

Optimal numerical strategy for unsteady natural convection in two and three dimensions

Gary B. Brassington *

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Abstract

Analyses of accuracy and computational cost of finite difference methods in computational fluid dynamics have illustrated a criterion for the minimum order for efficient calculations. This criterion favours the use of higher than second order methods when modelling greater than two space-time dimensions. These analyses have assumed the

*School of Civil and Environmental Engineering, JCU, Townsville, Qld 4811, AUSTRALIA. <mailto:gary.brassington@jcu.edu.au>

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dominant length scale to be homogeneous throughout the model domain. Natural convection in a cavity can exhibit inhomogeneity of the smallest dominant length scales in identifiable sub-domains. Any inhomogeneity of this nature is shown to have a significant impact on the computational efficiency. This extended analysis suggests that an optimally efficient numerical calculation for unsteady natural convection requires: a non-uniform grid that complements the distribution of length scales to obtain a homogeneous non-dimensional grid scale; and a numerical order equal to or greater than the space-time dimension.

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1 Introduction

Natural convection in a cavity has become one of the classical heat transfer problems with a large volume of research experimentally, analytically and numerically. There are several permutations of the cavity problem related to cavity shape, boundary conditions and fluid properties. Specifically the problem that is used to motivate the numerical analysis is that of a rectangular cavity that is instantaneously heated and cooled on opposing vertical walls with the remaining walls thermally insulated.

This problem is dynamically interesting with several regions having unique scales; and dominant physical processes. The vertical boundary upon startup, develops a thermal boundary layer by conduction, the buoyancy force overcomes viscous friction to develop a two or three dimensional viscous boundary layer. The buoyant fluid turns the corner and intrudes along the horizontal boundary as a low Reynolds number gravity current.

At low Rayleigh numbers (Ra), (a parameter important to the cavity problem [10]) many of the features mentioned reach a quasi-steady, quasi-two dimensional state over a relatively short time scale. A numerical calculation of the perturbation field acquires an effectively high order accuracy. As Rayleigh number is increased there is a transition in behaviour from steady state to turbulence. The nature of the transition and the properties in a turbulent state are of practical interest to many heat transfer applications. The perturbation methods are less useful in such regimes and genuinely higher

than second order calculations can provide some advantage.

The vertical boundary layer scales are reduced as Ra increases. Patterson and Imberger [10] demonstrate, through a two dimensional analysis, the Ra dependence for the thermal boundary layer as $\delta_T \sim \frac{h}{Ra^{1/4}}$ and $\tau_T \sim \frac{h^2}{\kappa Ra^{1/2}}$ where δ_T is the length scale, τ_T is the timescale and h is cavity height. Resolving these scales in a numerical model can be a significant design constraint.

Recent experimental studies, [14] and [15], have revealed the existence of three dimensional flow features in the gravity current intrusion, that may be related to roll structures. The development of a three dimensional cavity model would have the ability to capture these scales and would be of more general utility. Three dimensional, high Ra solutions demand the consideration of an optimal strategy.

Sanderson [12] demonstrated the minimum numerical order, m , for optimal efficiency should be equal or greater than the number of space-time dimensions, D , being modelled. Sanderson [12] also concluded that there was diminishing advantage in numerical orders greater than $D + 1$. The argument compared asymptotic behaviour of the accuracy (i.e., truncation error) and the computational cost. The models for the truncation error and computational cost to obtain it were based on a homogeneous non dimensional time and grid scale ($\Delta = \frac{\Delta t}{T} = \frac{\Delta x}{\ell}$), which are the ratio of the time step or grid scale ($\Delta t, \Delta x$) with the smallest dominant time and length scale (T, ℓ). The numerical design principal suggests that for unsteady two and three dimensional cavity problems $D = 3$ and $D = 4$ respectively the nu-

merical model should be constructed to be minimally third and fourth order respectively. This is greater than second order, which is typically used.

The natural convection problem has inhomogeneous smallest dominant length scales throughout the model domain. Armfield [1] developed a second order stretched coordinate model that concentrated the finest grid scales toward the boundaries. The coordinate system goes some way toward matching grid scales to dominant length scales. This results in a near homogeneous non dimensional grid scale for quasi-steady natural convection flows. In these flows the Armfield [1] model applied to the perturbation of the variables can be considered near optimal. However, at larger Ra where the field variables become unsteady and develop three space dimension features the space-time dimension D is more closely 3 or 4 and this can no longer be considered optimal.

In Section 2 we define the three dimensional cavity model equations, Section 3 discusses the optimal strategy for numerical models applied to problems with inhomogeneous length scales and Section 4 outlines an approach that might be used to fulfil the optimal strategy.

2 Equations

Natural convection in a cavity for Newtonian fluids (e.g., water) are well modelled by the Navier-Stokes equation under the Boussinesq and incompressible

assumptions. The three dimensional governing equations for a Cartesian coordinate system are written as

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla p - g\mathbf{k} + \nu\nabla^2\mathbf{u}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

$$\frac{DT}{Dt} = \kappa\nabla^2 T, \quad (3)$$

$$\rho = \rho(T), \quad (4)$$

where $\mathbf{u} \equiv (u, v, w)$, p , ρ , T are velocity, pressure, density and temperature, ν , κ are kinematic viscosity and thermal diffusivity, $\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ and the total derivative is, $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$.

An alternative buoyancy term can be derived for a fluid whose density is a function of temperature only, $\rho \equiv \rho(T)$, using the coefficient of thermal expansion β . Assuming that $\beta\delta T \ll 1$ ($\beta \approx 1 \times 10^{-5}$ at $T = 278^\circ K$ and $\beta \approx 7.1 \times 10^{-4}$ at $T = 373^\circ K$, [2]),

$$\beta\delta T \approx \frac{\rho}{\rho_0} - 1 = \frac{\rho'}{\rho_0}. \quad (5)$$

The boundary conditions are given by

$$\mathbf{u} = 0 \quad \text{on} \quad x = 0, l, \quad y = 0, b, \quad z = 0, h \quad (6)$$

$$\frac{\partial T}{\partial y}(x, 0, z) = \frac{\partial T}{\partial y}(x, b, z) = 0, \quad (7)$$

$$\frac{\partial T}{\partial z}(x, y, 0) = \frac{\partial T}{\partial z}(x, y, h) = 0, \quad (8)$$

$$T = T_0 + \frac{\Delta T}{2}, \quad x = 0, \quad t \geq 0; \quad T = T_0 - \frac{\Delta T}{2}, \quad x = l, \quad t \geq 0 \quad (9)$$

The pressure is composed of: the hydrostatic pressure P_0 due to the mean density $\rho_0 \equiv \rho(T_0)$; the hydrostatic pressure p_h due to density perturbations $\rho' = \rho - \rho_0$, which is due to temperature fluctuations; and nonhydrostatic pressure q . Note there is no pressure due to free surface fluctuations. We can remove the hydrostatic pressure due to the mean density and substitute the coefficient of thermal expansion to further simplify the momentum equations. Following [11] we non-dimensionalise the set of equations to obtain

$$\frac{D\mathbf{U}}{Dt^*} = -\widetilde{\nabla}_H P - \widetilde{\nabla} Q + \widetilde{\nabla}^2 \mathbf{U}, \quad (10)$$

$$\widetilde{\nabla} \cdot \mathbf{U} = 0, \quad (11)$$

$$\frac{DT^*}{Dt^*} = \frac{1}{\sigma} \widetilde{\nabla}^2 T^*, \quad (12)$$

where the non-dimensional variables are defined by $\mathbf{u} \equiv \frac{\nu}{h} \mathbf{U} = \frac{\nu}{h}(U, V, W)$, $(p_h, q) \equiv \frac{\rho_0 \nu^2}{h^2}(P, Q)$, $\delta T = \Delta T T^*$, $t = \frac{h^2}{\nu} t^*$, $(x, y, z) = h(X, Y, Z)$, $\widetilde{\nabla}_H \equiv \mathbf{i} \frac{\partial}{\partial X} + \mathbf{j} \frac{\partial}{\partial Y}$, $\widetilde{\nabla} \equiv \mathbf{i} \frac{\partial}{\partial X} + \mathbf{j} \frac{\partial}{\partial Y} + \mathbf{k} \frac{\partial}{\partial Z}$ and $\frac{D}{Dt^*} \equiv \frac{\partial}{\partial t^*} + U \frac{\partial}{\partial X} + V \frac{\partial}{\partial Y} + W \frac{\partial}{\partial Z}$. The parameter $\sigma = \frac{\nu}{\kappa}$ is the Prandtl number.

The hydrostatic pressure is similarly non-dimensionalised to

$$P = \frac{Ra}{\sigma} \int_Z^1 T^* dZ \quad (13)$$

where $Ra = \frac{g\beta\Delta Th^3}{\nu\kappa}$.

3 Efficient Strategy

The efficiency, in finite difference, is a comparison of the accuracy of the model calculation compared with the computational cost to attain it. Sander-son [12] developed a model to represent this efficiency based on the multiplication of the truncation error ϵ and the the computational cost C . The truncation error is dependent upon: the numerical order m ; the non-dimensional grid scale $\Delta = \frac{\Delta x}{\ell}$; and the numerical scheme. The computational cost is dependent upon: m ; Δ ; the numerical scheme; and the space-time dimension D . The asymptotic behaviour of ϵ and C as the resolution is increased (i.e., $\Delta \rightarrow 0$) compares the rate of decline in error with the rate of increase in computational cost. The efficiency can be considered optimal if the decline in error is dominant and the efficiency, ϵC , asymptotes to zero or in some cases a constant.

Provided the numerical scheme is stable and consistent (i.e., $\Delta < \gamma$ where $\gamma \in (0, 1)$ is a constant determined by the numerical scheme) the optimal efficiency occurs when $m > D$. There is little advantage asymptotically in using numerical orders greater than $D + 1$. These conclusions are based on the assumption of a uniform or homogeneous non-dimensional grid scale Δ throughout the model domain. In the case of an inhomogeneous Δ , the condition $\Delta < \gamma$ may be only just satisfied in some regions while in other

regions $\Delta \ll \gamma$. This results in a large inhomogeneity in accuracy, which is a disadvantage as the global accuracy would be dominated by the regions of lower accuracy forward in time. The non-uniform distribution of accuracy must be considered sub optimal.

It is proposed that the optimal strategy of any numerical model domain of interest resolves all points to the same level of accuracy. For finite difference schemes the accuracy or error is represented by the leading truncation term, in a Taylor series sense,

$$\epsilon = \alpha \Delta^m \quad (14)$$

where α is the proportionality constant. The constant of proportionality is typically a function of the order and the numerical scheme. Similar to [3] a simple example of second differences can be used to demonstrate the truncation error model.

Consider a second difference of a variable ϕ defined at points uniformly-spaced in the x -direction. A fourth order stencil for this second difference is given by

$$\begin{aligned} \frac{1}{\Delta x^2} \delta_{xx}^4(\phi_i) &= \frac{1}{24\Delta x^2} (-\phi_{i-2} + 16\phi_{i-1} - 30\phi_i + 16\phi_{i+1} - \phi_{i+2}) \\ &= \left. \frac{\partial^2 \phi}{\partial x^2} \right|_i + \frac{\Delta x^4}{180} \left. \frac{\partial^6 \phi}{\partial x^6} \right|_i + \text{h.o.t.} \end{aligned} \quad (15)$$

where δ_{xx}^4 represents the coefficients of a second difference stencil in the x -direction that has a numerical order of four.

The leading truncation term is the dominant error for a consistent scheme. To determine the relative magnitude of the error we normalise the leading truncation term by the second derivative being estimated

$$\epsilon = \frac{\Delta x^4}{180} \left(\frac{\partial^6 \phi}{\partial x^6} \Big|_i / \frac{\partial^2 \phi}{\partial x^2} \Big|_i \right). \quad (16)$$

Consider the dominant scales of the system as $x = \ell x'$ and $\phi = \Phi \phi'$, then the non-dimensional error reduces to

$$\epsilon = \alpha \left(\frac{\Delta x}{\ell} \right)^4 = \alpha \Delta^4 \quad (17)$$

where $\alpha = \frac{1}{180}$ and $\frac{\partial^2 \phi'}{\partial x'^2} \Big|_i, \frac{\partial^6 \phi'}{\partial x'^6} \Big|_i$ are $O(1)$ by definition.

If the model problem has a single dominant length scale throughout the entire domain, a homogeneous error can be obtained by a uniform mesh Δx and a uniform numerical order. The grid scale Δx can be determined by the consistency condition independent of the numerical order. The numerical order can be determined by the optimal computational efficiency condition, $m = D + 1$.

However, if the model has inhomogeneous length scales we need to introduce a non uniform distribution of the grid scales Δx and/or a non uniform distribution of numerical order m to obtain a uniform distribution of accuracy ϵ . In order to determine the efficiency for these problems we will

simplify the analysis to two methods. The first uses a uniform grid scale Δx and employs a non uniform distribution of numerical order to obtain uniform accuracy. Higher numerical orders are used for regions that are poorly resolved $\Delta < \gamma$ and lower numerical order can be used in regions that are well resolved $\Delta \ll \gamma$. The second method uses a uniform numerical order and a non-uniform distribution of grid scales to achieve a homogeneous non dimensional grid scale Δ . In both cases all Δ must be minimally resolved, $\Delta < \gamma$, so that the numerical solution is consistent. Let us consider the two approaches separately.

3.1 Order Selective Model

This provides obviously less utility as numerical orders are discrete and there is diminishing advantage for numerical orders $m > D + 1$.

Assume that the problem can be characterised by a set of sub-domains that have length scales

$$\ell_i \in [\ell_S, \ell_L] \quad (18)$$

where ℓ_i represents the smallest dominant length scale of sub-domain i , ℓ_S and ℓ_L represent the minimum and maximum of the smallest dominant length scales of all sub-domains. If we choose a uniform mesh $\Delta x \sim \text{const}$ then the non-dimensional grid scale is represented by

$$\Delta \in \left[\frac{\Delta x}{\ell_L}, \frac{\Delta x}{\ell_S} \right] = [\Delta_L, \Delta_S]. \quad (19)$$

We define a homogeneity scale as

$$\zeta = \frac{\Delta_L}{\Delta_S} \quad (20)$$

where $\zeta \ll 1$ represents low homogeneity or high inhomogeneity and $\zeta \sim 1$ represents a homogeneous model problem. If Δx is a constant the homogeneity scale reduces to

$$\zeta = \frac{\ell_S}{\ell_L} \quad (21)$$

where $\zeta \in (0, 1]$. From the behaviour of numerical stencils the largest non dimensional grid scale must be bounded to e.g., $\Delta_c = \gamma < 0.6$ [12]. We use this to select the grid scale of the mesh e.g., $\Delta x \sim \ell_S \Delta_c$ and the non-dimensional grid scale becomes

$$\Delta \in [\zeta \Delta_c, \Delta_c]. \quad (22)$$

The error also has a range given by,

$$\epsilon \in \alpha(\Delta_L^{m_L}, \Delta_S^{m_S}) \quad (23)$$

where m_L and m_S represent the numerical order used for the two regions. If we wish to have a homogeneous error over the model domain we require

$$\Delta_L^{m_L} = \Delta_S^{m_S} \quad (24)$$

or equivalently

$$\frac{m_L}{m_S} = \frac{\log_{10}(\Delta_S)}{\log_{10}(\zeta \Delta_S)}. \quad (25)$$

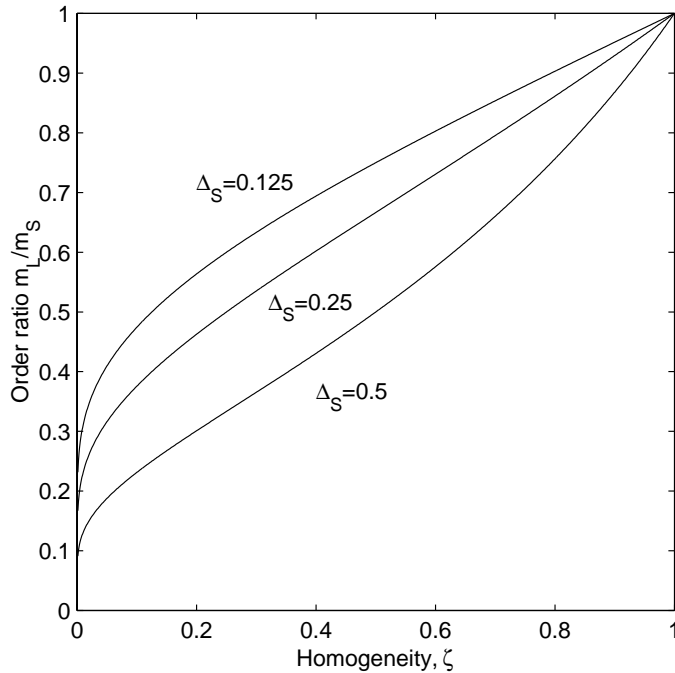


FIGURE 1: The ratio of numerical orders, $\frac{m_L}{m_S}$ as a function of homogeneity of length scale, ζ . The orders m_S and m_L correspond to the regions whose non-dimensional grid scale is Δ_S and Δ_L respectively where $\Delta_L \leq \Delta_S$.

The numerical order ratio is plotted in Figure 1 as a function of the homogeneity scale ζ . The three lines represent the ratio of orders for three different values of Δ_S . All three curves exhibit a near linear relationship, with the line corresponding to $\Delta_S = 0.5$ approximated by

$$m_L \approx (0.2 + 0.8\zeta)m_S \quad (26)$$

for $\zeta > 0.2$. Thus the order of the inhomogeneous regions can be reduced proportional to the homogeneity ζ to achieve comparable accuracy. If the non-dimensional grid scale Δ_S is reduced, while ζ is fixed, the impact of homogeneity is reduced and

$$\lim_{\Delta_S \rightarrow 0} \frac{\log_{10}(\Delta_S)}{\log_{10}(\zeta\Delta_S)} \rightarrow 1. \quad (27)$$

In this case the design restricts the model to be of uniform order, to achieve comparable accuracy. This can be interpreted as, regardless of the magnitude of the homogeneity scale provided the least well resolved region satisfies the condition $\Delta_S \ll \gamma$ the accuracy becomes indistinguishably uniform. At sufficiently high resolutions the numerical order should be uniform.

To determine the computational efficiency of the order selective method we must model the computational cost. The cost model developed by Sander-son [12] is given by

$$C = \beta m \Delta^{-D}. \quad (28)$$

This model contains three components. The first component is a proportionality constant that depends on the numerical methods used. The cost

is optimally directly proportional to the numerical order. Not all numerical methods have this proportionality however, for all operators of the Navier-Stokes equations there are optimal methods available (e.g., N-Cycle [9]; QUICK [8], and full-multi-grid [5, e.g.]). The third component expresses the proportionality with the total number of grid points in the model domain where $\Delta^{-D} = \left(\frac{\ell}{\Delta x}\right)^D$.

The total computational cost of the inhomogeneous length scale problem is proportional to the sum of the costs of the two regions having the extreme scales, Δ_L and Δ_S :

$$\begin{aligned} C &= \beta(m_L\Delta_L^{-D} + m_S\Delta_S^{-D}) \\ &= \beta m_S \Delta_S^{-D} \left(1 + \zeta^{-D} \left(\frac{\log_{10} \Delta_S}{\log_{10} \zeta \Delta_S}\right)\right) \end{aligned} \quad (29)$$

See that the more inhomogeneous the model the more expensive the computation of this method. As Δ_S is made small the limit of (27) would apply and the cost would be consistent with (28) the homogeneous case [12]. The cost component is plotted as a function of $\log_{10}(\Delta_S)$ and $\log_{10}(\zeta)$ in Figure 2. This demonstrates that the cost grows rapidly as a function of homogeneity compared with increased resolution.

The efficiency is modelled by the multiplication of error ϵ , which is homogeneous $\alpha\Delta_S^{m_S}$, with computational cost C , (29),

$$\epsilon C = \alpha\beta m_S \Delta_S^{(m_S-D)} \left(1 + \zeta^{-D} \left(\frac{\log_{10} \Delta_S}{\log_{10} \zeta \Delta_S}\right)\right). \quad (30)$$

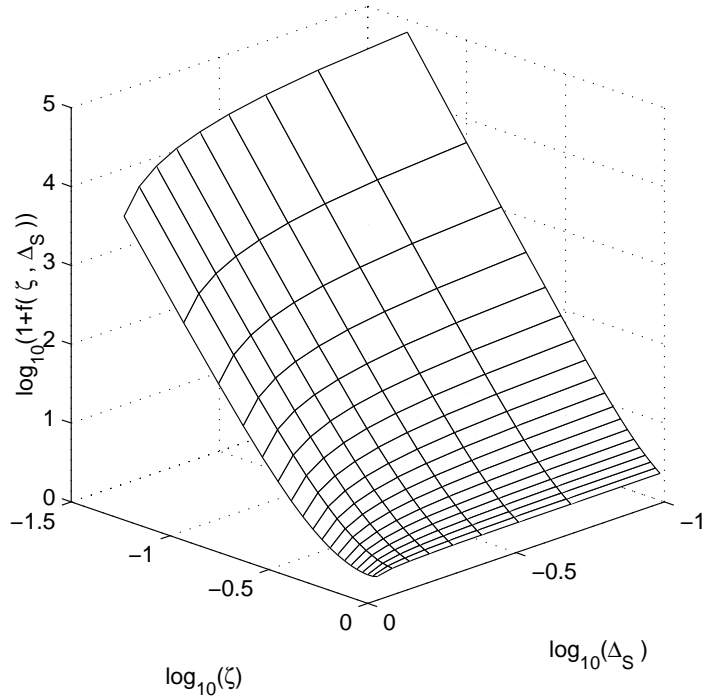


FIGURE 2: The logarithm base 10 of the efficiency component $(1 + \zeta^{-D} \frac{\log_{10} \Delta_S}{\log_{10} \zeta \Delta_S})$ as a function of $\log_{10}(\Delta_S)$ the largest non-dimensional grid scale and $\log_{10}(\zeta)$ the homogeneity scale, $D=4$.

Let us consider that for any given model the dominant scales are determined by the problem of interest and are independent of the numerical model i.e., $\zeta \approx \text{const}$. This is a good approximation for natural convection in a cavity at moderate Ra prior to the transition to turbulence. Note however, there is evidence related to high Reynolds number flows that fractality occurs [13]. In these cases the dominant scales cannot be considered independent of the model resolution.

The asymptotic behaviour of efficiency, (30), as the resolution is increased, $\Delta_S \rightarrow 0$, is given by

$$\lim_{\Delta_S \rightarrow 0} \epsilon C = \begin{cases} \infty, & m_S < D, \\ \alpha\beta m_S (1 + \zeta^{-D} (\frac{\log_{10}(\Delta_S)}{\log_{10}(\zeta\Delta_S)})), & m_S = D, \\ 0, & m_S > D. \end{cases} \quad (31)$$

If the problem is inhomogeneous we can afford to use the lower orders for the well resolved regions without sacrificing efficiency provided the order of the largest non-dimensional scale Δ_S is performed at an order $m_S > D$.

Note that for any fixed value of ζ the inhomogeneity can contribute significantly to the cost but if $m_S > D$ the accuracy for that cost should be maximised.

Consider the case where the resolution is fixed in the sense that $\Delta_S = \Delta_c < \gamma$ for the smallest scale but we decrease the homogeneity ζ . In respect to the cavity model this relates to changing the Ra such that increasing Ra

decreases homogeneity ζ . The asymptotic behaviour of efficiency as homogeneity reduces, $\zeta \rightarrow 0$, with Δ_S fixed is given by

$$\lim_{\zeta \rightarrow 0} \epsilon C \rightarrow \infty. \quad (32)$$

In this case, although the higher order methods, $m = D + 1$, provide greater efficiency the reduction in homogeneity can dominate the cost. As a consequence the magnitude of efficiency is also dominated when a uniform mesh is used over the entire domain and Δ_S is fixed.

In practice, the homogeneity and the resolution Δ_S are finite and the efficiency is some way from the asymptotic results. The efficiency is surface plotted as a function of numerical order and non-dimensional grid scale Δ_S for four different homogeneity values, $\zeta \equiv (1, 1E - 1, 1E - 2, 1E - 3)$. The numerical order is restricted to the discrete values, $m \equiv [2, \dots, 6]$ and the non-dimensional grid scale values are given by $\Delta_S \equiv [0.1, 0.2, \dots, 0.9]$. The values of $\min(\Delta_S) = 0.1$ represents a practical value of resolution since we would like to restrict the total computation time.

The four surface plots in Figure 3 demonstrate that given the non-dimensional grid scale satisfies the condition $\Delta_S < 0.6$, there is a clear efficiency gain obtained by using increased numerical orders. In all cases when the numerical order is less than the space-time dimension there is a loss of efficiency as the resolution is increased. In other words there is an impractical increase in cost for the relatively small gain in accuracy. The converse is true for $m > D$.

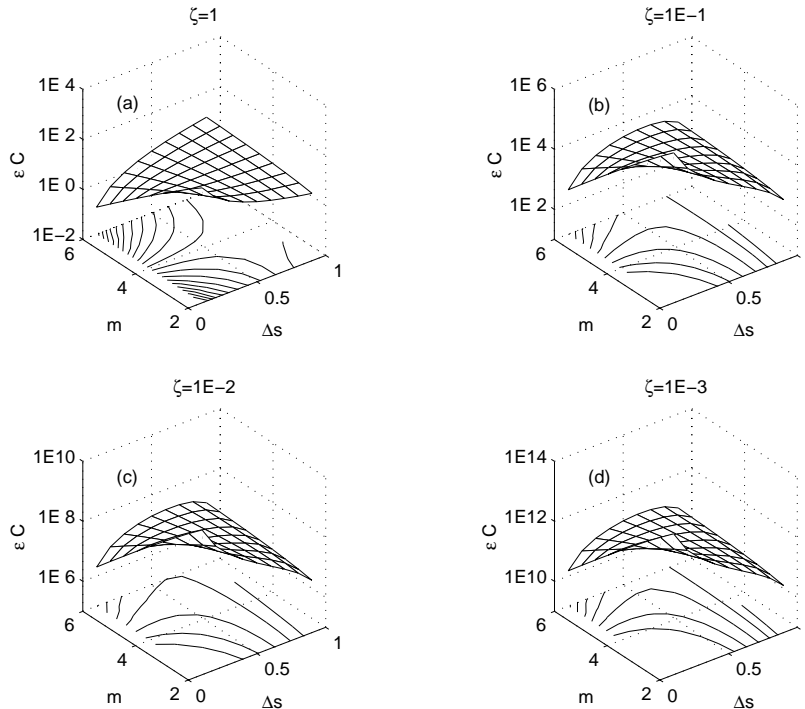


FIGURE 3: The efficiency ϵC as a function of numerical order m and non-dimensional grid scale Δ_S for a space-time dimension $D = 4$. The logarithm base 10 of efficiency is surface plotted for different homogeneity scales, (a) $\zeta = 1$, (b) $\zeta = 1E - 1$, (c) $\zeta = 1E - 2$ and (d) $\zeta = 1E - 3$. The contour of each surface is shown at the base of each figure.

As the homogeneity is reduced the total magnitude of the efficiency function is significantly increased. This behaviour, consistent with (32), demonstrates that modelling an inhomogeneous problem with a uniform mesh must be considered sub optimal even if the accuracy is greater than the dimension.

3.2 Resolution Selective Model

This approach uses a non-uniform distribution of mesh points that matches the distribution of dominant length scales that occur in different regions throughout the model. Assume that the problem can be characterised by a set of sub-domains that have dominant length scales ℓ_i

$$\ell_i \in [\ell_S, \ell_L]. \quad (33)$$

Choose a coordinate system so that the non-dimensional grid scale is approximately uniform throughout the entire domain

$$\Delta = \frac{\Delta x_S}{\ell_S} \approx \frac{\Delta x_L}{\ell_L}. \quad (34)$$

In this case the error is well modelled by (14) and there is no appearance of the homogeneity scale. The computational cost is proportional to the order m but is also proportional to the number of grid points. In this case the number of grid points can be approximated by

$$\left(\frac{\ell_S}{\Delta x_S}\right)^{-D} + \left(\frac{\ell_L}{\Delta x_L}\right)^{-D} \approx \beta \Delta^{-D}. \quad (35)$$

For this case the computational cost is identical with the equation developed by Sanderson [12], (28). Combining the truncation error and computational cost for a model with a non-uniform distribution of mesh points, the efficiency function ϵC is independent of the homogeneity scale ζ . The asymptotic behaviour is identical with that of a problem that has homogeneous length scales. This is a desirable strategy for natural convection in a cavity as efficiency of the model is independent of the Ra .

For finite non dimensional grid scale Δ_S , the efficiency is consistent with Figure 3a and [12]. This figure demonstrates that if $m < D$, increasing the resolution obtains increased accuracy inefficiently. However, for $m > D$ the increase in resolution improves accuracy in an efficient way.

We can now express the optimal numerical strategy for a general problem with inhomogeneous length scales and dimension D . The coordinate system should be distributed so that the non-dimensional grid scale is uniform and $\Delta < \gamma$. Also the numerical order of the model should be greater than the space-time dimension i.e., $m = D + 1$.

In the time dimension, the time step Δt may be more restricted due to stability and require $\frac{\Delta t}{T} \ll \Delta$ where Δ represents the non-dimensional grid scale of the rest of the model. This permits the numerical order of the time integration scheme to be relaxed to again produce a homogeneous ϵ throughout the domain [4]. It should be noted that this is not optimal but due to the stability restriction this may be unavoidable.

4 Method of Solution

The efficient computation of the cavity model in three and four space-time dimensions requires a numerical order $m \geq D$, which is higher than second order. The inhomogeneity of the length scales requires the use of a non-uniform distribution of grid scales that result in a homogeneous non dimensional grid scale.

Non uniform meshes can be constructed by, stretched coordinates [1], coordinate transforms, [6], or nesting [7, e.g.]. The stretched coordinates and transform coordinates can be used to good effect in the cavity problem as the smallest scales, for low Ra , are permanently confined to the boundary layer and the mesh can be fixed.

Stretched coordinates have been successfully applied to the two space dimension cavity at second order [11]. However higher than second order stencils can be problematic particularly for a control-volume mesh. Coordinate transforms introduce additional operators to the system of equations but are then able to take advantage of uniform mesh stencils. The model accuracy is sensitive to the calculation of these additional operators but in the cavity these terms need only be calculated to high accuracy once.

There are several permutations to a nesting algorithm, however for accuracy to be maintained the nesting would need to be two-way communicating. The nesting involves the embedding of a higher resolution model in a sub-domain of a larger scale model. The higher resolution model uses the large

scale model to specify boundary conditions and the higher resolution values are integrated and injected into the large scale model. The nested algorithm has the advantage of being adaptive and of more general utility as Ra is increased and the flows become turbulent. The full specification of the nesting algorithm is not detailed but the efficiency argument Section 3 suggests that the accuracy can be assured if the nesting boundary is located in a region of the larger mesh model where $\Delta \ll 0.6$. Both the nested and coordinate transform methods permit the use of uniform mesh stencils, which is a distinct programming advantage for higher than second order.

5 Conclusion

A model of accuracy and computational cost of a finite difference model has been used by Sanderson [12] to demonstrate that for efficient computation the numerical order should be greater than the space-time dimension of the model. In the context of a problem with inhomogeneous length scales e.g., natural convection in a cavity, the above conclusion was reaffirmed but with the additional criterion that the non-dimensional grid scale needs to be homogeneous throughout the model domain to be considered optimal.

Achieving a high order non-uniform mesh model can pose a challenge. The method advocated here is for the use of a uniform mesh model that takes advantage of the symmetric stencils and apply this model within either a coordinate transform or nested model context.

The problem of interest in this analysis is natural convection in a cavity however, the analysis applies more generally to any problem that exhibit inhomogeneity of the smallest dominant length scales throughout the model domain. For cavity problems, the unique sub-domains are largely spatially and temporally invariant and a non-uniform grid can be fixed, which is a significant programming advantage. For problems that exhibit inhomogeneity that is spatially and temporally variant an adaptive gridding should also be designed with a numerical order, $m = D + 1$.

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