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Constructing Dynamic Life Tables with a Single Factor Model

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ABSTRACT

The paper deals with the mortality risk evolution and presents a one factor model explaining the dynamics of all of the mortality rates. The selected factor will be the mortality rate at the key age, and an empirical study involving males and females in France and Spain will reveal that the present approach is not outperformed by more complex factor models. The key age seems to reflect several advantages with respect to other factors available in the literature. Actually, it is totally observable, and the methodology may be easily extended so as to incorporate more factors (more key ages), a cohort effect, specific mortality causes or specific ages. Furthermore, the choice of a key age as an explanatory factor is inspired by former studies about the interest rates dynamics, which allows us to draw on the model in order to address some longevity risk linked problems. Indeed, one only has to slightly modify some interest rate linked methodologies.

Keywords: Dynamic life tables; key mortality rate; Forecasting.

RESUMEN

En este trabajo se desarrollará un modelo unifactorial para explicar la dinámica de las tasas de mortalidad y abordar el riesgo relacionado con ésta. El factor seleccionado para explicar el comportamiento de la curva de mortalidad será la tasa de mortalidad correspondiente a una edad clave, y, mediante un análisis empírico de las poblaciones masculina y femenina de Francia y España, se pondrá de manifiesto que este enfoque produce resultados, al menos tan buenos, como los logrados por otros modelos bastante más complejos. Este planteamiento, basado en la edad clave, presenta varias ventajas frente a otras alternativas de la literatura. En efecto, el factor de riesgo (la edad clave) es totalmente observable, y la metodología puede extenderse fácilmente mediante la incorporación de factores adicionales (más edades clave), el estudio del efecto cohorte, el análisis de causas específicas de mortalidad o la consideración de sólo algunos tramos específicos dentro de la curva de mortalidad. Por otro lado, debe señalarse que este modelo, basado en edades clave, se ha inspirado en estudios previos sobre la dinámica de la curva de tipos de interés, por lo que gran parte de las metodologías desarrolladas para tipos de interés serían fácilmente adaptables al estudio de problemas relacionados con el riesgo de longevidad.

Palabras clave: dinámica de la tabla de mortalidad, tasa de mortalidad clave, GLM, Predicción

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ÍNDICE

1. INTRODUCCIÓN

Longevity risk is becoming more and more important in insurance industry. The evolution of the mortality table may provoke significant capital losses in the portfolio of long-term contracts, and consequently, the study of this evolution has became a major issue in life insurance.

Actuarial literature has focused on the dynamics of the mortality table by means of several complementary approaches. On the one hand, the seminal paper by Lee and Carter (1992) proposed to deal with a risk-factor model. They introduced their famous one factor model and, since then, many authors have extended the discussion by dealing with more factors (Booth et al. 2002; Brouhns et al. 2002; Cairns et al. (2006); Cairns et al. 2009) or cohort effects (Holford 1983; Renshaw and Haberman 2006; and Haberman and Renshaw 2009). On the other hand, the stochastic mortality modeling (Biffis 2005; Di Lorenzo 2006; Schrager 2006; Plat 2009), has become a second line of research providing us with suitable models which can be calibrated to market prices (Russo et al. 2011).

This paper attempts to capture the strengths of both approaches. Indeed, a one factor model in the line of Lee and Carter (1992) will be presented, but the significant difference with respect to former analyses is the chosen factor, which will equal the mortality rate at the "key age"'. Actually, every age might be selected as an explanatory factor, but the key age will equal that minimizing the in-the-sample error variance.

This methodology seems to reveal several advantages with respect to similar approaches. Indeed, the key age is totally observable, the methodology may be easily extended so as to incorporate more factors (more key ages), a cohort effect, specific mortality causes or specific ages. According to our empirical findings with respect to the French and the Spanish populations, our model is never outperformed by former ones. In other words, we have an easy way to estimate and extend this simple model involving a unique observable factor and reflecting a good enough empirical performance.

The use of a unique key age to explain the whole mortality table evolution is inspired in former studies about the Term Structure of Interest Rates or TSIR (Elton et al. 1990; Navarro and Nave; 2001). By dealing with a few maturities and spot rates, many authors were able to give tractable methods and algorithms in order to price interest rate derivatives and hedge the interest rate risk. In this sense, our analysis may complement the possibilities of a stochastic mortality modeling approach. In fact, by studying the mortality random behavior at the key age, one can address many topics in longevity risk management. For instance, one can estimate risk measures such as a longevity-V@R or a longevity-CV@R, one can diversify the

longevity risk by investing in sectors uncorrelated with the key age,[1] [2] one can create stochastic models explaining the mortality at the key age dynamics, or one can design, price and hedge mortality derivatives. There is a lot of experience about similar problems related to the TSIR.

The paper outline is as follows. Section 2 will be devoted to presenting the model and its practical implementation, Section Section 3 will summarize several factor models of previous literature, Section 4 will deal with model estimation issues for males and females in both France and Spain, Section 5 and Section 6 will compare our empirical results in both countries with those generated by other factor models, Section 7 will present some ideas related to risk management problems such as a longevity-V@R estimation, and Section 8 will conclude the paper.

2. SINGLE FACTOR MORTALITY MODEL

2.1 Model description

Following Elton et al. (1990) (EGM), in this model we will assume that changes in mortality rates are linearly related to a small number of factors. In this initial single factor version,[3] we will assume that the whole life table can be explained by one factor, namely, the mortality rate of a particular age. We will refer to this mortality rate as the "key" mortality rate. Specifically, we suppose that:

$$\Delta \ln\left(\hat{q}_{x,t}\right) = \alpha_{x,x^*} + b_{x,x^*}\left[\Delta \ln\left(\hat{q}_{x^*,t}\right)\right] + \varepsilon_{(x,x^*),t}. \qquad (1)$$

where:

- $\Delta ln(\hat{q}_{x,t})$ is the variation in the logarithm of the crude[4] mortality rate at age *x* from year *t-1* to year *t*.
- $\Delta ln(\hat{q}_{x*,t})$ is the change in the logarithm of the mortality rate at the key age *x** from *t-1* to *t*. This key age will be chosen to maximize the explanatory power of the model.
- $\alpha_{x,x*}$ is a constant term that captures the general tendency of a reduction (increment) in mortality rates and is assumed to be independent of the behavior of the key mortality rate $\hat{q}_{x,t}$. The value of this term may differ from one age to another indicating a differential behavior in the reduction of mortality rates over time.

[1] In the line of the Capital Asset Pricing Model, for instance.
[2] V@R and CV@R are very important risk measures in the insurance industry. Both regulatory (Solvency II) and internal (corporate) risk management constrains often impose the use of V@R and CV@R. See for instances Hardy (2003).
[3] The model can be easily generalized for two or more factors, in order to yield multifactor life table models.
[4] The model could be implemented using graduated mortality rates. Eventually, we decided to use crude mortality rates to avoid data manipulation.

- $b_{x,x*}$ is a parameter that describes the sensitivity of the logarithm of the mortality rate at age *x* to changes in the logarithm of the key morality rate and captures changes in the shape of the mortality curve over time.
- $\varepsilon_{(x,x*),t}$ is a random error term with zero mean and constant variance $\sigma^2_{\varepsilon,(x,x*)}$.

Although it might be expected that the variance of $\varepsilon_{(x,x*),t}$ depends on both the level of mortality and the size of population it should be pointed out that in equation (1) once the key age x* has been chosen, for each x, $\Delta ln(\hat{q}_{x,t})$ is regressed against $\Delta ln(\hat{q}_{x*,t})$ with t=(1975, …, 2006) and therefore both x and x* are fixed. Thus, to test the presence of heteroskedasticity, we apply a battery of test: Breusch-Pagan, White, Goldfelt-Quandt and Harrison-McCabe tests (Breusch and Pagan, 1979; White et al., 1980; Goldfelt and Quandt, 1965; and Harrison and McCabe, 1979). As an additional check we regressed the squared residuals of equation (1) against $\hat{q}_{x,t}$, $(\hat{q}_{x,t})^2$, $E_{x,t}$ and $(E_{x,t})^2$ ($E_{x,t}$ being the exposed to risk at age x during year t). None of these tests allowed us to reject the null hypothesis of homoscedasticity at 99%.

To obtain the key age x*, we proceed as follows. If we take into account that for a given age x*, the coefficient of determination between $\Delta ln(\hat{q}_{x,t})$ and $\Delta ln(\hat{q}_{x,t})$, $R^2_{x,x*}$ is given by:

$$R^2_{x,x^*} = 1 - \frac{\mathrm{var}\left(\varepsilon_{(x,x^*),t}\right)}{\mathrm{var}\left(\Delta \ln\left(\hat{q}_{x,t}\right)\right)}, \qquad (2)$$

then:

$$R^2_{x,x^*} \cdot \mathrm{var}\left(\Delta \ln\left(\hat{q}_{x,t}\right)\right) = \mathrm{var}\left(\Delta \ln\left(\hat{q}_{x,t}\right)\right) - \mathrm{var}\left(\varepsilon_{(x,x^*),t}\right). \qquad (3)$$

According to expression (3) minimizing the variance of the residual term is equivalent to maximizing the left-hand side of equation (3).
A measure of the explanatory power of $\hat{q}_{x*,t}$, with respect to the whole life table can then be obtained by adding up expression (3) across all ages x, that is:

$$\psi\left(x^*\right) = \sum_x R^2_{x,x^*} \cdot \mathrm{var}\left(\Delta \ln\left(\hat{q}_{x,t}\right)\right). \qquad (4)$$

Elton et al. (1990) suggest to use a weighting scheme when summing across ages, in order to deal with a subjective criterion allowing us to choose the key age in such a way that the explanatory power of the model focuses on certain ages or some range.

Thus, the key age will be obtained as the age that maximizes the objective function $\gamma(x^*)$, that is:[5]

[5] See also Navarro and Nave (2001).

$$\max_{x^*} \varphi(x^*) = \max_{x^*} \sum_x w_x \cdot R^2_{x,x^*} \cdot \mathrm{var}\left(\Delta \ln\left(\hat{q}_{x,t}\right)\right). \qquad (5)$$

For instance, for a life insurance company it may be of special interest to have a model with a good performance in a particular range of ages as retired people or people over 60 years old. Another possibility would be to fix w_x according to the size of the reserves of the company. The resulting model could be particularly accurate in the tranche of the mortality curve where the company concentrates most of its business. In this article, we decided to assign the values $w_i = 1/82$ for i =18, 19, …, 99 and $w_i = 0$ for i= 0 to 17 as the population of interest for insurance companies is mostly over 18 years old. Also, in this study we will always distinguish between male and female populations when obtaining the key age.[6]

2.2 Adjusting a sensitivity function to b_(x,x^*)

Once the key age is determined, we can use linear regression techniques to obtain estimates of the parameters $\alpha_{x,x*}$ and $b_{x,x*}$. The second step in the modeling process consists of finding a function to approximate the sensitivity of changes in the logarithm of mortality rates to changes in the logarithm of the resulting key mortality rate,[7] that is, a function that describes the values of $\hat{b}_{x,x*}$. This function must satisfy two constraints. Firstly, the function must be "smooth enough", since sensitivities of mortality rates must be quite similar for similar ages. Secondly, $b_{x,x*} = 1$ must hold.

We propose two approaches to find this function.

2.2.1. Parametric Approach

Under the first approach, we will use a very simple parametric function, denoted by $b^*(x)$ and inspired by a function suggested by Díaz et al. (2006). Thus:

$$\hat{b}_{x,x^*} = b^*(x) + u_x = \beta_1 \cdot \exp\left[-\beta_2 \left(x - x^*\right)^2\right] + \left(1 - \beta_1\right) + u_x. \qquad (6)$$

This function is symmetric with respect to x*, the key age. This implies that the sensitivity of changes in the mortality rates to changes in the key mortality rate depends on the distance between x and x*.

The parameters β_1 and β_2 capture the slope and width, respectively, of the hump of $\hat{b}_{x,x*}$ values around the key age (see in Figure 2).

[6] An alternative approach to determine the key age x* would be based on maximizing the likelihood function or its logarithm. We eventually decided to follow the original paper of Elton et al. (1990).

[7] We have not adjusted any function to describe the behavior of $\hat{\alpha}_{x,x*}$, although this could be easily done.

Notice that $b^*(x^*) = 1$ trivially follows from (6). Besides, u_x is a random error with zero mean. To prevent the presence of heterokedasticity in (6) we assume that σ_x^2, the variance of the error term, has the following structure: $\sigma_x^2 = k_0 + \frac{k_1}{x}$. Accordingly, parameters β_1 and β_2 were estimated by applying GLS distinguishing again between males and females. Under this variance structure, the hyphotesis of constant variance for $u_x^* = \frac{u_x}{\sigma_x}$ cannot be rejected.

2.2.2. Splines

The second approach to describe $\hat{b}_{x,x*}$ consists of using splines.[8] A function $S(x)$ is said to be a spline of degree *k* on $[a,b]$ if:

- $S(x) \in C^{k-1} \quad on \quad [a,b]$
- $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ and
- $S(x) = \begin{cases} S_0(x) & t_0 < x < t_1 \\ S_1(x) & t_1 < x < t_2 \\ & \dots \\ S_{n-1}(x) & t_{n-1} < x < t_n \end{cases}$

where $S(x) \in P^k$. If *k=3*, $S(x)$ is called *cubic spline*. If *k=3* and $S''(t_0) = S''(t_n) = 0$, holds then $S(x)$ is called *natural cubic spline*.

When using spline functions, two important problems must be addressed. First, the number of knots $(t_0, t_1, \dots, t_n)$ must be defined. The higher is the number of knots the greater is the accuracy of the adjustment. However, as we increase the number of knots the smoothness of the function worsens. Thus, there is a tradeoff between accuracy and smoothness. To solve this drawback, we employ the criterion proposed by McCulloch (1971) and Shea (1984), whereby the number of knots is set equal to the square root of the number of observations less two, that is, $n + 1 = \sqrt{N} - 2$, where *N* is the number of mortality rates analyzed in the study. The second problem is to choose the position of each knot. In this paper, we will select those knots that minimize the sum of the squared errors, testing all integer numbers between the minimum and maximum ages considered. In any case, the final spline selected must satisfy $S(x^*) = 1$.

The specific set of splines eventually selected to describe the sensitivities $b_{x,x*}$ will be characterized in Section 4.3, where we describe the calibration of the model.

2.3 Forecasting mortality rates

The final step in the process of constructing the dynamic life tables consists of developing a methodology to forecast future mortality rates.

[8] See, for instance, Keele (2008) for further details about these functions and their properties.

Let us recall that, according to equation (1), if $b_{x,x*}$ is replaced by $b^*(x)$ and one rearranges terms, then:

$$\ln\left(\hat{q}_{x,t}\right) = \ln\left(\hat{q}_{x,t-1}\right) + \alpha_{x,x^*} + b^*\left(x\right)\cdot z_t^* + \eta_{x,t}, \tag{7}$$

where:

- $z_t^* = ln(\hat{q}_{x*,t}) - ln(\hat{q}_{x*,t-1})$ represents the change in the logarithm of the mortality rate corresponding to the key age *x** from *t-1* to *t* or, alternatively, the relative change in the key mortality rate.
- $\eta_{x,t}$ is an error term with mean zero and variance σ_η^2.

To forecast future mortality rates we assume that the variable $ln(\hat{q}_{x*,t})$ follows a time series of the ARIMA family, which will allow us to forecast the future behavior of $ln(\hat{q}_{x*,t})$ and then, using equation (7), all the other mortality rates.

3. Alternative Dynamic Mortality models

There are many popular models used to forecast future mortality rates and construct dynamic life tables. In this paper, we describe and calibrate some of these models and then they will be used as benchmarks to be compared with the Single Factor Model (SFM).

Table 1

Description of main alternative models.

Model	Formula
Lee-Carter (LC)	$ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \alpha_x + \beta_x^{(1)}\cdot k_t^{(1)}$
Lee-Carter 2 (LC2)	$ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \alpha_x + \beta_x^{(1)}\cdot k_t^{(1)} + \beta_x^{(2)}\cdot k_t^{(2)}$
Age-Period-Cohort (APC)	$ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \alpha_x + k_t^{(1)} + \gamma_{t-x}$
Cairns-Blake-Dowd (M5)	$ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = k_t^{(1)} + (x-\bar{x})\cdot k_t^{(2)}$
Cairns-Blake-Dowd (M6)	$ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = k_t^{(1)} + (x-\bar{x})\cdot k_t^{(2)} + \gamma_{t-x}$
Cairns-Blake-Dowd (M7)	$ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = k_t^{(1)} + (x-\bar{x})\cdot k_t^{(2)} + k_t^{(3)}\cdot[(x-\bar{x})^2 - \hat{\sigma}_x^2] + \gamma_{t-x}$
Cairns-Blake-Dowd (M8)	$ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = k_t^{(1)} + (x-\bar{x})\cdot k_t^{(2)} + \gamma_{t-x}\cdot(x_c - x)$
Renshaw-Haberman (RH)	$ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \alpha_x + \beta_x^{(1)}\cdot k_t^{(1)} + \gamma_{t-x}$
Plat	$ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \alpha_x + k_t^{(1)} + (\bar{x}-x)\cdot k_t^{(2)} + (\bar{x}-x)^+\cdot k_t^{(3)} + \gamma_{t-x}$

These alternative models are described in Table 1, where:

- α_x; parameter that captures the average shape of the mortality curve.
- $k_t^{(i)}$; parameter that describes the general tendency of the whole life table.
- $\beta_x^{(i)}$; parameter that explains the age effect *x* with respect to the general trend in mortality rates.
- γ_{t-x}; parameter that accounts for the cohort effect.
- $\bar{x}$; is the average age over the range of ages *(0-99)*.
- $\hat{\sigma}_x^2$; is the average value of $(x-\bar{x})^2$.
- $(\bar{x}-x)^+ = max(0,\bar{x}-x)$.
- x_c; parameter to be estimated.

3.1 Lee-Carter model (LC)

One of the most popular models to estimate future mortality rates was developed by Lee and Carter (1992). This model suggests the adjustment of the central mortality rate $m_{x,t}$ by an exponential function that depends on age *x* and time *t*. Although different versions and proposals have been developed to implement the model in this paper, we decided to adjust the Lee-Carter model by using one of its most recent versions, Debón et al. (2008), where $q_{x,t}$ is adjusted by using a logit link and assuming that the number of deaths follows a binomial distribution, that is:

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \alpha_x + \beta_x^{(1)} \cdot k_t^{(1)}. \qquad (8)$$

3.2 Bi-Facor Lee-Carter model (LC2)

The Lee-Carter model was developed by Lee (2000) and since its publication many authors have tried to improve the adjustment and to eliminate the tendency in the residuals. Booth et al. (2002) and Renshaw and Haberman (2003) decided to add a second risk factor to (8). This variation of the model can be applied too with a logit version to the mortality rates:[9]

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \alpha_x + \beta_x^{(1)} \cdot k_t^{(1)} + \beta_x^{(2)} \cdot k_t^{(2)}. \qquad (9)$$

3.3 Age-Period-Cohort model (APC)

Another extension of the model consists of including a cohort effect, an improvement that provided very good results in fields like medicine or demography (Clayton and Schifflers, 1987; Holford, 1983; Hobcraft et al., 1985). In actuarial literature, Currie

[9] Following Debón et al. (2010) to estimate the parameters.

et al. (2006) and Renshaw and Haberman (2006) were the first authors to introduce this cohort effect. Particularly, the model of Renshaw and Haberman is commonly called Age-Period-Cohort (APC), where:

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right)=\alpha_x+\beta_x^{(1)}+\gamma_{t-x}. \qquad (10)$$

3.4 Cairns-Blake-Dowd model (M5)

Cairns et al. (2006) developed a structure with two terms to model the mortality rates:

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right)=\beta_x^{(1)}\cdot k_t^{(1)}+\beta_x^{(2)}\cdot k_t^{(2)}. \qquad (11)$$

A simplified version of the model assumes that $\beta_x^{(1)}=1$ and $\beta_x^{(2)}=(x-\bar{x})$, and so the model becomes:

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right)=k_t^{(1)}+\left(x-\bar{x}\right)\cdot k_t^{(2)}. \qquad (12)$$

3.5 Cairns-Blake-Dowd model (M6)

In Cairns et al. (2009) the former model was extended to include the cohort effect γ_{t-x} to (11), redefining the model as:

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right)=\beta_x^{(1)}\cdot k_t^{(1)}+\beta_x^{(2)}\cdot k_t^{(2)}+\beta_x^{(3)}\cdot\gamma_{t-x}. \qquad (13)$$

It is again required that $\beta_x^{(1)}=1$, $\beta_x^{(2)}=(x-\bar{x})$ and $\beta_x^{(3)}=1$. Thus,

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right)=k_t^{(1)}+\left(x-\bar{x}\right)\cdot k_t^{(2)}+\gamma_{t-x}. \qquad (14)$$

3.6 Cairns-Blake-Dowd model (M7)

Other alternative extension of the original M5 model Cairns et al. (2009) consisted of adding cohort and quadratic age effects:

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right)=k_t^{(1)}+\left(x-\bar{x}\right)\cdot k_t^{(2)}+k_t^{(3)}\cdot\left[\left(x-\bar{x}\right)^2-\hat{\sigma}_x^2\right]+\gamma_{t-x}. \qquad (15)$$

3.7 Cairns-Blake-Dowd model (M8)

Finally, Cairns et al. (2009) suggested a modification of the cohort term effect by introducing a decreasing effect with respect to the age:

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right)=k_t^{(1)}+\left(x-\bar{x}\right)\cdot k_t^{(2)}+\gamma_{t-x}\cdot\left(x_c-x\right). \qquad (16)$$

All models from M5 to M8 are widely known as generalized CBD-Perks models (Cairns-Blake-Dowd).

3.8 Renshaw-Haberman model (RH)

Renshaw and Haberman (2006) considered different substructures of (10). Among them, we have:

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right)=\alpha_x+\beta_x^{(1)}\cdot k_t^{(1)}+\gamma_{t-x}. \qquad (17)$$

Haberman and Renshaw (2011) suggested this very simple version that solves some stability issues of the original model.

3.9 PLAT model (PL)

Plat (2009) adds diverse characteristics of several models: It includes the parameter α_x of Lee and Carter (1992), incorporates the cohort effect Renshaw and Haberman (2006), the robust structure of the Renshaw-Haberman model and the multiple factor structure of Cairns et al. (2006, 2009):

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right)=k_t^{(1)}+\left(x-\bar{x}\right)\cdot k_t^{(2)}+k_t^{(3)}\cdot\left(\bar{x}-x\right)^+ +\gamma_{t-x}. \qquad (18)$$

4. Model Calibration

The data used in this study consist of estimates of crude mortality rates obtained from the Human Mortality Database (2018) (HMD) corresponding to the Spanish and French experiences. Data from *1975-2006* were used to calibrate the

models. Data from *2007-2015* were used for out of sample testing and ages covered in this study, range from *0* to *99*, distinguishing between male and female populations.

4.1 Data

To select the key age we proceed with the optimization process described in Section 2.1. The weighting scheme applied in function (4) implied that ages below *18* were not taken into consideration when selecting the key age. There are mainly two reasons for that decision. Firstly, theses ages are of minor interest for the insurance industry, and secondly the number of deaths at these ages is very small and so the corresponding estimates of the crude rate of mortality are extremely volatile, distorting the analysis.

4.2 Selection of the key age

This high volatility of the relative changes in crude mortality rates also affects (but not strongly) the older ages. However, we included them in the determination of the key age. Actually, the right side of the mortality curve is of special relevance for the insurance and pension industries. Although crude mortality rates for elderly people are also highly variable, we considered it inadequate to disregard them when determining the key age.

The functions $\varphi(x^*)$ in equation (5), which represent the explanatory power of each mortality rate to the whole mortality curve between *18* and *99* years old in Spain and France are depicted in Figure 1. In the Spanish case the mortality rate with the highest explanatory power is $x^*=29$ for both male and female. In the French case the key age is *83* for males and *91* for females.

Let us highlight the similar pattern of that in both pictures: a double hump that peaks at ages around *25-35* years and *80-95* years, respectively. The main difference is that the first hump dominates the second one in the Spanish population, while for the French population we can observe the opposite effect. Recall that the function $\varphi(x^*)$ represents the explanatory power of a singular age *x* in order to describe the changes in the whole mortality curve. These results can be justified by a more severe impact of AIDS in Spain, a disease that caused a very pronounced change in the shape of the mortality curve in the tranche of ages between *20* and *40* years during the sample period covered by this study. The influence of AIDS in the French population was by far less intense, although it can be also observed. By contrast, the other main change in shape of the mortality curve affected its right end and captured the steady decrease in mortality rates of people over *45* years old, a change that made it to increase the slope of the mortality curve in this tranche of ages.

To sum up, we can state that the key age gives a hint of the part of the mortality curve where the main changes in its shape are taking place. In the case of the

Spanish population, this part was the tranche of ages between 20 and 40 probably as a consequence of the dramatic irruption of AIDS.[10] In the case of the French population, though also affected by the disease, the main transformation in the shape of the mortality curve were observed for more advanced ages.

Figure 1

Values of the function $\varphi(x^*)$, covering the period 1975 to 2006.

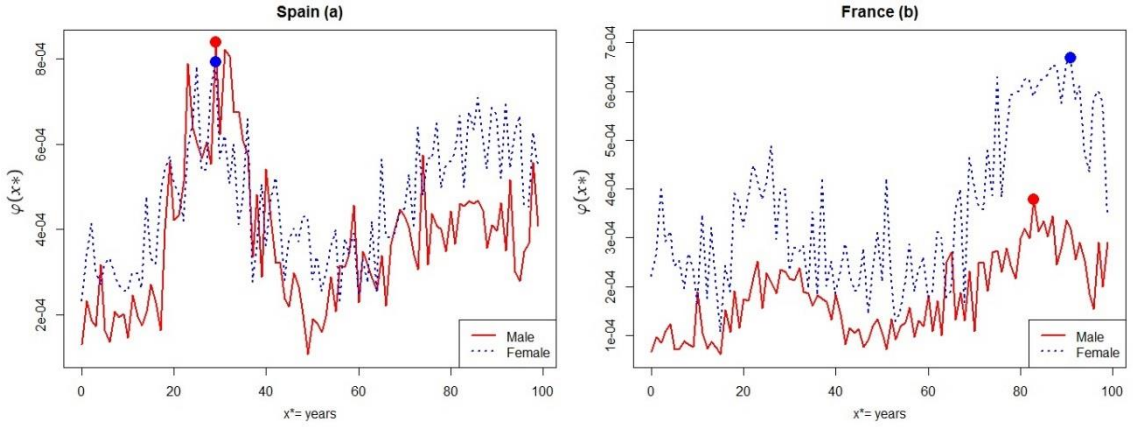


Table 1 shows the values of the parameters estimates $\hat{\alpha}_{x,x*}$ and $\hat{b}_{x,x*}$ corresponding to the sample period *(1975-2006)* for each country and population, as well as their corresponding key ages.

In the case of the Spanish population, the values of $\hat{b}_{x,x*}$ are represented in Figures 2 (a) and 2 (b) for males and females, respectively. As can be seen, the pattern in both figures is similar, with a maximum at the key rate *x*=29*, a symmetric hump around the key age and an approximately constant value close to zero from 35 years of age onwards. At the same time, the values of $\hat{\alpha}_{x,x*}$ are negative for all ages, except for the key age (where the value of $\hat{\alpha}_{x,x*}$ must equal zero) and the age of *98* in both cases. When $\hat{b}_{x,x*} \approx 0$, the main force driving mortality rates is the constant $\hat{\alpha}_{x,x*}$ which can be understood as a constant annual change in mortality rates. As seen in Table 1, the $\hat{\alpha}_{x,x*}$ values critically depend on *x*, and it is much more negative for females than for males, indicating a faster decrement in the mortality rates of women during the sample period. This phenomenon is particularly acute in the tranche between *50* and *80* years of age.

In the case of France, the values $\hat{b}_{x,x*}$ are plot in Figures 2 (c) and 2 (d). Again males and females follow a similar pattern: there is a hump at early ages, then the value of $\hat{b}_{x,x*}$ remains constant and close to zero between *20* and *60* years and finally, it

[10] This fact has been highlighted in Felipe et al. (2002), Guillen and Vidiella-i Anguera (2005) and Debón et al. (2008) where it is documented the particularly acute impact of this disease in the Spanish population in comparison to other European countries.

increases with age approaching a value equal to one at the key age. The values of $\hat{\alpha}_{x,x*}$ are negative for all ages up to the key age. At the key age, the value of $\hat{\alpha}_{x,x*}$ must equal zero. For older ages the $\hat{\alpha}_{x,x*}$ values are usually positive but close to zero. As in the case of the Spanish population, the values of $\hat{\alpha}_{x,x*}$ are close to zero around the key age. By contrast, when $\hat{b}_{x,x*}$ is close to zero the main force driving mortality rates are the constants $\hat{\alpha}_{x,x*}$. It is worth noting that the $\hat{\alpha}_{x,x*}$ values are smaller particularly between *75* and *85* years for the female population, indicating a faster decrement in mortality rate.

Table 2

$\hat{\alpha}_{x,x*}$ and $\hat{b}_{x,x*}$ (1975-2006).

	Spain				France			
	Male *(x*=29)*		Female *(x*=29)*		Male *(x*=83)*		Female *(x*=91)*	
	$\hat{\alpha}_{x,x*}$	$\hat{b}_{x,x*}$	$\hat{\alpha}_{x,x*}$	$\hat{b}_{x,x*}$	$\hat{\alpha}_{x,x*}$	$\hat{b}_{x,x*}$	$\hat{\alpha}_{x,x*}$	$\hat{b}_{x,x*}$
0	-0.0527	0.0580	-0.0553	-0.0338	-0.0431	-0.1227	-0.0438	-0.1807
10	-0.0384	-0.1767	-0.0254	0.1422	0.0069	2.5863	-0.0323	0.9545
15	-0.0154	0.1374	-0.0262	0.0504	-0.0306	-0.0842	-0.0249	0.2451
20	-0.0183	0.1489	-0.0250	-0.2318	-0.0208	0.4554	-0.0162	0.6571
25	-0.0084	0.4428	-0.0369	-0.2051	-0.0225	-0.2031	-0.0424	-0.7773
29	0.0000	1.0000	0.0000	1.0000	-0.0180	0.0031	-0.0284	0.1037
30	-0.0043	0.6932	-0.0301	0.1860	-0.0144	0.0758	-0.0301	0.3002
40	-0.0120	0.2446	-0.0164	-0.0303	-0.0140	0.2164	-0.0200	-0.0383
50	-0.0151	-0.0146	-0.0183	0.0294	-0.0142	0.0670	-0.0141	-0.0489
60	-0.0148	0.0380	-0.0249	0.0236	-0.0121	0.4106	-0.0151	0.1696
70	-0.0203	0.0605	-0.0306	0.0135	-0.0205	0.2402	-0.0184	0.5047
80	-0.0165	0.0726	-0.0234	0.0584	-0.0101	0.5208	-0.0167	0.6392
83	-0.0164	0.0720	-0.0232	-0.0138	0.0000	1.0000	-0.0132	0.6885
90	-0.0085	0.0198	-0.0135	0.0089	0.0000	0.6599	-0.0044	0.8262
91	-0.0071	0.0947	-0.0087	0.0527	0.0002	0.6613	0.0000	1.0000
95	-0.0052	0.0953	-0.0044	0.0613	0.0072	0.7510	-0.0054	0.5424
99	-0.0036	0.1336	-0.0005	0.1932	0.0218	1.3141	0.0052	0.6456

4.3 Calibration the sensitivity function $b^*(x)$

The next step consists of approximating a function to describe as accurately as possible the behavior of the estimates of $b_{x,x*}$. In Section 2.2, we proposed two options. The first consists of adjusting function (6), that is:

$$\hat{b}_{x,x^*} = b^*(x) + u_x = \beta_1 \cdot \exp\left[-\beta_2 (x - x^*)^2\right] + (1 - \beta_1) + u_x \qquad (19)$$

The parameters of this function were estimated by GLS according to weighting scheme, explained in Section 2.2.2, for each country (Spain and France) and each

group (Male and Female) and they are shown in Table 2. Figures 2 (a), 2 (b), 2 (c) and 2 (d) depict the resulting $b^*(x)$ functions together with the values of $\hat{b}_{x,x*}$.

Table 3

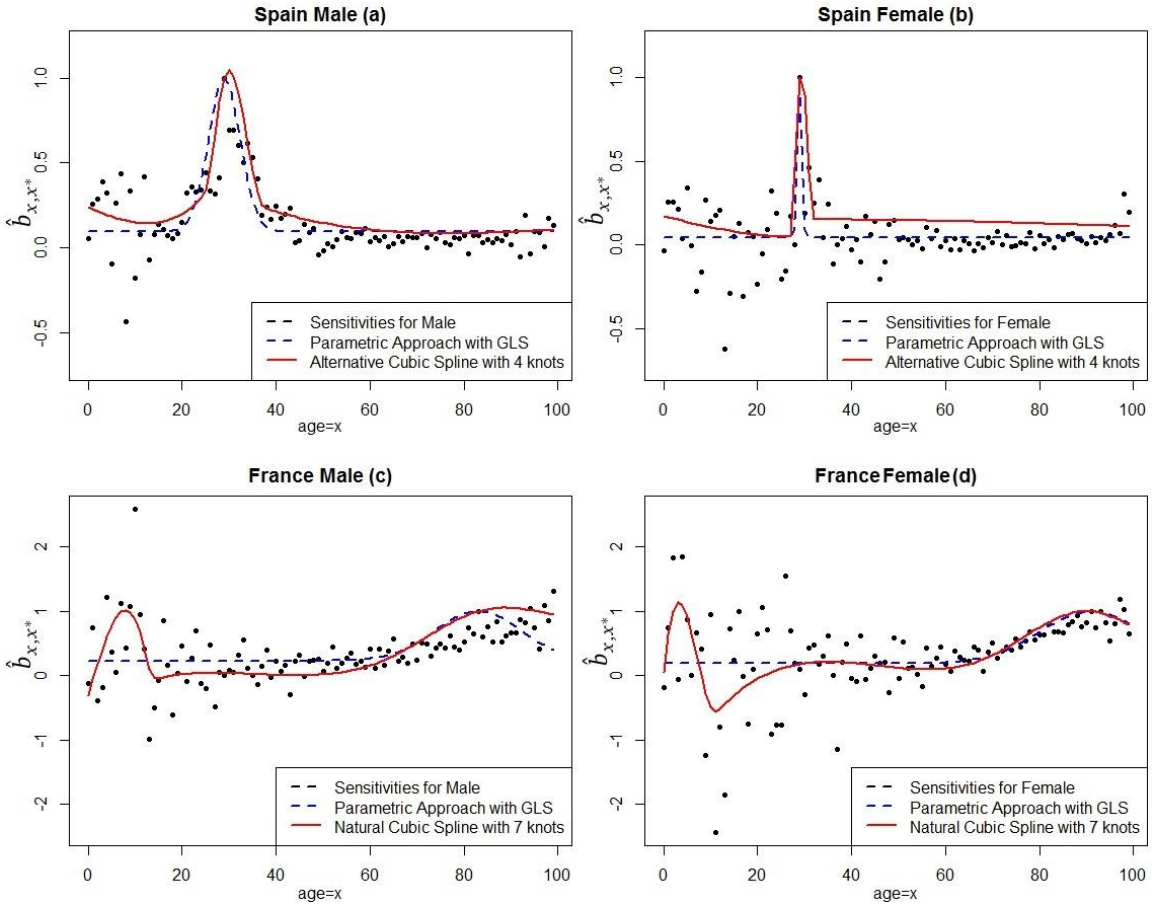
Parameter estimates of the function $b^*(x)$ of sensitivities to the key mortality rate for the sample period *(1975-2006).*

Country	Spain		France	
Number of obs = 100	Male	Female	Male	Female
Key age x*	29	29	83	91
β_1	0.9014*	0.9548*	0.7711*	0.8152*
β_2	0.0466*	2.9709*	0.0059*	0.041*
Adj-R^2	0.40	0.27	0.07	0.12
$\sum_x u_x^2$	2.37	2.28	20.41	38.90

*Significantly different from zero at 99%.

Figure 2

$\hat{b}_{x,x*}$ values and adjusted functions using the parametric approach and the best cubic splines for the sample period *(1975-2006)*. (a) Spain male, (b) Spain female, (c) France male and (d) France female.



The second option was to fit splines to describe the values of $\hat{b}_{x,x*}$.[11] We tried different types of splines, including quadratic splines, cubic splines, natural cubic splines and an alternative type of spline explained bellow. In particular, the set of splines that were tried to describe $\hat{b}_{x,x*}$ for each population were:

a) Cubic natural splines with 7 knots. Two knots are located at the initial and final ages *(0* and *99)*. Another one is located at the key age to ensure that $b^*(x) = 1$. The other four knots were selected among integer ages within the age interval *[0, 99]* to minimize the sum of squared errors. These were the optimal splines finally applied to the French population.
b) An alternative cubic spline, $S(x)$, specifically designed to describe the behaviour of $\hat{b}_{x,x*}$ around the key age:

Let $P(x)$ be a 3rd degree polynomial:

$$P(x) = a\left(x^3 - x*^3\right) + b\left(x^2 - x*^2\right) + c\left(x - x*\right) + 1$$

We add to the left and right of $x=x*$ cubic functions, such that the spline becomes:

$$S(x) = P(x) + \sum_{j=1}^{m} l_j \left(x - \varsigma_i\right)_-^3 + \sum_{i=1}^{n} k_i \left(x - \eta_i\right)_+^3$$

where $\varsigma_1 < \varsigma_2 < \cdots < \varsigma_m < x^* < \eta_1 < \eta_2 < \cdots < \eta_n$ are the knots of the spline function, and,

$$\left(x - \eta_i\right)_+ = Max\left\{x - \eta_i, 0\right\} \quad \text{and} \quad \left(x - \varsigma_j\right)_- = Min\left\{x - \varsigma_2, 0\right\}$$

In this way, we can ensure that $S(x^*) = 1$ and belongs to C^2. The knots were selected among integer ages between *0* and *99*, applying OLS, so as to minimize the sum of the squared errors. These splines were eventually selected to describe $\hat{b}_{x,x*}$ for the Spanish population.

The results and the splines selected are shown in Table 4 and depicted in Figure 2.

[11] Other types of splines could be applied. For instance, penalized splines (see Eilers and Marx (1996)) have been used for smoothing and forecasting mortality rates as in Currie et al. (2004b).

Table 4

Spline functions used to describe $b_{x,x*}$ *(1975-2006).*

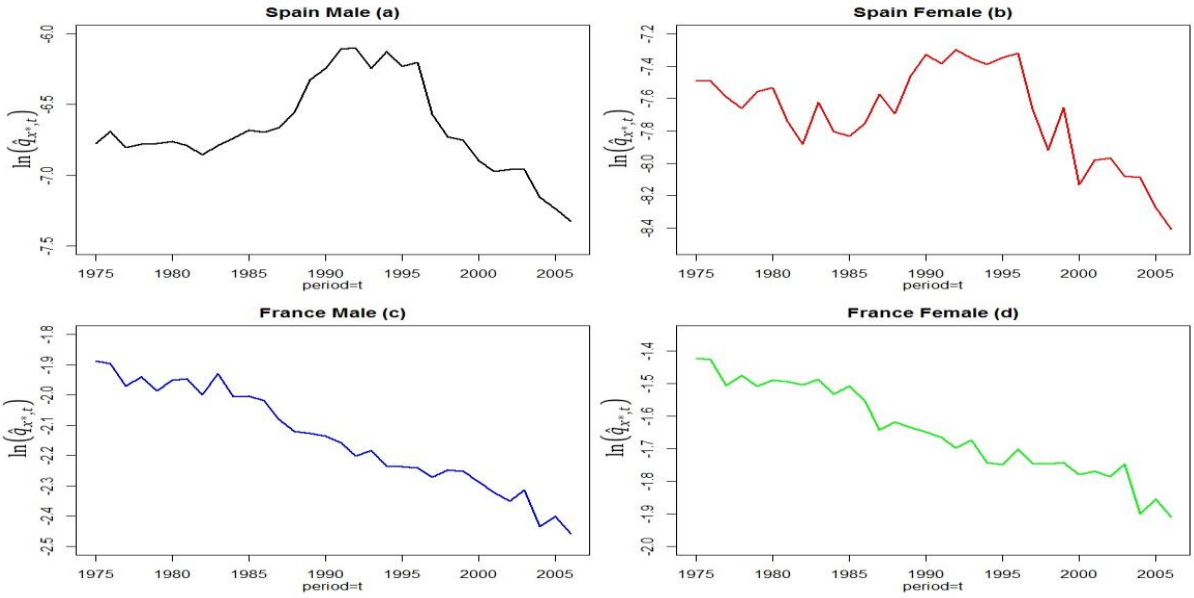
Country	Spain		France	
Number of obs=100	Male	Female	Male	Female
Key age *x**	29	29	83	91
Natural cubic Splines	Knots: 24,25,29, 37,38	Knots: 27,27,29,32,32	Knots: 12,13,15,65,89	Knots: 1,11,12,69,91
$\sum_x u_x^2$	1.9964	3.3615	14.6864	27.3379
Alternative Splines	Knots: 28,29,30,51	Knots: 28,29,30,31	Knots: 18,19,84,85	Knots: 11,12,92,93
$\sum_x u_x^2$	1.5770	3.1407	14.8134	27.6175

4.4 Forecasting mortality rates

The next step is to forecast future mortality rates. In Figure 3, we show the behavior of the logarithms of the key mortality rates during the sample period *(1975 - 2006)*. It is of interest to highlight the strong increment in mortality rates (particularly for the male and female Spanish populations) that occurred during the 1980s, followed by an extraordinary drop, due to the impact of AIDS and later discoveries of treatments for that disease. It should be noted that the key ages correspond to the population groups most strongly affected by the infection: the portion of the population roughly *30* years of age.

Figure 3

Evolution of $Ln(q_{x^*;t-1,t})$ in key ages, sample period *(1975-2006).*



To model the behavior of these key mortality rates, we examined different alternatives in the ARIMA family. They were selected using *auto.arima* functions and *forecast* from the R Core Team (2018) package developed by Hyndman and Khandakar (2008) which specifies the ARIMA (p,d,q) model that best fits the data based on the Akaike criterion. We employed four different sample windows to estimate the parameters of the model used to forecast mortality rates:

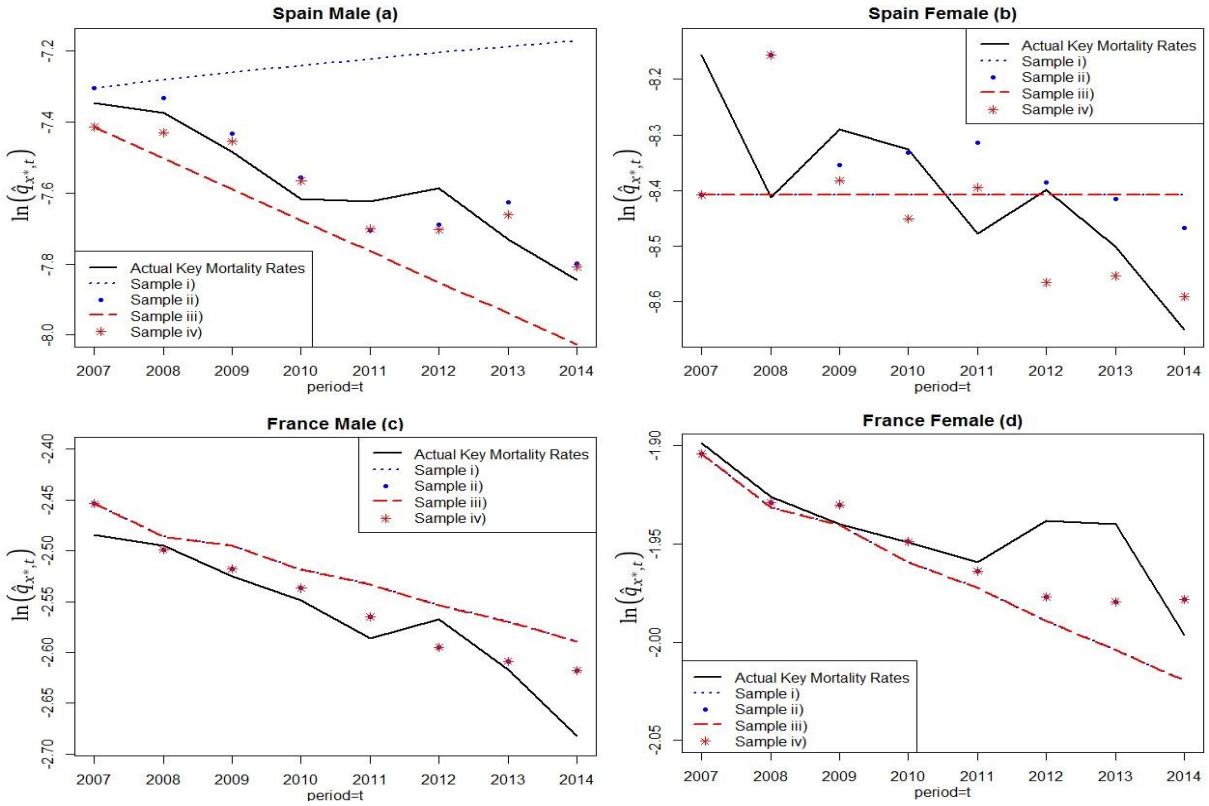
i. The sample period covers *1975-2006*; the model is used to forecast mortality rates from *2007* onwards without updating the sample.
ii. The time series parameters are re-estimated, enlarging the end of the sample window every year from *2006* through *2014*. In this case, the model is re-estimated every year and used to forecast the mortality rate only one year ahead.
iii. The sample period starts in the year when the key mortality rate reaches its maximum value through *2006*. Once the parameters are estimated, they are used to forecast mortality rates from *2007* onwards, without updating them.
iv. The forecasting sample period begins the same year as in iii), but the sample period is enlarged, and the parameters are re-estimated every year to forecast future mortality rates only one year ahead (similarly to ii)).

Using these four proposals, we proceeded to estimate the expected future values of the key mortality rates from *2007* to *2014*, a period used for out-of-sample testing. Figure 4 shows the key future mortality forecasts for the two countries considered in this study (France and Spain) and for the two groups of populations (male and female), along with the actual behavior of the key mortality rates.

As seen at a glance in Figure 4, using the samples that start in the early *1990s* produces, in the case of Spain, a much better forecast of mortality rates. This is due to the irregular behavior of mortality rates during the eighties were mortality rates around the key ages increased dramatically to reach a maximum during the nineties and then resuming their declining path.

Figure 4

Forecast of the key mortality rate for Spain male (a), Spain female (b), France male (c) and France female (d), (sample i, sample ii, sample iii and sample iv).



In the case of the French population, samples i) and iii) and samples ii) and iv) are equal as the key mortality rates reached their maximum at the beginning of the sample period *(1975)*.

Once forecasts of the key mortality rates were obtained, estimates of the remaining expected future mortality rates were determined by taking expectations on both sides of the equation:

$$\Delta \ln\left(\hat{q}_{x,t}\right) = \alpha_{x,x^*} + b_{x,x^*}\left[\Delta \ln\left(\hat{q}_{x^*,t}\right)\right] + \varepsilon_{(x,x^*),t}. \qquad (20)$$

In the case of the non-updating samples (i and ii) we have:

$$E_{t-1}\left[\ln\left(\hat{q}_{x,t+i}\right)\right] = \ln\left(\hat{q}_{x,t-1}\right) + (i+1)\cdot\alpha_{x,x^*} + b^*(x)\cdot E_{t-1}\left[\ln\left(\hat{q}_{x^*,t+i}\right) - \ln\left(\hat{q}_{x^*,t-1}\right)\right], \quad (21)$$

$$i = 0,1,2,....,7; \quad \text{t=2007}$$

In the case of the updating samples (ii and iv):

$$E_{t-1}\left[\ln\left(\hat{q}_{x,t}\right)\right]=\ln\left(\hat{q}_{x,t}\right)+\alpha_{x,x^*}+b^*(x)\cdot E_{t-1}\left[\ln\left(\hat{q}_{x^*,t}\right)-\ln\left(\hat{q}_{x^*,t-1}\right)\right], \qquad (22)$$
$$t=2007,2008,....,2014$$

where α_{x,x^*} and $b^*(x)$ are re-estimated every year *t*.

5. ALTERNATIVE MORTALITY MODELS

5.1 Model calibration

In this section, we calibrate the different versions and proposals, explained in Section 3, which will be used as benchmarks for comparison with the Single Factor Model developed above.

The first important mortality model by Lee and Carter (1992) proposed the use of Singular Value of Decompositions (SVD) to estimate the parameters. The second procedure consisted of maximising the log-likelihood using the Newton-Raphson method.[12] Currie et al. (2004) improved the estimation procedure by applying GLM (Generalized Linear Models) to the force of mortality, using the *gnm* library (Turner and Frith, 2018) from the R Core Team (2018) package developed by Hyndman and Khandakar (2008).

We decided to adjust the probability of death assuming a binomial distribution and a logit link for every model explained in Section 3.[13] In this paper, we apply *gnm* function of R Core Team (2018) to adjust LC and LC2 models.[14] The rest of models were calibrated with *StMoMo* package of R Core Team (2018) developed by Villegas et al. (2018).[15]
Additionally, APC, M6, M7, M8, RH and PLAT models include the cohort term $\hat{\gamma}_{t-x}$. In all cases, zero weights are applied to the *1886* cohort,[16] to the earliest and the latest 3 cohort years (Renshaw and Haberman (2006).

Furthermore, details with the fitting of LC2 and APC parameters are illustrated in Figures 5 and 6.

[12] See for instance, Brouhns et al. (2002), Renshaw and Haberman (2006) and Cairns et al. (2009).
[13] According to Lee (2000) the logit version avoids estimates of $m_{x,t}$ greater than one.
[14] Following Debón et al. (2008), where the authors compared the calibration of Lee-Carter model to the Spanish mortality data applying different methods: SVD, maximum likelihood and GLM, the latter providing the best outcomes.
[15] This library provides a tool for fitting stochastic mortality models.
[16] See, for instance, Haberman and Renshaw (2009).

Figure 5

Model LC2: Estimated values of α_x (a), $\beta_x^{(1)}$ (b), $k_t^{(1)}$ (c), $\beta_x^{(2)}$ (d), $k_t^{(2)}$ (e), over the sample period (1975-2006).

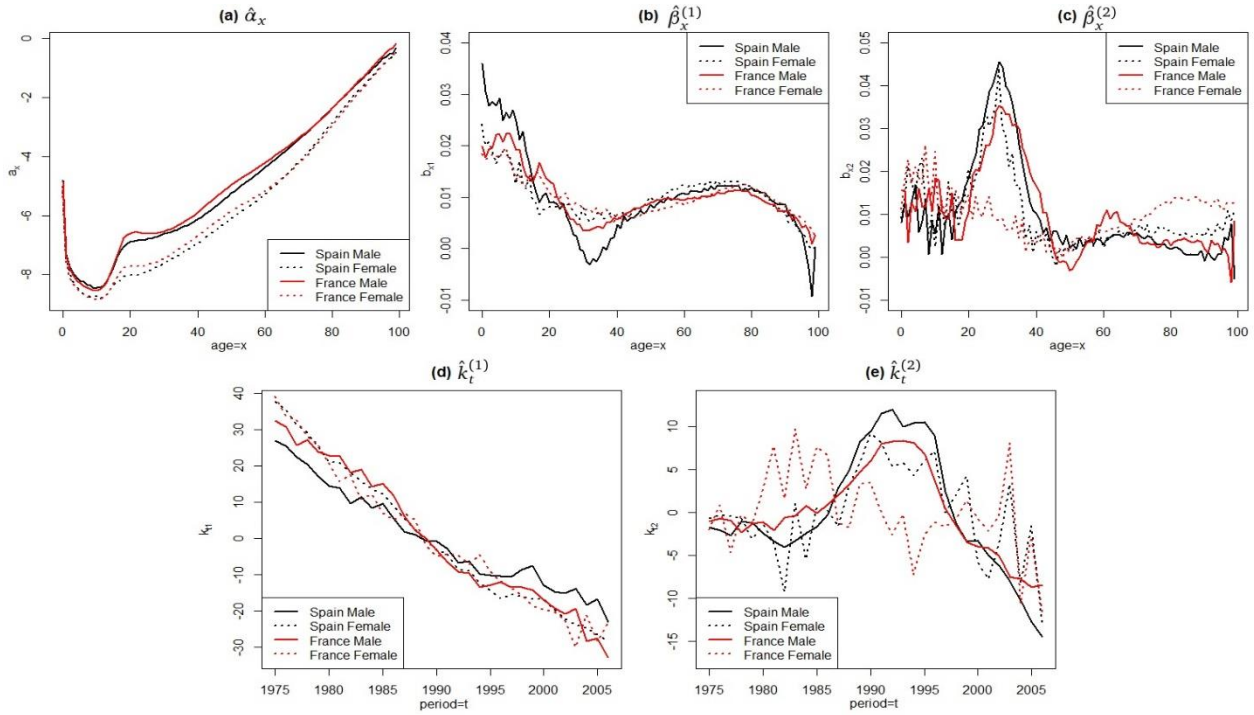
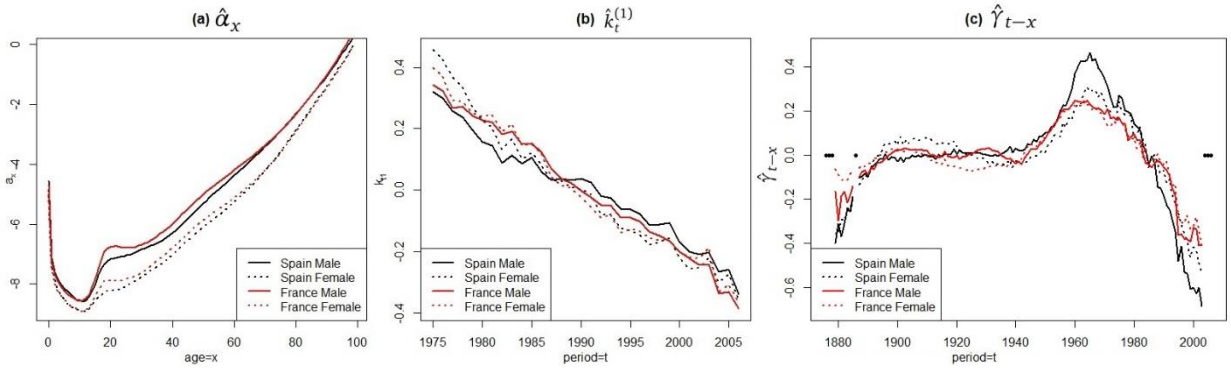


Figure 6

Model APC: Estimated values of α_x (a), $k_t^{(1)}$ (b) and $\hat{\gamma}_{t-x}$ (c), over the sample period (1975-2006).



Bearing in mind that α_x represents the average shape of the mortality rates as a function of the age *x*, \ Figures 5 (a) and 6 (a) show that women mortality rates were lower than men ones, and Spanish mortality rates are lower than French ones for

both men and women. With respect to $\beta_x^{(1)}$ and $\beta_x^{(2)}$, we should note that it reports how mortality rates react to changes in mortality trend. The higher is the value of $\beta_x^{(i)}$, the greater is the decrement in the mortality rate at age *x*. It is interesting to observe in Figures 5 (b) and 6 (c) that the values of $\beta_x^{(1)}$ are close to zero for ages around *29* (the key age) and $\beta_x^{(2)}$ values reach its maximum around the same age, indicating a different pattern of mortality around these ages. $\hat{\gamma}_{t-x}$ represents a cohort effect.

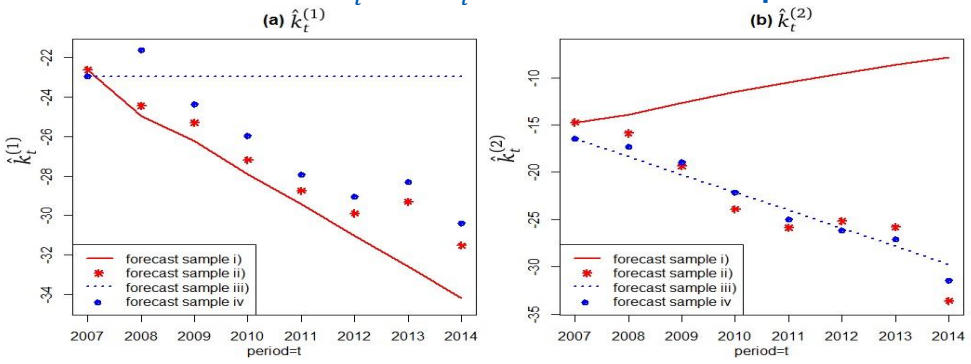
5.2 Lee-Carter model mortality rates forecast

In all the models explained above, the predictions of future mortality rates are based on the adjustment of a time series to the mortality period indexes $k_t^{(i)}$ and the cohort index $\hat{\gamma}_{t-x}$. We need to estimate these indexes with time series techniques to forecast the future mortality rates.

The main problem when forecasting mortality models is to determine the dynamics of period and cohort indexes. In the original model, Lee and Carter (1992) used an ARIMA (0,1,0) to adjust $k_t^{(i)}$. Following Renshaw and Haberman (2006), Cairns et al. (2011) and Lovász (2011), we assume that the cohort index $\hat{\gamma}_{t-x}$ can be described by an univariate ARIMA process which is independent of the period index $k_t^{(1)}$. So, we apply the StMoMo library which uses the function auto.arima of forecast library from the R Core Team (2018) package, to determine the ARIMA model that best fits $k_t^{(i)}$ and $\hat{\gamma}_{t-x}$ indexes.17 As in Section 4.4, we employ the same four alternative samples to forecast the values of $k_t^{(i)}$ and $\hat{\gamma}_{t-x}$.

As an example, Figures 7 and 8 present, respectively, the expected values of the mortality indexes for LC2 and APC in Spanish male population.

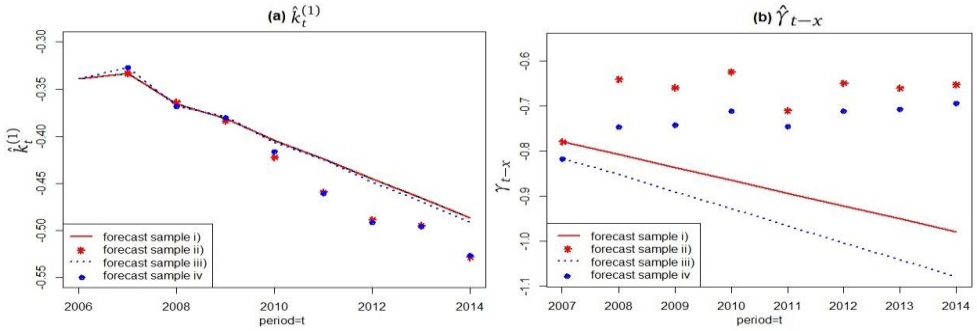
Figure 7
Forecast of $k_t^{(1)}$ and $k_t^{(2)}$ values for male in Spain with LC2.



[17] See Hyndman and Khandakar (2008) for further details.

Figure 8

Forecast of $k_t^{(1)}$ and $\hat{\gamma}_{t-x}$ values for male in Spain with APC.



6. Comparison between the Single Factor model and other Mortality models

In this section, we proceed to compare the Single-Factor Model with different mortality models. In Table 5 we see some relevant criteria to compare the characteristics of these ten models.[18] When a criterion is met, the model is labeled ✓; otherwise, the model is marked with ✗, both symbols are used when the model partly meets the criterion.

Table 5

Characteristics to evaluate mortality models

Model	SFM	LC	LC2	APC	M5	M6	M7	M8	RH	PLAT
Number of parameters	200	232	364	271	64	193	227	195	363	335
Number of constrains	0	2	4	2	0	3	3	3	3	6
Include the cohort effect	✗	✗	✗	✓	✗	✓	✓	✓	✓	✓
Number of indexes to forecast	1	1	2	2	2	3	4	3	2	4
Ease of implementation	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Observable variables	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗
Applicable for full range of variables	✓	✓/✗	✓/✗	✓/✗	✗	✗	✗	✗	✓/✗	✓

[18] Other comparison criteria can be seen in Cairns et al. (2009), Plat (2009) and Haberman and Renshaw (2011).

The characteristics are defined as:

- Number of parameters: effective number of parameters estimated in each model. Spain and France, males and females, ages *0-99* and period *(1975 - 2006)*, insample.
- Number of constraints: set of constraints necessary to ensure the identifiability of the model.
- Include the cohort effect: the model incorporates a cohort effect.
- Number of indexes to forecast: number of indexes that is necessary to model for estimating the future mortality rates.
- Ease of implementation: how easy is to implement the model.
- Observable variables: indexes can be directly observed.
- Applicable for full range of age, the model can be applied to all ages.

In Table 5 we have compared the characteristics of *9* different mortality models against our Single Factor Model. We would like to emphasize three main findings:

i. Our model and M5 do not need constraints to estimate the parameters, meanwhile the rest of the mortality models require at least two constraints.
ii. The cohort effect is not included in SFM but it would be possible to add it to the model.
iii. One of the main advantages of the model is that the variable that measures the general tendency of mortality is directly observable, we know exactly what it is, the key mortality rate. Meanwhile, in the other mortality models $k_t^{(i)}$ captures the general tendency but it has to be estimated.

6.1 Forecasting ability of the models

After describing the main features of each model, we proceed to compare their forecasting ability. In this paper, we estimate the forecasting mortality rate errors for each year from *2007* to *2014*, the out of sample period.

We denote the logarithm of the actual crude mortality rate for *t=2007, …,2014* by $ln(\hat{q}_{x,t})$ and the logarithms of the mortality rates forecasted by the Single Factor Model and different alternatives for *t=2007, …, 2014* by $ln(\hat{q}_{x,t}^{SFM})$, $ln(\hat{q}_{x,t}^{LC})$, $ln(\hat{q}_{x,t}^{LC2})$, $ln(\hat{q}_{x,t}^{APC})$, …, respectively. Then, the forecasting errors for each year and age in the Single Factor Model are given by:

$$\varepsilon_{x,t+i}^{SFM} = \ln\left(\hat{q}_{x,t+i}\right) - E_{t-1}\left[\ln\left(\hat{q}_{x^*,t+i}^{SFM}\right)\right] \qquad i = 0,1, ..., 8; \;\; t = 2007 \qquad (23)$$

in the case of non-updating samples (cases i) and iii) in Section 4.4) or,

$$\varepsilon_{x,t}^{SFM} = \ln\left(\hat{q}_{x,t}\right) - E_{t-1}\left[\ln\left(\hat{q}_{x^*,t}^{SFM}\right)\right] \qquad t = 2007, 2008, ..., 2014, \qquad (24)$$

in the case of updating samples (cases ii) and iv) in Section 4.4).

For the rest of models,[19] for instance Lee-Carter model,[20] the forecasting errors for each year and age are calculated as:

$$\varepsilon_{x,t+i}^{LC} = \ln\left(\hat{q}_{x,t+i}\right) - \ln\left(\frac{e^{\hat{a}_x+\hat{b}_x\cdot E_{t-1}[k_{t+i}]}}{1+e^{\hat{a}_x+\hat{b}_x\cdot E_{t-1}[k_{t+i}]}}\right) \qquad i = 0,1, ..., 8; \quad t = 2007 \qquad (25)$$

in the case of non-updating samples (cases i) and iii)) or,

$$\varepsilon_{x,t}^{LC} = \ln\left(\hat{q}_{x,t}\right) - \ln\left(\frac{e^{\hat{a}_x+\hat{b}_x\cdot E_{t-1}[k_t]}}{1+e^{\hat{a}_x+\hat{b}_x\cdot E_{t-1}[k_t]}}\right) \qquad t = 2007, 2008, ..., 2014, \qquad (26)$$

in the case of updating samples (cases ii) and iv)).

To illustrate the results, we plot the forecasting errors for LC2 against SFM for the Spanish male population[21] (Figures 9 and 10) as well as the forecasting errors for APC against SFM for the French female population (Figures 11 and 12).[22] Each figure represents the errors corresponding to the four samples considered in this paper (i), ii), iii) and iv) as described in Section 4.4).

Panels (a) and (c) in Figures 9, 10, 11 and 12 represent errors as a function of age. As we have forecasted mortality for *2007-2014*, for each model and age, there are eight errors, representing each of the out of sample years. By contrast, Panels (b) and (c) in Figures 9, 10, 11 and 12 depict errors as a function of the forecasting horizon. Therefore, we have *100* observations for each forecasting period and model, corresponding to ages from *0* to *99*.

As can be seen in Figures 9 and 10, the errors size in the Bi-Factorial Lee-Carter model are similar to the SFM. Nevertheless, for Spanish males older than *70* our model seems to produce smaller forecasting errors. Besides, if one compares SFM and APC (one of the mortality models that better adjusts the data), both models produce similar errors, except that SFM seems to be better for French women older than 70.

[19] All the errors for each model explained in Section 3 are calculated similarly to (25) and (26).
[20] Recall that we have estimated the logit version of each model as explained in Section 3.
[21] In this case, we apply the Parametric Adjustment for $\hat{b}_{x,x*}$ as explained in Section 2.2.1.
[22] In this case, we apply Spline Adjustment for $\hat{b}_{x,x*}$ as explained in Section 2.2.2.

Figure 9

Forecasting errors. Spanish male population. SFM with PA vs LC2 sample i) and sample ii).

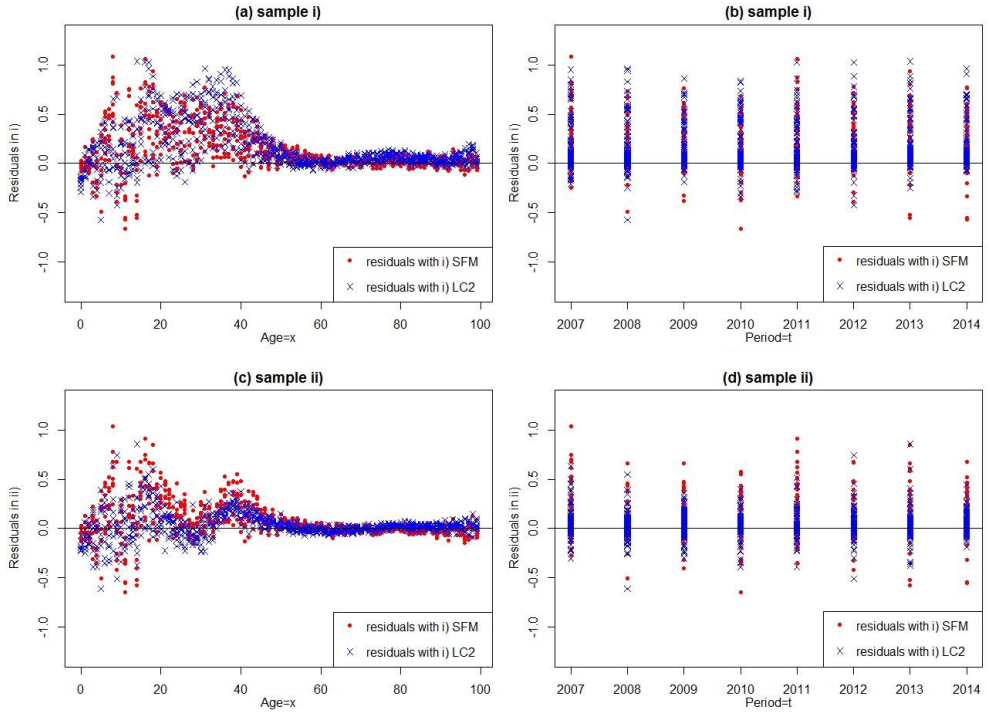


Figure 10

Forecasting errors. Spanish male population. SFM with PA vs LC2 sample iii) and sample iv).

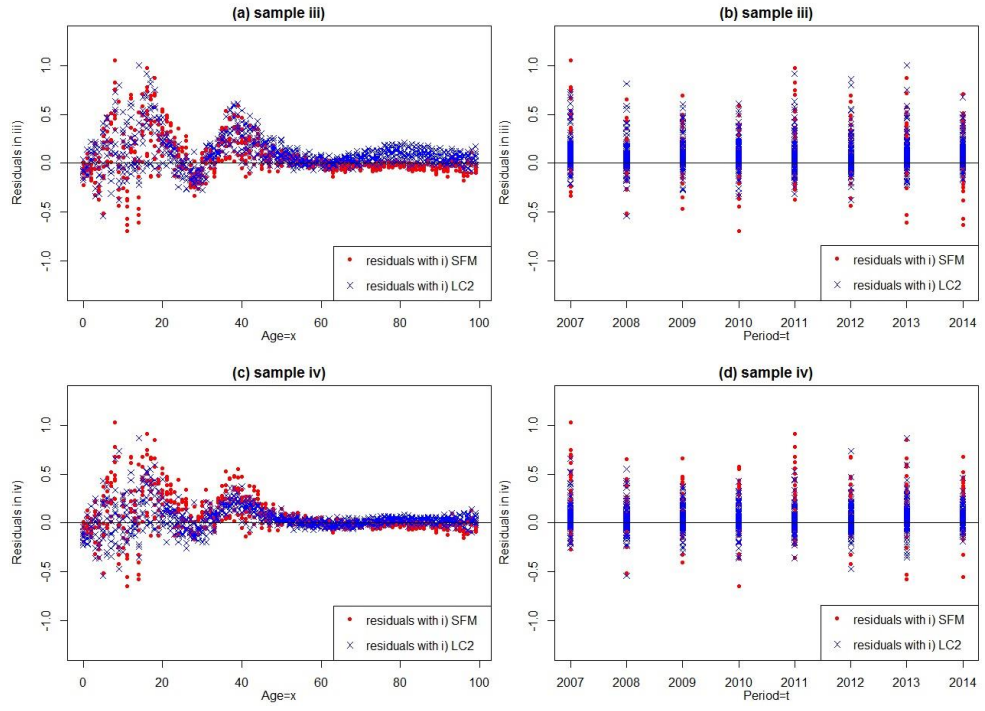


Figure 11

Forecasting errors. French female population. SFM with SA vs APC sample i) and sample ii).

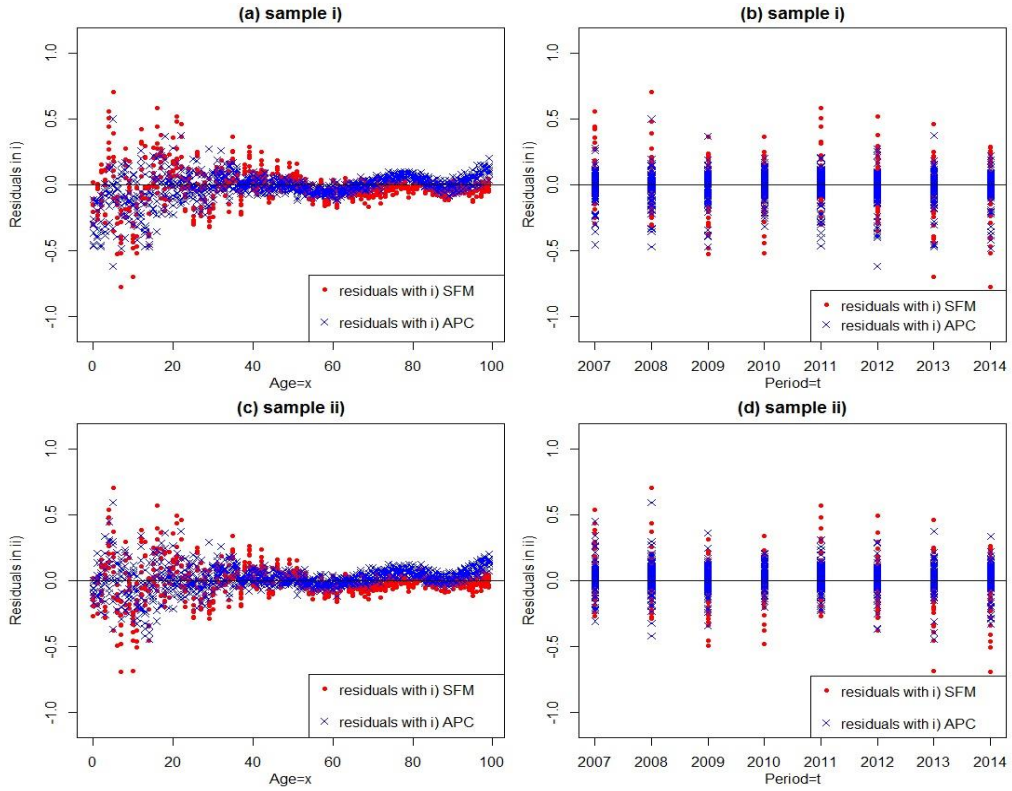
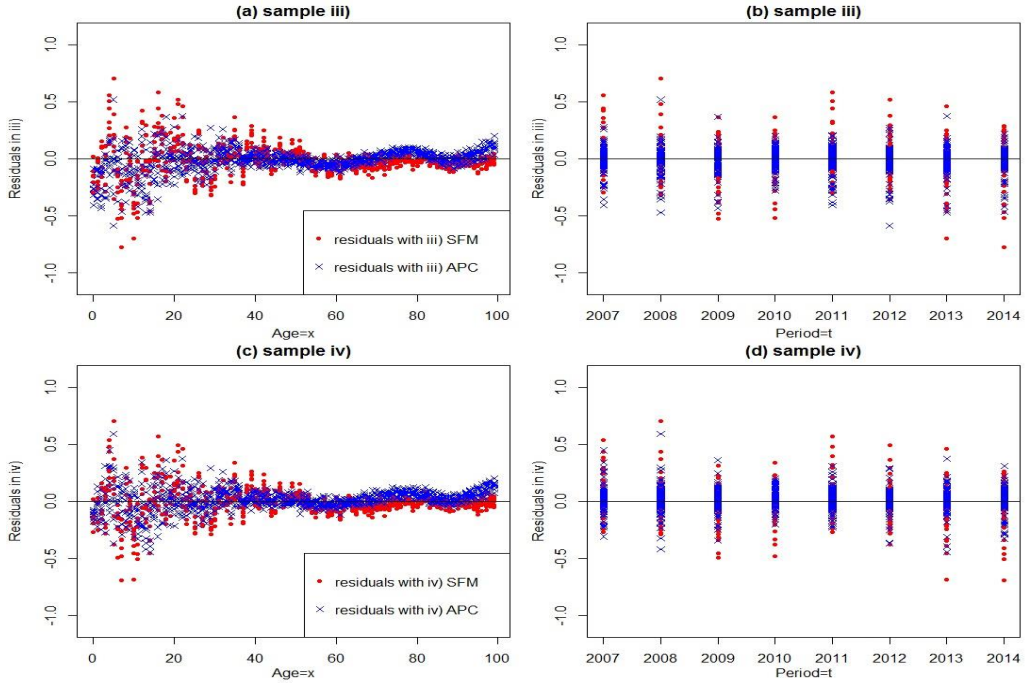


Figure 12

Forecasting errors. French female population. SFM with SA vs APC sample iii) and sample iv).



6.2 SSE, MAE and MSE

To compare the performance of the models, we have computed three standard metrics of accuracy: the Sum of Squared Errors (SSE), the Mean Absolute Error (MAE) and the Mean Square Error (MSE). The first is defined as:[23]

$$SSE = \sum_{x,t}\left(\ln\left(\hat{q}_{x,t}\right) - \ln\left(\hat{q}_{x,t}^{SFM/LC/LC2/\ldots}\right)\right)^2, \qquad \text{t=2007, 2008, . . . , 2014} \qquad (27)$$

The second is defined as:[24]

$$MAE(t) = \frac{\sum_{x=0}^{99}\left|\ln\left(\hat{q}_{x,t}\right) - \ln\left(\hat{q}_{x,t}^{SFM/LC/LC2/\ldots}\right)\right|}{100}, \qquad t = 2007, 2008, \ldots, 2014. \qquad (28)$$

The third is defined as:[25]

$$MSE(t) = \sqrt{\frac{\sum_{x=0}^{99}\left(\ln\left(\hat{q}_{x,t}\right) - \ln\left(\hat{q}_{x,t}^{SFM/LC/LC2/\ldots}\right)\right)^2}{100}}, \qquad t = 2007, 2008, \ldots, 2014. \qquad (29)$$

Errors are calculated for each year from *2007* to *2014*, using the four alternative sample windows and the best sensitivity adjustment functions for $\hat{b}_{x,x*}$ for each population.[26] The outcomes for Spanish male and French female population are summarized in Table 6 for SSE, in Tables 7 and 8 for MAE and Tables 9 and 10 for MSE, respectively.

[23] See, for instances Chen et al. (2009).
[24] See, for instance Booth et al. (2006).
[25] See, for instance Felipe et al. (2002), Debón et al. (2010), D´Amato et al. (2012) and Wang et al. (2018).
[26] The approach of $\hat{b}_{x,x*}$ which produces the best result for SSE, MAE and MSE is the Parametric Approach for Spanish male and female. In the French case they are the Parametric Adjustment for male and the Spline Adjustment for female population. Anyway, the rest of alternatives produce very similar outcomes.

Table 6: SSE for Spain and France for both male and female populations.

Mortality Model	Spanish male population				Spanish female population			
	Sample i)	Sample ii)	Sample iii)	Sample iv)	Sample i)	Sample ii)	Sample iii)	Sample iv)
SFM	55.05	32.14	33.71	31.64	27.02	23.65	27.02	23.59
LC1	121.01	61.85	121.69	99.62	46.12	30.43	47.53	30.96
LC2	67.91	15.68	31.83	15.81	45.84	18.02	23.66	16.56
APC	46.54	29.37	47.16	29.59	26.39	21.36	28.93	21.91
M5	461.18	525.04	517.66	533.21	2547.33	2300.13	2492.64	2292.79
M6	610.23	401.20	728.42	405.61	671.22	479.97	825.99	496.57
M7	356.04	270.52	527.37	274.90	413.69	330.98	643.47	363.09
M8	525.09	398.10	658.12	413.06	560.76	476.08	765.91	495.28
RH	30.77	16.86	25.92	16.44	23.75	16.80	35.09	15.63
PLAT	58.12	58.90	67.25	60.11	31.03	38.60	32.93	38.80
Mortality Model	France male population				France female population			
	Sample i)	Sample ii)	Sample iii)	Sample iv)	Sample i)	Sample ii)	Sample iii)	Sample iv)
SFM	9.21	8.49	9.19	8.49	16.41	14.33	16.41	14.33
LC1	18.68	11.63	18.68	11.63	16.54	12.22	16.54	12.22
LC2	12.97	7.75	12.97	7.75	16.81	13.11	16.81	13.11
APC	10.25	8.06	10.31	8.11	10.82	8.45	10.16	8.49
M5	394.62	379.81	394.62	379.81	1656.42	1541.16	1656.42	1541.16
M6	692.63	760.97	762.10	457.69	664.83	440.60	715.02	448.52
M7	653.56	375.82	811.57	391.67	452.55	383.54	629.94	374.07
M8	597.85	451.74	735.75	467.19	562.44	459.82	681.56	475.20
RH	9.40	7.31	9.60	7.31	21.44	8.67	19.64	8.02
PLAT	11.95	11.12	12.12	11.24	11.82	11.79	10.07	11.84

Table 7: MAE for Spanish male population.

Forecasting Horizon	Spanish male population									
	Sample i)									
	SFM	LC1	LC2	APC	M5	M6	M7	M8	RH	PLAT
2008	0.0963	0.1697	0.0872	0.1480	0.4142	0.4113	0.3289	0.4125	0.0934	0.1643
2010	0.1696	0.2530	0.1887	0.1854	0.3742	0.4894	0.3572	0.4867	0.1448	0.1733
2012	0.1994	0.2666	0.2210	0.1893	0.3820	0.5298	0.3704	0.5107	0.1510	0.1814
2014	0.2574	0.3313	0.3040	0.2146	0.3773	0.6219	0.4217	0.5843	0.2010	0.1826
Sample ii)										
2008	0.0922	0.1570	0.0744	0.1415	0.4314	0.3960	0.3237	0.4009	0.0938	0.1724
2010	0.1211	0.1995	0.0913	0.1490	0.4183	0.4224	0.3047	0.4296	0.1165	0.1701
2012	0.1387	0.1820	0.0875	0.1221	0.4402	0.4121	0.2894	0.4244	0.0908	0.1534
2014	0.1535	0.1906	0.1113	0.1296	0.4357	0.4235	0.3148	0.4672	0.1093	0.1490
Sample iii)										
2008	0.0922	0.1699	0.0772	0.1479	0.4322	0.4207	0.3404	0.4172	0.0920	0.1763
2010	0.1103	0.2541	0.1321	0.1852	0.4101	0.5186	0.3858	0.5001	0.1342	0.2004
2012	0.1387	0.2679	0.1495	0.1911	0.4331	0.5832	0.4318	0.5463	0.1342	0.2049
2014	0.1534	0.3338	0.2155	0.2173	0.4388	0.7092	0.5191	0.6463	0.1765	0.2354
Sample iv)										
2008	0.0890	0.1650	0.0791	0.1391	0.4397	0.3965	0.3271	0.4014	0.0900	0.1808
2010	0.1211	0.1991	0.0963	0.1527	0.4234	0.4222	0.3100	0.4327	0.1182	0.1783
2012	0.1383	0.1823	0.0858	0.1224	0.4474	0.4116	0.2963	0.4266	0.0893	0.1553
2014	0.1533	0.1905	0.1091	0.1315	0.4348	0.4527	0.3189	0.4707	0.1101	0.1549

Table 8: MAE for French female population.

Forecasting Horizon	French female population									
	Sample i)									
	SFM	LC1	LC2	APC	M5	M6	M7	M8	RH	PLAT
2008	0.0772	0.0879	0.0848	0.0636	0.9204	0.5143	0.4413	0.5274	0.0613	0.0614
2010	0.0940	0.1006	0.0964	0.0763	0.9477	0.5604	0.4703	0.5537	0.0941	0.0816
2012	0.1217	0.1219	0.1139	0.0799	0.9857	0.5906	0.4958	0.5548	0.1522	0.0897
2014	0.1224	0.1277	0.1250	0.0858	1.0043	0.6695	0.5261	0.5940	0.1986	0.0808
Sample ii)										
2008	0.0756	0.0837	0.0771	0.0586	0.9247	0.4862	0.4320	0.5106	0.0564	0.0618
2010	0.0878	0.0824	0.0873	0.0745	0.9127	0.4779	0.4445	0.5137	0.0688	0.0805
2012	0.1117	0.0955	0.0842	0.0687	0.9164	0.4512	0.4399	0.4911	0.0686	0.0807
2014	0.1074	0.0938	0.1169	0.0878	0.9504	0.4681	0.4829	0.5051	0.0899	0.1076
Sample iii)										
2008	0.0772	0.0879	0.0848	0.0626	0.9204	0.5164	0.4494	0.5338	0.0590	0.0598
2010	0.0940	0.1006	0.0964	0.0753	0.9477	0.5666	0.4979	0.5719	0.0906	0.0774
2012	0.1217	0.1219	0.1139	0.0783	0.9857	0.6051	0.5596	0.5915	0.1475	0.0835
2014	0.1224	0.1277	0.1250	0.0828	1.0043	0.6967	0.6355	0.6572	0.1914	0.0738
Sample iv)										
2008	0.0756	0.0837	0.0771	0.0598	0.9247	0.4871	0.4262	0.5119	0.0568	0.0629
2010	0.0878	0.0824	0.0873	0.0748	0.9127	0.4787	0.4262	0.5156	0.0694	0.0809
2012	0.1117	0.0955	0.0842	0.0688	0.9164	0.4520	0.4030	0.4930	0.0649	0.0803
2014	0.1074	0.0938	0.1169	0.0884	0.9504	0.4690	0.4275	0.5072	0.0881	0.1103

Table 9: MSE for Spanish male population.

Forecasting Horizon	Spanish male population									
	Sample i)									
	SFM	LC1	LC2	APC	M5	M6	M7	M8	RH	PLAT
2008	0.1623	0.2771	0.1376	0.2097	0.7731	0.7388	0.6045	0.7238	0.1315	0.2880
2010	0.2463	0.3856	0.2639	0.2466	0.7430	0.8471	0.6619	0.8025	0.1880	0.2711
2012	0.2869	0.4242	0.3283	0.2526	0.7559	0.9174	0.6789	0.8321	0.2025	0.2665
2014	0.3606	0.4885	0.4239	0.2710	0.7665	1.0597	0.7677	0.9299	0.2698	0.2596
Sample ii)										
2008	0.1562	0.2543	0.1218	0.1999	0.7941	0.7037	0.5996	0.6948	0.1367	0.2911
2010	0.1954	0.3024	0.1394	0.2070	0.7915	0.7103	0.5762	0.7069	0.1582	0.2728
2012	0.2055	0.2712	0.1285	0.1829	0.8206	0.6977	0.5590	0.6955	0.1300	0.2667
2014	0.2485	0.2774	0.1629	0.1753	0.8376	0.7558	0.6041	0.7527	0.1612	0.2413
Sample iii)										
2008	0.1524	0.2773	0.1223	0.2099	0.7941	0.7644	0.6471	0.7529	0.1290	0.2958
2010	0.1945	0.3865	0.1843	0.2470	0.7831	0.9063	0.7636	0.8723	0.1737	0.2890
2012	0.2177	0.4251	0.2051	0.2545	0.8170	1.0198	0.8694	0.9604	0.1817	0.2900
2014	0.2675	0.4901	0.2866	0.2737	0.8450	1.2113	1.0467	1.1198	0.2378	0.2994
Sample iv)										
2008	0.1513	0.2599	0.1229	0.1990	0.8041	0.7089	0.6159	0.7104	0.1295	0.2970
2010	0.1936	0.3020	0.1456	0.2089	0.7976	0.7111	0.5758	0.7190	0.1615	0.2770
2012	0.2048	0.2704	0.1261	0.1831	0.8289	0.7011	0.5626	0.7102	0.1285	0.2645
2014	0.2481	0.2773	0.1630	0.1765	0.8381	0.7557	0.5808	0.7663	0.1634	0.2471

Table 10: MSE for French female population.

Forecasting Horizon	French female population									
	Sample i)									
	SFM	LC1	LC2	APC	M5	M6	M7	M8	RH	PLAT
2008	0.1265	0.1182	0.1191	0.0961	1.3842	0.7931	0.6948	0.7895	0.0878	0.0884
2010	0.1485	0.1374	0.1392	0.1133	1.4150	0.8953	0.7620	0.8468	0.1211	0.1336
2012	0.1645	0.1496	0.1482	0.1231	1.4625	0.9717	0.7840	0.8676	0.1887	0.1407
2014	0.1753	0.1673	0.1698	0.1297	1.4913	1.0776	0.8161	0.9090	0.2477	0.1264
Sample ii)										
2008	0.1244	0.1122	0.1086	0.0857	1.3904	0.7495	0.6717	0.7557	0.0784	0.0880
2010	0.1416	0.1234	0.1311	0.1109	1.3691	0.7528	0.7034	0.7737	0.1038	0.1202
2012	0.1536	0.1231	0.1202	0.1062	1.3700	0.7402	0.6998	0.7620	0.1019	0.1264
2014	0.1561	0.1292	0.1543	0.1173	1.4228	0.7395	0.7275	0.7569	0.1214	0.1426
Sample iii)										
2008	0.1265	0.1182	0.1191	0.0937	1.3842	0.8024	0.7274	0.8129	0.0824	0.0858
2010	0.1485	0.1374	0.1392	0.1118	1.4150	0.9195	0.8560	09083	0.1176	0.1231
2012	0.1645	0.1496	0.1482	0.1197	1.4625	1.0158	0.9642	0.9812	0.1828	0.1258
2014	0.1753	0.1673	0.1698	0.1231	1.4913	1.1446	1.0966	1.0823	0.2377	0.1055
Sample iv)										
2008	0.1244	0.1122	0.1086	0.0872	1.3904	0.7552	0.6775	0.7635	0.0794	0.0899
2010	0.1416	0.1234	0.1311	0.1116	1.3691	0.7577	0.7044	0.7838	0.1052	0.1211
2012	0.1536	0.1231	0.1202	0.1062	1.3700	0.7482	0.6874	0.7784	0.0944	0.1257
2014	0.1561	0.1292	0.1543	0.1169	1.4228	0.7446	0.6986	0.7690	0.1182	0.1452

6.3 Sign Test

Finally, in this section we apply a sign test (Dixon and Mood, 1946) to analyze the significance of the differences in the forecasting abilities of the models. If the forecasting power of two models were similar, we would expect that the number of ages where one model anticipates mortality rates better than the other would be similar. Let $\delta_{X,T}$ be defined as follows:

$$\delta_{x,t} = \begin{cases} 1 & \text{if} \quad \left|\varepsilon_{x,t}^{SFM}\right| < \left|\varepsilon_{x,t}^{LC}\right| \\ 0 & \text{otherwise} \end{cases}$$
$$x = 0,1,...,99; \qquad t = 2007,2008,...,2014. \tag{30}$$

If both models had the same forecasting ability for a given horizon *t*, *(t=2007, 2008,… ,2014)*, then $X_t = \sum_{x=0}^{99} \delta_{X,T}$ would be a binomial random variable with parameters *N=100* and *p=0.5*, that is:

$$X_t \sim Bi(100, 0.5) \tag{31}$$

In a one-tail test the critical values of *X* are equal to *56, 58* and *62* at significance levels of 90%, *95%* and *99%*, respectively. If X_t is equal to or greater than these values that would indicate that the Single Factor Model provides significantly better estimates of future mortality rates than the competing model for each period *t (t=2007, 2008, …, 2014)*.

Figure 13: Number of items $\left|\varepsilon_{x,t}^{SFM}\right| < \left|\varepsilon_{x,t}^{LC/LC2/.}\right|$ In Spanish male population with PA, a) 2007 and b) 2014.

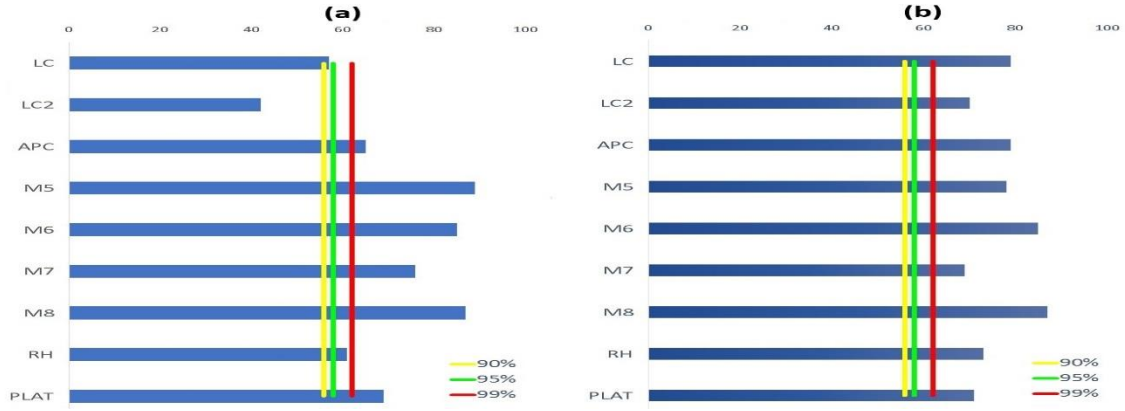
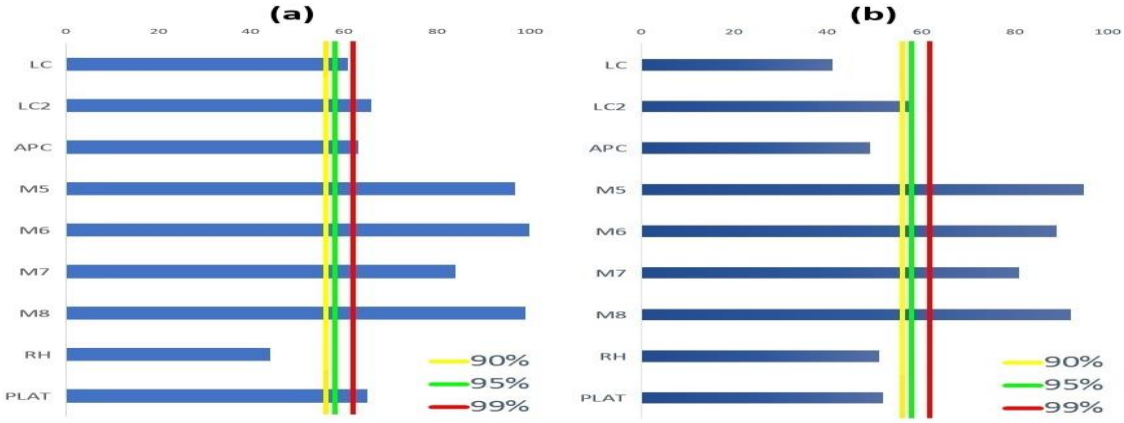


Figure 13

Number of items $\left|\varepsilon_{x,t}^{SFM}\right| < \left|\varepsilon_{x,t}^{LC/LC2/\cdot}\right|$ In French female population with SA, a) 2007 and b) 2014.



7. FINANCIAL LIKE APPROACHES IN LONGEVITY RISK.

As said in the introduction, the approach of this paper is inspired in former studies about the *TSIR*, and therefore this approach should be useful to address several financial and risk management problems involving the longevity risk. Potential examples are the estimation risk measures such as a longevity –*V@R*, the diversification of the longevity risk by investing in sectors uncorrelated with the key age, or the design, valuation and hedging of mortality derivatives in the line of Milevsky and Promislow (2001). Obviously, all of these topics are beyond the scope of this paper, though, with illustrative purposes only, we will summarize some general ideas about a longevity-*V@R* estimate.

A first approach in a longevity –*V@R* estimate may be related to Monte Carlo simulation. If the distribution of $\Delta ln\left(\hat{q}_{x*,t}\right)$ can be properly estimated then (1) allows us to simulate $\Delta ln\left(\hat{q}_{x,t}\right)$ for every age *x*, and consequently one can simulate the whole behavior of a portfolio of life-insurance linked products. In other words, a correct estimation of the distribution of $\Delta ln\left(\hat{q}_{x*,t}\right)$ for a given time horizon *T* will enable us to simulate a large sample of the global indemnification of a life insurance portfolio within this horizon *T*, and this sample will enable us to measure the portfolio risk according to several classical risk measures such as the *V@R*. This

measurement may be important to address classical problems such as the estimation of reserves or other risk management issues.

Under some assumptions the Monte Carlo simulation may be replaced by analytical approximations. For instance, following the approach of Vasicek (1987) about the estimation of a credit-*V@R* for a portfolio of loans (see also Hull 2014), one can give the approximation of a longevity-*V@R* by means of:

$$longevity - V@R_{\beta} \approx M\Phi\left(\frac{\Phi^{-1}\left(F\left(T\right)\right)+\sqrt{\rho}\Phi^{-1}\left(\beta\right)}{\sqrt{1-\rho}}\right) \qquad (32)$$

where β is the level of confidence, $M > 0$ is a parameter depending on the portfolio size, ϕ is the cumulative distribution function of the standard normal distribution, F is the cumulative distribution function of the variable "time till dead at the key age"', and ρ is a "copula correlation" connecting the variables "time till dead at age x"' and "time till dead at age y"', which is supposed to be independent of both x and y. Obviously, ideal assumptions in the line of those of Vasicek (1987) must be imposed in order to accept the expression above. If these assumptions do not hold, then one can still focus on more general credit risk approaches in order to estimate a longevity-*V@R*. In the limit case, if none of the assumptions hold in our particular problem, then, as indicated above, Monte Carlo simulation may provide us with useful numerical procedures.

8. Conclusions

The mortality evolution may be explained by a one factor model only involving a key age, since our empirical study for males and females in both France and Spain reveals that more complex factor models do not outperform our results. The advantage of this new approach seems to be clear. Indeed, the factor has nothing hidden and is totally observable, the methodology may be easily extended so as to incorporate more factors (more key ages), specific mortality causes or a cohort effect, and, according to former analyses related to the *TSIR* dynamics, our findings may complement the possibilities of a stochastic mortality modeling approach. In fact, by studying the mortality random behavior at the key age, one can address many topics in longevity risk management

such as the estimation of a longevity-*V@R*, the longevity risk diversification, or the design, valuation and hedging of mortality linked derivatives.

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