left(\gamma^* C^{-1}\sigma + \left(\varrho_0 T_0^{-1}\lambda + \gamma^* C^{-1}\gamma\right)\theta\right) - D^*\sqrt{\kappa}^{-1} q/T_0 &= \varrho_0 Q/T_0
\end{aligned}
$$

and finally

$$
\begin{aligned}
D(1+\varepsilon^2 D^* D)^{-1}\theta + \varepsilon DD^*\left(1+\varepsilon^2 DD^*\right)^{-1}\left((1+\varepsilon^2 DD^*)\sqrt{\kappa}^{-1} q/T_0 + \varepsilon D\theta\right) \\
+ T_0\left((1+\varepsilon^2 DD^*)\sqrt{\kappa}^{-1} q/T_0 + \varepsilon D\theta\right) = 0,
\end{aligned}
$$

The last one implies

$$\left(\varepsilon DD^* + T_0 + T_0\varepsilon^2 DD^*\right)\sqrt{\kappa}^{-1} q/T_0 + \left((1+\varepsilon^2 DD^*)^{-1} + \varepsilon^2 DD^*\left(1+\varepsilon^2 DD^*\right)^{-1} + \varepsilon T_0\right) D\theta = 0,$$

$$\text{i.e. } \left(1+\varepsilon^2 DD^*\right)\sqrt{\kappa}^{-1} q + \varepsilon DD^*\sqrt{\kappa}^{-1} q/T_0 + (1+\varepsilon T_0) D\theta = 0,$$

$$\text{i.e. } \left(1+\varepsilon^2 DD^*\right)\sqrt{\kappa}^{-1} q + D\left(\varepsilon D^*\sqrt{\kappa}^{-1} q/T_0 + \theta + \varepsilon T_0\theta\right) = 0.$$

Thus,

$$\sqrt{\kappa}^{-1} q + \left(1+\varepsilon^2 DD^*\right)^{-1} D\left(\varepsilon D^*\sqrt{\kappa}^{-1} q/T_0 + \theta + \varepsilon T_0\theta\right) = 0,$$

which implies

$$\sqrt{\kappa}^{-1} q + D\left(\left(1+\varepsilon^2 D^*D\right)^{-1}\varepsilon D^*\sqrt{\kappa}^{-1} q/T_0 + (1+\varepsilon T_0)\left(1+\varepsilon^2 D^*D\right)^{-1}\theta\right) = 0, \tag{5.2}$$

and hence, defining

$$\phi := (1+\varepsilon T_0)\left(1+\varepsilon^2 D^*D\right)^{-1}\theta + \varepsilon\left(1+\varepsilon^2 D^*D\right)^{-1} D^*\sqrt{\kappa}^{-1} q/T_0 + T_0$$

and recalling that $D = \sqrt{\kappa}\,\mathring{\operatorname{grad}}$, we end up with

$$\begin{aligned}\theta &= (1+\varepsilon T_0)^{-1}\left(1+\varepsilon^2 D^*D\right)(\phi - T_0) - \frac{\varepsilon}{T_0}(1+\varepsilon T_0)^{-1} D^*\sqrt{\kappa}^{-1} q \\ &= (1+\varepsilon T_0)^{-1}\left(1-\varepsilon^2 \operatorname{div}\kappa\,\mathring{\operatorname{grad}}\right)(\phi - T_0) + \frac{\varepsilon}{T_0}(1+\varepsilon T_0)^{-1}\operatorname{div} q\end{aligned}$$

and (5.2) gives the Fourier's law

$$q + \kappa\,\mathring{\operatorname{grad}}(\phi - T_0) = 0.$$

Thus, using $-\varepsilon^2 \operatorname{div}\kappa\,\mathring{\operatorname{grad}}(\phi - T_0) = \varepsilon^2 \operatorname{div} q$, we get that

$$\begin{aligned}\theta &= (1+\varepsilon T_0)^{-1}(\phi - T_0) + (1+\varepsilon T_0)^{-1}\varepsilon^2 \operatorname{div} q + (1+\varepsilon T_0)^{-1}\frac{\varepsilon}{T_0}\operatorname{div} q \\ &= (1+\varepsilon T_0)^{-1}(\phi - T_0) + \frac{\varepsilon}{T_0}\left((1+\varepsilon T_0)^{-1}\varepsilon T_0 \operatorname{div} q + (1+\varepsilon T_0)^{-1}\operatorname{div} q\right) \\ &= \phi - T_0 + \frac{\varepsilon}{T_0}\left(\operatorname{div} q - \frac{T_0^2}{1+\varepsilon T_0}(\phi - T_0)\right)\end{aligned}$$

This can also be written as

$$\theta - \left(1 - \frac{\varepsilon T_0}{1+\varepsilon T_0}\right)(\phi - T_0) = \frac{\varepsilon}{T_0}\operatorname{div} q. \tag{5.3}$$

Equation (5.3) represents the final relation satisfied by the two temperatures. The parameter ε would be an *alternative two-temperature parameter*.

Notes

1. Of course here $\sqrt{\alpha}\kappa\sqrt{\alpha} = \alpha\kappa$, but we prefer to write it in this more symmetric fashion, since in the eventual first-order model equations α can be chosen more generally, that is, as a continuous, selfadjoint, strictly positive-definite operator, without affecting well-posedness. Also κ will be allowed to be a continuous, selfadjoint, strictly positive-definite operator.

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