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Asymmetric TP and TN completion problems ☆

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ABSTRACT

We first give a complete list of polynomial conditions on the data for TP (TN) completability of partial TP (TN) matrices with just one or two unspecified entries in either or both of the upper right or lower left entries. These results are used to identify which such patterns, and related patterns, are TP (TN) completable. Then, the TN completable echelon and TP completable jagged patterns are characterized. This generalizes earlier work on combinatorially symmetric TN completable patterns.

1. Introduction

An m-by-n matrix is called *totally positive*, TP (*totally nonegative*, TN) if all its minors are positive (nonnegative). The many interesting properties of such matrices may be found in [4], as well as in some prior references [1,6,12]. A *partial matrix* [7] is one in which some entries are specified and the remaining unspecified entries are free to be chosen. A *completion* of a partial matrix is a choice of values for the unspecified entries, resulting in a conventional matrix. Since TP and TN are conditions inherited by all submatrices, a necessary condition that a partial matrix have a TP (TN) completion is that it be *partial TP (TN)*, i.e., that all the minors based entirely upon specified entries are positive (nonnegative). The *pattern* of a partial matrix is an inventory of the positions of the specified entries. A pattern $\mathcal{P}$ is TP (TN) *completable* if every partial TP (TN) matrix with pattern $\mathcal{P}$ has a TP (TN) completion.

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For other patterns $\mathcal{Q}$, some partial TP (TN) matrices will have a TP (TN) completion and others will not. Additional conditions on the data (specified entries) are needed to assure the existence of a TP (TN) completion. As a consequence of the Tarski–Seidenburg principle [2], each pattern $\mathcal{Q}$ will have a finite list of additional, besides partial TP (TN), polynomial inequality conditions on the data that are equivalent to the existence of a TP (TN) completion. Such conditions, however, are not generally easy to find. There are prior results about TP (TN) completable patterns [9,5,11]. Most notably, in [9] the combinatorially symmetric, TN-completable patterns were characterized. They are the monotonically labelled, block clique graphs (chordal graphs with all vertex separators of one vertex and multiplicity one, labelled so that maximal cliques occur in numerical order). They are easily seen to be a very special case of our asymmetric results in Section 3. In [10] it was shown that the combinatorially symmetric TN-completable patterns are also the TP- completable ones. We conjecture that our TN-completable patterns identified here are also TP-completable. There are additional completability conditions for other particular patterns, e.g., [8,3], in the literature.

Our purpose here is to augment both knowledge about (TN/TP) completable patterns and about the conditions for non-completable patterns. We first identify the additional polynomial conditions for TP completability of those patterns with one or two unspecified entries in the upper right and/or lower left corner. We also consider, in addition, the possibility of multiple unspecified entries in the upper left or lower right. Second, we then consider rather general completable patterns in which the specified entries are concentrated toward the main diagonal.

We call a (possibly rectangular) pattern *echelon* if, whenever a position is unspecified, either all positions north and east (NE) of it are unspecified or south and west (SW) of it are. For example,

$$\begin{bmatrix} * & * & * & ? & ? & ? & ? & ? \\ * & * & * & * & * & ? & ? & ? \\ ? & * & * & * & * & ? & ? & ? \\ ? & ? & ? & * & * & * & ? & ? \\ ? & ? & ? & * & * & * & * & ? \\ ? & ? & ? & ? & ? & * & * & * \end{bmatrix}$$

in which $*$ denotes a specified and ? an unspecified position, is echelon. Among the echelon patterns, we characterize those that are TN completable, using the results about completability conditions just mentioned.

We call a pattern that is a 90 rotation of an echelon pattern a *jagged* pattern, i.e. if a position is unspecified, then either every position north and west of it is, or every position south and east of it is. An example of a jagged pattern is

$$\begin{bmatrix} ? & ? & ? & * & * & * & * \\ ? & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & ? & ? \\ * & * & * & * & ? & ? & ? \end{bmatrix}$$

We also show that every jagged pattern is TP completable. Examples are given to show that such patterns need not be TN completable, so that there is a difference between TN and TP completability of patterns. (We guess that the TN completable patterns are contained in the TP completable ones, properly.) We also give conjectures about completability of jagged echelon patterns.

2. Patterns with few unspecified entries

Consider the m-by-n pattern P_1 with just one unspecified entry in the $(1, n)$ position and its transpose P_1'. We call these *one-sided patterns*. The m-by-n pattern P_2, with just 2 unspecified entries, in the $(1, n)$ and $(m, 1)$ positions, is the *two-sided pattern*. Since an m-by-n matrix is TP if and only if its *initial minors* [4] are positive, a one-sided partial TP matrix, say with pattern P_1, has a TP completion if and only if the upper right entry can be chosen so that the contiguous upper right minors are all positive. Call these minors
$UR_i(x) = \det A(x)[\{1, 2, \ldots, i\}; \{n-i+1, \ldots, n\}]$, in which x denotes the unspecified $(1, n)$ entry of the partial TP matrix A of pattern P_1, $i = 1, 2, \ldots, \min\{m, n\}$. Here, as throughout, we use the standard submatrix notation: $A[\alpha; \beta]$ denotes the submatrix of A lying in the rows indexed by α and the columns indexed by β. So, we study the conditions on x under which $UR_i(x) > (\geq) 0$. Since $UR_i(x)$ is a linear function of x, this is not difficult. We have $UR_i(x) = UR_i(0) + x(-1)^{i+1} detA[\{2, \ldots, i\}; \{n-i+1, \ldots, n-1\}]$. Thus, if i is odd, we have $UR_i(x) > (\geq) 0$ iff

$$x > (\geq) - UR_i(0)/ \det A[\{2, \ldots, i\}; \quad \{n-i+1, \ldots, n-1\}]$$

and if i is even, $UR_i(x) > (\geq) 0$ iff

$$x > (\geq) UR_i(0)/ \det A[\{2, \ldots, i\}; \{n-i+1, \ldots, n-1\}].$$

In case $i = 1$, the RHS lower bound above is 0. Call the lower bound in the first inequality $URL_i(A)$ and the upper bound in the second inequality $URU_i(A)$, so that the third symbol L indicates that i is odd and the third symbol U indicates that i is even. Now, let

$$URL(A) = \max_{i=1,3,\ldots} URL_i(A)$$

and

$$URU(A) = \min_{i=2,4,\ldots} URU_i(A)$$

Note that if $m \leq n$, then $URL(A)$ and $URU(A)$ depend only upon the last m columns of A, and if $m \geq n$, they depend only upon the first n rows of A. We then have the following characterization of those partial TP matrices of pattern P_1 that have a TP completion, in terms only of rational expressions (that could be converted to polynomial expressions) in the specified entries.

Theorem 1. *Let A be an m-by-n partial TP matrix of pattern P_1. Then, A has a TP completion if and only if $URL(A) < URU(A)$.*

Proof. The inequality holds iff there are values x in the interval
$(URL(A), URU(A))$ iff there are values x such that all upper right minors $URU_i(x) > 0$. Since these are the only initial minors containing x, it is necessary and sufficient that they be positive, as all other initial minors are fully specified and positive, because A is partial TP. □

There is an analogous result for the pattern P_1', obtained via transposition, If A is a partial TP matrix of pattern P_1', define

$$LLL(A) = URL(A^T)$$

and

$$LLU(A) = URU(A^T)$$

Then,

Theorem 2. *Let A be an m-by-n partial TP matrix of pattern P_1'. Then A has a TP completion if and only if*

$$LLL(A) < LLU(A)$$

There are corresponding statments about TN completion with non-strict inequalities when the relevant denominator minors are positive.

We next turn to the pattern P_2. Note that if P_2 is m-by-n with $m \neq n$, there is no incremental difficulty compared to two separate P_1 and P_1' problems, as there is no contiguous minor containing both the upper right and lower left entries. Thus, we consider only the case of P_2 being n-by-n. In this event, the only contiguous minor containing both unspecified entries is the determinant itself. Thus, there is a P_1 subproblem and a P_1' subproblem, which must both be solvable in such a way that their intervals permit a pair for which the determinant is positive. Let $A = A(x, y)$ be a partial TP matrix of pattern P_2, in which x is the unspecified entry in the $1, n$ position, and y is the unspecified entry in the $m, 1$ position. Then, $A_{n1} = A[\{1, 2, \ldots, n-1\}; \{2, 3, \ldots, n\}]$ is a partial TP matrix of pattern P_1, and $A_{1n} = A[\{2, \ldots, n\}; \{1, \ldots, n-1\}]$ is a partial TP matrix of pattern P_1'. If $URL(A_{n1}) < URU(A_{n1})$ and $LLL(A_{1n}) < LLU(A_{1n})$, then A will have a TP completion iff there is an $x \in (URL(A_{n1}), URU(A_{n1}))$ and a $y \in (LLL(A_{1n}), LLU(A_{1n}))$ such that $detA(x, y) > 0$.

A special case of Sylvester's determinantal identity [HJ] that has proven to be useful in the analysis of TP/TN matrices is the following. If A is n-by-n and partitioned as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & A_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

with A_{22} $(n-2)$-by-$(n-2)$ and nonsingular, then

$$\det A = \frac{\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & A_{22} \end{bmatrix} \det\begin{bmatrix} A_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - \det\begin{bmatrix} a_{12} & a_{13} \\ A_{22} & a_{23} \end{bmatrix} \det\begin{bmatrix} a_{21} & A_{22} \\ a_{31} & A_{32} \end{bmatrix}}{\det A_{22}} \tag{1}$$

Because of Sylvester's determinantal identity, $\det A(x, y)$ is maximized by choosing x so as to minimize $\det A_{in}$ and y so as to minimize $\det A_{n1}$, i.e., x and y at extremes of their intervals. We thus have.

Theorem 3. *Suppose that A is a partial TP matrix of pattern P_2. Then A has a TP completion if and only if*

(i) $URL(A_{n1}) < URU(A_{n1})$;
(ii) $LLL(A_{1n}) < LLU(A_{1n})$; *and*
(iii)(a) $detA(URL(A_{n1}), LLL(A_{1n})) > 0$ *if n is odd; or*

(b) $detA(URU(A_{n1}), LLU(A_{1n})) > 0$ *if n is even.*

Proof. Conditions (i) and (ii) are clearly necessary. In their presence (iii) is sufficient as the choice of $x = URL(A_{n1}) + \epsilon$ and $y = LLL(A_{1n}) + \epsilon$ for ϵ sufficiently small when n is odd and the choice of $x = URU(A_{n1}) - \epsilon$ and $y = LLU(A_{1n}) - \epsilon$ for ϵ sufficiently small when n is even will make $\det A(x, y) > 0$. Condition (iii) is necessary, as no other choice of x and y in their respective intervals can make $\det A(x, y)$ larger because of Sylvester's identity. □

We note that each of Theorems 1–3 may be extended to the TN case, easily when the denominator minors (coefficient of x and/or y) are positive, and, more subtly, when some are 0 (which may imply that others are). In the former case, the strict inequality simply becomes nonstrict, and in the latter, the resulting 0 coefficients lead to fewer conditions. Since, in general, more than just initial minors need be checked for TN, this may not fully settle the TN cases, but this will not be necessary for the completion results later.

If the upper left and/or lower right entry or entries of a P_1, P_1' or P_2 are unspecified, then the pattern may become TP completable. This is because, by Sylvester's identity, upper left or lower right principal minors may be made sufficiently large so as to overcome the product of upper right and lower left minors of the same size. Of course, if an upper right or lower left submatrix is a not completable P_1 or P_1' pattern, then the pattern in question is not completable.

Example 1. The following $A(x) = \begin{bmatrix} 1.1 & 1 & 0.56 & x \\ 0.95 & 1.012 & 0.99 & 1 \\ 0.45 & 0.5 & 0.7 & 1 \\ 0.2245 & 0.25 & 0.6 & 1.1 \end{bmatrix}$ 4-by-4 partial TP matrix has no TP, nor even TN, completion, though $A(x)$ is partial TP, as $\det(A(x)) = -0.002804 - 0.179850x$ is always negative for $x \geqslant 0$. Another example may be found on p. 187 of [4].

This means that the m-by-n P_1 and P_1' (by transposition) patterns are not TP completable when $m, n \geqslant 4$. They are also not TN completable.

Example 2. Let $A(x, y) = \begin{bmatrix} 1 & 1 & 0.4 & x \\ 0.4 & 1 & 1 & .4 \\ 0.2 & 0.8 & 1 & 1 \\ y & 0.2 & 0.4 & 1 \end{bmatrix}$, which is partial TP, but A has no TN completion because $\det(A(x, y)) = -.0016 - .008x - .328y - .2xy$ is always negative for $x, y \geqslant 0$. So, the m-by-n P_2 pattern is not TP completable for $m, n \geqslant 4$, and also not TN completable (see page 187 of [4]).

We may now summarize what we know about our special patterns P_1, P_1' and P_2. An m-by-n partial positive and partial TP (TN) matrix of type P_1, P_1' and P_2, with m or $n \leqslant 3$, always has a TP (TN) completion. If the minimum of m and n is at least 4, this is no longer true and these patterns are neither TP nor TN completable. However, we have given the necessary additional polynomial conditions for completability. If, in addition, in the 4-by-4 case, either the upper left, lower right position or a position adjacent to an already unspecified position is also unspecified, then such a pattern is completable.

3. Echelon and jagged patterns

A pattern is called *jagged* if, for each unspecified entry, either (1) all entries north and west of it (i.e., if in the (i, j) positive, then all positions (k, l) with $k \leqslant i, l \leqslant j$) are unspecified or (2) all entries south and east of it (i.e., if in the (i, j) position, then all positions (k, l) with $k \geqslant i, l \geqslant j$) are unspecified. We refer to case (1) only as *upper left jagged* and case (2) only as *lower right* jagged. Either of these is referred to as *singly jagged* and when both occur, we say *doubly jagged*.

A pattern is called *echelon* if, for each unspecified entry, either (3) all entries both north and east of it (i.e., if in the (i, j) position, then all positions (k, l) with $k \leqslant i, l \geqslant j)$ are unspecified or (4) all entries south and west of it (i.e., if in the (i, j) position, then all positions (k, l) with $k \geqslant i$ and $l \leqslant j)$ are unspecified. We refer to case (3) only as *upper right echelon* and case (4) only as *lower left echelon*. Either of these is referred to as *single echelon*, while when both occur, we say *double echelon*. Echelon refers to any of these possibilities.

Finally, a pattern in which we ask only that at least one of (1) and (2) and at least one of (3) or (4) occur, is called *jagged echelon*.

We now turn to general echelon and jagged patterns with the purpose of giving our two primary results, along with examples and lemmas that may be of general interest.

Theorem 4. *An echelon pattern is TN completable if and only if it contains no 4-by-4 P_1 , P_1' or P_2 as a subpattern.*

We note that an echelon or jagged echelon pattern contains a P_1, P_1' or P_2 subpattern if and only if it contains a contiguous (consecutive row and column index sets) one of the same size. This makes checking the condition in Theorem 4 straightforward, especially as the requirement of 4 may be replaced by $k \geqslant 4$, in any case, and, in particular for P_2. We also note that Theorem 4 asymmetricaly and significantly generalizes the ranking general TN completion result of [9], as monotonically labelled block clique graphs (and many more patterns) contain none of the forbidden subpatterns. For jagged patterns there are no combinational restrictions in the TP-completable case.

Theorem 5. *Each jagged pattern is TP completable.*

However, such patterns need not be TN-completable.

Examples. Neither singly nor doubly jagged patterns need be TN completable. (a) The pattern $\begin{bmatrix} ? & ? & * \\ * & * & * \\ * & * & * \end{bmatrix}$ is not TN-completable as shown by the partial TN data $\begin{bmatrix} ? & ? & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. The $\{1, 3\}$,$\{2, 3\}$ and $\{1, 2\}$,$\{1, 2\}$ submatrices are in conflict, as the former requires the second ? to be >1,while the latter requires it to be 0, in order to have a TN completion.

(b) The pattern $\begin{bmatrix} ? & ? & * \\ * & * & ? \\ * & * & ? \end{bmatrix}$ is not TN-completable as shown by the partial TN data $\begin{bmatrix} ? & ? & 1 \\ 2 & 0 & ? \\ 1 & 1 & ? \end{bmatrix}$. The previously mentioned conflict is still present.

The small size of these example indicates, as we shall see, that most jagged patterns are not TN-completable. However, we conjecture the following

Conjecture 1. *An echelon pattern is TP completable if and only if it contains no P_1,P_1' or P_2 as a subpattern.*

This conjecture would be to Theorem 4 what the [10] result is to the [9] result.

Then, we would also conjecture a statement that marries Theorem 5 and Conjecture 1 as a comprehensive result that nearly follows from Conjecture 1.

Conjecture 2. *A jagged echelon pattern is TP completable if and only if it contains no P_1, P_1' or P_2 as a subpattern.*

Because of the above examples an analogous statement for TN completability cannot be valid.

The strategy for proof of Theorems 4 and 5 is as follows. The necessity of the condition in both follows from a natural, more general, lemma that we prove (Lemma 1). For sufficiency, we first prove Theorem 4. This is by showing inductively that if an echelon pattern has none of the offending 4-by-4 patterns, it has sufficiently many unspecified entries that either its NE or SW submatrices with one fewer rows and one fewer columns can be completed to be rank deficient, while the remaining submatrices necessary for application of Sylvester's Identity are TN. Then, by Sylvester, the maximal determinants will be nonnegative. This all uses a uniform completion strategy that we call *generalized standard* GS, which conveys to submatrices (that are necessarily also echelon). This gives the proof of Theorem 4. Theorem 5 is then proven by ordering the upper left/lower right unspecified entries so that, for any data, they may, in order, be made sufficiently large that the partial TP matrix has a TP completion. This completes the proof of Theorem 5.

We now turn to the details.

Definition . An upper "corner" in an echelon pattern is a specified position, for which the positions above and to the right are specified, while the next position to the NE is not. A lower corner is defined similarly for the lower echelon portion of an echelon pattern. The *generalized standard* completion of a partial TN matrix with an echelon pattern is defined as follows. Consider an upper corner in the upper echelon portion of a partial TN matrix and choose values for those unspecified entries that complete the block determined by the corner, the specified entries in the row and in the column of the corner entry, so as to produce a rank 1, or less, if possible, block. Completion in the lower echelon portion is similar. This process continues, in the same way, until, with decreasing numbers of corners, all unspecified entries are chosen.

For example, if the corner data is positive, we complete

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & ? & ? \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ ? & ? & a_{33} & a_{34} & a_{35} \\ ? & ? & a_{43} & a_{44} & a_{45} \end{bmatrix} \text{ to } \begin{bmatrix} a_{11} & a_{12} & a_{13} & \frac{a_{13}a_{24}}{a_{23}} & \frac{a_{13}a_{25}}{a_{23}} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \frac{a_{21}a_{32}}{a_{23}} & \frac{a_{22}a_{33}}{a_{23}} & a_{33} & a_{34} & a_{35} \\ \frac{a_{21}a_{43}}{a_{23}} & \frac{a_{22}a_{43}}{a_{23}} & a_{43} & a_{44} & a_{45} \end{bmatrix}$$

If some corner datum is 0, then, the partial TN data guarantees that either all specified entries NE or SW of it are 0, as they also will be in the GS completion.

The general fact that implies the necessity of our conditions is the following.

Lemma 1. *Suppose that $\mathcal{P}$ is not a TN-completable (TP-completable) pattern and that $\mathcal{P}$ sits contiguously in $\mathcal{Q}$ as a subpattern, then $\mathcal{Q}$ is not TN-completable (TP-completable).*

Proof. Let A be a partial TN (TP) matrix of pattern $\mathcal{P}$ that has no TN (TP) completion. We wish to embed A as a contiguous submatrix in a partial TN (TP) matrix B of pattern $\mathcal{Q}$. Then, since A is not completable, B cannot be completable. We do this inductively as follows. Border A with a row at the top (if appropriate - otherwise a row at the bottom or a column to one side or the other) by choosing one entry at a time, working right to left (or appropriate order, otherwise). Choose each successive entry to be large enough to make any minor it completes nonnegative (positive). This may be done as this entry will be the upper left entry of any such minor and the complementary minor will be nonnegative (positive). If the complementary minor is 0, we may border to produce a larger minor of 0. Now, replace any entry that has been specified by an unspecified entry, as appropriate, and continue, as needed, until a partial TN (TP) B has been produced. (In the event of a row at the bottom, work left to right; a column at the left, work bottom to top; and a column the right, top to bottom.) $\square$

We now give a series of lemmas about rank, which, in view of the GS completion strategy, show that NE or SW blocks are rank deficient, which supports application of Sylvester's Identity. The first is known and elementary.

Lemma 2. *If $G \in M_{m,n}(\mathbb{F})$ has a p-by-q submatrix of rank r, then*

$$\operatorname{rank}(G) \leqslant r + (m+n) - (p+q) \tag{2}$$

Corollary 1. *If $G \in M_{m,n}(\mathbb{F})$, $m \leqslant n$, has a p-by-q submatrix of rank r, then, if $r+n < p+q$, $\operatorname{rank}(G) < m$, i.e., G is rank deficient.*

Lemma 3. *If $\mathcal{P}$ is an m-by-n, $m \leqslant n$, echelon pattern containing no 4-by-4 P_1, P_1' or P_2 subpattern, then $\mathcal{P}$ has a contiguous s-by-t block of unspecified entries with $s + t = n - 1$.*

Proof. As the claim can be verified by exhaustion for smaller values of m and n, we assume that $4 < m \leqslant n$. Suppose that the claimed conclusion does not hold. Then, for every contiguous block of unspecified entries $s + t \leqslant n - 2$. Among such blocks in the NE, ending before row $m - 3$, choose the one for which $s + t$ is a maximum. Suppose that the values are s_1 and t_1. Ties do not matter. Now pick that block with largest sum in the SW, beginning with now $s_1 + 3$. Because $s_i + t_i \leqslant n - 2$, $i = 1, 2$, there will either be a P_2 that includes row $s_1 + 1$ and $s_2 + 2$, or a P_1 or P_1' that includes $s_1 + 1$ or $s_1 + 2$, each k-by-k with $k \geqslant 4$. $\square$

Lemma 4. *Let $\mathcal{P}$ be an m-by-n echelon pattern that contains no 4-by-4 subpattern P_1, P_1' or P_2. Then in the GS completion A of any partial TN matrix of pattern $\mathcal{P}$, either $A(1; n)$ or $A(m; 1)$ is rank deficient.*

Proof. Since the NE (SW) block of the GS completion, determined inclusively by any corner entry of the upper (lower) echelon part has rank at most 1, Lemma 3, applied to Corollary 1 verifies the claim. Which of the two blocks $A(1; n)$ or $A(m; 1)$ is rank deficient is determined by whether the block guaranteed in Lemma 3 is upper or lower. Of course, both could occur. $\square$

It is clear that echelon patterns have the following useful feature.

Lemma 5. *If $\mathcal{Q}$ is the result of deleting a column or row from an echelon pattern $\mathcal{P}$, then $\mathcal{Q}$ is an echelon pattern. Moreover, if A is the GS completion of a partial TN matrix with pattern $\mathcal{P}$, then a similar deletion from A will leave blocks whose rank is limited by those of A.*

With the preceding lemmas we may now turn to a proof of Theorem 4. The proof is inductive, focusing upon the smaller of m and n, which, by the symmetry of echelon patterns, we may assume to be m. It is easy to check the validity of the theorem for values of $m \leqslant 4$, using the results of the prior section and the GS completion. This begins the induction. Now, suppose we consider a value of m and that the theorem is correct for all smaller values of m, using the GS completion technique. To show that maximal minors (m-by-m) in the GS completion are nonnegative, we use Sylvester's identity, the induction hypothesis (applied to NW and SE minors) and Lemma 4 (to see that either the NE or SW minor is 0). Smaller minors come more directly from the inductive hypothesis. They are minors of completions of echelon partial matrices, with smaller values of m, that are tantamount to the GS completion, by Lemma 5. This completes the proof of Theorem 4.

We now turn to Theorem 5, and first prove it in case there is only one unspecified entry.

Lemma 6. *A jagged pattern with just one unspecified entry is TP-completable.*

Proof. By the definition of jagged. The unspecified entry must be in the (1, 1) or final, lower right position. The argument in the two cases is the same. Suppose the (1, 1) position. Then, the unspecified entry enters positively into each minor it completes, as, in each, the complementary minor is positive, due to the partial TP assumption. Thus, this entry may be chosen sufficiently large that all minors it completes are also positive. □

A simple consequence is the singly jagged case, when Lemma 6 is applied in the proper order.

Lemma 7. *Any singly jagged pattern is TP-completable.*

Proof. Again the two cases are the same. Suppose the pattern is upper left jagged. Now the unspecified entries may be ordered lexicographically, beginning with the most southerly and easterly, so that each successive unspecified entry may be chosen by applying Lemma 2 to the submatrix of all entries south and east of it (including it). This application leaves an upper left jagged partial TP matrix that is successively completed. □

Now, to complete the proof of Theorem 5, distinguish two cases, when the pattern is doubly jagged. In the first, suppose that *no* unspecified entry has all the entries *both* NW *and* SE of it unspecified. In this event, Lemma 7 may be applied twice, first to, say, a maximal upper left submatrix that is singly echelon and then to the entire pattern after the upper left entries have been chosen. Otherwise, when there are such unspecified entries, the completion may again be divided into two stages via unspecified, central lines, again applying Lemma 7, insertion of such lines (using the result of [11]) and finally applying the first mentioned case. This completes the proof of Theorem 5.

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