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Different estimation methods and joint confidence region for the Inverse Burr Distribution based on progressively first-failure censored sample with application to the nanodroplet data

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In this article, the point and interval estimation of parameters for an inverse Burr distribution based on progressively first-failure censored sample is studied. In point estimation, the maximum likelihood and Bayesian methods are developed for estimating the unknown parameters. An expectation-maximization algorithm is applied for computing the maximum likelihood estimators. The Bayes estimates relative to both the symmetric and asymmetric loss functions are provided using the Lindley's approximation and the Metropolis-Hastings algorithm. In interval estimation, approximate and exact confidence intervals with the exact confidence region for the two parameters have been introduced. Moreover, the proposed methods are carried out to a real data set contains the spreading of nanodroplet impingement onto a solid surface in order to demonstrate the applicabilities.

keywords: Expectation-Maximization algorithm, Exact confidence interval, Exact confidence region, Metropolis-Hastings Algorithm, Nanodroplet, Progressively first-failure censoring.

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1 Introduction

Numerous parametric models are used in problems related to the analysis and modeling of failure processes. The inverse Burr distribution (*IBurr*), which was proposed by Burr (Burr, 1942), is widely used for modeling the real data in different sciences, including economics, meteorology, agricultural and medicine. This distribution can be used as an alternative model to many well known distributions such as Weibull, gamma and lognormal. The cumulative distribution function and the probability density function of the *IBurr* distribution with the shape parameters $\alpha > 0$ and $\beta > 0$ are given by, respectively

$$F(x;\alpha,\beta) = \left(1 + x^{-\beta}\right)^{-\alpha}; \qquad x > 0, \tag{1}$$

$$f(x;\alpha,\beta) = \alpha\beta x^{-\beta-1}\left(1 + x^{-\beta}\right)^{-(\alpha+1)}; \qquad x > 0.$$

It is to be noted that the inverse Burr (or Burr III) and the Burr XII (*BXII*) distributions are related by an inverse transformation. Recent past, the *IBurr* and *BXII* distributions have been focus of investigation for many authors, see for example, Kumar (2016), Al-Moisheer (2016), Panahi and Sayyareh (2016), Cordeiro et al. (2017), Mdlongwa et al. (2017) and Panahi (2017b). Moreover, in the recent decades the nanotechnology as a clean technology has been widely applied in various disciplines of science. The spreading of a nanodroplet on solid surfaces is important for a wide range of applications, including propulsion, surface coating, spray painting, spray cooling, ink-jet printing, agricultural sprays and biological sensors. We also know that the experimental data obtained in engineering practice may be removed during the experiment (for different reasons such as destruction of materials, precision of measurement and the high cost of experiment). This reason leads us into the area of progressively censoring which enables us to remove experimental units at points other than the terminal point of the experiment (Aggarwala and Balakrishnan, 1998). The progressively censoring scheme saves time and cost of the experiment. But, the progressively first-failure (*PFF*) censoring scheme has a shorter test time and a saving of resources than the progressively censoring scheme. A *PFF* censoring scheme can be described as follows:
Suppose n units groups with k items within each group are put on test. Now at the time of the first failure ($X_{1:m:n:k}$), R_1 groups and the group in which the first failure is observed are randomly removed from the experiment. Continuing on, at the time of the second failure ($X_{2:m:n:k}$), R_2 groups are randomly removed from the experiment along with the group which contains the second failure item and finally when the m^{th} failure ($X_{m:m:n:k}$) is observed, the remaining groups are removed from the test. The observed failure $X_{1:m:n:k} < X_{2:m:n:k} < ... < X_{m:m:n:k}$ are called *PFF* censored sample with the progressive censoring scheme $R = (R_1, ..., R_m)$. The *PFF* censoring scheme includes various kinds of censoring scheme as:

- The complete sample when $k = 1, m = n, R_i = 0;\ i = 1, ..., m$.

- The progressively censoring scheme when $k = 1$.

- The Type II first failure censoring scheme when $k \neq 1, R_m = n - m, R_i = 0;\ i = 1, ..., m-1$.
- The Type II censoring scheme when $k = 1, R_m = n - m, R_i = 0;\ i = 1, ..., m-1$.

Some of the earlier works on *PFF* censoring were conducted by Mohammed et al. (2017), Dube et al. (2016), Lio and Tsai (2012), Wang and Shi (2012) and Wu and Kuş (2009). Although several literatures have been studied about the *IBurr* distribution, but we have not come across any work about the point and interval methods for estimating the parameters of the *IBurr* distribution and examine its suitability in modeling the nanodroplet data under *PFF* censoring scheme. So in this paper, the estimation problem, point and interval, have been studied. It is observed that the maximum likelihood estimators cannot be obtained in explicit forms. So, we use the expectation-maximization (EM) algorithm to compute the MLEs of the unknown parameters which involves solving two one dimensional optimization problems rather than one two dimensional problem. Since closed form expressions for the Bayes estimators cannot be obtained, we utilize two approximations, namely Lindley's approximation and Monte Carlo Markov chain procedure to obtain them. Moreover, the asymptotic confidence interval for parameters, the exact confidence interval for and the joint confidence region for the parameters are constructed. The rest of the paper is organized as follows. In Section 2, the point estimations are computed. Different interval estimations are presented in Section 3. Analyses of nanodroplet spreading data appear in section 4 and finally we conclude the paper in Section 5.

2 Different Point Estimations

2.1 Maximum Likelihood Estimation

Let $X_{1:m:n:k}, ..., X_{m:m:n:k}$ denote the *PFF* order statistics obtained from an experimental test involving n units taken from a $IBurr(\alpha, \beta)$ distribution and $(R_1, ..., R_m)$ being the censoring scheme. To simplify the notation, we will use X_i in place of $X_{i:m:n:k}$. Then the likelihood function based on *PFF* censored sample can be obtained as:

$$l(\alpha, \beta) = Ak^m \prod_{i=1}^{m} f(x_i\,|\alpha, \beta)\ [1 - F(x_i\,|\alpha, \beta)]^{k(R_i+1)-1};\ 1 \leq i \leq m \leq n \tag{2}$$

where, $A = n(n - R_1 - 1)....(n - R_1 - R_2 - ... - R_{m-1} - m + 1)$. Ignoring the additive constant, the log-likelihood function from $IBurr(\alpha, \beta)$ can be written as

$$L(\alpha, \beta) = m \ln \alpha + m \ln \beta - (\beta + 1) \sum_{i=1}^{m} \ln x_i - (\alpha + 1) \sum_{i=1}^{m} \ln(1 + x_i^{-\beta})$$
$$+ \sum_{i=1}^{m} (k(R_i + 1) - 1) \ln(1 - (1 + x_i^{-\beta})^{-\alpha}) \tag{3}$$

The MLEs of the unknown parameters can be obtained by taking derivatives with respect to α and β of (3) and putting then equal to zero. It is observed that the MLE's cannot be obtained in closed form. Thus, we propose to use the EM algorithm to compute the MLEs of α and β, treating it as a missing value problem. (see, Dempster et al., 1977). Let us denote the observed, the censored and the complete data by $X = (X_{1:m:n}, ..., X_{m:m:n})$, $Z = (Z_{11}, ..., Z_{m1}, ..., Z_{m(kR_m+k-1)})$ and $W = (X, Z)$ respectively. Therefore, the log-likelihood function $L_c = (W; \alpha, \beta)$ of the complete data after ignoring the constants can be written as:

$$L_c(W; \alpha, \beta) = nk \ln \alpha + nk \ln \beta - (\alpha + 1) \sum_{i=1}^{m} \sum_{j=1}^{k(R_i+1)-1} \ln(1 + z_{ij}^{-\beta})$$

$$-(\alpha + 1) \sum_{i=1}^{m} \ln(1 + x_i^{-\beta}) - (\beta + 1) \sum_{i=1}^{m} \ln x_i - (\beta + 1) \sum_{i=1}^{m} \sum_{j=1}^{k(R_i+1)-1} \ln z_{ij}.$$

E-step:
This step involves the computation of the conditional expectation of the log-likelihood with respect to the incomplete data given the observed data. For this purpose, we compute the pseudo log-likelihood function as:

$$L_s(\alpha, \beta) = E\left(L_c(W; \alpha, \beta) \mid X\right) = nk \ln \alpha + nk \ln \beta - (\beta + 1) \sum_{i=1}^{m} \ln x_i$$

$$-(\alpha + 1) \sum_{i=1}^{m} \ln(1 + x_i^{-\beta}) - (\beta + 1) \sum_{i=1}^{m} \sum_{j=1}^{k(R_i+1)-1} E\left[\ln Z_{ij} \mid Z_{ij} > x_i\right]$$

$$-(\alpha + 1) \sum_{i=1}^{m} \sum_{j=1}^{k(R_i+1)-1} E\left[\ln(1 + Z_{ij}^{-\beta}) \mid Z_{ij} > x_i\right], \tag{4}$$

where, $E\left[\ln Z_{ij} \mid Z_{ij} > x_i\right] = A(x_i, \alpha, \beta)$ and $E\left[\ln(1 + Z_{ij}^{-\beta}) \mid Z_{ij} > x_i\right] = B(x_i, \alpha, \beta)$ are computed in Appendix.

M-step:
This step includes the maximization of the pseudo log-likelihood function (4). Therefore, if at the s^{th} stage the estimate of (α, β) is $(\alpha^{(s)}, \beta^{(s)})$, then $(\alpha^{(s+1)}, \beta^{(s+1)})$ can be obtained by maximizing

$$L_c^*(\alpha, \beta) = nk \ln \alpha + nk \ln \beta - (\beta + 1) \sum_{i=1}^{m} \ln x_i - (\alpha + 1) \sum_{i=1}^{m} \ln(1 + x_i^{-\beta})$$

$$-(\beta + 1) \sum_{i=1}^{m} (k(R_i + 1) - 1) A(x_i, \alpha^{(s)}, \beta^{(s)})$$

$$-(\alpha+1)\sum_{i=1}^{m}(k(R_i+1)-1)B(x_i,\alpha^{(s)},\beta^{(s)}),$$

with respect to α and β. For this purpose first we evaluate $\beta^{(s+1)}$ by solving the fixed point type equation (see, Kundu and Pradhan, 2009)

$$h(\beta)=\beta, \tag{5}$$

where,

$$h(\beta)=\left[\frac{1}{nk}\sum_{i=1}^{m}\ln x_i-\frac{\hat{\alpha}(\beta)+1}{nk}\sum_{i=1}^{m}\frac{x_i^{-\beta}\ln x_i}{1+x_i^{-\beta}}+\frac{1}{nk}\sum_{i=1}^{m}(k(R_i+1)-1)A(x_i,\alpha^{(s)},\beta^{(s)})\right]^{-1}$$

$$\hat{\alpha}(\beta)=\frac{nk}{\sum_{i=1}^{m}\ln(1+x_i^{-\beta})+\sum_{i=1}^{m}(k(R_i+1)-1)B(x_i,\alpha^{(s)},\beta^{(s)})},$$

Finally after finding $\beta^{(s+1)}$ from (5), the estimate $\alpha^{(s+1)}$ is derived as $\alpha^{(s+1)}=\hat{\alpha}(\beta^{(s+1)})$. Therefore if at the s^{th} Step, the estimate of α and β are $\alpha^{(s)}$ and $\beta^{(s)}$ respectively, then the following algorithm can be used to proceed from the s^{th} Step to $(s+1)^{th}$ Step.

Algorithm:
Step 1: Maximize $L_c^*(\alpha,\beta)$ using the fixed point type equation (5). Continue the process till it converges. At the $(s+1)^{th}$ Step, the value of β, that maximizes $L_c^*(\alpha,\beta)$ is $\beta^{(s+1)}$.
Step 2: For fixed $\beta^{(s+1)}$, obtain $\alpha^{(s+1)}$ as

$$\alpha^{(s+1)}=\frac{nk}{\sum_{i=1}^{m}\ln(1+x_i^{-\beta^{(s+1)}})+\sum_{i=1}^{m}(k(R_i+1)-1)B(x_i,\alpha^{(s)},\beta^{(s)})}.$$

Step 3: Check the convergence of $(\alpha^{(s+1)},\beta^{(s+1)})$, if the convergence is met then the current $\alpha^{(s+1)}$ and $\beta^{(s+1)}$ are the maximum likelihood estimates of α and β via EM algorithm; otherwise, order $s=s+1$ and go back to Step 1.

2.2 Bayesian Estimation

In this section, we compute the Bayes estimates of the unknown parameters of the $IBurr(\alpha,\beta)$ distribution under the squared error (SE) and Linex (LI) loss functions. If θ is the parameter to be estimated by an estimator $\hat{\theta}$, then the SE and LI loss functions can be defined by

$$L_{SE}(\theta,\hat{\theta})=(\hat{\theta}-\theta)^2,$$

and

$$L_{LI}(\theta,\hat{\theta})=e^{w(\hat{\theta}-\theta)}-w(\hat{\theta}-\theta)-1,\ \ \theta\neq 0.$$

respectively. The Bayes estimates of θ with respect to the SE and LI loss functions are obtained as the posterior mean and $-\frac{1}{w}\ln\left[E_\theta(e^{-w\theta}\,|data)\right]$ respectively. The independent prior distributions for the α and β are taken to be $Gamma(a,b)$ and $Gamma(c,d)$ respectively. So, the joint prior distribution of α and β is of the form

$$\pi(\alpha,\beta) \propto \alpha^{a-1}e^{-b\alpha}\beta^{c-1}e^{-d\beta};\ \alpha>0,\ \beta>0,\ a>0,\ b>0,\ c>0,\ d>0 \tag{6}$$

So, the the joint posterior distribution of α and β is derived as

$$\pi(\alpha,\beta\,|data) = \frac{\pi(\alpha,\beta)L(\alpha,\beta)}{\int\limits_0^\infty\int\limits_0^\infty \pi(\alpha,\beta)L(\alpha,\beta)d\alpha d\beta} = A\alpha^{m+a-1}e^{-\alpha(b+\sum\limits_{i=1}^m \ln(1+x_i^{-\beta}))}\beta^{m+c-1}$$

$$\times e^{-\beta(d+\sum\limits_{i=1}^m \ln x_i)}\prod_{i=1}^m\left(1-(1+x_i^{-\beta})^{-\alpha}\right)^{k(R_i+1)-1}(1+x_i^{-\beta})^{-1}.$$

where A is the normalizing constant. Therefore, the Bayes estimate of α and β under SE loss function are given, respectively, by

$$\hat{\alpha}_{SE} = E(\alpha\,|data) = A\int\limits_0^\infty\int\limits_0^\infty \alpha^{m+a}e^{-\alpha(b+\sum\limits_{i=1}^m \ln(1+x_i^{-\beta}))}\beta^{m+c-1}$$

$$\times e^{-\beta(d+\sum\limits_{i=1}^m \ln x_i)}\prod_{i=1}^m\left(1-(1+x_i^{-\beta})^{-\alpha}\right)^{k(R_i+1)-1}(1+x_i^{-\beta})^{-1}d\alpha d\beta. \tag{7}$$

and

$$\hat{\beta}_{SE} = E(\beta\,|data) = A\int\limits_0^\infty\int\limits_0^\infty \alpha^{m+a-1}e^{-\alpha(b+\sum\limits_{i=1}^m \ln(1+x_i^{-\beta}))}\beta^{m+c}$$

$$\times e^{-\beta(d+\sum\limits_{i=1}^m \ln x_i)}\prod_{i=1}^m\left(1-(1+x_i^{-\beta})^{-\alpha}\right)^{k(R_i+1)-1}(1+x_i^{-\beta})^{-1}d\alpha d\beta. \tag{8}$$

Furthermore, the desired Bayes estimator of α and β under LI loss function are derived as

$$\hat{\alpha}_{LI} = \frac{-1}{w}\ln\left[E(e^{-w\alpha}\,|data)\right] = \frac{-1}{w}\ln\{A\int\limits_0^\infty\int\limits_0^\infty \alpha^{m+a-1}e^{-\alpha(w+b+\sum\limits_{i=1}^m \ln(1+x_i^{-\beta}))}\beta^{m+c-1}$$

$$\times e^{-\beta(d+\sum\limits_{i=1}^m \ln x_i)}\prod_{i=1}^m\left(1-(1+x_i^{-\beta})^{-\alpha}\right)^{k(R_i+1)-1}(1+x_i^{-\beta})^{-1}d\alpha d\beta\,\}. \tag{9}$$

and

$$\hat{\beta}_{LI} = \frac{-1}{w}\ln\left[E(e^{-w\beta}\,|data)\right] = \frac{-1}{w}\ln\{A\int_0^\infty\int_0^\infty \alpha^{m+a-1} e^{-\alpha(b+\sum_{i=1}^{m}\ln(1+x_i^{-\beta}))}\beta^{m+c-1}$$

$$\times e^{-\beta(w+d+\sum_{i=1}^{m}\ln x_i)}\prod_{i=1}^{m}(1-(1+x_i^{-\beta})^{-\alpha})^{k(R_i+1)-1}(1+x_i^{-\beta})^{-1}d\alpha d\beta\;\}. \tag{10}$$

It is clear that Equations 7-10 cannot be obtained analytically. Therefore, we consider Lindley's approximation and Monte Carlo Markov chain Method to compute Bayes estimates.

2.2.1 Lindley's Method

Lindley (Lindley, 1980) proposed the procedure to approximate the ratio of two integrals such as Equations 7-10. Utilizing the Lindley's method, the Bayesian estimates (BLindley) of α under SE and LI loss functions can be written as:

$$\hat{\alpha}_{Lindley-SE} = \hat{\alpha} + \frac{1}{2}\left[(2(\frac{a-1}{\hat{\alpha}}-b)\;\hat{\sigma}_{\alpha\alpha} + 2(\frac{c-1}{\hat{\beta}}-d)\hat{\sigma}_{\alpha\beta} + \hat{\sigma}^2_{\alpha\alpha}\hat{L}_{\alpha\alpha\alpha}\right.$$

$$\left. +\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\beta\beta\alpha}\; + 2\hat{\sigma}_{\alpha\beta}\hat{\sigma}_{\beta\alpha}\hat{L}_{\alpha\beta\beta} + \hat{\sigma}_{\alpha\beta}\hat{\sigma}_{\beta\beta}\hat{L}_{\beta\beta\beta})\right],$$

and

$$\hat{\alpha}_{Lindley-LI} = e^{-c\hat{\alpha}} + \frac{1}{2}\left[c^2 e^{-c\hat{\alpha}}\hat{\sigma}_{\alpha\alpha} + -ce^{-c\hat{\alpha}}(2(\frac{a-1}{\hat{\alpha}}-b)\;\hat{\sigma}_{\alpha\alpha} + 2(\frac{c-1}{\hat{\beta}}-d)\hat{\sigma}_{\alpha\beta}\right.$$

$$\left. +\hat{\sigma}^2_{\alpha\alpha}\hat{L}_{\alpha\alpha\alpha} + \hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\beta}\hat{L}_{\beta\beta\alpha}\; + 2\hat{\sigma}_{\alpha\beta}\hat{\sigma}_{\beta\alpha}\hat{L}_{\alpha\beta\beta} + \hat{\sigma}_{\alpha\beta}\hat{\sigma}_{\beta\beta}\hat{L}_{\beta\beta\beta})\right],$$

respectively. Similarly, the Bayesian estimates of β under SE and LI loss functions are:

$$\hat{\beta}_{Lindley-SE} = \hat{\beta} + \frac{1}{2}\left[(2(\frac{c-1}{\hat{\beta}}-d)\;\hat{\sigma}_{\beta\beta} + 2(\frac{a-1}{\hat{\alpha}}-b)\hat{\sigma}_{\beta\alpha}\right.$$

$$\left. +\hat{\sigma}^2_{\beta\beta}\hat{L}_{\beta\beta\beta} + 3\hat{\sigma}_{\beta\beta}\hat{\sigma}_{\alpha\beta}\hat{L}_{\alpha\beta\beta} + \hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\alpha}\hat{L}_{\alpha\alpha\alpha})\right],$$

and

$$\hat{\beta}_{Lindley-LI} = e^{-c\hat{\beta}} + \frac{1}{2}\left[c^2 e^{-c\hat{\beta}}\hat{\sigma}_{\beta\beta} + -ce^{-c\hat{\beta}}(2(\frac{c-1}{\hat{\beta}}-d)\;\hat{\sigma}_{\beta\beta} + 2(\frac{a-1}{\hat{\alpha}}-b)\hat{\sigma}_{\beta\alpha}\right.$$

$$+\hat{\sigma}_{\beta\beta}^2\hat{L}_{\beta\beta\beta}+3\hat{\sigma}_{\beta\beta}\hat{\sigma}_{\alpha\beta}\hat{L}_{\alpha\beta\beta}+\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\beta\alpha}\hat{L}_{\alpha\alpha\alpha})\Big],$$

respectively. Also, $\hat{L}_{\alpha^n\beta^m}=\partial^{n+m}L(\alpha,\beta)/\partial\alpha^n\partial\beta^m;\ n,m=0,1,...$ and $\hat{\sigma}_{ij}$ are the $(ij)^{th}$ elements of matrix $\left[-\partial^2L(\alpha,\beta)/\partial\alpha\partial\beta\right]^{-1};\ i,j=1,2$. In our problem we have

$$\hat{\sigma}_{\alpha\alpha}=\frac{\hat{L}_{\beta\beta}}{\hat{L}_{\alpha\alpha}\hat{L}_{\beta\beta}-\hat{L}_{\alpha\beta}^2},\ \ \hat{\sigma}_{\beta\beta}=\frac{\hat{L}_{\alpha\alpha}}{\hat{L}_{\alpha\alpha}\hat{L}_{\beta\beta}-\hat{L}_{\alpha\beta}^2},\ \ \hat{\sigma}_{\alpha\beta}=\hat{\sigma}_{\beta\alpha}=-\frac{\hat{L}_{\alpha\beta}}{\hat{L}_{\alpha\alpha}\hat{L}_{\beta\beta}-\hat{L}_{\alpha\beta}^2}.$$

$$\hat{L}_{\alpha\beta}=\sum_{i=1}^{m}\frac{x_i^{-\beta}\ln x}{1+x_i^{-\beta}}+\sum_{i=1}^{m}k(R_i+1)-1)(\frac{\alpha x_i^{-\beta}(\ln x_i)(1+x_i^{-\beta})^{-\alpha-1}\ln(1+x_i^{-\beta})}{1-(1+x_i^{-\beta-\alpha})}$$

$$-\frac{(1+x_i^{-\beta})^{-\alpha-1}x_i^{-\beta}\ln x_i}{1-(1+x_i^{-\beta})^{-\alpha}}+\frac{\alpha x^{-\beta}(\ln x_i)(1+x_i^{-\beta})^{-2\alpha-1}\ln(1+x_i^{-\beta})}{(1-(1+x_i^{-\beta})^{-\alpha})^2}),$$

$$\hat{L}_{\alpha\alpha}=-\frac{m}{\alpha^2}-\sum_{i=1}^{m}(k(R_i+1)-1)\frac{(1+x_i^{-\beta})^{-\alpha}\ln^2(1+x_i^{-\beta})}{(1-(1+x_i^{-\beta})^{-\alpha})^2},$$

$$\hat{L}_{\beta\beta}=-\frac{m}{\beta^2}-(\alpha+1)\sum_{i=1}^{m}\frac{x_i^{-\beta}\ln^2x_i}{(1+x_i^{-\beta})^2}-\sum_{i=1}^{m}(k(R_i+1)-1)$$

$$\times(\frac{\alpha x_i^{-\beta}\ln^2x_i(1+x_i^{-\beta})^{-\alpha-1}\widehat{\Re}_i}{1-(1+x_i^{-\beta})^{-\alpha}}+\frac{\alpha^2x_i{}^{-2\beta}\ln^2x_i(1+x_i^{-\beta})^{-2\alpha-2}}{(1-(1+x_i^{-\beta})^{-\alpha})^2}),$$

$$\widehat{\Re}_i=[(\alpha+1)(1+x_i^{-\beta})^{-1}x_i^{-\beta}-1].$$

$$\hat{L}_{\alpha\alpha\alpha}=\frac{2m}{\alpha^3}+\sum_{i=1}^{m}(k(R_i-1)-1)(\frac{2(1+x_i^{-\beta})^{-2\alpha}\ln^3(1+x_i^{-\beta})}{(1-(1+x_i^{-\beta})^{-\alpha})^3}$$

$$+\frac{(1+x_i^{-\beta})^{-2\alpha}\ln^3(1+x_i^{-\beta})}{(1-(1+x_i^{-\beta})^{-\alpha})^3}),$$

$$\hat{L}_{\alpha\alpha\beta}=-\sum_{i=1}^{m}(k(R_i-1)-1)(\frac{-2\alpha(1+x_i^{-\beta})^{-2\alpha-1}x_i^{-\beta}\ln(x_i)\ln^2(1+x_i^{-\beta})}{(1-(1+x_i^{-\beta})^{-\alpha})^3}$$

$$+\frac{\alpha(1+x_i^{-\beta})^{-\alpha-1}x_i^{-\beta}\ln(x_i)\ln^2(1+x_i^{-\beta})}{(1-(1+x_i^{-\beta})^{-\alpha})^2}-\frac{2(1+x_i^{-\beta})^{-\alpha-1}x_i^{-\beta}\ln(x_i)\ln(1+x_i^{-\beta})}{(1-(1+x_i^{-\beta})^{-\alpha})^2}),...$$

2.2.2 Monte Carlo Markov chain Method

In this subsection, we apply the Metropolis-Hastings algorithm to compute the Bayes estimates. We generate random samples from the posterior distribution of α and β using the Monte Carlo Markov chain and Metropolis-Hastings (M-H) algorithm (Metropolis et al. (1953) and Hastings (1970)). We consider normal as the proposal distribution for α and β with prescribed means and variances. we use the following algorithm to compute the Bayes estimates.

Step 1: Choose $(\alpha^{(0)}, \beta^{(0)})$ as an initial value for stating the algorithm.

Step 2: Using M-H generate $\alpha^{(j)}$ and $\beta^{(j)}$ from $Normal(\alpha^{(j-1)}, \nu_\alpha)$ and $Normal(\beta^{(j-1)}, \nu_\beta)$ respectively.

Step 3: Evaluate the acceptance probabilities

$$P = \min\left[1, \frac{\pi(\alpha^{(j)}, \beta^{(j)}\,|data)}{\pi(\alpha^{(j-1)}, \beta^{(j-1)}\,|data)}\right].$$

Step 4: Repeat steps 2-3, M times.
Here, for remove the affection of the selection of initial values, the first M_0 samples have been burned-in. Finally based on this method, the Bayes estimates of α under SE as well as LI loss functions can be obtained as

$$\hat{\alpha}_{MH-SE} = \sum_{i=1}^{M-M_0} \frac{\alpha_i}{M-M_0}, \quad \hat{\beta}_{MH-SE} = \sum_{i=1}^{M-M_0} \frac{\beta_i}{M-M_0} \tag{11}$$

and

$$\hat{\alpha}_{MH-LI} = -\frac{1}{w}\ln\left[\sum_{i=1}^{M-M_0} \frac{e^{-w\alpha_i}}{M-M_0}\right], \quad \hat{\beta}_{MH-LI} = -\frac{1}{w}\ln\left[\sum_{i=1}^{M-M_0} \frac{e^{-w\beta_i}}{M-M_0}\right], \tag{12}$$

For more details about the MCMC technique, see, for example, Sel et al. (2018), Mohammed et al. (2017) and Panahi (2017a).

3 Interval Estimation

In this section, the asymptotic confidence interval, the exact confidence interval for β and the exact confidence region for the unknown parameters have been studied.

3.1 Asymptotic Confidence Interval

Let $X_{1:m,n}, ..., X_{m:m,n}$ denote the corresponding lifetimes from *IBurr* distribution. The observed Fisher information matrix $I_X(\theta)$ is obtained by using the missing information principle of Louis (Louis, 1982) as:

$$I_X(\theta) = I_W(\theta) - I_{W|X}(\theta), \tag{13}$$

where, $I_W(\theta)$ and $I_{W|X}(\theta)$ denote the complete and the missing information matrix respectively. It is to be noted that $I_W(\theta)$ can be expressed as:

$$I_W(\theta) = -E\left[\frac{\partial^2 L_c(W;\theta)}{\partial\theta^2}\right] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where,

$$a_{11} = \frac{nk}{\alpha^2},\ a_{12} = a_{21} = -nk\alpha\beta\int_0^\infty x^{-2\beta-1}\left(1+x^{-\beta}\right)^{-\alpha-2}\ln x dx,$$

$$a_{22} = \frac{nk}{\beta^2} + n\alpha\beta(\alpha+1)\int_0^\infty x^{-2\beta-1}\left(1+x^{-\beta}\right)^{-\alpha-2}\ln^2 x dx.$$

Also, the Fisher information in one observation is given by

$$I^{(i)}_{W|X}(\theta) = -E_{Z_{ij}|X_i}\left[\frac{\partial^2 \ln f_{Z_{ij}}(z_{ij}\,|x_i,\ \theta)}{\partial\theta^2}\right] = \begin{bmatrix} b_{11}(x_j;\alpha,\beta) & b_{12}(x_j;\alpha,\beta) \\ b_{21}(x_j;\alpha,\beta) & b_{22}(x_j;\alpha,\beta) \end{bmatrix}$$

where, $b_{11}(x_i;\alpha,\beta)$, $b_{12}(x_i;\alpha,\beta) = b_{21}(x_i;\alpha,\beta)$ and $b_{22}(x_i;\alpha,\beta)$ can be computed using the conditional distribution of $Z_{ij}\,|X_i = x_i$ which is given by

$$f_{Z|X}(z_{ij}\,|X_i = x_i) = \frac{f(z_{ij}\,|\alpha,\beta)}{1-F(x_i\,|\alpha,\beta)};\ z_{ij} > x_i, j = 1, ..., k(R_i+1)-1, i = 1, ..., m.$$

Thus, the total missing information is given by

$$I_{W|X}(\theta) = \sum_{i=1}^{m}(k(R_i+1)-1)I^{i}_{W|X}(\theta)$$

So, the $100(1-\gamma)\%$ asymptotic confidence intervals (ACI) for α and β can be obtained respectively, by

$$\hat\alpha \pm Z_{\gamma/2}\sqrt{\nu(\hat\alpha)} \quad \textit{and} \quad \hat\beta \pm Z_{\gamma/2}\sqrt{\nu(\hat\beta)}. \tag{14}$$

where $\nu(\hat\alpha)$ and $\nu(\hat\beta)$ are the elements on the main diagonal of $I_X^{-1}(\hat\theta)\,\big|_{\theta=(\alpha,\beta)}$ and $z_{\gamma/2}$ is the standard normal variate.

3.2 Exact confidence interval

Suppose that $X_{1:m:n:k}, ..., X_{m:m:n:k}$ is a *PFF* censored sample from the *IBurr* distribution. Let $Q^*_{i:m:n:k} = k\alpha \ln(1 + x^{-\beta}_{i:m:n:k})$; $i = 1, ..., m$. It is clear that the $Q^*_{1:m:n:k} > Q^*_{2:m:n:k} > ... > Q^*_{m:m:n:k}$ is a progressively censored order statistics from an exponential distribution with mean 1. Now we utilize the following transformation:

$$Y_1 = nQ^*_{m:m:n:k}, Y_2 = (n - R_1 - 1)(Q^*_{m-1:m:n:k} - Q^*_{m:m:n:k}), ...,$$

$$Y_m = (n - R_1 - ... - R_{m-1} - m + 1)(Q^*_{1::m:n:k} - Q^*_{2:m:n:k})$$

The generalized spacings $Y_1, Y_2, ..., Y_m$ are independent and identically distributed as an exponential distribution with mean 1. Using the independent property of $\eta = 2Y_1 \sim \chi^2(2)$ and $\upsilon = 2\sum_{i=2}^{m} Y_i \sim \chi^2(2m - 2)$, we can introduce the following pivotal quantities:

$$U = \frac{\upsilon}{(m-1)\eta} = \frac{\sum_{i=1}^{m} (R_i + 1)(Q^*_{i:m:n:k} - Q^*_{m:m:n:k})}{n(m-1)Q^*_{m:m:n:k}}, \tag{15}$$

and

$$V = 2\sum_{i=1}^{m} (R_i + 1)Q^*_{i:m:n:k} = 2k\alpha \sum_{i=1}^{m} (R_i + 1) \ln(1 + x^{-\beta}_{i:m:n:k}). \tag{16}$$

We know that that U has an F distribution with $(2(m - 1), 2)$ degrees of freedom and V has chi-squared distribution with $2m$ degrees of freedom. Also, U and V are independent.

Lemma: For a given sample $X_{1:m:n:k}, ..., X_{m:m:n:k}$, the U is a strictly increasing function of β. Also, the following equation has a unique solution for any $\beta > 0$.

$$U(\beta) = \frac{\sum_{i=1}^{m} (R_i + 1)(\frac{\ln(1+x^{-\beta}_{i:m:n:k})}{\ln(1+x^{-\beta}_{m:m:n:k})} - 1)}{n(m-1)} = a$$

Proof: It is clear that $\frac{\ln(1+x^{-\beta}_{i:m:n:k})}{\ln(1+x^{-\beta}_{m:m:n:k})}$ is a strictly increasing function of β when $\beta > 0$ (see, Wu and Kuş (2009)). So, $U(\beta)$ is a strictly increasing function of β. Moreover, we have

$$\lim_{\beta \to 0} U(\beta) = 0 \quad and \quad \lim_{\beta \to \infty} U(\beta) = \infty.$$

Therefore, $U(\beta) = a$; $a > 0$ has a unique solution for some $\beta > 0$.

3.2.1 Exact confidence interval for β

In this Subsection, we introduce the $100(1-\gamma)\%$ exact confidence interval for β using the following Theorems.

Theorem 1: Suppose that $X_{1:m:n:k},...,X_{m:m:n:k}$ is a *PFF* censored sample from a cumulative density function (1). Then for any $0<\gamma<1$,

$$\left[\phi\left(X_{1:m:n:k},...,X_{m:m:n:k},F_{1-\gamma/2}(2m-2,2)\right),\phi\left(X_{1:m:n:k},...,X_{m:m:n:k},F_{\gamma/2}(2m-2,2)\right)\right], \tag{17}$$

is a $100(1-\gamma)\%$ exact confidence interval for β, where, $F_\gamma(\iota_1,\iota_2)$ denote the upper percentile of F distribution with ι_1 and ι_2 degrees of freedom and $\phi\left(X_{1:m:n:k},...,X_{m:m:n:k},a\right)$ is a solution of β for the following equation

$$\frac{\sum_{i=1}^{m}(R_i+1)\left(\frac{\ln(1+x_{i:m:n:k}^{-\beta})}{\ln(1+x_{m:m:n:k}^{-\beta})}-1\right)}{n(m-1)}=a.$$

Proof: The proof follows obviously from the Lemma and equation (14).

3.2.2 Joint Confidence Region for the Parameters

The following Theorem has been used for constructing the $100(1-\gamma)\%$ joint confidence region for α and β.
Theorem 2: Let $X_{1:m:n:k},...,X_{m:m:n:k}$ be a *PFF* censored sample. Then the $100(1-\gamma)\%$ joint confidence region is given by:

$$\phi\left(X_{1:m:n:k},...,X_{m:m:n:k},F_{(1+\sqrt{1-\gamma})/2}(2m-2,2)\right)<\beta$$
$$<\phi\left(X_{1:m:n:k},...,X_{m:m:n:k},F_{(1-\sqrt{1-\gamma})/2}(2m-2,2)\right)$$
$$\frac{\chi^2_{(1+\sqrt{1-\gamma})/2}(2m)}{2k\alpha\sum_{i=1}^{m}(R_i+1)\ln(1+x_{i:m:n:k}^{-\beta})}<\alpha<\frac{\chi^2_{(1-\sqrt{1-\gamma})/2}(2m)}{2k\alpha\sum_{i=1}^{m}(R_i+1)\ln(1+x_{i:m:n:k}^{-\beta})}$$

Proof: Based on equations (14) and (15) and using the independent property of the quantities U and V, the proof is straight forward.

4 Nanodroplet Data Analysis

The data set consists of the spreading of nanodroplet after impact with a solid surface. Spreading of nanodroplets on solid surfaces is important in a wide variety of applications, including ink jet printing, DNA synthesis and etc. The quality of nano coated surface depends on the spreading of nanodroplets. For this reason, nanodroplet spreading data impacting on non-wettable surface has been used in this paper. The data is obtained by the computational fluid dynamics and molecular kinetic theory (CFD-MK) method.

In this method the volume-of-fluid (VOF) technique is used to track the free-surface of the nanodroplet. Molecular scale around the fluid solid contact line is simulated by molecular kinetic theory equation. Dynamic contact angles are applied as a boundary condition at the liquid- solid contact lines. The droplet with diameter 6nm has been considered. The wetting and spreading of this droplet can be examined on surfaces with high to low surface energies. This variation resulted in three different surfaces are wettable, partially wettable and non-wettable when the static contact angle are less than 40 ($\theta_0 < 40$), between 40 and 140 ($40 < \theta_0 < 140$) and greater than 140 ($\theta_0 > 140$) respectively. Figure 1 presents the images of a nanodroplet impacting with a velocity of 1.25 m/s on a flate surface (Asadi, 2012).

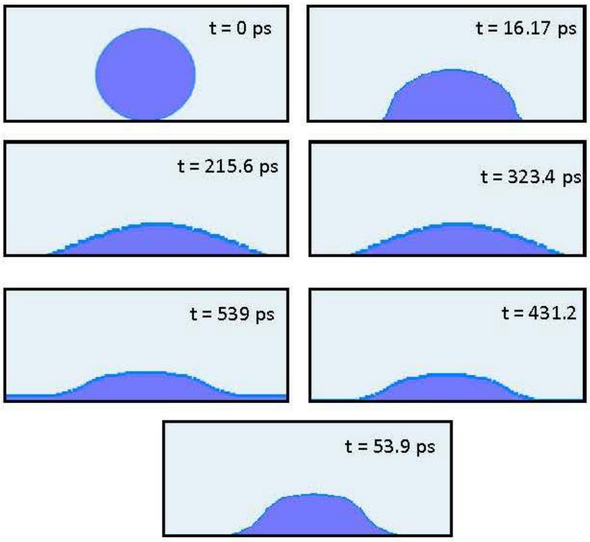

Figure 1: Images of a 6 nm droplet impacting with velocity of 1.25 m/s on a flat wettable surface.

The data are presented in Table 1. Before progressing, we have fitted the *IBurr* distribution to the uncensored data set and compute the K-S distance between the empirical and fitted distribution functions, it is 0.0803 and the corresponding p-value is 0.821. The high p-value indicates the reasonability of the *IBurr* model for fitting to this data. We have just presented the PP plot in Figure 2. We also want to check the fitting of the proposed data set for the *IBurr* model over the inverse gamma (*IG*), inverse Weibull (*IW*), inverse Lindley (*IL*) and inverse exponential (*IE*) models.

Table 2 lists the values of different adaptive measures for model discrimination, such as Akaike information criterion (*C1*), Bayesian information criterion (*C2*), Kolmogrov-Smirnov statistic (*S1*) and Cramer-von Mises statistic (*S2*).

These results show that the *IBurr* can be chosen as the best model more than any other proposed models. We further compute the chi-square goodness of fit for testing the following null hypothesis:

Table 1: Nanodroplet spreading data.

0.2289300	0.5810291	0.6935846	0.7221355	0.7357869	0.7389012
0.7486177	0.7491848	0.7688918	0.7689745	0.7857656	0.7882443
0.7962973	0.7972708	0.8094872	0.8342509	0.8451560	0.8527647
0.8744825	0.8832821	0.8905104	0.8928568	0.9603346	0.9624409
0.9677539	0.9792698	0.9926678	1.0297182	1.0890227	1.0972401
1.1235326	1.1559192	1.1755080	1.1764967	1.1836366	1.1975052
1.2171928	1.2456470	1.2475189	1.3245510	1.3485822	1.3796668
1.3932774	1.4432065	1.4697339	1.4974976	1.5356593	1.5375506
1.5426158	1.5430411	1.5679230	1.6287098	1.6744305	1.6838840
1.7235515	1.7685406	1.7980336	1.8073133		

Table 2: Values of different adaptive measures of model discrimination.

Models	*C1*	*C2*	*S1*	*S2*
$IBurr$	55.0184	59.1393	0.08026	0.11334
IG	68.8039	72.9248	0.11649	0.13208
IW	91.9350	96.0559	0.20527	0.51772
IL	125.8046	127.8651	0.38829	2.01735
IE	133.8998	135.9602	0.41932	2.39897

H_0 : The data are from the *IBurr* distribution.
In order to apply chi-square goodness of fit to this data set, the observed frequencies and the expected frequencies are presented in Table 3.

Table 3: Observed and expected frequencies and chi-squared statistic.

Intervals	*Observed* (O)	*Expected* (E)	$(O-E)^2/E$	χ^2
0.00-0.769	10	9.6686	0.01136	
0769-1.026	17	17.3747	0.00808	
1.026-1.283	12	14.8821	0.55814	1.18422
1.283-1.540	9	8.0766	0.10556	
$1.540-\infty$	10	7.9981	0.50108	

It is observed that the calculated chi-square (1.18422) is less than the critical value $\chi^2_{(2,0.095)} = 5.9915$. So, the null hypothesis cannot be rejected and the *IBurr* can be used reasonably good to obtain inferential results from the proposed data. Now we have created the *PFF* censored sample of size 19 out of 29 groups with 2 items within each group of nanodroplet data as:

$$n = 29,\ k = 2,\ m = 19,\ R_1 = R_2 = R_5 = R_7 = R_9 = 2;\ R_i = 0; i \neq 1, 2, 5, 7, 9$$

For this example, 10 groups of nanodroplet data are censored, and 19 first failures are observed.

Based on the above censored samples, we have estimated the unknown parameters using the MLEs and the Bayes estimators. To compute the MLEs of parameters,we have used the EM algorithm as described in Subsection 2.1 and stopped the iterative process when the difference between two consecutive iterates is less than 10^{-6}. For estimating the unknown parameters using the Bayes estimators, we have assumed non-informative priors, *i.e.*, $a = b = c = d = 0$. For obtaining the Bayesian estimates using the MCMC method (BMCMC), we generated the Markov chain with 10^4 observations and taken the corresponding maximum likelihood estimations for α and β as an initial values for running the algorithm. The results of different estimators for complete and censored data are presented in Table 4.

Furthermore, we obtain the Kolmogrov-Smirnov statistic between the empirical distribution function and the *IBurr* distribution functions based on MLEs and BMCMC. The results are presented in Table 5.
Based on the Table 5, it is observed that the *IBurr* provides a good fit for both complete and *PFF* censored samples.
Now, we consider the interval estimations of the parameters. First, we obtain the 95%

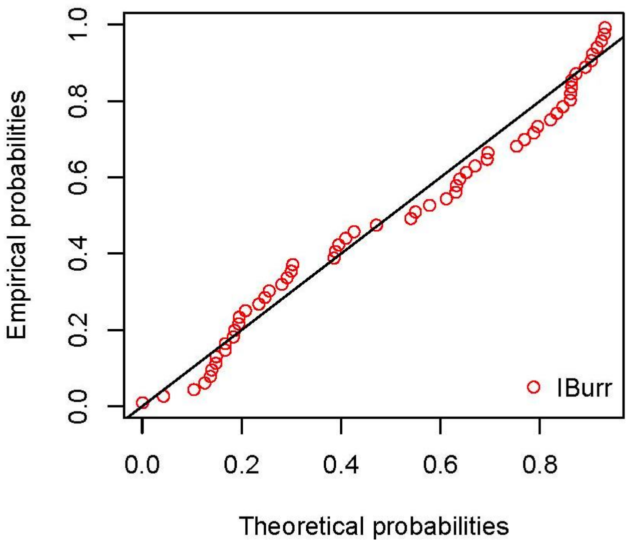

Figure 2: The PP plot for the nanodroplet data.

Table 4: Different estimations for the parameters for α and β.

Data	*Parameters*	*MLE*	*BLindley*	*BMCMC*
	α	1.20228	1.19663	1.21177
			(1.1883)	(1.2105)
Complete data				
	β	4.70108	4.68622	4.72102
			(4.6548)	(4.7035)
	α	1.18435	1.14456	1.19813
			(1.1276)	(1.1898)
Censored data				
	β	4.57145	4.51758	4.61012
			(4.4938)	(4.5973)

Note: In each cell of the two last columns, the first and second rows represent the Bayesian estimations based on the SE and LI loss functions.

Table 5: Kolmogrov-Smirnov distances and corresponding p-values for MLEs and BMCMC.

Data	*Methods*	*Kolmogrov-Smirnov*	*p-values*	*-Log-likelihood*
Complete data	MLE	0.0803	0.8200	25.5092
	BMCMC	0.0802	0.8207	25.5126
Censored data	MLE	0.1513	0.7774	12.4662
	BMCMC	0.1512	0.7778	12.5888

asymptotic intervals of α and β under *PFF* censored sample, they are (0.35426, 2.0175) and (3.88273, 9.01584) respectively. Now, we consider the 95% exact confidence interval of β and the 95% joint confidence region for α and β. For this purpose, we need the following percentiles:

$$F_{0.0127}(114, 2) = 0.220375, F_{0.9873}(114, 2) = 78.23032, F_{0.025}(36, 2) = 0.24426,$$

$$F_{0.975}(36, 2) = 39.470, F_{0.0127}(36, 2) = 0.202379, F_{0.9873}(36, 2) = 78.2113,$$

$$\chi^2_{(0.0127)}(116) = 84.646, \chi^2_{(0.9873)}(116) = 152.674,$$

$$\chi^2_{(0.0127)}(38) = 21.221, \chi^2_{(0.9873)}(38) = 60.083.$$

According to the equation (16), the 95% confidence interval of β under *PFF* censored is (3.27378, 10.35218). Furthermore, the 95% joint confidence region for α and β are given in two different cases as:

Case 1 (complete data,$R_i = 0$):

$$\begin{cases} 4.669079 < \beta < 13.35324 \\ \frac{84.646}{2\sum_{i=1}^{58}(R_i+1)\ln(1+x_{i:m:n:k}^{-\beta})} < \alpha < \frac{152.674}{2\sum_{i=1}^{58}(R_i+1)\ln(1+x_{i:m:n:k}^{-\beta})}, \end{cases}$$

and
Case 2 (censored data):

$$\begin{cases} 3.044495 < \beta < 11.36941 \\ \frac{21.221}{2k\sum_{i=1}^{19}(R_i+1)\ln(1+x_{i:m:n:k}^{-\beta})} < \alpha < \frac{60.083}{2k\sum_{i=1}^{19}(R_i+1)\ln(1+x_{i:m:n:k}^{-\beta})}. \end{cases}$$

Figure 3 show the 95% joint confidence regions of α and β based on *PFF* censored sample. It is observed that the region is small when β is large.

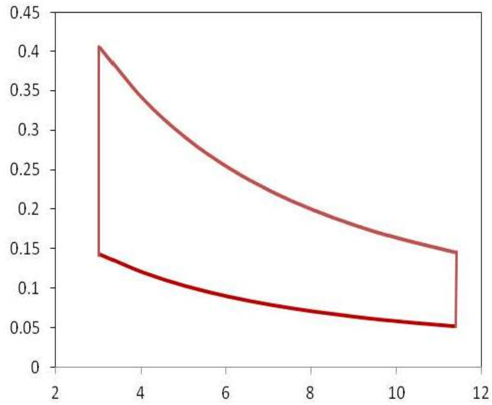

Figure 3: Joint confidence region for α and β under the *PFF* censored data.

5 Conclusions

In this article, the point estimations of the unknown parameters of the *IBurr* have been earned using the EM algorithm, BLindley and BMCMC (M-H algorithm) under *PFF* censored sample. In interval estimation viewpoint, we computed the asymptotic confidence intervals of the unknown parameters using the missing information principle and also introduced two pivotal quantities for constructing an exact confidence interval and an exact confidence region for the parameters based on *PFF* censoring scheme. The developed methods are also applied to a real data set based on the spreading of nanodroplet after impact with a solid surface. Using different graphical methods and testing, it is found that the *IBurr* distribution is suitable for the nanodroplet spreading data in contrast to any other proposed distribution.

Appendix:
From Ng et al. (2002), we have

$$f_{Z|X}(z_j\,|X_1 = x_1\,, ..., X_i = x_i) = f_{Z|X}(z_{ij}\,|X_i = x_i)$$

$$= \frac{f(z_{ij}\,|\alpha, \beta)}{1 - F(x_i\,|\alpha, \beta)};\ z_{ij} > x_i, j = 1, ..., k(R_i + 1) - 1, i = 1, ..., m$$

Now, we can write

$$E\,(\ln Z_{ij}\,|Z_{ij} > d) = \frac{\alpha\beta}{1 - F(d\,|\alpha, \beta)} \int_d^\infty \ln(x) x^{-\beta-1}(1 + x^{-\beta})^{-\alpha-1} dx,\ put\ \ x = d/y$$

$$= \frac{\alpha\beta}{d^{\beta}(1 - F(d\,|\alpha, \beta))} \int_{0}^{1} \ln(d/y) y^{-\alpha\beta-1} (y^{-\beta} + d^{-\beta})^{-\alpha-1} dy$$

and

$$E\left(\ln(1 + Z_{ij}^{-\beta})\,|Z_{ij} > d\right) = \frac{\alpha\beta}{1 - F(d\,|\alpha, \beta\,)} \int_{d}^{\infty} \ln(1 + x^{-\beta}) x^{-\beta-1} (1 + x^{-\beta})^{-\alpha-1} dx$$

$$= \frac{\alpha\beta}{d^{\beta}(1 - F(d\,|\alpha, \beta\,))} \int_{0}^{1} \ln(1 + (\frac{d}{y})^{-\beta}) y^{-\alpha\beta-1} (y^{-\beta} + d^{-\beta})^{-\alpha-1} dy.$$

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