# Singular controls in optimal collision avoidance for participants with unequal linear speeds

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(Received 9 January 2012; revised 6 March 2012)

**Abstract**

This article studies optimal collision avoidance strategies for participants with unequal linear speeds in a planar close proximity encounter. It is known that bang-bang collision avoidance strategies are optimal for encounters of participants with equal linear speeds. However, as shown recently, bang-bang collision avoidance strategies are not necessarily optimal when the linear speeds of the participants are not equal. We study the structure of optimal singular controls for collision avoidance of participants with unequal linear speeds, but equal turn capabilities. We prove that both controls cannot be singular simultaneously, and that the only possible singular control is a zero control. We use several optimization techniques to compute optimal state, control and adjoint variables. Numerical simulations suggest that a zero control strategy only exists for a slower participant and that, at most, one switching from a bang-bang to a singular control occurs. Different types of

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structural changes of the controls with change in the initial conditions are identified via the numerical simulations.

# Contents



# 1 Introduction

Close proximity encounters (that is, when the participants are sufficiently close in space and time to be of operational concern) occur in many situations in aviation, navigation and robotics. The problem of optimal cooperative collision avoidance of two participants in a planar close proximity encounter can be formulated as an optimal control problem characterized by a three-dimensional state vector, a two-dimensional control vector, a terminal cost functional, and a free terminal time [4, 5, 6, 7, 8, 9, 10, 11, 12]. The performance objective is to maximize the distance between the participants at the terminal time $\mathsf{T}$. In this problem, the *controls* are the non-dimensional angular speeds of the two participants, which are bounded functions of time on a time interval $[0, T]$. Three types of controls that appear in this problem are *bang-bang*, *singular* and *bang-singular* controls. A control function is called *bang-bang* on an interval $I \subset [0, T]$ when it takes values at its bounds on I. A *singular* control on an interval $I \subset [0, T]$ is a control function that takes values in the interior of the control region. A control is called *bang-singular* on $[0, T]$ if it consists of a combination of bang-bang and singular arcs.

Merz, Tarnopolskaya et al. [4, 5, 6, 7, 8, 9, 10, 11, 12] showed that optimal collision avoidance strategies for participants with equal linear speeds require bang-bang control strategies that remain constant for the whole duration of the encounter. This means that, in order to avoid a collision, each participant should turn with maximum allowed angular speed. However, we recently showed [11, 12] that such a strategy is not necessarily optimal for the case when the participants have unequal linear speeds. Combinations of both the model parameter and the initial conditions, under which bang-bang controls are no longer optimal at the terminal time, were established [11, 12].

Bang-bang and singular controls appear in various areas of application (for example, see recent applications in biomedical engineering [1, 2]). In this article, we study bang-bang and singular control strategies for collision avoidance of participants with unequal linear speeds in a planar close proximity encounter. First, we establish the elementary properties of singular controls. We prove that both controls cannot be singular simultaneously, and that the only possible singular control is a zero control. We then study the optimal strategies numerically. To determine the structure of optimal controls, that is, the sequence of bang-bang and singular arcs, we apply nonlinear programming methods to the discretized control problem. Then in a refinement step, switching times of bang-singular controls were computed with high accuracy using the arc-parametrization method of Maurer et al. [3].

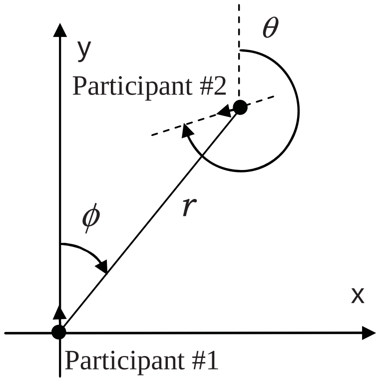


Figure 1: Schematics of a close proximity encounter.

# 2 Problem formulation

We make the same assumptions as in the close proximity encounter models [4, 5, 6, 7, 8, 9, 10, 11, 12]. Because of a short encounter time, a common assumption in such models is that the linear speeds of the participants are constant. The performance criterion is to maximize the terminal miss distance, which is the minimal distance between the participants during the manoeuvre. The problem of optimal collision avoidance of two participants with unequal linear speeds but equal turn capabilities is viewed as an optimal control problem [11, 12] with the state vector $\rho^* = (r, \phi, \theta)$, which satisfies the differential equations

$$\dot{\rho} = \begin{bmatrix} -\cos\phi + \gamma\cos(\theta - \phi) \\ -u_1 + [\sin\phi + \gamma\sin(\theta - \phi)]/r \\ -u_1 + u_2 \end{bmatrix} \equiv f(\rho, u), \tag{1}$$

with the control function $u^* = (u_1, u_2)$, $u : [0, T] \to U$ and $U = [-1, 1] \times [-1, 1] \subseteq R^2$. Here, $r$ specifies the non-dimensional instantaneous relative displacement between the participants (the non-dimensional $r$ is obtained by dividing a relative distance by the lower bound on the turn radius of the first participant); $\phi$ denotes the relative bearing; and $\theta$ is the relative heading, the instantaneous angle defining the relative direction of their motion (see Figure 1). The *controls* $u_1$ and $u_2$ are the non-dimensional *angular speeds* of the participants which are scaled so that $u_1, u_2 \in [-1, 1]$. The scaling is $u_1 = \omega_1/\omega_1^*$ and $u_2 = \omega_2/\omega_2^*$, where $\omega_1$ and $\omega_2$ are the angular speeds, and $\omega_1^*$ and $\omega_2^*$ are the absolute values of physical bounds on the angular speeds of the participants. The values $u_k = 1$, $k = 1, 2$, correspond to *right turns* of participants, while $u_k = -1$, $k = 1, 2$, correspond to *left turns*. The parameter $\gamma = V_2/V_1$ is a non-dimensional parameter with $0 < \gamma \leqslant 1$, where $V_1$ and $V_2$ are the linear speeds of the participants. Note that the faster participant is located at the origin (Figure 1). The derivatives with respect to the non-dimensional time $t$ are denoted with dots. The superscript $*$ denotes a transpose of a vector. The domains for the variables $\phi$ and $\theta$ are $-\pi \leqslant \phi < \pi$, $0 \leqslant \theta < 2\pi$.

The non-dimensional maneuver time $T$, the *terminal time*, is defined as the time to closest approach between the participants. It is determined by the conditions

$$\dot{r}(t) < 0 \quad \text{for } t \in [0, T), \quad \text{and} \quad \dot{r}(T) = 0 . \tag{2}$$

The objective is to maximize the terminal miss distance over all admissible controls:

$$\max_{u \in U_{ad}} \left[ J(\rho, u) = r(T) \right] . \tag{3}$$

Thus, the control problem is a Mayer problem with free terminal time.

For the purpose of this article, it is convenient to consider the problem in Cartesian coordinates. Using the transformation of variables,

$$r = \sqrt{x^2 + y^2}\,, \quad r \sin\phi = x\,, \quad r\cos\phi = y\,,$$

Equation (1) is rewritten in Cartesian coordinates,

$$\dot{X} = \begin{bmatrix} -u_1 y + \gamma \sin\theta \\ -1 + u_1 x + \gamma\cos\theta \\ -u_1 + u_2 \end{bmatrix} = f(X, u), \tag{4}$$

where $X^* = (x, y, \theta)$. The objective function in this case is

$$J(X, u) = \sqrt{x_T^2 + y_T^2}\,. \tag{5}$$

Here and henceforth, a subscript $T$ refers to the value at the terminal time, whereas a subscript $0$ refers to the values at the initial time.

# 3 Necessary optimality conditions: maximum principle

In Cartesian coordinates, the Hamiltonian takes the form

$$\begin{aligned} H(X, \lambda, u) &= \lambda^* \cdot f(X, u) \\ &= \lambda_x[-u_1 y + \gamma \sin\theta] + \lambda_y[-1 + u_1 x + \gamma\cos\theta] + \lambda_\theta[-u_1 + u_2] \\ &= u_1[-\lambda_x y + \lambda_y x - \lambda_\theta] + u_2\lambda_\theta + \lambda_x\gamma\sin\theta - \lambda_y + \lambda_y\gamma\cos\theta\,, \end{aligned} \tag{6}$$

where the adjoint variables $\lambda^* = (\lambda_x, \lambda_y, \lambda_\theta)$ satisfy the equation

$$\dot\lambda = -\nabla_X H(X, \lambda, u) = \begin{bmatrix} -\lambda_y u_1 \\ \lambda_x u_1 \\ -\lambda_x\gamma\cos\theta + \lambda_y\gamma\sin\theta \end{bmatrix}. \tag{7}$$

Henceforth we assume that all trajectories are normal. In view of the objective (5) and the terminal constraint (2), the endpoint Lagrangian is

$$l(X, \nu) = \sqrt{x^2 + y^2} + \nu\frac{x\dot x + y\dot y}{\sqrt{x^2 + y^2}}\,, \quad \nu \in R\,, \tag{8}$$

and the transversality condition is $\lambda(T) = \nabla_X l(X(T), \nu)$. For bang-bang controls, Tarnopolskaya et al. [12] showed that the multiplier $\nu = 0$. Numerical results confirm $\nu = 0$ also holds for bang-singular controls (a rigorous proof is outside the scope of this article). Hence, the transversality condition reduces to

$$\lambda(T) = \begin{bmatrix} \sin\phi_T \\ \cos\phi_T \\ 0 \end{bmatrix}, \quad x_T = r_T \sin\phi_T\,, \quad y_T = r_T\cos\phi_T\,. \tag{9}$$

Note that both controls, $u_1$ and $u_2$, *appear linearly* in the Hamiltonian. The *switching functions* are defined as the partial derivatives of the Hamiltonian with respect to the control components,

$$\Phi_1 = \Phi_1(X,\lambda) = \frac{\partial H}{\partial u_1} = -\lambda_x y + \lambda_y x - \lambda_\theta\,, \quad \Phi_2 = \Phi_2(X,\lambda) = \frac{\partial H}{\partial u_2} = \lambda_\theta\,. \tag{10}$$

As customary, we use the notation $\Phi_k(t) = \Phi_k(X(t),\lambda(t))$, $k = 1,2$, along trajectories. Due to the transversality condition (9), the switching functions have zero value at the terminal time, $\Phi_1(T) = \Phi_2(T) = 0$. Moreover, since the terminal time $T$ is free, the Hamiltonian vanishes along the trajectory,

$$H(X,\lambda,u) = u_1\,\Phi_1 + u_2\Phi_2 + \lambda_x\gamma\sin\theta - \lambda_y + \lambda_y\gamma\cos\theta = 0 \quad \text{on } [0,T]. \tag{11}$$

It follows from the Pontryagin Maximum Principle that optimal controls $u_1$ and $u_2$ maximize the Hamiltonian on the control set. This implies that the switching functions determine the control functions according to, for $k = 1,2$,

$$u_k(t) = \begin{cases} 1, & \text{if } \Phi_k(t) > 0\,, \\ -1, & \text{if } \Phi_k(t) < 0\,, \\ \text{undetermined}, & \text{if } \Phi_k(t) = 0\,. \end{cases} \tag{12}$$

The control $u_k$ is called *bang-bang* on an interval $I \subset [0,T]$ if the switching function $\Phi_k(t)$ has only isolated zeros on I, whereas the control $u_k$ is called *singular* on I if $\Phi_k(t) = 0$ holds for all $t \in I$. This suggests four possible bang-bang control strategies:

1. $u_1 = u_2 = 1$, the right-right (RR) strategy;
2. $u_1 = u_2 = -1$, the left-left (LL) strategy;

3. $u_1 = -u_2 = 1$, the right-left (RL) strategy; and
4. $u_1 = -u_2 = -1$, the left-right (LR) strategy.

In all cases the participants are turning with maximum possible angular speeds. The first letter in the strategy notation, R or L, indicates the strategy of the faster participant (located at the origin of Figure 1). As was shown by Merz, Tarnopolskaya et al. [4, 6, 7, 8], for given initial conditions, one of these strategies is optimal for the whole duration of the encounter if the linear speeds of the participants are equal.

# 4 Singular controls

When the linear speeds of the participants are *not equal*, values of the model parameter $\gamma$ and the initial relative heading angle $\theta_0$ exist such that bang-bang strategies are no longer optimal at the terminal time [12]. We repeat this result here for convenience.

**Proposition 1.** *For the participants with unequal linear speeds ($\gamma \neq 1$), bang-bang RR and LL strategies are not optimal at the terminal time if*

$$\gamma < \cos\theta_0 \quad \textit{for} \quad (0 < \theta_0 < \pi/2) \cup (3\pi/2 < \theta_0 < 2\pi). \tag{13}$$

We now study the structure of singular controls. First, we show that both controls cannot be singular simultaneously.

**Proposition 2.** *There does not exist an interval $I \subset [0, T]$ such that $\Phi_1(t) = \Phi_2(t) = 0$ on $I$; that is, the controls $u_1$ and $u_2$ can not be singular simultaneously.*

**Proof:** Assume that

$$\Phi_1 = -\lambda_x y + \lambda_y x - \lambda_\theta = 0\,, \quad \Phi_2 = \lambda_\theta = 0 \quad \text{on } I.$$

Using the adjoint equations (7) the derivatives of the switching functions are computed as

$$\dot{\Phi}_1 = \lambda_x\,, \quad \dot{\Phi}_2 = \gamma(-\lambda_x \cos\theta + \lambda_y \sin\theta)\,. \tag{14}$$

The equations $\Phi_1 = \Phi_2 = 0$ and $\dot{\Phi}_1 = \dot{\Phi}_2 = 0$ together with the transversality condition (11) then imply that $\lambda_x = \lambda_y = \lambda_\theta = 0$ holds on the interval I. By uniqueness of solutions to the adjoint equation, we then would have $\lambda = 0$ on the whole interval $[0, T]$. This contradicts the transversality condition (9) and proves Proposition 2. ♠

We now assume that one control is singular and show that the only possible value for this control is zero. First we investigate the case that the control $u_1$ is singular while the control $u_2 = \pm 1$ is bang-bang.

**Proposition 3.** *Let the control $u_1$ be singular on an interval $I \subset [0, T]$ and $u_2 = \pm 1$ be bang-bang on I. Then the singular control $u_1 = 0$.*

**Proof:** Since $u_1$ is singular on I, the switching function $\Phi_2$ vanishes on I. Differentiating the identity $\Phi_1(t) = 0$ twice using the adjoint equations (7), we obtain the following relations:

$$\Phi_1 = -\lambda_x y + \lambda_y x - \lambda_\theta = 0\,, \quad \dot{\Phi}_1 = \lambda_x = 0\,, \quad \ddot{\Phi}_1 = -\lambda_y u_1 = 0\,. \tag{15}$$

The third equation in (15) yields the singular control $u_1 = 0$, since otherwise $\lambda_y = 0$ would imply $\lambda_\theta = 0$ and, therefore, $\lambda_x = \lambda_y = \lambda_\theta = 0$ on I. Again, this would imply $\lambda_x = \lambda_y = \lambda_\theta = 0$ on the whole interval $[0, T]$, which contradicts the transversality condition (9). ♠

**Proposition 4.** *Let the control $u_2$ be singular on an interval $I \subset [0, T]$ and $u_1 = \pm 1$ be bang-bang on I. Then the singular control $u_2 = 0$.*

**Proof:** By definition of a singular control $u_2$ we have $\Phi_2(t) = 0$ on I. Differentiating this relation twice, the adjoint equations (7) imply

$$\dot{\Phi}_2/\gamma = -\lambda_x \cos\theta + \lambda_y \sin\theta = 0\,, \quad \ddot{\Phi}_2 = \gamma(\lambda_x \sin\theta + \lambda_y \cos\theta)u_2 = 0\,. \quad (16)$$

It follows from the second equation in (16) that the singular control $u_2 = 0$. Otherwise, equations $-\lambda_x \cos\theta + \lambda_y \sin\theta = 0$ and $\lambda_x \sin\theta + \lambda_y \cos\theta = 0$ would hold, which have only the trivial solution $\lambda_x = \lambda_y = 0$, a contradiction. This completes the proof of Proposition 4. ♠

If the control $u_1$ does not become singular, then Propositions 1 and 2 assert that the control $u_2$ must be singular on a terminal interval $[t_s, T]$. The control $u_2$ is discontinuous at $t_s$ and thus $t_s$ is called a *switching time*. A more detailed analysis reveals that there does not exist another singular arc in $[0, T]$; however, such analysis is outside the scope of this article. Hence, the singular control

$$u_2(t) = \begin{cases} \pm 1 & \text{for } 0 \leqslant t < t_s\,, \\ 0 & \text{for } t_s \leqslant t \leqslant T\,. \end{cases} \quad (17)$$

In this notation, the case $t_s = 0$ represents a *totally singular* control $u_2 = 0$ on $[0, T]$, whereas $t_s = T$ gives a bang-bang control $u_2 = \pm 1$ on $[0, T]$.

# 5 Case studies for singular controls

This section uses several optimization techniques to calculate optimal state, control and adjoint variables. First, we discretize the control problem using a sufficiently fine grid. Then the resulting large scale nonlinear programming problem is solved by Interior Point methods or Sequential Quadratic Programming methods. In a second step, the switching times of bang-singular controls are determined with high accuracy using the arc parametrization method presented by Maurer et al. [3]. All computations indicate that only the control $u_2$ of the slower participant can have a singular arc on a terminal interval $[t_s, T]$ (compare with (17)), while $u_1 = \pm 1$ is bang-bang with no switch. The optimal structure of both controls depends on the initial state $(x_0, y_0, \theta_0)$ and the parameter $\gamma$. To get an insight into the structural changes of the controls, we fix the model parameter values

$$\gamma = 0.2\,, \quad y_0 = 6\,, \quad \theta_0 = \pi/5\,,$$

and then compute optimal solutions for initial values of the $x$-coordinate in the range $-10 \leqslant x_0 \leqslant 10$. The values of $\gamma$ and $\theta_0$ comply with the conditions in Proposition 1 for the non-existence of optimal RR or LL bang-bang controls.

We denote a *control strategy* by *LL–L0*, when $u_1(t) = u_2(t) = -1$ holds on the interval $[0, t_s]$ and $u_1(t) = -1$ and $u_2(t) = 0$ on the terminal interval $[t_s, T]$. Control strategies *LR–L0* or *RL–R0* are defined in a similar way. We did not observe *RR–R0* strategies in our computations. In case $0 < t_s < T$, the control $u_2$ is called a *bang-singular* control. The computations were carried out using the arc parametrization method [3], where the arc lengths of bang-bang or singular arcs are optimized. We obtained several types of strategies and transitions between bang-bang and singular strategies, as described below.

**LR strategy** For $8.72 \leqslant x_0$ we find pure bang-bang controls $u_1 = -1$ and $u_2 = 1$. The limiting point $x_0 = 8.72$ is characterized by $\dot{\Phi}_2(T) = 0$ holds. Together with the condition $\Phi_2(T) = 0$ this indicates that we expect a singular arc of $u_2$ for $x_0 < 8.72$.

**LR–L0 strategy** In the range $3.75 < x_0 < 8.72$ we obtain *LR–L0* strategies with a singular control structure (17). The solution for the initial value $x_0 = 6$ is depicted in Figure 2. The following numerical results were obtained:

$$r_0 = 8.4853\,, \quad r_T = 8.3431\,, \quad t_s = 0.21162\,, \quad T = 0.52950\,,$$
$$\lambda_x(0) = 0.7447\,, \quad \lambda_y(0) = 0.6674\,, \quad \lambda_\theta(0) = 0.004436\,.$$

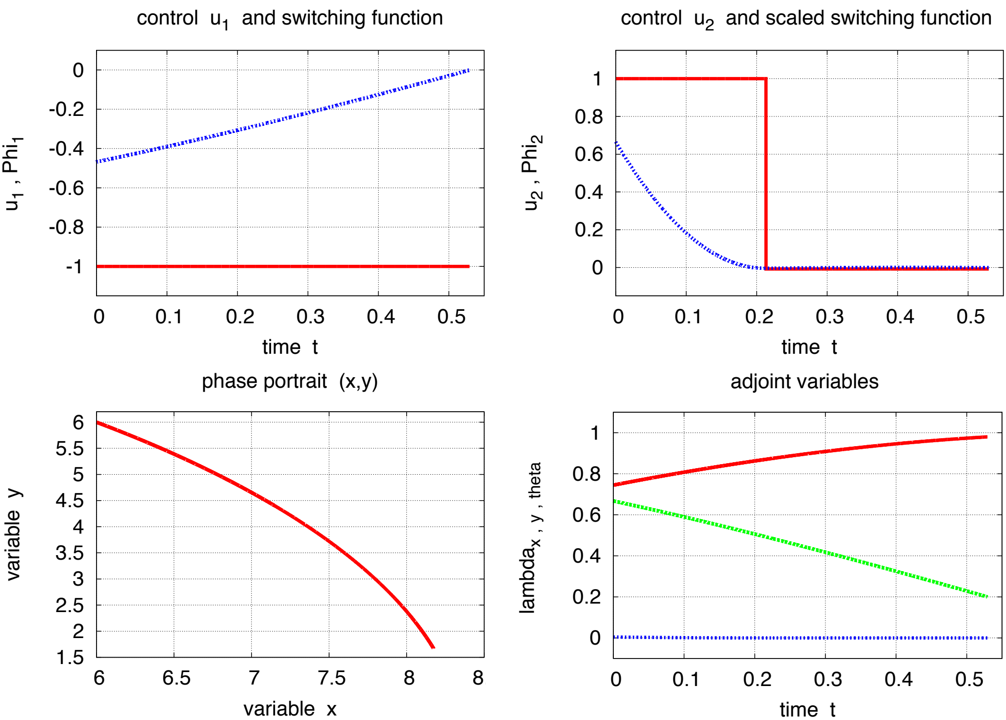


Figure 2: *LR–L0* strategy for $x_0 = 6$ and $y_0 = 6$. Top row: control $u_1$ and switching function $\Phi_1$ (left); control $u_2$ and switching function $\Phi_2$ (right). Control functions are shown with red lines, while switching functions are shown with blue lines. Bottom row: phase portrait $(x, y)$ (left) and plot of adjoint variables $\lambda_x$ (red), $\lambda_y$ (blue) and $\lambda_\theta$ (green).

The numerical results for $x_0 = 4$,

$$r_0 = 7.2111\,, \quad r_T = 6.9679\,, \quad t_s = 0.040793\,, \quad T = 0.70046\,,$$
$$\lambda_x(0) = 0.6203\,, \quad \lambda_y(0) = 0.7844\,, \quad \lambda_\theta(0) = 0.0001653\,,$$

exhibit a switching time $t_s$ near zero. Indeed, we observed that $t_s \to 0$ for $x_0 \to 3.75^+$. Hence, for $x_0 \approx 3.75$ the control $u_2 = 0$ is *totally singular* with $\Phi_2(t) = 0$ on $[0, T]$. This allows for a change in the structure of the control at $x_0 = 3.75$, namely, the structure *LR–L0* changes to *LL–L0*.

**LL–L0 strategy** This type of strategy was obtained in the range $-0.05 < x_0 < 3.75$ of initial values. For $x_0 = 3.4$ we obtained the numerical results

$$r_0 = 6.8964\,, \quad r_T = 6.6107\,, \quad t_s = 0.022436\,, \quad T = 0.76393\,,$$
$$\lambda_x(0) = 0.5695\,, \quad \lambda_y(0) = 0.8220\,, \quad \lambda_\theta(0) = -0.00005105\,.$$

The next structural change of strategies occurs at $x_0 \approx -0.05$. Here, we encountered the phenomenon that the two different strategies *LL–L0* and *RL–R0* produce the same functional value $r_T = 5.345$. We also observed such transitional points for pure bang-bang controls and called them dispersal points [6, 7, 8, 9, 10, 11, 12, cf.]. In the economic literature, such a point is called a Skiba-point. This type of transition is depicted in Figure 3, where the different control strategies are shown for $x_0 = 0$ and $x_0 = -0.1$.

**RL–R0 strategy** This strategy was obtained in the range $-1.61 < x_0 < -0.05$. For $x_0 = -0.1$ we obtained the numerical results

$$r_0 = 6.0008\,, \quad r_T = 5.3462\,, \quad t_s = 0.76749\,, \quad T = 1.2299\,,$$
$$\lambda_x(0) = -0.1391\,, \quad \lambda_y(0) = 0.9903\,, \quad \lambda_\theta(0) = -0.05606\,.$$

Another structural change occurs at $x_0 \approx -1.61$, when the *RL–R0* strategy gives way to the pure bang-bang strategy *RL* for $x_0 < -1.61$. The singular

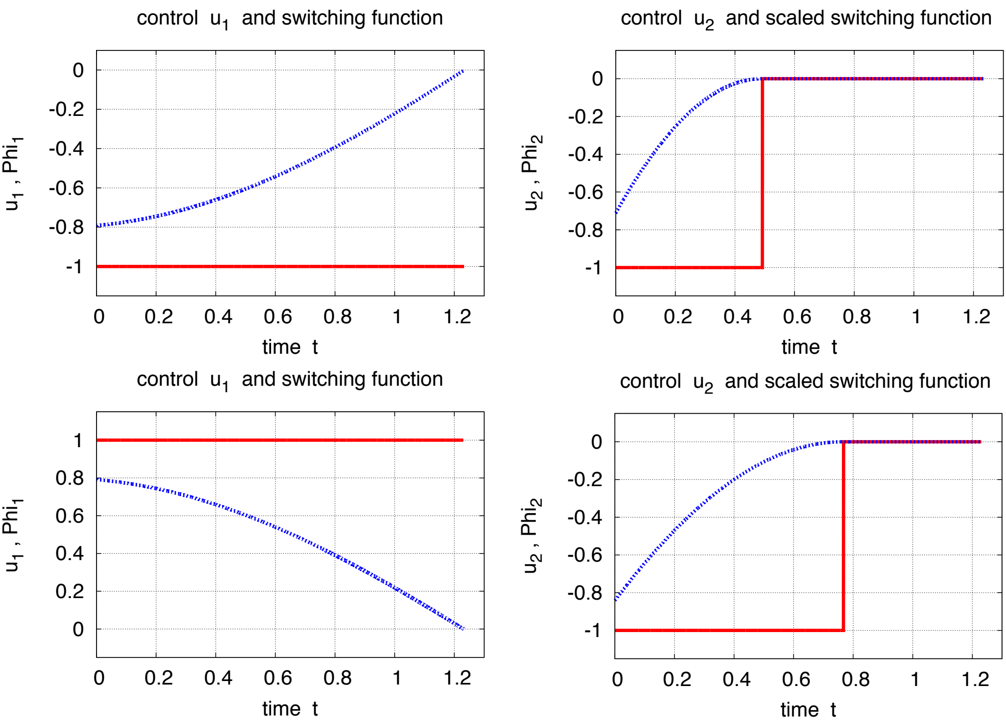


Figure 3: Top row: *LL-L0* strategy for $x_0 = 0$ and $y_0 = 6$: control $u_1$ and switching function $\Phi_1$ (left), control $u_2$ and switching function $\Phi_2$ (right); Bottom row: *RL-R0* strategy for $x_0 = -0$ and $y_0 = 6$: controls $u_1$ and $u_2$ and switching functions $\Phi_1$ and $\Phi_2$.

arc disappears, since we have $t_s \to T$ for $x_0 \to -1.61^+$. This is illustrated by the numerical results for $x_0 = -1.6$ where the switching time $t_s$ is close to the final time $T$ :

$$r_0 = 6.2097\,, \quad r_T = 5.7228\,, \quad t_s = 0.98335\,, \quad T = 1.0045\,,$$
$$\lambda_x(0) = -0.3566\,, \quad \lambda_y(0) = 0.9342\,, \quad \lambda_\theta(0) = -0.09071\,.$$

# 6 Conclusions

This article establishes, for the first time, several features of the structure of a singular control in the optimal collision avoidance for a planar close proximity encounter of participants with unequal linear speeds but equal turn capabilities. We showed that both controls cannot be singular simultaneously, and that the only possible singular control is a zero control. Several combinations of initial conditions and model parameters that result in conditions for which no optimal *RR* or *LL* bang-bang control exists at the terminal time (Proposition 1 [11, 12]) have been studied via the optimization methods developed by Maurer et al. [3].

The results of the study suggest that no more than one switching point is possible, and that switching to a zero control occurs only for the slower participant. Thus, the only possible structure of optimal singular control is bang-bang control switching to bang-singular control. Several types of such controls were observed. Optimization methods allow us to detect changes in the structure of optimal controls. Two types of structural change were observed. The first type is characterized by a totally singular arc with $u_2 \equiv 0$ separating two bang-singular controls $u_2$, one with initial value $u_2(0) = 1$ and the other with $u_2(0) = -1$. In the second type of structural transition, two different bang-singular strategies produce the same value of the objective functional.

A complete synthesis of optimal singular control, for feasible initial conditions $x_0$ and $y_0$ and given values of the model parameter $\gamma$ and $\theta_0$, awaits further study.

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