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A CLOSED-FORM SOLUTION FOR FREE VIBRATION OF MULTIPLE CRACKED TIMOSHENKO BEAM AND APPLICATION

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Abstract. A closed-form solution for free vibration is constructed and used for obtaining explicit frequency equation and mode shapes of Timoshenko beams with arbitrary number of cracks. The cracks are represented by the rotational springs of stiffness calculated from the crack depth. Using the obtained frequency equation, the sensitivity of natural frequencies to crack of the beams is examined in comparison with the Euler–Bernoulli beams. Numerical results demonstrate that the Timoshenko beam theory is efficiently applicable not only for short or fat beams but also for the long or slender ones. Nevertheless, both the theories are equivalent in sensitivity analysis of fundamental frequency to cracks and they get to be different for higher frequencies.

Keywords: Timoshenko beams, multiple cracked beams, natural frequencies, sensitivity analysis.

1. INTRODUCTION

About a century has passed from the date when Timoshenko Beam Theory (TBT) was proposed and although it is not straightforward as the classical Euler–Bernoulli Beam Theory (EBT) the TBT is not less popular nowadays than the classical one [1]. Generalized by taking into account the shear deformation and rotary inertia the TBT has extended applicability of beam theory to analysis of short or fat beams which are more widely encountered in the practice of structural engineering. Basics for dynamic analysis of Timoshenko beams are provided in numerous publications on structural dynamics, for example, the References [2–4]. Recently, because of potential hazards produced by presence of a crack in a structure, dynamic analysis of cracked structures gets an enormous attention of researchers and engineers. Numerous methods were proposed for modal analysis and crack detection based on the classical EBT [5–10]. Among the obtained results it is worth to note that a closed-form solution for the vibration mode of Euler–Bernoulli beams with multiple cracks was conducted and used for obtaining an explicit expression for characteristic equation of the beam [11, 12]. Vibration of cracked Timoshenko beams has been studied by numerous authors in [13–18] among that the studies by Li [15] and Aydin [18] are noteworthy by their achievements in modal analysis of cracked Timoshenko beams. Some particular solutions of the crack detection problem for Timoshenko beam were obtained earlier in [13, 14, 17] and recently by Khaji and his coworkers in [19, 20] using the conventional methods.

In the present paper a closed-form solution for free vibration of Timoshenko beam with arbitrary number of cracks is conducted and used for constructing an explicit expression for both frequency equation and mode shape of the beam. Using the obtained frequency equation, the sensitivity of natural frequencies to crack of the beams is examined in comparison with the Euler–Bernoulli beams [12]. Numerical results demonstrate that the Timoshenko beam theory is efficiently applicable not only for short or thick beams but also for the long or slender one. Nevertheless, both the theories are equivalent in sensitivity analysis of fundamental frequency to cracks and they get to be different in the analysis of higher frequencies.

2. A CLOSED-FORM SOLUTION FOR FREE VIBRATION OF TIMOSHENKO BEAM WITH MULTIPLE CRACKS

Consider a uniform beam of length ℓ; material density (ρ); elasticity (E) and shear (G) modulus; area $A = b \times h$ and moment of inertia $I = bh^3/12$ of cross section. Assuming first order shear deformation (Timoshenko) theory of beam, the displacement field in cross-section at x and height z from the neutral axis is

$$u(x, z, t) = u_0(x, t) - z\theta(x, t); \;\; w(x, z, t) = w_0(x, t), \tag{1}$$

with $u_0(x, t), w_0(x, t), \theta(x, t)$ being respectively the displacements and slope at central axis.

Using the constituting equations

$$\varepsilon_x = \partial u_0/\partial x - z\partial\theta/\partial x; \;\; \gamma_{xz} = \partial w_0/\partial x - \theta; \;\; \sigma_x = E\varepsilon_x; \;\; \tau_{xz} = \kappa G\gamma_{xz}. \tag{2}$$

and Hamilton principle, the equations for free vibration of the beam can be established as

$$\rho A\ddot{w} - \kappa GA(w'' - \theta') = 0; \;\; \rho I\ddot{\theta} - EI\theta'' - \kappa GA(w' - \theta) = 0, \tag{3}$$

where $\ddot{w} = \partial^2 w/\partial t^2, w' = \partial w/\partial x, w'' = \partial^2 w/\partial x^2$ and $\ddot{\theta} = \partial^2\theta/\partial t^2, \theta' = \partial\theta/\partial x, \theta'' = \partial^2\theta/\partial x^2$.

Seeking solution of (3) in the form

$$w(x, t) = W(x)e^{i\omega t}; \;\; \theta(x, t) = \Theta(x)e^{i\omega t}, \tag{4}$$

one gets

$$\omega^2\rho W(x) + \kappa G(W'' - \Theta') = 0; \;\; \omega^2\rho I\Theta(x) + EI\Theta''(x) + \kappa GA(W' - \Theta) = 0. \tag{5}$$

Furthermore, it is assumed that the beam has been cracked at positions $e_j, j = 1, \ldots, n$ and the cracks are modeled by rotational springs of stiffness K_j calculated from crack depth [8]. Therefore, conditions that must be satisfied at the crack section are

$$\begin{aligned} &W(e_j+0) = W(e_j-0); \;\; \Theta(e_j+0) = \Theta(e_j-0) + M(e_j)/K_j\,; \\ &Q(e_j+0) = Q(e_j-0) = Q(e_j); \;\; M(e_j+0) = M(e_j-0) = M(e_j), \end{aligned} \tag{6}$$

where N, Q, M are respectively internal axial, shear forces and bending moment at section x

$$M = EI\Theta'_x; \;\; Q = \kappa GA(W'_x - \Theta). \tag{7}$$

Substituting (7) into (6) one can rewrite the latter conditions as

$$\begin{aligned} &W(e_j+0) = W(e_j-0) = W(e_j); \;\; \Theta'_x(e_j+0) = \Theta'_x(e_j-0) = \Theta'(e_j); \\ &\Theta(e_j+0) = \Theta(e_j-0) + \gamma_j\Theta'_x(e_j); \; W'_x(e_j+0) = W'_x(e_j-0) + \gamma_j\Theta'_x(e_j); \; \gamma_j = EI/K_j. \end{aligned} \tag{8}$$

Seeking solution of Eq. (5) in the form $W_0(x) = C_w e^{\lambda x}, \Theta_0(x) = C_\theta e^{\lambda x}$ one is able to obtain so-called characteristic equation

$$\lambda^4 + b\lambda^2 - c = 0, \tag{9}$$

$$b = \alpha(1+\beta); \;\; c = \alpha(\tau - \alpha\beta); \;\; \alpha = \rho\omega^2/E; \;\; \beta = E/\kappa G; \;\; \tau = A/I. \tag{10}$$

This is a square algebraic equation with respect to $\eta = \lambda^2$ that can be elementarily solved to give roots

$$\eta_1 = (-b + \sqrt{b^2+4c})/2; \, \eta_2 = -(b + \sqrt{b^2+4c})/2. \tag{11}$$

Note that in the case if $c = 0$ the Eq. (9) has the roots

$$\lambda_{1,2} = \pm i\sqrt{b} = \pm i\omega\sqrt{\rho(1+\beta)/E}; \lambda_{3,4} = 0. \tag{12}$$

This occurs when $\omega = \omega_c = \sqrt{12\kappa G/\rho h^2}$ acknowledged as cut-off frequency of the beam. Otherwise, the Eq. (9) has the roots

$$\lambda_{1,2} = \pm k_1; \;\; \lambda_{3,4} = \pm ik_2; \;\; k_1 = \sqrt{(\sqrt{b^2+4c} - b)/2}, \;\; k_2 = \sqrt{(\sqrt{b^2+4c} + b)/2}, \tag{13}$$

for frequency less than cut-off one, $\omega < \omega_c = \sqrt{\kappa GA/\rho I}$. Since the cut-off frequency is very high, vibration of the beam is often investigated in the lower frequency range $(0, \omega_c)$. Thus, in the frequency range, general continuous solution of Eq. (5) can be represented as

$$W_0(x) = C_1 \cosh k_1 x + C_2 \sinh k_1 x + C_3 \cos k_2 x + C_4 \sin k_2 x, \tag{14}$$

$$\Theta_0(x) = r_1 C_1 \sinh k_1 x + r_1 C_2 \cosh k_1 x + r_2 C_3 \sin k_2 x - r_2 C_4 \cos k_2 x, \tag{15}$$

$$r_1 = (\rho\omega^2/\kappa G k_1 + k_1); r_2 = (\rho\omega^2/\kappa G k_2 - k_2). \tag{16}$$

Particularly, solution (14) and (15) satisfying the conditions

$$W_0(0) = 0; W'_0(0) = 1; \Theta_0(0) = 1; \Theta'_0(0) = 0,$$

is

$$S_w(x) = S_1 \sinh k_1 x + S_2 \sin k_2 x; \; S_\theta(x) = r_1 S_1 \cosh k_1 x - r_2 S_2 \cos k_2 x, \tag{17}$$

$$S_1 = (r_2 + k_2)/(r_1 k_2 + r_2 k_1); \; S_2 = (r_1 - k_1)/(r_1 k_2 + r_2 k_1). \tag{18}$$

Using obtained above particular solution, general solution of Eq. (5) satisfying conditions (8) at cracks is represented by

$$W(x,\omega) = C_1W_1(k_1,x) + C_2W_2(k_1,x) + C_3W_3(k_2,x) + C_4W_4(k_2,x), \tag{19}$$

$$\Theta(x,\omega) = C_1\Theta_1(k_1,x) + C_2\Theta_2(k_1,x) + C_3\Theta_3(k_2,x) + C_4\Theta_4(k_2,x), \tag{20}$$

where

$$\begin{aligned}
&W_1(x) = \cosh k_1x + \sum_{j=1}^{n}\mu_{1j}K_w(x-e_j);\;\; W_2(x) = \sinh k_1x + \sum_{j=1}^{n}\mu_{2j}K_w(x-e_j),\\
&W_3(x) = \cos k_2x + \sum_{j=1}^{n}\mu_{3j}K_w(x-e_j);\;\; W_4(x) = \sin k_2x + \sum_{j=1}^{n}\mu_{4j}K_w(x-e_j),\\
&\Theta_1(x) = r_1\sinh k_1x + \sum_{j=1}^{n}\mu_{1j}K_\theta(x-e_j);\;\; \Theta_2(x) = r_1\cosh k_1x + \sum_{j=1}^{n}\mu_{2j}K_\theta(x-e_j),\\
&\Theta_3(x) = r_2\sin k_2x + \sum_{j=1}^{n}\mu_{3j}K_\theta(x-e_j);\;\; \Theta_4(x) = -r_2\cos k_2x + \sum_{j=1}^{n}\mu_{4j}K_\theta(x-e_j),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&K_w(x) = \begin{cases} 0: & x<0;\\ S_w(x): & x\ge 0;\end{cases} \quad K'_w(x) = \begin{cases} 0: & x<0;\\ S'_w(x): & x\ge 0;\end{cases}\\
&K_\theta(x) = \begin{cases} 0: & x<0;\\ S_\theta(x): & x\ge 0;\end{cases} \quad K'_\theta(x) = \begin{cases} 0: & x<0;\\ S'_\theta(x): & x\ge 0;\end{cases}\\
&\mu_{kj} = \gamma_j\left\{L_k(e_j) + \sum_{i=1}^{j-1}\mu_{ki}S'_\theta(e_j-e_i)\right\};\; k=1,2,3,4; j=1,2,\ldots,n.
\end{aligned} \tag{22}$$

$$L_1(x) = k_1r_1\cosh k_1x; L_2(x) = k_1r_1\sinh k_1x; L_3(x) = k_2r_2\cos k_2x; L_4(x) = k_2r_2\sin k_2x. \tag{23}$$

3. NATURAL FREQUENCIES AND MODE SHAPES

In this section, frequency equation is obtained for beam with classical boundary conditions such as simply supported (SS), clamped (CC) beam and cantilever (clamped and free (CF) end beam). The boundary conditions are expressed as follow:

• For SS-beam: $W(0) = M(0) = W(\ell) = M(\ell) = 0$. In this case

$$C_1W_1(k_1,0) + C_2W_2(k_1,0) + C_3W_3(k_2,0) + C_4W_4(k_2,0) = 0 \Rightarrow C_1 + C_3 = 0,$$

$$C_1\Theta'_1(k_1,0) + C_2\Theta'_2(k_1,0) + C_3\Theta'_3(k_2,0) + C_4\Theta'_4(k_2,0) = 0 \Rightarrow r_1k_1C_1 + r_2k_2C_3 = 0,$$

from that we have got $C_1 = C_3 = 0$ and

$$C_2W_2(k_1,\ell) + C_4W_4(k_2,\ell) = 0;\;\; C_2\Theta'_2(k_1,\ell) + C_4\Theta'_4(k_2,\ell) = 0. \tag{24}$$

Therefore, frequency equation is obtained in the form $W_2(k_1,\ell)\Theta'_4(k_2,\ell)$ $-W_4(k_2,\ell)\Theta'_2(k_1,\ell) = 0$ or

$$d_{SS}(\omega) + \sum_{j=1}^{n}\left[\mu_{2j}\bar{S}_2(\ell-e_j) + \mu_{4j}\bar{S}_4(\ell-e_j)\right] + \sum_{j,k=1}^{n}\mu_{2j}\mu_{4j}\bar{S}_{24}(e_j,e_k) = 0, \tag{25}$$

where

$$\begin{aligned} d_{SS}(\omega) &= (r_2k_2 - r_1k_1)\sinh k_1\ell \sin k_2\ell, \\ \bar{S}_2(\ell - e_j) &= [r_2k_2S_w(\ell - e_j) - S'_\theta(\ell - e_j)]\sin k_2\ell, \\ \bar{S}_4(\ell - e_j) &= [S'_\theta(\ell - e_j) - r_1k_1S_w(\ell - e_j)]\sinh k_1\ell, \\ \bar{S}_{24}(e_j, e_k) &= S_w(\ell - e_j)S'_\theta(\ell - e_k) - S_w(\ell - e_k)S'_\theta(\ell - e_j). \end{aligned} \tag{26}$$

For uncracked beam, the frequency equation (24) is reduced to $\sin k_2\ell = 0$ that leads to $k_2\ell = j\pi; j = 1, 2, 3, \ldots$ and in the case of single crack, $n = 1$, Eq. (25) is

$$f_{SS}(\omega) + \gamma g_{ss}(e, \omega) = 0, \tag{27}$$

with

$$f_{SS}(\omega) = \sinh k_1\ell \sin k_2\ell,$$

$$g_{ss}(e, \omega) = r_2k_2S_1 \sinh k_1e \sinh k_1(\ell - e) \sin k_2\ell + r_1k_1S_2 \sin k_2e \sin k_2(\ell - e) \sinh k_1\ell.$$

The latter equation has been obtained in [19].

• For CC-beam: $W(0) = \Theta(0) = W(\ell) = \Theta(\ell) = 0$. The conditions at $x = 0$ lead to $C_1 + C_3 = 0$ and $r_1C_2 - r_2C_4 = 0$. So that

$$W(x) = C_1L_{w1}(x) + C_2L_{w2}(x);\;\; \Theta(x) = C_1L_{\theta 1}(x) + C_2L_{\theta 2}(x) \tag{28}$$

with

$$L_{w1}(x) = L_{01}(x) + \sum_{j=1}^{n} \bar{\mu}_{1j}K_w(x - e_j);\;\; L_{w2}(x) = L_{02}(x) + \sum_{j=1}^{n} \bar{\mu}_{3j}K_w(x - e_j); \tag{29}$$

$$L_{\theta 1}(x) = L_{03}(x) + \sum_{j=1}^{n} \bar{\mu}_{1j}K_\theta(x - e_j);\;\; L_{\theta 2}(x) = L_{04}(x) + \sum_{j=1}^{n} \bar{\mu}_{3j}K_\theta(x - e_j);$$

$$L_{01} = \cosh k_1x - \cos k_2x;\; L_{02} = r_2 \sinh k_1x + r_1 \sin k_2x;\; \bar{\mu}_{1j} = \mu_{1j} - \mu_{3j};\; \bar{\mu}_{3j} = r_2\mu_{1j} + r_1\mu_{3j};$$

$$L_{03} = r_1 \sinh k_1x - r_2 \sin k_2x;\;\; L_{04} = r_1r_2(\cosh k_1x - \cos k_2x).$$

Therefore, frequency equation for clamped beam is derived from the conditions

$$W(\ell) = C_1L_{w1}(\ell) + C_2L_{w2}(\ell) = 0;\;\; \Theta(\ell) = C_1L_{\theta 1}(\ell) + C_2L_{\theta 2}(\ell) = 0 \tag{30}$$

as $L_{w1}(\ell)L_{\theta 2}(\ell) - L_{w2}(\ell)L_{\theta 1}(\ell) = 0$ or

$$d_{CC}(\omega) + \sum_{j=1}^{n} [\bar{\mu}_{1j}\bar{S}_1(\ell - e_j) + \bar{\mu}_{3j}\bar{S}_3(\ell - e_j)] + \sum_{j,k=1}^{n} \bar{\mu}_{1j}\bar{\mu}_{3k}\bar{S}_{13}(e_j, e_k) = 0, \tag{31}$$

where

$$\begin{aligned} d_{CC}(\omega) &= L_{01}(\ell)L_{04}(\ell) - L_{02}(\ell)L_{03}(\ell), \\ \bar{S}_1(\ell - e_j) &= L_{04}(\ell)S_w(\ell - e_j) - L_{02}(\ell)S_\theta(\ell - e_j), \\ \bar{S}_3(L - e_j) &= L_{01}(\ell)S_\theta(\ell - e_j) - L_{03}(\ell)S_w(\ell - e_j), \\ \bar{S}_{13}(e_j, e_k) &= S_w(\ell - e_j)S_\theta(\ell - e_k) - S_w(\ell - e_k)S_\theta(\ell - e_j). \end{aligned} \tag{32}$$

In the case of single crack, Eq. (31) is simplified [19]

$$d_{CC}(\omega) + \gamma\{[L_1(e) - L_3(e)]\bar{S}_1(\ell - e) + [r_2L_1(e) + r_1L_3(e)]\bar{S}_3(\ell - e)\} = 0,$$

or

$$f_{cc}(\omega) + \gamma g_{cc}(e, \omega) = 0, \tag{33}$$

with

$$f_{cc}(\omega) = 2r_1 r_2(1 - \cosh k_1\ell \cos k_2\ell) + (r_2^2 - r_1^2)\sinh k_1\ell \sin k_2\ell;$$

$$\begin{aligned} g_{cc}(e, \omega) = [L_1(e) - L_3(e)]\{L_{04}(\ell)S_w(\ell - e) - L_{02}(\ell)S_\theta(\ell - e)\} + \\ + [r_2 L_1(e) + r_1 L_3(e)]\{L_{01}(\ell)S_\theta(\ell - e)s - L_{03}(\ell)S_w(\ell - e)\}. \end{aligned}$$

• For CF-beam: $W(0) = \Theta(0) = M(\ell) = Q(\ell) = 0$. It is found above that conditions for clamp at $x = 0$ lead the solutions (19), (20) to expressions (27), so that conditions for free end at $x = \ell$ now yield

$$C_1[L'_{w1}(\ell) - L_{\theta 1}(\ell)] + C_2[L_{w2}(\ell) - L_{\theta 2}(\ell)] = 0; \;\; C_1 L'_{\theta 1}(\ell) + C_2 L'_{\theta 2}(\ell) = 0, \tag{34}$$

that allow one to obtain frequency equation for CF-beam in the form

$$[L'_{w1}(\ell)L'_{\theta 2}(\ell) - L'_{w2}(\ell)L'_{\theta 1}(\ell)] - [L_{\theta 1}(\ell)L'_{\theta 2}(\ell) - L_{\theta 2}(\ell)L'_{\theta 1}(\ell)] = 0. \tag{35}$$

The latter equation can be rewritten as

$$d_{CF}(\omega) + \sum_{j=1}^{n} [\bar{\mu}_{1j}\bar{G}_1(\ell - e_j) + \bar{\mu}_{3j}\bar{G}_3(\ell - e_j)] + \sum_{j,k=1}^{n} \bar{\mu}_{1j}\bar{\mu}_{3k}\bar{G}_{13}(e_j, e_k) = 0, \tag{36}$$

where

$$\begin{aligned} &d_{CF}(\omega) = [L'_{01}(\ell) - L_{03}(\ell)]L'_{04}(\ell) - [L'_{02}(\ell) - L_{04}(\ell)]L'_{03}(\ell), \\ &\bar{G}_1(\ell - e_j) = L'_{04}(\ell)[S'_w(\ell - e_j) - S_\theta(\ell - e_j)] - [L'_{02}(\ell) - L_{04}(\ell)]S'_\theta(\ell - e_j), \\ &\bar{G}_3(L - e_j) = [L'_{01}(\ell) - L_{03}(\ell)]S'_\theta(\ell - e_j) - L'_{03}(\ell)[S'_w(\ell - e_j) - S_\theta(\ell - e_j)], \\ &\bar{G}_{13}(e_j, e_k) = [S'_w(\ell - e_j) - S_\theta(\ell - e_j)]S'_\theta(\ell - e_k) - [S'_w(\ell - e_k) - S_\theta(\ell - e_k)]]S'_\theta(\ell - e_j). \end{aligned} \tag{37}$$

Similarly, one can obtain frequency for beam with single crack in the form [19]

$$d_{CF}(\omega) + \gamma\{[L_1(e) - L_3(e)]\bar{G}_1(\ell - e) + [r_2 L_1(e) + r_1 L_3(e)]\bar{G}_3(\ell - e)\} = 0, \tag{38}$$

Solving Eqs. (25), (31) and (36) with respect to ω gives rise natural frequencies $\omega_j, j = 1, 2, 3, \ldots$ Every natural frequency ω_j allows one to calculate first the wave numbers k_{1j}, k_{2j} by using (13) and then associated mode shape as

$$\varphi_j(x) = D_j[W_4(k_{2j}, L)W_2(k_{1j}, x) - W_2(k_{1j}, L)W_4(k_{2j}, x)]. \tag{39}$$

In latter equation arbitrary constant D_j is determined by a chosen condition for normalization.

4. NUMERICAL ANALYSIS

To validate the theoretical development, natural frequencies computed by different methods (analytical method [3]; Galerkin's method [17] and the present method) for simply supported beam are compared and given in Tabs. 1–2. The Tables show that the analytical method, Galerkin's and present methods give the same results in computing natural frequencies of intact (uncracked) beam structures with different slenderness ratios. However, disagreement of the methods is apparent when they are applied for

Table 1. Comparison of frequency parameter ($\lambda_k = [\omega_k^2 \rho A/EI]^{1/4}$) computed by using different beam theories and methods for simply supported uniform intact beam

Eigenvalue No	1	2	3	4	5
EBT [3]	π	2π	3π	4π	5π
EBT – Present	3.1416	6.2832	9.4248	12.5664	15.7080
TBT [3]	3.1155	6.0867	8.8180	11.2766	13.4740
TBT – Present	3.1157	6.0907	8.8405	11.3431	13.6132
Beam parameters: $E = 200$ GPa; $\rho = 7855$ kg/m^3;$\nu = 0.3$; $\kappa = 5/6; L = 1.0; b = 0.1; h = 0.1$ (m)					

Table 2. Comparison of natural frequencies computed by using different beam theories and methods for simply supported cracked beam with various slenderness (L/h) and single crack at the beam middle

Frequency No	1		2		3		4	
	ω_0	ω_c/ω_0	ω_0	ω_c/ω_0	ω_0	ω_c/ω_0	ω_0	ω_c/ω_0
$L/h = 15$								
EBT – GM [17]	303.64	0.8836	1214.56	0.9801	2732.77	0.9185	4808.26	0.9673
EBT – Present	303.64	0.8383	1213.10	1.0000	2732.80	0.8740	4851.50	1.0000
TBT – GM [17]	301.34	0.8844	1179.28	0.9806	2565.03	0.9234	4366.67	0.9707
TBT – Present	301.30	0.8397	1179.30	1.0000	2565.00	0.8827	4367.70	1.0000
$L/h = 10$								
EBT – GM [17]	455.46	0.8268	1821.85	0.9588	4099.15	0.8906	7287.39	0.9397
EBT – Present	455.46	0.7319	1819.70	1.0000	4099.20	0.8430	7277.30	1.0000
TBT – GM [17]	447.84	0.8293	1710.02	0.9613	3599.00	0.9030	5918.77	0.9509
TBT – Present	447.80	0.7857	1710.00	1.0000	3599.00	0.8628	5918.80	1.0000
$L/h = 5$								
EBT – GM [17]	910.92	0.6855	3643.72	0.8721	8198.31	0.8245	14574.77	0.8545
EBT – Present	910.92	0.6631	3639.30	1.0000	8198.30	0.7922	14555.00	1.0000
TBT – GM [17]	855.01	0.6985	2959.38	0.8936	5643.70	0.8686	8551.50	0.9069
TBT – Present	855.00	0.6799	2959.40	1.0000	5643.70	0.8484	8511.50	1.0000
Beam parameters	$E = 62.1$ GPa; $G = 23.3$ Gpa; $\rho = 2770$ kg/m^3; $\nu = 0.3$; $\kappa = 5/6; e/L = 0.5; a/h = 0.5$							
EBT–Euler Beam Theory; TBT–Timoshenko Beam Theory; GM–Galerkin Method; ω_0 - natural frequency (Rad/s) of intact beam; ω_c/ω_0 - frequency of cracked beam/frequency of intact beam								

cracked beam and miscalculation of Galerkin's method can be observed from that it results in reduction of second and fourth frequencies as the crack appeared at the middle of beam whereas the frequencies should be unchanged due to crack. Finally, it can be seen from Tab. 2 that Timoshenko beam model is more useful to apply for calculating natural frequencies of cracked beam. Note, all the results related to Euler-Bernoulli beam provided herein as EBT-present are obtained for corresponding beam parameters by using the theory developed in Ref. [12].

Effect of slenderness ratio on natural frequencies computed by different beam theories is demonstrated in Tab. 3. The data depicted in Tab. 3 show that Timoshenko beam

Table 3. Comparison of frequency parameter computed by using different beam theories for simply supported uniform intact beam with various slenderness (L/h)

Eigenvalue No	1	2	3	4	5
$L/h = 100(L = 10, h = 0.1)$					
Euler-Bernoulli beam [12]	0.3142	0.6283	0.9425	1.2566	1.5708
Timoshenko beam - present	0.3141	0.6281	0.9418	1.2549	1.5675
$L/h = 50(L = 5, h = 0.1)$					
Euler–Bernoulli beam [12]	0.6283	1.2566	1.8850	2.5133	3.1416
Timoshenko beam - present	0.6281	1.2549	1.8793	2.4999	3.1157
$L/h = 30(L = 3, h = 0.1)$					
Euler–Bernoulli beam [12]	1.0472	2.0944	3.1416	4.1888	5.2360
Timoshenko beam - present	1.0462	2.0866	3.1157	4.1286	5.1213
$L/h = 20(L = 2, h = 0.1)$					
Euler–Bernoulli beam [12]	1.5708	3.1416	4.7124	6.2832	7.8540
Timoshenko beam - present	1.5675	3.1157	4.6277	6.0907	7.4963
$L/h = 15(L = 1.5, h = 0.1)$					
Euler–Bernoulli beam [12]	2.0944	4.1888	6.2832	8.3776	10.4720
Timoshenko beam - present	2.0866	4.1286	6.0907	7.9513	9.7019
$L/h = 10(L = 1, h = 0.1)$					
Euler–Bernoulli beam [12]	3.1416	6.2832	9.4248	12.5664	15.7080
Timoshenko beam - present	3.1157	6.0907	8.8405	11.3431	13.6132
$L/h = 5(L = 0.5, h = 0.1)$					
Euler–Bernoulli beam [12]	6.2832	12.5664	18.8496	25.1328	31.4160
Timoshenko beam - present	6.0907	11.3431	15.6790	19.3142	22.4441
Beam parameters: $E = 200$ GPa; $\rho = 7855$ kg/m^3; $\nu = 0.3$; $\kappa = 5/6$; $\lambda_k = [\omega_k^2 \rho A/EI]^{1/4}$					

theory gives rise almost the same natural frequencies as the Euler–Bernoulli beam theory for the beams with slenderness ratio greater 20 (acknowledged as long or slender beams). This fact allows one to make a conclusion that Timoshenko beam theory is useful not only for short or thick beams but also for long or slender ones while the Euler–Bernoulli theory is applicable only for the long or slender beams. In case of cracked beam, natural frequencies of Timoshenko beam with single crack computed by the present method are compared to those given in Ref. [19] that are obtained by the conventional transfer matrix method (see Tab. 4). The comparison demonstrates very good agreement of the results, especially, some frequencies are computed identically (when they are unaffected by presence of crack). Thus the proposed in this study method is validated not only in the case of uncracked beam but also for beam with cracks. Furthermore, natural frequencies of cracked Timoshenko beam normalized by those of intact one are computed as function of crack position along the beam span for various slenderness ratios $(10, 20, 30)$. The frequency ratios (cracked to intact) acknowledged as sensitivity of natural frequencies to

Table 4. Comparison of natural frequencies with those given in Khaji et al. [19] for Timoshenko beam in case of single crack and various boundary conditions (SS, CC and CF-beams)

Freq.	$L/h = 3$			$L/h = 5$			$L/h = 7$			$L/h = 9$		
No	Present	Khaji	%	Present	Khaji	%	Present	Khaji	%	Present	Khaji	%
Simply supported beam (SS)												
1	6909.92	6781.8	1.8	2919.21	2877.3	1.4	1599.09	1580.3	1.1	1006.68	996.9	0.9
2	**27316.17**	**27316.2**	0.0	**12222.26**	**12222.2**	0.0	**6803.30**	**6803.3**	0.0	**4293.28**	**4293.3**	0.0
3	42979.87	42718.8	0.6	21132.25	21006.4	0.6	12504.58	12430.6	0.6	8633.80	8177.0	0.5
4	**64534.74**	**64534.6**	0.0	**35435.66**	**35435.8**	0.0	21681.69	**21681.7**	0.0	**14556.97**	**14557.0**	0.0
Clamped end beam (CC)												
1	13700.28	13628.4	0.5	6244.02	6205.3	0.6	3530.10	3509.1	0.6	2255.5	2243.2	0.5
2	**31009.05**	**31009.0**	0.0	**15650.89**	**15650.9**	0.0	**9335.63**	**9335.6**	0.0	**6134.70**	**6134.7**	0.0
3	44150.76	43315.8	1.8	23731.61	23599.0	0.5	14875.08	14796.1	0.5	10149.80	10098.1	0.5
4	**64534.74**	**64534.6**	0.0	**37268.75**	**37268.75**	0.0	**23764.42**	**23764.4**	0.0	**16475.53**	**16475.5**	0.0
Cantilever beam (CF)												
1	3157.3	3134.9	0.7	1228.7	1222.9	0.5	645.2	642.9	0.3	396.0	394.9	0.3
2	12826.6	12664.4	1.2	5954.8	5888.2	1.1	3386.0	3352.9	0.9	2170.8	2152.2	0.8
3	33422.2	33381.4	0.1	**16510.6**	**16504.4**	0.0	**9684.8**	**9683.4**	0.0	**6296.1**	**6295.6**	0.0
4	49075.2	48898.9	0.3	5443.6	25326.0	0.4	15583.1	15506.8	0.5	10490.0	0438.8	0.5
Beam parameters: $E = 210$ GPa, $\rho = 7860$ kg/m^3, $b = 12.5$ mm, $h = 25$ mm, $\kappa = 5/6$, $f_k = \omega_k/2\pi$ (Hz), $k = 1, 2, 3, 4$												

crack are compared to those obtained by using Euler–Bernoulli beam theory and shown in Figs. 1–3 corresponding to the simply supported, clamped-clamped and clamped-free boundary conditions. Obviously, the frequency sensitivities computed for both the beam theories are identical as the slenderness ratio equals to 30 and they get to be apparently deviated for the ratio $L/h = 10$. In the latter case, natural frequencies computed by Euler–Bernoulli beam theory are more sensitive to crack. Note, that the sensitivity of fundamental frequency of cantilever beam is independent on which beam theory is applied.

To investigate combined effect of the beam theories, slenderness ratio and multiple cracks on natural frequencies, the frequency parameter $\lambda_k = [\omega_k^2 \rho A/EI]^{1/4}, k = 1, 2, 3, 4, 5$ are computed for various slenderness $L/h = 5, 10, 20 (h = 0.2, 0.1, 0.05, L = 1)$, number of cracks $n = 0, 1, 2, 3$ using the different beam theories (EBT and TBT). Deviation of the frequency parameters computed by the EBT and TBT is calculated and measured in percent (%).

Obviously, the frequency parameter computed for Euler–Bernoulli intact beam is independent on the beam thickness (h) for the beam length (L) fixed, as can be seen in Tab. 5, it is dependent only on the beam length. However, as a crack occurred in beam the parameter decreases with increasing beam thickness and number of cracks. Unlikely, the frequency parameter of Timoshenko beam is always decreasing as the beam thickness and number of cracks are growing. Difference between the beam theories measured by

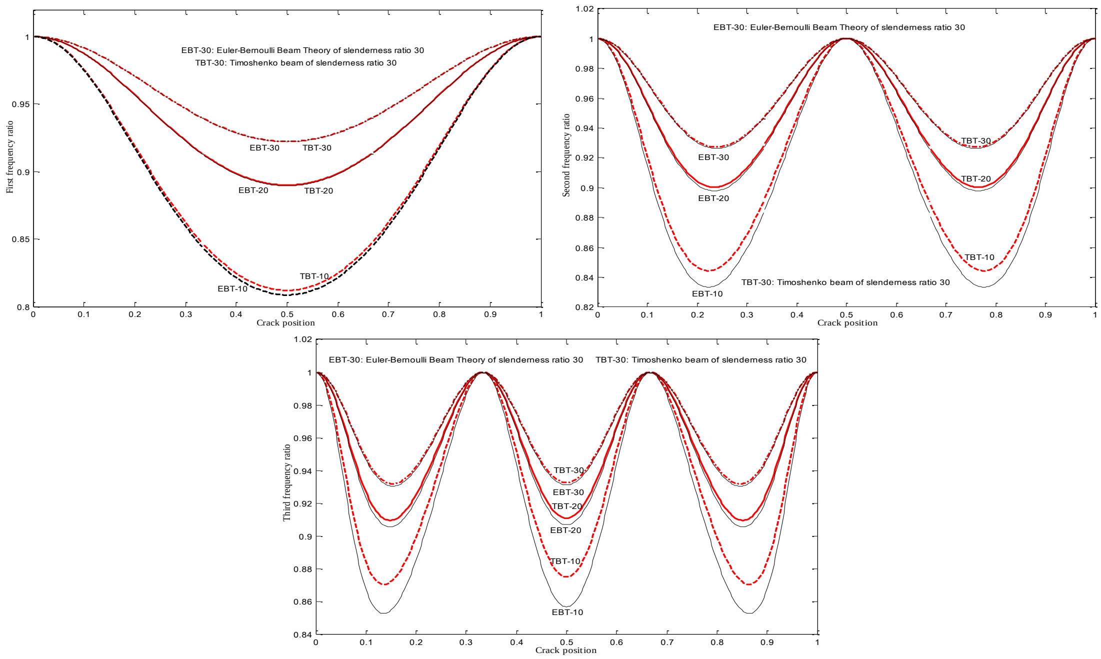

Fig. 1. Change (induced by single crack of depth 30%) in natural frequencies computed by EBT and TBT in various slenderness of uniform simply supported beam

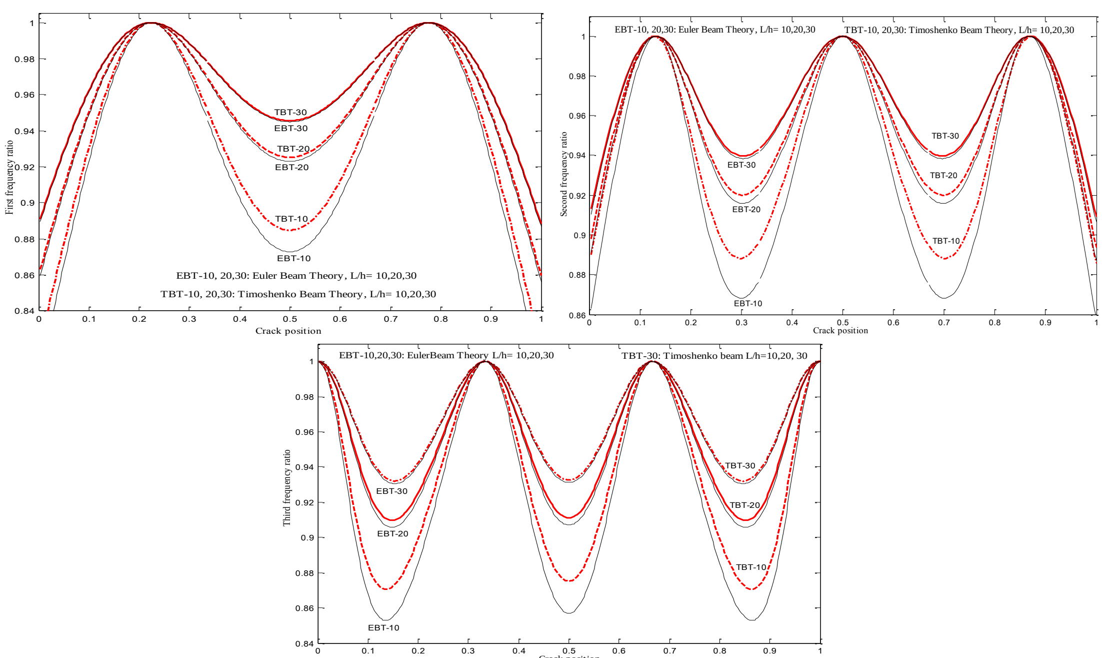

Fig. 2. Change (induced by single crack of depth 30%) in natural frequencies computed by EBT and TBT in various slenderness of uniform clamped end beam

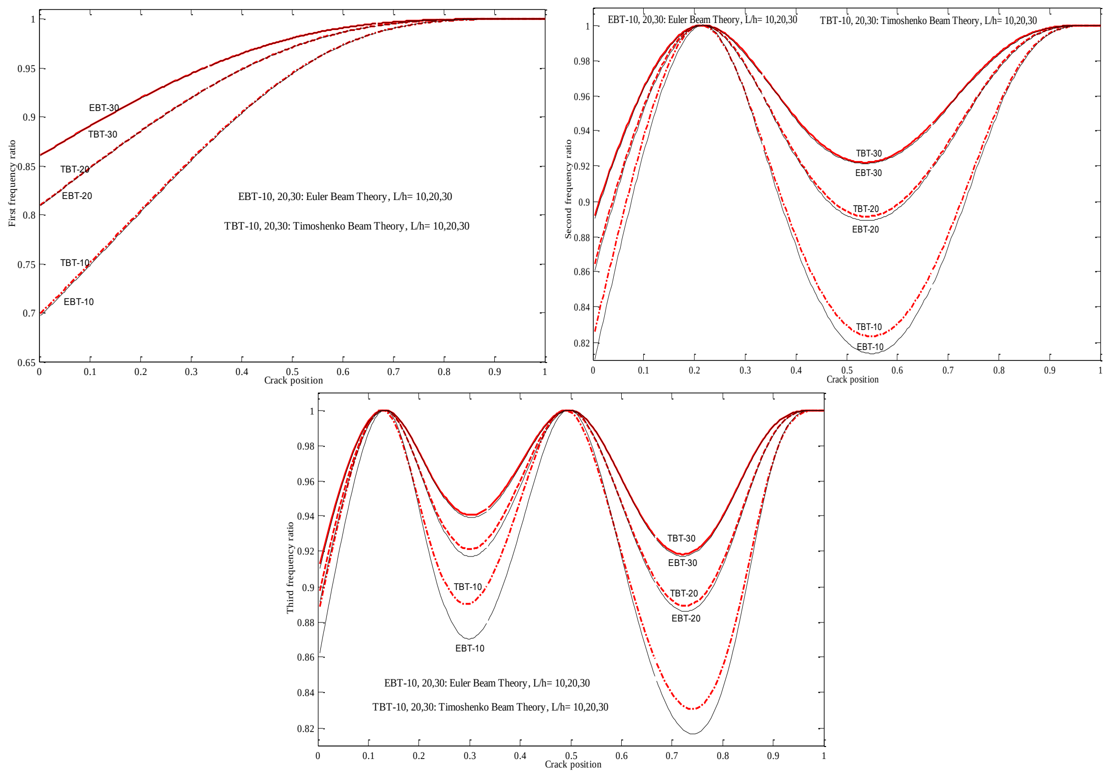

Fig. 3. Change (induced by single crack of depth 30%) in natural frequencies computed by EBT and TBT in various slenderness of uniform cantilevered beam

deviation of the frequency parameter is significant and rising with mode number. However, the deviation decreases not only as usually for increasing slenderness ratio but also when number of cracks rises, except the case of fundamental frequency.

Table 5. Comparison of natural frequencies computed by Timoshenko and Euler–Bernoulli beams with various number of cracks (equal depth 30%), slenderness ratio and boundary conditions

L/h	No. Freq.	No crack			Single crack at (1/6)			Double crack at (1/6;1/2)			Triple crack at (1/6;1/2;5/6)		
		TBT	EBT	δ(%)	TBT	EBT	δ(%)	TBT	EBT	δ(%)	TBT	EBT	δ(%)
Simply supported beam													
5	1	3.0453	3.1416	3.0	2.8769	2.9545	2.6	2.7202	2.5438	-6.9	2.6269	2.4783	-6.0
	2	5.6716	6.2832	9.7	5.0824	5.4966	7.5	5.0342	.3615	6.1	4.6248	4.9248	6.1
	3	7.8395	9.4248	16.8	7.2807	.5199	4.5	.9992	.7166	9.3	6.2958	6.5851	4.4
	4	9.6571	12.5664	23.1	9.3386	11.9524	21.8	9.3165	11.8327	21.2	8.9259	11.1797	20.1
	5	1.2220	15.7080	28.5	11.1138	15.4740	28.1	10.8266	4.2329	23.9	10.6653	13.7340	22.3

10	1	3.1157	3.1416	0.8	3.0202	3.0431	0.7	2.9163	2.7670	-5.4	2.8493	2.7157	-4.9
	2	.0907	.2832	.0	.6488	.7939	2.5	5.6352	5.7569	2.1	5.2980	5.4130	2.1
	3	8.8405	9.4248	6.2	8.2847	8.7481	5.3	8.0456	8.1319	1.1	7.4634	7.4260	-0.5
	4	11.3431	12.5664	9.7	10.9777	12.0688	9.0	10.9611	12.0125	8.7	10.5372	11.4781	8.2
	5	13.6132	15.7080	13.3	13.4888	15.5165	13.0	13.1670	14.5410	9.4	12.9827	14.1942	8.5
20	1	3.1350	3.1416	0.2	3.0849	3.0911	0.2	3.0250	2.9253	-3.4	2.9839	2.8906	-3.2
	2	6.2314	6.2832	0.8	5.9637	6.0074	0.7	5.9611	6.0000	0.6	5.7353	5.7730	0.6
	3	9.2554	.4248	1.8	8.8385	8.9811	1.5	8.6795	8.5669	-1.3	8.2761	8.1421	-1.6
	4	12.1813	12.5664	3.0	11.8588	12.2080	2.8	11.8522	12.1895	2.7	11.4967	11.8076	2.6
	5	14.9926	15.7080	.5	14.8731	15.5678	4.4	14.6178	14.8937	1.8	14.4615	4.6810	1.5
Clamped beam													
5	1	4.2420	4.7300	10.3	4.2028	4.6668	9.9	4.0313	4.1188	2.1	3.9593	3.9522	-0.1
	2	6.4179	7.8532	18.2	6.3354	7.7581	18.3	6.3327	7.7544	18.3	6.2671	7.6823	18.4
	3	8.2853	10.9956	24.6	7.9838	10.2966	22.4	7.6980	9.4961	18.9	7.4429	9.1124	18.3
	4	9.9037	14.1372	29.9	9.6529	13.2401	27.1	9.6393	13.0124	25.9	9.3531	12.3131	24.0
	5	11.3487	17.2788	34.3	11.2675	16.6622	32.3	11.0219	15.5326	29.0	10.9035	14.4893	24.7
10	1	4.5795	4.7300	3.2	4.5418	4.6862	3.1	4.4159	4.3432	-1.6	4.3631	4.2598	-2.4
	2	7.3312	7.8532	6.6	7.2712	7.7887	6.6	7.2707	7.7875	6.6	7.2196	7.7334	6.6
	3	9.8561	10.9956	10.3	9.5050	10.4967	9.4	9.2695	9.8594	5.9	9.0049	9.5540	5.7
	4	12.1454	14.1372	14.1	11.7489	13.4203	12.4	11.7274	13.3097	11.8	11.3212	2.7100	10.9
	5	14.2324	17.2788	17.6	14.0334	16.7587	16.2	13.7334	15.8277	13.2	13.4489	15.0624	10.7
20	1	4.6899	4.7300	0.8	4.6639	4.7029	0.8	4.5917	4.5064	-1.9	4.5605	4.4663	-2.1
	2	7.7035	7.8532	1.9	7.6652	7.8140	1.9	7.6651	7.8137	1.9	7.6303	7.7783	1.9
	3	10.6401	10.9956	3.2	10.3589	10.6819	3.0	10.1896	10.2332	0.4	9.9690	10.0127	0.4
	4	13.4611	14.1372	4.7	13.0478	13.6321	4.3	13.0363	13.5950	4.1	12.6374	13.1440	3.8
	5	16.1590	17.2788	6.5	15.8783	16.8852	5.9	15.6257	16.2090	3.6	15.2800	15.7069	2.7
Cantilever beam													
5	1	1.8466	1.8751	1.5	1.5168	1.5303	0.9	1.5038	1.4918	-0.8	1.5031	1.4912	-0.8
	2	4.2853	4.6941	8.7	4.2661	4.6437	8.1	3.9871	3.8567	-3.4	3.9121	3.8132	-2.6
	3	6.6113	7.8548	15.8	6.5092	7.7613	16.1	6.479	7.7536	16.4	5.9321	6.8610	13.5
	4	8.5186	10.9955	22.5	8.1882	10.296	20.5	7.9608	9.5163	16.3	7.1856	8.6368	16.8
	5	10.1584	14.1372	28.1	9.9341	13.2402	24.9	9.8889	13.014	24.0	9.2159	12.0138	23.3
10	1	1.8677	1.8751	0.4	1.6557	1.6603	0.3	1.6457	1.6308	-0.9	1.6452	1.6303	-0.9
	2	4.5724	4.6941	2.6	4.5450	4.6590	2.4	4.3632	4.178	-4.4	4.3153	4.1448	-4.1
	3	7.4154	7.8548	5.6	7.3492	7.7913	5.6	7.3447	7.7881	5.7	6.9612	7.3250	4.9
	4	9.9873	10.9955	9.1	9.6142	10.4963	8.4	9.3903	9.8721	4.8	8.7555	9.1851	4.6
	5	12.3224	14.1372	12.8	11.9264	3.4203	11.1	11.8842	13.3107	10.7	11.2005	12.4725	0.2
20	1	1.8732	1.8751	0.1	1.7502	1.7516	0.1	1.7435	1.7320	-0.7	1.7432	1.7316	-0.7
	2	4.6620	4.6941	0.7	4.6415	4.6722	0.6	4.5414	4.4002	-3.2	4.5139	4.3782	-3.1
	3	7.7305	7.8548	1.6	7.6915	7.8162	1.6	7.6907	7.8151	1.6	7.4669	7.5792	1.5
	4	10.6862	10.9955	2.8	10.3983	10.6816	2.6	10.2317	10.2403	0.1	9.7774	9.7825	0.1
	5	13.5319	14.1372	4.3	13.1149	13.6321	3.8	3.0995	13.5955	3.6	12.5261	12.9795	3.5

Beam parameters: $E = 210$ GPa; $\rho = 7860$ kg/m^3; $\nu = 0.3$; $\kappa = 5/6$; $L = 1$; $b = 0.1$; $h = 0.2; 0.1; 0.05$ (m)

5. CONCLUSION

Summarizing results obtained in the present study the conclusions can be made as follow:

A closed form solution has been conducted for free vibration of Timoshenko beam with arbitrary number of cracks. This solution is straightforward to derive an explicit expression for frequency equation and mode shapes of multiple cracked Timoshenko beams;

Analysis of natural frequencies obtained from the frequency equation shows that the Timoshenko beam theory is useful for vibration analysis of not only short or thick cracked beams but also the long or slender ones, while the Euler-Bernoulli beam theory is applicable only for long and slender beams;

Nevertheless, sensitivities of natural frequencies to cracks computed by the EBT and TBT are the same for beams of slenderness ratio (L/h) greater than 20 and for the ratio less than 20 natural frequencies computed by the EBT are more sensitive to cracks than those computed by TBT;

The obtained closed-form solution can be used for vibration analysis and crack identification of more complicated structures such as stepped multispan beams or framed structures with cracks that is a subject of next studies of the authors.

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