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Daniel Corey

Charles R. Johnson
William & Mary, crjohn@WM.EDU

Ilya Spitkovsky
William & Mary, imspitkovsky@gmail.com

Ryan Kirk

Brian Lins

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The product field of values[☆]

Daniel Corey [a], Charles R. Johnson [b], Ryan Kirk [c], Brian Lins [d,*], Ilya Spitkovsky [b]

[a] University of Notre Dame, United States
[b] The College of William & Mary, United States
[c] University of North Carolina at Chapel Hill, United States
[d] Hampden-Sydney College, United States

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ABSTRACT

For two n-by-n matrices, A, B, the product field of values is the set $P(A, B) = \{\langle Ax, x\rangle\langle Bx, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$. In this paper, we establish basic properties of the product field of values. The main results are a proof that the product field is a simply connected subset of the complex plane and a characterization of matrix pairs for which the product field has nonempty interior.

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1. Introduction

Let $M_n(\mathbb{C})$ denote the set of complex n-by-n matrices and suppose $A, B \in M_n(\mathbb{C})$. Recall that the classical field of values (also known as the numerical range) of the matrix A is the subset of the complex plane defined by

$$F(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}.$$

This is a compact, convex set for each A [5].

The class of generalizations to two matrices given by

$$Q_{k,l}(A, B) = \{\langle Ax, x\rangle^k\langle Bx, x\rangle^l : x \in \mathbb{C}^n, \|x\| = 1\},$$

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* Corresponding author. Address: Hampden-Sydney College, Box 131, Hampden-Sydney, VA 23943, United States. Tel.: +1 434 223 6264.

E-mail addresses: dcorey@nd.edu (D. Corey), crjohnso@math.wm.edu (C.R. Johnson), rtkirk01@gmail.com (R. Kirk), blins@hsc.edu (B. Lins), ilya@math.wm.edu.

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in which k and l are integers was first considered in [2]; we assume that $0 \notin F(A)$ ($0 \notin F(B)$) if $k < 0$ (respectively, $l < 0$).

A connected subset Ω of the complex plane is said to be *ray convex* if for each $z \notin \Omega$, there is a ray anchored at z that does not intersect Ω. Note that ray convexity implies simple connectivity (so long as the set in question is connected), but not conversely. Motivated by a numerical application [7], the *ratio field of values* $R(A, B) = Q_{-1,1}$ was studied in [2,9]. In particular, it was shown in [2] that the ratio field, while seldom convex, is always ray convex (and therefore simply connected), while according to [9] the simple connectedness fails if the condition $0 \notin F(B)$ dropped. Also [2], $Q_{k;l}(A; B)$ is not generally simply connected whenever $|k| + |ll > 2$. The remaining case $k = l = 1$ was left unresolved.

Here we consider this case. It is natural to call $Q_{1,1}(A, B)$ the *product field of values*, and thus to denote it $P(A, B)$. We show below that the product field is also ray convex (and not generally convex), completing a classification of the $Q_{k,l}$'s with respect to simple connectivity, ray convexity, and convexity. Interestingly, the product field lacks certain structural characteristics that supported the proof of ray convexity in the ratio field case. As a result, the product field requires more topological and analytical considerations.

We will begin with a discussion on the basic properties of the product field, followed by a comparison of $P(A, B)$ and $R(A, B)$. Next, we will prove that the product field is ray convex and consequently simply connected. We follow with a description of the product field in certain special cases. In Section 5 we characterize the matrix pairs for which the product field has empty interior. An Appendix A with images of several example product fields of values is included at the end (Figs. 1–7). These images were computed numerically using MATLAB. As shown in Appendix A, the product field of values of two matrices can take on a wide variety of shapes and nonconvex product fields are typical.

2. Basic properties

This section is primarily concerned with the immediate observations about the product field. Afterwards, we will highlight significant differences between the product field and the ratio field.

Proposition 1. *Suppose A and B are complex $n \times n$ matrices and $P(A, B)$ is their associated product field of values.*

(1) (*Compactness*) *$P(A, B)$ is a compact subset of $\mathbb{C}$.*
(2) (*Connectedness*) *$P(A, B)$ is a connected subset of $\mathbb{C}$.*
(3) (*Homogeneity*) *$P(\alpha A, \beta B) = \alpha\beta P(A, B)$ for complex numbers α and β.*
(4) (*Symmetry*) *$P(A, B) = P(B, A)$.*
(5) (*Subadditivity*) *$P(A_1 + A_2, B) \subset P(A_1, B) + P(A_2, B)$.*
(6) *If $U \in U_n(\mathbb{C})$, then $P(U^*AU, U^*BU) = P(A, B)$ ($U_n(\mathbb{C})$ is the group of $n \times n$ unitary matrices).*
(7) *$P(A, B) \subset F(A)F(B)$ where $F(A)$ ($F(B)$) is the field of values for A (B).*
(8) *$\bigcup_{U \in U_n(\mathbb{C})} P(A, U^*BU) = F(A)F(B)$.*
(9) *$P(A, B)$ is symmetric with respect to the real axis if the entries of A and B are real.*
(10) *If A and B are both Hermitian or both skew-Hermitian, then $P(A, B)$ is a compact interval on the real axis.*
(11) *If A is Hermitian and B is skew-Hermitian (or vice-versa), then $P(A, B)$ is a compact interval on the imaginary axis.*

Proof. Let $\mathbb{C}S^n$ denote the complex unit n-sphere $\mathbb{C}S^n = \{x \in \mathbb{C}^n : ||x|| = 1\}$. Note that $\mathbb{C}S^n$ is topologically equivalent to S^{2n-1}. The function $f : \mathbb{C}S^n \to \mathbb{C}$,

$$f(x) = \langle Ax, x\rangle\langle Bx, x\rangle$$

is continuous and $f(\mathbb{C}S^n) = P(A, B)$. This means that $P(A, B)$ is compact and connected, proving (1) and (2). Homogeneity, symmetry, subadditivity and (7) are immediate from set theoretic considerations.

For (6), observe that

$$P(U^*AU, U^*BU) = f(U(\mathbb{C}S^n)).$$

Since U is a bijection of $\mathbb{C}S^n$ onto itself, $f(U(\mathbb{C}S^n)) = f(\mathbb{C}S^n) = P(A, B)$. Next we will show (8). Consider $z \in F(A)F(B)$, say

$$z = \langle Ax, x\rangle\langle By, y\rangle$$

for some $x, y \in \mathbb{C}S^n$. Since $U_n(\mathbb{C})$ acts transitively on $\mathbb{C}S^n$, there is a unitary matrix U so that $Ux = y$. This means that

$$\langle Ax, x\rangle\langle By, y\rangle = \langle Ax, x\rangle\langle U^*BUx, x\rangle,$$

and therefore $z \in F(A)F(B)$ if and only if $z \in \bigcup_{U\in U_n(\mathbb{C})} P(A, U^*BU)$, as required.

Now we will prove (9). Suppose that the entries of A and B are real. Then $\bar{A} = A$, $\bar{B} = B$, and

$$(\langle Ax, x\rangle\langle Bx, x\rangle)^* = \langle \bar{A}\bar{x}, \bar{x}\rangle\langle \bar{B}\bar{x}, \bar{x}\rangle = \langle Ay, y\rangle\langle By, y\rangle,$$

where $y = \bar{x} \in \mathbb{C}S^n$.

For (10), observe that

$$(\langle Ax, x\rangle\langle Bx, x\rangle)^* = \langle \pm Ax, x\rangle\langle \pm Bx, x\rangle = \langle Ax, x\rangle\langle Bx, x\rangle,$$

if A and B are Hermitian (respectively, skew-Hermitian). This means that $P(A, B) \subset \mathbb{R}$. Likewise for (11),

$$(\langle Ax, x\rangle\langle Bx, x\rangle)^* = \langle \pm Ax, x\rangle\langle \mp Bx, x\rangle = -\langle Ax, x\rangle\langle Bx, x\rangle,$$

if A is Hermitian and B is shew Hermitian (respectively, A is skew-Hermitian and B is Hermitian). This means that $P(A, B) \subset i\mathbb{R}$ ($i\mathbb{R} = \{ix : x \in \mathbb{R}\}$). □

There are two noteworthy differences between the product field of values $P(A, B)$ and the ratio field of values $R(A, B)$ (whenever we talk about $R(A, B)$, it is always assumed that 0 is not in $F(B)$). For $P(A, B)$, we may assume that B is upper triangular by property (6) in the previous proposition (every matrix $B \in M_n(\mathbb{C})$ is unitarily similar to a triangular matrix, see [4]). However, $R(A, B)$ satisfies a property stronger than (6). If $C \in GL_n(\mathbb{C})$, then [2]

$$R(C^*AC, C^*BC) = R(A, B).$$

Since any square matrix B with $0 \notin F(B)$ is congruent to a diagonal matrix [6], we may assume that B is diagonal. In particular, this property greatly simplifies the study of the ratio field in the 2×2 case.

The second notable difference between $P(A, B)$ and $R(A, B)$ is translatability. For the ratio field [2],

$$R(A + \beta B, B) = R(A, B) + \beta,$$

with $\beta \in \mathbb{C}$. This means that if the ratio field is translated in the complex plane, the resulting set is a ratio field for some different pair of matrices. This property is key for the proof that $R(A, B)$ is simply connected in [2]. This proof can be summarized as follows. Take an α not in $R(A, B)$, it is sufficient to show that there is a ray emanating from α not intersecting $R(A, B)$. Translate the ratio field so that α becomes the origin. Finally, use convexity of the regular field of values and the estimate

$R(A, B) \subset F(A)/F(B)$ to produce a ray not intersecting $F(A)/F(B)$ to complete the proof. However, it is not immediately clear that translation of the product field $P(A, B)$ results in a product field of a different pair of matrices. This means we cannot adapt a proof that $P(A, B)$ is simply connected from that of $R(A, B)$.

3. Simple connectivity of the product field of values

First, let us discuss the relationship between ray convexity and simple connectivity.

Definition 1. A connected set Ω in the complex plane is *ray convex* if for each α not in Ω there is a ray emanating from α not intersecting Ω. If such a ray exists, it is called an *escape ray* for α with respect to Ω.

Observe that Ω is ray convex if and only if its complement with respect to the extended plane $\mathbb{C} \cup \{\infty\}$ is star shaped at ∞. In particular, the complement of Ω with respect to the extended plane is connected. This means that Ω is simply connected. However, the converse is not true. Consider the spiral

$$S = \{z = \theta e^{i\theta} : \theta \in \mathbb{R}, \theta > 0\}.$$

This set is simply connected, but not ray convex. In fact, no point outside of S has an escape ray with respect to S.

Theorem 1. *For $A, B \in M_n(\mathbb{C})$, $P(A, B)$ is ray convex and therefore simply convex.*

As in the previous section, let $f : \mathbb{C}S^n \to \mathbb{C}$ be the function

$$f(x) = \langle Ax, x\rangle\langle Bx, x\rangle. \tag{1}$$

Fix a point α not in $P(A, B)$ and let $g_\alpha : \mathbb{C}S^n \to \mathbb{R}$ be the map

$$g_\alpha(x) = \mathrm{Im}\left(\int_{f(\gamma)} \frac{dz}{z-\alpha}\right),$$

where $\gamma : [0, 1] \to \mathbb{C}S^n$ is any smooth path from x_0 to x in $\mathbb{C}S^n$. To check that this is well defined, suppose γ_1 and γ_2 are two such paths and g_α^1 (g_α^2) is the function g_α determined by the curve γ_1 (γ_2). Consider the cycle $\Gamma = \gamma_1 - \gamma_2$

$$\Gamma(t) = \begin{cases} \gamma_1(2t) & 0 \leqslant t \leqslant 1/2, \\ \gamma_2(1-2t) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Since α is not in the image of f we may consider the winding number of $f(\Gamma)$ with respect to α:

$$n(f(\Gamma), \alpha) = \frac{1}{2\pi i}\int_{f(\Gamma)} \frac{dz}{z-\alpha} = \frac{1}{2\pi i}\left(\int_{f(\gamma_1)} \frac{dz}{z-\alpha} - \int_{f(\gamma_2)} \frac{dz}{z-\alpha}\right) = \frac{1}{2\pi i}(g_\alpha^1 - g_\alpha^2).$$

Recall that a curve is contractible in a topological space if it can be continuously deformed to a single point in that space.

Lemma 1. *If γ is a closed curve in $\mathbb{C}S^n$, then $f(\gamma)$ is a contractible closed curve in $P(A, B)$*

Proof. Since f is a continuous function from $\mathbb{C}S^n$ to $P(A, B)$, it induces a homomorphism

$$f_* : \pi_1(\mathbb{C}S^n) \to \pi_1(P(A,B)).$$

Since $\pi_1(\mathbb{C}S^n)$ is trivial, f_* is the trivial homomorphism. Therefore $f_*(\pi_1(\mathbb{C}S^n))$ is trivial, proving the lemma. □

By the previous lemma, $f(\Gamma)$ is a contractible curve in $P(A,B)$, so $n(\Gamma(f),\alpha)$ necessarily vanishes and $g_\alpha^1 = g_\alpha^2$. This proves that g_α is well defined.

Lemma 2. *The following are equivalent:*

(1) *The product field of values $P(A,B)$ is ray convex.*
(2) *For each x_1 and x_2 in $\mathbb{C}S^n$ and each α not in $P(A,B)$,*

$$|g_\alpha(x_2) - g_\alpha(x_1)| < 2\pi.$$

(3) *For each pair of orthonormal vectors x_1 and x_2 in $\mathbb{C}S^n$, any point α not in $P(A,B)$ has an escape ray with respect to the curve $\Gamma : [0,2\pi] \to \mathbb{C}$*

$$\Gamma(t) = f(x_1 \cos t + x_2 \sin t).$$

Proof. Suppose (2) is true. Fix α not in $P(A,B)$. Then a ray R whose angle is not in $g_\alpha(\mathbb{C}S^n)$ (but is within 2π of any point in this set) is an escape ray for α. This proves that $P(A,B)$ is ray convex.

Now suppose (1) is true, and R is an escape ray for α with respect to $P(A,B)$. Since each curve $\gamma(t)$ defined in (3) lies entirely in $P(A,B)$, R is an escape ray for α with respect to $\Gamma(t)$. Thus (1) implies (3).

Finally, we will show that (3) implies (2). Suppose x_1 and x_2 are in $\mathbb{C}S^n$ and α is not in $P(A,B)$. If x_1 and x_2 are linearly dependent, say $x_2 = e^{i\theta}x_1$ for some real number θ, then $f(x_1) = f(x_2)$ (this is an easy calculation). This means we can assume that x_1 and x_2 are linearly independent. Now set $y_1 = x_1$ and

$$y_2 = \frac{x_2 - \langle x_1, x_2\rangle x_1}{\|x_2 - \langle x_1, x_2\rangle x_1\|}.$$

So $\{y_1, y_2\}$ is an orthonormal basis for the linear span of x_1 and x_2. Consider the curve

$$\Gamma(t) = f(y_1 \cos t + y_2 \sin t).$$

From (3), the point α has an escape ray with respect to Γ, so

$$g_\alpha(y_1 \cos t + y_2 \sin t)$$

is an interval of length less than 2π. Since x_1 and x_2 lie on $y_1 \cos t + y_2 \sin t$ (this curve is a parametrization of the circle which is precisely $\mathbb{C}S^n \cap \operatorname{span}\{x_1, x_2\}$), x_1 and x_2 satisfy the inequality in (2), as required. □

Now we will prove that $P(A,B)$ is ray convex by demonstrating (3) in the above Lemma.

Proof of Theorem 1. Suppose α is not in $P(A,B)$ and x_1 and x_2 are orthonormal vectors in $\mathbb{C}S^n$. Let Γ be the function defined in Lemma 2 part (3). Applying the translation $z \mapsto z - \alpha$ takes $\Gamma(t)$ to $\Gamma_0(t) = \Gamma(t) - \alpha$ and α to the origin. The curve Γ_0 can be written as

$$\Gamma_0(t) = \beta_1 \cos^4(t) + \beta_2 \cos^3(t)\sin(t) + \beta_3 \cos^2(t)\sin^2(t) + \beta_4 \cos(t)\sin^3(t) + \beta_5 \sin^4(t), \quad (2)$$

where the coefficients β_i are complex numbers (to see this, explicitly compute $f(x_1 \cos t + x_2 \sin t)$ and apply the translation by α using the relationship $1 = (\cos^2 t + \sin^2 t)^2$). Note that β_1 and β_5 in (2) must be nonzero, otherwise the curve passes through the origin (remember that $n(\Gamma_0, \alpha) = 0$). By the fundamental theorem of algebra, we may factor $\Gamma_0(t)$ into linear factors

$$c(\cos t+b_1 \sin t)(\cos t+b_2 \sin t)(\cos t+b_3 \sin t)(\cos t+b_4 \sin t) = c\gamma_1(t)\gamma_2(t)\gamma_3(t)\gamma_4(t). \quad (3)$$

We will now digress into the geometry of these linear factors. It is well known that a curve of the form

$$c(t) = \cos t + k \sin t$$

for some complex number k is an ellipse centered at the origin (when $\operatorname{Im} k = 0$, this curve is just a line segment from -1 to 1). Thus $n(c, 0) = \operatorname{sign}(\operatorname{Im} k)$ assuming $\operatorname{Im} k \neq 0$.

The following lemma shows that the product of two such linear factors is also an ellipse (so long as neither factor is a line segment).

Lemma 3. *Suppose k_1 and k_2 are complex numbers. Then the curve*

$$c_2(t) = (\cos t + k_1 \sin t)(\cos t + k_2 \sin t), \quad 0 \leqslant t \leqslant 2\pi$$

is an ellipse (so long as it does not degenerate).

Proof. By an appropriate affine transformation, c_2 turns into $\tilde{c}_2(t) = \cos^2 t + k_3 \cos t \sin t$. Now the curve

$$t \mapsto \cos^2 t + i \cos t \sin t$$

is a circle (in fact, it is the circle of radius 1/2 centered at 1/2). The linear transformation on $\mathbb{R}^2 \cong \mathbb{C}$ that takes the ordered basis $\{1, i\}$ to $\{1, k_3\}$ will take this circle to $\tilde{c}_2$, so it is an ellipse (linear transformations take ellipses to ellipses). $\square$

Now we will return to the proof of Theorem 1. Each b_i in Eq. (3) has a nonvansihing imaginary part. Indeed, suppose that $\operatorname{Im} b_i = 0$ for some $i = 1, 2, 3, 4$. Then the curve γ_i passes through the origin, so Γ_0 also passes through the origin, a contradiction. Next, observe that $n(\gamma_i, 0) = \operatorname{sign}(\operatorname{Im} b_i)$ and

$$n(\Gamma_0, 0) = \sum_{i=1}^{4} n(\gamma_i, 0) = \sum_{i=1}^{4} \operatorname{sign}(\operatorname{Im} b_i).$$

The above expression is necessarily 0; this only happens when exactly two b_i's have positive imaginary parts and two have negative imaginary parts. After reordering the factors, suppose b_1 and b_2 have positive imaginary parts and b_3 and b_4 have negative imaginary parts. Then

$$n(\gamma_1\gamma_3, 0) = n(\gamma_1, 0) + n(\gamma_3, 0) = 0.$$

So $\gamma_1\gamma_3$ is an ellipse contained in some open half plane H_1. Likewise, $\gamma_2\gamma_4$ is an ellipse contained in an open half plane H_2. Therefore, the curve Γ_0 is contained in the set

$$H = H_1H_2 = \{z_1z_2 \colon z_1 \in H_1, z_2 \in H_2\}.$$

The set H is contained in $\mathbb{C}\backslash R$ where R is a ray emanating from the origin determined by the sets H_1 and H_2. This ray R is an escape ray for the origin with respect to the curve Γ_0, as required. $\square$

We conclude with a remark on the above proof. Let A be a $n \times n$ matrix with complex entries. The classical field is the image of the function $\psi_1 : \mathbb{C}S^n \to \mathbb{C}$, $\psi_1(x) = \langle Ax, x\rangle$. If ψ_2 is the function $\psi_2(x) = \langle Bx, x\rangle$, then $P(A, B)$ is the image of the function $\psi_1\psi_2$, which is simply connected. This raises the following question: if two functions from $\mathbb{C}S^n$ to $\mathbb{C}$ individually produce convex sets, does their product produce a simply connected set? If this is true, the simple connectivity of $P(A, B)$ would be an immediate consequence. However, this is not the case; we will construct two such functions as follows. There is a continuous map π from $\mathbb{C}S^n$ whose image is $[0, 1]$. Let a and b be two complex numbers that are linearly independent when regarded as elements of $\mathbb{R}^2 \cong \mathbb{C}$. Define $\varphi_1, \varphi_2 : [0, 1] \to \mathbb{C}$ by

$$\varphi_1(t) = \begin{cases} 1-4t & 0 \leqslant t < \frac{1}{4} \\ 0 & \frac{1}{4} \leqslant t < \frac{1}{2} \\ 4t-2 & \frac{1}{2} \leqslant t < \frac{3}{4} \\ 1 & \frac{3}{4} \leqslant t \leqslant 1 \end{cases}, \qquad \varphi_2(t) = \begin{cases} a & 0 \leqslant t < \frac{1}{4} \\ 4a\left(t-\frac{1}{4}\right) + 4b\left(\frac{1}{2}-t\right) & \frac{1}{4} \leqslant t < \frac{1}{2} \\ b & \frac{1}{2} \leqslant t < \frac{3}{4} \\ 4b\left(t-\frac{3}{4}\right) + 4a(1-t) & \frac{3}{4} \leqslant t \leqslant 1 \end{cases}.$$

So $\Phi_1 = \varphi_1\pi$ and $\Phi_2 = \varphi_2\pi$ are continuous functions on $\mathbb{C}S^n$. The image of Φ_1 is the interval $[0, 1]$ and the image of Φ_2 is the line segment joining a and b, both of which are convex. However, the product $\Phi_1\Phi_2$ produces the triangle with vertices $0, a, b$ which is obviously not simply connected.

4. Explicit descriptions of the product field in special cases

The simplest situation occurs when one of the matrices A, B is a scalar multiple of the identity matrix I. Then apparently $P(A, B) = F(C)$, where

$$C = \begin{cases} \lambda B & \text{if } A = \lambda I, \\ \lambda A & \text{if } B = \lambda I. \end{cases}$$

So, in this (trivial) case all possible shapes of product fields are exactly the same as those of the fields of values for matrices of the same size. Note that a complete description of those was obtained recently in [3].

The next in complexity case is when the triple A, B, I is linearly dependent. Without loss of generality, let

$$B = \lambda A + \mu I, \tag{4}$$

where $\lambda \neq 0$ ($\lambda = 0$ being covered by the preceding observation).

Proposition 2. *Under condition* (4),

$$P(A, B) = \lambda F(C)^2 - \frac{\mu^2}{4\lambda}.$$

Here $C = A + \frac{\mu}{2\lambda}I$, and we use the convention $Z^2 = \{z^2 : z \in Z\}$ for subsets $Z \subset \mathbb{C}$.

Proof. Directly from the definition of the product field and (4) we see that

$$\begin{aligned} P(A, B) &= \{\langle Ax, x\rangle\langle Bx, x\rangle : \ \|x\| = 1\} \\ &= \{\langle Ax, x\rangle(\lambda\langle Ax, x\rangle + \mu) : \ \|x\| = 1\} = \{z(\lambda z + \mu) : z \in F(A)\} \\ &= \left\{\lambda\left(z + \frac{\mu}{2\lambda}\right)^2 - \frac{\mu^2}{4\lambda} : z \in F(A)\right\} = \left\{\lambda\zeta^2 - \frac{\mu^2}{4\lambda} : \zeta \in F(C)\right\}. \qquad \square \end{aligned}$$

Consequently, all possible shapes of $P(A, B)$ in the setting of Proposition 2 are affine transformations of (point-wise) squares of classical fields of values.

Suppose that A is a linear combination of I and a Hermitian matrix, but not a scalar multiple of the identity:

$$A = \alpha_1 I + \alpha_2 H \quad \text{with } \alpha_2 \neq 0,\ \ H = H^* \notin \mathbb{C}I.$$

Then (4) implies a similar representation for B:

$$B = \beta_1 I + \beta_2 H, \quad \beta_2 \neq 0.$$

The computation from the proof of Proposition 2 reveals that then

$$P(A, B) = \alpha_2\beta_2 \left(F(H) + \frac{\alpha_1\beta_2 + \alpha_2\beta_1}{2\alpha_2\beta_2} \right)^2 + \alpha_1\beta_1 - \frac{(\alpha_1\beta_2 + \alpha_2\beta_1)^2}{4\alpha_2\beta_2}.$$

Since $F(H)$ is a line segment, we immediately obtain:

Corollary 1. *Let A and B both be linear combinations of a Hermitian matrix H and the identity. Then $P(A, B)$ is a parabolic arc, possibly degenerating into a line segment.*

In the next statement, we list some other cases in which $P(A, B)$ is a line segment.

Proposition 3. *Suppose that either* (i) *one of the matrices A, B is a scalar multiple of the conjugate transposed of the other, or* (ii) *both A and B are scalar multiples of Hermitian matrices. Then $P(A, B)$ is a closed subinterval of a line passing through the origin.*

Proof. (i) If say $B = \lambda A^*$, then

$$P(A, B) = \{\langle Ax, x\rangle\langle\lambda A^* x, x\rangle :\ \|x\| = 1\} = \lambda\{|z|^2 : z \in F(A)\}.$$

But $\{|z|^2 : z \in F(A)\}$ is a closed interval in $\mathbb{R}_+$, due to the convexity of the field of values.

(ii) If $A = \alpha H_1$, $B = \beta H_2$, with H_1, H_2 being Hermitian, then $P(A, B) \subset \alpha\beta\mathbb{R}$. Since $P(A, B)$ is closed, bounded and connected, it is therefore a closed subinterval of the line $\alpha\beta\mathbb{R}$. $\square$

For $n = 2$, condition (4) holds (possibly after switching A with B) if and only if A and B commute. So, for two commuting 2-by-2 matrices A, B the product field $P(A, B)$ is either a parabolic arc or an affine image of the square of an elliptical disk (see Figs. 4 and 7 in Appendix A). Note that already in this setting the product field may fail to be convex.

5. Empty interior

The main result of this section is the criterion for the interior of $P(A, B)$ to be empty.

Theorem 2. *The interior of the product field of values $P(A, B)$ is empty if and only if one of the following* (*overlapping*) *conditions holds:*

(i) *A and B are scalar multiples of Hermitian matrices,*

(ii) *A and B are linear combinations of the identity matrix and some Hermitian matrix:*

$$A = \alpha_1 I + \alpha_2 H, \quad B = \beta_1 I + \beta_2 H, \tag{5}$$

(iii) *one of the matrices A, B is a scalar multiple of the conjugate transposed of the other.*

Sufficiency is of course a direct consequence of Corollary 1 and Proposition 3. Moreover, as soon as the necessity in Theorem 2 has been established, the description of all possible shapes of $P(A, B)$ follows immediately.

Corollary 2. *For $A, B \in M_n(\mathbb{C})$, if $P(A, B)$ has empty interior, then $P(A, B)$ is either a point, a line segment, or a parabolic arc.*

Before proving necessity, we establish a number of auxiliary results in the case $n = 2$.

In what follows, we treat $\mathbb{C}$ as a real inner product space, with inner product $\langle z_1, z_2\rangle = \mathrm{Re}(z_1\bar{z}_2)$. Note that two complex numbers z_1 and z_2 are *orthogonal* if and only if $\langle z_1, z_2\rangle = 0$. Two complex numbers are *real linearly dependent* if and only if $\langle z_1, iz_2\rangle = 0$ or equivalently $\mathrm{Im}(z_1\bar{z}_2) = 0$.

Lemma 4. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be 2-by-2 matrices with complex entries. Assume neither matrix is normal. If $P(A, B)$ has empty interior, then there is a unitary U such that UAU^* and UBU^* both have constant diagonal.*

Proof. Suppose that no such U exists. We may assume without loss of generality that $a_{11} = a_{22}$ but $b_{11} \neq b_{22}$. Let $x = x(\theta) = \begin{bmatrix} \cos(\theta/2) \\ \omega\sin(\theta/2)\end{bmatrix}$ where $\omega \in \mathbb{C}$, $|\omega| = 1$. The expression $p(\theta) = (x(\theta)^*Ax(\theta))(x(\theta)^*Bx(\theta))$ is a second degree trigonometric polynomial with complex coefficients of the form:

$$p(\theta) = \alpha_0 + \alpha_1\cos\theta + \beta_1\sin\theta + \alpha_2\cos 2\theta + \beta_2\sin 2\theta.$$

Expanding the polynomial, we compute:

$$\alpha_2 = -\frac{1}{2}\left(\frac{a_{12}\omega + a_{21}\bar{\omega}}{2}\right)\left(\frac{b_{12}\omega + b_{21}\bar{\omega}}{2}\right), \quad \text{and} \quad \beta_2 = \frac{1}{2}\left(\frac{a_{12}\omega + a_{21}\bar{\omega}}{2}\right)\left(\frac{b_{11} - b_{22}}{2}\right).$$

Since B is not normal, $\left(\frac{b_{12}\omega+b_{21}\bar{\omega}}{2}\right)$ and $\left(\frac{b_{11}-b_{22}}{2}\right)$ must be real linearly independent for some ω. When α_2 and β_2 are real linearly independent, it follows from the proof of [8, Theorem 1] that $p(\theta)$ is a closed curve with at most finitely many self-intersections (that is, points $\theta_1 \neq \theta_2$ such that $p(\theta_1) = p(\theta_2)$). The proof given in [8] is for polynomials in $e^{i\theta}$ but essentially the same argument applies to trigonometric polynomials with complex coefficients. Since a continuous closed curve with only finitely many self-intersections must enclose a nonempty subset of $\mathbb{C}$ and since the range of $p(\theta)$ is contained in $P(A, B)$, it follows that $P(A, B)$ has nonempty interior. □

Lemma 5. *Let $\Phi \in \mathbb{C}[X, Y]$ be the polynomial*

$$\Phi(X, Y) = c_0 + c_1X + c_2Y + c_3X^2 + c_4XY + c_5Y^2,$$

and assume that c_1, c_2, and c_4 are not zero. If the image of an open domain $D \subset \mathbb{R}^2$ under Φ has empty interior in $\mathbb{C}$, then $c_1X + c_2Y$ factors as $\alpha(s_1X + s_2Y)$ where $\alpha \in \mathbb{C}$ and $s_1, s_2 \in \mathbb{R}$ and one of the following conditions is also true.

(1) *$c_3X^2 + c_4XY + c_5Y^2$ factors as $\beta(s_1X + s_2Y)^2$, $\beta \in \mathbb{C}$.*
(2) *$c_3 = s_3\alpha$, $c_4 = s_4\alpha$, and $c_5 = s_5\alpha$ where each $s_i \in \mathbb{R}$.*

Proof. We calculate the Jacobian determinant

$$D\Phi = \begin{vmatrix} \frac{\partial \operatorname{Re} \Phi(X,Y)}{\partial X} & \frac{\partial \operatorname{Re} \Phi(X,Y)}{\partial Y} \\ \frac{\partial \operatorname{Im} \Phi(X,Y)}{\partial X} & \frac{\partial \operatorname{Im} \Phi(X,Y)}{\partial Y} \end{vmatrix}.$$

The constant term of the Jacobian determinant is $\operatorname{Im} c_2 \operatorname{Re} c_1 - \operatorname{Im} c_1 \operatorname{Re} c_2 = 2\operatorname{Im}(c_2\bar{c}_1)$. If $\Phi(D)$ has empty interior, then the Jacobian determinant is identically zero, so $\operatorname{Im} c_2\bar{c}_1 = 0$. Therefore c_2 and c_1 are real linearly dependent. The coefficient of the X^2 term of the Jacobian determinant is $2(\operatorname{Im} c_4 \operatorname{Re} c_3 - \operatorname{Im} c_3 \operatorname{Re} c_4) = 4\operatorname{Im} c_4\bar{c}_3$. The Y^2 coefficient is $2(\operatorname{Im} c_5 \operatorname{Re} c_4 - \operatorname{Im} c_4 \operatorname{Re} c_5) = 4\operatorname{Im} c_5\bar{c}_4$. Thus, if $\Phi(D)$ has empty interior, c_4 and c_5 must be real linearly dependent, as are c_4 and c_3. This proves that there exist $s_1, s_2, s_3, s_4, s_5 \in \mathbb{R}$ such that $c_1 = \alpha s_1$, $c_2 = \alpha s_2$ and $c_3 = \beta s_3$, $c_4 = \beta s_4$, and $c_5 = \beta s_5$ for some $\alpha, \beta \in \mathbb{C}$.

If we substitute s_i, α, and β into the Jacobian determinant formula, we get

$$D\Phi = (\operatorname{Re}\alpha \operatorname{Im}\beta - \operatorname{Im}\alpha \operatorname{Re}\beta)(s_1s_4X - 2s_2s_3X + 2s_1s_5Y - s_2s_4Y).$$

Consequently either, α and β are real linearly dependent, or $s_1s_4 - 2s_2s_3 = 0$ and $2s_1s_5 - s_2s_4 = 0$. In the later case, $\frac{s_3}{s_4} = \frac{s_1}{2s_2}$ and $\frac{s_5}{s_4} = \frac{s_2}{2s_1}$. Thus

$$s_3X^2 + s_4XY + s_5Y^2 = \frac{s_4}{2}\left(\frac{s_1}{s_2}X^2 + 2XY + \frac{s_2}{s_1}Y^2\right) = \frac{s_4}{2s_1s_2}(s_1X + s_2Y)^2. \quad \square$$

In [1], it was observed that the field of values of a 2-by-2 matrix is the affine image of a 2-sphere. In the following proposition we give an explicit formula for this affine linear transformation.

Proposition 4. *Let $A = [a_{ij}] \in M_2(\mathbb{C})$. The field of values $F(A)$ is the image of the unit sphere $S^2 = \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$ under the affine linear transformation*

$$\hat{f}(X, Y, Z) = \frac{1}{2}\operatorname{tr}(A) + \left(\frac{a_{12} + a_{21}}{2}\right)X + \left(\frac{a_{12} - a_{21}}{2}\right)iY + \left(\frac{a_{22} - a_{11}}{2}\right)Z. \tag{6}$$

Proof. For any $x \in \mathbb{C}^2$ with $||x|| = 1$, let

$$q(x) = (2|x_1||x_2|\cos\theta, 2|x_1||x_2|\sin\theta, |x_2|^2 - |x_1|^2), \tag{7}$$

where x_1 and x_2 are the entries of x and $\theta = \arg(x_2/x_1)$. Note that $q(x)$ is the stereographic projection of x_2/x_1 onto the Riemann sphere represented as the unit sphere S^2 in $\mathbb{R}^3$. In particular, q maps two unit vectors to the same point if and only if they are scalar multiples. Thus we may consider the map $\hat{f}$ defined by the commutative diagram below.

$$\begin{tikzcd} S^3 \arrow[r, "f"] \arrow[d, "q"'] & F(A) \\ S^2 \arrow[ur, dashed, "\hat{f}"'] & \end{tikzcd}$$

In order to derive an equation for $\hat{f}$, we expand $f(x)$. By scalar multiplication, we may assume that x has entries $(|x_1|, |x_2|e^{i\theta})$ with $\theta = \arg(x_2/x_1)$. Then,

$$
\begin{aligned}
f(x) &= \begin{bmatrix} |x_1| & |x_2|e^{-i\theta} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2|e^{i\theta} \end{bmatrix} \\
&= a_{11}|x_1|^2 + a_{22}|x_2|^2 + (a_{12}+a_{21})|x_1||x_2|\cos\theta + (a_{12}-a_{21})|x_1||x_2|i\sin\theta \\
&= \left(\frac{a_{11}+a_{22}}{2}\right) + \left(\frac{a_{22}-a_{11}}{2}\right)Z + \left(\frac{a_{12}+a_{21}}{2}\right)X + \left(\frac{a_{12}-a_{21}}{2}\right)iY,
\end{aligned}
$$

where (X, Y, Z) are the entries of $q(x)$. This expression in terms of (X, Y, Z) is the formula for $\hat{f}$. In particular, $\hat{f}$ is an affine linear map from $\mathbb{R}^3$ into the complex plane. □

Lemma 6. *Let $\alpha, \beta \in \mathbb{C}^2\backslash\{0\}$ and define $f(x) = x^*(\alpha\beta^T)x$. Let D denote the real unit disk $\{x \in \mathbb{R}^2 : x^Tx \leqslant 1\}$. If $f(D)$ has empty interior, then the matrix $\alpha\beta^T$ is normal or a multiple of a matrix with real entries.*

Proof. Note that $f(D) = \operatorname{conv}(f(S^1) \cup \{0\})$ where S^1 is the unit circle $S^1 = \{x \in \mathbb{R}^2 : x^Tx = 1\}$. The image $f(S^1)$ is a subset of the field of values $F(\alpha\beta^T)$.

By Proposition 4, $F(\alpha\beta^T)$ is the image of the unit sphere in $\mathbb{R}^3$ under an affine linear transformation $\hat{f}$ given by (6). If $\alpha\beta^T$ is normal, then $F(\alpha\beta^T)$ is a line segment connecting the eigenvalues of $\alpha\beta^T$. Since $\alpha\beta^T$ is rank 1, it follows that $F(\alpha\beta^T)$ is contained in a line passing through the origin, so $f(D)$ must have empty interior. Therefore, let us assume that $\alpha\beta^T$ is not normal. The field of values $F(\alpha\beta^T)$ is a filled ellipse and the rank of the linear part of $\hat{f}$ is two. Thus the nullity of the linear part of $\hat{f}$ is one and any element of the nullspace is mapped by $\hat{f}$ to $\operatorname{tr}(\alpha\beta^T)/2$. The intersection of the nullspace with the Riemann sphere consists of two points, which we call the *North and South poles* of $\hat{f}$.

The set $q(S^1)$ is a great circle on the Riemann sphere. The image of a great circle under the map $\hat{f}$ will be an ellipse, unless the great circle passes through both the North and South poles of $\hat{f}$. In that case, the image of the great circle is a nontrivial line segment passing through $\operatorname{tr}(A)/2$.

The eigenvectors of $A = \alpha\beta^T$ are α with eigenvalue $\lambda = \beta^T\alpha$ and $\beta^\perp = \begin{bmatrix} \beta_2 \\ -\beta_1 \end{bmatrix}$ with eigenvalue 0. It is possible that $\beta^\perp$ and α are multiples, in which case 0 is an eigenvalue with algebraic multiplicity 2, but geometric multiplicity 1.

Since f maps an eigenvector to its eigenvalue, it follows that a great circle passing through both $q(\beta^\perp)$ and $q(\alpha)$ must also pass through the North and South poles of $\hat{f}$. Note that a great circle is defined by any two points it passes through. This implies that if a great circle passes through the North pole or the South pole and also passes through $q(\beta^\perp)$ or $q(\alpha)$, then all four points are on the great circle. In that case both $\beta^\perp$ and α will be multiples of real vectors, and so is β, which implies that $\alpha\beta^T$ is a multiple of a real matrix. We will now show that this is the only way $\alpha\beta^T$ can not be normal and still have $f(D)$ with empty interior.

If the great circle $q(S^1)$ passes through the North and South pole of $\hat{f}$ and does not pass through either $q(\alpha)$ or $q(\beta^\perp)$, then $f(S^1)$ will be a nontrivial line segment that is not contained in a the line passing through the origin. Let z_1 and z_2 denote the endpoints of this line segment. Then $f(D)$ is the convex hull of $\{0, z_1, z_2\}$ which has nonempty interior. □

Lemma 7. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be 2-by-2 matrices with complex entries and constant diagonals. Assume neither matrix is normal. Let*

$$\alpha = \begin{bmatrix} a_{12} + a_{21} \\ i(a_{12} - a_{21}) \end{bmatrix}, \quad \beta = \begin{bmatrix} b_{12} + b_{21} \\ i(b_{12} - b_{21}) \end{bmatrix}.$$

Define $f(x) = x^*(\alpha\beta^T)x$ and $D = \{x \in \mathbb{R}^2 : ||x|| \leqslant 1\}$. If $f(D)$ has empty interior, then $b_{12} = \gamma\bar{a}_{21}$ and $b_{21} = \gamma\bar{a}_{12}$ for some $\gamma \in \mathbb{C}\backslash\{0\}$.

Proof. By Lemma 6, the rank 1 matrix $\alpha\beta^T$ must be either normal or a multiple of a real matrix. We will show that $\alpha\beta^T$ cannot be a multiple of a real matrix. Note that $\alpha\beta^T$ is a multiple of a real matrix if and only if both α and β are multiples of vectors with real entries. We may treat $\mathbb{C}$ as a real inner-product space, with $\langle z_1, z_2\rangle = \mathrm{Re}(z_1\bar{z_2})$. Then α is a multiple of a real vector if and only if $\langle \alpha_1, i\alpha_2\rangle = 0$. Note that $\langle \alpha_1, i\alpha_2\rangle = \langle a_{12} + a_{21}, a_{21} - a_{12}\rangle = |a_{21}|^2 - |a_{12}|^2$. Thus α is a multiple of a real vector if and only if $|a_{12}| = |a_{21}|$ which cannot happen since A is not normal. We conclude that $\alpha\beta^T$ is normal.

Since $\alpha\beta^*$ must have zero as an eigenvalue, it is normal if and only if it is a multiple of a 2-by-2 Hermitian matrix. Therefore, $\beta = \gamma\bar{\alpha}$ for some $\gamma \in \mathbb{C}$.

Thus

$$\begin{aligned} b_{12} + b_{21} &= \gamma(\bar{a}_{12} + \bar{a}_{21}), \\ i(b_{12} - b_{21}) &= -\gamma i(\bar{a}_{12} - \bar{a}_{21}). \end{aligned}$$

Solving for b_{12} and b_{21} gives $b_{12} = \gamma\bar{a}_{21}$ and $b_{21} = \gamma\bar{a}_{12}$. □

Proposition 5. *Given $A, B \in M_2(\mathbb{C})$ such that neither A nor B is normal, $P(A, B)$ has empty interior if and only if B is a multiple of A^*.*

Proof. If B is a multiple of A^*, then by property 7 of Proposition 1, $P(A, B)$ is contained in a line passing through the origin. Suppose $P(A, B)$ has empty interior. By Lemma 4 we may assume that both A and B have constant diagonal. Using (6), we see that $P(A, B)$ is the image of the unit disk $D = \{(X, Y) \in \mathbb{R}^2 : X^2 + Y^2 \le 1\}$ under the following map

$$P(X, Y) = \frac{1}{4}(c_0 + c_1X + c_2Y + c_3X^2 + c_4XY + c_5Y^2),$$

where

$$\begin{aligned} c_0 &= (a_{11} + a_{22})(b_{11} + b_{22}), \\ c_1 &= (a_{12} + a_{21})(b_{11} + b_{22}) + (a_{11} + a_{22})(b_{12} + b_{21}), \\ c_2 &= i(a_{12} - a_{21})(b_{11} + b_{22}) + i(a_{11} + a_{22})(b_{12} - b_{21}), \\ c_3 &= (a_{12} + a_{21})(b_{12} + b_{21}), \\ c_4 &= i(a_{12} + a_{21})(b_{12} - b_{21}) + i(a_{12} - a_{21})(b_{12} + b_{21}), \\ c_5 &= -(a_{12} - a_{21})(b_{12} - b_{21}). \end{aligned}$$

By Lemma 5 the coefficients c_1 and c_2 are real linearly dependent as are c_3 with c_4 and c_5 with c_4. In particular the image of $c_3X^2 + c_4XY + c_5Y^2$ has empty interior. Note that if

$$\alpha = \begin{bmatrix} a_{12} + a_{21} \\ i(a_{12} - a_{21}) \end{bmatrix}, \quad \beta = \begin{bmatrix} b_{12} + b_{21} \\ i(b_{12} - b_{21}) \end{bmatrix},$$

then $c_3X^2 + c_4XY + c_5Y^2 = x^*(\alpha\beta^T)x$ where $x = \begin{bmatrix} X \\ Y \end{bmatrix}$. Therefore, by Lemma 7 there exists $\gamma \in \mathbb{C}$ such that $b_{12} = \gamma\bar{a}_{21}$ and $b_{21} = \gamma\bar{a}_{12}$ and therefore $\beta = \gamma\bar{\alpha}$. It follows that $c_3X^2 + c_4XY + c_5Y^2 = \gamma x^*(\alpha\alpha^*)x = \gamma||x^*\alpha||^2 = \gamma||(a_{12} + a_{21})X + i(a_{12} - a_{21})Y)||^2$. As noted in the proof of Lemma 7, $(a_{12} + a_{21})$ and $i(a_{12} - a_{21})$ are real linearly independent since A is not normal. Therefore,

$c_3X^2 + c_4XY + c_5Y^2$ cannot factor as $c(s_1X + s_2Y)^2$ where $c \in \mathbb{C}$ and $s_1, s_2 \in \mathbb{R}$. Thus the constants c_3, c_4, c_5 must all lie on the same line passing through the origin as c_1 and c_2, according to Lemma 5.

Substituting into the formulas for c_1 and c_2 gives

$$c_1 = (a_{12} + a_{21})\text{tr}\,(B) + \gamma\,\text{tr}\,(A)(\bar{a}_{12} + \bar{a}_{21}) = \alpha_1\text{tr}\,(B) + \gamma\,\text{tr}\,(A)\bar{\alpha}_1,$$
$$c_2 = i(a_{12} - a_{21})\text{tr}\,(B) - i\gamma\,\text{tr}\,(A)(\bar{a}_{12} - \bar{a}_{21}) = \alpha_2\text{tr}\,(B) + \gamma\,\text{tr}\,(A)\bar{\alpha}_2,$$

Since c_3, c_4, and c_5 are all real multiples of γ, it follows that c_1 and c_2 are also real multiples of γ. Thus both

$$\frac{\alpha_1}{\gamma}\text{tr}\,(B) + \text{tr}\,(A)\bar{\alpha}_1 \quad \text{and} \quad \frac{\alpha_2}{\gamma}\text{tr}\,(B) + \text{tr}\,(A)\bar{\alpha}_2$$

are real. Consequently

$$\text{Im}\left(\frac{\alpha_1}{\gamma}\text{tr}\,(B)\right) = -\,\text{Im}\,(\text{tr}\,(A)\bar{\alpha}_1)\,,$$

$$\text{Im}\left(\frac{\alpha_2}{\gamma}\text{tr}\,(B)\right) = -\,\text{Im}\,(\text{tr}\,(A)\bar{\alpha}_2)\,.$$

Expanding in terms of the real and imaginary parts of tr (B) gives

$$\text{Im}\left(\frac{\alpha_1}{\gamma}\right)\text{Re}(\text{tr}\,(B)) + \text{Re}\left(\frac{\alpha_1}{\gamma}\right)\text{Im}(\text{tr}\,(B)) = -2\,\text{Im}\,(\text{tr}\,(A)\bar{\alpha}_1)\,,$$

$$\text{Im}\left(\frac{\alpha_2}{\gamma}\right)\text{Re}(\text{tr}\,(B)) + \text{Re}\left(\frac{\alpha_2}{\gamma}\right)\text{Im}(\text{tr}\,(B)) = -2\,\text{Im}\,(\text{tr}\,(A)\bar{\alpha}_2)\,.$$

Since α_1/γ and α_2/γ are real linearly independent, the matrix

$$\begin{bmatrix} \text{Im}\left(\frac{\alpha_1}{\gamma}\right) & \text{Re}\left(\frac{\alpha_1}{\gamma}\right) \\ \text{Im}\left(\frac{\alpha_2}{\gamma}\right) & \text{Re}\left(\frac{\alpha_2}{\gamma}\right) \end{bmatrix}$$

is nonsingular. Therefore there is a unique solution for tr (B), which is tr $(B) = \gamma\overline{\text{tr}\,(A)}$. Since both A and B have constant diagonals, $b_{11} = b_{22} = \gamma\bar{a}_{11}$. This completes the proof that B is a multiple of A^*. □

Proposition 6. *For $A, B \in M_2(\mathbb{C})$ such that A is normal, if $P(A, B)$ has empty interior, then either A and B are simultaneously unitarily diagonalizable or they are both multiples of Hermitian matrices.*

Proof. First, suppose that B is not normal. We will show that $P(A, B)$ must have nonempty interior. In this case, we may assume without loss of generality that A is a diagonal matrix

$$A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Let $x = \begin{bmatrix} \sqrt{t} \\ \sqrt{1-t}e^{i\theta} \end{bmatrix}$ where $t \in (0, 1)$ is fixed and $\theta \in [0, 2\pi)$ is allowed to vary. Then

$$\langle Ax, x\rangle \langle Bx, x\rangle = (ta_{11} + (1-t)a_{22}) \left(tb_{11} + (1-t)b_{22} + (b_{12}e^{i\theta} + b_{21}e^{-i\theta})\sqrt{t(1-t)} \right),$$

and as θ varies, this curve will be a nondegenerate ellipse since B is not normal and therefore $|b_{12}| \neq |b_{21}|$.

We may therefore assume that both A and B are normal. We will assume also that neither matrix is a multiple of the identity since that case is trivial. Then both matrices have two distinct eigenvectors up to scalar multiplication. Using Proposition 4, the field of values of both matrices is the image of the unit sphere in $\mathbb{R}^3$ under an affine linear map. In both cases, the affine linear map must be rank one since A and B are both normal, and the field of values of a normal matrix is a line segment connecting the two eigenvalues. Let λ_1^A, λ_2^A denote the eigenvalues of A with corresponding eigenvectors x_1^A and x_2^A respectively. Similarly let λ_1^B, λ_2^B denote the eigenvalues of B corresponding to x_1^B and x_2^B. Let $f_A(x) = \langle Ax, x\rangle$ and $f_B(x) = \langle Bx, x\rangle$ and let $\hat{f}_A$ and $\hat{f}_B$ denote the corresponding affine linear maps as in Proposition 4. Since $f_A(x_1^A) = \lambda_1^A$ and $f_A(x_2^A) = \lambda_2^A$ which are the endpoints of $F(A)$, it follows for q given by (7) that $q(x_1^A)$ and $q(x_2^A)$ are antipodal points on S^2. Using the inner-product on $\mathbb{R}^3$, we may write the rank one affine linear map $\hat{f}_A$ as

$$\hat{f}_A(y) = \tfrac{1}{2}\mathrm{tr}\,(A) + \tfrac{1}{2}(\lambda_1^A - \lambda_2^A)\langle y, q(x_1^A)\rangle,$$

where $y \in S^2 \subset \mathbb{R}^3$. Similarly we may write

$$\hat{f}_B(y) = \tfrac{1}{2}\mathrm{tr}\,(B) + \tfrac{1}{2}(\lambda_1^B - \lambda_2^B)\langle y, q(x_1^B)\rangle.$$

Let $\hat{X} = \langle y, q(x_1^A)\rangle$ for any $y \in S^2 \subset \mathbb{R}^3$ and let $\hat{Y} = \langle y, q(x_1^B)\rangle$ and note that the image of S^2 under the map $y \mapsto (\hat{X}, \hat{Y})$ contains an open set around 0 if and only if $q(x_1^A)$ and $q(x_1^B)$ are linearly independent. Assume for now that this is the case. Then $P(A, B)$ is the image of a set in $\mathbb{R}^2$ with nonempty interior under the map

$$\Phi(\hat{X}, \hat{Y}) = \left(\tfrac{1}{2}\mathrm{tr}\,(A) + \tfrac{1}{2}(\lambda_1^A - \lambda_2^A)\hat{X}\right)\left(\tfrac{1}{2}\mathrm{tr}\,(B) + \tfrac{1}{2}(\lambda_1^B - \lambda_2^B)\hat{Y}\right).$$

Note that

$$\Phi(\hat{X}, \hat{Y}) = \frac{1}{4}(c_0 + c_1\hat{X} + c_2\hat{Y} + c_3\hat{X}^2 + c_4\hat{X}\hat{Y} + c_5\hat{Y}^2),$$

where

$$\begin{aligned} c_0 &= \mathrm{tr}\,(A)\mathrm{tr}\,(B),\\ c_1 &= \mathrm{tr}\,(B)(\lambda_1^A - \lambda_2^A),\\ c_2 &= \mathrm{tr}\,(A)(\lambda_1^B - \lambda_2^B),\\ c_3 &= c_5 = 0,\\ c_4 &= (\lambda_1^A - \lambda_2^A)(\lambda_1^B - \lambda_2^B). \end{aligned}$$

By Lemma 5, $c_2 = \mathrm{tr}\,(A)(\lambda_1^B - \lambda_2^B)$ and $c_4 = (\lambda_1^A - \lambda_2^A)(\lambda_1^B - \lambda_2^B)$ must be real linearly dependent. Factoring out $(\lambda_1^B - \lambda_2^B)$ we find that $\lambda_1^A + \lambda_2^A$ and $\lambda_1^A - \lambda_2^A$ are real linearly dependent. Therefore λ_1^A and λ_2^A both lie on the same line as $\mathrm{tr}\,(A)$ and $\lambda_1^A - \lambda_2^A$, and A must be a multiple of a Hermitian matrix. A similar argument holds for B.

Recall that we temporarily assumed that $q(x_1^A)$ and $q(x_1^B)$ were linearly independent. If that is not the case, then the eigenvectors of A are multiples of the eigenvectors of B and the two matrices can be simultaneously diagonalized by the same unitary congruence. □

We are ready now to prove the necessity part of Theorem 2.

Necessity. For $n = 2$ the result follows from Propositions 5 and 6 as soon as one observes that commuting normal 2-by-2 matrices satisfy condition (ii) of Theorem 2.

Let now A and B be $n \times n$ matrices such that $P(A, B)$ has empty interior. Then the same is true for $P(A', B')$, where $A' = V^*AV$, $B' = V^*BV$ for any n-by-2 isometry V. From the already obtained result for the case $n = 2$ it follows that the 2-by-2 matrices A', B' satisfy one of the conditions (i)–(iii) of Theorem 2. Recalling the standard notation

$$\mathrm{Re}\,X = \frac{1}{2}(X + X^*), \quad \mathrm{Im}\,X = \frac{1}{2i}(X - X^*),$$

we observe that among the four matrices $\mathrm{Re}\,A'$, $\mathrm{Im}\,A'$, $\mathrm{Re}\,B'$, $\mathrm{Im}\,B'$ there are at most two linearly independent. Since this is true for any V as described above, Theorem 2 of [10] allows us to conclude that actually there are at most two linearly independent matrices amongst $\mathrm{Re}\,A$, $\mathrm{Re}\,B$, $\mathrm{Im}\,A$, $\mathrm{Im}\,B$.

Now, if both pairs $\{\mathrm{Re}\,A, \mathrm{Im}\,A\}$ and $\{\mathrm{Re}\,B, \mathrm{Im}\,B\}$ are linearly dependent, condition (i) is apparently satisfied. Therefore, suppose that at least one of these pairs is linearly independent. Without loss of generality, let it be $\{\mathrm{Re}\,A, \mathrm{Im}\,A\}$. Then $\mathrm{Re}\,B$, $\mathrm{Im}\,B$ must be linear combinations of $\mathrm{Re}\,A$, $\mathrm{Im}\,A$. In other words,

$$B = \lambda A + \mu A^*. \tag{8}$$

If $\lambda = 0$ in (8), then (iii) holds. Otherwise, by scaling we may suppose that $\lambda = 1$, and from (8)

$$P(A, B) = \{z^2 + \mu\,|z|^2 : z \in F(A)\} = \{\zeta + \mu\,|\zeta| : \zeta \in \Omega\},$$

where $\Omega = \{\zeta = z^2 : z \in F(A)\}$.

Note that the Jacobian of the mapping

$$\zeta \mapsto \zeta + \mu\,|\zeta|$$

on $\mathbb{C} \setminus \{0\}$ is

$$1 + \frac{\langle \zeta, \mu \rangle}{|\zeta|}.$$

Since it cannot equal zero identically on any open set, from the emptyness of the interior of $P(A, B)$ it follows that the interior of Ω, and therefore of $F(A)$, must be empty.

Due to the convexity of $F(A)$ the latter constrain means that $F(A)$ is actually a line segment, that is, A is as prescribed by the first formula in (5). From this and (8) we conclude that B also is as prescribed by (5). Thus, the remaining case (ii) holds.

Appendix A. Numerical examples

The following are images of the product field of values for the specified matrices A and B. All images were generated from MATLAB. Inside most of these product fields is a curve of the form $f(x_1 \cos t + x_2 \sin t)$ for fixed x_1 and x_2 where $f : \mathbb{C}S^n \to \mathbb{C}$ is the function $f(x) = \langle Ax, x\rangle\langle Bx, x\rangle$.

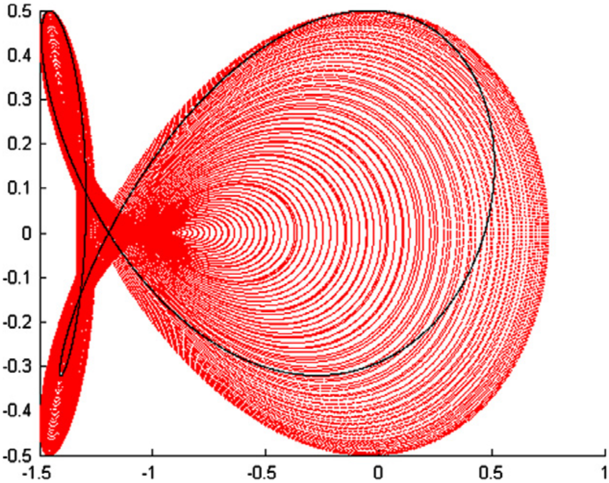

Fig. 1. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/5} \end{bmatrix}.$

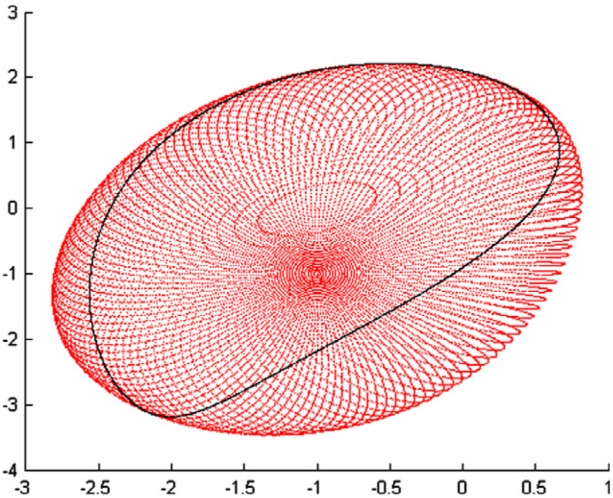

Fig. 2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1+i \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ 4i & -1 \end{bmatrix}.$

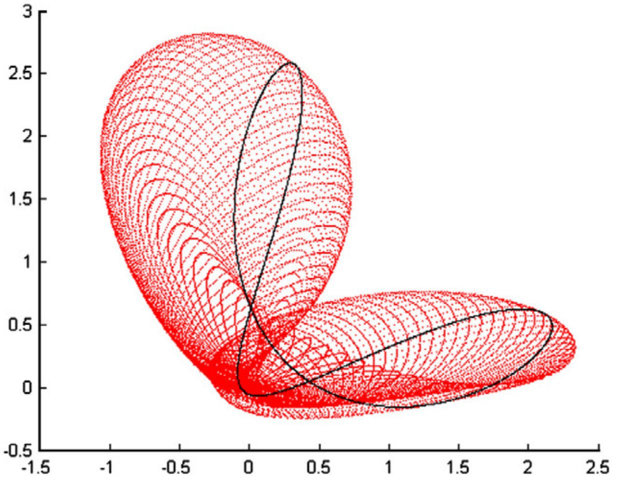

Fig. 3. $A = \begin{bmatrix} 1 & 1-3i \\ 0 & 2i \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1+i & 0 \end{bmatrix}$.

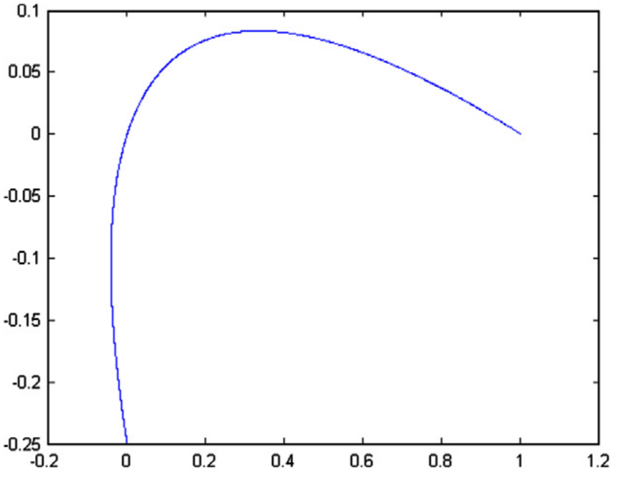

Fig. 4. $A = \begin{bmatrix} 1 & 0 \\ 0 & i/2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$

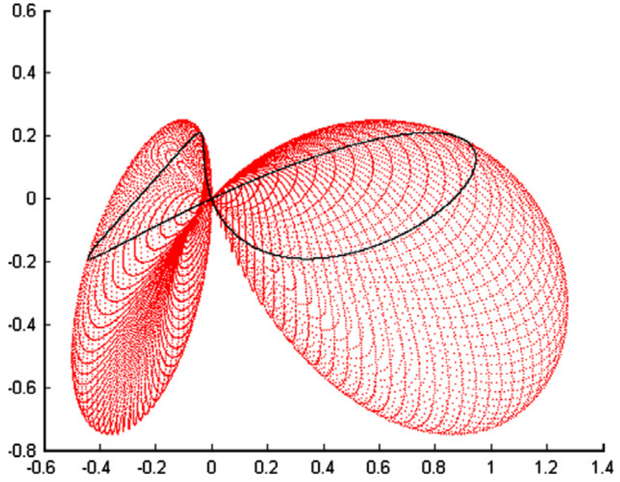

Fig. 5. $A = \begin{bmatrix} 1 & 1+i \\ 0 & i \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -i & 0 \end{bmatrix}$.

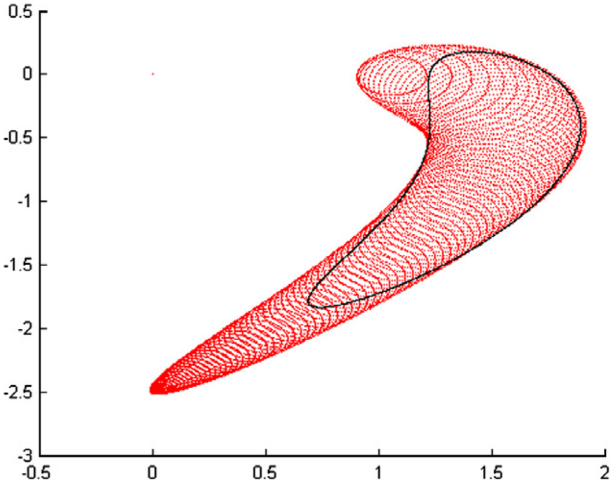

Fig. 6. $A = \begin{bmatrix} 1 & 0 \\ 0 & -i/2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1-i/2 \\ 0 & 5 \end{bmatrix}$.

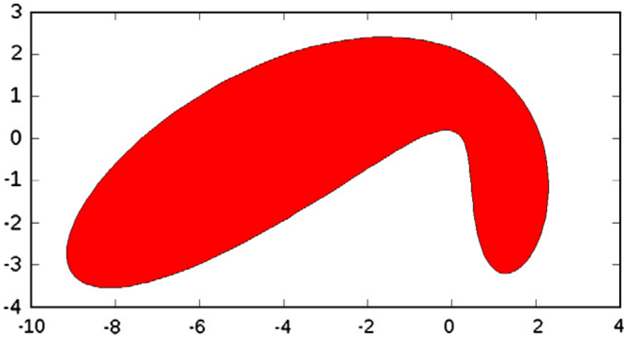

Fig. 7. $A = B = \begin{bmatrix} 2i & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$.

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