On the stability function of functionally fitted Runge–Kutta methods

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Abstract

Classical collocation Runge–Kutta methods are polynomially fitted in the sense that they integrate an ODE problem exactly if its solution is an algebraic polynomial up to some degree. Functionally fitted Runge–Kutta methods are collocation techniques that generalize this principle to solve an ODE problem exactly if its solution is a linear combination of a chosen set of arbitrary basis functions. Given for example a periodic or oscillatory ODE problem with a known frequency, it might be advantageous to tune a trigonometric functionally fitted Runge–Kutta method targeted at such a problem. However, functionally fitted Runge–Kutta methods lead to variable coefficients that depend on the parameters of the problem, the time, the step size, and the basis functions in a non-trivial manner that inhibits any in-depth analysis of the behavior of the methods in general. We present

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the class of so-called separable basis functions and show that it is possible to characterize the stability region of some special methods in this particular class. Explicit stability functions are given for some representative examples.

Contents

1 Introduction

Consider the initial value problem of dimension d

$$\begin{aligned} &\boldsymbol{y}'(x) = \boldsymbol{f}(x, \boldsymbol{y}(x))\,, \quad \boldsymbol{y}(x_0) = \boldsymbol{y}_0\,, \\ &\boldsymbol{y} : \mathbb{R} \to \mathbb{R}^d\,, \quad \boldsymbol{f} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d\,, \quad x \in [x_0,\, x_0 + X]\,, \end{aligned} \tag{1}$$

with the usual assumption that $\boldsymbol{f}$ is continuous and satisfies a Lipschitz condition in $[x_0,\, x_0 + X] \times \mathbb{R}^d$. Classical Runge–Kutta methods are very popular for solving this problem. A given s-stage RK method to solve (1) is defined by its Butcher tableau [1]

$$\begin{array}{c|c} \boldsymbol{c} & \boldsymbol{A} \\ \hline & \boldsymbol{b}^T \end{array}, \quad \boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{s\times s}, \quad \boldsymbol{b} = [b_1, \ldots, b_s]^T, \quad \boldsymbol{c} = \boldsymbol{A}\boldsymbol{e} = [c_1, \ldots, c_s]^T,$$

where $\boldsymbol{e} = [1, \ldots, 1]^T$. At each step, using the current value $\boldsymbol{y}_n \approx \boldsymbol{y}(x_n)$ and taking an appropriate step size h, the iteration for $\boldsymbol{y}_{n+1} \approx \boldsymbol{y}(x_n + h)$ is represented compactly using a Kronecker tensor product notation:

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{e} \otimes \boldsymbol{y}_n + h(\boldsymbol{A} \otimes \boldsymbol{I}_d)\boldsymbol{F}(\boldsymbol{e}x_n + \boldsymbol{c}h, \boldsymbol{Y})\,, \\ \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + h(\boldsymbol{b}^T \otimes \boldsymbol{I}_d)\boldsymbol{F}(x_n + \boldsymbol{c}h, \boldsymbol{Y})\,, \end{aligned} \tag{2}$$

where $\boldsymbol{F}(x_n + \boldsymbol{c}h, \boldsymbol{Y}) = [\boldsymbol{f}(x_n + c_1 h, \boldsymbol{Y}_1)^T, \ldots, \boldsymbol{f}(x_n + c_1 h, \boldsymbol{Y}_s)^T]^T$ and $\boldsymbol{Y} = [\boldsymbol{Y}_1^T, \ldots, \boldsymbol{Y}_s^T]^T \approx \boldsymbol{y}(x_n + \boldsymbol{c}h)$ are intermediate stage values. In the scalar case $(d = 1)$, this becomes

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{e}y_n + h\boldsymbol{A}f(\boldsymbol{e}x_n + \boldsymbol{c}h, \boldsymbol{Y}) \in \mathbb{R}^s\,, \\ y_{n+1} &= y_n + h\boldsymbol{b}^T f(\boldsymbol{e}x_n + \boldsymbol{c}h, \boldsymbol{Y}) \in \mathbb{R}\,, \end{aligned} \tag{3}$$

where $\boldsymbol{Y} = [Y_1, \ldots, Y_s]^T$ and $f(\boldsymbol{e}x_n + \boldsymbol{c}h, \boldsymbol{Y}) = [f(x_n + c_1 h, Y_1), \ldots, f(x_n + c_s h, Y_s)]^T$. For ease of presentation we shall retain this scalar form, but it should be stressed that the discussion also holds in vector form.

For an explicit RK, $\boldsymbol{A}$ is strictly lower-triangular and $c_1 = 0$, allowing to directly compute the iterates. However, for an implicit RK, $\boldsymbol{A}$ is general and so the iteration scheme is non-linear and needs special solution techniques such as Newton root-finding techniques or fixed-point iterations [1].

2 Functionally fitted RK methods

Functionally fitted RK (FRK) methods [6] are collocation techniques based on the idea of demanding to solve (1) exactly if its solution is a linear combination of a chosen set of basis functions $\{u_k(x)\}_{k=1}^s$. When the basis functions are the monomials, classical methods are recovered. The advantage of FRK methods is that the basis functions are chosen to exploit suitable properties of the problem that may be known in advance.

Definition 1 (Functionally fitted RK) *An s-stage Runge–Kutta method is a* FRK *(or a generalized collocation* RK*) method with respect to the basis functions $\{u_k(x)\}_{k=1}^s$ if the following relations are satisfied for all $k = 1, \ldots, s$:*

$$\begin{aligned} u_k(x+h) &= u_k(x) + h\boldsymbol{b}(x,h)^T u_k'(\boldsymbol{e}x + \boldsymbol{c}h)\,, \\ u_k(\boldsymbol{e}x + \boldsymbol{c}h) &= \boldsymbol{e}u_k(x) + h\boldsymbol{A}(x,h)u_k'(\boldsymbol{e}x + \boldsymbol{c}h)\,. \end{aligned} \tag{4}$$

This immediately gives a linear system to be solved for $\boldsymbol{A}$ and $\boldsymbol{b}$, yielding a method with variable coefficients that generally depend on x and h. The collocation points $(c_i)_{i=1}^s$ are usually taken in $[0,1]$ and we assume they are distinct.

2.1 The collocation condition

Not all arbitrary fitting functions satisfy (4); that is, we cannot solve for $\boldsymbol{A}$ and $\boldsymbol{b}$ with any choice of basis. We now discuss this aspect.

Definition 2 (Collocation condition) *A set of sufficiently smooth functions $\{u_1(x), u_2(x), \ldots, u_s(x)\}$ is said to satisfy the collocation condition if the matrices*

$$\boldsymbol{E}(x,h) = \Big[u_1(\boldsymbol{e}x + \boldsymbol{c}h) - u_1(\boldsymbol{e}x),\ \ldots,\ u_s(\boldsymbol{e}x + \boldsymbol{c}h) - u_s(\boldsymbol{e}x)\Big],$$

$$\boldsymbol{F}(x,h) = \Big[u_1'(\boldsymbol{e}x + \boldsymbol{c}h),\ \ldots,\ u_s'(\boldsymbol{e}x + \boldsymbol{c}h)\Big],$$

satisfy the condition that for any given value x_0, both $\boldsymbol{E}(x_0,h)$ and $\boldsymbol{F}(x_0,h)$ are nonsingular almost everywhere on the interval $h \in [0, X]$.

As indicated by Hoang et al. [4], the practical implication here is that the coefficients of an FRK method based on basis functions that satisfy the collocation condition are uniquely determined almost everywhere on the integration domain. Additionally, Ozawa [6] and Hoang et al. [4] showed that an s-stage FRK method has a stage order s and a step order at least s and at most $2s$. Superconvergent methods that attain the maximum order of $2s$ are constructed by specifically choosing the collocation points $(c_i)_{i=1}^s$ to satisfy some orthogonality condition, as is the case with Gauss–Legendre points.

2.2 The collocation solution

The *collocation solution* is a fundamental function reminiscent of the *collocation polynomial* found in classical algebraic collocation techniques. Given the basis functions $\{u_1, \ldots, u_s\}$, let $\boldsymbol{H} = \operatorname{Span}\{1, u_1, \ldots, u_s\}$, that is

$$\boldsymbol{H} = \left\{ v \in C[x_0,\, x_0 + X] : v(x) = a_0 + \sum_{i=1}^{s} a_i u_i(x), a_i \in \mathbb{R}, i = 0, \ldots, s \right\}.$$

Choose $(c_i)_{i=1}^s$ distinct and non-zero. We call $u(x)$ the *collocation solution* if it is an element of $\boldsymbol{H}$ that satisfies the differential equation at the collocation points, that is,

$$u(x_0) = y_0\,, \quad u'(x_0 + c_i h) = f(x_0 + c_i h, u(x_0 + c_i h))\,, \quad i = 1, \ldots, s\,. \tag{5}$$

If $u(x)$ exists, the numerical solution after one step is taken as

$$y_1 = u(x_0 + h). \tag{6}$$

This indirect way of doing so is usually referred to as the *collocation method.* While $u(x)$ is only defined implicitly, its existence can be assumed by the following result established by Hoang et al. [4].

Theorem 3 *The collocation method (6) is equivalent to the s-stage* FRK *method $(\boldsymbol{c},\, \boldsymbol{b}(x,h),\, \boldsymbol{A}(x,h))$, with respect to the given basis functions.*

3 Separable methods

Generalized FRK methods suffer from the drawback that their coefficients depend on x and h in a non-trivial manner that inhibits a deeper analysis of the behavior of the methods. However, there exists a particular class with coefficients that are *independent* of x. We refer to this class as *separable* methods [5].

Definition 4 *The set* $\{u_k\}_{k=1}^{s}$ *is said to be separable if* $\boldsymbol{u} = [1, u_1, \ldots, u_s]^T$ *satisfies*

$$\boldsymbol{u}(x+y) = \boldsymbol{\mathcal{F}}(y)\boldsymbol{u}(x)\,, \quad \textit{for all } x, y \in \mathbb{R}\,, \tag{7}$$

where $\boldsymbol{\mathcal{F}} \in \mathbb{R}^{(s+1)\times(s+1)}$ *with entries that are univariate functions.*

Theorem 5 *Assume that* $\{u_k\}_{k=1}^{s}$ *are sufficiently smooth functions that satisfy (4) at* $x = 0$. *Assume furthermore that they can be separated according to (7). Then, they generate a* FRK *method with coefficients* $(\boldsymbol{c}, \boldsymbol{b}(0,h), \boldsymbol{A}(0,h))$.

Remark 6 The collocation condition in §2.1 simplifies when the basis functions can be separated according to (7) in the sense that if the condition is satisfied at $x = 0$ then it is satisfied at any x.

Remark 7 Theorem 5 shows that the generalized RK coefficients depend only on h for the class of basis functions that can be separated according to (7). Examples include:

1. $\{u_k(x)\}_{k=1}^{2n} = \{\sin(\omega_1 x), \cos(\omega_1 x), \ldots, \sin(\omega_n x), \cos(\omega_n x)\}$;
2. $\{u_k(x)\}_{k=1}^{2m+n} = \{\sin(\omega x), \cos(\omega x), \ldots, \sin(m\omega x), \cos(m\omega x)\} \cup \{x^i\}_{i=1}^{n}$;
3. $\{u_k(x)\}_{k=1}^{2(n+1)} = \{\sin(\omega x), \cos(\omega x), \ldots, x^n \sin(\omega x), x^n \cos(\omega x)\}$; and

4. $\{u_k(x)\}_{k=1}^{2(m+1)} = \{\exp(\pm wx), x\exp(\pm wx), \dots, x^m \exp(\pm wx)\}$.

Hence polynomials, exponentials, sine-cosine and hyperbolic sine-cosine functions, and various combinations belong to this class. Note that collocation methods based on a combination of functions of different type are also called *mixed-collocation*. Coleman and Duxbury [2], for example, use the particular set of basis functions $\{\sin(\omega x), \cos(\omega x)\} \cup \{1, x, \dots, x^{s-1}\}$.

Theorem 8 *If $\boldsymbol{u} = [1, u_1, \dots, u_s]^T$ is separable according to (7), then there exists a constant matrix $\boldsymbol{S} = \boldsymbol{\mathcal{F}}'(0)$ such that $\boldsymbol{\mathcal{F}}(x) = e^{\boldsymbol{S}x}$.*

Proof: Since $\boldsymbol{u}(x+y+z) = \boldsymbol{\mathcal{F}}(x+y)\boldsymbol{u}(z) = \boldsymbol{\mathcal{F}}(x)\boldsymbol{\mathcal{F}}(y)\boldsymbol{u}(z)$, and this holds for any z, we necessarily have $\boldsymbol{\mathcal{F}}(x+y) = \boldsymbol{\mathcal{F}}(x)\boldsymbol{\mathcal{F}}(y)$, for all x, y, because the components of $\boldsymbol{u}(z)$ are linearly independent since $\boldsymbol{u}(z)$ satisfies the collocation condition. Similarly, $\boldsymbol{\mathcal{F}}(0) = \boldsymbol{I}$.

We claim that $\boldsymbol{\mathcal{F}}(x)$ is differentiable. First, we prove that $\boldsymbol{\mathcal{F}}'(0)$ exists. By definition,

$$\boldsymbol{u}'(x) = \lim_{h\to 0} \frac{\boldsymbol{u}(x+h) - \boldsymbol{u}(x)}{h} = \left(\lim_{h\to 0} \frac{\boldsymbol{\mathcal{F}}(h) - \boldsymbol{I}}{h}\right) \boldsymbol{u}(x).$$

The resulting limit factor, which by definition is $\boldsymbol{\mathcal{F}}'(0)$, is an independent term. If it was not defined, it would contradict the fact that $\boldsymbol{u}'$ exists everywhere. Now, by definition,

$$\begin{aligned}\boldsymbol{\mathcal{F}}'(x) &= \lim_{h\to 0} \frac{\boldsymbol{\mathcal{F}}(x+h) - \boldsymbol{\mathcal{F}}(x)}{h} \\ &= \boldsymbol{\mathcal{F}}(x)\left(\lim_{h\to 0} \frac{\boldsymbol{\mathcal{F}}(h) - \boldsymbol{I}}{h}\right) = \left(\lim_{h\to 0} \frac{\boldsymbol{\mathcal{F}}(h) - \boldsymbol{I}}{h}\right)\boldsymbol{\mathcal{F}}(x).\end{aligned}$$

Thus the existence of $\boldsymbol{\mathcal{F}}'(0)$ implies the existence of

$$\boldsymbol{\mathcal{F}}'(x) = \boldsymbol{\mathcal{F}}'(0)\boldsymbol{\mathcal{F}}(x) = \boldsymbol{\mathcal{F}}(x)\boldsymbol{\mathcal{F}}'(0).$$

From there, $\boldsymbol{\mathcal{F}}(x) = e^{\boldsymbol{\mathcal{F}}'(0)x}\boldsymbol{\mathcal{F}}(0) = e^{\boldsymbol{\mathcal{F}}'(0)x}$. ♠

The following result provides an effective procedure for identifying and constructing separable methods.

Theorem 9 *Let $\boldsymbol{H}' = \operatorname{Span}\{1, u_1', \ldots, u_s'\}$. Then $\boldsymbol{u} = [1, u_1, \ldots, u_s]^T$ is separable if and only if $\boldsymbol{H}' \subset \boldsymbol{H}$.*

Proof:

- $\Leftarrow$ $\boldsymbol{H}' \subset \boldsymbol{H}$ means that there exists a constant matrix $\boldsymbol{S}$ such that $\boldsymbol{u}'(x) = \boldsymbol{S}\boldsymbol{u}(x)$. Therefore $\boldsymbol{u}(x) = e^{\boldsymbol{S}x}\boldsymbol{u}(0)$, and so $\boldsymbol{u}(x+y) = e^{\boldsymbol{S}(x+y)}\boldsymbol{u}(0) = e^{\boldsymbol{S}y}\boldsymbol{u}(x)$, which means that $\boldsymbol{u}$ is separable.
- $\Rightarrow$ From the proof of Theorem 8, we have $\boldsymbol{u}'(x) = \boldsymbol{\mathcal{F}}'(0)\boldsymbol{u}(x)$. Hence $\boldsymbol{H}' \subset \boldsymbol{H}$.

♠

Remark 10 From the analysis above we see that any separable system of functions $\boldsymbol{u}(x) = [1, u_1(x), \ldots, u_s(x)]^T$ is of the form $\boldsymbol{u}(x) = e^{\boldsymbol{S}x}\boldsymbol{u}_0$, and this characterizes completely what a separable system of functions is.

3.1 Stability function

We now limit ourselves to separable methods and set $\boldsymbol{A}(h) = \boldsymbol{A}(0, h)$, $\boldsymbol{b}(h) = \boldsymbol{b}(0, h)$. Once the formal Butcher tableau of the method is constructed, it should be noted that it is possible to deliberately generate the coefficients with a value different from the actual step size. Consequently, we shall use a different parameter, ν, and refer to the method as $(c, \boldsymbol{b}(\nu), \boldsymbol{A}(\nu))$. This makes it possible to view the method as a family of methods because each choice of ν generates a particular method. But usually in practice, ν is fixed in advance or determined by the step size and some other parameter of the problem under consideration (for example, the fitted frequency in the case of a periodic problem).

Applying the FRK method $(c, \boldsymbol{b}(\nu), \boldsymbol{A}(\nu))$ to the Dahlquist test equation $y' = \lambda y$, we get the iteration scheme

$$y_{n+1} = R_\nu(\lambda h) y_n = \cdots = (R_\nu(\lambda h))^{n+1} y_0 \,,$$

where

$$R_\nu(z) = 1 + z\boldsymbol{b}(\nu)^T (I - z\boldsymbol{A}(\nu))^{-1}\boldsymbol{e} = \frac{\det[I - z\boldsymbol{A}(\nu) + z\boldsymbol{e}\boldsymbol{b}(\nu)^T]}{\det[I - z\boldsymbol{A}(\nu)]}$$

is the well known stability function [1, p.86, e.g.], which in our case depends on ν (and the underlying basis functions) as well.

Definition 11 *The stability region of an* FRK *method for a given parameter ν is*

$$S_\nu := \{z \in \mathbb{C} : |R_\nu(z)| \leq 1\} \,. \tag{8}$$

The stability region is difficult to characterize in general, except for some special cases [5, cf.]. Also, as first pointed out by Coleman and Ixaru [3], classical definitions need to be recast to suit the variable coefficient context in a practically useful way while still remaining consistent with the spirit of the original definitions. Thus we define a method as A-stable if it can withstand any Dahlquist test equation; that is, whatever the value of λ in the negative half-plane.

Definition 12 (A-stable method) *A* FRK *method with variable coefficients $(\boldsymbol{c}, \boldsymbol{b}(\nu), \boldsymbol{A}(\nu))$, with the stability function $R_\nu(z)$ above, is said to be A-stable at $\nu = \nu_0$ if $|R_{\nu_0}(z)| \leq 1$, for all $z \in \mathbb{C}^-$.*

Remark 13 We recommend to choose $\boldsymbol{c}$ so that the algebraic method is A-stable because, as $\nu \to 0$, $\boldsymbol{A}(\nu)$ and $\boldsymbol{b}(\nu)$ converge to the classical constant coefficients defined by $\boldsymbol{c}$ [6, Corollary 1], and so the stability function $R_\nu(z) = \det[I - z\boldsymbol{A}(\nu) + z\boldsymbol{e}\boldsymbol{b}(\nu)^T]/\det[I - z\boldsymbol{A}(\nu)]$ converges to the classical function.

To characterize the stability function of FRK methods in general, we turn to the collocation solution $u(x)$, which was shown in §2.2 to be a linear combination of the basis functions that satisfies the differential equation at the collocation points, which means for the test equation that

$$u(x_0) = y_0\,, \quad u'(x_0 + c_i h) = \lambda u(x_0 + c_i h)\,, \quad i = 1, \ldots, s\,. \tag{9}$$

For brevity, we set $x_0 = 0$, $x_1 = 1$, $h = 1$, $y_0 = 1$, $\lambda = z$, and use a similar way as classical methods to get the following generalization.

Theorem 14 *Suppose* $\operatorname{Span}\{1, u_1', \ldots, u_s'\} = \boldsymbol{H}' \subset \boldsymbol{H} = \operatorname{Span}\{1, u_1, \ldots, u_s\}$, *and assume that the collocation condition is satisfied. Then the stability function* $R(z) = u(1)$ *where*

$$u(x) = e^{zx}\left(K\int_0^x e^{-\xi z}M(\xi)d\xi + 1\right) \tag{10}$$

$$= \frac{K\left(z^s M(x) + z^{s-1}M'(x) + \cdots + M^{(s)}(x)\right) - u^{(s+1)}(x)}{K\left(z^s M(0) + z^{s-1}M'(0) + \cdots + M^{(s)}(0)\right) - u^{(s+1)}(0)}\,, \tag{11}$$

where K *is a constant and* $M \in \boldsymbol{H}$ *is the interpolation function such that* $M(c_i) = 0$, $i = 1, \ldots, s$.

Proof: The collocation solution $u(x) \in \boldsymbol{H}$. Therefore the assumption that $\boldsymbol{H}' \subset \boldsymbol{H}$ implies that $u'(x) - zu(x) \in \boldsymbol{H}$ as well (albeit understood as a linear combination with complex coefficients due to z), and furthermore it vanishes at the collocation points. Hence, there is a constant K such that

$$u'(x) - zu(x) = KM(x)\,, \quad u(x_0) = y_0 = 1\,,$$

where $M(x)$ is an interpolation function in $\boldsymbol{H}$ such that $M(x_0) \neq 0$ and $M(x_0 + c_i h) = 0$, $i = 1, \ldots, s$, whose existence is guaranteed by Hoang et al. [4, Lemma 2.2] under the collocation condition. Now, this inhomogeneous equation can be solved by the variation of constant formula to obtain the first equality. The other equality is obtained by differentiating $u' - zu = KM$ s times. ♠

The result is developed further by writing that $\boldsymbol{H} \ni M(x) = \boldsymbol{\mu}^T \boldsymbol{u}(x) = \boldsymbol{\mu}^T \exp[\boldsymbol{\mathcal{F}}'(0)x]\boldsymbol{u}_0$, with a constant vector $\boldsymbol{\mu} \in \mathbb{R}^{s+1}$, but this will be done elsewhere. Note that we recover the classical result in the case of the monomials where $\boldsymbol{H} = \operatorname{Span}\{1, x, \ldots, x^s\}$, $u^{(s+1)}(x) = 0$, and $M(x) = \frac{1}{s!}\prod_{i=1}^{s}(x - c_i)$.

3.2 Example of trigonometric methods

Consider the basis $\{\cos(\omega x), \sin(\omega x)\}$. This leads to a so-called 2-stage Trigonometric Implicit RK (TIRK) method studied by Hoang et al. [5]. Letting $\nu = \omega h$, Theorem 5 indicates how to obtain the coefficients by solving the systems

$$\nu \boldsymbol{b}^T \begin{bmatrix} \cos(c_1\nu) & \sin(c_1\nu) \\ \cos(c_2\nu) & \sin(c_2\nu) \end{bmatrix} = \begin{bmatrix} \sin\nu & 1 - \cos\nu \end{bmatrix},$$
$$\nu \boldsymbol{A} \begin{bmatrix} \cos(c_1\nu) & \sin(c_1\nu) \\ \cos(c_2\nu) & \sin(c_2\nu) \end{bmatrix} = \begin{bmatrix} \sin(c_1\nu) & 1 - \cos(c_1\nu) \\ \sin(c_2\nu) & 1 - \cos(c_2\nu) \end{bmatrix}.$$

From Theorem 14 above, the interpolation function $M \in \boldsymbol{H}$, and is represented in Fourier series form as

$$M(x) = \sin\frac{(x - c_1)\nu}{2} \sin\frac{(x - c_2)\nu}{2} = a_0 + a_1\cos(\nu x) + b_1 \sin(\nu x),$$

where $a_0 = \frac{1}{2}\cos\frac{(c_2 - c_1)\nu}{2}$, $a_1 = -\frac{1}{2}\cos\frac{(c_1 + c_2)\nu}{2}$, $b_1 = -\frac{1}{2}\sin\frac{(c_1 + c_2)\nu}{2}$, with the

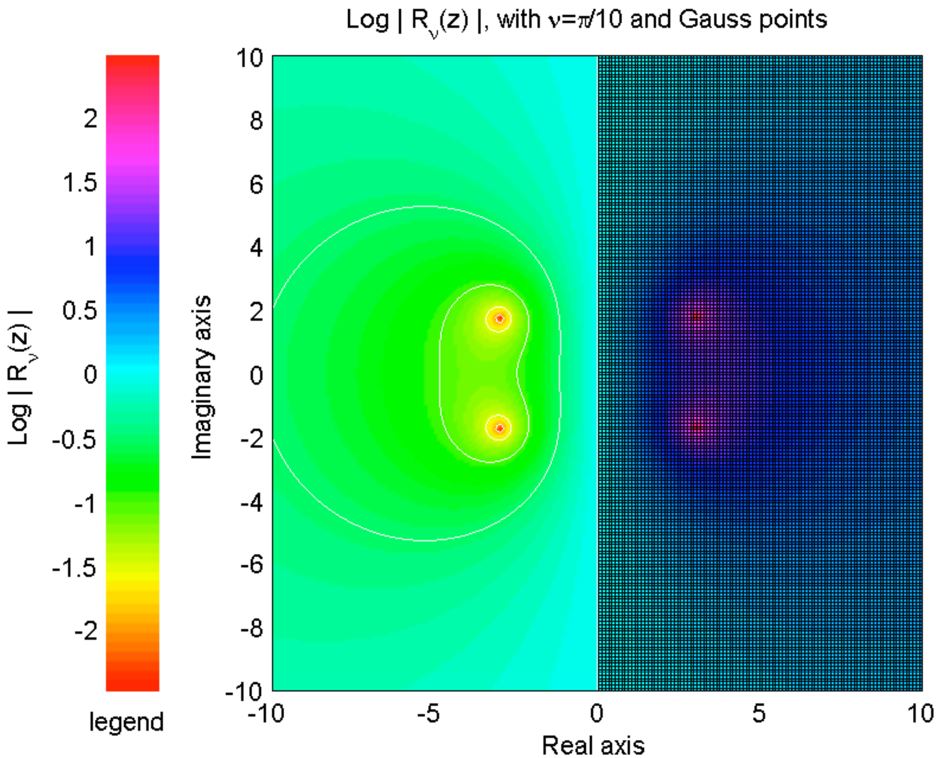

FIGURE 1: Plots of Log $|R_\nu(z)|$ for the trigonometric basis using Gauss points $((3-\sqrt{3})/6, (3+\sqrt{3})/6)$.

stability function

$$R_\nu(z) = \frac{z^2(a_0 + a_1\cos\nu + b_1\sin\nu) + z\nu(-a_1\sin\nu + b_1\cos\nu) + \nu^2 a_0}{z^2(a_0 + a_1) + z\nu b_1 + \nu^2 a_0} . \quad (12)$$

Based on this formula, the stability region of the method at a certain value of ν is plotted without any difficulty (see Figures 1–2).

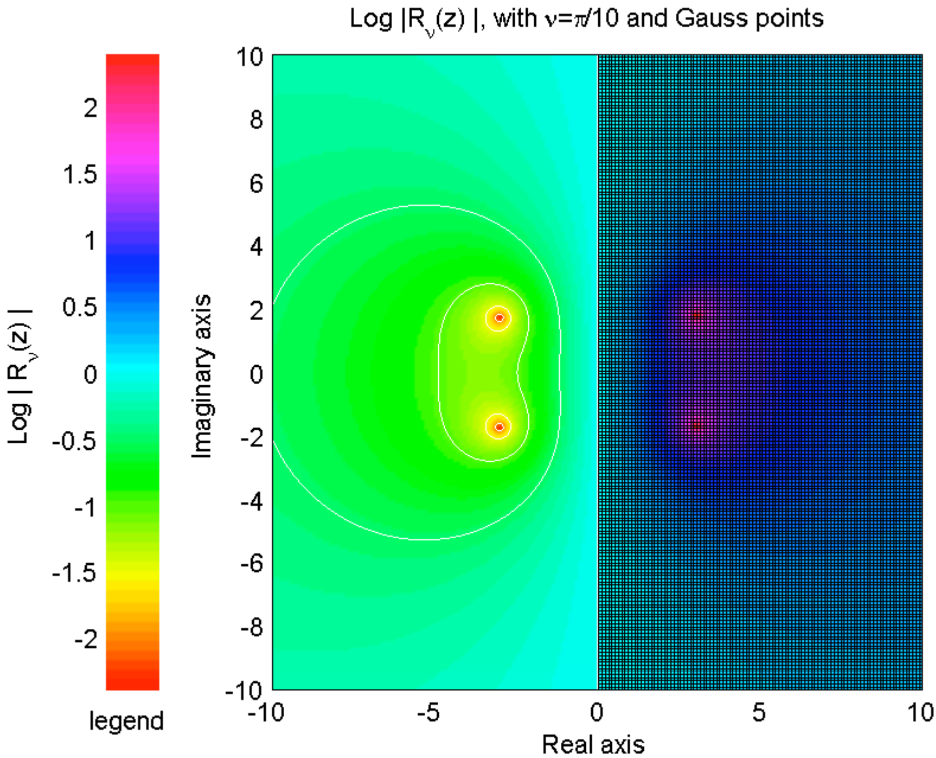

Figure 2: Plots of the exponential basis using Gauss points $((3-\sqrt{3})/6, (3+\sqrt{3})/6)$.

3.3 Example of exponential methods

Consider the basis $\{e^{\omega x}, e^{-\omega x}\}$. Let $\nu = \omega h$. Similarly to the previous example, using Theorem 5, the coefficients are found by solving algebraic systems. As for the stability function, first note that the interpolation function here is

$$M(x) = (e^{x\nu} - e^{c_1\nu})(e^{-x\nu} - e^{-c_2\nu}) = \alpha + \beta e^{-x\nu} + \gamma e^{x\nu} \,,$$

where $\alpha = 1 + e^{(c_1-c_2)\nu}$, $\beta = -e^{c_1\nu}$, $\gamma = -e^{-c_2\nu}$. Hence Theorem 14 gives

$$\begin{aligned} u(x) &= e^{zx}\left(K\int_0^x e^{-\xi z}(\alpha + \beta e^{-\xi\nu} + \gamma e^{\xi\nu})d\xi + 1\right) \\ &= e^{zx}\left(K\left[\alpha\frac{e^{-zx}-1}{-z} + \beta\frac{e^{-zx-x\nu}-1}{-(z+\nu)} + \gamma\frac{e^{-zx+x\nu}-1}{-(z-\nu)}\right] + 1\right). \end{aligned}$$

Since $u(x) \in \boldsymbol{H} = \operatorname{Span}\{1, e^{\omega x}, e^{-\omega x}\}$, irrelevant terms must vanish, meaning $K = -\left(\alpha\frac{1}{z} + \beta\frac{1}{z+\nu} + \gamma\frac{1}{z-\nu}\right)^{-1}$. And taking $u(1)$ gives the stability function

$$R_\nu(z) = \frac{\alpha\frac{1}{z} + \beta\frac{e^{-\nu}}{z+\nu} + \gamma\frac{e^{\nu}}{z-\nu}}{\alpha\frac{1}{z} + \beta\frac{1}{z+\nu} + \gamma\frac{1}{z-\nu}} \,. \tag{13}$$

Again, using Matlab, the stability function of the method with Gauss points at $\nu = \pi/10$ is plotted and presented in Figures 1–2.

One can check that when using Gauss points, the stability functions in (12) and (13) satisfy $|R_\nu(iy)| = 1$, for all $y \in \mathbb{R}$. And from the plots of the stability regions in Figures 1–2, we see that these stability functions have two poles in $\mathbb{C}^+$. Therefore, we conclude that these methods are A-stable at $\nu = \pi/10$ because they have no poles in $\mathbb{C}^-$. More generally, by plotting the stability regions of these methods, we found that the methods are A-stable for $\nu \in (0, \pi]$.

4 Conclusion

After providing the key ideas underlying functionally fitted methods, we detailed the class of so-called separable methods in particular. A general characterization of their stability function was given and illustrated explicitly with some representative examples. Theorem 14 enables a more detailed study of the stability regions of separable methods. The first formula in (10) is useful for studying the stability properties of separable FRK methods numerically or even analytically. The second formula (11) generalizes the representation seen in algebraic and trigonometric methods. It makes clear that in general the stability function of a separable method is a rational function of z.

References

[1] K. Burrage. *Parallel and sequential methods for ordinary differential equations.* Oxford University Press, Oxford, 1995. C153, C159

[2] J. P. Coleman and S. C. Duxbury. Mixed collocation methods for $y'' = f(t, y)$. *J. Comput. Appl. Math.*, 126:47–75, 2000. doi:10.1016/S0377-0427(99)00340-4 C157

[3] J. P. Coleman and L. Gr. Ixaru. P-stability and exponential-fitting methods for $y'' = f(x, y)$. *IMA J. Numer. Anal.*, 16:179–199, 1996. doi:10.1093/imanum/16.2.179 C159

[4] N. S. Hoang, R. B. Sidje, and N. H. Cong. On functionally-fitted Runge–Kutta methods. *BIT*, 46:861–874, 2006. doi:10.1007/s10543-006-0092-x C155, C161

[5] N. S. Hoang, R. B. Sidje, and N. H. Cong. Analysis of trigonometric implicit Runge–Kutta methods. *J. Comput. Appl. Math.*, 198:187–207, 2007. doi:10.1016/j.cam.2005.12.006 C156, C159, C161

[6] K. Ozawa. Functional fitting Runge–Kutta method with variable coefficients. *Japan J. Indust. App. Math*, 18:105–128, 2001. C153, C155, C160

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