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On Certain Type of Sequence Spaces Defined by φ −Function

E Herawati[1] and S N R Gultom[1]

[1]*Department of Mathematics, Universitas Sumatera Utara, Medan, 20155, Indonesia*

Abstract. In this paper, we introduce non-negative real valued φ −function on $\mathbb{R}$. Using φ −function, we define the sequence spaces $W(f), W_0(f),$ and $W_\infty(f)$. We will study some topological properties defined by certain paranorm of these spaces.

Keywords: Paranorm, Sequence Space

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1. Introduction

The space of all sequence take value on real numbers we denote by ω. Any non-empty linear subspace of ω is called a sequence space. The sequence spaces l_1, cs and bs, we use for the meaning of the spaces of all absolutely convergent series, convergent series, and bounded series, respectively.

A linear topological space X over the real field $\mathbb{R}$ is said to be a *paranormed space* if there is a function $g : X \longrightarrow \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$, and scalar multiplication is continuous, that is $|\lambda_n - \lambda| \longrightarrow 0$ and $g(x_n - x) \longrightarrow 0$ imply $g(\lambda_n x_n - \lambda x) \longrightarrow 0$ for every λ in $\mathbb{R}$ and x in X, where θ is the zero in the linear space X.

A paranorm g is called *total paranorm* if $g(x) = 0$ implies $x = 0$ and the pair $\mathbb{X} = (X, g)$ is called *total paranormed space*. Wilansky [1, p. 183] showed that by given some total paranorm, any set become a linear space or vector space.

Nakano [2] and Simons [3] introduced the notion of paranormed sequence space. Later on it was further investigated by some author, like Maddox [4,5], Lascarides [6], Rath and Tripathy [7], Tripathy and Sen [8] and many others ([9], [10], [11]).

A function $M: \mathbb{R} \longrightarrow [0, \infty)$ is said to be an *Orlicz function* if M is even, convex, continuous, $M(0) = 0$, and $M(u) \longrightarrow \infty$ as $u \longrightarrow \infty$.

W. Orlicz [12] used the idea of Orlicz function to construct the space L^M. Lindenstrauss-Tzafriri [13] construct the sequences space $\ell^\exists(M)$ make use of the Orlicz function M;

$$\ell^\exists(M) = \left\{x = (x_k) \in \omega : (\exists\rho > 0)\left(\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty\right)\right\}$$

*Corresponding author at: Universitas Sumatera Utara, Medan, 20155, Indonesia

E-mail address: elvina@usu.ac.id

Journal Homepage: https://talenta.usu.ac.id/index.php/jormtt

The set denoted by $\ell^{\exists}(M)$ is called an Orlicz sequence space. Lindenstrauss-Tzafriri proved that $\ell^{\exists}(M)$ is a Banach space respected with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

The Orlicz sequence space $\ell^{\exists}(M)$ with $M(x) = x^p$ is closely link to the space ℓ_p for $1 \leq p < \infty$,

$$\ell_p = \left\{x = (x_k) \in \omega : \sum_{k=1}^{\infty} |x_k|^p < \infty\right\}.$$

Various modifications of these definitions have been studied in the mathematical literature, like ℓ is changed to another solid sequence space. If any sequence (x_k) in a sequence space X and for all sequence (λ_k) of scalar with $|\lambda_k| \leq 1$ for all natural numbers k, implies $(\lambda_k x_k) \in X$, then the sequence space X is said to be *solid* (or normal) [14].

The $\Delta_2 - condition$ be valid for an Orlicz function M, if there exists positive real number K such that for every positive real number x implies $M(2x) \leq KM(x)$.

A continuous function $f : \mathbb{R} \longrightarrow [0, \infty)$ is called a φ *−function* if $f(t) = 0$ if and only if $t = 0$, even and non-decreasing on $[0, \infty)$. Using φ −function, we define the following sets

$$W(f) = \left\{x = (x_k) \in \omega \ : \ (\exists \rho > 0)\ (\exists l > 0)\ \ \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k - l|}{\rho}\right) \longrightarrow 0, n \longrightarrow \infty\right\}$$

$$W_0(f) = \left\{x = (x_k) \in \omega \ : \ (\exists \rho > 0)\ \ \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k|}{\rho}\right) \longrightarrow 0, n \longrightarrow \infty\right\}$$

$$W_\infty(f) = \left\{x = (x_k) \in \omega \ : \ (\exists \rho > 0)\ \ \sup_n \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k|}{\rho}\right) < \infty\right\}$$

In this work we will study some of topological properties of the set $W(f), W_0(f)$ and $W_\infty(f)$.

2. Main Results

In this section we prove some results involving the set $W(f), W_0(f)$ and $W_\infty(f)$.

Theorem 1. *The set* $W(f)$, $W_0(f)$ *and* $W_\infty(f)$ *are linear space, if* f *as* φ*-function fills the* Δ_2*-condition.*

Proof. Let $x = (x_k)$ and $y = (y_k)$ be sequences in $W(f)$ so there exists $\rho_1, \rho_2 > 0$ and $l_1, l_2 > 0$ with the result that

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k - l_1|}{\rho_1}\right) \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty \qquad (1)$$

and

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|y_k - l_2|}{\rho_2}\right) \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty \qquad (2)$$

Let $\rho = max\,\{\rho_1, \rho_2\}$ and assume that $l = l_1 + l_2$. since f is a non-decreasing function on $[0, \infty)$, then

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k + y_k - (l_1 + l_2)|}{\rho}\right) \le \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k - l_1|}{\rho}\right) + \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|y_k - l_2|}{\rho}\right)$$
$$\le \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k - l_1|}{\rho_1}\right) + \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|y_k - l_2|}{\rho_2}\right)$$

From (1) and (2), we get

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k + y_k - (l_1 + l_2)|}{\rho}\right) \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty.$$

Thus, the addition of $x + y$ closed in $W(f)$. (3)

Let $\alpha \in \mathbb{R}$ and sequence x in $W(f)$, then we can possess $\rho > 0$ and $l > 0$ so as

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k - l|}{\rho}\right) \longrightarrow 0, \text{as } n \longrightarrow \infty$$

Let $l = \alpha l_1$. For $\alpha = 0$, it can be easily verified that

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\alpha x_k - l|}{\rho}\right) \longrightarrow 0, as\; n \longrightarrow \infty$$

Then we assume that $\alpha \neq 0$. Since $0 < |\alpha|$, then by the Archimedian, there exists $n_0 \in \mathbb{N}$ so that $|\alpha| \le 2^{n_0}$, and because of f is a non-decreasing function on $[0, \infty)$ and satisfy the Δ_2-condition, then there exists $M > 0$ such that $f(|\alpha| x_k) \le f(2^{n_0} x_k) \le M^{n_0} f(x_k)$, for any natural numbers k. Thus,

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\alpha x_k - l|}{\rho}\right) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|\alpha||x_k - l_1|}{\rho}\right)$$
$$\le \frac{M^{n_0}}{n}\sum_{k=1}^{n} f\left(\frac{|x_k - l_1|}{\rho}\right) \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty \qquad (4)$$

From (3) and (4), we can take the conclusion that the set $W(f)$ is a linear space.

The proof of the rest cases, $W_0(f)$ and $W_\infty(f)$ will follow similarly. ■

Theorem 2. *A real function $g : W(f) \longrightarrow \mathbb{R}$ becomes a paranorm if we define $g(x)$ as*

$$g(x) = \inf\left\{\rho > 0 : \frac{1}{n}\sum_{k=1}^{\mathrm{n}} f\left(\frac{|x_k|}{\rho}\right) \le 1, n \in \mathbb{N}\right\}.$$

Proof. It is not hard to show that $g(x) \geq 0$ and $g(-x) = g(x)$, for every $x \in W(f)$. Let a sequence $x = (x_k), y = (y_k) \in W(f)$, then there is positive real numbers ρ_1, ρ_2 and $l_1, l_2 > 0$ with the result

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k - l_1|}{\rho_1}\right) \longrightarrow 0, \text{ as } n \longrightarrow \infty$$

and

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|y_k - l_2|}{\rho_2}\right) \longrightarrow 0, \text{ as } n \longrightarrow \infty$$

Since f is a non-decreasing function on $[0, \infty)$, we get

$$\begin{aligned} g(x+y) &= inf\left\{\rho > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k + y_k|}{\rho}\right) \leq 1\right\} \\ &\leq inf\left\{\rho_1 > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k|}{\rho_1}\right) \leq 1\right\} + inf\left\{\rho_2 > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|y_k|}{\rho_2}\right) \leq 1\right\} \\ &\leq g(x) + g(y) \end{aligned}$$

So the following inequality $g(x+y) \leq g(x) + g(y)$ holds, for every $x, y \in W(f)$. Furthermore, for any scalar sequence (λ_n) and $\left(x_k^{(n)}\right) \subset W(f)$, where

$$|\lambda_n - \lambda| \to 0 \text{ and } g\left(x_k^{(n)} - x_k\right) \to 0 \text{ for } x \in W(f) \text{ and } n \to \infty$$

we have

$$\begin{aligned} g\left(\lambda_n x_k^{(n)} - \lambda x_k\right) &= inf\left\{\rho > 0: \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{\left|\lambda_n x_k^{(n)} - \lambda x_k\right|}{\rho}\right) \leq 1\right\} \\ &\leq inf\left\{\rho > 0: \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{\left|\lambda_n x_k^{(n)} - \lambda x_k^{(n)}\right|}{\rho}\right) \leq 1\right\} \\ &\quad + inf\left\{\rho > 0: \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{\left|\lambda x_k^{(n)} - \lambda x_k\right|}{\rho}\right) \leq 1\right\} \\ &= inf\left\{\rho = \left(\frac{|\lambda_n - \lambda|}{|\lambda_n - \lambda|}\rho\right) > 0: \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{\left|x_k^{(n)}\right|}{\rho/|\lambda_n - \lambda|}\right) \leq 1\right\} \\ &\quad + inf\left\{\rho = \left(\frac{|\lambda|}{|\lambda|}\rho\right) > 0: \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{\left|x_k^{(n)} - x_k\right|}{\rho/|\lambda|}\right) \leq 1\right\} \\ &= |\lambda_n - \lambda| inf\left\{\rho^* = \left(\frac{\rho}{|\lambda_n - \lambda|}\right) > 0: \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{\left|x_k^{(n)}\right|}{\rho^*}\right) \leq 1\right\} \\ &\quad + |\lambda| inf\left\{\rho^{**} = \left(\frac{\rho}{|\lambda|}\right) > 0: \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{\left|x_k^{(n)} - x_k\right|}{\rho^{**}}\right) \leq 1\right\} \end{aligned}$$

$$= |\lambda_n - \lambda| g\left(x_k^{(n)}\right) + |\lambda| g(x_k^{(n)} - x_k)$$

Since $|\lambda_n - \lambda| \longrightarrow 0$ and $g\left(x_k^{(n)} - x_k\right) \longrightarrow 0$, it follows that $g\left(\lambda_n x_k^{(n)} - \lambda x_k\right) \longrightarrow 0$. This is a complete proof of the theorem. ■

Theorem 3. *The linear space $W(f)$ is a complete paranormed sequence space, whenever f as φ-function satisfies the convex property and Δ_2-condition.*

Proof. Let an Cauchy real sequence $\left(x^{(n)}\right)$ in $W(f)$ with

$$(x^{(n)}) = \left(x_k^{(n)}\right) = (x_1^{(n)}, x_2^{(n)}, \dots)$$

It's mean for any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ so that for every $m \geq n \geq n_0$, we get

$$g(x^{(m)} - x^{(n)}) < \varepsilon$$

Thus,

$$\frac{1}{r}\sum_{k=1}^{\mathrm{r}} f\left(\frac{\left|x_k^{(m)} - x_k^{(n)}\right|}{\varepsilon}\right) \leq 1$$

Since f is convex, then

$$\frac{1}{r}\sum_{k=1}^{\mathrm{r}} f\left(\left|x_k^{(m)} - x_k^{(n)}\right|\right) \leq \varepsilon \frac{1}{r}\sum_{k=1}^{\mathrm{r}} f\left(\frac{\left|x_k^{(m)} - x_k^{(n)}\right|}{\varepsilon}\right) \leq \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, then $f\left(\left|x_k^{(m)} - x_k^{(n)}\right|\right) = 0$ for every $\mathrm{m} \geq \mathrm{n} \geq n_0$. This implies that $\left|x_k^{(m)} - x_k^{(n)}\right| < \varepsilon$ for every $\mathrm{m} \geq \mathrm{n} \geq n_0$. It follows that $\left(x_k^{(n)}\right)$ becomes a Cauchy sequence on $\mathbb{R}$ for every $k \in \mathbb{N}$. Since $\mathbb{R} = (\mathbb{R}, |.|)$ is a complete normed space, then there exists $x_k \in \mathbb{R}$ for every $k \in \mathbb{N}$ with $\lim_{n\to\infty} x_k^{(n)} = x_k$. Thus for every $\mathrm{n} \geq n_0$, we get

$$\left|x_k^{(m)} - x_k\right| = \left|x_k^{(m)} - \lim_{n\to\infty} x_k^{(n)}\right| = \lim_{n\to\infty}\left|x_k^{(m)} - x_k^{(n)}\right| < \varepsilon^2$$

Let $x = (x_k) \in \omega$. Since $\left(x^{(n)}\right) \subset W(f)$, then there exists $l > 0$ and $\rho > 0$ implies

$$\frac{1}{r}\sum_{k=1}^{\mathrm{r}} f\left(\frac{\left|x_k^{(n)} - l\right|}{\rho}\right) \to 0, \quad r \longrightarrow \infty$$

Using the continuity of f

$$\frac{1}{r}\sum_{k=1}^{\mathrm{r}} f\left(\frac{|x_k - l|}{\rho}\right) = \frac{1}{r}\sum_{k=1}^{\mathrm{r}} f\left(\frac{\left|\lim_{n\to\infty} x_k^{(n)} - l\right|}{\rho}\right)$$

$$= \lim_{n\to\infty} \frac{1}{r}\sum_{k=1}^{\mathrm{r}} f\left(\frac{\left|x_k^{(n)} - l\right|}{\rho}\right) = 0, \qquad r \longrightarrow \infty$$

This implies that

$$\frac{1}{r}\sum_{k=1}^{r} f\left(\frac{|x_k - l|}{\rho}\right) \longrightarrow 0, \qquad \text{as } r \longrightarrow \infty$$

As a result, the sequence x in $W(f)$. Furthermore, we will show that $g(x^{(n)} - x) \longrightarrow 0$ as $n \longrightarrow \infty$. Because of the continuous property of φ-function, then

$$\frac{1}{r}\sum_{k=1}^{r} f\left(\frac{\left|x_k^{(n)} - x_k\right|}{\rho}\right) = \frac{1}{r}\sum_{k=1}^{r} f\left(\frac{\left|x_k^{(n)} - \lim_{m\to\infty} x_k^{(m)}\right|}{\rho}\right)$$

$$= \frac{1}{r}\sum_{k=1}^{r} f\left(\frac{\left|x_k^{(n)} - x_k^{(m)}\right|}{\rho}\right) \leq 1$$

Thus,

$$g\left(x^{(n)} - x\right) = inf\left\{\rho > 0 : \frac{1}{r}\sum_{k=1}^{r} f\left(\frac{\left|x_k^{(n)} - x_k\right|}{\rho}\right) \leq 1\right\}$$

This implies that $g\left(x^{(n)} - x\right) < \rho$ for every $\rho > 0$. It follows that there exists a real sequence $\left(\frac{c}{2^m}\right), m \geq 1$, for a real number c together with

$$g\left(x^{(n)} - x\right) < \frac{c}{2^m}, m \geq 1$$

Thus we get $g\left(x^{(n)} - x\right) \longrightarrow 0$ as $n \longrightarrow \infty$. We can deduce that the linear space $W(f)$ satisfies complete property with paranorm. ■

Furthermore, in the similar way, we can conclude that the spaces $W_0(f)$ and $W_\infty(f)$ are complete paranormed spaces equipped with the same paranorm, i.e.,

$$g(x) = \inf\left\{\rho > 0 : \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{|x_k|}{\rho}\right) \leq 1, n \in \mathbb{N}\right\}$$

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