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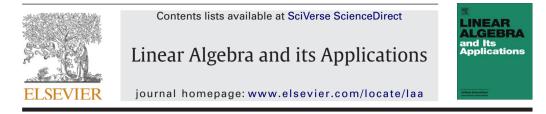*William & Mary*, crjohn@WM.EDU

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The relation between the diagonal entries and the eigenvalues of a symmetric matrix, based upon the sign pattern of its off-diagonal entries ☆

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ABSTRACT

It is known that majorization is a complete description of the relationships between the eigenvalues and diagonal entries of real symmetric matrices. However, for large subclasses of such matrices, the diagonal entries impose much greater restrictions on the eigenvalues. Motivated by previous results about Laplacian eigenvalues, we study here the additional restrictions that come from the off-diagonal sign-pattern classes of real symmetric matrices. Each class imposes additional restrictions. Several results are given for the all nonpositive and all nonnegative classes and for the third class that appears when $n = 4$. Complete description of the possible relationships are given in low dimensions.

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1. Introduction

It is known that among all real symmetric (complex Hermitian) n-by-n matrices, the complete relationship between the diagonal entries and eigenvalues is characterized by majorization [5]. However, for even very large subclasses of the real symmetric matrices, there may be additional restrictions (in addition to majorization). For example, classes of additional inequalities have recently been identified for graph Laplacians [1]. Our purpose here is to examine additional restrictions based upon the sign pattern class of the off-diagonal entries. As none of the diagonal entries, eigenvalues or symmetry is changed by signature or permutation similarity, we are interested in sign pattern classes, up to these symmetries. Of course, majorization implies that the largest (smallest) eigenvalue is at least (at most) the largest (smallest) diagonal entry, and we are primarily interested in inequalities between the ith largest eigenvalue and the kth largest diagonal entry. We give a new (universal) such inequality for matrices with nonpositive off-diagonal entries that generalizes an inequality recently proven for graph Laplacians [1], but further graph Laplacian inequalities do not generalize to the nonpositive off-diagonal case. In low dimensions, necessary and sufficient inequalities are given, though these involve more complicated inequalities, and interesting sufficient conditions for both nonnegative and for nonpositive off-diagonal entries are given for general n.

2. Known results and definitions

First, we present several definitions that will help us later to divide matrices into types, and also several well known theorems and lemmas that deal with eigenvalues of symmetric or nonnegative matrices.

Definition 1. A *signature matrix* is a diagonal matrix whose diagonal entries are ± 1.

Definition 2. A *signature similarity* of a square matrix A is a product of the form SAS, with S is a signature matrix.

We continue with the definition for *majorization*.

Definition 3. Let $\alpha = [\alpha_i] \in \mathbb{R}^n$ and $\beta = [\beta_i] \in \mathbb{R}^n$ be given. The order of the entries of α and β is as follows:

$$\alpha_{j_n} \leqslant \alpha_{j_{n-1}} \leqslant \cdots \leqslant \alpha_{j_1}, \quad \beta_{m_n} \leqslant \beta_{m_{n-1}} \leqslant \cdots \leqslant \beta_{m_1}.$$

The vector β is said to *majorize* the vector α if $\sum_{i=1}^{k} \beta_{m_i} \geqslant \sum_{i=1}^{k} \alpha_{j_i}$ for all $k = 1, 2, \ldots, n$ with equality for $k = n$.

The following lemmas and theorems are from [5].

Lemma 2.1. *Let A be an n-by-n real symmetric matrix with eigenvalues $\lambda_n \leqslant \lambda_{n-1} \leqslant \cdots \leqslant \lambda_1$. Then $\lambda_k = max\{\frac{\langle Ag,g \rangle}{\langle g,g \rangle} | g \perp v_{k-1}, v_{k-2}, \ldots, v_1\}$, when $v_{k-1}, v_{k-2}, \ldots, v_1$ are eigenvectors of eigenvalues $\lambda_{k-1}, \lambda_{k-2}, \ldots, \lambda_1$ respectively.*

Lemma 2.2. *Let n be a given positive integer, and let $\{\lambda_i | i = 1, 2, \ldots, n\}$ and $\{\hat{\lambda}_i | i = 1, 2, \ldots, n+1\}$ be two given sequences of real numbers such that $\hat{\lambda}_{n+1} \leqslant \lambda_n \leqslant \hat{\lambda}_n \leqslant \lambda_{n-1} \leqslant \cdots \leqslant \lambda_1 \leqslant \hat{\lambda}_1$. Let $\Lambda = diag(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$. There exists a real number a and a real vector $y \in \mathbb{R}^n$ such that $\{\hat{\lambda}_{n+1}, \hat{\lambda}_n, \ldots, \hat{\lambda}_1\}$ is the set of the eigenvalues of the real symmetric matrix*

$$\hat{A} \equiv \begin{pmatrix} \Lambda & y \\ y^T & a \end{pmatrix} \in M_{n+1}(\mathbb{R}).$$

Furthermore, a and y may be constructed in the following way:

$$a = \sum_{i=1}^{n+1} \hat{\lambda}_i - \sum_{i=1}^{n} \lambda_i.$$

In order to construct y, define the polynomials

$$f(t) = \prod_{i=1}^{n+1}(t - \hat{\lambda}_i), \quad g(t) = \prod_{i=1}^{n}(t - \lambda_i).$$

Define $s(t) = f(t)/g'(t)$ ($s(t)$ is in lowest terms). For all $1 \leqslant i \leqslant n$, y_i can be chosen to be some solution of the equation $y_i^2 = -s(\lambda_{n-i+1})$.

Remark. By the proof of this lemma, $-s(\lambda_{n-i+1})$ is nonnegative, hence this equation always has a solution, and if this solution is nonzero, then both of the options for the solution may be chosen in order to construct the vector y.

The following theorem shows that majorization is a complete description of the relationships between the eigenvalues and diagonal entries of a general real symmetric matrix.

Theorem 2.3 [5]**.** *Let $n \geqslant 1$ and let $a_n \leqslant a_{n-1} \leqslant \cdots \leqslant a_1$ and $\lambda_n \leqslant \lambda_{n-1} \leqslant \cdots \leqslant \lambda_1$ be given real numbers. If the vector $\lambda = [\lambda_i]$ majorizes the vector $a = [a_i]$, then there exists a real symmetric matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ with spectrum $\{\lambda_i\}$ such that $a_{ii} = a_i$ for $i = 1, 2, \ldots, n$.*

Next, we recall the Interlacing Theorem [5].

Theorem 2.4 (Interlacing Theorem)**.** *Let $A \in M_n$ be a given Hermitian matrix, and let $B \in M_{n-1}$ be a principal submatrix of A. Let the eigenvalues of A and B be denoted by $\{\lambda_i\}$ and $\{\hat{\lambda}_i\}$, respectively, and assume that they have been arranged in nonincreasing order $\lambda_1 \geq \cdots \geq \lambda_n$ and $\hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_{n-1}$. Then $\lambda_i \geq \hat{\lambda}_i \geq \lambda_{i+1}$ for all $1 \leq i \leq n - 1$.*

Finally, we also recall part of the Perron–Frobenius Theorem for nonnegative matrices [5]. We use $\rho(A)$ to denote the spectral radius of A.

Theorem 2.5 (Perron–Frobenius)**.** *If $A \in M_n$ and A is nonnegative matrix, then $\rho(A)$ is an eigenvalue of A and there is a nonnegative vector $x \geqslant 0$, $x \neq 0$, such that $Ax = \rho(A)x$.*

3. Matrix types

Here, we consider the connections between the sign pattern of the off-diagonal entries, and inequalities that involve diagonal entries and eigenvalues. Nonpositive off-diagonal entries will be denoted by "$-$", and nonnegative off-diagonal entries will be denoted by "$+$". If some off-diagonal entry is zero, we may denote it either with "$+$" or "$-$", and so if a matrix has some zero off-diagonal entries, there are several sign patterns that may be associated with it. We now divide matrices into several types, based upon their off-diagonal sign pattern. It may be that if a matrix has a zero off-diagonal entry, it will be associated with more than 1 type.

As we mentioned in Section 1, the operations permutation and signature similarity do not change the diagonal entries, eigenvalues or symmetry. We will say that all the matrices from some set are of the same type if for each two matrices from the set, we can move from the off-diagonal sign pattern of

the first one to the off-diagonal sign pattern of the other one, using only the operations permutation and signature similarity. We do not take into account the signs of the entries that are on the main diagonal.

We will define now two important types:

Definition 4. A real symmetric matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ is of *Type Z*, if it is possible to bring it to the form for which all the off-diagonal entries are nonpositive, using only the operations permutation and signature similarity.

Definition 5. A real symmetric matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ is of *Type P*, if it is possible to bring it to the form for which all the off-diagonal entries are nonnegative, using only the operations permutation and signature similarity.

In the next lemma, we give a full characterization of the types for $n = 3$, 4. This characterization will be also useful later, when we examine the relations between diagonal entries and eigenvalues of special types.

Lemma 3.1. *There are exactly two types of real symmetric matrices of size 3-by-3: Type P and Type Z.*
For 4-by-4 real, symmetric matrices, there are exactly three types: Type P and Type Z, and another type, which we call Type 3.

Proof. We start with the 3-by-3 case. If A is a real symmetric 3-by-3 matrix with nonnegative off-diagonal entries, then by performing signature similarity with each one of the matrices

$$\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right]$$

we get a matrix with two nonpositive off-diagonal entries, and one nonnegative (and there are three options for the place of the nonnegative entry, depends on which signature matrix of the three above we chose). By the definition, all these three kinds of matrices, together with the matrix with the same off-diagonal sigh pattern as A, are of Type P. Similarly, if we start with a real symmetric 3-by-3 matrix with nonpositive off-diagonal entries, and perform the same operations, we get a Type Z matrix. Since there are exactly eight different off-diagonal sign patterns, it is clear that all of them appeared above, so for the 3-by-3 case, there are exactly these two Types P and Z.

Consider the 4-by-4 case. There are 16 different signature matrices of size 4-by-4, and 64 different off-diagonal sign patterns that a real symmetric 4-by-4 matrix can have. Now, if $A = [a_{ij}] \in M_4(\mathbb{R})$, and S is a signature matrix, then $SAS = (-S)A(-S)$. It is easy to check that if we go through all the matrices S of the form $\begin{pmatrix} \pm1 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & 0 \\ 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, then we may divide the 64 different off-diagonal sign

patterns to eight different equivalence classes, each equivalence class has eight different off-diagonal sign patterns matrices, and inside each class we can move from one off-diagonal sign pattern to another by applying signature similarity with the matrix S of the form above, so each one of the eight different signature matrices above is associated with a different class. The class that contains the off-diagonal sign pattern in which all the off-diagonal entries are nonnegative would be Type P. Similarly, the class that contains the off-diagonal sign pattern in which all the off-diagonal entries are nonpositive would be Type Z (and in both classes, it is easy to check that performing permutation similarity on some off-diagonal sign pattern from the class will leave us inside the class). Now, there are six different off-diagonal sign patterns, in which one entry (above the main diagonal) is nonpositive, and

the others are nonnegative. It is impossible to move from one such off-diagonal sign pattern to the other with signature similarity, hence these six off-diagonal sign patterns appear each one in a different equivalence class. On the other hand , we can move from each one with such off-diagonal sign pattern to the other by applying permutation similarity. Therefore, all these six classes, which have 48 different off-diagonal sign patterns in total, are of the same type, and we name it Type 3. The following pattern is an example of Type 3:

$$\begin{pmatrix} ? & + & + & + \\ + & ? & + & + \\ + & + & ? & - \\ + & + & - & ? \end{pmatrix}$$

we refer to this pattern as the canonical form of Type 3. $\square$

Note that A is of Type P if and only if $-A$ is of Type Z, and also if A is of Type 3 then $-A$ is also of Type 3. In this paper, we are primarily interested in the relations among eigenvalues, diagonal entries and types of symmetric matrices. In order to make the wordings more clear, we have the following notation: Let $\lambda = [\lambda_i]_{i=1}^n, d = [d_i]_{i=1}^n$ be two vectors of length n. We say that $\{\lambda, d\} \in E(R, n)$ if there exists a real symmetric matrix of order n and of type R for which λ is the set of eigenvalues and d is the set of diagonal entries. Generally, R may be one of types P, Z and Type 3, or some different type if $n > 4$. An interesting question that one may ask is what is the number of the different types of matrices of order n. In [3], the following is proven:

Theorem 3.2. *The number of sign patterns of totally nonzero symmetric n-by-n matrices, up to conjugation by permutation and signature matrices and negation, is equal to the number of unlabeled graphs on n vertices.*

Note that the definition of "types" in this theorem is slightly different from ours (we do not allow negation while the theorem does, and we do not care about the signs of the off-diagonal entries while the theorem does), but that can be a good starting point for one that is interested in calculating the number of different types.

4. Bounds for eigenvalues of special types

We start with a lower bound for the second largest eigenvalue of a Type Z matrix. This bound is a generalization of the bound for Laplacian matrices that appears in [1].

Theorem 4.1. *Let A be an n-by-n ($n \geqslant 3$) real symmetric Type Z matrix, with diagonal entries $d_n \leqslant d_{n-1} \leqslant \cdots \leqslant d_1$, and eigenvalues $\lambda_n \leqslant \lambda_{n-1} \leqslant \cdots \leqslant \lambda_1$. Then $d_3 \leqslant \lambda_2$.*

Proof. Suppose at first that $n = 3$. Let A be a 3-by-3 symmetric Z-matrix, and let h be an eigenvector associated with the eigenvalue λ_1. From Lemma 2.1 we have

$$(*) \quad \lambda_2 = max \left\{ \frac{\langle Ag, g \rangle}{\langle g, g \rangle} | g \perp h \right\}.$$

There are two possibilities:

  (1) One of the entries of h is zero.
  (2) All the entries of h are different from zero.

In case (1), we assume that $h_t = 0$ for some $1 \leqslant t \leqslant 3$. We take a vector g such that

$$g_i = \begin{cases} 0 \ \text{if } i \neq t \\ 1 \ \text{if } i = t \end{cases}.$$

Since g is orthogonal to h, we get from (*) that $\lambda_2 \geqslant \frac{\langle Ag, g \rangle}{\langle g, g \rangle}$, and hence

$$\lambda_2 \geqslant \frac{\langle Ag, g \rangle}{\langle g, g \rangle} = a_{tt} \geqslant min\{d_1, d_2, d_3\} \geqslant d_3$$

and we are done.

In case (2), at least two of the entries of h have the same sign. Suppose without loss of generality that h_s, h_t have the same sign for some $1 \leqslant s, t \leqslant 3, s \neq t$. Define a vector g by

$$g_i = \begin{cases} 0 \ \ \text{if } i \neq t, s \\ 1 \ \ \text{if } i = t \\ -\delta \ \ \text{if } i = s \end{cases}$$

with $\delta > 0$ such that g is orthogonal to h (since h_s, h_t are with the same sign there exists such positive δ). Therefore

$$\lambda_2 \geqslant \frac{\langle Ag, g \rangle}{\langle g, g \rangle} = \frac{a_{tt} - \delta a_{ts} - \delta a_{st} + \delta^2 a_{ss}}{1 + \delta^2}.$$

A is Z-matrix, and hence a_{ts} and a_{st} are nonpositive. In addition, δ is positive, and hence

$$\lambda_2 \geqslant \frac{a_{tt} - \delta a_{ts} - \delta a_{st} + \delta^2 a_{ss}}{1 + \delta^2}$$
$$\geqslant \frac{a_{tt} + \delta^2 a_{ss}}{1 + \delta^2} \geqslant min\{a_{tt}, a_{ss}\}$$
$$\geqslant min\{d_1, d_2, d_3\} \geqslant d_3$$

and we are done. Suppose now that A is an n-by-n symmetric Z-matrix, $n > 3$, with diagonal entries $d_n \leqslant d_{n-1} \leqslant \cdots \leqslant d_1$. Let A_1 be a principal 3-by-3 submatrix of A whose diagonal entries are d_1, d_2, d_3. It follows that $\lambda_2(A_1) \geqslant d_3$, and hence by the Interlacing Theorem, $\lambda_2(A) \geqslant \lambda_2(A_1) \geqslant d_3$. □

Corollary 4.2. *Let A be an n-by-n ($n \geqslant 3$) symmetric Type P matrix, with diagonal entries $d_n \leqslant d_{n-1} \leqslant \cdots \leqslant d_1$, and eigenvalues $\lambda_n \leqslant \lambda_{n-1} \leqslant \cdots \leqslant \lambda_1$. Then $d_{n-2} \geqslant \lambda_{n-1}$.*

Proof. By the definition, $-A$ is a Z-matrix, and hence from Theorem 4.1, $\lambda_2(-A) \geqslant d_3(-A)$. Since $\lambda_2(-A) = -\lambda_{n-1}(A)$ and $d_3(-A) = -d_{n-2}(A)$ we get $-\lambda_{n-1}(A) \geqslant -d_{n-2}(A)$, and hence $d_{n-2}(A) \geqslant \lambda_{n-1}(A)$. □

Note that Theorem 4.1 and Corollary 4.2 are, of course, not valid for general, symmetric matrices. In the next section, we give a comprehensive description of the relation between λ and d in the 3-by-3 case.

5. Full characterization of the 3-by-3 case

We start with the following corollary, which is a direct consequence of Lemma 3.1, Theorem 4.1 and Corollary 4.2.

Corollary 5.1. *Let A be a 3-by-3 real symmetric matrix, with diagonal entries $d_3 \leqslant d_2 \leqslant d_1$, and eigenvalues $\lambda_3 \leqslant \lambda_2 \leqslant \lambda_1$. Then one of the following happens:*

(1) *A is of Type Z, and $\lambda_2 \geqslant d_3$.*
(2) *A is of Type P, and $\lambda_2 \leqslant d_1$.*

The next theorem will help us to give a full characterization of the 3-by-3 case:

Theorem 5.2. *Let $\{\lambda_1, \lambda_2, \lambda_3\}$ and $\{d_1, d_2, d_3\}$ be two given sequences of real numbers such that $d_3 \leqslant d_2 \leqslant d_1$ and $\lambda_3 \leqslant \lambda_2 \leqslant \lambda_1$. Suppose that the vector $\lambda = [\lambda_i]$ majorizes the vector $d = [d_i]$. Then:*

(1) *If $\lambda_2 \geqslant d_3$, then $\{\lambda, d\} \in E(Z, 3)$.*
(2) *If $\lambda_2 \leqslant d_1$, then $\{\lambda, d\} \in E(P, 3)$.*

Proof. We start by proving (1). From Theorem 2.3 and the fact that $[\lambda_i]$ majorizes $[d_i]$, there exists a 2-by-2 symmetric matrix with diagonal entries d_1, d_2 and eigenvalues $\lambda_1, d_1 + d_2 - \lambda_1$. If the off-diagonal entries of this matrix are positive, we modify the matrix by multiplying it from both of the sides with the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. This multiplication does not change neither the eigenvalues nor the diagonal entries.

Let $B = \begin{pmatrix} d_1 & c \\ c & d_2 \end{pmatrix}$ be such matrix. As we mentioned before, the eigenvalues of B are $\lambda_1, d_1 + d_2 - \lambda_1$, and c is nonpositive. There exists a real orthogonal matrix $Q \in M_2(\mathbb{R})$ such that $B = Q \begin{pmatrix} d_1 + d_2 - \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} Q^T$, and the columns of Q are the eigenvectors of B.

From majorization and the assumption in (1), we have the following equalities and inequalities:

$$d_1 + d_2 \leqslant \lambda_1 + \lambda_2, \tag{1}$$

$$d_3 \leqslant \lambda_2, \tag{2}$$

$$d_1 + d_2 + d_3 = \lambda_1 + \lambda_2 + \lambda_3. \tag{3}$$

Combining (3) and (3) yields

$$\lambda_1 + \lambda_3 \leqslant d_1 + d_2. \tag{4}$$

Hence, from (4) we have

$$\lambda_3 \leqslant d_1 + d_2 - \lambda_1, \tag{5}$$

and from (2) we have

$$d_1 + d_2 - \lambda_1 \leqslant \lambda_2. \tag{6}$$

Therefore, $\{\lambda_1, \lambda_2, \lambda_3\}$ are interlaced with $\{\lambda_1, d_1 + d_2 - \lambda_1\}$, since from (5) and (6) we have

$$\lambda_3 \leqslant d_1 + d_2 - \lambda_1 \leqslant \lambda_2 \leqslant \lambda_1 \leqslant \lambda_1. \tag{7}$$

Using Lemma 2.2, there exists a real number a and a vector $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$ such that $\{\lambda_1, \lambda_2, \lambda_3\}$

is the set of the eigenvalues of the matrix $K = \begin{pmatrix} d_1 + d_2 - \lambda_1 & 0 & y_1 \\ 0 & \lambda_1 & y_2 \\ y_1 & y_2 & a \end{pmatrix}$ where

$$
\begin{aligned}
a &= \lambda_1 + \lambda_2 + \lambda_3 - \lambda_1 - (d_1 + d_2 - \lambda_1) \\
&= \lambda_1 + \lambda_2 + \lambda_3 - d_1 - d_2 \\
&= d_1 + d_2 + d_3 - d_1 - d_2 \\
&= d_3.
\end{aligned}
$$

Define

$$
A = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} K \begin{pmatrix} Q^T & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B & Qy \\ (Qy)^T & a \end{pmatrix} = \begin{pmatrix} B & Qy \\ (Qy)^T & d_3 \end{pmatrix}.
$$

Notice that the eigenvalues of A are $\{\lambda_1, \lambda_2, \lambda_3\}$, and that its diagonal entries are $\{d_1, d_2, d_3\}$. Therefore we only need to show that A is of Type Z, and then we are done. Since c is nonpositive, it is enough to show that y may be chosen in such way that Qy will be nonpositive. From the definition of Q, the first column of Q is the eigenvector that corresponds to the smallest eigenvalue of B. Since B is of Type Z, from Perron–Frobenius Theorem we may conclude that the first column of Q is nonnegative.

Using the notation from Lemma 2.2 we have

$$
f(t) = (t - \lambda_1)(t - \lambda_2)(t - \lambda_3), g'(t) = [(t - \lambda_1)(t - d_1 - d_2 + \lambda_1)]' = 2t - d_1 - d_2.
$$

Now, from the remark after Lemma 2.2, y_1 can be chosen to be some nonpositive number. Since $f(\lambda_1) = 0$ there are two options:

- $g'(\lambda_1) \neq 0$.
- $g'(\lambda_1) = 0$.

In the first case, from Lemma 2.2, $y_2{}^2 = -f(\lambda_1)/g'(\lambda_1) = 0$, and hence $y_2 = 0$. Therefore,

$$
Qy = Q \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} Q_{11}y_1 \\ Q_{21}y_1 \end{pmatrix},
$$

and since the first column of Q is nonnegative, and y_1 is nonpositive, we get that Qy is nonpositive, and therefore A is of Type Z.

In the second case, $g'(\lambda_1) = 0$ implies $2\lambda_1 = d_1 + d_2$. Now,

$$
2\lambda_1 = d_1 + d_2 \leqslant \lambda_1 + \lambda_2 \leqslant 2\lambda_1,
$$

and therefore $d_1 + d_2 = \lambda_1 + \lambda_2$, which implies that $d_3 = \lambda_3$. In this case we may take $A = \begin{pmatrix} B & 0 \\ 0 & d_3 \end{pmatrix}$

and again, A is of Type Z, with eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ and diagonal entries $\{d_1, d_2, d_3\}$.

In order to prove (2), consider the lists $-\lambda = \{-\lambda_3, -\lambda_2, -\lambda_1\}$, and $-d = \{-d_3, -d_2, -d_1\}$. We have $-\lambda_1 \leqslant -\lambda_2 \leqslant -\lambda_3$, and $-d_1 \leqslant -d_2 \leqslant -d_3$. Now, since λ majorizes d then $-\lambda$ majorizes $-d$. In addition, according to (2), $\lambda_2 \leqslant d_1$, and hence $-\lambda_2 \geqslant -d_1$. In conclusion, the lists $-\lambda$ and $-d$ satisfies the requirements of (1), and hence it is possible to construct a Type Z matrix A with eigenvalues $\{-\lambda_1, -\lambda_2, -\lambda_3\}$ and diagonal entries $-d_1 \leqslant -d_2 \leqslant -d_3$. Hence $-A$ is a Type P matrix which satisfies all the requirements of (2). $\square$

Finally, we are ready now to give a full characterization in the 3-by-3 case, as is presented in the following two theorems:

Theorem 5.3. *Let $\{\lambda_1, \lambda_2, \lambda_3\}$ and $\{d_1, d_2, d_3\}$ be two given sequences of real numbers such that $d_3 \leqslant d_2 \leqslant d_1$ and $\lambda_3 \leqslant \lambda_2 \leqslant \lambda_1$. Suppose that the vector $\lambda = [\lambda_i]$ majorizes the vector $d = [d_i]$. Then:*

(1) *$\{\lambda, d\} \in E(Z, 3)$ if and only if $\lambda_2 \geqslant d_3$.*
(2) *$\{\lambda, d\} \in E(P, 3)$ if and only if $\lambda_2 \leqslant d_1$.*

Theorem 5.4. *Let $\{\lambda_1, \lambda_2, \lambda_3\}$ and $\{d_1, d_2, d_3\}$ be two given sequences of real numbers such that $d_3 \leqslant d_2 \leqslant d_1$ and $\lambda_3 \leqslant \lambda_2 \leqslant \lambda_1$. Suppose that the vector $\lambda = [\lambda_i]$ majorizes the vector $d = [d_i]$. Then:*

(1) *If $\lambda_2 > d_1$, then $\{\lambda, d\} \in E(Z, 3)$ and $\{\lambda, d\} \notin E(P, 3)$.*
(2) *If $d_1 \geqslant \lambda_2 \geqslant d_3$, then $\{\lambda, d\} \in E(Z, 3)$ and $\{\lambda, d\} \in E(P, 3)$.*
(3) *If $\lambda_2 < d_3$, then $\{\lambda, d\} \notin E(Z, 3)$ and $\{\lambda, d\} \in E(P, 3)$.*

Remark. One important family of Type P matrices is the family of symmetric nonnegative matrices. For such 3-by-3 matrices, our results in parts (2) of Theorems 5.2 and 5.3 are analogous to the following result due to Fiedler [4]:

Lemma 5.5. *Let $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$ and $d_1 \geqslant d_2 \geqslant d_3 \geqslant 0$. Then There exists a 3-by-3 symmetric nonnegative matrix B with eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ and diagonal entries $\{d_1, d_2, d_3\}$ if and only if the following conditions are satisfied:*

- $\lambda_1 \geqslant d_1$.
- $\lambda_1 + \lambda_2 \geqslant d_1 + d_2$.
- $\lambda_1 + \lambda_2 + \lambda_3 = d_1 + d_2 + d_3$.
- $\lambda_2 \leqslant d_1$.

6. Special inequalities for general n

In this section, we present a large class of sufficient conditions for the existence of matrices of Types P and Z for general n.

Theorem 6.1. *Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\{d_1, d_2, \ldots, d_n\}$ $(n \geqslant 3)$ be two given sequences of real numbers such that $d_n \leqslant d_{n-1} \leqslant \cdots \leqslant d_1$ and $\lambda_n \leqslant \lambda_{n-1} \leqslant \cdots \leqslant \lambda_1$. Suppose that the vector $\lambda = [\lambda_i]$ majorizes the vector $d = [d_i]$. Then:*

(1) *If $\lambda_j \geqslant d_{j+1}$ for all $j = 2, 3, \ldots, n - 1$, then $\{\lambda, d\} \in E(Z, n)$.*
(2) *If $\lambda_j \leqslant d_{j-1}$ for all $j = 2, 3, \ldots, n - 1$, then $\{\lambda, d\} \in E(P, n)$.*

Proof. We start with case (1). In order to prove it, we will use induction on n. The case $n = 3$ is proven in Theorem 5.2. Suppose then that the statement is true for $n-1$, and we will show that it is true also for n. Consider the lists $\hat{d} = \{d_1, d_2, \ldots, d_{n-2}, d_{n-1}\}$, $\gamma = \left\{\lambda_1, \lambda_2, \ldots, \lambda_{n-3}, \lambda_{n-2}, \sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-2} \lambda_i\right\}$. Since λ majorizes d, and

$$\lambda_1 + \lambda_2 + \cdots + \lambda_{n-3} + \lambda_{n-2} + \sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-2} \lambda_i = d_1 + d_2 + \cdots + d_{n-2} + d_{n-1}$$

we get that γ majorizes $\hat{d}$. In addition, using the assumption in (1) we have $\lambda_j \geqslant d_{j+1}$ for all $j = 2, 3, \ldots, n - 2$. Hence, by the inductive assumption, there exists a real symmetric matrix $B = [B_{ij}] \in M_{n-1}(\mathbb{R})$ of Type Z such that $b_{ii} = d_i$ for $i = 1, 2, \ldots, n - 1$ and such that γ

is the set of the eigenvalues of B. There exists a real orthogonal matrix $Q \in M_{n-1}(\mathbb{R})$ such that $B = Q diag \left(\sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-2} \lambda_i, \lambda_{n-2}, \lambda_{n-3}, \ldots, \lambda_2, \lambda_1 \right) Q^T$, and the columns of Q are the eigenvectors of B. From majorization and the assumption in (1), we have the following equalities and inequalities:

$$\sum_{i=1}^{n-1} d_i \leqslant \sum_{i=1}^{n-1} \lambda_i, \tag{8}$$

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \lambda_i, \tag{9}$$

$$d_n \leqslant \lambda_{n-1}. \tag{10}$$

From (8) we get

$$\sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-2} \lambda_i \leqslant \lambda_{n-1}. \tag{11}$$

From (9) and (10) we have

$$\sum_{i=1}^{n-1} d_i \geqslant \sum_{i=1}^{n-2} \lambda_i + \lambda_n, \tag{12}$$

and hence

$$\sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-2} \lambda_i \geqslant \lambda_n. \tag{13}$$

Therefore, using (11), (13) and the definition of γ we get that λ is interlaced with γ. Using Lemma 2.2, there exists a real number a and a vector $y \in \mathbb{R}^{n-1}$ such that λ is the set of the eigenvalues of the matrix

$$K = \begin{pmatrix} diag \left(\sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-2} \lambda_i, \lambda_{n-2}, \lambda_{n-3}, \ldots, \lambda_2, \lambda_1 \right) & y \\ y^T & a \end{pmatrix} \in M_n(\mathbb{R}),$$

where

$$a = \sum_{i=1}^{n} \lambda_i - \left(\sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-2} \lambda_i \right) - \sum_{i=1}^{n-2} \lambda_i$$

$$= \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} d_i$$

$$= d_n.$$

Define

$$A = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} K \begin{pmatrix} Q^T & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B & Qy \\ (Qy)^T & a \end{pmatrix} = \begin{pmatrix} B & Qy \\ (Qy)^T & d_n \end{pmatrix}.$$

Now, since B is of Type Z, it is enough to show that y may be chosen in such way that Qy will be nonpositive. We denote $y^T = (y_1, y_2, \ldots, y_n)$. The first column of Q is the eigenvector that corresponds to the smallest eigenvalue of B, and again from Perron–Frobenius Theorem we know that the first column of Q is nonnegative. Using the notation from Lemma 2.2 we have

$$f(t) = \prod_{i=1}^{n} (t - \lambda_i), g'(t) = \left(\left(t - \left(\sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-2} \lambda_i \right) \right) \prod_{i=1}^{n-2} (t - \lambda_i) \right)'.$$

Now, y_1 can be chosen to be some nonpositive number. Since $f(\lambda_i) = 0$ for all $i = 1, 2, \ldots, n-2$, there are two options:

- $g'(\lambda_i) \neq 0$ for all $i = 1, 2, \ldots, n-2$.
- There exists at least one $i \in 1, 2, \ldots, n-2$ such that $g'(\lambda_i) = 0$.

In the first case, similarly to the proof of Theorem 5.2, we get $y_j = 0$ for all $j = 2, \ldots, n-1$. Therefore Qy is equal to the first column of Q (which is a nonnegative vector), multiplied by y_1 (a nonpositive number). Hence Qy is a nonpositive vector, and so A is of Type Z and we are done.

In the second case, let us pick some $j \in \{1, 2, \ldots, n-2\}$ for which $g'(\lambda_j) = 0$. First, consider the case

$$\lambda_j = \sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-2} \lambda_i. \tag{14}$$

In this case we have

$$\lambda_j + \sum_{i=1}^{n-2} \lambda_i = \sum_{i=1}^{n-1} d_i. \tag{15}$$

Now, since

$$\lambda_{n-1} \leqslant \lambda_j, \tag{16}$$

we get

$$\sum_{i=1}^{n-1} \lambda_i \leqslant \lambda_j + \sum_{i=1}^{n-2} \lambda_i = \sum_{i=1}^{n-1} d_i. \tag{17}$$

On the other hand, we have

$$\sum_{i=1}^{n-1} \lambda_i \geqslant \sum_{i=1}^{n-1} d_i. \tag{18}$$

From (17) and (18) we conclude that

$$\sum_{i=1}^{n-1} \lambda_i = \sum_{i=1}^{n-1} d_i, \tag{19}$$

and hence

$$\lambda_n = d_n. \tag{20}$$

Note that the inequality in (16) has to be an equality (otherwise we get a contradiction to the assumption that λ majorizes d). Hence the eigenvalues of B are $\{\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, \ldots, \lambda_2, \lambda_1\}$. Define the matrix A to be $\begin{pmatrix} B & 0 \\ 0 & d_n \end{pmatrix}$. A is of Type Z, and it satisfies all the requirements of the theorem, and hence in this case we are done.

Assume now that (14) does not hold. Since $g(\lambda_j) = 0, g'(\lambda_j) = 0, \lambda_j$ is as a root $h(t)$ of multiplicity x, such that $x \geqslant 2$, when $h(t) = \prod_{i=1}^{n-2} (t - \lambda_i)$. The multiplicity of λ_j as a root of $f(t)$ and $g(t)$ is then at leat x, and exactly x respectively. Therefore, the term $t - \lambda_j$ appears at least one time more in $f(t)$ than in $g'(t)$. Hence, using the notation $s(t)$ from Lemma 2.2, we get $s(\lambda_j) = 0$, and from here the solution continues in the same way as in the first case and we are done.

Regarding the proof of (2), we can follow similar argument to the one we did at Theorem 5.2. $\quad\square$

Remark. A related result, which has a bit different point of view, may be found in [2] (Theorem 2.25).

To conclude this section we present a relation to the following Theorem, which is due to Suleimanova [7].

Theorem 6.2. *Let $\{\lambda_i\}_{i=1}^n$ be a collection of n real numbers such that $\lambda_j < 0$ for all $j \in \{2, 3, \ldots, n\}$ and $\sum_{i=1}^n \lambda_i \geqslant 0$. Then there exists a real symmetric nonnegative matrix with the spectrum $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$.*

Note that we may say that not only there exists a real symmetric nonnegative with the given spectrum, but according to Theorem 6.1, there exists such matrix whose all diagonal entries are zero except one of them which is equal to $\sum_{i=1}^n \lambda_i$. More generally, using Theorem 6.1 we can prove the following:

Theorem 6.3. *Let $\lambda_1 \geqslant 0 > \lambda_2 \geqslant \cdots \geqslant \lambda_n$, and let $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ such that $d_{n-2} \geqslant 0$. Suppose that the vector $\lambda = [\lambda_i]$ majorizes the vector $d = [d_i]$. Then $\{\lambda, d\} \in E(P, n)$.*

Note that this theorem generalizes the theorem of Suleimanova.

7. The 4-by-4 case

So far, we have given a full characterization in the 3-by-3 case, and some partial results for general n. We next discuss the 4-by-4 case. A wide range of possibilities is covered. We start with some necessary conditions for eigenvalues of matrices of given types.

Theorem 7.1. *Let A be a 4-by-4 real symmetric matrix with diagonal entries $d_4 \leqslant d_3 \leqslant d_2 \leqslant d_1$, and eigenvalues $\lambda_4 \leqslant \lambda_3 \leqslant \lambda_2 \leqslant \lambda_1$. Then, at least one of the following happens:*

1. *A is of Type Z, and then must satisfy*
   $\lambda_2 \geqslant d_3$
   $2d_1 \geqslant \lambda_1 + \lambda_4$
   $d_2 + d_3 \geqslant \lambda_2 + \lambda_4$.
2. *A is of Type P, and then must satisfy*
   $\lambda_3 \leqslant d_2$
   $2d_4 \leqslant \lambda_1 + \lambda_4$
   $d_2 + d_3 \leqslant \lambda_1 + \lambda_3$.
3. *A is of Type 3, and then must satisfy*
   $\lambda_3 \leqslant d_1$
   $\lambda_2 \geqslant d_4$.
   In addition, A satisfies at least one of the following:
   (a) $\lambda_2 \geqslant d_3$
       $d_2 + d_3 \geqslant \lambda_2 + \lambda_4$.
   (b) $\lambda_2 \geqslant d_3$
       $\lambda_3 \leqslant d_2$
       $d_1 + d_2 \geqslant \lambda_2 + \lambda_4$.
   (c) $d_2 + d_3 \leqslant \lambda_1 + \lambda_3$
       $d_2 + d_3 \geqslant \lambda_2 + \lambda_4$.
   (d) $\lambda_3 \leqslant d_2$
       $d_2 + d_3 \leqslant \lambda_1 + \lambda_3$.

Proof. From Lemma 3.1 we know that any 4-by-4 matrix has to be either of Type Z, P or 3. First, let us suppose that A is of Type Z. From Theorem 4.1 we have $\lambda_2 \geqslant d_3$. Now, consider the 3-by-3 principal

submatrix B of A whose diagonal entries are $\{d_2, d_3, d_4\}$. Denote its eigenvalues by $\mu_3 \leqslant \mu_2 \leqslant \mu_1$. By using the Interlacing Theorem, we have

$$\lambda_4 \leqslant \mu_3 \leqslant \lambda_3 \leqslant \mu_2 \leqslant \lambda_2 \leqslant \mu_1 \leqslant \lambda_1. \tag{21}$$

B is of Type Z, and hence, from Theorem 4.1 we have

$$\mu_2 \geqslant d_4. \tag{22}$$

In addition,

$$\mu_3 + \mu_2 + \mu_1 = d_2 + d_3 + d_4. \tag{23}$$

From (22) and (23) we get

$$\mu_3 + \mu_1 \leqslant d_2 + d_3. \tag{24}$$

And then, from (21) and (24) we have

$$\lambda_4 + \lambda_2 \leqslant d_2 + d_3. \tag{25}$$

Let us look at $-A + d_1 I$. This matrix is a nonnegative matrix, and hence from Perron–Frobenius Theorem we get

$$-(-\lambda_1 + d_1) \leqslant -\lambda_4 + d_1, \tag{26}$$

and therefore

$$2d_1 \geqslant \lambda_1 + \lambda_4. \tag{27}$$

Suppose now that A is of Type P. Hence $-A$ is of Type Z, and we finish by applying on $-A$ the results from above. The last case is the assumption that A is of Type 3. From the proof of Lemma 3.1, using only permutation and signature similarity we can bring A to the form

$$B = \begin{pmatrix} d_{\sigma(1)} & + & + & + \\ + & d_{\sigma(2)} & + & + \\ + & + & d_{\sigma(3)} & - \\ + & + & - & d_{\sigma(4)} \end{pmatrix}$$

$\sigma \in S_4$ and each one of the signs $+, -$ includes also the option of having zero in that entry. Now, by applying Corollary 4.2 on $B[1, 2, 3|1, 2, 3]$ and using the Interlacing Theorem, we get

$$\lambda_3 \leqslant max\{d_{\sigma(1)}, d_{\sigma(2)}, d_{\sigma(3)}\} \leqslant d_1. \tag{28}$$

Consider $B[2, 3, 4|2, 3, 4]$. We can apply Theorem 4.1 and the Interlacing Theorem, in order to get

$$\lambda_2 \geqslant min\{d_{\sigma(2)}, d_{\sigma(3)}, d_{\sigma(4)}\} \geqslant d_4. \tag{29}$$

Finally, let us look at $A[1, 2, 3|1, 2, 3]$ and $A[2, 3, 4|2, 3, 4]$. Without loss of generality, we may assume that $a_{ii} = d_i$ for all $i = 1, 2, 3, 4$. Using Lemma 3.1, here is the list of the possibilities:

- $A[1, 2, 3|1, 2, 3]$ is of Type Z and $A[2, 3, 4|2, 3, 4]$ and is of Type Z.
- $A[1, 2, 3|1, 2, 3]$ is of Type Z and $A[2, 3, 4|2, 3, 4]$ and is of Type P.
- $A[1, 2, 3|1, 2, 3]$ is of Type P and $A[2, 3, 4|2, 3, 4]$ and is of Type Z.
- $A[1, 2, 3|1, 2, 3]$ is of Type P and $A[2, 3, 4|2, 3, 4]$ and is of Type P.

In the first case, by looking at $A[1, 2, 3|1, 2, 3]$, $A[2, 3, 4|2, 3, 4]$ and applying Theorem 4.1 and the Interlacing Theorem we have

$$\lambda_2 \geqslant d_3 \tag{30}$$

and

$$\lambda_2(A[2, 3, 4|2, 3, 4]) \geqslant d_4 \tag{31}$$

respectively. Since

$$\lambda_1(A[2, 3, 4|2, 3, 4]) + \lambda_2(A[2, 3, 4|2, 3, 4]) + \lambda_3(A[2, 3, 4|2, 3, 4]) = d_4 + d_3 + d_2 \tag{32}$$

from (31) and (32) we get

$$\lambda_1(A[2, 3, 4|2, 3, 4]) + \lambda_3(A[2, 3, 4|2, 3, 4]) \leqslant d_3 + d_2. \tag{33}$$

Therefore, using The Interlacing Theorem we have

$$\lambda_2 + \lambda_4 \leqslant d_3 + d_2. \tag{34}$$

In the second case, by looking at $A[1, 2, 3|1, 2, 3]$, $A[2, 3, 4|2, 3, 4]$ and applying Theorem 4.1, Corollary 4.2, and The Interlacing Theorem we get

$$\lambda_2 \geqslant d_3 \tag{35}$$

and

$$\lambda_3 \leqslant d_2. \tag{36}$$

Now, since

$$\lambda_2(A[1, 2, 3|1, 2, 3]) \geqslant d_3, \tag{37}$$

we have

$$\lambda_1(A[2, 3, 4|2, 3, 4]) + \lambda_3(A[2, 3, 4|2, 3, 4]) \leqslant d_1 + d_2, \tag{38}$$

and hence by The Interlacing Theorem,

$$\lambda_2 + \lambda_4 \leqslant d_1 + d_2. \tag{39}$$

In the third case, by looking at $A[1, 2, 3|1, 2, 3]$ and applying Corollary 4.2, and the Interlacing Theorem, we get

$$\lambda_2(A[1, 2, 3|1, 2, 3]) \leqslant d_1. \tag{40}$$

Now,

$$\lambda_1(A[1, 2, 3|1, 2, 3]) + \lambda_2(A[1, 2, 3|1, 2, 3]) + \lambda_3(A[1, 2, 3|1, 2, 3]) = d_1 + d_2 + d_3, \tag{41}$$

and therefore from (40) and (41) we get

$$\lambda_1(A[1, 2, 3|1, 2, 3]) + \lambda_3(A[1, 2, 3|1, 2, 3]) \geqslant d_3 + d_2. \tag{42}$$

Finally,

$$\lambda_1 + \lambda_3 \geqslant d_3 + d_2. \tag{43}$$

The statement

$$\lambda_2 + \lambda_4 \leqslant d_3 + d_2 \tag{44}$$

can be proven in the same way as (34). Finally, consider the last case. The statements

$$\lambda_3 \leqslant d_2 \tag{45}$$

and

$$\lambda_1 + \lambda_3 \geqslant d_3 + d_2 \tag{46}$$

can be proven in the same way as (36) and (43) respectively. In conclusion, we proved statements (28)–(30), (34)–(36), (39), (43), (44)–(46). Therefore, if A is of Type 3, we are done. $\square$

Before discussing the 4-by-4 case further, we present the following useful lemma.

Lemma 7.2. *Let A be a 4-by-4 real symmetric matrix with diagonal entries $d_4 \leqslant d_3 \leqslant d_2 \leqslant d_1$, and eigenvalues $\lambda_4 \leqslant \lambda_3 \leqslant \lambda_2 \leqslant \lambda_1$. Then, exactly one of the following happens:*

- $\lambda_2 \geqslant d_3$ *and* $\lambda_3 > d_2$.
- $\lambda_2 < d_3$ *and* $\lambda_3 \leqslant d_2$.
- $\lambda_2 \geqslant d_3$ *and* $\lambda_3 \leqslant d_2$.

Proof. We need to show that it is impossible to have $\lambda_2 < d_3$ and $\lambda_3 > d_2$. Suppose in contradiction that it could happen. Then

$$\lambda_2 < d_3 \leqslant d_2 < \lambda_3$$

which is clearly impossible, so we get a contradiction. $\square$

We may now give a full characterization associated with each of the first two cases from Lemma 7.2. The following theorem covers a wide range of possibilities.

Theorem 7.3. *Let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $\{d_1, d_2, d_3, d_4\}$ be two given sequences of real numbers such that $d_4 \leqslant d_3 \leqslant d_2 \leqslant d_1$ and $\lambda_4 \leqslant \lambda_3 \leqslant \lambda_2 \leqslant \lambda_1$. Suppose that the vector $\lambda = [\lambda_i]$ majorizes the vector $d = [d_i]$. Then,*

1. *If $\lambda_3 > d_2$, we have:*
   (a) $\{\lambda, d\} \in E(Z, 4)$.
   (b) $\{\lambda, d\} \notin E(P, 4)$.
   (c) *If in addition $\lambda_3 \leqslant d_1$ then $\{\lambda, d\} \in E(\text{Type } 3, 4)$. If $\lambda_3 > d_1$, then $\{\lambda, d\} \notin E(\text{Type } 3, 4)$.*
2. *If $\lambda_2 < d_3$, we have:*
   (a) $\{\lambda, d\} \notin E(Z, 4)$.
   (b) $\{\lambda, d\} \in E(P, 4)$.
   (c) *If, in addition, $\lambda_2 \geqslant d_4$ then $\{\lambda, d\} \in E(\text{Type } 3, 4)$. If $\lambda_2 < d_4$, then $\{\lambda, d\} \notin E(\text{Type } 3, 4)$.*

Proof. We start with case number 1. From Lemma 7.2, since $\lambda_3 > d_2$, we have $\lambda_2 \geqslant d_3$. In addition, $\lambda_3 \geqslant d_4$. Hence by using Theorem 6.1 we can deduce that $\{\lambda, d\} \in E(Z, 4)$. Now, since $\lambda_3 > d_2$, then according to Corollary 4.2, $\{\lambda, d\} \notin E(P, 4)$. Similarly, from Theorem 7.1, $\lambda_3 \leqslant d_1$ is a necessary condition for being Type 3, so if $\lambda_3 > d_1$, then $\{\lambda, d\} \notin E(\text{Type } 3, 4)$. Let us assume now that $\lambda_3 \leqslant d_1$. We will show that there exists a real symmetric matrix $B = [b_{ij}] \in M_3(\mathbb{R})$ of Type Z with diagonal entries $\{d_2, d_3, d_4\}$ and eigenvalues $\{\lambda_2, d_2 + d_3 + d_4 - \lambda_2 - \lambda_4, \lambda_4\}$. At first, since

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = d_1 + d_2 + d_3 + d_4 \tag{47}$$

and

$$\lambda_1 \geqslant d_1, \tag{48}$$

we have

$$\lambda_2 + \lambda_3 + \lambda_4 \leqslant d_2 + d_3 + d_4. \tag{49}$$

And therefore

$$\lambda_3 \leqslant d_2 + d_3 + d_4 - \lambda_2 - \lambda_4. \tag{50}$$

Now, we do not know which of λ_2 and $d_2 + d_3 + d_4 - \lambda_2 - \lambda_4$ is bigger, but we do know (using (50)) that both of them are equal to or greater than λ_4. Therefore, since

$$d_2 < \lambda_3 \leqslant \lambda_2 \leqslant max\{\lambda_2, d_2 + d_3 + d_4 - \lambda_2 - \lambda_4\} \tag{51}$$

and

$$d_2 + d_3 \leqslant \lambda_2 + \lambda_3 \leqslant \lambda_2 + d_2 + d_3 + d_4 - \lambda_2 - \lambda_4, \tag{52}$$

we get that $\{\lambda_2, d_2 + d_3 + d_4 - \lambda_2 - \lambda_4, \lambda_4, \}$ majorizes $\{d_2, d_3, d_4\}$. In addition, since

$$d_4 \leqslant \lambda_3 \leqslant min\{\lambda_2, d_2 + d_3 + d_4 - \lambda_2 - \lambda_4\}, \tag{53}$$

then from Theorem 5.2, there exists a matrix B which satisfies the conditions that were described above. Now, from (47) and the inequality $\lambda_3 \leqslant d_1$, we get

$$\lambda_1 \geqslant d_2 + d_3 + d_4 - \lambda_2 - \lambda_4. \tag{54}$$

Hence, from (50) and (54), one of the two following inequalities is satisfied:

$$\lambda_1 \geqslant d_2 + d_3 + d_4 - \lambda_2 - \lambda_4 \geqslant \lambda_2, \tag{55}$$

$$\lambda_2 \geqslant d_2 + d_3 + d_4 - \lambda_2 - \lambda_4 \geqslant \lambda_3. \tag{56}$$

Therefore, in both cases, $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ are interlaced with $\{\lambda_2, d_2 + d_3 + d_4 - \lambda_2 - \lambda_4, \lambda_4\}$. Using Lemma 2.2, there exists a real number a (which is, in this case, d_1) and a real vector $y \in \mathbb{R}^3$ such that the matrix

$$C = \begin{pmatrix} diag(\lambda_4, d_2 + d_3 + d_4 - \lambda_2 - \lambda_4, \lambda_2) & y \\ y^T & d_1 \end{pmatrix}$$

has eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. In addition, there exists a real orthogonal matrix $Q \in M_3(\mathbb{R})$ such that $B = Q diag(\lambda_4, d_2 + d_3 + d_4 - \lambda_2 - \lambda_4, \lambda_2)Q^T$, and the columns of Q are the eigenvectors of B. Define

$$A = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} C \begin{pmatrix} Q^T & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B & Qy \\ (Qy)^T & d_1 \end{pmatrix}.$$

Let us observe the structure of Q. From Perron–Frobenius Theorem, the first column of Q is nonnegative. The second and the third columns are orthogonal to the first one, hence, the three entries of each one of them cannot be of the same sign. We can assume that each one of them has two nonnegative entries and one nonpositive entry (we can do it since if v is an eigenvector then $-v$ is also an eigenvector). In addition, since the second column of Q is orthogonal to the third one, the places of the nonpositive entry in each one of them are different. So up to permutation of the rows, the sign pattern of Q is of

the form: $\begin{pmatrix} + & + & + \\ + & + & - \\ + & - & + \end{pmatrix}$ where each one of the signs $+, -$ includes also the option of having zero in that

entry. Since B is of Type Z, in order to show that A is of Type 3, it is enough to show that y can be chosen in such way that Qy will have at least one nonpositive entry, and at least one nonnegative entry. Since this property (of having at least one row from each kind) is not changed if the rows of Q are permuted, we can assume that the sign pattern of Q is as given above. Using the notation from Lemma 2.2 we have

$$f(t) = (t - \lambda_1)(t - \lambda_2)(t - \lambda_3)(t - \lambda_4), g'(t) = [(t - \lambda_4)(t - (d_2 + d_3 + d_4 - \lambda_2 - \lambda_4))(t - \lambda_2)]'.$$

We can see that $f(\lambda_4) = 0$. Now, from (53), if $g'(\lambda_4) = 0$ then in particular $\lambda_4 = \lambda_3$. But since $\lambda_3 > d_2$, we get $d_4 \geqslant \lambda_4 > d_2$ which is impossible. Hence, $g'(\lambda_4) \neq 0$, so from Lemma 2.2, we can choose $y_1 = 0$. Regarding λ_2, we have $f(\lambda_2) = 0$. If $g'(\lambda_2) \neq 0$, then according to Lemma 2.2, one of y_2, y_3 is zero (it depends on whether λ_2 is smaller or bigger than $d_2 + d_3 + d_4 - \lambda_2 - \lambda_4$). The other one can be chosen to be nonnegative, and hence the sign pattern of Qy equals either to the sign pattern of the second column of Q, or to the sign pattern of the third column. In both cases A is of Type 3 and we are done. The last case that we need to check is $g'(\lambda_2) = 0$. This case may appear only if $\lambda_2 = d_2 + d_3 + d_4 - \lambda_2 - \lambda_4$. Therefore, from Lemma 2.2, y_2 and y_3 can be chosen such that $y_2^2 = y_3^2$.

Let us choose $-y_2 = y_3$, y_2 is nonnegative. Hence Qy is of the form $\begin{pmatrix} + & + & + \\ + & + & - \\ + & - & + \end{pmatrix} \begin{pmatrix} 0 \\ y_2 \\ -y_2 \end{pmatrix}$, and it is easy to see that its second entry is nonnegative, while its third entry is nonpositive, and hence we get again that A is of Type 3, and we are done. Consider case number 2. We have now the assumption that $\lambda_2 < d_3$. In a very similar way to the proof of case number 1, we can prove (a),(b) and (c) for the assumption $\lambda_2 < d_4$. Hence the only case that is left is $\lambda_2 \geqslant d_4$. The lists $\{-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4\}$ and $\{-d_1, -d_2, -d_3, -d_4\}$ satisfy the assumptions of case number 1, and hence it is possible to construct a Type 3 matrix A with these given lists of diagonal entries and eigenvalues. By the proof of Lemma 3.1, $-A$ is also Type 3, so we can look at $-A$ and we are done. $\square$

We conclude with a full characterization in the case in which all diagonal entries are equal (we assume that they are all equal to d).

Theorem 7.4. *Let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a given sequence of real numbers such that $\lambda_4 \leqslant \lambda_3 \leqslant \lambda_2 \leqslant \lambda_1$. Let d be a given real number, and suppose that the vector $\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{pmatrix}$ majorizes the vector $D = \begin{pmatrix} d & d & d & d \end{pmatrix}$. Then we have the following cases:*

1. $\lambda_2 < d$.
   Then $\{\lambda, D\} \in E(P, 4)$, $\{\lambda, D\} \notin E(Z, 4)$, $\{\lambda, D\} \notin E(Type\ 3, 4)$.
2. $\lambda_3 > d$.
   Then $\{\lambda, D\} \in E(Z, 4)$, $\{\lambda, D\} \notin E(P, 4)$, $\{\lambda, D\} \notin E(Type\ 3, 4)$.
3. $\lambda_2 \geqslant d$ *and* $\lambda_3 \leqslant d$.
   Then $\{\lambda, D\} \in E(Type\ 3, 4)$. In addition, there are several options:
   (a) $\lambda_1 + \lambda_4 > 2d$.
       Then $\{\lambda, D\} \in E(P, 4)$ and $\{\lambda, D\} \notin E(Z, 4)$.
   (b) $\lambda_1 + \lambda_4 < 2d$.
       Then $\{\lambda, D\} \in E(Z, 4)$ and $\{\lambda, D\} \notin E(P, 4)$.
   (c) $\lambda_1 + \lambda_4 = 2d$.
       Then $\{\lambda, D\} \in E(Z, 4)$ and $\{\lambda, D\} \in E(P, 4)$.

Proof. We start with case 1. Since $\lambda_2 < d$, then from Theorems 4.1 and 7.1, we have $\{\lambda, D\} \notin E(Z, 4)$ and $\{\lambda, D\} \notin E(Type\ 3, 4)$. By Theorem 2.3, there exists a real symmetric matrix $A = [a_{ij}] \in M_4(\mathbb{R})$ with the given lists of eigenvalues and diagonal entries. Hence this matrix A has to be of Type P. Case number 2 can be proven in a very similar way to case number 1, by using Theorems 2.3 and 7.1, and Corollary 4.2. Let us consider case number 3, part (a). From Theorem 7.1, $\{\lambda, D\} \notin E(Z, 4)$. Now, since $\lambda_1 + \lambda_4 > 2d$, we get

$$2d - \lambda_4 < \lambda_1. \tag{57}$$

In addition, we have

$$2\lambda_2 \leqslant \lambda_1 + \lambda_2 = 4d - \lambda_3 - \lambda_4 = 2d - \lambda_3 + 2d - \lambda_4 \leqslant 2(2d - \lambda_4), \tag{58}$$

and therefore

$$\lambda_2 \leqslant 2d - \lambda_4. \tag{59}$$

Hence, from (57), (59) and the assumptions in case 3, $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ are interlaced with $\{2d - \lambda_4, d, \lambda_4\}$. Consider the matrix $B = \begin{pmatrix} d & \lambda_4 - d & 0 \\ \lambda_4 - d & d & 0 \\ 0 & 0 & d \end{pmatrix}$. Its eigenvalues are

$\{2d - \lambda_4, d, \lambda_4\}$. Define $Q = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$, and we have $B = Q diag(\lambda_4, d, 2d - \lambda_4) Q^T$.

Using Lemma 2.2, there exists a vector $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$ such that $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is the set of the

eigenvalues of the matrix $K = \begin{pmatrix} diag(\lambda_4, d, 2d - \lambda_4) & y \\ y^T & d \end{pmatrix} \in M_4(\mathbb{R})$. Define

$$A = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} K \begin{pmatrix} Q^T & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B & Qy \\ (Qy)^T & d \end{pmatrix}.$$

Using the notation from Lemma 2.2 we have

$$f(t) = (t - \lambda_1)(t - \lambda_2)(t - \lambda_3)(t - \lambda_4), g'(t) = [(t - \lambda_4)(t - d)(t - (2d - \lambda_4))]'.$$

Now, $f(\lambda_4) = 0$. If $g'(\lambda_4) = 0$, then either $\lambda_4 = d$ or $\lambda_4 = 2d - \lambda_4$, which implies again $\lambda_4 = d$. Therefore, Since $\lambda_3 \leqslant d$, we have $\lambda_3 = d$. Hence $\lambda_1 + \lambda_2 = 2d$, and since $\lambda_2 \geqslant d$ we get $\lambda_1 = \lambda_2 = d$. In this case, $\lambda_1 + \lambda_4 = 2d$ which contradicts the assumptions. Therefore, $g'(\lambda_4) \neq 0$, and hence we can chose $y_1 = 0$. Therefore A is of the form

$$A = \begin{pmatrix} d & \lambda_4 - d & 0 & y_3/\sqrt{2} \\ \lambda_4 - d & d & 0 & -y_3/\sqrt{2} \\ 0 & 0 & d & y_2 \\ y_3/\sqrt{2} & -y_3/\sqrt{2} & y_2 & d \end{pmatrix}.$$

Now, each one of y_2, y_3 can be chosen to be either nonpositive, or nonnegative. Since $\lambda_4 - d < 0$, if we chose y_2 and y_3 to be nonpositive, Then A is of both Types 3 and P (this is because we can decide whether we look at zero as "+" or as "−". For the Type 3 case we will look at both zeros as "−", and for the Type P case we will look at the zero in the first line as "+", and on the other one as "−"). The next case is number 3, part (b). We can apply the proof from above on $-A$, and we are done. The last case is number 3, part (c). Since $\lambda_1 + \lambda_4 = 2d$, then $\lambda_2 + \lambda_3 = 2d$. Define

$$A = \begin{pmatrix} d & d - \lambda_4 & 0 & 0 \\ d - \lambda_4 & d & 0 & 0 \\ 0 & 0 & d & d - \lambda_2 \\ 0 & 0 & d - \lambda_2 & d \end{pmatrix}.$$

The eigenvalues of this matrix are $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, and A is of Types Z, P and 3. $\square$

We would like to emphasize some of the advantages of Theorem 7.4. First, we present the following result, due to Fiedler [4]:

Theorem 7.5. *Let* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n, a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ *satisfy*

1. $\sum_{i=1}^s \lambda_i \geqslant \sum_{i=1}^s a_i, \quad s = 1, 2, \ldots, n - 1.$
2. $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n a_i.$
3. $\lambda_k \leqslant a_{k-1}, \quad k = 2, 3, \ldots, n - 1.$

Then there exists an n-by-n symmetric nonnegative matrix B with eigenvalues λ_1, λ_2, ..., λ_n *and diagonal entries* a_1, a_2, ..., a_n.

In [6] it is shown that for $n \geqslant 4$, the conditions in Theorem 7.5 are only sufficient. The authors provide the following example:

$$B = \begin{pmatrix} 5 & 2 & \frac{1}{2} & \frac{1}{2} \\ 2 & 5 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 5 & 2 \\ \frac{1}{2} & \frac{1}{2} & 2 & 5 \end{pmatrix}.$$

B is a symmetric nonnegative matrix with eigenvalues 8, 6, 3, 3 and diagonal entries 5, 5, 5, 5 in which $\lambda_2 > a_1$, and hence condition 3 in Theorem 7.5 is not satisfied. However, these eigenvalues and diagonal entries satisfy the conditions of Theorem 7.4 part 3(a).

In conclusion, we have investigated various relations between diagonal entries and eigenvalues of different types of matrices. Still, the general question of describing the different types and giving the exact relations between diagonal entries and eigenvalues that correspond to these types is still open for $n > 5$. In addition, for $n = 4$, the third case in Lemma 7.2 is also still open. For example, through majorization it is easy to show that there exists a 4-by-4 symmetric matrix with eigenvalues 5, 4, 0, −3 and diagonal entries 3, 3, 0, 0. However, since this case falls down in the open case of Lemma 7.2, it is unclear at this point which types exactly can be constructed with this eigenvalues and diagonal entries.

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