A Noncooperative Model of Contest Network Formation*

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Abstract

In this paper we study a model of weighted network formation. The bilateral interaction is modeled as a Tullock contest game with the possibility of a draw. We describe stable networks under different concepts of stability. We show that a Nash stable network is either the empty network or the complete network. The complete network is not immune to bilateral deviations. When we allow for limited farsightedness, stable networks immune to bilateral deviations must be complete M-partite networks, with partitions of different sizes. The empty network is the efficient network. We provide several comparative statics results illustrating the importance of network structure in mediating the effects of shocks and interventions. In particular, we show that an increase in the likelihood of a draw has a non-monotonic effect on the level of wasteful contest spending in the society. To the best of our knowledge, this paper is the first attempt to model weighted network formation when the actions of individuals are neither strategic complements nor strategic substitutes.

Key Words: network formation; weighted network; contest; limited farsightedness.
JEL: D85; D74; C72.

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1 Introduction

A contest is a strategic interaction in which opposing parties make costly investments in order to increase their chances of gaining control over scarce resources. Contests have been studied in different settings, including political rent seeking (Hillman and Riley, 1989), discretionary spending of top managers (Inderst et al., 2007), competition for funding (Pfeffer and Moore, 1980), sport (Szymanski, 2003), litigation (Sytch and Tatarynowicz, 2014), and armed conflict (König et al., 2017). Agents often compete with several opponents simultaneously. In this case, the set of bilateral contest relations in a population can be described as a network, in which each agent is a node, and a link indicates the contest between two agents. Contest networks emerge in many situations. For instance, (Sytch and Tatarynowicz, 2014) studies the observed network of patent infringements and antitrust lawsuits among US pharmaceutical firms. (König et al., 2017) theoretically and empirically demonstrates the importance of the network structure of conflicts among groups in the Second Congo War. One may also expect that the structure of a contest network has important implications in other settings, including distributional conflicts in a federation as in (Wärneryd, 1998), lobbying for discretionary spending of top managers as in (Inderst et al., 2007), and appropriation of property rights as in (MacKenzie and Ohndorf, 2013).

In this paper we propose a model in which players make costly investments (exert costly effort) to extract resources from other players in the society. It is a model of weighted network formation, in which *players choose with whom to engage in a bilateral contest and how much to invest in each of their contests.* Our starting point is the model introduced in (Franke and Ozturk, 2015). In their model, the set of bilateral contests in the population is given, hence the contest network is exogenous. The prize of a contest is a fixed transfer from the loser to the victor. Our first departure from (Franke and Ozturk, 2015) is in the definition of the bilateral contest game, where we use a different specification which, being more general than one used in (Franke and Ozturk, 2015), allows ties. The main difference between our paper and (Franke and Ozturk, 2015) is that we propose a model in which the structure of the contest network is determined endogenously. In our model, agents decide both with whom to fight and how much effort to exert in each of their contests. We say that a link between two players exists or that they are engaged in a contest when at least one of them invests a nonzero effort in fighting the other. In this setting, our first task is to describe the *efficient* network architecture that maximizes the sum of benefits of all agents in the population. We proceed by providing a characterization of *stable* network structures under different notions of stability. Finally, we provide several comparative statics results which highlight the importance of the network structure when assessing how changes in parameters of the model affect individual and aggregate outcomes. In the next few paragraphs we discuss our main results.

We start our analysis of stability by defining Nash stable networks. We show that the

Nash stable network is, generically, the complete network in which players exert the same effort in all contests.[1] The Nash stable network is complete, even though every player would prefer not to be engaged in any of her contests. The reason is the coordination problem when two players contemplate destroying the link between them. For any contest in the complete network, both players would be better off if they destroyed the link between them. However, if one player unilaterally deviates and chooses investment 0, the other player is strictly better off if she invests a non-zero effort in the contest between them. The complete network is not immune to bilateral deviations - and thus a strong pairwise stable network (Bloch and Dutta, 2009) generically does not exist.[2]

The lack of forward looking is implied when using the Nash equilibrium as a stability concept. Starting a contest is always a profitable action for a player because she does not take into account that the new opponent will fight back. We consider an alternative stability concept where we relax this assumption and allow limited forward looking. We assume that a player, when forming a link, takes into account that the new opponent will fight back. However, we still assume that players do not take into account further adjustments in other players' strategies that may be a consequence of the new link creation. In that sense, players are *limited farsighted*. We define a *limited farsighted pairwise stable network* (LFPS) as a network that is immune to both unilateral and bilateral deviations of limited farsighted players.

The limited farsightedness assumption provides tractability, and we believe it is also sensible. Indeed, calculating all the effects of a change in the network structure on the equilibrium investment profiles is a highly nonlinear problem even when the number of nodes in the network is small. Assuming that players are able to make these calculations, for any contemplated choice of opponents and efforts, would be a very strong assumption about their cognitive abilities. Moreover, recent experimental results suggest that, even in a simple bilateral Tullock contest game, players find it very difficult to anticipate opponents' best responses to their actions. Furthermore, even when the action of an opponent is known, they fail to calculate their own best response correctly (Masiliunas et al., 2014). In (Kirchsteiger et al., 2016) authors find evidence in favor of the limited farsightedness in an experimental investigation of much simpler network formation games.

We show that in every LFPS non-empty network, players are partitioned in $M \geq 2$ partitions of *unequal* sizes. Members of the same partition do not have links with each other, but have links with all other players in the network. So, even though players are ex-ante homogeneous, a stable non-empty network is necessarily asymmetric. To understand this result, the concept of a player's strength is useful. In the model, a player is strong when her opponents are weak. Thus, the strength of a player can be seen

[1] The empty network is Nash stable, for instance, in the case when the marginal cost of effort, for any level of effort, is so high that a non-zero investment against an opponent who invests 0 is still not profitable. We explicitly state this condition in Proposition 7.

[2] The empty network is immune to bilateral deviations.

as a recursive measure of her position in the contest network. In the model, a strong player[3] has an incentive to form a link with a weak player, provided that the difference in their strengths is large enough. This is because it is cheaper to win a contest against a weak player than against a strong player. As the number of opponents of a weak player increases, she becomes relatively weaker and therefore a more attractive opponent for other strong players. This mechanism leads to network configurations with three types of players in a stable network. The strongest players in the society (*attackers*) win all of their contests. *Hybrid* type players are strong enough to win against the weakest players, but are, at the same time, weak enough to be attractive opponents for the strongest players. The weakest players are *victims*. They lose all of their contests. We find that there will always be a single class of attackers and a single class of victims in a stable non-empty network. The remaining $M-2$ classes, if they exist, are classes of hybrids. There are no links between the members of the same class in a LFPS network, whereas there is a link between any two players from different classes. The class of attackers is the largest class, while the class of victims is the smallest class.

Finally, we examine how the level of inefficiency in a stable network, as measured by the total contest (wasteful) spending, depends on the parameters of the model. We mention a few interesting results. When the stable network is asymmetric enough, an increase in the likelihood of a draw (i.e. a third party mediation intervention) may actually lead to an increase in the overall contest spending. On the other hand, when the network is not very asymmetric, an increase in the likelihood of a draw will always lead to a decrease in the contest spending. We also describe how an idiosyncratic cost shock (i.e. a third party intervention affecting only one player in the network) propagates through the network, and affects the investments of other players.

1.1 Related work

Our paper contributes to the literature of weighted network formation in which players choose their investment levels specifically for each link. Several other papers study network formation with link-specific actions. (Goyal et al., 2008) studies the formation of R&D networks between firms that also compete in a market. (Bloch and Dutta, 2009) and the follow-up work by (Deroïan, 2009) study a model of network formation in which agents choose how much to invest in each of their communication links. (Baumann, 2017) develops a model of friendship formation in which players choose how much time to devote to socializing with each of their friends, and how much time to spend alone. All of these papers consider a bilateral interaction which is directly beneficial to both parties (i.e. collaboration, communication, socializing). Our model deals with a qualitatively different type of interactions - contests. Moreover, in the above mentioned papers, neighbors' actions are either strategic complements or strategic substitutes. In the model

[3] Strength is an endogenous concept in our model, and it is a function of the global network structure.

presented in this paper, neighbors' actions are neither strategic substitutes nor strategic complements.

Our paper also contributes to the literature on contests. Studying contests has a long tradition in economics, starting from seminal works on rent seeking (Tullock, 1967), and lobbying (Krueger, 1974). A recent comprehensive review of the literature on contests can be found in (Corchón and Serena, 2018). This literature is mostly concerned with the analysis of n-lateral contest games. In this paper, we consider an environment in which a population of players plays interrelated bilateral contests. We model the bilateral contest game following (Nti, 1997, Amegashie, 2006) and (Blavatskyy, 2010). Since, in our model, the transfer size does not depend on the number of opponents (same as in (Franke and Ozturk, 2015)), our model captures the situations in which the prize is relational. For instance, this is may be the case in lobbying (Hillman and Riley, 1989), appropriation of property rights (MacKenzie and Ohndorf, 2013), and litigation (Sytch and Tatarynowicz, 2014). In this paper we show that accounting for the network structure of bilateral contests when studying the effects of changes in the parameters of the model on the equilibrium outcomes (as done in (Nti, 1997) for example), may lead to qualitatively different results compared to the case when the network structure is ignored.

The importance of the structure of a contest network has recently been acknowledged in the literature, both theoretically and empirically. There are several papers that study contests on a given network. (Franke and Ozturk, 2015) develops a model in which players play bilateral contests with their neighbors in a given network. (Dziubiński et al., 2016) studies a model in which the network of connections between players determines potential conflicts, and agents sequentially choose if they wish to start a conflict with their neighbors and the effort level they are going to exert. (König et al., 2017) studies a model of conflict on a given network with two types of links: enmity links and alliance links. All agents participate in a single n-lateral contest and the network structure is built in the payoff function. They also conduct an econometric analysis using data on the Second Congo War, and find that there are significant fighting externalities across contests. (Matros and Rietzke, 2018) studies a model in which there are two types of nodes: players and contests contests. Players connected to the same contest play an n-lateral contest game. None of these models consider network formation. The model in this paper endogenizes the network structure in the model of (Franke and Ozturk, 2015), and provides new comparative static results.

There are a few papers that are concerned with formation of contest networks. (Jackson and Nei, 2015) studies the impact of trade on the formation of interstate alliances and on the onset of war. They show that trade can mitigate conflict. (Grandjean et al., 2017) studies a network formation model in which agents form a network of collaboration links and then engage in a single n-lateral contest. The position of a player in the collaboration network determines her valuation of the contest prize. The closest paper to ours is (Hiller, 2016), which develops a model of network formation in which players form positive links (friendship) and negative links (enmity). A negative link indicates that players are involved in a contest. However, in (Hiller, 2016) players do not choose the fighting effort like they do in our model, and therefore the model in (Hiller, 2016) is not a model of weighted network formation. (Goyal et al., 2016) provides a comprehensive review of the literature on conflict and networks.

The rest of the paper is organized in 5 sections. Section 2 lays out the model. In Section 3 we characterize efficient and LFPS networks. In Section 4 we present comparative static results. Section 5 provides a characterization of Nash stable networks and strongly pairwise stable networks. We conclude in Section 6. All the proofs are given in Appendix A.

2 Model

In this section we describe our network formation model. In the next paragraph we informally summarize the model. In Subsection 2.1 we formally introduce the notion of a contest network, and describe the model. In Subsection 2.2 we define the concepts of stability and efficiency we use in this paper.

Informally, we consider a population composed of a finite number of ex-ante identical players. Players can engage in bilateral contests. The outcome of a contest is probabilistic, and depends on costly investments by both parties. The prize of the contest is a fixed transfer from the defeated to the victor. Individuals choose both with whom to engage in a contest and how much to invest in each of their contests. We are interested in stable and efficient social structures that arise from this type of interaction, and how the structure of a stable contest network mediates the effects of various types of shocks and third party interventions.

2.1 Setup

Denote with $N = \{1, 2, ..., n\}$ the set of players. Each player $i \in N$ chooses how much to invest in bilateral contests with other players. Strategy of player i is vector $\boldsymbol{s}_i = (s_{i1}, s_{i2}, ..., s_{i,i-1}, s_{i,i+1}, ..., s_{in}) \in \mathbb{R}^{n-1}_{\geq 0}$, where s_{ij} denotes the investment of player i in bilateral contest with j.

The expected payoff of a bilateral contest between players i and j, $\pi_{ij}(s_{ij}, s_{ji})$, is defined by:

$$\pi_{ij}(s_{ij}, s_{ji}; r) = \frac{\phi(s_{ij})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - \frac{\phi(s_{ji})}{\phi(s_{ij}) + \phi(s_{ji}) + r}. \tag{1}$$

The expression $\frac{\phi(s_{ij})}{\phi(s_{ij})+\phi(s_{ji})+r} \in [0, 1]$ determines the probability with which i wins the transfer $T = 1$ from j, and it defines the Contest Success Function (CSF) $F : \mathbb{R}^2_{\geq 0} \to [0, 1]$. The specific form of CSF we use in this paper is introduced in (Nti, 1997). The technology function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ in (1) transforms the investment in the contest (i.e. money, effort) into actual means of fighting (i.e. guns, lawyers). We assume that ϕ is: *(i) continuous and twice differentiable, (ii) strictly increasing and weakly concave, and (iii)* $\phi(0) = 0$. Point (ii) imposes non-increasing returns to scale, and (iii) guarantees that zero investment implies zero actual means of fighting. The parameter $r \geq 0$ captures the likelihood of a draw (there is no transfer between players in the event of a draw). There are many situations in which contests can end without a winner. For instance, a litigation can end in a mistrial, sport contests often end in a tie, etc. Alternatively, one can interpret r as noise in a transferable contest, using CSF proposed in (Blavatskyy, 2010) and modeling noise as in (Amegashie, 2006). In this paper we refer to r simply as *the likelihood of a draw.*[4] A comprehensive review of contest models that allow ties can be found in (Corchón and Serena, 2018). The CSF used in (1) is fairly general, and includes CSFs studied in (Tullock, 1980, Loury, 1979, Dixit, 1987) as special cases. In particular, by setting ϕ to be identity mapping and $r = 0$ we get the CSF used in (Franke and Ozturk, 2015).

We say that there is a contest between two players, i and j, whenever $s_{ij} + s_{ji} > 0$. Players i and j are said to be connected when there is a contest between them. Therefore, strategy profile $\boldsymbol{s}$ defines (induces) *weighted and non-directed network* $g(\boldsymbol{s})$. Weight $s_{ij} + s_{ji}$ is assigned to link $ij = ji$.[5] When i and j are connected we write $ij \in g$. It is clear that different strategy profiles $\boldsymbol{s}$ can induce the same weighted network. In this paper we use the terms *link* and *contest* as synonyms when talking about network $g(\boldsymbol{s})$. We will use N_i to denote the neighborhood of node i, so $N_i = \{j \in N : ij \in g\}$, and $d_i = |N_i|$ to denote the degree of node i. The expected payoff of agent i from network $g(\boldsymbol{s})$ is defined by:

$$\pi_i(g(\boldsymbol{s})) = \sum_{j \in N_i} \pi_{ij}(s_{ij}, s_{ji}; r) - c(w_i), \tag{2}$$

where

$$w_i = \sum_{j \in N_i} s_{ij}$$

is the total investment of player i in all of her contests. Function $c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the cost function that we assume to be *continuous, twice continuously differentiable, strictly increasing and strictly convex, with* $c(0) = 0$.

We conclude this section by specifying what it means to form or destroy a link. Consider strategy profile $\boldsymbol{s}$. Suppose that strategies $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$ are such that $s_{ij} = s_{ji} = 0$. This means $ij \notin g(\boldsymbol{s})$. We say that player i *starts a contest* with j or that i *forms link* ij, when i deviates from strategy $\boldsymbol{s}_i$ to strategy $\hat{\boldsymbol{s}}_i$ such that $\hat{s}_{ij} > 0$. If, strategies $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$ are such that $s_{ij} + s_{ji} > 0$, and after a (potentially bilateral) deviation of players i and

[4] For other interpretations of r see (Nti, 1997).

[5] To simplify notation, we omit dependence on $\boldsymbol{s}$ whenever there is no danger of ambiguity.

j to strategies $\hat{\boldsymbol{s}}_i$ and $\hat{\boldsymbol{s}}_j$, we have $\hat{s}_{ij} + \hat{s}_{ji} = 0$, we say that *players i and j ended contest ij* or *deleted link ij*.

2.2 Efficiency and stability

In this subsection we first define efficient networks. Then we introduce the concepts of network stability which we employ in this paper. We define Nash stable networks, and point out why using this standard equilibrium notion may be inadequate for the model we study. Finally, we introduce limited farsighted pairwise stability (LFPS), which circumvents the shortcomings of Nash stability while still allowing for a reasonable tractability in the analysis. In Section 5 we discuss how LFPS relates to other stability concepts usually employed when stuyding the formation of weighted networks, namely Nash stability and strong pairwise stability (Bloch and Dutta, 2009). We also provide a characterization of Nash stable networks and strongly pairwise stable networks.

Define the value of network $g(\boldsymbol{s})$ with:

$$V(g(\boldsymbol{s})) = \sum_{i=1}^{n} \pi_i(g(\boldsymbol{s})). \tag{3}$$

We say that network $g(\boldsymbol{s})$ is efficient if it is a maximizer of the value function V.

Definition 1
Network $g(\boldsymbol{s})$ is efficient if $V(g(\boldsymbol{s})) \geq V(g(\boldsymbol{s}'))$ for any $\boldsymbol{s}'$.

We define Nash stable networks as in (Bloch and Dutta, 2009, Definition 2):

Definition 2 (Nash stable networks)
A network $g(\boldsymbol{s})$ is Nash stable if there is no individual i and strategy $\boldsymbol{s}'_i$ such that

$$\pi_i \left(g(\boldsymbol{s}'_i, \boldsymbol{s}_{-i}) \right) > \pi_i \left(g(\boldsymbol{s}) \right).$$

So, a network $g(\boldsymbol{s})$ is Nash stable if no player can alter her investment pattern and obtain a higher payoff. The Nash equilibrium may not be the most suitable stability concept for our model. There are at least two reasons for this. First, we show that starting a contest is profitable for any player, except in extreme cases.[6] Thus, a deviation which leads to the formation of a new link is always profitable. Second, a deviation which results in the destruction of a link is never profitable. The former is a consequence of the lack of forward looking when starting a contest. When players are not farsighted, they do not take into account that the opponent will *fight back*. The latter is a consequence of the fact that Nash stability deals only with unilateral deviations. We discuss these points in more detail in Section 5, where we provide a characterization of Nash stable networks in Proposition 7.

[6]See Section 5 for more details.

To address the issues pointed out in the previous paragraph, we consider a model in which (i) we assume that when i decides to form a link with j, she takes into account the immediate reaction from j (i.e. anticipates that j will fight back), and (ii) we allow for bilateral deviations of players. In the following paragraphs we discuss (i) in more detail.

Models of network formation usually assume either pure myopia or complete farsightedness (Kirchsteiger et al., 2016). In our model, pure myopia implies that starting a contest is always profitable. Given the complexity of the network effects, full farsightedness is too strong of an assumption to make. Indeed, even for networks with a small number of nodes, solving for the equilibrium requires finding the roots of a high order polynomial. Thus, calculating all future adjustments in other players' strategies after a deviation is computationally extremely demanding. Moreover, experimental results suggest that limited farsightedness may be the most accurate way to describe players' behavior in network formation games (Kirchsteiger et al., 2016). In this paper we adopt a specific form of limited farsightedness, described in the next paragraph.

Consider strategy profile $\boldsymbol{s}$. Let $F_i = \{j \in N|\ ij \notin g\}$. Thus, F_i is the set of players with whom player i does not have a contest. Consider a situation in which i contemplates initiating contests with players $j \in L_i \subseteq F_i$. We assume that, when assessing the payoff of starting contest ij with action s_{ij}, player i expects that j will fight back by choosing the best response $BR(s_{ij})$, given $j's$ current contest investments $\boldsymbol{s}_j$. This means that, when i forms links to players from set L_i by deviating from $\boldsymbol{s}_i$ to $\boldsymbol{s}'_i$, her expected payoff is $\pi_i\left(g(\boldsymbol{s}'_i, \hat{\boldsymbol{s}}_{L_i}, \boldsymbol{s}_{-i-L_i})\right)$ where $\hat{\boldsymbol{s}}_{L_i} = \left(\hat{\boldsymbol{s}}_j\right)_{j\in L_i}$ is such that for each $j \in L_i$:

$$\pi_j\left(g(\boldsymbol{s}'_i, \hat{\boldsymbol{s}}_j, \boldsymbol{s}_{L_i-j}, \boldsymbol{s}_{-i-L_i})\right) \geq \pi_j\left(g(\boldsymbol{s}'_i, \hat{\boldsymbol{s}}'_j, \boldsymbol{s}_{L_i-j}, \boldsymbol{s}_{-i-L_i})\right), \tag{4}$$

for each $\hat{\boldsymbol{s}}'_j$ with $\hat{s}'_{jk} = \hat{s}_{jk} = s_{jk}$ for $k \neq i$. Here we use $-i - L_i$ to denote all players, except i and players from L_i. We write $L_i - j$ to denote players in L_i except player j.

We are now ready to state the stability concept we use in this paper.

Definition 3 (Limited Farsighted Pairwise Stable Networks)
Weighted network $g = g(\boldsymbol{s}^*)$ is stable if conditions (U) and (B) hold.

(U) For any player $i \in N$, and any, potentially empty, set $L_i \subseteq F_i$, and any strategy $\boldsymbol{s}_i \in \mathbb{R}^{n-1}_{\geq 0}$,

$$\pi_i\left(g(\boldsymbol{s}^*)\right) \geq \pi_i\left(g\left(\boldsymbol{s}_i, \hat{\boldsymbol{s}}_{L_i}, \boldsymbol{s}^*_{-i-L_i}\right)\right).$$

(B) For any pair of players (i, j) such that $ij \in g(\boldsymbol{s}^*)$, any two sets $L_i \subseteq F_i$ and $L_j \subseteq F_j$, and any two strategies $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$ such that $ij \notin g(\boldsymbol{s}_i, \boldsymbol{s}_j, \boldsymbol{s}^*_{-i-j})$,

$$\pi_i(g(\boldsymbol{s}_i, \boldsymbol{s}_j, \hat{\boldsymbol{s}}_{L_i}, \boldsymbol{s}^*_{-i-j-L_i}) \geq \pi_i(g(\boldsymbol{s}^*)) \Rightarrow \pi_j(g(\boldsymbol{s}_j, \boldsymbol{s}_i, \hat{\boldsymbol{s}}_{L_j}, \boldsymbol{s}^*_{-j-i-L_j}) < \pi_j(g(\boldsymbol{s}^*)).$$

Part (U) of Definition 3 states that no player $i \in N$ has an incentive to unilaterally deviate and change her pattern of contest investments. The important assumption there is that if the deviation entails the onset of a contest with player j, player i takes into account that j may fight back, as discussed in the paragraph preceeding equation (4). Part (B) of Definition 3 states no two players find it profitable to jointly deviate by deleting the link between them, while at the same time potentially adjusting their strategies in other contests or forming new links. In part, the motivation for the equilibrium concept in Definition 3 is dynamic. In online Appendix B we propose a process of network formation which rests in equilibria as defined in Definition 3. In Section 5 we discuss how LFPS relates to other stability concepts employed in the literature on weighted network formation.

It is clear that, in order to start a contest (create a link), the action of one party suffices. This is a natural property, since, for instance, to start a litigation process it is sufficient that one side files a lawsuit. On the other hand, to end contest ij, both players i and j must choose zero investment. In other words, to make peace, both sides must choose not to fight. Therefore, in our model, the creation of a link is the result of an unilateral action, while the destruction of a link is a result of a bilateral action.

3 Analysis

In this section we first outline some properties of the network formation game. Then we show that the unique efficient network is the empty network. This is not surprising, given the negative sum nature of the bilateral contest game. We then turn our attention to the analysis of stable networks. We state and discuss a series of important intermediate results that lead to one of the main results of this paper - a LFPS network must be a complete M-partite graph with partitions of different sizes. We then provide the sufficient and necessary conditions for stability when $M = 2$.

3.1 Preliminary considerations

We begin our analysis by outlining the properties of the payoff function and the nature of strategic interactions. It is straightforward to verify that the payoff function (2) of player i is increasing and concave in $\boldsymbol{s}_i$, and decreasing and convex in $\boldsymbol{s}_{-i}$. The sign of the first and the sign of the second derivative of the payoff function with respect to r depend on $\boldsymbol{s}_i$ and $\boldsymbol{s}_{-i}$. When a player's probability of winning is greater than the probability of losing in all contests, the payoff function will be decreasing and convex in r. Similarly, if the probability of winning is lower than the probability of losing in all of her contests, the payoff function is increasing and concave in r.[7] The best reply curves of the bilateral contest game are nonlinear and non-monotonic. The bilateral contest game is neither a game of strategic complements nor strategic substitutes. To the best of our knowledge,

[7]When $r = 0$ the payoff function is not defined at the point $s_{ij} = s_{ji} = 0$, however, this does not affect our results.

the only papers that consider this type of bilateral strategic interactions on networks are (Franke and Ozturk, 2015) and (Bourlès et al., 2017). Neither of these papers studies network formation.

3.2 Efficient Networks

We briefly discuss the efficient network structure. It is easy to show that the unique network structure which maximizes the total utility of the society is the empty network. This is a direct consequence of the transferable nature of the contest game and the fact that effort is costly. Indeed, the total payoff that society obtains from network $g(\boldsymbol{s})$ can be expressed as:

$$V\left(g(\boldsymbol{s})\right) = \sum_{i \in N} \sum_{j \in N} \left(\frac{\phi(s_{ij}) - \phi(s_{ji})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - c(w_i) \right) = -\sum_{i \in N} c(w_i).$$

In Section 4 we discuss the welfare properties of stable networks. Since contest spending is wasteful, we focus on the total contest investment as a measure of the inefficiency associated with a stable network. We discuss how this measure behaves when we vary the parameters of the model, and how this depends on the structure of the stable contest network.

3.3 Stable networks

In this section we identify LFPS network architectures. We start with some useful observations, and then through a series of intermediate results arrive at our main result in this section - a description of stable networks. We show that a non-empty stable network must be connected, and must have a complete M-partite structure. We describe Nash stable networks and strongly pairwise stable networks in Section 5.

Under structure of contest network $g(\boldsymbol{s})$ we think of unweighted network $(N, \bar{g})$ with the set of nodes N and the set of links $\bar{g} \subseteq \{\{i, j\} : i \in N \wedge j \in N\}$ such that $ij \in \bar{g}$ if and only if $ij \in g(\boldsymbol{s})$. In this paper we will always use $\bar{g}$ to denote unweighted and undirected network, and $g = g(\boldsymbol{s})$ to denote the weighted contest network.

As hinted in Section 2.1, there are infinitely many strategy profiles $\boldsymbol{s}^*$ that result in the same network structure. The first result we present in this paper is that any stable network structure is induced by one and only one strategy profile $\boldsymbol{s}$.

Proposition 1
Let $g(\boldsymbol{s}^*)$ be a LFPS network. If $g(\boldsymbol{s}')$ is a LFPS and such that $ij \in g(\boldsymbol{s}')$ if and only if $ij \in g(\boldsymbol{s}^*)$ then $\boldsymbol{s}' = \boldsymbol{s}^*$. If additionally $\phi'(0) = \infty$, then $s^*_{ij} > 0$ and $s^*_{ji} > 0$ whenever $ij \in g(\boldsymbol{s}^*)$. In special case: $\phi(x) = x$ and $c(x) = \alpha x^2$, inequalities $s^*_{ij} > 0$ and $s^*_{ji} > 0$ hold when r is small enough.

For simplicity, additional to the assumptions stated in Section 2.1, in the remaining part of Section 3 and in Section 4 we will require that $\phi'(0) = \infty$. For some results in these *sections we will focus on the special case with* $\phi(x) = x$ *and* $c(x) = \alpha x^2$, $\alpha > 0$, *which will be explicitly stated. Then we will also assume* $r \to 0$. This additional assumption simply guarantees that whenever there is a contest ij, both players will invest non-zero amount of resources in fighting.

Proposition 1 states that if $g(\boldsymbol{s}^*)$ is a LFPS network then there does not exist another a LFPS network $g(\boldsymbol{s}')$ such that $\boldsymbol{s}^* \neq \boldsymbol{s}'$ with the property that $ij \in g(\boldsymbol{s}^*) \Leftrightarrow ij \in g(\boldsymbol{s}')$. Hence, without ambiguity, we can talk about LFPS stability of unweighted and undirected network $\bar{g}$.

Definition 4
Unweighted and undirected network $\bar{g} = (N, \bar{g})$ *is said to be stable when there exists a strategy profile* $\boldsymbol{s}$ *such that* $g(\boldsymbol{s})$ *is LFPS, and* $ij \in g(\boldsymbol{s}) \Leftrightarrow ij \in \bar{g}$.

It is clear from Proposition 1 that stable $\bar{g}$ can be induced with one and only one strategy profile $\boldsymbol{s}^*$.

We now define the strength of a player in a stable network.

Definition 5
Consider stable network $g = g(\boldsymbol{s}^*)$. *Player* $i \in N$ *is said to be stronger than player* $j \in N$ *in* g *whenever* $w_i^* < w_j^*$.

Definition 5 is motivated with the result that for two players i and j, such that $ij \in g(\boldsymbol{s}^*)$ and $g(\boldsymbol{s}^*)$ is LFPS, i wins contest ij whenever $w_i^* < w_j^*$. We state and prove this result formally in Proposition 10 in Appendix A.[8] This seemingly counter-intuitive result is a direct consequence of the convexity of the cost function - when w_i^* is high, the resources are more costly at the margin.

We now introduce a useful way to partition players in a stable network with respect to their strengths. Sort $(w_i^*)_{i \in N}$ starting from the lowest $(w_1^* < w_2^* < ... < w_M^*)$, where $M \leq n$ is the number of different total equilibrium investment levels. We use W_i to denote the class of players that have the i-th lowest total investment level, and with $|W_i|$ the cardinality of class i.

Definition 6
Player $a \in W_i$ *is an attacker if all of her contests are with agents from* $\overline{W}_i = \{W_j | j > i\}$. *Player* $a \in W_i$ *is a hybrid if there exist players* b *and* c *such that* $ab, ac \in g$ *and* $w_b^* > w_a^* > w_c^*$. *Player* $a \in W_i$ *is a victim if she has all of her contests with players from* $\underline{W}_i = \{W_j | j < i\}$.

Definition 6 acknowledges the fact that a contest between two players of the same strength is not profitable to any of the players involved, and hence cannot be part of a stable network.

[8]This result, in the context of the game on a fixed network, appears in (Franke and Ozturk, 2015, Proposition 2) for the case when $r = 0$, $\phi(x) = x$ and $c(x) = x^2$.

If j is weaker than i in stable network $g(\boldsymbol{s}^*)$ and $ij \in g(\boldsymbol{s}^*)$, there exists a bilateral deviation which is profitable for j in which i and j destroy link ij. This is simply because j loses the contest ij and thus prefers not to engage in it (see Proposition 10 in Appendix A). Therefore, we say that i controls link ij if i is stronger than j. This in particular implies that in a stable network every attacker must receive a positive payoff. If this were not true for some attacker i and contest ij, then a joint deviation in which i and j choose $s_{ij} = s_{ji} = 0$ (delete link ij) would be profitable for both i and j.

In order to study the network formation, it is important to be able to compare contests in the network. We now state a result which enables us to do that.

Proposition 2
Let $g(\boldsymbol{s}^*)$ be a LFPS network. Suppose $a \in W_i$, $b \in W_j$, $c \in W_k$ such that $i < j < k$ and $ab \in g$, $ac \in g$, $bc \in g$. Then $s^*_{ab} > s^*_{ac}$, $s^*_{ba} > s^*_{ca}$, $s^*_{ca} < s^*_{cb}$ and $s^*_{ac} > s^*_{bc}$.

Proposition 2 states that a strong player which is engaged in contests with two players spends less, and has a less intensive contest with the weaker of the two opponents.

Our first intermediate result is that $g(\boldsymbol{s})$ cannot be stable if for some player i and two players j and k, such that $w_i < w_j \leq w_k$ we have $ij \in g(\boldsymbol{s})$ and $ik \notin g(\boldsymbol{s})$. If this were the case, a bilateral deviation in which i and j destroy link ij, and i forms link ik, would be profitable for both i and j. Intuitively, a strong player prefers to have contests with the weakest players in the network.[9] A consequence of this result is that a non-empty stable network g must be connected. Indeed, if there are two components in the stable network g, then there must be at least one attacker that is not connected to the weakest player in the network. This will be an attacker that does not belong to the same component as the weakest player in the network. We state this result as Corollary 2 in Appendix A. In the rest of the paper we focus on connected networks.

We now discuss, in turn, some properties of classes of attackers, hybrids and victims in a stable network. We begin by arguing that all members of the same class of attackers must have the same neighborhood in a stable network. Then we show that there can be only one class of attackers in a stable network, and that members of this class are connected to all other players in the network, except to the members of their own class. In the next two paragraphs we outline the main intuition behind these results.

If attackers i and j from the same class W in stable network $g(\boldsymbol{s}^*)$ have different neighborhoods ($N_i \neq N_j$), it cannot be that $N_i \subset N_j$ nor $N_j \subset N_i$. If this were true, w^*_i and w^*_j would not be the same, hence i and j would not belong to the same class.[10] Moreover, together with $N_i \neq N_j$, this implies that there exists $k \in N_i \setminus (N_i \cap N_j)$. If k is stronger than every neighbor of player j, then there exists $h \in N_j \setminus N_i$ which is weaker

[9]See Lemma 1 in Appendix A for the formal statement and the proof.
[10]In Corollary 1 in Appendix A we show that the strength of a player decreases with her neighborhood, with respect to the set inclusion.

than k. As we argued earlier in this section, in this case there is a profitable deviation in which ik is destroyed, and ih is formed. If k is not stronger than every neighbor of j, there is an analogous profitable deviation for j. The incentive to start contests with weak players in the network is the main mechanism at work.

To show that there is only one class of attackers, we consider two representative players from two different classes of attackers, $i \in W_\ell$ and $j \in W_m$. We first show that in a stable network it cannot be that $N_i \subset N_j$. Indeed, if this were the case, then $w_j^* \geq w_i^*$. Furthermore, if j does have an incentive to break any of her links (so all contests with $N_j \setminus N_i$ are profitable for j), i will have an incentive to form links with all players $N_j \setminus N_i$. This is the case because after such a deviation players from $N_j \setminus N_i$ will become even weaker, and since i is stronger than j, each of the newly formed links will increase i's expected payoff.[11] If neighborhoods of i and j are not nested, the argument proceeds analogously to the discussion in the previous paragraph. Furthemore, all members of the unique class of attackers W_1 are connected to all nodes in a stable network that do not belong to W_1. For the formal statement and the proof see Lemma 4 in Appendix A.

We now turn our attention to the classes of hybrids and victims. We find that, in a stable network, all members of the same class of hybrids must have the same neighborhood. To show this, we first partition the neighborhood of a hybrid player into two sets: the set of stronger opponents and the set of weaker opponents. To be more precise, consider LFPS network g. Let $\bar{N}_i = \{j \in N_i \wedge w_j^* < w_i^*\}$ and $\underline{N}_i = \{j \in N_i \wedge w_j^* \geq w_i^*\}$, and refer to these sets as the *strong neighborhood* of i and the *weak neighborhood* of i respectively. Consider now the strongest class of hybrids, W_2. As we have seen before, all members of W_2 must be connected to every member of W_1. This means that all members of W_2 have the same strong neighborhood. To show that they also have the same weak neighborhood, we use the same argument we have used when arguing that attackers have the same neighborhood. Proceeding analogously, we show that the claim holds for members of all other hybrid classes $W_k : 2 \leq k \leq K$. We formalize this intuition in Lemma 5 in Appendix A.

Since there is a finite number of players, there exists the weakest player in a stable network (not necessarily just one player). From Lemma 1 we know that a player who wins at least one contest must be connected to the weakest players in the network. This, in particular, holds for the weakest class of hybrids. Players that are not connected to the weakest players must be the weakest players themselves. The set of the weakest players in the network constitutes the class of victims.

So far we have argued that in a non-empty stable network we can partition players into $M < n$ classes with respect to their strength. There is only one class of attackers and only one class of victims. The remaining $M - 2$ classes, provided that they exist, are classes of hybrids of different strength. Each player $i \in W_\ell$ is in a contest with all players

[11] For the formal statement and proof see Lemma 3 in Appendix A.

outside W_ℓ. This means that a stable network must have a complete M-partite structure. We are now ready to state the main result about LFPS networks, which follows directly from the intermediate results discussed above.

Proposition 3
A non-empty stable network $g(\boldsymbol{s}^*)$ has a complete M-partite network structure with $|W_k| > |W_{k+1}|$ $\forall k \in \{1, ..., M-1\}$. The empty network is stable.

It is clear that not all complete M-partite networks with property $|W_k| > |W_{k+1}|$ are stable. The difference in strengths, and consequently in the class sizes, must be at least large enough to ensure that every bilateral contest in the network is profitable for the stronger opponent. For the sake of simplicity, we discuss this issue focusing on the particular class of networks from Proposition 3 with $M = 2$. For the remaining part of this section, and Section 4*we assume that $\phi(x) = x$ and $c(x) = \alpha x^2$, with $\alpha > 0$, and $r > 0$ is small enough such that each $s^*_{ij} > 0$ and $s^*_{ji} > 0$ for each $ij \in g(\boldsymbol{s}^*)$* (see Proposition 1). We denote two partitions by A and V, and the sizes of those partitions by a and v respectively. Class A is the class of attackers, and class V is the class of victims. We use $K_{a,v}$ to denote a complete (unweighted) bipartite network with partitions of sizes a and v. We keep the number of players in the population fixed ($a+v = n$). The following proposition holds.

Proposition 4
Consider population with n players. There exists v^* such that complete bipartite network $K_{a,v} = K_{n-v,v}$ is LFPS only when $v < v^*$.

We first show is a unique strategy profile $\boldsymbol{s}$ such that condition (U) from Definition 3 holds, and that $g(\boldsymbol{s})$ has complete bipartite network structure $K_{n-v,v}$. Then we show that there exists v^* such that $g(\boldsymbol{s})$ also satisfies condition (B) from Definition 3 only when $v < v^*$. To understand the intuition, it is illustrative to think about how the payoff of an attacker $i \in A$ behaves when we move from $g(\boldsymbol{s})$ which is $K_{a,v}$ to $g(\boldsymbol{s}')$ which is $K_{a-1,v+1}$. There are two effects on π_i. A higher v means that there are more contests. This means that the amount of resources that can be appropriated increases which is beneficial for $i \in A$, but at the same time the number of opponents of $i \in A$ increases, and therefore it is more difficult for i to *defend* herself. The trade-off between these two effects is illustrated in Figure 1. The net effect depends on the size of v in a nonlinear way - for a small v the first effect dominates, while for higher values of v the second effect dominates. Similar reasoning holds when $i \in A$ contemplates to end the contest with $j \in V$. After deleting ij, i can relocate some of the freed resources in her other contests, and earn a higher payoff from the remaining contests. On the other hand, i will be involved in a smaller number of contests, and therefore the maximal amount of resources she can extract will decrease. Note that $j \in V$ always prefers to delete the link with $i \in A$.

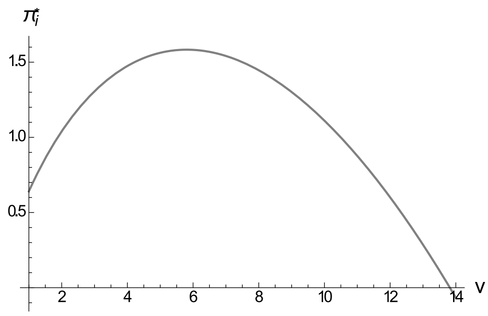

Figure 1: The payoff of $i \in A$ for fixed $n = 50$ when v varies and players play the unique $\boldsymbol{s}$ consistent with (U). $r = 0$, $c(x) = x^2$, and $\phi(x) = x$.

4 Comparative statics

In this section we are primarily interested in the inefficiencies associated with stable networks. We focus on the total wasteful spending $w^* = \sum_i w_i^*$. We analyze the effects of small changes in the parameters of the model on w^* and $\boldsymbol{s}^*$ while keeping the network structure fixed, and the role of the network structure in mediating the propagation of small shocks hitting a player in the network. We start by analyzing how changes in the likelihood of a draw, the marginal cost, and transfer size affect w^*. Not surprisingly, we find that when the effort becomes less expensive at the margin for all players, or when the transfer T increases in all contests, w^* increases. Interestingly, when the likelihood of a draw r increases, the total spending in the equilibrium may both increase and decrease. The direction of the effect crucially depends on how *asymmetric* the stable network is, and on the value of r. The following proposition summarizes these comparative static findings:

Proposition 5
Let $\phi(x) = x$ and $c(x) = \alpha x^2$. Let $K_{a,v} : \ v < a$ be a stable network, then:

1. *w^* decreases with α, and increases with transfer size T.*
2. *w^* may both increase and decrease with r. When $r \to 0$ w^* will increase in r when $a > 37v$.*

The non-monotonic effect of a change in r on w^* is a consequence of the non-monotonicity of the best reply function in r. When r is small enough, the best reply function of $i \in A$, $BR_i(\cdot)$, will be increasing in r as long as $BR_i(s_{ji}; r) > 3s_{ji} + r$. Therefore, a priori it is not clear if an increase in r will result in an increase or a decrease in the equilibrium spending per contest. To illustrate this point, Figure 2 depicts the best response curves for a contest $ij \in K_{a,v}$ when r changes from 0 to 0.05. The left panel is the plot for $K_{4,1}$. In this case the change r from 0 to 0.05 will lead to the new equilibrium (intersection of dotted lines) in which both $i \in A$ and $j \in V$ spend less, and therefore the intensity of contest ij decreases. The situation is different on the right panel, where we consider the effect of the same change but for $K_{40,1}$. In this case, in the new equilibrium i invests more, and the intensity of each contest ij is larger when $r = 0.05$ than when $r = 0$.

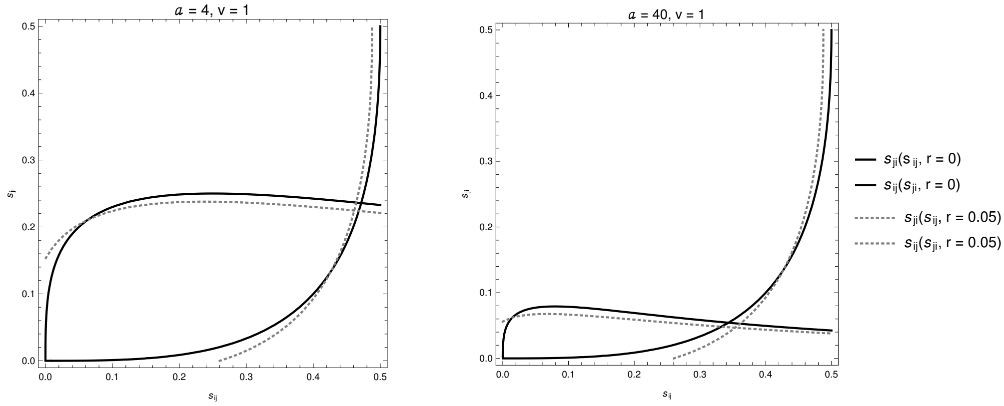

Figure 2: The equilibrium for $r = 0$ and $r = 0.05$. $i \in A$ and $j \in V$. When $K_{a,1}$ is stable then $s^*_{ji} = s^*_{jk}$ and $s^*_{ij} = s^*_{kj}$, $k \in A$.

When r increases, keeping the contest efforts fixed, the probability of losing for weak players (members of V) decreases. Since weak players already have a high marginal cost of spending at their current total investment level, they will have an incentive to decrease their spending. On the other hand, an increase in r will lead to a decrease in the probability of winning for stronger players (members of A). When strong players' total effort is not high, this will lead to an increase in their per contest effort. An increase in the investment of strong players will further increase the incentive of weak players to spend less. What will be the final effect on w^* depends on the relative magnitudes of the two effects discussed above. In Figure 3 we consider network $K_{200,1}$ in which an increase in r can lead to an increase in w^*.

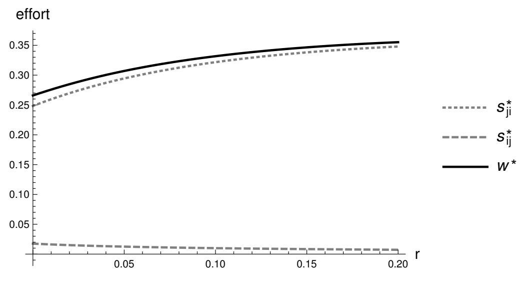

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Figure 3: Star network ($a = 200$, $v = 1$): Graph depicts the equilibrium efforts of the center node i and the periphery node j in a single contest, and w^*, as functions of r.

The effects of changes in the likelihood of a draw on the equilibrium outcomes in contest games have been already studied in (Nti, 1997) and (Acemoglu and Jensen, 2013).

Both of these papers find that a decrease in the likelihood of a draw unambiguously leads to an increase in the total equilibrium effort. The reason why we find qualitatively different results is that we take into account asymmetries implied by the network structure. In (Nti, 1997) the author studies symmetric n-lateral contests. In (Acemoglu and Jensen, 2013) the authors consider changes in r which are a *positive shock* to a player. When the network is asymmetric enough, a decrease in r is a negative shock for weak players, and positive shock for strong players. Hence, the results from (Acemoglu and Jensen, 2013) cannot be applied.

In Proposition 5 we have considered changes that simultaneously affect all players in the network. Now we discuss the effects of a change that affects only one player. We contemplate a scenario in which the cost function of player k for an exogenous reason changes to $c_k = (\alpha + \epsilon_k)x^2$. We will refer to this change as the *cost shock* hitting player k.[12] in case of conflict, for instance, the shock can be a third party intervention which makes it more costly for a party to acquire weapons. We are interested to see how $\boldsymbol{s}^*$ and w^* change in response to the shock, and how this depends on the structure of the network. We focus on small shocks, $\epsilon_k \to 0$. As before, we focus on stable $K_{a,v}$. To answer this question we note that the total equilibrium spending is implicitly defined with a system of equations (5), where d_i denotes the degree of node i (see Lemma 8 in Appendix A).

$$\begin{aligned} w_k^* &= \sum_{j \in N_k} \frac{\alpha w_j^*}{(\alpha w_j^* + (\alpha + \epsilon_k) w_k^*)^2} - d_k \frac{r}{2}, \\ w_i^* &= \sum_{j \in N_i, j \neq k} \frac{\alpha w_j^*}{(\alpha w_j^* + \alpha w_i^*)^2} + \frac{(\alpha + \epsilon_k) w_k^*}{(\alpha w_i^* + (\alpha + \epsilon_k) w_k^*)^2} \mathbb{1}_{ik \in g} - d_i \frac{r}{2}, \ i \neq k. \end{aligned} \tag{5}$$

System (5) provides the expression for the strength of player i as a function of the strengths of her neighbors. It is interesting to note that, even though the actions of a player are link-specific, the equilibrium payoff of a player can be expressed as a function of her total equilibrium spending (see Lemma 8 in Appendix A). Taking derivatives of (5) with respect to ϵ_k and solving for $\frac{\partial w_i^*}{\partial \epsilon_k}$, $i \in N$ we get the following result:

Proposition 6

Let $K_{a,v}$ be a stable network. Suppose player k experiences a cost shock.

(i) If $k \in A$ then $\frac{\partial w_k^*}{\partial \epsilon_k} < 0$, $\frac{\partial w_i^*}{\partial \epsilon_k} < 0$ $i \in A, i \neq k$, and $\frac{\partial w_j^*}{\partial \epsilon_k} > 0$, $j \in V$. If $k \in V$ then $\frac{\partial w_k^*}{\partial \epsilon_k} < 0$, $\frac{\partial w_j^*}{\partial \epsilon_k} < 0$ $j \in V, j \neq k$, and $\frac{\partial w_i^*}{\partial \epsilon_k} < 0$, $i \in A$.

(ii)

$$\frac{\partial w^*}{\partial \epsilon_k} < 0, \ k \in N.$$

[12] Other types of small shocks can be studied using the same approach.

To understand (i) from Proposition 6, notice that, when $k \in A$, the direct effect of the shock hitting k will be that k will decrease her contest investment w_k^*. Because members of V are weaker than k, their effort in contests with k will increase. At the same time, they will decrease their investment in contests with other players from A. When $k \in V$, the direct effect of the shock will again cause a decrease in w_k^*. Since all opponents of k are stronger than k, they will also decrease their investment in contests with k, but will increase their investment in contests with other members of V. This will, in turn, lead to a decrease in the total equilibrium effort of other members of V. This result is a consequence of the network structure of interactions, and the property of the best reply function, which increases with the effort of a weaker opponent and decreases with the effort of a stronger opponent. Even though some players may spend more in contests after the shock, w^* still decreases after the shock.

5 Discussion

In this section we discuss the relation between LFPS and other concepts of stability used in the analysis of the formation of weighted networks. We point out some issues when these equilibrium concepts are applied to the formation of contest networks, and argue that LFPS addresses some of these issues. Two stability concepts employed in the literature on weighted network formation are: the Nash stability (Rogers, 2006, Bloch and Dutta, 2009, Baumann, 2017), and the strong pairwise stability (Bloch and Dutta, 2009, Baumann, 2017). In this section we do not rely on the additional assumption stated in Section 3.3 that $\phi'(0) = \infty$ or that, in case when $\phi(x) = x$, we have $r \to 0$.

We first discuss Nash stable networks in our model (Definition 2). In case when, at zero investment level, the marginal benefit of investing in a contest against player who does not defend herself is greater than the marginal cost, the complete network will be the only Nash stable network structure. Otherwise, the empty network is the only Nash stable network structure. The following proposition holds:

Proposition 7
The Nash stable network is the empty network, when $\frac{\phi'(0)}{r} \leq c'(0)$. *Otherwise the unique Nash stable network* $g(\boldsymbol{s})$ *is the complete network, with* $s_{ij} = s_{ji} > 0$, $\forall i, j \in N$.

We note that the condition $\frac{\phi'(0)}{r} > c'(0)$ will be satisfied in the special case when ϕ is the identity mapping and c is a quadratic function defined with $c(x) = \alpha x^2$, for any finite $r > 0$ and $\alpha > 0$.

Proposition 7 states that a non-empty Nash stable network is the complete network. This is true even though no contest in the complete network is profitable for any player, and any two players i and j would benefit from ending contest ij. However, the destruction of a link is never a profitable unilateral deviation. This is a consequence of a coordination problem which often arises in non-cooperative models of network formation in which the link formation is a bilateral decision (Bloch and Dutta, 2009). In our model, the link destruction is essentially a bilateral decision, which creates similar coordination problem. To address this issue (Bloch and Dutta, 2009, Definition 3) introduces the concept of strong pairwise stability, which considers both unilateral and bilateral deviations. We show that a non-empty strongly pairwise stable contest network does not exist. To see why, recall that the strong pairwise stability is a refinement of the Nash stability. According to Proposition 7 the unique non-empty Nash stable network is the complete network. In the complete network, each pair of players has an incentive to bilaterally deviate by destroying the link between them, since they have the same strength. Therefore, the complete network is not immune to bilateral deviations.

Proposition 8
The strong pairwise stable network is the empty network if $\frac{\phi'(0)}{r} \leq c'(0)$*. Otherwise, it does not exist.*

When initiating a contest, one may expect that a player takes into account that the opponent will fight back. For instance, this is the case in litigation, lobbying, and conflict. Therefore, in the definition of LFPS networks, we assume that a player takes into account the expected effort that a new opponent will devote to this contest. In particular, we assume that, when calculating the expected payoff of starting contest ij with action s_{ij}, player i assumes that j will fight back by choosing the best response $s_{ji} = BR(s_{ij})$, given $j's$ current total spending w_j. Thus, i is limited farsighted, since she does not take into account further adjustments in investments that will take place in the network once ij is formed. Since calculating all the adjustments in equilibrium strategies when forming a link is equivalent to solving a highly nonlinear system of equations, which is even numerically a very difficult problem, we believe that this is a reasonable assumption. Experimental results suggest that in network formation games players are limited farsighted (Kirchsteiger et al., 2016), even in models that are much simpler than the model considered in this paper. Furthermore, experimental evidence indicates that the difficulty in forming correct beliefs about the opponent's best response may be one of the main reasons behind the fact that in experiments subjects rarely play Nash strategies in Tullock contest games (Masiliunas et al., 2014).

6 Conclusion

To the best of our knowledge this is the first model of weighted network formation in which the interaction between neighbors is an antagonistic one. Moreover, in the model, actions of neighbors are neither strategic substitutes nor strategic complements. This type of strategic interaction has not been considered in the literature on weighted network formation so far. In the paper, we describe efficient and stable networks using different notions of stability. We also derive several comparative statics results illustrating the fact that taking into account the structure of the contest network may lead to very different results compared to cases when the network structure is ignored. We believe that the qualitative insights of the model are applicable to many situations, including competitions between divisions in companies, lobbying, and allocation of property rights.

There are several promising directions for further research . First, our model considers only enmity links. It would be interesting to extend the model by allowing the formation of weighted friendship links that imply positive spillovers (i.e. reduction of cost of fighting), and see if this leads to different stable network configurations. Introducing heterogeneity is a step which is necessary to make the model's predictions empirically testable. Heterogeneity in the effectiveness of the contest technology (function ϕ), cost of fighting, and transfers can be directly included in the model. Furthermore, one could consider a position in the network as a source of heterogeneity. For instance, we can imagine that the amount of resources each enemy of a country expects to extract decreases with the number of opponents of that country. Finally, we focus on bilateral contests. It would be interesting to study contest network formation allowing also for multilateral contests. A starting point for this may be the model presented in this paper and (Matros and Rietzke, 2018).

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Appendix A: Proofs

Contest Game on a Given Network

To understand the proofs in Appendix A, it is useful to revisit the case when the set contests is exogeneously given and fixed. This is the case studied in Franke and Ozturk (2015). So, let the set of possible contest in the society be defined with a binary, undirected network $\bar{g}$. The contest game on network $\bar{g}$ is defined by:

$$C(\bar{g}) = \{N, \{S_i(\bar{g})\}_{i=1}^n, \{\pi_i\}_{i=1}^n\}. \tag{6}$$

In (6), N is the set of players, payoff functions π_i are defined in (2), and the strategy space of player i is given by:

$$S_i(\bar{g}) \equiv \{\boldsymbol{s}_i \in \mathbb{R}_{\geq 0}^{n-1} : s_{ij} = 0 \text{ whenever } ij \notin \bar{g}\}.$$

The following proposition, which is a version of the existence and uniqueness result for the contest game on a given network (Franke and Ozturk, 2015, Proposition 1 and Lemma 1), holds as well when the payoff function are given with (2).

Proposition 9
For any binary network $\bar{g}$, there exists a unique pure strategy Nash equilibrium of game $C(\bar{g})$, $\bar{\mathbf{s}}$. The equilibrium $\bar{\mathbf{s}}$ is interior ($\bar{s}_{ij} > 0\ \forall ij \in \bar{g}$) if $\phi'(0) = \infty$. When $\phi(x) = x$ and $c(x) = x^2$ the equlibrium will be interior for r small enough.

***Proof of Proposition** 9.*
Existence and Uniqueness. It is enough to follow the same steps as in the proof of (Franke and Ozturk, 2015, Proposition 1 and Lemma 1) when the payoff function are given with (2). The main part of the proof is showing that game $C(\bar{g})$ is a concave game, as defined in Rosen (1965), and then directly applying Rosen's result.

Interiority. Assume that $ij \in \bar{g}$, in the Nash equilibrium, $\bar{\boldsymbol{s}}$, of $C(\bar{g})$, and that $\bar{s}_{ij} = 0 \vee \bar{s}_{ji} = 0$. We show that when this logical disjunction is true, there is a profitable deviation for either player i or player j. Hence, $\bar{s}_{ij} = 0 \ \vee \bar{s}_{ji} = 0$ cannot be a part of the Nash equilibrium of game $C(\bar{g})$ when $ij \in \bar{g}$. We consider the case when $\phi'(0) = \infty$, and the case $\phi(x) = x,\ c(x) = x^2$ separately.

Case 1: $\phi'(0) = \infty$.

Suppose, without loss of generality, that $\bar{s}_{ij} = 0$. There is a profitable deviation in which i invests $\epsilon > 0$ in contest with j. The marginal cost of this deviation $c'(\bar{w}_i + \epsilon)$. The marginal benefit of the deviation is $\frac{r+2\phi(\bar{s}_{ji})}{(\phi(0)+\phi(\bar{s}_{ji})+r)^2}\phi'(0)$ (which becomes $\frac{r}{(r+\phi(\epsilon))^2}\phi'(\epsilon)$ in case when also $\bar{s}_{ji} = 0$). It is clear that the marginal benefit at $\epsilon = 0$ is infinite, while the marginal cost remains bounded.

Case 2: $\phi(x) = x$ and $c(x) = x^2$.

(i) Suppose first that $\bar{s}_{ij} = \bar{s}_{ji} = 0$. We show that this cannot happen for any finite $r > 0$.

(a) If $\bar{w}_i = 0$ consider a deviation in which player i invests $\epsilon > 0$ in contest ij. The cost of this deviation is ϵ^2. The benefit is $\frac{\epsilon}{\epsilon + r}$. It is easy to see that the benefit is larger than the cost, for ϵ small enough, since

$$\frac{\epsilon}{\epsilon + r} - \epsilon^2 = \epsilon \left(\frac{1 - r\epsilon - \epsilon^2}{r + \epsilon} \right).$$

(b) If $\bar{w}_i > 0$ then there exists contest ik such that $\bar{s}_{ik} > 0$. Consider a deviation in which player i reallocates $\epsilon > 0$ from ik to ij (keeping $\bar{w}_i$ fixed). The marginal benefit of this deviation for player i, calculated at $\epsilon = 0$ is:

$$\left. \frac{\partial}{\partial \epsilon} \frac{\epsilon}{\epsilon + r} \right|_{\epsilon = 0} = \frac{1}{r}.$$

The marginal cost of the deviation is:

$$\left. \frac{\partial}{\partial \epsilon} \frac{(\bar{s}_{ik} - \epsilon) - \bar{s}_{ki}}{\bar{s}_{ik} - \epsilon + \bar{s}_{ki} + r} \right|_{\epsilon = 0} = -\frac{r + 2\bar{s}_{ki}}{(r + \bar{s}_{ik} + \bar{s}_{ki})^2}.$$

It is easy to check that the marginal benefit outweights the marginal cost. Indeed

$$\frac{1}{r} - \frac{r + 2\bar{s}_{ki}}{(r + \bar{s}_{ik} + \bar{s}_{ki})^2} = \frac{2r\bar{s}_{ik} + (\bar{s}_{ik} + \bar{s}_{ki})^2}{r\left(r + \bar{s}_{ik} + \bar{s}_{ki}\right)^2} > 0.$$

(ii) We now show that when $ij \in \bar{g}$ it cannot be that $\bar{s}_{ij} = 0$ and $\bar{s}_{ji} > 0$ when r becomes infinitesimal. Suppose otherwise, so suppose that this is the case for some two players i and j.

(a) If $\bar{w}_i = 0$ then a profitable deviation for player i is to exert $\epsilon > 0$ in contest ij. The marginal cost of this deviation, 2ϵ, approaches to 0 when $\epsilon \to 0$. The marginal benefit of the proposed deviation, $\frac{r + 2\bar{s}_{ji}}{r + \bar{s}_{ji} + \epsilon}$, is positive and bounded away from 0. Hence for ϵ small enough, the proposed deviation is profitable.

(b) Finally, we consider the case when $\bar{w}_i > 0$. First we show that when $r \to 0$ then $\bar{s}_{ji} \to 0$. Using that, we show that the marginal benefit for player i of investing in contest ij calculated at 0 becomes unbounded when r approaches 0.

Since, by assumption, $\bar{s}_{ji}$ is greater than zero, it must satisfy the first order optimality (sufficient and necessary) conditions. Thus, the following holds:

$$\frac{r}{(\bar{s}_{ji} + r)^2} = 2 \sum_{\ell} \bar{s}_{j\ell} \Rightarrow$$

$$r = \left(2 \sum_{\ell \neq i} \bar{s}_{j\ell} + 2\bar{s}_{ji} \right) (r + \bar{s}_{ji})^2.$$

From the last equation above, it is clear that when $r \to 0$ then $\bar{s}_{ji} \to 0$. In this case ($r \to 0$), the marginal benefit of player i of investing in contest against player j calculated at 0 (equal to $\frac{r+2\bar{s}_{ji}}{(r+\bar{s}_{ji})^2}$) becomes unbounded. Indeed, it can be verified that:

$$\lim_{r\to 0}\frac{r+2\bar{s}_{ji}(r)}{(r+\bar{s}_{ji}(r))^2}=+\infty, \tag{7}$$

since $\lim_{r\to 0}\bar{s}_{ji}(r)=0$. The marginal cost of this deviation is obviously bounded from above. Therefore, it cannot be that $\bar{s}_{ij}=0$ and $\bar{s}_{ji}>0$ when r is small enough.

□

Proofs of Claims from Section 3

Proof of Proposition 1.

Uniqueness. Since both $g(\boldsymbol{s}^*)$ and $g(\boldsymbol{s}')$ are stable, condition (U) from Definition 3 must hold. In particular, it must hold for any player i and $L_i=\emptyset$. We note that weighted network $g(\boldsymbol{s})$ with structure $\bar{g}$ will satisfy the condition (U) if and only if $\boldsymbol{s}=\bar{\boldsymbol{s}}$, where $\bar{\boldsymbol{s}}$ is the Nash equilibrium of game $C(\bar{g})$. Then, Proposition 9 implies that, if $g(\boldsymbol{s}^*)$ and $g(\boldsymbol{s}')$ are two stable networks with the same network structure, $\bar{g}$, then $\boldsymbol{s}'=\boldsymbol{s}^*=\bar{\boldsymbol{s}}$.

Interiority. Follows directly from the proof of the interiority part of Proposition 9.

□

The following proposition is an extension of Proposition (Franke and Ozturk, 2015, Proposition 2) and provides a foundation for definition of strength (Definition 5).

Proposition 10

Suppose that conditions for the interiority from Proposition 1 are satisfied. Then: $w_i^*\geq w_j^*\Rightarrow s_{ij}^*\leq s_{ji}^*$, *with equality when* $w_i^*=w_j^*$.

Proof of Proposition 10. The following first-order conditions for contest $ij\in g(\boldsymbol{s}^*)$ must hold:

$$\left(\frac{(r+2\phi(s_{ji}^*))\phi'(s_{ij}^*)}{(r+\phi(s_{ij}^*)+\phi(s_{ji}^*))^2}-c'(w_i^*)=0\right)\wedge\left(\frac{(r+2\phi(s_{ij}^*))\phi'(s_{ji}^*)}{(r+\phi(s_{ij}^*)+\phi(s_{ji}^*))^2}-c'(w_j^*)=0\right). \tag{8}$$

From (8) we get:

$$\frac{(r+2\phi(s_{ji}^*))\phi'(s_{ij}^*)}{(r+2\phi(s_{ij}^*))\phi'(s_{ji}^*)}=\frac{c'(w_i^*)}{c'(w_j^*)}.$$

Since $\phi'(x)>0$, $\phi''(x)\leq 0$ and $c''(x)>0$:

$$w_i^*\geq w_j^*\Rightarrow\frac{c'(w_i^*)}{c'(w_j^*)}\geq 1\Rightarrow\frac{(r+2\phi(s_{ji}^*))\phi'(s_{ij}^*)}{(r+2\phi(s_{ij}^*))\phi'(s_{ji}^*)}\geq 1\Rightarrow s_{ji}^*\geq s_{ij}^*, \tag{9}$$

where the last implication in (9) follows from the facts that ϕ is an increasing function and ϕ' is a decreasing function, and the equality holds when $\bar{w}_i=\bar{w}_j$. □

Proof of Proposition 2. To prove the claim, we compare the solutions of the FOC system associated to links ab and ac. To do this, it is helpful to first consider the following parameterized system of equations on $\mathbb{R}^2_{\geq 0}$ with unknowns x and y, and positive parameters β_1 and β_2:

$$\frac{(r+2\phi(y))\phi'(x)}{(r+\phi(x)+\phi(y))^2} - c'(\beta_1) = 0, \; \frac{(r+2\phi(x))\phi'(y)}{(r+\phi(x)+\phi(y))^2} - c'(\beta_2) = 0. \tag{10}$$

It is easy to verify that (10) satisfies the conditions of the implicit function theorem. Note that when $\beta_1 = w_a^*$ and $\beta_2 = w_b^*$, then $x = s_{ab}^*$ and $y = s_{ba}^*$ is the unique solution of system (10). Taking the derivative of x and y defined by (10) with respect to β_1 we get:

$$\frac{\partial x}{\partial \beta_1} = \frac{c''(\beta_1)\,(r+2\phi(x))\,(r+\phi(x)+\phi(y))^2\left[\phi''(y)(r+\phi(x)+\phi(y)) - 2\phi'(y)^2\right]}{Den},$$
$$\frac{\partial y}{\partial \beta_1} = \frac{2c''(\beta_1)(\phi(x)-\phi(y))\phi'(x)\phi'(y)(r+\phi(x)+\phi(y))^2}{Den},$$

where

$$\begin{aligned} Den =& 2\phi'(x)^2\left(2\phi'(y)^2(r+\phi(x)+\phi(y)) - (r+2\phi(x))(r+2\phi(y))\phi''(y)\right) \\ &+ (r+2\phi(x))(r+2\phi(y))\phi''(x)\left(\phi''(y)(r+\phi(x)+\phi(y)) - 2\phi'(y)^2\right). \end{aligned}$$

For positive x and y, expression Den will be positive, given the properties of functions ϕ and c stated in Subsection 2.1. Furthermore, the numerator of $\frac{\partial x}{\partial \beta_1}$ is negative, while the numerator of $\frac{\partial y}{\partial \beta_1}$ will be negative when $\phi(x) < \phi(y)$ (and therefore when $x < y$), and otherwise positive. Therefore, for the unique solution (x, y) of system 10 the following holds comparative statics result holds:

$$\begin{aligned} &\frac{\partial x}{\partial \beta_1} < 0, \\ &\frac{\partial y}{\partial \beta_1} \leq 0 \text{ when } x \leq y, \\ &\frac{\partial y}{\partial \beta_1} > 0 \text{ when } x > y. \end{aligned} \tag{11}$$

We prove now that $s_{ab}^* > s_{ac}^*$. The other inequalities stated in the claim of the Proposition are proven analogously. Consider (8) associated to ab and (8) associated to ac, which must hold in an interior equilibrium $\boldsymbol{s}^*$.

$$\frac{(r+2\phi(s_{ba}^*))\phi'(s_{ab}^*)}{(r+\phi(s_{ab}^*)+\phi(s_{ba}^*))^2} - c'(w_a^*) = 0, \; \frac{(r+2\phi(s_{ab}^*))\phi'(s_{ba}^*)}{(r+\phi(s_{ab}^*)+\phi(s_{ba}^*))^2} - c'(w_b^*) = 0. \tag{8 ab}$$

$$\frac{(r+2\phi(s_{ca}^*))\phi'(s_{ac}^*)}{(r+\phi(s_{ac}^*)+\phi(s_{ca}^*))^2} - c'(w_a^*) = 0, \; \frac{(r+2\phi(s_{ac}^*))\phi'(s_{ca}^*)}{(r+\phi(s_{ac}^*)+\phi(s_{ca}^*))^2} - c'(w_c^*) = 0. \tag{8 ac}$$

We can think of (8 ab) as a system of equations (10) with unknowns s_{ab}^*, s_{ba}^*, where w_a^* and w_b^* are playing a role of β_1 and β_2, and analogously for (8 ac). By assumption $w_a^* < w_b^*$ and $w_a^* < w_b^*$. Then, Proposition 10 implies that $s_{ab}^* > s_{ba}^*$, and $s_{ac}^* > s_{ca}^*$ respectively. Taking this into account and comparing systems (8 ab) and (8 ac), the second inequality in (11) implies that $s_{ab}^* > s_{ac}^*$. □

We now state and prove an important corollary of Proposition 2 which states that the total equilibrium investment w^* is increasing with the neighborhood of a player, with respect to the relation of set inclusion.

Corollary 1 (of Proposition 2)
Let $N_i \subsetneq N_j$ in stable network g, then $w_i^* < w_j^*$.

Proof of Corollary 1. Suppose the claim does not hold. So, suppose that $N_i \subsetneq N_j$ and $w_i^* \geq w_j^*$. Then, from Proposition 2 it follows that for every $k \in N_i \cap N_j$ $s_{ik}^* \leq s_{jk}^*$. But then $w_i^* = \sum_{k\in N_i} s_{ik}^* \leq \sum_{k\in N_i} s_{jk}^* < \sum_{k\in N_j} s_{jk}^* = w_j^*$, which is in contradiction with $w_i^* \geq w_j^*$. □

We now state and prove Lemmas 1 to 4 which are concerned with attackers in a stable network. Our main goal is to show that there can be only one class of attackers in LFPS network. For clarity, we do this in several steps, each step being a separate lemma. We first show that attackers always have links with weakest players in the network (Lemma 1). We use Lemma 1 extensively in proofs of subsequent claims in the paper. An useful corollary of this lemma is that a stable network must be connected. We continue by showing that members of the same class of attackers must have the same neighborhood (Lemma 2), and that two different class of attackers cannot have nested neighborhoods (Lemma 3). Finally, using Lemmas 1 - 3 we show that there can be only one class of attackers (Lemma 4).

Lemma 1
If $ij \in g(\boldsymbol{s}^*)$, and g is LFPS, then $ik \in g\ \forall(k \in N) : w_k^* \geq w_j^*$.

Proof of Lemma 1. Assume that $g(\boldsymbol{s}^*)$ is stable, and such that for some player i and two other players j, k with $w_j^* < w_k^*$ we have $ij \in g(\boldsymbol{s}^*)$ and $ik \notin g(\boldsymbol{s}^*)$. We show that in this case there exists a profitable deviation for players i and j, hence $g(\boldsymbol{s}^*)$ cannot be stable.

First note that if contest ij is not profitable for i, then it cannot be part of the stable network ((B) does not hold).

When ij is profitable for i, it must be $w_i^* < w_j^*$. We show that there is a profitable bilateral deviation for i and j. Consider a deviation in which j deviates from $\boldsymbol{s}_j^*$ to $\boldsymbol{s}_j'$ such that $s_{ji}' = 0$ and $s_{j\ell}' = s_{j\ell}^*$ for all $\ell \neq i$. At the same time, i deviates to $\boldsymbol{s}_i'$ such that $s_{ik}' = s_{ij}^*$, $s_{ij}' = 0$ and $s_{i\ell}' = s_{i\ell}^*$ for all $\ell \notin \{j, k\}$. It is clear that this deviation is profitable for j. We prove that it is also profitable for i. It is enough to prove that the expected reaction of k to the proposed deviation, denoted by $\hat{s}_{ki}$, is such that $\hat{s}_{ki} < s_{ji}^*$. To do this, we note that s_{ji}^* must satisfy the following optimality condition:

$$\frac{(r + 2\phi(s_{ij}^*))\phi'(s_{ji}^*)}{(r + \phi(s_{ij}^*) + \phi(s_{ji}^*))^2} = c'(w_j^*). \tag{12}$$

The expected reaction of player k to the proposed deviation is determined with the following condition:

$$\frac{(r + 2\phi(s_{ij}^*))\phi'(\hat{s}_{ki})}{(r + \phi(s_{ij}^*) + \phi(\hat{s}_{ki}))^2} = c'(w_k^* + \hat{s}_{ki}). \tag{13}$$

Since $w_k^* + \hat{s}_{ki} > w_k^* \geq w_j^*$ it must be that $c'(w_k^* + \hat{s}_{ki}) > c'(w_j^*)$. This is due to strict convexity of c. Thus the right hand side of (13) is strictly larger than the right hand side of (12). The same relation must hold for the left hand sides of (12) and (13). Since ϕ is an increasing function and ϕ' is a decreasing function, this holds only when $\hat{s}_{ki} < s_{ji}^*$. □

Corollary 2 (of Lemma 1)
A non-empty stable network $g(\boldsymbol{s}^*)$ is connected.

Proof of Corollary 2: We use a proof by contradiction. Assume that the claim does not hold, so there are at least two components in stable network g. Choose two components (C_1 and C_2) from g such that the weakest player in the network (v_1) belongs to C_1. All opponents of v_1 must find the contest with v_1 profitable, otherwise the network would not be stable ((B) would not hold). Then, the strongest player in C_2 (denote her with a_2) by Lemma 1 has an incentive to form a link with v_1 instead of a link with one of her current opponents, who by definition is not weaker than v_1. If $|C_2| = 1$, a_2 does not have any opponents. Then, she has an incentive to form a link with v_1 with action s_{a_1,v_1}^*, since $a_1 v_1 \in g$ is a profitable contest for a_1. □

Lemma 2
Two players that belong to the same class of attackers W_a have the same neighborhood in stable network g.

Proof of Lemma 2: Let g be a stable network. Consider any two attackers $i, j \in W_a$, and suppose, contrary to what is asserted, that $N_i \neq N_j$. It cannot be that $N_i \subset N_j$ because then the total spending of i and j would not be equal (by Corollary 1). Since $N_i \neq N_j$, there exist nodes $h \in N_i \backslash N_j$ and $k \in N_j \backslash N_i$. Suppose that, without loss of generality, $w_k^* \geq w_h^*$. Then it is profitable for player i to replace ih with link ik according to Lemma 1. This is in contradiction with the assumption that g is stable. □

Lemma 3
Let i and j be two attackers in stable network $g(\boldsymbol{s}^*)$. It cannot be that $N_i \subset N_j$.

Proof of Lemma 3. If i and j belong to the same class, then Lemma 2 implies $N_i = N_j$. Consider now the case when i and j belong to different classes of attackers. We assume that $N_i \subset N_j$ and show that there will always exist a profitable deviation. We will use N_i to denote the neighborhood of i in network $g(\boldsymbol{s}^*)$.

Since $N_i \subset N_j$, by Corollary 1 it must be $w_i^* < w_j^*$.

Suppose first that $\pi_j(g(\boldsymbol{s}^*)) \geq \pi_i(g(\boldsymbol{s}^*))$. We show that in this case i can form links to all players in $L_i = N_j \setminus N_i$, and obtain a payoff greater than $\pi_j(g(\boldsymbol{s}^*))$. To show this, consider the deviation in which player i deviates to $\tilde{\boldsymbol{s}}_i = \boldsymbol{s}_j^*$. Let us denote the payoff of player i after this deviation with $\pi_i(g(\tilde{\boldsymbol{s}}_i, \hat{\boldsymbol{s}}_{L_i}, \boldsymbol{s}_{-i-L_i}^*))$ where $\hat{\boldsymbol{s}}_{L_i}$ is defined in (4). We proceed by showing that $\pi_i(g(\tilde{\boldsymbol{s}}_i, \hat{\boldsymbol{s}}_{L_i}, \boldsymbol{s}_{-i-L_i}^*)) > \pi_j(g(\boldsymbol{s}^*))$.

Because $w_i^* < w_j^*$, Proposition 2 implies that $s_{ki}^* < s_{kj}^*$ $k \in N_i \cap N_j$. The convexity of the cost function implies that $\hat{s}_{ki} < s_{kj}^*$ for all $k \in L_i$ under the contemplated deviation. This means that after the deviation the expected cost of i will be equal to the cost of j, i and j will have the same set of opponents, and $\frac{\phi(\tilde{s}_{ik})-\phi(\hat{s}_{ki})}{\phi(\tilde{s}_{ik})+\phi(\hat{s}_{ki})+r} > \frac{\phi(s^*_{jk})-\phi(s^*_{kj})}{\phi(s^*_{jk})+\phi(s^*_{kj})+r}$ $\forall k \in N_j$. Therefore $\pi_i(g(\tilde{\boldsymbol{s}}_i, \hat{\boldsymbol{s}}_{L_i}, \boldsymbol{s}^*_{-i-L_i})) > \pi_j(g(\boldsymbol{s}^*)) \geq \pi_j(g(\boldsymbol{s}^*))$.

Suppose now that $\pi_i(g(\boldsymbol{s}^*)) > \pi_j((\boldsymbol{s}^*))$, and suppose that j does not have an incentive to update her strategy (otherwise the network would not be stable).[13] From $\pi_i(g(\boldsymbol{s}^*)) > \pi_j((\boldsymbol{s}^*))$ it follows that:

$$\sum_{k \in N_i} \pi_{ik}(s^*_{ik}, s^*_{ki}; r) > -c(w^*_j) + c(w^*_i) + \sum_{k \in N_j} \pi_{jk}(s^*_{jk}, s^*_{kj}; r). \tag{14}$$

Consider now the same deviation of player i, as contemplated in the first part of the proof. We get (using N_i to denote neighborhood of i in network $g(\boldsymbol{s}^*)$):

$$\begin{aligned}
&\pi_i(g(\tilde{\boldsymbol{s}}_i, \hat{\boldsymbol{s}}_{L_i}, \boldsymbol{s}^*_{-i-L_i})) - \pi_i(g(\boldsymbol{s}^*)) = \\
&\sum_{k \in N_i} \pi_{ik}(s^*_{jk}, s^*_{ki}; r) + \sum_{k \in L_i} \pi_{ik}(s^*_{jk}, \hat{s}_{ki}; r) - \sum_{k \in N_i} \pi_{ik}(s^*_{ik}, s^*_{ki}; r) - c(w^*_j) + c(w^*_i) > \\
&\sum_{k \in N_i} \pi_{ik}(s^*_{jk}, s^*_{ki}; r) + \sum_{k \in L_i} \pi_{ik}(s^*_{jk}, \hat{s}_{ki}; r) - \left(-c(w^*_j) + c(w^*_i) + \sum_{k \in N_j} \pi_{jk}(s^*_{jk}, s^*_{kj}; r)\right) - c(w^*_j) + c(w^*_i) = \\
&\sum_{k \in N_i} \pi_{ik}(s^*_{jk}, s^*_{ki}; r) + \sum_{k \in L_i} \pi_{ik}(s^*_{jk}, \hat{s}_{ki}; r) - \sum_{k \in N_j} \pi_{jk}(s^*_{jk}, s^*_{kj}; r) > 0,
\end{aligned}$$

where the first inequality comes directly from (14) and the last inequality comes from the fact that $\hat{s}_{ki} < s^*_{kj}$ for all $k \in L_i$. This completes the proof. □

Lemma 4

There is only one class of attackers (W_1) in stable network $g(\boldsymbol{s}^*)$. Members of W_1 are connected to all players outside W_1.

Proof of Lemma 4: Suppose, contrary to what is asserted, that there are two different classes of attackers W_1 and W_2 in LFPS network $g(\boldsymbol{s}^*)$. Since Lemma 2 implies that all members of the same class of attackers have the same neighborhood, we restrict our attention to representative nodes $i \in W_1$ and $j \in W_2$.

Since $w^*_j > w^*_i$ there are 2 possible situations that we need to consider:

(i) $N_i \subset N_j$ is ruled out by Lemma 3.

(ii) $N_i \not\subset N_j \implies (\exists k \in N_i \backslash N_j \wedge \exists h \in N_j \backslash N_i)$. If $w^*_k \geq w^*_h$ Lemma 1 implies that j has a profitable deviation. If $w^*_k < w^*_h$ the same lemma implies that i has a profitable deviation.

□

We now prove a lemma which is concerned with hybrids. In the proof we rely on arguments which are analogous to those used in the proof of Lemma 4.

[13]Recall that since j is an attacker, any of her opponents would be better off by destroying a link with j.

Lemma 5

In a stable network $g(\boldsymbol{s}^*)$ all members of a hybrid class are connected to all other nodes in the network that do not belong to their class.

Proof of Lemma 5: If there are only two classes of nodes in a stable network (W_1 and W_2) then there are no hybrid types. Suppose there are more than two classes of nodes in a stable network. First, let us consider the strongest mixed type class (W_2). A node $h \in W_2$ must be connected to all nodes from W_1. This is because hybrid h must be connected to at least one player that is stronger than her, who must be an attacker since $h \in W_2$. Then, Lemma 4 implies that h must be connected to all players from W_1, since all nodes in W_1 have the same neighborhood. This holds for any $h \in W_2$.

Let us now prove that all members of the class W_2 have the same neighborhood. Suppose this is not true. Let h_1 and h_2 be two players from W_2 such that $N_{h_1} \neq N_{h_2}$. The following implication holds: $(W_1 \subset N_{h_1} \wedge W_1 \subset N_{h_2}) \Rightarrow ((N_{h_1}/N_{h_2}) \cup (N_{h_2}/N_{h_1})) \cap W_1 = \emptyset$. Thus, $\bar{N}_{h_1} = \bar{N}_{h_2}$ and $\underline{N}_{h_1} \neq \underline{N}_{h_2}$. It cannot be $\underline{N}_{h_1} \subset \underline{N}_{h_2} \vee \underline{N}_{h_2} \subset \underline{N}_{h_1}$ because then it would be $w^*_{h_1} \neq w^*_{h_2}$ by Corollary 1. Consider two nodes, $k \in \underline{N}_{h_1} \setminus \underline{N}_{h_2}$ and $\ell \in \underline{N}_{h_2} \setminus \underline{N}_{h_1}$. If $w^*_k \geq w^*_\ell$ then h_2 and ℓ have a profitable deviation (link $h_2\ell$ is destroyed, link h_2k is formed). If $w^*_k < w^*_\ell$, then h_1 and k have an analogous profitable deviation.

Let W_3 be the third strongest class in the network. If $M = 3$ then, by definition, all players in W_2 must be connected to some players from W_3, because otherwise they would not be hybrid types. Note that if player $i \in W_3$ is connected to some player from class W_2 then she is connected to all players from class W_2 - because we have shown that all members of class W_2 have the same neighborhood. If there exists player $j \in W_3$ who is not connected to all players from W_2, then j is only connected to all players from W_1. But then i and j cannot belong to the same class. So, for $K = 3$ the claim of the lemma holds.

Suppose $M > 3$. Lemma 1 implies that all members of W_1 must be connected to all members of W_3 since they are connected to all members of W_2. We now show that all players from W_2 are connected to all players from W_3. Again we proceed by using a proof by contradiction. Suppose that there exist players $i \in W_2$ and $j \in W_3$ such that $ij \notin g(\boldsymbol{s}^*)$. We show that in this case there is a profitable deviation. Since all players from W_2 have the same neighborhood there are no links between members of class W_2 and j. This means that j loses only in contests with players from W_1. Hence, j has control over all of her links except links with players from W_1. Furthermore, $w^*_i < w^*_j \Rightarrow N_i \neq N_j$. There are two possibilities for relation between N_i and N_j that we need to consider:

(i) $N_i \subset N_j$ case can be ruled out by applying the same argument as in Lemma 3 to $\underline{N}_i$ and $\underline{N}_j$.

(ii) $N_i \not\subset N_j \Rightarrow (\exists k \in N_i \backslash N_j \wedge \exists h \in N_j \backslash N_i)$. But then, if $w^*_k \geq w^*_h$ Lemma 1 implies that j has a profitable deviation, and if $w^*_k < w^*_h$, the same Lemma implies that i has a profitable deviation.

We have shown that in a stable network it cannot happen that there are no links between members of W_2 and W_3. If two players from W_2 and W_3 are connected, than all players from W_2 and W_3 are connected, because all players from W_2 have the same neighborhood, and because of Lemma 1.

Using the same reasoning as above, we can show that all players from W_k must be connected to all players from W_{k+1}. Since the number of nodes in the network is finite, the number of classes is finite and this procedure reaches W_M in a finite number of steps. □

Corollary 3

There is only one class of victims in a stable network g and all victims have the same neighborhood

Proof of Corollary 3: Follows from Lemma 4 and Lemma 5. □

We show now that classes must be of different sizes, and that stronger players belong to more numerous classes.

Lemma 6

Let $|W_k|$ *denote the number of nodes that belong to class* W_k *in stable network* $g(\boldsymbol{s}^*)$. *Then* $|W_k| > |W_{k+1}|$ $\forall k \in \{1, 2, ..., M-1\}$.

Proof of Lemma 6: Suppose that the claim does not hold, so $|W_k| \leq |W_{k+1}|$ for some $k = 1, ..., M-1$. The system (8) implies that for any player a, $s^*_{ac} = s^*_{ad}$ whenever d and c belong to the same class. Therefore, for any two players a, b such that $a \in W_k$ and $b \in W_{k+1}$, we have that $w^*_a = \sum_{i \neq k, c \in W_i} |W_i| s^*_{ac}$ and $w^*_b = \sum_{i \neq k+1, c \in W_i} |W_i| s^*_{bc}$. Since $w^*_a < w^*_b$, Proposition 2 implies $s^*_{ac} > s^*_{bc}$, $c \in \{W_1, W_2 .., W_K\} \setminus \{W_k, W_{k+1}\}$. Furthermore, since $w^*_a < w^*_b$ we have that $s^*_{ab} > s^*_{ba}$ according to Proposition 10. But then $|W_k| < |W_{k+1}| \Rightarrow \sum_{i \neq k, c \in W_i} |W_i| s^*_{ac} > \sum_{i \neq k+1, c \in W_i} |W_i| s^*_{bc} \Rightarrow w^*_a > w^*_b$. This is in contradiction with $a \in W_k$ and $b \in W_{k+1}$. □

Proof of Proposition 3: From Lemma 4, Lemma 5 and Corollary 3 it directly follows that a nonempty stable network g must be a complete M-partite network. Lemma 6 directly implies the asymmetry in sizes. □

Proof of Proposition 4: Consider game $C(K_{n-v,v})$. Proposition 9 states that there is a unique pure strategy Nash equilibrium $\bar{\boldsymbol{s}}$ of game $C(K_{n-v,v})$. The equilibrium is interior, under maintained assumptions. Since $\bar{\boldsymbol{s}}$ is the NE of $C(K_{n-v,v})$, $g(\bar{\boldsymbol{s}})$ satisfies (U) for $L_i = \emptyset$. The only new links that can possibly be formed in $g(\bar{\boldsymbol{s}})$ are with members of own partition. It is easy to see that no player will have an incentive to form a link with a member of own partition in $g(\bar{\boldsymbol{s}})$, since all members of the same partition have the same total spending. Hence, $\bar{\boldsymbol{s}}$ satisfies condition (U) from Definition 3. In the remaining part of the proof we show that part (B) of Definition 3 will be satisfied when $v < v^*$.

First note that a deviation in which players $i \in A$ and $j \in V$ destroy link ij is profitable for player $j \in V$, simply because she is a victim. We will now show that there exists $v^* > 0$ such that $i \in A$ prefers to destroy link with $j \in V$ in $K_{n-v,v}$ whenever $v \geq v^*$. To this end, let us define functions $h : \mathbb{R}^3_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $f : \mathbb{R}^3_{\geq 0} \to \mathbb{R}_{\geq 0}$ by:

$$h(v, s, r) = \max_x \left\{ \frac{x - s}{x + s + r} v - \alpha (vx)^2 \right\}, \tag{15}$$

$$f(n,v,r) = h(v-1, \bar{s}_{v,n-v}, r) - h(v, \bar{s}_{v,n-v}, r), \tag{16}$$

where $\bar{s}_{v,n-v}$ denotes the Nash equilibrium per-contest investments of a member of V in $C(K_{n-v,v})$. Due to symmetry, all members of the same partition will play the same strategy in $\bar{\boldsymbol{s}}$. Note that $f(n,v,r)$ is the expected benefit of destroying a link of an attacker in network $g(\bar{\boldsymbol{s}})$, where $\bar{\boldsymbol{s}}$ is the Nash equilibrium of $C(K_{n-v,v})$.

We now show that function f is monotonically increasing in $v \in [1,a]$ and that it takes a positive value when v is big. We will treat v as a continuous variable in the remaining part of the proof.

We show now that for $v \in [1,a] = [1, n-v]$,

$$f(n, v-1, r) < f(n, v, r).$$

In order to do this, we first show that h decreases with s, and that it decreases faster with s for higher values of v ($\frac{\partial h}{\partial s}$ decreases with v). Indeed, taking the derivative of h with respect to s we get:

$$\frac{\partial h}{\partial s} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial h}{\partial s} = -\frac{2x+r}{(x+s+r)^2}v, \tag{17}$$

where we used the fact that $\frac{\partial h}{\partial x} = 0$, since x is the maximizer of h. Differentiating with respect to v we get:

$$\frac{\partial^2 h}{\partial v \partial s} = 2\left[-\frac{x+\frac{r}{2}}{(x+s+r)^2} + v\frac{x-s}{(x+s+r)^3}\frac{\partial x}{\partial v}\right]. \tag{18}$$

The above derivative will be negative for all positive values of s and x such that $x \geq s$ and $\frac{\partial x}{\partial v} < 0$. This will hold in particular when $v \in [1,a]$ - since in the Nash equilibrium of $C(K_{n-v,v})$, the attackers exert a higher effort than the victims ($x \geq s$) and the investment of members of A decreases with v ($\frac{\partial x}{\partial v} < 0$).

From (17) and (18) we have that when $v \in [1,a)$

$$\frac{\partial\left[h(v-1,s,r) - h(v,s,r)\right]}{\partial s} > 0. \tag{19}$$

Since $\bar{s}_{v-1,n-v+1} < \bar{s}_{v-1,n-v} < \bar{s}_{v,n-v}$ from (19) directly follows that:

$$h(v-1, \bar{s}_{v,n-v}, r) - h(v, \bar{s}_{v,n-v}, r) > h(v-1, \bar{s}_{v-1,n-v+1}, r) - h(v, \bar{s}_{v-1,n-v+1}, r) \Rightarrow$$
$$f(n,v,r) > h(v-1, \bar{s}_{v-1,n-v+1}, r) - h(v, \bar{s}_{v-1,n-v+1}, r).$$

Finally, using the fact that h is concave in v (directly follows from the concavity of payoff function $\frac{x-s}{x+s+r}v - (vx)^2$ in v, see (De la Fuente, 2000, Theorems 2.12. and 2.13) for the formal argument), the following holds:

$$h(v-1, \bar{s}_{v-1,n-v+1}, r) - h(v, \bar{s}_{v-1,n-v+1}, r) > h(v-2, \bar{s}_{v-1,n-v+1}, r) - h(v-1, \bar{s}_{v-1,n-v+1}, r),$$

and therefore:

$$f(n, v, r) > f(n, v-1, r),$$

which is what we wanted to prove.

If for $v = 1$, f takes a positive value, than no $K_{n-v,v}$ is stable, and $v^* = 1$. If for $v = 1$ f takes a negative value, this means that star network is stable. We also know that when $v = a$ no player earns positive payoff from any contest, so f takes a positive value in this case. The fact that f is strictly monotone, and that it changes sign implies that there exists $v^* \in [1, a]$ such that $f(n, v, r) \geq 0$ for $v \geq v^*$ and $f(n, v, r) < 0$ for $v < v^*$, which completes the proof. □

Proofs of Claims from Section 4

We first show that the contest game on a complete bipartite network is a nice aggregative game (Acemoglu and Jensen, 2013), so we can use results from that paper for some of our comparative statics exercises. For the cases when results from (Acemoglu and Jensen, 2013) cannot be directly applied, we rely on the implicit function derivation of the equilibrium conditions.

Lemma 7
The contest game on a complete bipartite network $C(K_{a,v})$ can be represented as a nice aggregative game as defined in Acemoglu and Jensen (2013).

Proof of Lemma 7: The pure strategy Nash equilibrium of game $C(K_{a,v})$ is such that all players from the same class play the same strategy and invest the same amount of effort in each of their contest. When with $\phi(x) = x$ the conditions which determine the equilibrium investments in $C(K_{a,v})$ are equivalent to the system of FOCs that pins down the pure strategy Nash equilibrium of two players game in which the strategy space of each player is the set of nonnegative real numbers and the payoffs are defined by:

$$\pi_i(s_{ij}, s_{ji}; r) = \frac{s_{ij} - s_{ji}}{s_{ij} + s_{ji} + r} - \frac{1}{v} c(v s_{ij}),$$
$$\pi_j(s_{ji}, s_{ij}; r) = \frac{s_{ji} - s_{ij}}{s_{ij} + s_{ji} + r} - \frac{1}{a} c(a s_{ij}).$$

Since $\frac{s_{ij} - s_{ji}}{s_{ij} + s_{ji} + r} = -1 + \frac{2 s_{ij} + r}{s_{ij} + s_{ji} + r}$ it is straightforward to verify that this game is a *nice aggregative game* studied in (Acemoglu and Jensen, 2013). □

Proof of Proposition 5.

1. According to Lemma 7, the contest game on a complete bipartite network can be represented as a nice aggregative game. To prove that w^* decreases with α, it is sufficient to show that a decrease in α is a *positive shock* (Acemoglu and Jensen, 2013, Definition

9). A decrease in α will lead to a new cost function $\tilde{c}$ such that $\tilde{c}'(x) < c'(x)\ \forall x \in \mathbb{R}_{\geq 0}$. Denote with $\tilde{\pi}_i$ the payoff function of player $i \in A$ (and symmetrically for $j \in V$) after c becomes $\tilde{c}$. It is straightforward to see that $\frac{\partial \tilde{\pi}_i}{\partial s_{ij}} \leq \frac{\partial \pi_i}{\partial s_{ij}}$ when $\tilde{c}'(vs_{ij}) \leq c'(vs_{ij})$. Therefore a decrease in α is a positive shock. Analogously, a change in transfer T from $T = 1$ to $T = \tilde{T} > 1$ is a positive shock to both players.

2. To conduct a comparative statics exercise with respect to r we cannot apply the result for aggregative games, as an increase in r can be a positive shock for one player, and, at the same time, a negative shock for some other player. Indeed,

$$\frac{\partial^2 \pi_i}{\partial s_{ij} \partial r} = \frac{s_{ij} - 3s_{ji} - r}{(s_{ij} + s_{ji} + r)^3},$$

does not have the same sign for all non-negative arguments. Therefore, we rely on the implicit function theorem. The strategy profile $\boldsymbol{s}^*$ satisfies the first order optimality conditions:

$$\begin{aligned} \frac{r + 2s^*_{ji}}{(s^*_{ij} + s^*_{ji} + r)^2} &= c'\left(\sum_{k \in V} s^*_{ik}\right), \quad i \in A, \\ \frac{r + 2s^*_{ij}}{(s^*_{ij} + s^*_{ji} + r)^2} &= c'\left(\sum_{k \in A} s^*_{jk}\right), \quad j \in V. \end{aligned} \tag{20}$$

Taking the derivative of (20) with respect to r we get the following system of equations:

$$\frac{1 + 2s^*_{ji}(r)}{(r + s^*_{ij}(r) + s^*_{ji}(r))^2} - 2\frac{(r + 2s^*_{ji}(r))(1 + s^{*\prime}_{ij}(r) + s^{*\prime}_{ji}(r))}{(1 + s^*_{ij}(r) + s^*_{ji}(r))^2} = c''\left(\sum_{k \in V} s^*_{ik}(r)\right) \sum_{k \in V} s^{*\prime}_{ik}(r), \ i \in A$$
$$\frac{1 + 2s^*_{ij}(r)}{(r + s^*_{ij}(r) + s^*_{ji}(r))^2} - 2\frac{(r + 2s^*_{ij}(r))(1 + s^{*\prime}_{ij}(r) + s^{*\prime}_{ji}(r))}{(1 + s^*_{ij}(r) + s^*_{ji}(r))^2} = c''\left(\sum_{k \in A} s^*_{jk}(r)\right) \sum_{k \in A} s^{*\prime}_{jk}(r), \ j \in V$$

Using the symmetry, and solving for $s^{*\prime}_{ij}(r)$ and $s^{*\prime}_{ji}(r)$ we get:

$$\begin{aligned} s^{*\prime}_{ij}(r) &= -\frac{2 + a(r + s^*_{ij} + s^*_{ji})(r - s^*_{ij} + 3s^*_{ji})c''_2}{\Omega} \\ s^{*\prime}_{ji}(r) &= -\frac{2 + v(r + s^*_{ij} + s^*_{ji})(r - s^*_{ji} + 3s^*_{ij})c''_1}{\Omega} \end{aligned} \tag{21}$$

where

$$\Omega = 4 + (r + s^*_{ij} + s^*_{ji}) \times$$
$$\left[2vc''_1(r + s^*_{ij}) + ac''_2\left(r^3 vc''_1 + 3r^2 vc''_1(s^*_{ij} + s^*_{ji}) + r(2 + 3v(s^*_{ij} + s^*_{ji})^2 c''_1) + 4s^*_{ji} + v(s^*_{ij} + s^*_{ji})^3 c''_1\right)\right],$$

and $c''_1 = c''\left(\sum_{k \in V} s^*_{ik}\right)$ and $c''_2 = c''\left(\sum_{k \in A} s^*_{jk}\right)$.

The expression Ω is positive, since c is a convex function. Furthermore $a > v$ and $s^*_{ij} > s^*_{ji}$ imply that $s^{*\prime}_{ji}(r)$ in (21) is always negative. On the other hand, $s^{*\prime}_{ij}(r)$ can be both positive and negative. It will take a positive value whenever:

$$s_{ij}^* > \frac{2}{a(r + s_{ij}^* + s_{ji}^*)c_2''} + r + 3s_{ji}^*.$$

We now discuss the sign of $\frac{\partial w^*}{\partial r}$. From (21) we get:

$$\frac{\partial w^*(r)}{\partial r} > 0 \Leftrightarrow -\left[v(r + 3s_{ij}^* - s_{ji}^*)c_1'' + a(r + 3s_{ji}^* - s_{ij}^*)c_2''\right] > \frac{4}{r + s_{ij}^* + s_{ji}^*}. \quad (22)$$

When $c(x) = \alpha x^2$, equation (22) simplifies to

$$\frac{\partial w^*(r)}{\partial r} > 0 \Leftrightarrow -\alpha\left[v(r + 3s_{ij}^* - s_{ji}^*) + a(r + 3s_{ji}^* - s_{ij}^*)\right] > \frac{2}{r + s_{ij}^* + s_{ji}^*}.$$

The above inequality will hold when $a(r + 3s_{ji}^* - s_{ij}^*)$ is sufficiently small and negative. This will happen when a is large enough relative to v, and r is small enough.

In a specific case when $r \to 0$ (22) becomes

$$-\left[a(3s_{ji}^* - s_{ij}^*) + v(3s_{ij}^* - s_{ji}^*)\right] > \frac{2}{\alpha(s_{ij}^* + s_{ji}^*)}.$$

In this case (see Franke and Ozturk (2015) for derivation) $s_{ji}^* = \left(\frac{v}{a}\right)^{\frac{1}{2}} s_{ij}^*$ and $s_{ij}^* + s_{ji}^* = \frac{1}{\alpha^{\frac{1}{2}}(av)^{\frac{1}{4}}}$, so the above inequality can be written as

$$\left[(a - 3v) + \left(\frac{v}{a}\right)^{\frac{1}{2}}(v - 3a)\right] s_{ij}^* > \frac{2}{\alpha^{\frac{1}{2}}}(av)^{\frac{1}{4}}. \quad (23)$$

From (21) it follows that (23) can hold only when $s_{ij}^* > 3s_{ji}^*$, as otherwise ${s_{ij}^*}'(r) \leq 0$. Using the fact that $s_{ij}^* + s_{ji}^* = \frac{1}{\alpha^{\frac{1}{2}}(av)^{\frac{1}{4}}}$, we rewrite this condition as $s_{ij}^* > \frac{3}{4\alpha^{\frac{1}{2}}}(av)^{-\frac{1}{4}}$ which, together with (23), gives that $\frac{\partial w^*}{\partial r} > 0$ if

$$\left[(a - 3v) + \left(\frac{v}{a}\right)^{\frac{1}{2}}(v - 3a)\right] \frac{3}{4\alpha^{\frac{1}{2}}}(av)^{-\frac{1}{4}} > \frac{2}{\alpha^{\frac{1}{2}}}(av)^{\frac{1}{4}},$$

which is true whenever $a \geq 37v$.

□

Lemma 8

The total spending of each node in the equilibrium is defined as a solution of system (5)*. Furthemore*

$$\pi_i(\boldsymbol{s}^*) = \sum_{j \in N_i} \frac{w_j^* - w_i^*}{w_j^* + w_i^*} - \alpha {w_i^*}^2. \quad (24)$$

Proof of Lemma 8. Expressing s_{ij}^* from (8), when ϕ is identity mapping, and $c_i(x) = \alpha_i x^2$ we get that in the equilibrium:

$$s_{ij}^* = \frac{4\alpha_j w_j^*}{(2\alpha_i w_i^* + 2\alpha_j w_j^*)^2} - \frac{r}{2}. \tag{25}$$

Summing over all contests of player i, and setting $\alpha_i = \alpha$ $i \neq k$ and $\alpha_k = \alpha + \epsilon_k$ we get (5).

Plugging in (25) in (2) we get (24). □

Proof of Proposition 6. Suppose first that $k \in A$. Due to the symmetry, (5) is reduced to the following system of equations:

$$\begin{aligned} w_k^* &= v\frac{\alpha w_j^*}{(\alpha w_j^* + (\alpha+\epsilon_k)w_k^*)^2} - v\frac{r}{2}, \\ w_i^* &= v\frac{\alpha w_j^*}{(\alpha w_j^* + \alpha w_i^*)^2} - v\frac{r}{2}, \ i \in A \text{ and } i \neq k, \\ w_j^* &= (a-1)\frac{\alpha w_i^*}{(\alpha w_j^* + \alpha w_i^*)^2} + \frac{(\alpha+\epsilon_k)w_k^*}{(\alpha w_j^* + (\alpha+\epsilon_k)w_k^*)^2} - a\frac{r}{2}, \ j \in V. \end{aligned} \tag{26}$$

Differentiating with respect to ϵ_k, and letting $\epsilon_k \to 0$ we get the following linear system in first derivatives:

$$\begin{aligned} \left(1 + \frac{2}{\alpha}v\frac{w_j^*}{(w_k^* + w_j^*)^3}\right) w_k^{*\prime} &= \frac{v}{\alpha}\frac{w_k^* - w_j^*}{(w_k^* + w_j^*)^3} w_j^{*\prime} - \frac{v}{2\alpha^2}\frac{w_k^* w_j^*}{(w_k^* + w_j^*)^3}, \\ \left(1 + \frac{2}{\alpha}v\frac{w_j^*}{(w_i^* + w_j^*)^3}\right) w_i^{*\prime} &= \frac{v}{\alpha}\frac{w_i^* - w_j^*}{(w_i^* + w_j^*)^3} w_j^{*\prime}, \\ \left(1 + \frac{2}{\alpha}\frac{(a-1)w_i^* + w_k^*}{(w_i^* + w_j^*)^3}\right) w_j^{*\prime} &= \frac{a-1}{\alpha}\frac{w_j^* - w_i^*}{(w_j^* + w_i^*)^3} w_i^{*\prime} + \frac{1}{\alpha}\frac{w_j^* - w_k^*}{(w_j^* + w_i^*)^3} w_k^{*\prime} - \frac{1}{\alpha^2}\frac{w_k^{*2} - w_j^* w_k^*}{(w_j^* + w_k^*)^3}. \end{aligned} \tag{27}$$

Using the fact that, when $\epsilon_k \to 0$, then $w_i^* = w_k^*$, we get

$$\begin{aligned} \left((w_i^* + w_j^*) + \frac{2}{\alpha}v\frac{w_j^*}{(w_i^* + w_j^*)^2}\right) w_k^{*\prime} &= \frac{v}{\alpha}\frac{w_i^* - w_j^*}{(w_i^* + w_j^*)^2} w_j^{*\prime} - \frac{v}{2\alpha^2}\frac{w_i^* w_j^*}{(w_i^* + w_j^*)^2}, \\ \left((w_i^* + w_j^*) + \frac{2}{\alpha}v\frac{w_j^*}{(w_i^* + w_j^*)^2}\right) w_i^{*\prime} &= \frac{v}{\alpha}\frac{w_i^* - w_j^*}{(w_i^* + w_j^*)^2} w_j^{*\prime}, \\ \left((w_j^* + w_i^*) + \frac{2}{\alpha}a\frac{w_i^*}{(w_i^* + w_j^*)^2}\right) w_j^{*\prime} &= \frac{a-1}{\alpha}\frac{w_j^* - w_i^*}{(w_i^* + w_j^*)^2} w_i^{*\prime} + \frac{1}{\alpha}\frac{w_j^* - w_i^*}{(w_i^* + w_j^*)^2} w_k^{*\prime} - \frac{1}{\alpha^2}\frac{w_i^{*2} - w_j^* w_i^*}{(w_i^* + w_j^*)^2}. \end{aligned}$$

When $r \to 0$, w_i^* and w_j^* simplify to

$$w_i^* = \frac{v}{\alpha}\frac{w_j^*}{(w_i^* + w_j^*)^2}, \quad w_j^* = \frac{a}{\alpha}\frac{w_i^*}{(w_i^* + w_j^*)^2},$$

and therefore we can write (27) as

$$\begin{aligned}
(3w_i^* + w_j^*)w_k^{*\prime} &= \frac{vw_j^* - aw_i^*}{a} w_j^{*\prime} - \frac{2}{\alpha} w_i^{*2}, \\
(3w_i^* + w_j^*)w_i^{*\prime} &= \frac{vw_j^* - aw_i^*}{a} w_j^{*\prime}, \\
(w_i^* + 3w_j^*)w_j^{*\prime} &= \frac{aw_i^* - vw_j^*}{va} w_k^{*\prime} + (a-1)\frac{aw_i^* - vw_j^*}{va} w_i^{*\prime} + \frac{w_i^*}{\alpha}\frac{aw_i^* - vw_j^*}{va}.
\end{aligned} \tag{28}$$

We note that the equilibrium strengths in $K_{a,v}$ can be expressed as (see Franke and Ozturk (2015) for derivation):

$$\begin{aligned}
w_i^* &= \frac{1}{\sqrt{\alpha}} \frac{v\sqrt{a}}{\sqrt{v} + \sqrt{a}} \frac{1}{(av)^{\frac{1}{4}}}, \\
w_j^* &= \frac{1}{\sqrt{\alpha}} \frac{a\sqrt{v}}{\sqrt{v} + \sqrt{a}} \frac{1}{(av)^{\frac{1}{4}}}.
\end{aligned} \tag{29}$$

Plugging (29) in (28) and solving the resulting linear system we get:

$$\begin{aligned}
w_k^{*\prime} &= -\frac{(av)^{\frac{3}{4}} \left((\sqrt{a} - \sqrt{v})^2 + 8a\sqrt{v}\,(\sqrt{a} + \sqrt{v})\right)}{4a\sqrt{a}(\sqrt{a} + 3\sqrt{v})(\sqrt{a} + \sqrt{v})^2 \alpha\sqrt{\alpha}} < 0, \\
w_i^{*\prime} &= -\frac{(av)^{\frac{3}{4}} (\sqrt{a} - \sqrt{v})^2}{4a\sqrt{a}(\sqrt{a} + 3\sqrt{v})(\sqrt{a} + \sqrt{v})^2 \alpha\sqrt{\alpha}} < 0, \\
w_j^{*\prime} &= \frac{\sqrt{v}(\sqrt{a} - \sqrt{v})}{4\alpha\sqrt{\alpha}(va)^{\frac{1}{4}}(\sqrt{a} + \sqrt{v})^2} > 0.
\end{aligned} \tag{30}$$

The case when $k \in V$ is analogous (or just change switch v and a).

Finally from (30) and (29) we get:

$$\frac{\partial w^*}{\partial \epsilon_k}|_{\epsilon_k = 0} = \frac{1}{av}\left(w_k^{*\prime} + (a-1)w_i^{*\prime} + vw_j^{*\prime}\right) = -\frac{v}{4\alpha^{\frac{3}{2}}(av)^{\frac{5}{4}}} < 0,$$

which completes the proof. □

Proofs of Claims from Section 5

Proof of Proposition 7. Consider contest network $g(\boldsymbol{s})$ such that $ij \notin g$ for some players i and j. We show that $g(\boldsymbol{s})$ is not Nash stable when $\frac{\phi'(0)}{r} > c'(0)$.

(i) Consider first the case when player i is not involved in any contest, thus $w_i = 0$. The marginal benefit of investing $\epsilon > 0$ in contest ij calculated at $\epsilon = 0$ is $\frac{\phi'(0)}{r}$. The marginal cost of this action is $c'(0)$. As long as $\frac{\phi'(0)}{r} > c'(0)$ player i will wish to start a contest with player j.

(ii) When $w_i > 0$, there must exist some some k such that $s_{ik} > 0$. We discuss two possible cases:

(a) There exists a contest $ik \in g(\boldsymbol{s})$ such that $s_{ik} \geq s_{ki}$. Consider a deviation in which i

reallocates $\epsilon > 0$ from contest ik to start contest with j. The marginal benefit of this action for i is $\frac{\phi'(0)}{r}$. The marginal cost of a proposed deviation is $\frac{(r+2\phi(s_{ki}))\phi'(s_{ik})}{(r+\phi(s_{ik})+\phi(s_{ki}))^2}$. The following chain of inequalities holds:

$$\frac{(r+2\phi(s_{ki}))\phi'(s_{ik})}{(r+\phi(s_{ik})+\phi(s_{ki}))^2} \leq \frac{r+2\phi(s_{ki})}{(r+2\phi(s_{ki}))^2}\phi'(s_{ik}) \leq \frac{1}{r+2\phi(s_{ki})}\phi'(0) < \frac{1}{r}\phi'(0), \quad (31)$$

where we have used the fact that ϕ is increasing and concave function. So, in this case, the marginal benefit of the proposed deviation is greater than it's marginal cost.

(b) There is no $ik \in g(\boldsymbol{s})$ such that $s_{ik} \geq s_{ki}$. In this case consider a deviation in which i reallocates s_{ik} from contest ik to ij. The change in payoff due to this deviation is equal to $\left(\frac{\phi(s_{ik})}{\phi(s_{ik})+r} - \frac{\phi(s_{ki})}{\phi(s_{ki})+r}\right) - \frac{\phi(s_{ik})-\phi(s_{ki})}{\phi(s_{ik})+\phi(s_{ki})+r}$. Simplifying we get:

$$\frac{\phi(s_{ik})}{\phi(s_{ik})+r} - \frac{\phi(s_{ki})}{\phi(s_{ki})+r} = \frac{\phi(s_{ik})-\phi(s_{ki})}{\frac{\phi(s_{ik})\phi(s_{ki})}{r}+\phi(s_{ik})+\phi(s_{ki})+r} > \frac{\phi(s_{ik})-\phi(s_{ki})}{\phi(s_{ik})+\phi(s_{ki})+r},$$

where for the last inequality we used the fact that $s_{ik} < s_{ki}$, and ϕ is increasing.

Hence, provided that $\frac{\phi'(0)}{r} > c'(0)$, Nash stable network $g(\boldsymbol{s})$ must be such that $s_{ij} + s_{ji} > 0$, for any pair of players i and j, that is $ij \in g, \forall i, j \in N$.

We have proved that a Nash stable network must be the complete network. We now argue that there is a unique strategy profile $\boldsymbol{s}$ such that the complete network $g(\boldsymbol{s})$ is Nash stable. Moreover, $\boldsymbol{s}$ is such that $s_{ij} = s_{ji} = s > 0$, for any two players i and j.

To do that, we recall that there exists a unique pure strategy Nash equilibrium of the game $C(\bar{g})$ when $\bar{g}$ is the complete network. In this equilibrium each player must play the symmetric strategy, as otherwise the uniqueness result would not hold.[14] The condition $\frac{\phi'(0)}{r} > c'(0)$ ensures that $\bar{\boldsymbol{s}} \neq \boldsymbol{0}$, by same argument as used in (i) of this proof. It directly follows from the definition of a Nash stable network that it must be $s_{ij} = \bar{s}_{ij}$, where $\bar{\boldsymbol{s}}$ is the Nash equilibrium of the contest game on the complete network.

Finally, when $\frac{\phi'(0)}{r} \leq c'(0)$ exerting positive amount of resources in contest against opponent who invests 0 is never profitable. Furthermore, if for any pair of players we have $s_{ij} > 0$ and $s_{ji} > 0$ and, without loss of generality, $s_{ij} \geq s_{ji}$, then the marginal loss of i in decreasing s_{ij} is always smaller then the marginal gain measured by the cost decrease, as long as $\frac{\phi'(0)}{r} \leq c'(0)$. Indeed, the following chain of inequalities hold:

$$\frac{(r+2\phi(s_{ji}))\phi'(s_{ij})}{(r+\phi(s_{ij})+\phi(s_{ji}))^2} < \frac{1}{r}\phi'(0) \leq c'(0) < c'(w_i),$$

where the first inequality comes from (31). This completes the proof. □

Proof of Proposition 8. Omitted. □

[14] If $\bar{\boldsymbol{s}}$ was asymmetric, by relabeling players we could find more than one pure strategy NE of the contest game on the complete network, which would contradict the uniqueness result.

Online Appendix B

LPFS as a resting point of a dynamic process of network formation

We can think of the stable networks from Definition 3 as stable states of the coupled dynamic process we present in this section. Players make decisions about their links and about actions assigned to these links. We assume that a link between players i and j is formed if one player decides to form it (unilateral), while link ij is destroyed if both agents agree to destroy it (bilateral). Time is indexed with $t \in \mathbb{N} \cup \{0\}$. In $t = 0$ an arbitrary contest network $g(\boldsymbol{s})$ is given.

For each period t:

(i) At the beginning of period t strategy profile $\boldsymbol{s}_{t-1}$ is a pure strategy NE of game $C(\bar{g}_{t-1})$, where network $\bar{g}_{t-1}$ describes the set of contests from the end of period $t-1$.

(ii) Players i and j are chosen randomly from the population. They jointly choose their linking patterns which leads to a network of interactions $\bar{g}_t$. Players calculate the expected benefit from forming a link as described in Subsection 2.2.

(iii) The second dynamic process (*action adjustment process*[15]) starts, and all agents update their actions given $\bar{g}_t$ according to the *action adjustment process* formally described below. This process settles at the pure strategy NE of game $C(\bar{g}_t)$.

We now formally describe the action adjustment process mentioned in (iii) above. Let $\nabla_i \pi_i$ denote the gradient of the payoff function with respect to $\boldsymbol{s}_i$. Define function $J : \prod_i \mathbb{R}^n_{\geq 0} \to \prod_i \mathbb{R}^n_{\geq 0}$ with:

$$J(\mathbf{s}) = \begin{pmatrix} \nabla_1 \pi_1(\mathbf{s}) \\ \nabla_2 \pi_2(\mathbf{s}) \\ \cdots \\ \nabla_n \pi_n(\mathbf{s}) \end{pmatrix}.$$

The action adjustment process is defined with:

$$\dot{\mathbf{s}} = \lambda J(\mathbf{s}), \tag{32}$$

where λ is a constant. It is clear that $\bar{\boldsymbol{s}}$ is the stable state of this process. We also prove that $\bar{\boldsymbol{s}}$ is a globally asymptotically stable state of (32). To show this, we show that the rate of change of $||J|| = JJ'$ is always negative (and equal to 0 in the equilibrium). Denote with $\mathbf{H}$ the Jacobian of J. The following holds:

$$\dot{\overline{JJ'}} = (\mathbf{H}\dot{\mathbf{s}})'J + J'\mathbf{H}\dot{\mathbf{s}} = (J'\mathbf{H}'J + J'\mathbf{H}J) = J'(\mathbf{H}' + \mathbf{H})J < 0,$$

[15]We assume that this process takes place in continuous time and therefore on a faster time-scale than the network formation process. That is players infinitely more often revise their investment in ongoing contests compared to contemplating starting/ending a contest

where the last inequality follows from the fact that $(\mathbf{H}' + \mathbf{H})$ is a negative definite matrix, and $\mathbf{H}'$ is the transposed matrix $\mathbf{H}$. Thus, if every player adjusts her actions according to the adjustment process in (32), the action adjustment process converges, irrespective of the initial conditions. Thus, we have proved Proposition 11.

Proposition 11
The action adjustment process given by equation (32) *is globally asymptotically stable.*

We do not study the properties of the dynamical process of network formation. However, it is clear from the definition that if this process settles on a single network configuration, then this network must be LPFS. It is interesting to note that Proposition 11 has a very practical application. It provides an efficient way to numerically calculate the Nash equilibrium of game $C(\bar{g})$.