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ON THE APPROXIMATE CONTROLLABILITY OF SOME SEMILINEAR PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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This work concerns the study of the approximate controllability for some nonlinear partial functional integrodifferential equation with infinite delay arising in the modelling of materials with memory, in the framework of Hilbert spaces. We give sufficient conditions that ensure the approximate controllability of the system by supposing that its linear undelayed part is approximately controllable, admits a resolvent operator in the sense of Grimmer, and by making use of the measure of noncompactness and the Mönch fixed-point Theorem. As a result, we obtain a generalization of several important results in the literature, without assuming the compactness of the resolvent operator. An example of applications is given for illustration.

1. Introduction

Control theory arises in many modern applications in engineering and environmental sciences, it is one of the most interdisciplinary research areas. While

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studying a control system, one of the most common problems that appear is the controllability problem, which consists in checking the possibility of steering the control system from an initial state (initial condition) to a desired terminal one (boundary condition), by an appropriate choice of the control u. The controllability problem has two distinguished notions which are the exact and the approximate controllability problems. Several authors have studied the concept of exact controllability for systems represented by nonlinear evolutions equations, in which the authors have effectively used fixed point technique (see e.g., [2, 5, 10, 23, 26, 27] and the references contained in them). In infinite-dimensional spaces the notion of exact controllability is usually too strong and, therefore has limited applicability (see [17] and references therein). The notion of approximate controllability is very often completely adequate in applications (see [17] and references therein). It is therefore important to study this weaker version of controllability for nonlinear integrodifferential systems. Approximate controllability of nonlinear differential and integrodifferential systems with and without delays in infinite dimensional spaces has been extensively studied (see e.g., [11, 12, 17, 18, 20–22, 25] and the references contained in them).

In [22], the authors studied the following nonlinear impulsive integrodifferential equation with unbounded delay:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t G(t-s)x(s)ds + f(t,x_t) + Bu(t) \ \text{ for } t \in J = [0,b] \\ x_0 = \phi \in \mathcal{P}, \\ \Delta x(t_k) = I_K(x_{t_k}), k = 1,\cdots,m \end{cases} \tag{1}$$

where $A : \mathcal{D}(A) \to X$ and $G(t) : \mathcal{D}(G(t)) \supset \mathcal{D}(A)$ are closed linear operators. Assuming the compactness of the resolvent operator for the associated linear system, they obtained existence and approximate controllability results for system (1).

In [25], the author considered the following unbounded delayed integrodifferential systems:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_{-\infty}^0 a(s)A_1x(t+s)ds + f(t,x_t) + Bu(t) \ \text{ for } t \in J = [0,b] \\ x_0 = \phi \in \mathcal{B}, \end{cases} \tag{2}$$

where $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a compact C_0-semigroup, and $A_1 : X \to X$ is a bounded linear operator. Using the compactness of the semigroup generated by A, the author obtained the existence and approximate controllability results for equation (2).

In [11], the authors considered the following semi-linear neutral evolution system with infinite delay:

$$\begin{cases} \dfrac{d}{dt}[x(t)+L_1(x_t)] = -Ax(t) + L_2(x_t) + f(t,x_t)+Bu(t) \;\text{ for } t \in J=[0,b] \\ x_0 = \phi \in \mathcal{BC}_\alpha, \end{cases} \tag{3}$$

where $-A:\mathcal{D}(-A)\to X$ is the infinitesimal generator of an analytic semigroup, $L_1:\mathcal{D}(A)\to X$, $L_2:\mathcal{B}\to X$ are bounded linear operators, and the spaces $\mathcal{B},\mathcal{BC}_\alpha$ are aximatically defined as we shall see later. Assuming the compactness of the analytic semigroup generated by $(-A,\mathcal{D}(-A))$ and using the theory of α-norm, the authors obtained existence and approximate controllability results for equation (3).

In [12], the authors considered the following semilinear neutral integrodifferential equations with finite delay:

$$\begin{cases} \dfrac{d}{dt}[x(t)+F(t,x_t)] = -Ax(t) + \displaystyle\int_0^t \gamma(t-s)x(s)ds + G(t,x_t)+Bu(t) \\ \text{for } t \in J=[0,b] \\ x_0 = \varphi, \;\; t \in [-r,0], \end{cases} \tag{4}$$

where $-A:\mathcal{D}(-A)\to X$ is the infinitesimal generator of an analytic semigroup, and $\gamma(\cdot)$ is a family of closed linear operators. Using the theory of α-norm and the fractional power, and without assuming the compactness of the resolvent operator, the authors obtained the existence and approximate controllability results for system (4).

More recently, Alka *et al* in [1], studied the approximate controllability of nonlocal fractional differential inclusions involving the Caputo fractional derivative of order $q\in(1,2)$ in a Hilbert space. Utilizing measure of noncompactness and multivalued fixed point strategy, they obtained a new set of sufficient conditions to ensure the approximate controllability of the nonlocal fractional differential inclusions. Also, Avadhesh *et al* in [3] established sufficient condition for the controllability of a control problem represented by second-order nonlinear differential equation with non-instantaneous impulses in a Hilbert space. They obtained their results using the strongly continuous cosine family of linear operators and Banach fixed point method.

Motivated by the above works, we study in this paper, the approximate controllability for some systems that arise in the analysis of heat conduction in materials with memory and viscosity [13, 16]. The interesting thing about materials with memory is that they act adaptively to their environment. They can easily be shaped into different forms at a low temperature, but return to their original shape on heating. Steering such systems from an initial state (initial condition) to a desired terminal one (boundary condition) by choosing appropriately a control, is of interest to many engineers and scientists. Such systems take the form of the following abstract model of partial functional integrodifferential equation with infinite delay in a Banach space $(X, \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t \gamma(t-s)x(s)ds + f(t,x_t) + Cu(t) \ \text{ for } t \in I = [0,b] \\ \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{5}$$

where $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a Hilbert space X; for $t \geq 0$, $\gamma(t)$ is a closed linear operator with domain $\mathcal{D}(\gamma(t)) \supset \mathcal{D}(A)$. The control u belongs to $L^2(I,U)$ which is a Banach space of admissible controls, where U is another Hilbert space. The operator $C \in \mathcal{L}(U,X)$, where $\mathcal{L}(U,X)$ denotes the Banach space of bounded linear operators from U into X, and the phase space $\mathcal{B}$ is a linear space of functions mapping $]-\infty,0]$ into X satisfying axioms which will be described later, for every $t \geq 0$, x_t denotes the history function of $\mathcal{B}$ defined by $x_t(\theta) = x(t+\theta)$ for $-\infty \leq \theta \leq 0$, $f : I \times \mathcal{B} \to X$ is a continuous function satisfying some conditions. In the literature devoted to equations with finite delay, the phase space is the space of continuous functions on $[-r,0]$, for some $r > 0$, endowed with the uniform norm topology. But when the delay is unbounded, the selection of the phase space $\mathcal{B}$ plays an important role in both qualitative and quantitative theories. A usual choice is a normed space satisfying some suitable axioms, which was introduced by Hale and Kato [14].

Integrodifferential equations appear in many areas of applications such as Electronics, Engineering, Physical Sciences, Fluid Dynamics, etc, During the last decades, these integrodifferential systems have received considerable attention. In recent years, many authors have worked on the existence and regularity of solutions of nonlinear functional integrodifferential equations with infinite delay, using the resolvent operator theory, see e.g., [9] and the references contained in it.

R. Grimmer in [13], proved the existence and uniqueness of resolvent operators that give the variation of parameters formula for the solutions, for these integrodifferential equations. In [7], W. Desch, R. Grimmer and W. Schappacher proved that the compactness of the resolvent operator is equivalent to that of the

semigroup.
In this work, we use the fact that the operator-norm continuity of the resolvent operator is equivalent to that of the semigroup [10]. In fact, we assume that the resolvent operator admitted by the linear undelayed part of equation (5) is operator-norm continuous. This property allows us to drop the compactness assumption on the operator semigroup, considered by the authors in [11, 22, 25], and prove that the operator solution satisfies the Mönch condition. We prove the approximate controllability result using the Mönch's fixed-point Theorem and the Hausdorff measure of noncompactness. This method enables us overcome the resolvent operator case considered in this work. Here the semigroup property can not be used because resolvent operators in general are semigroups.
So, as contribution, compared to [11, 22, 25] and many other references in the literature, this paper considers a broader class of functional differential equations; the compactness condition on the operator semigroup is dropped and replaced by a weaker and more realistic condition which is continuity in the operator norm topology; the variation of parameter formula for the solution is written using the resolvent operators which do not satisfy the semigroup condition and are therefore more general than semigroups. Also, our technique of proof, using measure of noncompactness is of particular interest, as measure of noncompactness is an important tool in the wide areas of functional analysis and differential equations [1]. To the best of our knowledge, up to now no work has reported on approximate controllability of partial functional integrodifferential equation (5) with infinite delay in Hilbert spaces. It has been an untreated topic in the literature, and this fact also motivates the present work.
The rest of the work is organized as follows: Section 2 is devoted to preliminary results. In this section, we give the definition of resolvent operator. This allows us to define the mild solution of equation (5). In section 3, we study the existence of mild solutions to equation (5). In section 4, we prove the approximate controllability of the control system (5), assuming the approximate controllability of the associated linear undelayed part. In section 5, we give an example to illustrate the obtained results.

2. Integrodifferential equations and measure of noncompactness

Integrodifferential equations have applications in many problems arising in physical systems, the following one-dimensional model in viscoelasticity is one of the applications of that theory

$$\begin{cases} \alpha \dfrac{\partial^2 \omega}{\partial t^2}(t,\xi) + \beta \dfrac{\partial \omega}{\partial t}(t,\xi) = \dfrac{\partial \varphi}{\partial \xi}(t,\xi) + h(t,\xi), \\ \gamma \dfrac{\partial \omega}{\partial \xi}(t,\xi) + \displaystyle\int_0^t a(t-s) \dfrac{\partial \omega}{\partial \xi}(s,\xi)ds = \varphi(t,\xi), \ (t,\xi) \in \mathbb{R}^+ \times [0,1], \\ \omega(t,0) = \omega(t,1) = 0, \quad t \in \mathbb{R}^+, \\ \omega(0,\xi) = \omega_0(\xi), \quad \xi \in [0,1], \end{cases}$$

where, ω is the displacement, φ is the stress, h is the external force, $\alpha, \ \gamma > 0$ and β are constants. In this model, the first equation describes the linear momentum while the second equation describes the constitutive relation between stress and strain. Setting $\gamma = 1, \ v = \frac{\partial \omega}{\partial t}$, and $u = \frac{\partial \omega}{\partial \xi}$, the above equations can be rewritten as follows

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & \partial_\xi \\ \frac{\partial_\xi}{\alpha} & 0 \end{bmatrix} \left\{ \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + \int_0^t \begin{bmatrix} a(t-s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} ds \right\}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\beta}{\alpha} \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{h(t)}{\alpha} \end{bmatrix}, \ t \geq 0.$$

Setting

$$x(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \partial_\xi \\ \frac{\partial_\xi}{\alpha} & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} a(t-s) & 0 \\ 0 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\beta}{\alpha} \end{bmatrix}, \quad \text{and} \quad p(t) = \begin{bmatrix} 0 \\ \frac{h(t)}{\alpha} \end{bmatrix},$$

we can rewrite the above equation into the following abstract form

$$\begin{cases} x'(t) = A\left[x(t) + \displaystyle\int_0^t G(t-s)x(s)\right]ds + Kx(t) + p(t) \ \text{ for } \quad \mathrm{t} \geq 0 \\ x(0) = x_0. \end{cases}$$

The operator A here is unbounded, while K and $G(t)$ are bounded operators for $t \geq 0$ on a Banach space X. When $AG(t) = G(t)A$, we obtain the following equation

$$\begin{cases} x'(t) = Ax(t) + \int_0^t G(t-s)Ax(s)ds + Kx(t) + p(t) \ \text{ for } \quad \text{t} \geq 0 \\ \\ x(0) = x_0. \end{cases}$$

which has been studied in [8]. We note that in general, the equality $AG(t) = G(t)A$ does not hold.

Let $I = [0,b]$, $b > 0$ and let X be a Banach space. A measurable function $x : I \to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable. We denote by $L^1(I,X)$ the Banach space of Bochner integrable functions $x : I \to X$ normed by

$$\|x\|_{L^1} = \int_0^b \|x(t)\|dt.$$

Consider the following linear homogeneous equation:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t \gamma(t-s)x(s)ds \ \text{ for } \quad t \geq 0 \\ \\ x(0) = x_0 \in X. \end{cases} \tag{6}$$

where A and $\gamma(t)$ are closed linear operators on a Banach space X.
In the sequel, we assume A and $\big(\gamma(t)\big)_{t\geq 0}$ satisfy the following conditions:

$(\mathbf{H_1})$ A is a densely defined closed linear operator in X. Hence $\mathcal{D}(A)$ is a Banach space equipped with the graph norm defined by, $|y| = \|Ay\| + \|y\|$ which will be denoted by $(X_1, |\cdot|)$.

$(\mathbf{H_2})$ $\big(\gamma(t)\big)_{t\geq 0}$ is a family of linear operators on X such that $\gamma(t)$ is continuous when regarded as a linear map from $(X_1, |\cdot|)$ into $(X, \|\cdot\|)$ for almost all $t \geq 0$ and the map $t \mapsto \gamma(t)y$ is measurable for all $y \in X_1$ and $t \geq 0$, and belongs to $W^{1,1}(\mathbb{R}^+, X)$. Moreover there is a locally integrable function $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\gamma(t)y\| \ \leq \ b(t)|y| \quad \text{and} \quad \left\|\frac{d}{dt}\gamma(t)y\right\| \ \leq \ b(t)|y| \ .$$

Remark 2.1. Note that $(\mathbf{H_2})$ is satisfied in the modelling of Heat Conduction in materials with memory and viscosity. More details can be found in [15].

Let $\mathcal{L}(X)$ be the Banach space of bounded linear operators on X.

Definition 2.2. [9] A resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (6) is a bounded operator valued function

$$R\,:\,[0,+\infty)\;\longrightarrow\;\mathcal{L}(X)$$

such that

(i) $R(0)=Id_X$ and $\|R(t)\|\leq Ne^{\beta t}$ for some constants N and β.

(ii) For all $x\in X$, the map $t\mapsto R(t)x$ is continuous for $t\geq 0$.

(iii) Moreover for $x\in X_1$, $R(\cdot)x\in\mathcal{C}^1(\mathbb{R}^+;X)\cap\mathcal{C}(\mathbb{R}^+;X_1)$ and

$$\begin{aligned} R'(t)x &= AR(t)x+\int_0^t\gamma(t-s)R(s)xds\\ &= R(t)Ax+\int_0^t R(t-s)\gamma(s)xds. \end{aligned}$$

Observe that the map defined on $\mathbb{R}^+$ by $t\mapsto R(t)x_0$ solves equation (6) for $x_0\in\mathcal{D}(A)$.

Theorem 2.3. *[13] Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then, the linear equation (6) has a unique resolvent operator* $\big(R(t)\big)_{t\geq 0}$.

Remark 2.4. In general, the resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (6) does not satisfy the semigroup law, namely,

$$R(t+s)\neq R(t)R(s)\qquad\text{for some }\ t,s>0\,.$$

We have the following theorem that establishes the equivalence between the operator-norm continuity of the C_0-semigroup and the resolvent operator for integral equations.

Theorem 2.5. *[10] Let A be the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ and let $\big(\gamma(t)\big)_{t\geq 0}$ satisfy* $(\mathbf{H_2})$. *Then the resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation* (6) *is operator-norm continuous (or continuous in the uniform operator topology) for $t>0$ if and only if $\big(T(t)\big)_{t\geq 0}$ is operator-norm continuous for $t>0$.*

In this work, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [14]. Thus, $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ will be a normed linear space of functions mapping $]-\infty,0]$ into X and satisfying the following axioms:

$(\mathbf{A_1})$ There exist positive constant H and functions $K:\mathbb{R}^+\to\mathbb{R}^+$ continuous and $M:\mathbb{R}^+\to\mathbb{R}^+$ locally bounded, such that for $a>0$, if $x:]-\infty,a]\to X$ is continuous on $[0,a]$ and $x_0\in\mathcal{B}$, then for every $t\in[0,a]$, the following conditions hold:

(i) $x_t\in\mathcal{B}$,

(ii) $\|x(t)\|\le H\|x_t\|_{\mathcal{B}}$, which is equivalent to $\|\varphi(0)\|\le H\|\varphi\|_{\mathcal{B}}$ for every $\varphi\in\mathcal{B}$,

(iii) $\|x_t\|_{\mathcal{B}}\le K(t)\sup\limits_{0\le s\le t}\|x(s)\|+M(t)\|x_0\|_{\mathcal{B}}$.

$(\mathbf{A_2})$ For the function x in $(\mathbf{A_1})$, $t\to x_t$ is a $\mathcal{B}$-valued continuous function for $t\in[0,a]$.

$(\mathbf{A_3})$ The space $\mathcal{B}$ is complete.

Example [9] Let the spaces
BC the space of bounded continuous functions defined from $(-\infty,0]$ to X;
BUC the space of bounded uniformly continuous functions defined from $(-\infty,0]$ to X;
$C^\infty:=\Big\{\phi\in BC:\ \lim_{\theta\to-\infty}\phi(\theta)\,\text{exists}\Big\}$;
$C^0:=\Big\{\phi\in BC:\ \lim_{\theta\to-\infty}\phi(\theta)=0\Big\}$, be endowed with the uniform norm

$$\|\phi\|=\sup_{\theta\le 0}\|\phi(\theta)\|.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions $(\mathbf{A_1})-(\mathbf{A_3})$.

Definition 2.6. Let $u\in L^2(I,U)$ and $\varphi\in\mathcal{B}$. A function $x:]-\infty,b]\to X$ is called a mild solution of equation (5) if $x\in\mathcal{C}([0,b];X)$ and satisfies the following integral equation

$$x(t)=\begin{cases} R(t)\varphi(0)+\displaystyle\int_0^t R(t-s)\,[f(s,x_s)+Cu(s)]\,ds\ \text{ for } t\in I\\[2ex] \varphi(t)\ \text{ for }\ -\infty\le t\le 0.\end{cases}\tag{7}$$

Let's define the notion of approximate controllability which is the main topic of this paper.
Let $x(b,x_0,u)$ be the state value of (5) at terminal time b corresponding to the control u and the initial value $x_0=\varphi\in\mathcal{B}$. Introduce the set

$$\mathcal{R}(b,x_0)=\{x(b,x_0,u),\,u\in L^2(I;U)\},$$

which is called the reacheable set of system (5) at terminal time b.

Definition 2.7. Equation (5) is said to be approximately controllable on the interval $I=[0,b]$ if $\mathcal{R}(b,x_0)$ is dense in X, i.e., $\overline{\mathcal{R}(b,x_0)}=X$.

We introduce the following operators to study the approximate controllability of system (5).

$$\Gamma_b^0=\int_0^b R(b-s)CC^*R^*(b-s)\,ds, \quad W(\lambda,\Gamma_b^0)=(\lambda Id+\Gamma_b^0)^{-1},$$

where C^* and $R^*(t)$ denote the adjoints of the operators of C and $R(t)$ respectively, and we assume that the operator $W(\lambda,\Gamma_T)$ satisfies
$(\mathbf{H_0})$ $\lambda W(\lambda,\Gamma_b^0)\to 0$ as $\lambda\to 0^+$ in the strong operator topology.
From [12], hypothesis $(\mathbf{H_0})$ is equivalent to the fact that the linear control system:

$$\begin{cases} x'(t)=Ax(t)\;+\;\displaystyle\int_0^t \gamma(t-s)x(s)ds\;+Cu(t) \qquad \text{for } t\in I=[0,b] \\ x_0=\varphi\in\mathcal{B}, \end{cases} \tag{8}$$

corresponding to system (5), is approximately controllable on $[0,b]$.

For proving the results of the paper we recall some properties of the measure of noncompactness and the Mönch fixed-point Theorem.

Definition 2.8. [4] Let D be a bounded subset of a normed space Y. The Hausdorff measure of noncompactness (shortly MNC) is defined by

$$\beta(D)=\inf\Big\{\varepsilon>0: D\ has\ a\ finite\ cover\ by\ balls\ of\ radius\ less\ than\ \varepsilon\Big\}.$$

Theorem 2.9. *[4] Let D, D_1, D_2 be bounded subsets of a Banach space Y. The Hausdorff MNC has the following properties:*

(i) *If $D_1\subset D_2$, then $\beta(D_1)\le\beta(D_2)$, (monotonicity).*

(ii) $\beta(D)=\beta(\overline{D})$.

(iii) $\beta(D)=0$ *if and only if D is relatively compact.*

(iv) $\beta(\lambda D)=|\lambda|\beta(D)$ *for any $\lambda\in\mathbb{R}$, (Homogeneity)*

(v) $\beta(D_1+D_2) \leq \beta(D_1)+\beta(D_2)$, *where* $D_1+D_2 = \{d_1+d_2 : d_1 \in D_1,\, d_2 \in D_2\}$, *(subadditivity)*

(vi) $\beta(\{a\} \cup D) = \beta(D)$ *for every* $a \in Y$.

(vii) $\beta(D) = \beta(\overline{co}(D))$, *where* $\overline{co}(D)$ *is the closed convex hull of* D.

(viii) *For any map* $G : \mathcal{D}(G) \subseteq X \to Y$ *which is Lipschitz continuous with a Lipschitz constant* k, *we have*

$$\beta(G(D)) \ \leq \ k\beta(D),$$

for any subset $D \subseteq \mathcal{D}(G)$.

Let

$$R_b = \sup_{t\in[0,b]} \|R(t)\|, \ \ K_b = \sup_{t\in[0,b]} \|K(t)\|, \ \ M_b = \sup_{t\in[0,b]} \|M(t)\|.$$

We now state the following useful result for equicontinuous subsets of $\mathcal{C}([a,b];X)$, where X is a Banach space.

Lemma 2.10. *[4] Let* $M \subset \mathcal{C}([a,b];X)$ *be bounded and equicontinuous. Then* $\beta(M(t))$ *is continuous and*

$$\beta(M) = \sup\{\beta(M(t)); t \in [a,b]\}, \quad \text{where } M(t) = \{x(t); x \in M\}.$$

Lemma 2.11. *[4] Let* $M \subset \mathcal{C}([a,b];X)$ *be bounded and equicontinuous. Then the set* $\overline{co}(M)$ *is also bounded and equicontinuous.*

To prove the existence of mild solutions to equation (5), we shall need the following results.

Lemma 2.12. *[26] If* $(u_n)_{n\geq 1}$ *is a sequence of Bochner integrable functions from* I *into a Banach space* Y *with the estimation* $\|u_n(t)\| \leq \mu(t)$ *for almost all* $t \in I$ *and every* $n \geq 1$, *where* $\mu \in L^1(I,\mathbb{R})$, *then the function*

$$\psi(t) \ = \ \beta(\{u_n(t) : n \geq 1\})$$

belongs to $L^1(I,\mathbb{R}^+)$ *and satisfies the following estimation*

$$\beta\left(\left\{\int_0^t u_n(s)ds \ : \ n \geq 1\right\}\right) \ \leq \ 2\int_0^t \psi(s)ds.$$

We now state the following nonlinear alternative of Mönch's type for selfmaps, which we shall use in the proof of existence of mild solutions to equation (5).

Theorem 2.13. *[19](Mönch, 1980) Let $\mathcal{K}$ be a closed and convex subset of a Banach space Z and $0 \in \mathcal{K}$. Assume that $F : \mathcal{K} \to \mathcal{K}$ is a continuous map satisfying Mönch's condition, namely,*

$$D \subseteq \mathcal{K} \text{ be countable and } D \subseteq \overline{co}(\{0\} \cup F(D)) \implies \overline{D} \text{ is compact}.$$

Then F has a fixed point.

3. Existence result

In this section, we prove the existence of solutions using Mönch's fixed point theorem. First, we show that for any $x_b \in X$, by choosing a proper control u^λ (for any given $\lambda \in (0,1]$), there is a mild solution $x^\lambda(\cdot, x_0, u) \in \mathcal{C}(I,X)$ of system (5). For that goal, we need to assume that:

$(\mathbf{H_3})$ Equation (6) has a resolvent operator $\big(R(t)\big)_{t\geq 0}$ that is continuous in the operator-norm topology .

$(\mathbf{H_4})$ The function $f : I \times \mathcal{B} \longrightarrow X$ satisfies the following two conditions:

(i) $f(\cdot,\varphi)$ is measurable for $\varphi \in \mathcal{B}$ and $f(t,\cdot)$ is continuous for a.e $t \in I$,

(ii) for every positive integer q, there exists a function $l_q \in L^1(I,\mathbb{R}^+)$ such that

$$\sup_{\|\varphi\|_{\mathcal{B}} \leq q} \|f(t,\varphi)\| \leq l_q(t) \text{ for a.e. } t \in I \text{ and } \liminf_{q \to +\infty} \int_0^b \frac{l_q(t)}{q} dt = l < +\infty,$$

(iii) there exists a function $h \in L^1(I,\mathbb{R}^+)$ such that for any bounded and equicontinuous set $D \subset \mathcal{B}$,

$$\beta(f(t,D)) \leq h(t) \sup_{-\infty < \theta \leq 0} \beta(D(\theta)) \text{ for a.e } t \in I,$$

where

$$D(\theta) = \{\phi(\theta) : \phi \in D\}.$$

$(\mathbf{H_5})$

$$\tau = 2R_b\|h\|_{L^1}\Big(1+\frac{2(M_2R_b)^3}{\lambda^2}\Big) < 1,$$

where $R_b = \sup_{0\le t\le b}\|R(t)\|$ and M_2 is such that $M_2 = \|C\|$.

For any $x_b \in X$, $x :]-\infty, b] \to X$ mild solution of equation (5) and $\lambda \in (0,1]$, we define the control $u^\lambda(t)$ as follows.

$$u^\lambda(t) = C^*R^*(b-t)W(\lambda,\Gamma_b^0)\Big\{x_b - R(b)(x_0) - \int_0^b R(b-s)f(s,x_s)\,ds\Big\}.$$

Using this control, we define the operator $P^\lambda : \mathcal{C}(I,X) \to \mathcal{C}(I,X)$ as follows.

$$P^\lambda x(t) = R(t)\varphi(0) + \int_0^t R(t-s)f(s,x_s)\,ds$$

$$+\int_0^t R(t-s)CC^*R^*(t-s)W(\lambda,\Gamma_b^0)\Big\{x_b - R(b)\varphi(0) - \int_0^b R(b-s)f(s,x_s)\,ds\Big\}.$$

One can see that the a fixed point of P^λ is a mild solution of equation (5). So to show the existence of mild solutions, it suffices to show that P^λ has a fixed point. We have the following theorem.

Theorem 3.1. *Suppose hypotheses* $(\mathbf{H_3})-(\mathbf{H_5})$ *are satisfied. Then, equation (5) has a solution, provided that*

$$\Big(1+\frac{b}{\lambda}(R_bM_2)^2\Big)R_bK_bl < 1 \tag{9}$$

.

Proof. Let $K_b = \sup_{t\in[0,b]}\|K(t)\|$. For any $x_b \in X$ and $\lambda \in (0,1]$, we define the control $u^\lambda(t)$ as usual by the following formula:

$$u_x^\lambda(t) = C^*R^*(b-t)W(\lambda,\Gamma_b^0)\Big\{x_b - R(b)\varphi(0) - \int_0^b R(b-s)f(s,x_s)\,ds\Big\}.$$

For each $x \in \mathcal{C}([0,b],X)$ such that $x(0) = \varphi(0)$, we define its extension $\widetilde{x}$ from $]-\infty,b]$ to X as follows

$$\widetilde{x}(t) = \begin{cases} x(t) & \text{if } t\in[0,b] \\ \varphi(t) & \text{if } t\in]-\infty,0] \end{cases}$$

We define the following space

$$E_b = \Big\{x :]-\infty,b] \to X \text{ such that } x|_I \in \mathcal{C}([0,b],X) \text{ and } x_0 \in \mathcal{B}\Big\}.$$

where $x|_I$ is the restriction of x to I.
We show using this control that the operator $P^\lambda : E_b \to E_b$ defined by

$$(P^\lambda x)(t) = R(t)\varphi(0) + \int_0^t R(t-s)\left[f(s,\widetilde{x}_s) + Cu_x^\lambda(s)\right] ds \quad \text{for} \quad t \in I = [0,b]$$

has a fixed-point. This fixed point is then a mild solution of equation (5). For each $\varphi \in \mathcal{B}$, we define the function $y \in \mathcal{C}([0,b],X)$ by $y(t) = R(t)\varphi(0)$ and its extension $\widetilde{y}$ on $]-\infty,0]$ by

$$\widetilde{y}(t) = \begin{cases} y(t) & \text{if } t \in [0,b] \\ \varphi(t) & \text{if } t \in]-\infty,0] \end{cases}$$

For each $z \in \mathcal{C}([0,b],X)$, set $\widetilde{x}(t) = \widetilde{z}(t) + \widetilde{y}(t)$, where $\widetilde{z}$ is the extension by zero of the function z on $]-\infty,0]$. Observe that x satifies (7) if and only if $z(0) = 0$ and

$$z(t) = \int_0^t R(t-s)\left[f(s,\widetilde{z}_s + \widetilde{y}_s) + Cu_z^\lambda(s)\right] ds \quad \text{for } t \in [0,b],$$

where
$u_z^\lambda(t)$
$= C^*R^*(b-t)W(\lambda,\Gamma_b^0)\left\{x_b - R(b)\varphi(0) - \int_0^b R(b-s)f(s,\widetilde{z}_s + \widetilde{y}_s)\, ds\right\}(t).$
Now let

$$E_b^0 = \left\{z \in E_b \text{ such that } z_0 = 0\right\}.$$

Thus E_b^0 is a Banach space provided with the supremum norm. Define the operator $\Gamma^\lambda : E_b^0 \to E_b^0$ by

$$(\Gamma^\lambda z)(t) = \int_0^t R(t-s)\left[f(s,\widetilde{z}_s + \widetilde{y}_s) + Cu_z^\lambda(s)\right] ds \quad \text{for} \quad t \in [0,b]$$

Note that the operator P^λ has a fixed point if and only if Γ^λ has one. So to prove that P^λ has a fixed point, we only need to prove that Γ^λ has one.
For each positive number q, let $B_q = \{z \in E_b^0 : \|z\| \leq q\}$. Then, for any $z \in B_q$, we have by axiom $(\mathbf{A_1})$ that

$$\begin{aligned} \|z_s + y_s\| &\leq \|z_s\|_{\mathcal{B}} + \|y_s\|_{\mathcal{B}} \\ &\leq K(s)\|z(s)\| + M(s)\|z_0\|_{\mathcal{B}} + K(s)\|y(s)\| + M(s)\|y_0\|_{\mathcal{B}} \\ &\leq K_b\|z(s)\| + K_b\|R(t)\|\|\varphi(0)\| + M_b\|\varphi\|_{\mathcal{B}} \\ &\leq K_b\|z(s)\| + K_bR_bH\|\varphi\|_{\mathcal{B}} + M_b\|\varphi\|_{\mathcal{B}} \\ &\leq K_b\|z(s)\| + \Big(K_bR_bH + M_b\Big)\|\varphi\|_{\mathcal{B}} \\ &\leq K_b q + \Big(K_bR_bH + M_b\Big)\|\varphi\|_{\mathcal{B}} \end{aligned}$$

Thus,

$$\|z_s + y_s\| \leq K_b\, q + \Big(K_b R_b H + M_b\Big)\|\varphi\|_{\mathcal{B}} =: q'.$$

We shall prove the theorem in the following steps.

Step1. We claim that there exists $q > 0$ such that $\Gamma^\lambda(B_q) \subset B_q$. We proceed by contradiction. Assume that it is not true. Then for each positive number q, there exists a function $z^q \in B_q$, such that $\Gamma^\lambda(z^q) \notin B_q$, *i.e.*, $\|(\Gamma^\lambda z^q)(t)\| > q$ for some $t \in [0,b]$. Now we have that

$$\begin{aligned}
q &< \Big\|(\Gamma^\lambda z^q)(t)\Big\| \\
&\leq R_b \int_0^b \Big\|f(s, \tilde{z}_s^q + \tilde{y}_s)\Big\|\, ds + R_b \int_0^b \|C u_{z^q}^\lambda(s)\|\, ds \\
&\leq R_b \int_0^b \Big\|f(s, \tilde{z}_s^q + \tilde{y}_s)\Big\|\, ds \\
&+ R_b \int_0^b \Big\|CC^* R^*(b-t) W(\lambda, \Gamma_b^0)\Big[x_b - R(b)\varphi(0) - \int_0^b R(b-s) f(s, \tilde{z}_s^q)\, ds\Big]\Big\|\, ds \\
&\leq \frac{b}{\lambda}(R_b M_2)^2 \left(\|x_b\| + R_b\|\varphi(0)\| + R_b \int_0^b \|f(s, \tilde{z}_s^q)\|\, ds\right) + R_b \int_0^b \Big\|f(s, \tilde{z}_s^q + \tilde{y}_s)\Big\|\, ds \\
&\leq \frac{b}{\lambda}(R_b M_2)^2 \left(\|x_b\| + R_b H\|\varphi\|_{\mathcal{B}} + R_b \int_0^b l_{q'}(s)\, ds\right) + R_b \int_0^b l_{q'}(s)\, ds,
\end{aligned}$$

where $q' := K_b\, q + q_0$, with $q_0 := \Big(K_b R_b H + M_b\Big)\|\varphi\|_{\mathcal{B}}$.
Hence

$$q \leq \Big(1 + \frac{b}{\lambda}(R_b M_2)^2\Big) R_b \int_0^b l_{q'}(s)\, ds + \frac{b}{\lambda}(R_b M_2)^2 \Big(\|x_b\| + R_b H\|\varphi\|_{\mathcal{B}}\Big).$$

Dividing both sides by q and noting that $q' = K_b q + q_0 \to +\infty$ as $q \to +\infty$, we obtain that

$$1 \leq \Big(1 + \frac{b}{\lambda}(R_b M_2)^2\Big) R_b \left(\frac{\int_0^b l_{q'}(s)\, ds}{q}\right) + \frac{\frac{b}{\lambda}(R_b M_2)^2 \Big(\|x_b\| + R_b H\|\varphi\|_{\mathcal{B}}\Big)}{q}$$

and

$$\liminf_{q\to+\infty} \left(\frac{\int_0^b l_{q'}(s)\, ds}{q}\right) = \liminf_{q\to+\infty} \left(\frac{\int_0^b l_{q'}(s)\, ds}{q'} \frac{q'}{q}\right) = l K_b.$$

Thus we have, $1 \leq \left(1+\frac{b}{\lambda}(R_bM_2)^2\right)R_bK_bl$, and this contradicts (9). Hence for some positive number q, we must have $\Gamma^{\lambda}(B_q) \subset B_q$.

Step2. $\Gamma^{\lambda} : E_b^0 \to E_b^0$ is continuous. In fact let $\Gamma^{\lambda} := \Gamma_1^{\lambda} + \Gamma_2^{\lambda}$, where

$$(\Gamma_1^{\lambda} z)(t) = \int_0^t R(t-s) f(s, \widetilde{z}_s + \widetilde{y}_s)\, ds \quad \text{and} \quad (\Gamma_2^{\lambda} z)(t) = \int_0^t R(t-s) C u_z^{\lambda}(s)\, ds.$$

Let $\{z^n\}_{n\geq 1} \subset E_b^0$ with $z^n \to z$ in E_b^0. Then there exists a number $q > 1$ such that $\|z^n(t)\| \leq q$ for all n and $a.e.$ $t \in I$. So $z^n, z \in B_q$. By $(\mathbf{H_4}) - (\mathbf{i})$, $f(t, \widetilde{z}_t^n + \widetilde{y}_t) \to f(t, \widetilde{z}_t + \widetilde{y}_t)$ for each $t \in [0,b]$. And by $(\mathbf{H_4}) - (\mathbf{ii})$,

$$\|f(t, \widetilde{z}_t^n + \widetilde{y}_t) - f(t, \widetilde{z}_t + \widetilde{y}_t)\| \leq 2l_{q'}(t).$$

Then we have

$$\|\Gamma_1^{\lambda} z^n - \Gamma_1^{\lambda} z\|_{\mathcal{C}} \leq R_b \int_0^b \|f(s, \widetilde{z}_s^n + \widetilde{y}_s) - f(s, \widetilde{z}_s + \widetilde{y}_s)\|\, ds \longrightarrow 0, \ as\ n \to +\infty$$

by dominated convergence Theorem. Also we have that

$$\|\Gamma_2^{\lambda} z^n - \Gamma_2^{\lambda} z\|_{\mathcal{C}} \leq R_b \frac{b}{\lambda} (R_b M_2)^2 \int_0^b \|f(s, \widetilde{z}_s^n) - f(s, \widetilde{z}_s)\|\, ds \longrightarrow 0, \ as\ n \to +\infty$$

by dominated convergence Theorem. Thus

$$\|\Gamma^{\lambda} z^n - \Gamma^{\lambda} z\| \leq \|\Gamma_1^{\lambda} z^n - \Gamma_1^{\lambda} z\| + \|\Gamma_2^{\lambda} z^n - \Gamma_2^{\lambda} z\| \longrightarrow 0, \ as\ n \to +\infty.$$

Hence Γ^{λ} is continuous on E_b^0.

Step3. $\Gamma^{\lambda}(B_q)$ is equicontinuous on $[0,b]$. In fact let $t_1, t_2 \in I$, $t_1 < t_2$ and $z \in B_q$, we have

$$\begin{aligned}
\|(\Gamma^{\lambda}z)(t_2)-(\Gamma^{\lambda}z)(t_1)\| &\leq \int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|\|f(s,\widetilde{z}_s+\widetilde{y}_s)+Cu_z^{\lambda}(s)\|\,ds\\
&+ \int_{t_1}^{t_2}\|R(t_2-s)\|\|f(s,\widetilde{z}_s+\widetilde{y}_s)+Cu_z^{\lambda}(s)\|\,ds\\
&\leq \int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|l_{q'}(s)\,ds\\
&+ \int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|\frac{1}{\lambda}R_b(M_2)^2\times\\
&\left(\|x_b\|+R_bH\|\varphi\|_{\mathcal{B}}+R_b\int_0^b l_{q'}(\tau)\,d\tau\right)ds\\
&+ \int_{t_1}^{t_2}\|R(t_2-s)\|l_{q'}(s)\,ds\\
&+ \int_{t_1}^{t_2}\|R(t_2-s)\|\frac{1}{\lambda}R_b(M_2)^2\times\\
&\left(\|x_b\|+R_bH\|\varphi\|_{\mathcal{B}}+R_b\int_0^b l_{q'}(\tau)\,d\tau\right)ds
\end{aligned}$$

By the continuity of $\big(R(t)\big)_{t\geq 0}$ in the operator-norm toplogy, the dominated convergence Theorem, we conclude that the right hand side of the above inequality tends to zero and independent of z as $t_2 \to t_1$. Hence $\Gamma^{\lambda}(B_q)$ is equicontinuous on I.

Step4. We show that the Mönch's condition holds.
Suppose that $D \subseteq B_q$ is countable and $D \subseteq \overline{co}(\{0\} \cup \Gamma(D))$. We shall show that $\beta(D) = 0$, where β is the Hausdorff MNC. Without loss of generality, we may assume that $D = \{z^n\}_{n\geq 1}$. Since Γ^{λ} maps B_q into an equicontinuous family, $\Gamma^{\lambda}(D)$ is also equicontinuous on I.

By $(\mathbf{H_3})-(\mathbf{ii})$, $(\mathbf{H_4})-(\mathbf{iii})$ and Lemma 2.12, we have that

$$
\begin{aligned}
&\beta\left(\{u^{\lambda}_{z^n}(t)\}_{n\geq 1}\right)\\
&= \beta\left(C^*R^*(b-t)W(\lambda,\Gamma^0_b)\left\{x_b - R(b)\varphi(0)\int_0^b R(b-t)f\left(s,\{\tilde{z}^n_s+\widetilde{y}_s\}_{n\geq 1}\right)ds\right\}_{n\geq 1}(t)\right)\\
&\leq \frac{1}{\lambda}R_bM_2\beta\left(\{x_b - R(b)\varphi(0)\}\right) + \frac{1}{\lambda}R_bM_2\beta\left(\left\{\int_0^b R(t-b)f\left(s,\{\tilde{z}^n_s+\widetilde{y}_s\}_{n\geq 1}\right)ds\right\}_{n\geq 1}\right)\\
&\leq \frac{2M_2(R_b)^2}{\lambda}\left(\int_0^b h(s)\beta\left(\{\tilde{z}^n_s\}_{n\geq 1}+\{\widetilde{y}_s\}\right)ds\right)\\
&\leq \frac{2M_2(R_b)^2}{\lambda}\left(\int_0^b h(s)\left[\beta\left(\{\tilde{z}^n_s\}_{n\geq 1}\right)+\beta\left(\{\widetilde{y}_s\}\right)\right]ds\right)\\
&\leq \frac{2M_2(R_b)^2}{\lambda}\left(\int_0^b h(s)\beta\left(\{\tilde{z}^n_s\}_{n\geq 1}\right)ds\right), \text{since } \left\{\widetilde{y}_s:\ s\in[0,b]\right\} \text{ is compact}\\
&\leq \frac{2M_2(R_b)^2}{\lambda}\left(\int_0^b h(s)\sup_{-\infty<\theta\leq 0}\beta\left(\{\tilde{z}^n_s(\theta)\}_{n\geq 1}\right)ds\right)\\
&\quad\left(\text{by Lemma 2.10, since } D=\{z^n\}_{n\geq 1} \text{ is equicontinuous}\right)\\
&\leq \frac{2M_2(R_b)^2}{\lambda}\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\beta\left(\{(\Gamma^{\lambda}z^n)(t)\}_{n\geq 1}\right) &\leq \beta\left(\left\{\int_0^t R(t-s)f(s,\{\tilde{z}^n_s+\widetilde{y}_s\}_{n\geq 1})\,ds\right\}_{n\geq 1}\right)\\
&+ \beta\left(\left\{\int_0^t R(t-s)Cu^{\lambda}_{z^n}(s)\,ds\right\}_{n\geq 1}\right)\\
&\leq 2R_b\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)\\
&+ \frac{4R_b(M_2R_b)^3}{\lambda^2}\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)\\
&\leq 2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)\\
&+ \frac{4R_b(M_2R_b)^3}{\lambda^2}\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right).
\end{aligned}
$$

It follows that

$$\begin{aligned}\beta\Big(\Gamma^\lambda(D)(t)\Big) &\leq 2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)+\frac{4R_b(M_2R_b)^3}{\lambda^2}\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)\\ &\leq \Big(1+\frac{2(M_2R_b)^3}{\lambda^2}\Big)2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)\\ &= \tau\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)\end{aligned}$$

That is $\beta\Big(\Gamma^\lambda(D(t))\Big)\leq\tau\beta\Big(D(t)\Big)$. But from Mönch's condition, we have

$$\beta\Big(D(t)\Big)\ \leq\ \beta\Big(\overline{co}\Big(\{0\}\cup\Gamma^\lambda(D(t))\Big)\Big)\ =\ \beta\Big(\Gamma^\lambda(D(t))\Big)\ \leq\ \tau\beta\Big(D(t)\Big).$$

This implies that $\beta\Big(D(t)\Big)=0$, since $\tau<1$, which implies that $\beta\Big(\Gamma^\lambda(D)(t)\Big)=0$. This shows that $\overline{\Gamma^\lambda(D)(t)}$ is compact, that is $\overline{\{\Gamma^\lambda(x)(t);x\in D\}}$ is compact as desired. So $K^\lambda(D)$ is equicontinuous and equibounded for all $0<\lambda\leq 1$ and therefore by Ascoli-Arzela's Theorem, we have that $\Gamma^\lambda(D)$ is relatively compact.
But

$$\beta\Big(D\Big)\ \leq\ \beta\Big(\overline{co}\Big(\{0\}\cup\Gamma^\lambda(D)\Big)\Big)\ =\ \beta\Big(\Gamma^\lambda(D)\Big).$$

Since D and $\Gamma^\lambda(D)$ are equicontinuous on $[0,b]$ and D is bounded, it follows by Lemma 2.10 that $\beta\Big(\Gamma^\lambda(D)\Big)\leq\tau\beta\Big(D\Big)$, where τ is as defined in ($\mathbf{H_5}$). Thus from the Mönch condition, we get that

$$\beta\Big(D\Big)\leq\beta\Big(\overline{co}(\{0\}\cup\Gamma^\lambda(D)\Big)=\beta\Big(\Gamma^\lambda(D)\Big)\leq\tau\beta\Big(D\Big),$$

and since $\tau<1$, this implies $\beta\Big(D\Big)=0$, which implies that D is relatively compact as desired in B_q and the Mönch condition is satisfied. We conclude by Theorem 2.13, that for each $0<\lambda\leq 1$, Γ^λ has a fixed point z in B_q. Then $x=z+y$ is a fixed point of P^λ in E_b which is a mild solution of equation (5). And the proof is complete.

■

4. Approximate Controllability Results

We are now in the position to prove the approximate controllability of equation (5). We show that under certain assumptions, the approximate controllability of (5) is implied by the approximate controllability of the corresponding linear system (8). We prove that $x^\lambda(b)\to x_b$ in X, which implies the approximate controllability of (5). We have the following result.

Theorem 4.1. *Assume that the hypotheses* $(\mathbf{H_0})$, $(\mathbf{H_3})$ *are satisfied and in addition, the function* f *is continuous and uniformly bounded. Then, equation (5) is approximately controllable on* $[0,b]$.

Proof: Let $\overline{x}_b \in X,\ \lambda \in (0,1)$ and $\overline{x}^\lambda \in B_q$ be the mild solution of equation (5) obtained in Theorem 3.1 under the control u^λ given above. So $\overline{x}^\lambda$ satisfies:

$$\overline{x}^\lambda(b) = R(b)\varphi(0) + \int_0^b R(b-s)f(s,\overline{x}_s^\lambda)\,ds$$

$$+\int_0^b R(b-s)CC^*R^*(b-t)W(\lambda,\Gamma_b^0)\times$$

$$\left\{\overline{x}_b - R(b)\varphi(0) - \int_0^b R(b-s)f(s,\overline{x}_s^\lambda)\,ds\right\}ds.$$

From the definition of Γ_b^0, it follows that

$$\begin{aligned}\overline{x}^\lambda(b) &= \overline{x}_b + \left(\Gamma_b^0 W(\lambda,\Gamma_b^0) - Id\right)\left[\overline{x}_b - R(b)\varphi(0) - \int_0^b R(b-s)f(s,\overline{x}_s^\lambda)\,ds\right]ds \\ &= \overline{x}_b + \lambda W(\lambda,\Gamma_b^0)\left[\overline{x}_b - R(b)\varphi(0) - \int_0^b R(b-s)f(s,\overline{x}_s^\lambda)\,ds\right]ds\end{aligned}$$

We obtain that

$$\left\|\overline{x}^\lambda(b) - \overline{x}_b\right\| = \left\|\lambda W(\lambda,\Gamma_b^0)\left[\overline{x}_b - R(b)\varphi(0) - \int_0^b R(b-s)f(s,\overline{x}_s^\lambda)\,ds\right]ds\right\|$$

By the uniform boundedness of f, we have that there exists $N^* > 0$ such that

$$\int_0^b \|f(s,\overline{x}_s^\lambda)\|^2 \le b(N^*)^2,$$

and consequently the sequence $\{f(s,\overline{x}_s^\lambda)\}_\lambda$ is bounded (uniformly in λ) in $L^2(I,X)$. Then there is a subsequence still denoted by $\{f(s,\overline{x}_s^\lambda)\}_\lambda$, that weakly converges to, say, $F(s)$ in $L^2(I,X)$. Then, we have that:

$$\left\|\int_0^T R(T-s)\left[f(s,\overline{x}^\lambda(s))\,ds - F(s)\right]\right\| \to 0 \text{ as } \lambda \to 0^+.$$

It follows from $(\mathbf{H_0})$ that

$$\begin{aligned}
\left\|\bar{x}^{\lambda}(b)-\bar{x}_{b}\right\| &= \left\|\lambda W(\lambda,\Gamma_{b}^{0})\left[\bar{x}_{b}-R(b)\varphi(0)\right]-\lambda W(\lambda,\Gamma_{b}^{0})\left[\int_{0}^{b}R(b-s)f(s,\bar{x}_{s}^{\lambda})\,ds\right]ds\right\| \\
&\leq \left\|\lambda W(\lambda,\Gamma_{b}^{0})\left[\bar{x}_{b}-R(b)\varphi(0)\right]\right\|+\left\|\lambda W(\lambda,\Gamma_{b}^{0})\left(\int_{0}^{b}R(b-s)F(s)\,ds\right)\right\| \\
&+ \left\|\lambda W(\lambda,\Gamma_{b}^{0})\left(\int_{0}^{b}R(b-s)\left[f(s,\bar{x}_{s}^{\lambda})-F(s)\right]ds\right)\right\| \\
&\leq \left\|\lambda W(\lambda,\Gamma_{b}^{0})\left[\bar{x}_{b}-R(b)\varphi(0)\right]\right\|+\left\|\lambda W(\lambda,\Gamma_{b}^{0})\left(\int_{0}^{b}R(b-s)F(s)\,ds\right)\right\| \\
&+ \left\|\int_{0}^{b}R(b-s)\left[f(s,\bar{x}_{s}^{\lambda})-F(s)\right]ds\right\| \longrightarrow 0 \text{ as } \lambda\to 0^{+}.
\end{aligned}$$

So, $\bar{x}^{\lambda}(b)\to\bar{x}_{b}$ holds in X and therefore, we obtain the approximate controllability of equation (5), and the proof is complete.

■

we now illustrate our main result by the following example.

5. Example

Let Ω be bounded domain in $\mathbb{R}^n$ with smooth boundary and consider the following nonlinear integrodifferential equation.

$$\begin{cases}
\frac{\partial v(t,\xi)}{\partial t}=\Delta v(t,\xi)+\int_{0}^{t}\zeta(t-s)\Delta v(s,\xi)\,ds+\int_{-\infty}^{0}\alpha(\theta)g(t,v(t+\theta,\xi))\,d\theta \\
\qquad +\eta\,\omega(t,\xi) \text{ for } t\in I=[0,1] \text{ and } \xi\in\Omega \\
v(t,\xi)=0 \text{ for } t\in[0,1] \text{ and } \xi\in\partial\Omega \\
v(\theta,\xi)=\phi(\theta,\xi) \text{ for } \theta\in]-\infty,0] \text{ and } \xi\in\Omega,
\end{cases} \tag{10}$$

where $\eta>0$, $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and Lipschitzian with respect to the second variable, the initial data function $\phi:\mathbb{R}^{-}\times\Omega\to\mathbb{R}$ is a given function, $\omega:[0,1]\times\Omega\to\mathbb{R}$ continuous in t, $\alpha:\mathbb{R}^{-}\to\mathbb{R}$ is continuous, $\alpha\in L^{1}(\mathbb{R}^{-},\mathbb{R})$ and $\zeta\in W^{1,1}(\mathbb{R}^{+},\mathbb{R}^{+})$.

Let $X = U = L^2(\Omega)$ and the phase space $\mathcal{B} = BUC(\mathbb{R}^-, X)$, the the space of uniformly bounded continuous functions endowed with the following norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta \le 0} \|\varphi(\theta)\|.$$

Then, the space $BUC(\mathbb{R}^-, X)$ satisfies axioms $(\mathbf{A_1})$, $(\mathbf{A_2})$ and $(\mathbf{A_3})$.

We define $A : \mathcal{D}(A) \subset X \to X$ by:

$$\begin{cases} \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \\ \\ Av = \Delta v \ \text{ for } v \in \mathcal{D}(A). \end{cases}$$

Theorem 5.1. *(***Theorem 4.1.2, p. 79 of [24]***) A is the infinitesimal generator of a C_0-semigroup on $L^2(\Omega)$.*

A generates a C_0-semigroup $\big(T(t)\big)_{t \ge 0}$ on $L^2(\Omega)$.
Moreover, $\big(T(t)\big)_{t \ge 0}$ generated by A above, is compact for $t > 0$ and therefore is operator-norm continuous for $t > 0$. Thus by Theorem 2.5, the corresponding resolvent operator is operator-norm continuous for $t > 0$.
Define

$$x(t)(\xi) \ = \ v(t,\xi), \quad x'(t)(\xi) \ = \ \frac{\partial v(t,\xi)}{\partial t}, \quad \omega(t,\xi) = u(t)(\xi).$$

$$\varphi(\theta)(\xi) = \phi(\theta,\xi) \ \text{ for } \theta \in]-\infty, 0] \text{ and } \xi \in \Omega.$$

$$f(t,\psi)(\xi) \ = \ \int_{-\infty}^{0} \alpha(\theta) g(t, \psi(\theta)(\xi))\, d\theta \ \text{ for } \theta \in]-\infty, 0] \text{ and } \xi \in \Omega.$$

$C : X \to X$ be defined by $\Big(Cu(t)\Big)(\xi) = Cu(t)(\xi) = \eta\,\omega(t,\xi)$.

$$(\gamma(t)x)(\xi) \ = \ \zeta(t)\Delta v(t,\xi) \ \text{ for } \ t \in [0,1], \ x \in \mathcal{D}(A) \text{ and } \ \xi \in \Omega.$$

We suppose that $\varphi \in BUC(\mathbb{R}^-, X)$. Then, equation (10) is then transformed into the following form

$$\begin{cases} x'(t) = Ax(t) \ + \ \displaystyle\int_0^t \gamma(t-s)x(s)ds \ + \ f(t, x_t) + Cu(t) \ \text{ for } t \in I = [0,1], \\ \\ x_0 = \varphi \in \mathcal{B}. \end{cases} \tag{11}$$

Suppose there exists a continuous function $p \in L^1(I; \mathbb{R}^+)$ such that

$$|g(t,y)| \le p(t)|y| \ \text{ for } t \in I \text{ and } y \in \mathbb{R}.$$

One can see that f is Lipschitz continuous with respect to the second variable and moreover for $\varphi \in \mathcal{B}$, we have we have

$$\sup_{\|\varphi\|_{\mathcal{B}} \leq q} \left\| f(t,\varphi) \right\| \leq q \|\alpha\| p(t).$$

So f satisfies $(\mathbf{H_4}) - (\mathbf{i})$ and $(\mathbf{H_4}) - (\mathbf{ii})$ with $l_q(t) = q\|\alpha\| p(t)$. Also f satisfies $(\mathbf{H_4}) - (\mathbf{iii})$ by condition $(\mathbf{viii})$ of Theorem 2.9, since f is Lipschitz. Moreover, f is uniformly bounded, and b, R_b, and λ can be chosen such that $(\mathbf{H_5})$ and condition (9) are satisfied.
To obtain the approximate controllability for equation (10), it suffices for us to verify that hypothesis $(\mathbf{H_0})$ is satisfied. We have the following result:

Lemma 5.2. *([12]) Let $\gamma(t) \in L^1(\mathbb{R}^+) \cap \mathcal{C}^1(\mathbb{R}^+)$ with primitive $B(t) \in L^1_{loc}(\mathbb{R}^+)$ such that $B(t)$ is non-positive, non-decreasing and $B(0) = -1$. If the operator A is self-adjoint and positive semi-definite, then the resolvent operator $R(t)$ associated to (6) is self-adjoint as well.*

By Lemma 5.2 above, the resolvent operator $R(t)$ of (10) is self-adjoint. So it follows that
$C^*R^*(t)\xi = R(t)\xi$, for any $\xi \in X$.
Let now $C^*R^*(t)\xi = 0$, for all $t \in [0,b]$. Then, $C^*R^*(t)\xi = R(t)\xi = 0$, $t \in [0,b]$. It follows from the fact that $R(0) = Id$ that $\xi = 0$, so from [6] (Theorem 4.1.7) that the linear control system corresponding to (10) is approximately controllable on $[0,b]$, and therefore $(\mathbf{H_0})$ holds. Hence by Theorem 3.1 and Theorem 4.1, equation (10) is approximately controllable on $[0,b]$ provided that condition $(\mathbf{H_3})$ is fulfilled.

Conclusion

This paper contains the approximate controllability of some partial functional integrodifferential differential equation with infinite delay in Hilbert spaces. We use the resolvent operator theory, the Hausdorff Measure of Noncompactness and the Mönch fixed point theorem techniques to prove the existence of mild solutions. The result shows that without assuming the compactness of the resolvent operator for the associated linear homogeneous part, the Mönch fixed point theorem can effectively be used to obtain approximate controllability results under some sufficient conditions such as the approximate controllability of the associated linear homogeneous part. Moreover, the example presented in section 5 illustrates an application of the obtained results. The results can be extended to partial functional integrodifferential equation with nonlocal and impulsive conditions whose linear parts admit a resolvent operator in the sense of

R. Grimmer. We shall investigate their approximate controllability in our future work.

REFERENCES

[1] Alka Chadha, Rathinasamy Sakthivel, Swaroop Nandan Bora, *Solvability of control problem for fractional nonlinear differential inclusions with nonlocal conditions*, Nonlinear Analysis: Modelling and Control, Vol. 24 (2019), No. 4, 503-522.

[2] R. Atmania and S. Mazouzi, *Controllability of semilinear integrodifferential equations with nonlocal conditions*, Electr. J. Diff. Equ. **2005** (75) (2005), 1-9.

[3] Avadhesh Kumar, M. Muslim, R. Sakthivel, *Controllability of the Second-Order Nonlinear Differential Equations with Non-instantaneous Impulses*, Journal of Dynamical and Control Systems 24 (2018), 325-342.

[4] J. Banas and K. Goebel, Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, vol. 60 (Marcel Dekker, New York, 1980).

[5] Y.K. Chang, J.J. Nieto and W.S. Li, *Controllability of semilinear differential systems with nonlocal initial conditions in Banach spaces*, J. Optim. Theory Appl. **142** (2009), 267-273.

[6] R. Curtain, H.J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.

[7] W. Desch, R. Grimmer and W. Schappacher, *Some Considerations for Linear Integrodifferential Equations*, Journal of Mathematical analysis and Applications 104, (1984), 219-234.

[8] W. Desch, R. Grimmer and W. Schappacher, *Well-posedness and wave propagation for a class of integrodifferential equations in Banach space*, J. Diff. Equ. **74** (2) (1988), 391-411.

[9] K. Ezzinbi, H. Toure and I. Zabsonre, *Existence and regularity of solutions for some partial functional integrodifferential equations in Banach spaces*, Nonlinear Analysis, 70, (2009) 2761-2771, .

[10] Ezzinbi, K., Degla, G., Ndambomve, P. *Controllability for some partial functional integrodifferential equations with nonlocal condition in Banach spaces*, Discussiones Mathematicae Differ. Incl. Control Optim. 35(1), (2015). 1-22.

[11] Fatima Zahra Mokkedem and Xianlong Fu, *Approximate controllability of a semilinear neutral evolution system with infinite delay*, Int. J. Robust Nonlinear Control (2016), DOI: 10.1002/rnc.3619.

[12] Fatima Zahra Mokkedem, Xianlong Fu, *Approximate controllability of semilinear neutral integro-differential systems with finite delay*, Applied Mathematics and Computation 242 (2014) 202-215.

[13] R. Grimmer, *Resolvent operators for integral equations in a Banach space*, Transaction of American Mathematical society **273** (1982), 333-349.

[14] Hale J and Kato J. Phase space for retarded equations with infinite delay, Funkcial. Ekvac. 21 (1978), 11-41.

[15] J. Liang, J. H. Liu and Ti-Jun Xiao, *Nonlocal Problems for Integrodifferential Equations*, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 15, (2008), 815-824.

[16] A. Lorenzi and E. Sinestrari, *An inverse problem in the theory of materials with memory*, Nonlinear Analysis: Theory, Methods & Applications, Vol. 12, No. 12, (1988), 1317-1335.

[17] N. I. Mahmudov, *Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces*, SIAM Journal on Control and Optimization, 42 (2003), 1604-1622.

[18] N.I. Mahmudov and S Zorlu, *Approximate controllability of fractional integro-differential equations involving nonlocal initial conditions*, Boundary Value Problems 2013, 2013:118.

[19] H. Mönch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Analysis, Theory, Methods and Applications **4** (5) (1980), 985-999.

[20] Nazim I. Mahmudov, *Approximate controllability of evolution systems with non-local conditions*, Nonlinear Analysis 68 (2008), 536-546.

[21] R. Sakthivel, Yong Ren, N.I. Mahmudov, *On the approximate controllability of semilinear fractional differential systems*, Computers and Mathematics with Applications 62 (2011), 1451-1459.

[22] R. Sakthivel, Juan J. Nieto and N. I. Mahmudov, *Approximate controllability of nonlinear deterministic and stochastic systems with unbounded delay*,Taiwanese Journal of Mathematics, Vol. 14, No. 5, (2010), 1777-1797.

[23] S. Selvi and M.M. Arjunan, *Controllability results for impulsive differential systems with finite delay*, J. Nonlinear Sci. Appl. **5** (2012), 206-219.

[24] I. I. Vrabie, *C_0-Semigroups and Applications*, Mathematics Studies, 191, (2003).

[25] Wang L. W., *Approximate controllability results of semilinear integrodifferential equations with infinite delays*. Sci China Ser F-Inf Sci, 2009, 52(7): 1095-1102, doi: 10.1007/s11432-009-0127-4.

[26] J. Wang, Z. Fan and Y. Zhou, *Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces*, J. Optim. Theory Appl. **154** (2012), 292-302.

[27] J. Wang and W. Wei, *Controllability of integrodifferential systems with nonlocal initial conditions in Banach spaces*, J. Math. Sci. **177** (3) (2011), 459-465.

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