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NON-LINEAR ANALYSIS OF LAMINATED COMPOSITE DOUBLY CURVED SHALLOW SHELLS

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Abstract. This paper deals with governing equations and approximate analytical solutions based on some wellknown assumptions to the non-linear buckling and vibration problems of laminated composite doubly curved shallow shells. Obtained results will be presented by analytical expressions of the lower critical load, the postbuckling load-deflection curve and the fundamental frequency of non-linear free vibration of the shell.

1. INTRODUCTION

Linear analysis of laminated composite plates and shells was investigated by many authors. However for non-linear analysis of these structures we are concerned with more difficulties, because the non-linearly partial differential equations governing composite laminates of arbitrary geometries and boundary conditions cannot be solved exactly. Approximate analytical solution to the large deflection theory of laminated composite plates and non-linear bending and buckling analysis of cylindrical composite shells were considered, for example, in [2-6, 8-11,19,22...]. For plates of complicated geometries and shells one develops non-linear finite element models of laminated structures, more results were received, especially results of Reddy and the others [1, 7, 12-18, 21...] .

The problem of postbuckling behaviour of shell structures under loading and non-linear vibration of shells is of significant practical interest. The present paper is conerned with governing equations and approximate analytical solutions based on some wellknown assumptions to the non-linear buckling and vibration problems of laminated composite doubly curved shallow shells. Obtained results will be presented by analytical expression of the lower critical load, the postbuckling load-deflection curve and the fundamental frequency of non-linear free vibration of the shell.

2. GOVERNING EQUATIONS OF LAMINATED DOUBLY CURVED SHALLOW SHELL

2.1. Strain – displacement relations

Consider the strain state of a shallow shell when the deflection of middle surface is compared with the shell thickness. Using the cartesian coodinates, where axes x_1 and x_2 coincide with principal curves of the middle surface, axis $x_3 \equiv z$ is in the thickness direction and according to the Kirchoff – Love's theory the non-linear strain-displacement relationships for a doubly curved shallow shell are given by:

$$\varepsilon_{11} = \varepsilon_1^o + z\chi_1,$$
$$\varepsilon_{22} = \varepsilon_2^o + z\chi_2,$$
$$\gamma_{12} = \varepsilon_6^o + z\chi_6,$$

where

$$\varepsilon_1^o = \frac{\partial u}{\partial x_1} - k_1 w + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2, \tag{2.1}$$

$$\varepsilon_2^o = \frac{\partial v}{\partial x_2} - k_2 w + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2,$$

$$\varepsilon_6^o = \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} + \frac{\partial w.\partial w}{\partial x_1.\partial x_2},$$

$$\chi_1 = -\frac{\partial^2 w}{\partial x_1^2},$$

$$\chi_2 = -\frac{\partial^2 w}{\partial x_2^2},$$

$$\chi_6 = -2\frac{\partial^2 w}{\partial x_1 \partial x_2},$$

where $k_1 = \dfrac{1}{R_1}$, $k_2 = \dfrac{1}{R_2}$ are principal curvatures of the shell; R_1, R_2 are radii of curvatures; u, v and w are displacements of the middle surface along x_1, x_2 and z axes respectively. The strains in the middle surface and the changes of curvatures and twist are denoted by ε_i^0 and χ_i (i= $1, 2, 6$) respectively.

Note that strains (2.1) are not independent, they must be relative in the deformation compatibility equation:

$$\frac{\partial^2 \varepsilon_1^o}{\partial x_2^2} + \frac{\partial^2 \varepsilon_2^o}{\partial x_1^2} - \frac{\partial^2 \varepsilon_6^o}{\partial x_1 \partial x_2} = \left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} - k_1\frac{\partial^2 w}{\partial x_2^2} - k_2\frac{\partial^2 w}{\partial x_1^2}. \tag{2.2}$$

The classical lamination theory in which the transverse shear effects are neglected, is often used to analyze laminated composite structures.

2.2. Laminate constitutive relations

Consider a shell of total thickness h composed of N orthotropic layers perfectly bonded together with the principal material coodinates $\left(X_1^{(k)}, X_2^{(k)}, X_3^{(k)}\right)$ of the k-th lamina oriented at an angle θ_k to the shell coodinate x_1 in the counterlockwise sense and $X_3^{(k)} \equiv z$. Stress-strain relation of the k-th lamina in the shell coodinate system (x_1, x_2, z) are given as:

$$\{\sigma\}^{(k)} = \left[\overline{Q}\right]^{(k)} \{\varepsilon\}^{(k)}, \tag{2.3}$$

where

$$\{\sigma\} = \{\sigma_{11}, \ \sigma_{22}, \ \sigma_{12}\}^T,$$

$$\{\varepsilon\} = \{\varepsilon_{11}, \ \varepsilon_{22}, \ \gamma_{12}\}^T,$$

$$\left[\overline{Q}\right] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix},$$

{.} denotes a column vector and [...] denotes a matrix. $\overline{Q}_{ij}^{(k)}$ are the transformed stiffnesses, which are relative with the surface stress reduced stiffnesses Q_{ij}^{k} referred to the principal material coodinates of the k-th lamina [see Reddy 18],

$$
\begin{aligned}
\overline{Q}_{11} &= Q_{11}\cos^4\theta + 2\left(Q_{12}+2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta, \\
\overline{Q}_{12} &= \left(Q_{11}+Q_{22}-4Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{12}\left(\sin^4\theta+\cos^4\theta\right), \\
\overline{Q}_{22} &= Q_{11}\sin^4\theta + 2\left(Q_{12}+2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta, \\
\overline{Q}_{16} &= \left(Q_{11}-Q_{12}-2Q_{66}\right)\sin\theta\cos^3\theta + \left(Q_{12}-Q_{22}+2Q_{66}\right)\sin^3\theta\cos\theta, \\
\overline{Q}_{26} &= \left(Q_{11}-Q_{12}-2Q_{66}\right)\sin^3\theta\cos\theta + \left(Q_{12}-Q_{22}+2Q_{66}\right)\sin\theta\cos^3\theta, \\
\overline{Q}_{66} &= \left(Q_{11}+Q_{22}-2Q_{12}-4Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{66}\left(\sin^4\theta+\cos^4\theta\right)
\end{aligned}
\tag{2.4}
$$

Q_{ij} **can** be expressed in terms of engineering constants of a lamina

$$
Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}; \quad Q_{66} = G_{12}, \tag{2.5}
$$

where E_i is the modulus in the X_i direction, G_{12} is the shear modulus in the (X_1, X_2) plane, ν_{ij} are the associated ratios.

Using the lamina constitutive equations the stress resultants

$$
N_1 = \int_{-h/2}^{h/2} \sigma_{11}\left(1+\frac{z}{R_2}\right)dz, \quad N_2 = \int_{-h/2}^{h/2} \sigma_{22}\left(1+\frac{z}{R_1}\right)dz,
$$

$$
N_{12} = \int_{-h/2}^{h/2} \sigma_{12}\left(1+\frac{z}{R_2}\right)dz, \quad N_{21} = \int_{-h/2}^{h/2} \sigma_{21}\left(1+\frac{z}{R_1}\right)dz,
$$

$$
M_1 = \int_{-h/2}^{h/2} \sigma_{11}\left(1+\frac{z}{R_2}\right)zdz, \quad M_2 = \int_{-h/2}^{h/2} \sigma_{22}\left(1+\frac{z}{R_1}\right)zdz,
$$

$$
M_{12} = \int_{-h/2}^{h/2} \sigma_{12}\left(1+\frac{z}{R_2}\right)zdz, \quad M_{21} = \int_{-h/2}^{h/2} \sigma_{21}\left(1+\frac{z}{R_1}\right)zdz,
$$

can be expressed in terms of the membrane strains ε_i^o and curvature changes χ_i. For thin shallow shells we can omit the terms $\dfrac{z}{R_i}$ in the definition of the stress resultants and assume constant radii of the shell curvatures, the lamina equations can be simplified:

$$
N_{12} = N_{21} = N_6, \quad M_{12} = M_{21} = M_6,
$$

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^o\} \\ \{\chi\} \end{Bmatrix}, \tag{2.6}$$

where

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}, \quad [B] = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}, \quad [D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$$

$$\{\varepsilon^o\} = \{\ \varepsilon_1^o,\ \ \varepsilon_2^o,\ \ \varepsilon_6^o\ \}^T, \quad \{\chi\} = \{\ \chi_1,\ \chi_2,\ \chi_6\ \}^T,$$

$$\{N\} = \{\ N_1,\ \ N_2,\ \ N_6\ \}^T, \quad \{M\} = \{\ M_1,\ \ M_2,\ \ M_6\ \}^T,$$

{N} are called the force resultants and {M} are called the moment resultants. The laminate stiffness coefficients A_{ij}, B_{ij}, D_{ij} are defined by:

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \overline{Q}_{ij}^{(k)} \left(1,\, z,\, z^2\right) dz, \qquad (i,\, j = 1,\, 2,\, 6).$$

Note that in a multilayered symmetrically laminated material the coupling stiffnesses B_{ij} are equal to zero and the extensional A_{16}, A_{26} and bending stiffnesses D_{16}, D_{26} are negligible compared to the other stiffnesses. This means that the constitutive equations are identical to those for a specially orthotropic material. It leads to idea to express these relations to ones for orthotropic elastic material:

$$N_1 = A_{11}\varepsilon_1^o + A_{12}\varepsilon_2^o, \quad N_2 = A_{12}\varepsilon_1^o + A_{22}\varepsilon_2^o, \quad N_6 = A_{66}\varepsilon_6^o, \tag{2.7}$$

and conversely

$$\varepsilon_1^o = \frac{1}{E_1^*}\left(N_1 - \nu_1^* N_2\right), \quad \varepsilon_2^o = \frac{1}{E_2^*}\left(N_2 - \nu_2^* N_1\right), \quad \varepsilon_6^o = \frac{1}{G^*} N_6, \tag{2.8}$$

where

$$E_1^* = \frac{A_{11}A_{22} - A_{12}^2}{A_{22}}, \quad E_2^* = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}}, \quad G^* = A_{66},$$

$$\nu_1^* = \frac{A_{12}}{A_{22}}, \quad \nu_2^* = \frac{A_{12}}{A_{11}}, \quad \frac{\nu_1^*}{E_1^*} = \frac{\nu_2^*}{E_2^*}, \tag{2.9}$$

and

$$\begin{aligned} M_1 &= D_{11}\chi_1 + D_{12}\chi_2 = -D_1\left(\frac{\partial^2 w}{\partial x_1^2} + \mu_2 \frac{\partial^2 w}{\partial x_2^2}\right), \\ M_2 &= D_{12}\chi_1 + D_{22}\chi_2 = -D_2\left(\frac{\partial^2 w}{\partial x_2^2} + \mu_1 \frac{\partial^2 w}{\partial x_1^2}\right), \\ M_6 &= D_{66}\chi_6 = -2D_K \frac{\partial^2 w}{\partial x_1 \partial x_2}, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} &D_1 = D_{11}, \quad D_2 = D_{22}, \quad D_K = D_{66}, \\ &\mu_2 = \frac{D_{12}}{D_{11}}, \quad \mu_1 = \frac{D_{12}}{D_{22}}, \quad \frac{\mu_1}{D_1} = \frac{\mu_2}{D_2}, \\ &D_3 = 2D_K + D_1\mu_2 = 2D_K + D_2\mu_1. \end{aligned} \tag{2.11}$$

2.3. Equations of motion of laminated shallow shell

The equations of motion of a laminated doubly curved shallow shell are

$$\frac{\partial N_1}{\partial x_1}+\frac{\partial N_6}{\partial x_2}=J_O\frac{\partial^2 u}{\partial t^2}-J_1\frac{\partial^3 w}{\partial x_1\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1}+\frac{\partial N_2}{\partial x_2}=J_O\frac{\partial^2 v}{\partial t^2}-J_1\frac{\partial^3 w}{\partial x_2\partial t^2},$$

$$\begin{aligned}&\frac{\partial^2 M_1}{\partial x_1^2}+2\frac{\partial^2 M_6}{\partial x_1\partial x_2}+\frac{\partial^2 M_2}{\partial x_2^2}+k_1N_1+k_2N_2+\frac{\partial}{\partial x_1}\left(N_1\frac{\partial w}{\partial x_1}+N_6\frac{\partial w}{\partial x_2}\right)+\\&+\frac{\partial}{\partial x_2}\left(N_6\frac{\partial w}{\partial x_1}+N_2\frac{\partial w}{\partial x_2}\right)=J_O\frac{\partial^2 w}{\partial t^2}+J_1\left(\frac{\partial^3 u}{\partial x_1\partial t^2}+\frac{\partial^3 v}{\partial x_2\partial t^2}\right)-\\&-J_2\left(\frac{\partial^4 w}{\partial x_1^2\partial t^2}+\frac{\partial^4 w}{\partial x_2^2\partial t^2}\right)-q,\end{aligned}\tag{2.12}$$

where J_i are the mass inertia terms defined as

$$J_i=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}\rho^{(k)}z^i dz\qquad(i=0,1,2),\tag{2.13}$$

$\rho^{(k)}$ is the material mass density of the k-th layer, q is the transverse load.

A combination of boundary conditions may be assumed to exist at the edges of the shell. The shell panel considered in the following analysis is simply supported and displacements of its end cross sections are not restrained. Morever for dynamical analysis it is necessary to give initial conditions.

3. SOLUTION TO THE PROBLEM

For analytical analysis of the mentioned problem we introduce some well-known assumptions:

- The transverse load q is uniform

- The mass density of k-th layer is constant, such that for a multilayered symmetrically laminated material

$$J_1=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}\rho^{(k)}zdz=0.$$

- If the dynamical process can be considered without propagation of elastic waves, inertia terms in two first equations (2.12) can be neglected [see Volmir 20], then motion equations are simplified

$$\frac{\partial N_1}{\partial x_1}+\frac{\partial N_6}{\partial x_2}=0,\tag{3.1}$$

$$\frac{\partial N_6}{\partial x_1}+\frac{\partial N_2}{\partial x_2}=0,\tag{3.2}$$

$$\frac{\partial^2 M_1}{\partial x_1^2}+2\frac{\partial^2 M_6}{\partial x_1\partial x_2}+\frac{\partial^2 M_2}{\partial x_2^2}+k_1N_1+k_2N_2+\frac{\partial}{\partial x_1}\left(N_1\frac{\partial w}{\partial x_1}+N_6\frac{\partial w}{\partial x_2}\right)+$$
$$+\frac{\partial}{\partial x_2}\left(N_6\frac{\partial w}{\partial x_1}+N_2\frac{\partial w}{\partial x_2}\right)=J_O\frac{\partial^2 w}{\partial t^2}-J_2\left(\frac{\partial^4 w}{\partial x_1^2\partial t^2}+\frac{\partial^4 w}{\partial x_2^2\partial t^2}\right)-q. \tag{3.3}$$

Equations (3.1), (3.2) are satisfied by introducing the stress function φ in the form:

$$N_1=h\sigma_{11}=\frac{\partial^2\varphi}{\partial x_2^2},\quad N_2=h\sigma_{22}=\frac{\partial^2\varphi}{\partial x_1^2},\quad N_6=h\sigma_{12}=-\frac{\partial^2\varphi}{\partial x_1\partial x_2}. \tag{3.4}$$

By using (2.8) and (3.4) the compatibility equation (2.2) becomes

$$\frac{1}{E_2^*}\frac{\partial^4\varphi}{\partial x_1^4}+\left(\frac{1}{G^*}-2\frac{\nu_1^*}{E_1^*}\right)\frac{\partial^4\varphi}{\partial x_1^2\partial x_2^2}+\frac{1}{E_1^*}\frac{\partial^4\varphi}{\partial x_2^4}=$$
$$=\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)^2-\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2}-k_1\frac{\partial^2 w}{\partial x_2^2}-k_2\frac{\partial^2 w}{\partial x_1^2}. \tag{3.5}$$

According to relation (2.10) and (3.4) the motion equation (3.3) is reduced to

$$D_1\frac{\partial^4 w}{\partial x_1^4}+2D_3\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}+D_2\frac{\partial^4 w}{\partial x_2^4}-\left(k_1\frac{\partial^2\varphi}{\partial x_2^2}+k_2\frac{\partial^2\varphi}{\partial x_1^2}\right)-$$
$$-\left(\frac{\partial^2\varphi}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2}+\frac{\partial^2\varphi}{\partial x_2^2}\frac{\partial^2 w}{\partial x_1^2}-2\frac{\partial^2\varphi}{\partial x_1\partial x_2}\frac{\partial^2 w}{\partial x_1\partial x_2}\right)+J_o\frac{\partial^2 w}{\partial t^2}-$$
$$-J_2\left(\frac{\partial^4 w}{\partial x_1^2\partial t^2}+\frac{\partial^4 w}{\partial x_2^2\partial t^2}\right)-q=0, \tag{3.6}$$

where $D_3=2D_K+D_1\mu_2=2D_K+D_2\mu_1$.

Finally, the problem consists of two non-linear partial differential equations (3.5) and (3.6) governing composite laminated shallow shell. The boundary simply supported condition can be satisfied if the deflection is chosen as

$$w=f(t)\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}, \tag{3.7}$$

where a and b are the lengths of in-plane edges of the shallow shell, $f(t)$ is the maximum deflection.

Substituting (3.7) into the right side of the equation (3.5) yields

$$\frac{1}{E_2^*}\frac{\partial^4\varphi}{\partial x_1^4}+\left(\frac{1}{G^*}-2\frac{\nu_1^*}{E_1^*}\right)\frac{\partial^4\varphi}{\partial x_1^2\partial x_2^2}+\frac{1}{E_1^*}\frac{\partial^4\varphi}{\partial x_2^4}=$$
$$=f^2\frac{\pi^4}{2a^2b^2}\left(\cos\frac{2\pi x_1}{a}+\cos\frac{2\pi x_2}{b}\right)+f^2\pi^2\left(\frac{k_1}{b^2}+\frac{k_2}{a^2}\right)\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}.$$

Taking into account no-restrained displacements of end cross shell sections a solution to this equation can be received

$$\varphi=A\cos\frac{2\pi x_1}{a}+B\cos\frac{2\pi x_2}{b}+C\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}, \tag{3.8}$$

where

$$A = \frac{E_2^* f^2}{32}\frac{a^2}{b^2},\quad B = \frac{E_1^* f^2}{32}\frac{b^2}{a^2},\quad C = \frac{f\left(k_1 a^2 + k_2 b^2\right)}{\pi^2\left(\frac{1}{E_2^*}\frac{b^2}{a^2} + \frac{1}{G^*} - 2\frac{\nu_1^*}{E_1^*} + \frac{1}{E_1^*}\frac{a^2}{b^2}\right)}. \tag{3.9}$$

Substitution of (3.7) and (3.8) into equation (3.6) gives

$$\begin{aligned} W =& \pi^4 f\left(\frac{D_1}{a^4} + \frac{2D_3}{a^2b^2} + \frac{D_2}{b^4}\right)\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b} + \\ & k_1\left[B\left(\frac{2\pi}{b}\right)^2\cos\frac{2\pi x_2}{b} + C\left(\frac{\pi}{b}\right)^2\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}\right] + \\ & + k_2\left[A\left(\frac{2\pi}{a}\right)^2\cos\frac{2\pi x_1}{a} + C\left(\frac{\pi}{a}\right)^2\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}\right] - \\ & - \frac{\pi^4 f}{a^2b^2}\left[\left(4A\cos\frac{2\pi x_1}{a} + 2C\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b} + 4B\cos\frac{2\pi x_2}{b}\right)\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b} - \right. \\ & \left. -2C\left(\cos\frac{\pi x_1}{a}\cos\frac{\pi x_2}{b}\right)^2\right] + \left[J_o + J_2\pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right]\frac{d^2 f}{dt^2}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b} - q = 0. \end{aligned}$$

Applying the Bubnov – Galerkin's method

$$\int_0^a\int_0^b W\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}dx_1dx_2 = 0$$

and taking into account (3.9) results

$$\begin{aligned} &\left[J_0 + J_2\pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right]\frac{d^2 f}{dt^2} + \left[\pi^4\left(\frac{D_1}{a^4} + \frac{2D_3}{a^2b^2} + \frac{D_2}{b^4}\right) + \frac{\left(k_1a^2 + k_2b^2\right)^2}{\frac{b^4}{E_2^*} + \left(\frac{1}{G^*} - 2\frac{\nu_1^*}{E_1^*}\right)a^2b^2 + \frac{a^4}{E_1^*}}\right]f - \\ &- \frac{2}{3}\left[\frac{E_1^*k_1}{a^2} + \frac{E_2^*k_2}{b^2} + 16\frac{k_1a^2 + k_2b^2}{\frac{b^4}{E_2^*} + \left(\frac{1}{G^*} - 2\frac{\nu_1^*}{E_1^*}\right)a^2b^2 + \frac{a^4}{E_1^*}}\right]f^2 + \frac{\pi^4}{16}\left(\frac{E_2^*}{b^4} + \frac{E_1^*}{a^4}\right)f^3 - \frac{16q}{\pi^2} = 0. \end{aligned} \tag{3.10}$$

Denote $m = J_0 + J_2\pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$,

$$m_1 = \pi^4\left(\frac{D_1}{a^4} + \frac{2D_3}{a^2b^2} + \frac{D_2}{b^4}\right) + \frac{\left(k_1a^2 + k_2b^2\right)^2}{\dfrac{b^4}{E_2^*} + \left(\dfrac{1}{G^*} - 2\dfrac{\nu_1^*}{E_1^*}\right)a^2b^2 + \dfrac{a^4}{E_1^*}},$$

$$m_2 = \frac{2}{3}\left[\frac{E_1^*k_1}{a^2} + \frac{E_2^*k_2}{b^2} + 16\frac{k_1a^2 + k_2b^2}{\dfrac{b^4}{E_2^*} + \left(\dfrac{1}{G^*} - 2\dfrac{\nu_1^*}{E_1^*}\right)a^2b^2 + \dfrac{a^4}{E_1^*}}\right],$$

$$m_3 = \frac{\pi^4}{16}\left(\frac{E_2^*}{b^4} + \frac{E_1^*}{a^4}\right),$$

equation (3.10) is rewritten as

$$m\frac{d^2f}{dt^2} + m_1f - m_2f^2 + m_3f^3 - \frac{16q}{\pi^2} = 0. \tag{3.11}$$

In particular case for laminated plates k_1=k_2=0, equation gets form

$$m\frac{d^2f}{dt^2} + m_1^*f + m_3f^3 - \frac{16q}{\pi^2} = 0, \tag{3.12}$$

where $m_1^* = \pi^4\left(\dfrac{D_1}{a^4} + \dfrac{2D_3}{a^2b^2} + \dfrac{D_2}{b^4}\right)$.

4. NON-LINEAR BUCKLING ANALYSIS

Suppose that the shell is acted on by static transverse load, from (3.11) we can get relation between maximum deflection and transverse load

$$q = \frac{\pi^2}{16}\left(m_1f - m_2f^2 + m_3f^3\right), \tag{4.1}$$

it is the elastic equilibrium curve for the shell.

Particularly for a plate

$$q = \frac{\pi^2}{16}\left(m_1^*f + m_3f^3\right). \tag{4.2}$$

An interesting characteristics of composite laminated shell is its behavior under transverse load. Most often the critical buckling loads are determined through an eigenvalue analysis. The critical buckling loads can also be determined from geometric non-linear analysis, where the critical buckling loads are taken to be the so-called limit loads. Taking

$$\frac{dq}{df} = \frac{\pi^2}{16}\left(m_1 - 2m_2f + 3m_3f^2\right) = 0,$$

we have

$$f_{1,2} = \frac{m_2 \mp \sqrt{m_2^2 - 3m_1m_3}}{3m_3},$$

with condition $m_2^2 - 3m_1m_3 > 0$.

Critical buckling loads are determined by

$$q_{cr}^{(1)} = \frac{\pi^2}{16}\left(m_1 f_1 - m_2 f_1^2 + m_3 f_1^3\right), \tag{4.3}$$

$$q_{cr}^{(2)} = \frac{\pi^2}{16}\left(m_1 f_2 - m_2 f_2^2 + m_3 f_2^3\right). \tag{4.4}$$

We can show that $q_{cr}^{(1)}$ is maximum and and $q_{cr}^{(2)}$ is minimum. Consequently, because of

$$\left.\frac{d^2q}{df^2}\right|_{f_1} = \frac{\pi^2}{16}\left(-2m_2 + 6m_3 f_1\right) = -2\sqrt{m_2^2 - 3m_1m_3} < 0$$

and $\left.\dfrac{d^2q}{df^2}\right|_{f_2} = \dfrac{\pi^2}{16}\left(-2m_2 + 6m_3 f_2\right) = 2\sqrt{m_2^2 - 3m_1m_3} > 0.$

The typical graphs of load-deflection curves of a laminated composite shell and a laminated composite plate are presented in Fig. 1.

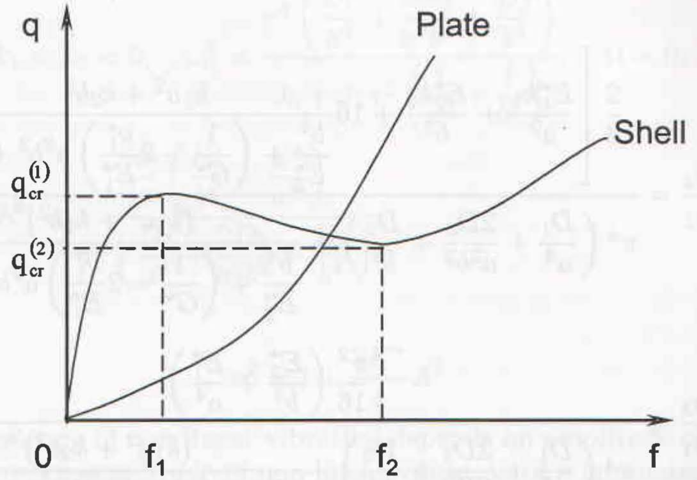

Fig. 1. Load-deflection curves

If $m_2^2 - 3m_1m_3 = 0$ the load-deflection curve of a laminated composite shell has only an inflexion point at $f_0 = \dfrac{m_2}{3m_3}$, because $\left.\dfrac{dq}{df}\right|_{f_0} = 0$ and

$$\left.\frac{d^2q}{df^2}\right|_{f_0} = \frac{\pi^2}{16}\left(-2m_2 + 6m_3 f_0\right) = \frac{\pi^2}{16}\left(-2m_2 + 6m_3\frac{m_2}{3m_3}\right) = 0.$$

5. NON-LINEAR DYNAMICAL ANALYSIS

Consider non-linear free vibration of a laminated composite shell by putting $q = 0$ in the equation (3.11)

$$m\frac{d^2 f}{dt^2} + m_1 f - m_2 f^2 + m_3 f^3 = 0. \tag{5.1}$$

For linear vibration

$$m\frac{d^2f}{dt^2}+m_1f=0, \qquad \text{or} \qquad \frac{d^2f}{dt^2}+\omega^2f=0,$$

where

$$\omega_0^2=\frac{m_1}{m}=\frac{\pi^4\left(\dfrac{D_1}{a^4}+\dfrac{2D_3}{a^2b^2}+\dfrac{D_2}{b^4}\right)+\dfrac{\left(k_1a^2+k_2b^2\right)^2}{\dfrac{b^4}{E_2^*}+\left(\dfrac{1}{G^*}-2\dfrac{\nu_1^*}{E_1^*}\right)a^2b^2+\dfrac{a^4}{E_1^*}}}{J_0+J_2\pi^2\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)}, \tag{5.2}$$

ω_o^2 is the fundamental frequency of linear vibration of the shell.

Equation (5.1) can be rewritten

$$\frac{d^2f}{dt^2}+\omega_0^2\left(f-\Omega f^2+Kf^3\right)=0, \tag{5.3}$$

where

$$\Omega=\frac{m_2}{m_1}=\frac{\dfrac{2}{3}\left[\dfrac{E_1^*k_1}{a^2}+\dfrac{E_2^*k_2}{b^2}+16\dfrac{k_1a^2+k_2b^2}{\dfrac{b^4}{E_2^*}+\left(\dfrac{1}{G^*}-2\dfrac{\nu_1^*}{E_1^*}\right)a^2b^2+\dfrac{a^4}{E_1^*}}\right]}{\pi^4\left(\dfrac{D_1}{a^4}+\dfrac{2D_3}{a^2b^2}+\dfrac{D_2}{b^4}\right)+\dfrac{\left(k_1a^2+k_2b^2\right)^2}{\dfrac{b^4}{E_2^*}+\left(\dfrac{1}{G^*}-2\dfrac{\nu_1^*}{E_1^*}\right)a^2b^2+\dfrac{a^4}{E_1^*}}},$$

$$K=\frac{m_3}{m_1}=\frac{\dfrac{\pi^2}{16}\left(\dfrac{E_2^*}{b^4}+\dfrac{E_1^*}{a^4}\right)}{\pi^4\left(\dfrac{D_1}{a^4}+\dfrac{2D_3}{a^2b^2}+\dfrac{D_2}{b^4}\right)+\dfrac{\left(k_1a^2+k_2b^2\right)^2}{\dfrac{b^4}{E_2^*}+\left(\dfrac{1}{G^*}-2\dfrac{\nu_1^*}{E_1^*}\right)a^2b^2+\dfrac{a^4}{E_1^*}}},$$

For seeking amplitude-frequency characteristics of non-linear vibration we substitute

$$f=A\cos\omega t,$$

into (5.3) to give

$$X\equiv A\left(\omega_0^2-\omega^2\right)\cos\omega t-\omega_0^2\Omega A^2\cos^2\omega t+K\omega_0^2A^3\cos^3\omega t=0.$$

Integrating over a quarter of vibration period

$$\int_0^{\frac{\pi}{2\omega}}X\cos\omega t\,dt=0,$$

leads to

$$\frac{\pi}{4\omega}A\left(\omega_0^2-\omega^2\right)-\frac{2\omega_0^2}{3\omega}A^2\Omega+\frac{3\pi}{16\omega}\omega_0^2A^3=0.$$

Because of A $\neq 0$, $\omega \neq 0$ we have

$$\omega_0^2-\omega^2-\frac{8}{3\pi}\Omega\omega_0^2A+\frac{3K}{4}\omega_0^2A^2=0,$$

or

$$\omega^2=\omega_0^2\left(1-\frac{8}{3\pi}\Omega A+\frac{3K}{4}A^2\right). \tag{5.4}$$

Denote $\nu^2=\dfrac{\omega^2}{\omega_o^2}$ - frequency of non-linear vibration of the shell. Equation (5.4) can be rewritten

$$\nu^2=1-\frac{8}{3\pi}\Omega A+\frac{3K}{4}A^2. \tag{5.5}$$

For a plate

$$k_1=k_2=0,\quad \omega_0^2=\frac{\pi^4\left(\dfrac{D_1}{a^4}+\dfrac{2D_3}{a^2b^2}+\dfrac{D_2}{b^4}\right)}{J_0+J_2\pi^2\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)},\quad \Omega=0,$$

$$K^*=\frac{\dfrac{\pi^2}{16}\left(\dfrac{E_2^*}{b^4}+\dfrac{E_1^*}{a^4}\right)}{\pi^4\left(\dfrac{D_1}{a^4}+\dfrac{2D_3}{a^2b^2}+\dfrac{D_2}{b^4}\right)},$$

and

$$\nu^2=1+\frac{3K^*}{4}A^2. \tag{5.6}$$

It is clear that frequency of non-linear vibration depends on amplitude of vibration. Typical graphs of frequency-amplitude of non-linear vibration of a laminated composite shell (??) and plate (5.6) are illustrated in Fig. 2.

We can see that for non-linear vibration of a shell

$$\nu_{\min}^2=1-\frac{64\Omega^2}{27K\pi^2},$$

with amplitude $A=\dfrac{16\Omega}{9K\pi}$.

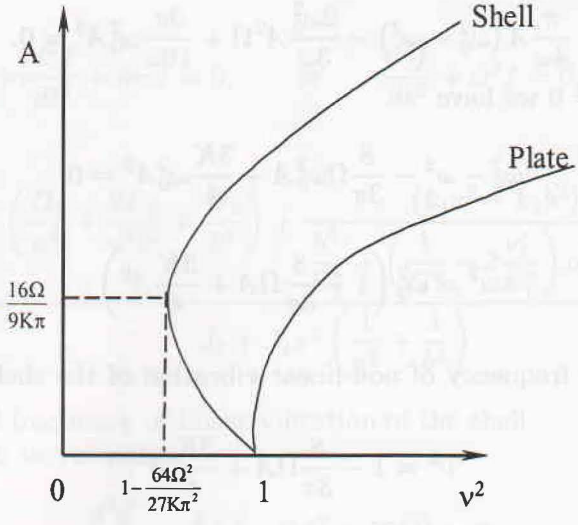

Fig. 2. Graphs of frequency vibration-amplitute

6. CONCLUSION

Governing equations for a laminated composite doubly curved shallow shell are derived.

Some wellknown assumptions proposed allow to solve non-linear buckling and vibration problems by analytically approximate method. The advantage of this approach is to obtain analytical expressions of non-linear buckling critical loads, elastic equilibrium curves and frequency of the non-linear free vibration of laminate composite doubly curved shell and plate.

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PHÂN TÍCH PHI TUYẾN VỎ THOẢI COMPOSITE LỚP CÓ HAI ĐỘ CONG

bài báo đề cập đến việc thiết lập các phương trình cơ bản và nghiệm giải tích gần đúng dựa trên một vài giả thiết quen biết của bài toán ổn định phi tuyến và dao động phi tuyến của vỏ ngoài composite lớp có hai độ cong. Các kết quả được trình bày dưới dạng các biểu thức giải tích của lực tới hạn dưới, đường cong lực -độ võng sau tới hạn và tần số cơ bản của dao động tự do phi tuyến của vỏ composite lớp.