Modified fractional-step methods for the Navier-Stokes equations

S. Armfield* R. Street†

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Abstract

Fractional step methods integrate the Navier-Stokes equations in a non-iterative fashion. The commonly used P2 pressure correction fractional step method introduces a second order in time error into the solution as a result of the non-iterative approach. In this paper two alternative fractional step methods are examined in which the additional error is third order in time. One of the methods extends the standard P2 method to increase the accuracy of the approximate pressure included in the momentum equations, and is denoted the P3 method. The other method solves a Poisson pressure equation prior to the solution of the momentum, and is denoted the pressure method. Both alternative methods are shown to reduce the overall error and increase the efficiency as compared to the standard method.

*School of Aerospace, Mechanical & Mechatronic Engineering, Sydney University, Sydney, AUSTRALIA. mailto:armfield@aeromech.usyd.edu.au

†Environmental Fluid Mechanics Laboratory, Stanford University, Stanford, California 94305-4020, USA

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1 Introduction

Fractional step projection methods integrate the Navier-Stokes equations in time at each time-step by first solving the momentum equations using an approximate pressure field to yield an intermediate velocity field that will not, in general, satisfy continuity. A Poisson equation is then solved with the divergence of the intermediate velocity as a source term to provide a pressure or pressure correction, which is then used to correct the intermediate velocity field, providing a divergence free velocity. The pressure is updated and integration then proceeds to the next time step. Details of the range of fractional step methods developed for the Navier-Stokes equations are given in the references [9, 5, 11, 18, 10, 14, 17, 4, 7, 8, 16, 6, 2, 3]. The P1 method sets the pressure field to zero in the momentum equation and the Poisson equation is then solved for the new pressure, while the P2 method sets the pressure in the momentum equation to that obtained at the previous time-step, and the Poisson equation is then solved for a pressure correction. Both the P1 method, with appropriate intermediate velocity boundary conditions, and the P2 method provide second order in time accuracy for the velocity and pressure fields, provided the momentum equation is integrated using a second order accurate scheme [2, 3]. However both of these schemes still have

an error that is approximately four times the magnitude of the equivalent iterative scheme, as a result of the non-iterative fractional step approach. Although the non-iterative schemes are still more efficient than the iterative schemes, it is clear that a non-iterative scheme with equivalent error to the iterative scheme would be preferable. In this paper two approaches are examined which increase the order of the fractional step error from second to third order in time, reducing the magnitude of the error and considerably increasing the efficiency of the fractional step methods.

2 Method

The governing equations are the Navier-Stokes equations in unsteady incompressible non-dimensional form:

$$u_t + (u \cdot \nabla)u = -\nabla P + \frac{1}{\mathrm{Re}}\nabla^2 u\,, \tag{1}$$

$$\nabla \cdot u = 0\,, \tag{2}$$

where u is the velocity, P the pressure and Re the Reynolds number.

The continuous equations are discretised using Adams-Bashforth for the advective terms and Crank-Nicolson for the diffusive terms, giving the system

$$\frac{v^{n+1} - v^n}{\Delta t} + \left[\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1})\right] = -Gp^{n+1/2} + \frac{1}{2\mathrm{Re}}L(v^{n+1} + v^n)\,, \tag{3}$$

$$Dv^{n+1} = 0\,, \tag{4}$$

where (v, p) are the discrete velocity and pressure respectively, H is the discrete advection operator, G the discrete gradient, L the discrete Laplace operator and D the discrete divergence. This is a second order in time discretisation, using an explicit scheme for the advection terms and an implicit scheme for the diffusion terms [11].

The P2 fractional step projection method integrates this equation set by first solving an approximate form of the momentum equations for v^*,

$$\frac{v^* - v^n}{\Delta t} + \left[\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1})\right] = -Gp^{n-1/2} + \frac{1}{2\mathrm{Re}}L(v^* + v^n)\,, \tag{5}$$

This approximate velocity will not initially satisfy continuity. A correction is then applied of the form

$$v^{n+1} = v^* - \Delta t G\pi\,, \tag{6}$$

where π is a pressure correction, such that the resulting v^{n+1} does satisfy continuity. An equation for π is constructed by substituting equation (6) into the continuity equation (4) to give

$$L\pi = Dv^*/\Delta t\,.$$

Once π is obtained, the velocity is corrected and the pressure is updated using the pressure correction as

$$p^{n+1/2} = p^{n-1/2} + \pi\,, \tag{7}$$

and the integration proceeds to the next time step. The P2 method will not in general provide an exact solution to the discrete equations, regardless of the accuracy to which the individual equations are solved as, although the velocity field is divergence free, the divergence free velocity field together with the updated pressure will not satisfy the discrete momentum equations. This resulting projection error has been shown for the P2 method to be [2],

$$\Delta t^2 G p_t\,. \tag{8}$$

The error is seen to be second order in time, and thus is consistent with the basic discretisation used for the momentum equations. The effect of this error has been determined by comparing the P2 solution to an iterative solution, where the momentum/Poisson pressure correction equations are repeatedly

cycled through at each time-step until the solution satisfies both the momentum and continuity equations. The projection error has been shown to increase the magnitude of the error by a factor of four times when compared to that of an iterative scheme. Although the P2 scheme is still more efficient than the iterative scheme as a result of its non-iterative form, it would clearly be desirable to construct a non-iterative scheme with the same error magnitude as that of the iterative scheme. In this paper two methods of constructing non-iterative schemes with reduced error are considered. The first is to improve the accuracy of the pressure used in the momentum equations by using a second order extrapolation to obtain $\tilde{p}^{n+1/2}$, which is then included in the momentum equations:

$$\frac{v^* - v^n}{\Delta t} + \left[\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1})\right] = -G\tilde{p}^{n+1/2} + \frac{1}{2\mathrm{Re}}L(v^* + v^n)\,, \qquad (9)$$

with $\tilde{p}^{n+1/2} = 2p^{n-1/2} - p^{n-3/2}$. The pressure in the momentum equation is now approximated to second order in time, which will result in the projection error being third order and therefore in principle considerably reduced for small Δt. This method will be denoted the P3 method, after [7], where a similar method was suggested without implementation.

An alternative method is also considered, whereby a pressure Poisson equation is first constructed for the pressure by taking the divergence of the momentum equations, setting the divergence of the $n+1$ velocity occuring in the time derivative to zero, and using a second order extrapolation for the $n+1/2$ time step velocity occuring in the viscous term, giving

$$Lp^{n+1/2} = \frac{Dv^n}{\Delta t} - D\left[\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1})\right] + \frac{D}{\mathrm{Re}}L(\frac{3}{2}v^n - \frac{1}{2}v^{n-1})\,, \qquad (10)$$

at all fully interior points, while at boundary adjacent points the normal component of the momentum equation lying on a boundary node is replaced with the normal component of velocity at that point. Once the pressure is obtained it is substituted into the momentum equations, which are solved for the velocity, and the integration continues to the next time step. The

resultant velocity and pressure will exactly satisfy the momentum equation, while an error in the divergence will arise from the difference between the Poisson pressure equation (10), and that constructed from the momentum equations with the final velocity included. The only difference is that between the Crank-Nicolson discretisation of the viscous terms, used in the final momentum equations, and the Adams-Bashforth discretisation used in the pressure equation. As both these discretisations are second order in time, the difference will be second order, and this will result in a third order in time error in the divergence. In practice equation (10) is solved in pressure correction form, with the $n - 1/2$ level pressure included on the right hand side. This method is denoted the pressure scheme.

The above schemes are defined on the standard MAC staggered grid using finite volumes. Results are presented below for natural convection flow in a square cavity which requires fixed velocity boundary conditions. The normal component of velocity, which has a node on the boundary, is set to the required value at that boundary, while the tangential component, which does not have a node on the boundary, has the average of the values at the immediate interior and exterior nodes set to the required value. The normal gradient of the pressure correction is set to zero at the boundary for all methods. No explicit boundary conditions are required for the pressure. For the iterative and P2 solvers the boundary conditions for the * velocity field are set to be the same as the physical boundary conditions, given above.

The equations are discretised using standard second-order central differences for the viscous terms, the pressure gradient and divergence terms. The QUICK third-order upwind scheme is used to obtained face values for use in the advective terms [12]. The momentum equations are inverted using an ADI scheme in which terms are shifted to the right hand side of the system to enable a series of tridiagonal matrices to be inverted in each direction. The terms shifted to the right hand side contain the latest available estimate for the unknown, allowing the domain to be repeatedly swept until an accurate solution is obtained. For all the methods tested four sweeps of the ADI solver

were used, where a single sweep consists of solving the series of tridiagonal systems associated with each coordinate direction once. Four sweeps of the solver gave solutions with residuals of less than 1×10^{-8} for all cases. A preconditioned restarted GMRES method is used to solve the pressure correction equations for all the methods. Other solvers, such as preconditioned conjugate gradient, incomplete LU, ADI and Jacobi have also been tested and found not to affect the overall accuracy or relative performance of the methods. Of the solvers tested GMRES was found to be the most efficient. The number of sweeps of the GMRES solver used varied with each of the methods tested and with the time-step and convergence criterion prescribed. For the non-iterative schemes for the smallest convergence criterion up to a hundred sweeps were required while for the largest convergence criterion as few as five were sufficient. For the iterative scheme the Poisson solver was limited to five sweeps. For the iterative, P2 and P3 schemes at each time step the solution was considered converged when the integral of the absolute divergence over the domain was less than a pre-set value. The convergence is applied to the iterative scheme after the solution of the momentum equations, with a minimum of two momentum/Poisson pressure correction iterations required. For the pressure scheme the pressure equation was considered converged when the integral of the absolute residual over the domain was less than a pre-set value.

3 Results

Results have been obtained for start-up natural convection cavity flow in a square cavity. Initially the fluid in the square cavity is stationary and isothermal at temperature $T = 0$. At time $t = 0$ the left and right walls are instantaneously heated and cooled to $\Delta T/2$ and $-\Delta T/2$ respectively, with the top and bottom boundaries adiabatic. All boundaries are no-slip. The control parameters for this flow are the Rayleigh number Ra and the Prandtl number Pr. The Rayleigh number $\mathrm{Ra} = g\alpha\Delta T H^3/\nu\kappa$, with g gravity, α the

coefficient of thermal expansion, H the height of the cavity, ν the kinematic viscosity and the diffusivity $\kappa = \nu/\mathrm{Pr}$. The results presented were obtained with $\mathrm{Ra} = 6 \times 10^5$ and $\mathrm{Pr} = 7.5$.

The two dimensional equations are used with x the horizontal co-ordinate, U the corresponding horizontal velocity component, y the vertical coordinate and V the corresponding vertical velocity component. The natural convection flow requires the inclusion and solution of a temperature equation, in addition to the Navier-Stokes equations. The temperature equation is solved using Adams-Bashforth and Crank Nicolson schemes in exactly the same manner as the momentum equations, and for brevity is not presented. Further details of the natural convection flow may be found in Patterson and Armfield [13] and Armfield and Patterson [1], and for brevity will not be presented here.

A 50×50 uniform mesh has been used. The 50×50 solution was compared to that obtained on a 200×200 mesh and the variation was found to be less than one percent. The 50×50 mesh is therefore considered to provide a sufficiently accurate resolution for this flow. To test the behaviour of the methods the flow was integrated from $t = 0$ to $t = 2$ for time-steps in the range $\Delta t = 0.003125$ to 0.1, and the 'error' expressed as the L_2 norm of the difference between a test solution obtained at a given Δt and a benchmark solution obtained with a time step of $\Delta t = 7.8125 \times 10^{-4}$, also integrated from $t = 0$ to $t = 2$. Times have been non-dimensionalised using the boundary layer start-up time for the natural convection cavity. Total time to steady state for the cavity is orders of magnitude greater than the boundary layer start up time. The maximum time step selected, $\Delta t = 0.1$, was chosen to be near to the empirically obtained stability limit of $\Delta t = 0.2$ for all methods.

For the P2, P3 and iterative methods at each time step results have been obtained with convergence criterion ranging from 1.0×10^{-4} to 1.0×10^{-9} in order-of-magnitude steps. The solution was considered converged at each time step when the integral over the domain of the absolute residual of the continuity equation was less than the convergence criterion. In this way it was possible to determine which was the appropriate convergence criterion

for each method and time step to ensure that as accurate as possible a solution was obtained. The results presented are those for which a further reduction of the convergence criterion by an order of magnitude led to a less than one percent change in the solution accuracy. This degree of accuracy was obtained with different criteria for each method and each time step, ranging from 1×10^{-4} for the P2 method with time step $\Delta t = 0.1$ to 1×10^{-9} for the iterative method with time step $\Delta t = 0.003125$. For the iterative method each integration of the Poisson pressure correction equation was halted after five iterations of the GMRES procedure, regardless of the accuracy of the solution at that stage of integration. The divergence test is applied to the iterative method after the momentum equations have been solved, and it was required that at each time step at least two iterations of the momentum/pressure correction cycle were carried out, regardless of the divergence after the first solution of the momentum equations. For the pressure method the convergence criteria is applied to the pressure equation by integrating the absolute residual of equation [8] over the domain. Solutions were obtained for the pressure method for a range of convergence criteria. It was found that for all time-steps tested a criteria of 1.0×10^{-4} gave an error that varied by less than one percent.

Figure 1 contains the error plotted against the time step for the iterative, P2, P3 and pressure methods, with the error being the average of the pressure, U, V and temperature errors. A correct representation of the pressure error, which for these schemes requires extrapolation to the $n + 1$ time step location, as shown in Armfield and Street [3], has been used here. The increased magnitude of the P2 projection error, when compared with the iterative scheme, is apparent. For both the P3 and pressure methods the error is almost identical to the iterative scheme. Clearly both of the new schemes have negligible additional error.

Figure 2, which contains plots of the error against CPU time, demonstrates the comparitive efficiency of the schemes. The relatively poor efficiency of the iterative scheme is seen, with the P2 scheme being more efficient, even though

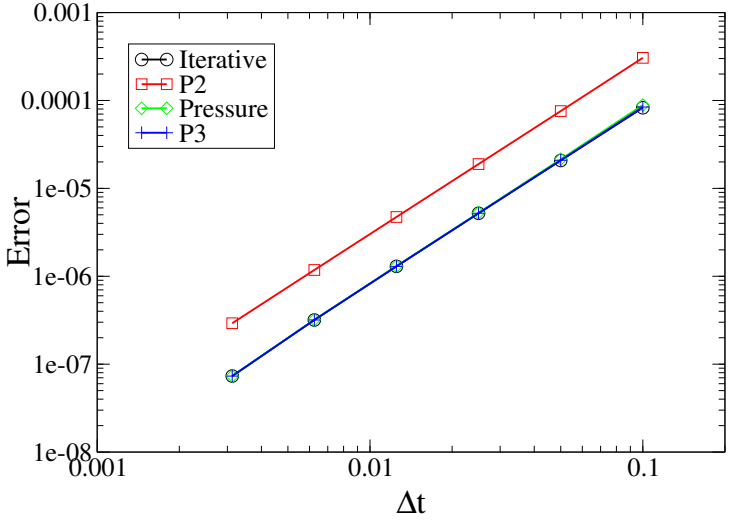

FIGURE 1: Comparison of accuracy for the iterative, P2, pressure and P3 methods

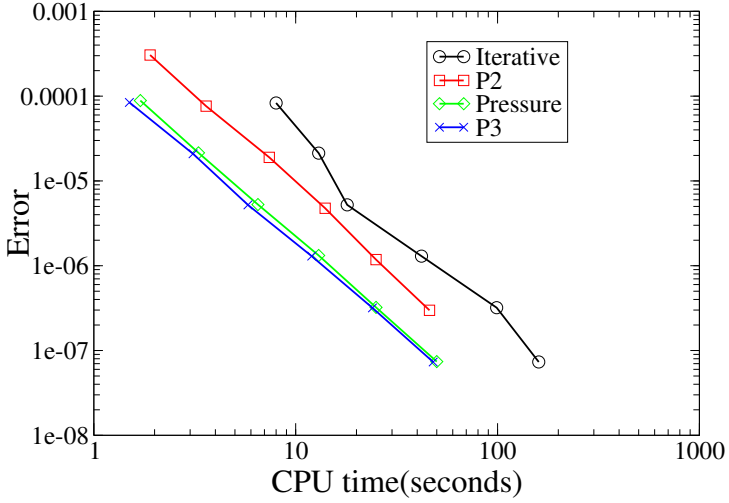

FIGURE 2: Comparison of efficiency for the iterative, P2, pressure and P3 methods

for a given time step size the error is larger. The P3 and pressure schemes are significantly more efficient than both the P2 and iterative schemes, with equivalent performance. The new schemes require half the CPU time of the iterative scheme, and 60% that of the P2 scheme, to achieve an equivalent error.

The divergence error for the pressure scheme has also been monitored, and was found to range from 3.2×10^{-6} for the largest time step, to 8.6×10^{-11} for the smallest time step, with a Δt^3 variation. An accurate solution of the pressure equation therefore ensures a small divergence error.

4 Conclusions

Both the P3 and pressure schemes are clearly more efficient than the P2 and iterative methods. In particular the new schemes have almost entirely eliminated the error associated with using a non-iterative scheme for the flow considered here. The improved accuracy has been achieved by including an additional extrapolation, of the velocity in the viscous terms for the pressure method and of the pressure for the P3 method. Empirical tests have shown that these extrapolations do not adversely affect the stability of either method, with the largest stable time step size being the same for all four methods. This may be compared with using a P2 method with Adams-Bashforth extrapolation for the viscous terms, where the stable time step is 0.025, an order-of-magnitude less than that of the schemes tested here.

There is clearly little to choose between the P3 and pressure methods, however it is noted that for the P3 method optimal results require setting a different convergence criterion for each time step size, whereas for the pressure method the optimum convergence criterion is the same for all time step sizes. It is also noted that in [15] an analysis of a P3 method, similar to that used here, was carried out and it was shown that such a scheme could lead to solutions unbounded in time. Although the empirical tests carried

out here have not shown this instability, this is a possible problem with the P3 method.

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