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Ken-ichi Kawarabayashi

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Gexin Yu
*William & Mary*, gyu@wm.edu

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## Notes

# Connectivities for *k*-knitted graphs and for minimal counterexamples to Hadwiger's Conjecture

Ken-ichi Kawarabayashi [a,1], Gexin Yu [b,2]

[a] *National Institute of Informatics and JST, ERATO, Kawarabayashi Large Graph Project,*
*2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan*
[b] *Department of Mathematics, College of William and Mary, Williamsburg, VA 23185, United States*

**A B S T R A C T**

For a given subset $S \subseteq V(G)$ of a graph $G$, the pair $(G, S)$ is knitted if for every partition of $S$ into non-empty subsets $S_1, S_2, \ldots, S_t$, there are disjoint connected subgraphs $C_1, C_2, \ldots, C_t$ in $G$ so that $S_i \subseteq C_i$. A graph $G$ is $\ell$-knitted if $(G, S)$ is knitted for all $S \subseteq V(G)$ with $|S| = \ell$. In this paper, we prove that every $9\ell$-connected graph is $\ell$-knitted.

Hadwiger's Conjecture states that every $k$-chromatic graph contains a $K_k$-minor. We use the above result to prove that the connectivity of minimal counterexamples to Hadwiger's Conjecture is at least $k/9$, which was proved to be at least $2k/27$ in Kawarabayashi (2007) [4].

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## 1. Introduction

One of the most interesting problems in graph theory is Hadwiger's Conjecture, which states that *every k-chromatic graph has a $K_k$-minor*, where a graph $H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges.

It is known that Hadwiger's Conjecture holds for $k \leqslant 6$. Wagner [11] in 1937 proved that the case $k = 5$ is equivalent to Four Color Theorem. About 60 years later, Robertson, Seymour and Thomas [8] proved that the case $k = 6$ is also equivalent to the Four Color Theorem. In their proof, minimal

counterexamples, which are also called "contraction-critical non-complete graphs", play an important role. Kawarabayashi and Toft [5] showed that 7-chromatic graphs contain a $K_7$-minor or a $K_{4,4}$-minor, in which the connectivity property of minimal counterexamples are, again, really important.

Many researchers have considered the connectivity property of contraction-critical graphs. Dirac [2] proved that every $k$-contraction-critical graph is 5-connected for $k \geqslant 5$, and Mader [7] extended 5-connectivity to the deep result that every $k$-contraction-critical graph is 7-connected for $k \geqslant 7$ and every 6-contraction-critical graph is 6-connected. Toft [10] proved that $k$-contraction-critical graphs are $k$-edge-connected. Kawarabayashi [4] proved the first general result on the vertex connectivity of minimal counterexamples to Hadwiger's Conjecture.

**Theorem 1.** *(See Kawarabayashi [4].) For all positive integers $k$, every minimal (with respect to the minor relation) $k$-chromatic counterexample to Hadwiger's Conjecture is $\lceil \frac{2k}{27} \rceil$-connected.*

In the proof of the above theorem, the main tool used was so-called $k$-linked graphs. A graph $G$ is *k-linked* if for every $2k$ distinct vertices $u_1, v_1, u_2, v_2, \ldots, u_k, v_k$ in $G$, there are $k$ disjoint paths $P_1, P_2, \ldots, P_k$ such that $P_i$ connects $u_i$ and $v_i$. $k$-linked graphs are very well-studied and play a very important role in the study of graph structures.

In this paper, we improve the result in Theorem 1, by studying a notion called "knitted graph" introduced by Bollobás and Thomason [1].

For $1 \leqslant m \leqslant k \leqslant |V(G)|$, a graph is *(k, m)-knit* if whenever $S$ is a set of $k$ vertices of $G$ and $S_1, \ldots, S_t$ is a partition of $S$ into $t \geqslant m$ non-empty parts, $G$ contains vertex-disjoint connected subgraphs $C_1, \ldots, C_t$ such that $S_i \subseteq V(C_i)$, $1 \leqslant i \leqslant t$. Clearly, a $(2k, k)$-knit graph is $k$-linked. In [1], Bollobás and Thomason proved that *if a $k$-connected graph $G$ contains a minor $H$, where $H$ is a graph with minimum degree at least $0.5(|H| + \lfloor 5k/2 \rfloor - 2 - m)$, then $G$ is $(k, m)$-knit.* They used this result to show that $22k$-connected graphs are $k$-linked, which is the first linear upper bound of connectivity for a graph to be $k$-linked.

We consider a slightly more general notion than $(k, m)$-knit. For a set $S \subseteq V(G)$ of a graph $G$, the pair $(G, S)$ *is knitted* if for every partition of $S$ into non-empty subsets $S_1, S_2, \ldots, S_t$, there are disjoint connected subgraphs $C_1, C_2, \ldots, C_t$ in $G$ so that $S_i \subseteq C_i$. A graph $G$ is *ℓ-knitted* if $(G, S)$ is knitted for all $S \subseteq V(G)$ with $|S| = \ell$. It is clear that an $\ell$-knitted graph is $(\ell, m)$-knit for all $m \leqslant \ell$.

In this paper, we give a connectivity condition for a graph to be $\ell$-knitted.

**Definition 1.** The pair $(A, B)$ is a *separation* of $G$ if $V(G) = A \cup B$ and there is no edge between $A - B$ and $B - A$. The *order* of a separation $(A, B)$ is $|A \cap B|$. If $S \subseteq A$, then we say that $(A, B)$ is a separation of $(G, S)$.

We shall prove the following theorem.

**Theorem 2.** *Let $k$ and $\ell$ be positive integers and $S \subseteq V(G)$ with $|S| = \ell < k/9$. If there is no separation of $(G, S)$ of size less than $\ell$, and every vertex in $G - S$ has degree at least $k - 1$, then $(G, S)$ is knitted.*

The theorem we will prove, Theorem 7, on edge-density in Section 3 is actually stronger than Theorem 2.

We are now ready to state and prove our result on connectivity of minimal counterexamples to Hadwiger's Conjecture.

**Theorem 3.** *For all positive integer $k$, every $k$-chromatic minimal (with respect to the minor relation) counterexample to Hadwiger's Conjecture is $\lceil \frac{k}{9} \rceil$-connected.*

**Proof.** Assume by contradiction that the statement fails. Then we have a minimal $k$-chromatic graph $G$ that has no $K_k$-minor and is not $k/9$-connected. Take a minimum cutset $S$. Then $|S| < k/9$. Let $A_1$ be a component of $G - S$ and $A_2 = G - S - A_1$. Then both $G[A_1 \cup S]$ and $G[A_2 \cup S]$ have the chromatic number at most $k - 1$.

Let $S_1$ be a maximum independent set in $G[S]$, and let $S_i$ be a maximum independent set in $G[S - \bigcup_{j=1}^{i-1} S_j]$ for $i \geqslant 2$. Let $v_1, v_2, \ldots, v_{|S|}$ be the set of vertices in $S$ such that $v_1, \ldots, v_{|S_1|} \in S_1, v_{|S_1|+1}, \ldots, v_{|S_1|+|S_2|} \in S_2$, and so on. Observe that if we contract each of the subgraph induced by $S_i$ into one vertex, then the resulting graph in $S$ is a clique.

Note that the minimum degree of $G$ is at least $k - 1$, thus each vertex in $A_p$ has at least $k - 1$ neighbors in $A_p \cup S$ for $p \in \{1, 2\}$. Note also that a separation in $(A_1 \cup S, S)$ or $(A_2 \cup S, S)$ is a separation in $(G, S)$, thus $(A_1 \cup S, S)$ and $(A_2 \cup S, S)$ have no separation of size less than $\ell$. By Theorem 2, both $(A_1 \cup S, S)$ and $(A_2 \cup S, S)$ are knitted. So there are disjoint connected subgraphs $C_i \subseteq A_1 \cup S$'s and $D_i \subseteq A_2 \cup S$ so that $S_i \subseteq C_i$ and $S_i \subseteq D_i$. Hence we can contract $A_1 \cup S$ into $S_1, S_2, \ldots$ such that the resulting graph on $S$ is complete. Let $G_1$ be the resulting graph plus $A_2$. Similarly, we can also contract $A_2 \cup S$ into $S_1, S_2, \ldots$ such that the resulting graph on $S$ is complete (let $G_2$ be the resulting graph plus $A_1$).

Then $\chi(G_1), \chi(G_2) \leqslant k - 1$ by minimality of $G$. But clearly we can combine the colorings of $G_1$ and $G_2$ to the whole graph $G$ using at most $k - 1$ colors. This is a contradiction. This completes the proof of the theorem. □

The rest of the paper is to prove Theorem 2. We will do this in two steps: in the first step (Section 3), we will show a graphs under study either is knitted or has a dense subgraph; in the second step (Section 2), we find a knitted subgraph in the dense subgraph. Note that this approach is very much similar to the one used by Thomas and Wollan [9].

## 2. Dense graphs are knitted

In this section, we study when a small dense graph contains a knitted subgraph. This is needed in our proof of Theorem 2 in Section 3.

To show a small dense graph is $k$-knitted, we use a result by Faudree et al. [3] on $k$-ordered graphs, where a graph is $k$-ordered if for every $k$ vertices of given order, there is a cycle containing the $k$ vertices of the given order. It is clear that a $k$-ordered graph is $k$-knitted. Throughout the paper, we will use $d(x, H)$ to denote the number of neighbors (degree) of $x$ in subgraph $H$ of $G$.

**Theorem 4.** *(See Faudree et al. [3].) For every graph $G$ with order $n \geqslant 2\ell \geqslant 2$, if $d(x, G) + d(y, G) \geqslant n + \frac{3\ell - 9}{2}$ for every pair of non-adjacent vertices $x$ and $y$, then $G$ is $\ell$-ordered.*

Note that for $n \geqslant 5\ell$, Kostochka and Yu [6] showed that a graph $G$ with minimum degree at least $\frac{n+\ell}{2} - 1$ is $\ell$-ordered. Since we do not know if the minimum degree condition still holds for $n < 5\ell$, we are unable to use this less demanding degree conditions in our proof.

**Theorem 5.** *Let $\alpha \geqslant 4.5$. A graph $H$ with minimum degree $\delta(H) \geqslant \alpha\ell + 1$ and $|V(H)| \leqslant 2\alpha\ell$ contains an $\ell$-knitted subgraph.*

**Proof of Theorem 5.** Assume by contradiction that $H$ is not $\ell$-knitted. Then there is a subset $S \subseteq V(H)$ with $|S| = \ell$, and a partition $S = \bigcup_{i=1}^{t} S_i$ such that we cannot find disjoint connected subgraphs containing $S_i$'s.

We consider partial $(\ell, t)$-knit $C = \bigcup_{i=1}^{t} C_i$, which is a subgraph of $G$ in which $S_i \subseteq C_i$ but $C_i$s are not necessarily connected.

An optimal $(\ell, t)$-knit $C = \bigcup_{i=1}^{t} C_i$ is a partial $(\ell, t)$-knit such that

(a) $|C| \leqslant \alpha\ell$;
(b) the number of components of $C$ is minimized; and
(c) subject to (a) and (b), $|C|$ is minimized.

We observe that the components in $C$ containing exactly one vertex in $S$ consist of one vertex, and a component with two vertices in $S$ is a path.

We may assume that $S_1 \subseteq C_1$, but $C_1$ is not connected. Then there exists $x, y \in S_1$ such that $x$ and $y$ belong to different components of $C_1$. Note that $H - C \neq \emptyset$, since $d(x, H - C) = d(x) - |C| \geq (\alpha \ell + 1) - \alpha \ell = 1$.

Now we show that for every $u \in H - C$ and for every component $P$ in $C$ with $|V(P) \cap S| \geq 2$, $d(u, P) \leq |V(P) \cap S| + 1$. We actually will give the following more general statement, which might be of independent interest.

**Lemma 1.** *Let $W$ be a graph. Let $S'$ be a subset of $V(W)$ with $|S'| \geq 2$, and let $F$ be subtree of $W$ such that $F \supseteq S'$ and all leaves of $F$ belong to $S'$. Let $u \in W - F$, and suppose that $d(u, F) \geq |S'| + 2$. Then $W[V(F) \cup \{u\}]$ contains a subtree $F_0$ with $u \in F_0$ such that $|F_0| < |F|$, $F_0 \supseteq S'$ and all leaves of $F_0$ belong to $S'$.*

**Proof.** Let $k = |S'|$. When $k = 2$, $F$ is a path with both leaves in $S'$, then since $d(u, F) \geq 4$, we can replace a segment of $F$ by $u$ to get a smaller subtree $F_0$ so that the leaves of $F_0$ belong to $S'$. So let $k \geq 3$.

Now we use induction on $|F|$. Note that $F$ has at least two leaves, and let $u_1, u_2 \in S$ be two of them. For $i = 1, 2$, let $P_i$ be maximal paths such that $u_i \in P_i$ and the subtree $F - V(P_i)$ contains $S' - \{u_i\}$. Note that $P_1 \cap P_2 = \emptyset$. For each $i$, let $x_i$ be the vertex in $F - P_i$ which is adjacent (in $F$) to an endpoint of $P_i$.

Let $i = 1$ or $2$. First assume $d(u, P_i) = 0$. Then by the induction assumption, $W[V(F - P_i) \cup \{u\}]$ contains a subtree $F'$ with $u \in F'$ such that $|F'| < |F - P_i|$, $F' \supseteq (S' - \{u'\}) \cup \{x_i\}$ and all leaves of $F'$ belong to $(S' - \{u_i\}) \cup \{x_i\}$. Adding $P_i$ to $F'$, we obtain a desired tree. Next assume $d(u, P_i) = 1$. Then by the induction assumption, $W[V(F - P_i) \cup \{u\}]$ contains a subtree $F'$ with $u \in F'$ such that $|F'| < |F - P_i|$, $F' \supseteq S' - \{u_i\}$ and all leaves of $F'$ belong to $S' - \{u_i\}$. Adding $P_i$ to $F'$, we obtain a desired tree. Thus we may assume $d(u, P_i) \geq 2$ for each $i = 1, 2$. Let $P_i = u_i P_i v_i v_i' P_i x_i'$ so that $x_i'$ is adjacent to $x_i$ and $v_i$ is the only neighbor of $u$ on $u_i P_i v_i$. Then $|V(v_i' P_i x_i')| \geq 1$. Now $F_0 = (F - \bigcup_{i=1}^{2} V(v_i' P_i x_i')) \cup \{u\}$ is a subtree (note that $k \geq 3$, so $F_0$ is connected) with desired properties. $\square$

Let $\delta^*$ be the minimum degree of $H - C$. We have the following

**Lemma 2.** $\delta^* \geq (\alpha - 1.5)\ell$.

**Proof.** For every $u \in H - C$,

$$d(u, H - C) = d(u, H) - d(u, C) \geq \delta(H) - d(u, C) \geq \alpha \ell + 1 - d(u, C).$$

So we just need to prove that $d(u, C) \leq 1.5\ell$ for every $u \in H - C$.

Let $P_j$, $1 \leq j \leq c_i$, be the components of $C_i$ in which $u$ has neighbors. If $|P_j \cap S| \geq 2$, then by Lemma 1 we have $d(u, P_j) \leq |P_j \cap S| + 1 \leq 3|P_j \cap S|/2$ and, if $|P_j \cap S| = 1$ then $|P_j| = 1$, and hence $d(u, P_j) = |P_j \cap S| \leq 3|P_j \cap S|/2$, which implies

$$d(u, C_i) = \sum_{j=1}^{c_i} d(u, P_j) \leq \sum_{j=1}^{c_i} 3|P_j \cap S|/2 \leq 3|C_i \cap S|/2.$$

Therefore $d(u, C) = \sum_{C_i} d(u, C_i) \leq 1.5|S| = 1.5\ell$, and the lemma is proven. $\square$

**Lemma 3.** *The subgraph $H - C$ is connected.*

**Proof.** Let $H_1, \ldots, H_p$ with $p \geq 1$ be the components of $H - C$. Then $H_i$ is not $\ell$-knitted, thus not $\ell$-ordered. So by Theorem 4, $2\delta^* < |H_i| + \frac{3\ell - 9}{2}$. Therefore we have

$$|H_i| > (2\alpha - 4.5)\ell + 4.5.$$

If $p \geq 2$, then $|H| \geq |C| + |H_1| + |H_2| > \ell + 2(2\alpha - 4.5)\ell + 9$, that is, $2\alpha \ell > (4\alpha - 8)\ell + 9$. So $(8 - 2\alpha)\ell > 9$, a contradiction to $\alpha \geq 4$. $\square$

**Lemma 4.** $|C| \leqslant \alpha \ell - 5$.

**Proof.** For otherwise, $|H - C| \leqslant 2\alpha\ell - |C| \leqslant 2\alpha\ell - (\alpha\ell - 4) = \alpha\ell + 4$. Then $2\delta^* - (|H - C| + \frac{3\ell - 9}{2}) \geqslant (2\alpha - 3)\ell - (\alpha\ell + 4) - \frac{3\ell - 9}{2} = (\alpha - 4.5)\ell + 0.5 > 0$. By Theorem 4, $H - C$ is $\ell$-ordered, thus $\ell$-knitted, a contradiction.  □

Let $A = N(x) \cap (H - C)$ and $B = N(y) \cap (H - C)$. Furthermore, let $A' = N(A) \cap (H - C) - A$ and $B' = N(B) \cap (H - C) - B$. Let $D = (H - C) - (A \cup A' \cup B \cup B')$. Then there is no path of length at most 6 from $x$ to $y$ through $A \cup A' \cup D \cup B' \cup B$, for otherwise, we may get $C'$ by adding this path to $C$. Note that $C'$ has less components than $C$, and $|C'| \leqslant |C| + 5 \leqslant (\alpha\ell - 5) + 5 = \alpha\ell$, a contradiction to the assumption that $C$ is optimal.

Take $u \in D - N(A')$, then $u$ has no neighbors in $A' \cup A$. Take $v \in A$, then every pair of $u, v, y$ has no common neighbors in $H - C$. Thus $|H| \geqslant d(y) + d(u, H - C) + d(v, H - C) \geqslant \delta(H) + 2\delta^* > \alpha\ell + (2\alpha - 3)\ell = (3\alpha - 3)\ell$, and it follows that $2\alpha\ell > (3\alpha - 3)\ell$, or $\alpha < 3$, a contradiction.

## 3. Proof of Theorem 2

We first introduce some notations.

**Definition 2.** A separation $(A, B)$ of $(G, S)$ is rigid if $(G[B], A \cap B)$ is knitted.

For a set $H \subseteq V(G)$, let $\rho(H)$ be the number of edges with at least one endpoint in $H$.

**Definition 3.** Let $G$ be a graph and $S \subseteq V(G)$, and $\alpha > 1$ be a real number. The pair $(G, S)$ is $\alpha\ell$-*massed* if

  (i) $\rho(V(G) - S) > \alpha\ell|V(G) - S| - 1$, and
 (ii) every separation $(A, B)$ of $(G, S)$ of order at most $|S| - 1$ satisfies $\rho(B - A) \leqslant \alpha\ell|B - A|$.

**Definition 4.** Let $G$ be a graph and $S \subseteq V(G)$, and let $\alpha > 1$ be a real number. The pair $(G, S)$ is $(\alpha, \ell)$-minimal if

  1. $(G, S)$ is $\alpha\ell$-massed,
  2. $|S| \leqslant \ell$ and $(G, S)$ is not knitted,
  3. subject to above two, $|V(G)|$ is minimum,
  4. subject to above three, $\rho(G - S)$ is minimum, and
  5. subject to above four, the number of edges of $G$ with both ends in $S$ is maximum.

**Theorem 6.** Let $\ell \geqslant 1$ be an integer and $\alpha \geqslant 2$ be a real number. Let $G$ be a graph and $S \subseteq V(G)$ such that $(G, S)$ is $(\alpha, \ell)$-minimal. Then $G$ has no rigid separation of order at most $|S|$, and $G$ has a subgraph $H$ with $|V(H)| \leqslant 2\alpha\ell$ and minimum degree at least $\alpha\ell + 1$.

With Theorem 6 and Theorem 5, we can actually obtain the following result, which is a little stronger than Theorem 2.

**Theorem 7.** Let $\ell$ be an integer. Let $G$ be a graph and $S \subseteq V(G)$ be an $\ell$-subset such that $(G, S)$ is $(4.5, \ell)$-*massed*. Then $(G, S)$ is knitted.

**Proof.** Suppose that some $(4.5, \ell)$-massed graph is not knitted and take such a graph $G$ so that $(G, S)$ is $(4.5, \ell)$-minimal. By Theorems 6 and 5, the graph $G$ has no rigid separation of order at most $\ell$ and has an $\ell$-knitted subgraph $K$.

If there are $|S| = \ell$ disjoint paths from $S$ to $K$ (we may suppose that each path uses one vertex in $K$), then for every partition of $S$, there is a corresponding partition of the endpoints of the paths in $K$; since $K$ is knitted, there are disjoint connected subgraphs in $K$ containing the parts of the endpoints, thus we have disjoint connected subgraph containing the parts of $S$.

If there is no $|S|$ disjoint paths from $S$ to $K$, then there is separation $(A, B)$ with $S \subseteq A, K \subseteq B$ of order at most $\ell - 1$. We may assume $(A, B)$ is a separation with smallest order. Then there are $|A \cap B|$ disjoint paths from $A \cap B$ to $K$. Similar to the above, for every partition of $A \cap B$, we have disjoint connected subgraph containing the parts of $A \cap B$. So $G[B, A \cap B]$ is knitted, that is, $(A, B)$ is a rigid separation of order at most $\ell - 1$, a contradiction. $\square$

**Proof of Theorem 6.** We prove this theorem in the following three claims.

**Claim 1.** *$G$ has no rigid separation of order at most $|S|$.*

**Proof.** For otherwise, take a rigid separation $(A, B)$ with minimum $A$.

We first assume that $|A \cap B| < |S|$. Let $G^*[A]$ be the resulting graph from $G[A]$ by adding all missing edges in $A \cap B$. Consider $(G^*[A], S)$. If it also satisfies both (i) and (ii), then $(G^*[A], S)$ is knitted, and a knit in $G^*[A]$ can be easily converted into a knit in $G$ since $(A, B)$ is a rigid separation. Since $G$ is $\alpha\ell$-massed, $\rho(B - A) \leqslant \alpha\ell|B - A|$, hence $\rho(A - S) > \alpha\ell|A - S| - (\alpha - 0.5)\ell^2$. So it satisfies (i), and thus does not satisfy (ii).

Let $(A', B')$ be a separation of $G^*[A]$ such that $S \subseteq A'$ and $B'$ is minimal. If $A \cap B \subseteq A'$, then $(A' \cup B, B')$ is a separation in $G$ violating (ii). So $A \cap B \nsubseteq A'$. Since $A \cap B$ forms a cliques, $A \cap B \subseteq B'$. Consider $(G^*[B'], A' \cap B')$. The minimality of $B'$ implies that it satisfies (ii), and $\rho(B' - A') > \alpha\ell|B' - A'| > \alpha\ell|B' - A'| - 1$ means that it satisfies (i) as well. So $(G^*[B'], A' \cap B')$ is knitted. Then $(G^*[B \cup B'], A' \cap B')$ is knitted, which means that $A' \cap B'$ is a rigid separation of $(G, S)$, a contradiction to the minimality of $A$.

Now assume that $|A \cap B| = |S|$. If there exist $|S|$ disjoint paths from $S$ to $A \cap B$, then the paths together with the rigidity of $(A, B)$ show that $(G, S)$ is knitted, a contradiction. So there is a separation $(A'', B'')$ of $(G[A], S)$ of order less than $|S|$ with $A \cap B \subseteq B''$. Choose such a separation with minimum $|A'' \cap B''|$. Then there are $|A'' \cap B''|$ disjoint paths from $A'' \cap B''$ to $A \cap B$, from the rigidity of $(A, B)$ we have $(A'', B \cup B'')$ is a rigid separation of $(G, S)$ with $|A''| < |A|$, a contradiction to the minimality of $A$. $\square$

**Claim 2.** *For every edge $uv$ with $v \notin S$, the vertices $u$ and $v$ have at least $\alpha\ell$ common neighbors.*

**Proof.** Consider the graph $G' = G/uv$, the resulting graph from $G$ by contradicting the edge $uv$. If $(G', S)$ is knitted, then $(G, S)$ is knitted. So $(G', S)$ violates (i) or (ii).

If $(G', S)$ violates (i), then

$$\rho(G' - S) \leqslant \alpha\ell|G' - S| - 1 = \alpha\ell|G - S| - 1 - \alpha\ell < \rho(G - S) - \alpha\ell.$$

Thus $u$ and $v$ have at least $\alpha\ell$ common neighbors, which gives the difference of sizes of $G$ and $G'$.

So we may assume that $(G', S)$ violates (ii). Let $(A', B')$ be a separation of $G'$ violating (ii) with $B'$ minimal. By minimality, the pair $(G'[B'], A' \cap B')$ is knitted. So $(A', B')$ is a rigid separation of $(G', S)$ (of order at most $|S| - 1$). Note that the separation induces a separation $(A, B)$ in $G$. If $\{u, v\} \nsubseteq A \cap B$, then $(A, B)$ is a rigid separation of $(G, S)$ of order at most $|S| - 1$, which a contradiction to Claim 1. So we assume that $u, v \in A \cap B$. Then by minimality of $B'$, $(G[B], A \cap B)$ is $\alpha\ell$-massed thus knitted, so $(A, B)$ is a rigid separation of size at most $|A' \cap B'| + 1 \leqslant |S|$, a contradiction to Claim 1 again. $\square$

**Claim 3.** *Let $\delta'$ be the minimum degree in $G$ among the vertices in $V(G) - S$. Then $\alpha\ell + 1 \leqslant \delta' < 2\alpha\ell$.*

**Proof.** We only need to prove that $\delta' < 2\alpha\ell$. Take an edge $e = uv$ in $G$, and consider $G_1 = G - e$. Then $G_1$ fails (i) or (ii).

If $G_1$ fails (ii), then $(G - e, S)$ contains a separation $(A, B)$ with $|A \cap B| < |S|$. It follows that $u \in A - B$ and $v \in B - A$, lest $(A, B)$ is a separation in $(G, S)$ violating (ii). Then $|N(u) \cap N(v)| \leqslant |A \cap B| < |S| \leqslant \ell < \alpha \ell$, a contradiction to Claim 2. So $G_1$ fails (i), that is, $\rho(G - S) \leqslant \alpha \ell |V(G) - S| - 1$.

If $\delta' \geqslant 2\alpha \ell$, then

$$2\big(\alpha \ell |V(G) - S| - 1\big) \geqslant 2\rho(G - S) \geqslant \sum_{v \in V(G) - S} deg(v) \geqslant 2\alpha \ell |V(G) - S|,$$

a contradiction.   □

Now let $v \in V(G) - S$ be a vertex with degree $\delta'$ in $G$. Let $H$ be the graph induced by $v$ and its neighbors. Then $H$ has at most $2\alpha \ell$ vertices, and $H$ has minimum degree at least $\alpha \ell + 1$.

## Acknowledgments

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