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WEAK SELF-SUSTAINED SYSTEM UNDER THE ACTIONS OF LESS WEAK EXCITATIONS

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SUMMARY. It has been known that, in several cases, to study quasi-linear oscillating system, the degrees of smallness of various factors must be distinguished in detail [2-7]. To affirm again this interesting remark, we shall examine a weak (of order ε^2) self-sustained system subjected to less weak (of order ε) excitations in resonance cases. It will be seen that the system considered is enhanced.

§1. SYSTEM UNDER CONSIDERATION AND ITS APPROXIMATE SOLUTION

Let us consider a quasi-linear oscillating system described by the following differential equation:

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \omega t) + \varepsilon^2(\Delta x - \gamma x^3 - h_0\dot{x} + h\dot{x} - k\dot{x}^3), \tag{1.1}$$

$$\varepsilon^2\Delta = \omega^2 - 1, \quad f(x,\omega t) = \begin{cases} f_1 &= \beta x^2 + 3p\cos 2\omega t \\ f_2 &= 2px\cos\omega t \end{cases} \tag{1.2}$$

where: x - an oscillatory variable; $\varepsilon > 0$ - small parameter; overdots denote differentiation with respect to time t; β, γ - coefficients of the quadratic and cubic non-linearities, respectively; $h_0 > 0$ - damping viscous coefficient; $h > 0$, $k > 0$, $p \geq 0$ - constants; 1 - the natural frequency, $\varepsilon^2\Delta$ - the detuning parameter assumed to be of order ε^2.

If $p = 0$, we have a "pure" self-sustained system with the positive friction force $(h\dot{x} - k\dot{x}^3)$. If $p > 0$, the mentioned system is subjected to the external excitation $3p\cos 2\omega t$ in subharmonic resonance of order one-half or to the parametric one $2px\cos\omega t$ in principal resonance (by fundamental we mean the cases where the natural frequency is near that of the external excitation or one-half of that of the parametric one).

The damping (negative friction) force is introduced to facilite the analyses and, as it will be shown below, the quadratic non-linearity βx^2 is necessary in the case of external excitation.

Using the asymptotic method [1], the solution of the differential equation (1.1) will be found in the form:

$$\begin{aligned} x &= a\cos\psi + \varepsilon u_1(a,\theta,\psi) + \varepsilon^2 u_2(a,\theta,\psi), \\ \dot{a} &= \varepsilon A_1(a,\theta) + \varepsilon^2 A_2(a,\theta), \\ \dot{\theta} &= \varepsilon B_1(a,\theta) + \varepsilon^2 B_2(a,\theta), \quad \psi = \omega t + \theta \end{aligned} \tag{1.3}$$

where: a, θ - slowly varying amplitude and dephase, respectively; u_1, $u_2(A_1, B_1, A_2, B_2)$ - unknown functions of a, θ, ψ (a, θ) which are periodic in θ, $\psi(\theta)$ with period 2π; u_1, u_2 do not contain the first harmonics $\sin\psi$, $\cos\psi$.

Substituting (1.2) into (1.1), equating the terms of like powers of ε, then identifying the terms of same harmonics, we obtain:

$$A_1 = B_1 = 0; \quad \omega^2\frac{\partial^2 u_1}{\partial\psi^2} + \omega^2 u_1 = f(a\cos\psi, \psi - \theta) \tag{1.4}$$

and

$$\dot{a} = \varepsilon^2 A_2(a,\theta) = \varepsilon^2 a\overline{A}_2(a^2, \sin 2\theta),$$
$$a\dot{\theta} = \varepsilon^2 aB_2(a,\theta) = \varepsilon^2 a\overline{B}_2(a^2, \cos 2\theta). \tag{1.5}$$

where $\overline{A}_2$, $\overline{B}_2$ are first degree polynomials relative to a^2.

The trivial solution $a = 0$ corresponds to the equilibrium regime (oscillation of order ε in the case of external excitation). Following the method presented in [8], the conditions for asymptotic stability of the equilibrium regime are of the form:

$$\mathrm{Re}\{\overline{A}_2(0, \sin 2\theta_*)\} < 0, \tag{1.6}$$

where θ_* are solution of the trigonometrical equation:

$$\overline{B}_2(0, \cos 2\theta_*) = 0 \tag{1.7}$$

The amplitude a_0 and the dephase θ_0 of the stationary oscillation satisfy the equation:

$$A_2(a_0, \theta_0) = 0, \qquad B_2(a_0, \theta_0) = 0 \tag{1.8}$$

Eliminating θ_0 from (1.8) and neglecting the terms of order greater than ε^5, we obtain the relationship:

$$W(A_0^2, \Delta) = 0, \quad A_0^2 = \frac{3}{4}a_0^2 \tag{1.9}$$

where W is a second degree polynomial in A_0^2.

Introducing the perturbations $\delta a = a - a_0$, $\delta\theta = \theta - \theta_0$ we can establish the variational system:

$$(\delta a)^{\cdot} = \varepsilon^2 \frac{\partial A_2}{\partial a_0}\delta a + \varepsilon^2 \frac{\partial A_2}{\partial \theta_0}\delta\theta$$
$$(\delta\theta)^{\cdot} = \varepsilon^2 \frac{\partial B_2}{\partial a_0}\delta a + \varepsilon^2 \frac{\partial B_2}{\partial \theta_0}\delta\theta \tag{1.10}$$

The characteristic equation of the system (1.10) is of the form:

$$\rho^2 - \varepsilon^2\Big(\frac{\partial A_2}{\partial a_0} + \frac{\partial B_2}{\partial \theta_0}\Big)\rho + \varepsilon^4\Big(\frac{\partial A_2}{\partial a_0}\frac{\partial B_2}{\partial \theta_0} - \frac{\partial A_2}{\partial \theta_0}\frac{\partial B_2}{\partial a_0}\Big) = 0 \tag{1.11}$$

The conditions for stability are:

$$\frac{\partial A_2}{\partial a_0} + \frac{\partial B_2}{\partial \theta_0} < 0 \tag{1.12a}$$

$$\frac{\partial A_2}{\partial a_0}\frac{\partial B_2}{\partial \theta_0} - \frac{\partial A_2}{\partial \theta_0}\frac{\partial B_2}{\partial a_0} > 0 \tag{1.12b}$$

The first condition (1.12a) leads to the inequality:

$$A_0^2 > \frac{1}{2}A_*^2, \quad A_2^2 = \frac{\lambda}{k}, \quad \lambda = h - h_0 \tag{1.13a}$$

The second condition (1.12b) can be written as:

$$\frac{\partial W}{\partial A_0^2} > 0 \tag{1.13b}$$

§2. SELF-SUSTAINED SYSTEM UNDER EXTERNAL EXCITATION

For the case $f = f_1$ (external excitation) we have:

$$A_1 = 0; \quad B_1 = 0; \quad u_1 = \frac{1}{\omega^2}\left\{\frac{\beta a^2}{2} - \frac{\beta a^2}{6}\cos 2\psi - p\cos(2\psi - 2\theta)\right\} \tag{2.1}$$

and:

$$\begin{aligned} \dot{a} &= -\frac{\varepsilon^2 a}{2\omega}\left\{\omega\left(\frac{3}{4}k\omega^2 a^2 - \lambda\right) - \frac{\beta p}{\omega^2}\sin 2\theta\right\} \\ a\dot{\theta} &= \frac{\varepsilon^2 a}{2\omega}\left\{\left[\Delta - \left(\frac{3\gamma}{4} - \frac{5\beta^2}{6\omega^2}\right)a^2\right] - \frac{\beta p}{\omega^2}\cos 2\theta\right\} \end{aligned} \tag{2.2}$$

The trigonometrical equation (1.7) becomes:

$$\Delta - \frac{\beta p}{\omega^2}\cos 2\theta_* = 0 \tag{2.3}$$

therefore, the stability conditions (1.6) for the equilibrium regime are:

$$\mathrm{Re}\left\{-\lambda\omega \pm \sqrt{\frac{\beta^2 p^2}{\omega^2} - \Delta^2}\right\} > 0 \tag{2.4a}$$

or approximatively:

$$\mathrm{Re}\left\{-\lambda \pm \sqrt{\beta^2 p^2 - \Delta^2}\right\} > 0 \tag{2.4b}$$

The relationship (1.9) is:

$$W = W_1(A_0^2, \Delta) = (k^2 + \xi^2)(A_0^2)^2 - 2(k\lambda + \xi\Delta)A_0^2 + (\lambda^2 - \beta^2 p^2 + \Delta^2) = 0 \tag{2.5}$$

from which, we obtain:

$$A_0^2 = \frac{k\lambda}{k^2 + \xi^2} + \frac{\xi\Delta}{k^2 + \xi^2} \pm \frac{\sqrt{D_1}}{k^2 + \xi^2} \tag{2.6}$$

where

$$\xi = \frac{4}{3}\left(\frac{3\gamma}{4} - \frac{5\beta^2}{6}\right) \tag{2.7}$$

$$D_1 = (k^2 + \xi^2)\beta^2 p^2 - (k\Delta - \xi\lambda)^2 \leq 0 \tag{2.8}$$

in the interval:

$$\frac{\lambda\xi}{k} - \frac{p|\beta|}{k}\sqrt{k^2 + \xi^2} \leq \Delta \leq \frac{\lambda\xi}{k} + \frac{p|\beta|}{k}\sqrt{k^2 + \xi^2} \tag{2.9}$$

Let us analyse the stability conditions (1.13).

It is easy to verify that A_*^2 is the "amplitude" of the pure self-sustained system. Indeed, if $p = 0$, from (2.8) we deduce $\Delta = \frac{\xi\lambda}{k}$, then, from (2.6), we obtain $A_0^2 = \frac{\lambda}{k} = A_*^2$. Thus, the first stability condition (1.13a) requires that the amplitude of the stable stationary oscillation must be large enough (greater than one-half of that of the pure self-sustained system).

The second stability condition (1.13b) is:

$$\frac{\partial W_1}{\partial A_0^2} > 0 \quad \text{or} \quad A_0^2 > \frac{k\lambda}{k^2 + \xi^2} + \frac{\xi\Delta}{k^2 + \xi^2} \tag{2.10}$$

Comparing (2.10) and (2.6) we conclude that only the oscillation whose amplitude corresponds to the sign + in (2.6) may be stable.

To estimate the influence of the external excitation, let us compare the resonance curves $A_0^2 = A_0^2(\Delta)$ of the pure self-sustained when $p = 0$ with those of the combined one when $p > 0$.

First, we suppose that $\lambda > 0$ i.e. $h > h_0$ (the positive friction is greater than the negative one). The equilibrium regime is always unstable. For $\xi = 0$ i.e. $\frac{3\gamma}{4} = \frac{5\beta^2}{6}$ the system is neutralized, the resonance curve is an ellipse of center C_0 ($\Delta = 0$, $A^2 = A_*^2$) and its backbone curve C_0C_0' is the abscissa line $A^2 = A_*^2$. Increasing (decreasing) ξ, the ellipse is deformed, its center C ($\Delta = \frac{\xi\lambda}{k}$, $A^2 = A_*^2$) moves to the right (left) along C_0C_0', the backbone curve CC' has positive (negative) slope $\frac{\xi}{k^2 + \xi^2}$. If $p = 0$ (pure self-sustained system), these ellipses degenerate to their corresponding centers (C_0 or C).

Figure 1 shows the resonance curves for the case $\lambda = 0.00012 > 0$; $k = 0.0024$; $\beta^2 = 0.0036$; $p = 0.03$ and $\xi = 0(a)$; $\xi = 0.0024(b, c)$. Heavy (dashed) curves correspond to stable (unstable) regimes. Obviously, the influence of the external excitation is significant: the maximum amplitude of the "combined" system is greater enough in comparison with that of the pure self-sustained one.

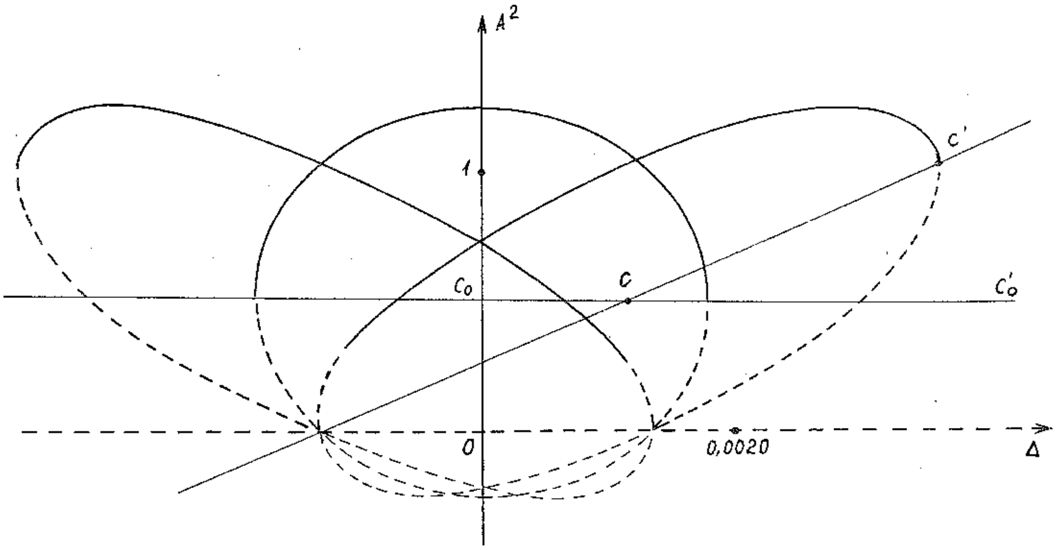

Fig. 1

Suppose now that $\lambda < 0$ ($h < h_0$) but $\lambda^2 < \beta^2 p^2$ (the resulting damping force is small enough). The equilibrium regime is unstable in the interval $|\Delta| < \sqrt{\beta^2p^2 - \lambda^2}$ and stable if $|\Delta| > \sqrt{\beta^2p^2 - \lambda^2}$. The center C_0 lies below the abscissa axis $O\Delta$, the center C moves along the abscissa line C_0C_0', on the left (right) if $\xi > 0$ ($\xi < 0$). In figure 2, the resonance curves correspond to the case where $\lambda = -0.0006$, other coefficients remain unchanged. In this case, the oscillation appears under the action of the external excitation.

At last, if $\lambda < 0$ and $\lambda^2 > \beta^2 p^2$, the equilibrium regime is always stable, the system is not excited.

It is noted that, if $\beta = 0$, the intensity p of the external excitation is absent in (2.2) and the forced oscillation $\varepsilon u_1 = -\varepsilon p \cos(2\psi - 2\theta)$ is of order ε. So without quadratic non-linearity, the interaction between external and self-sustained excitations can be neglected.

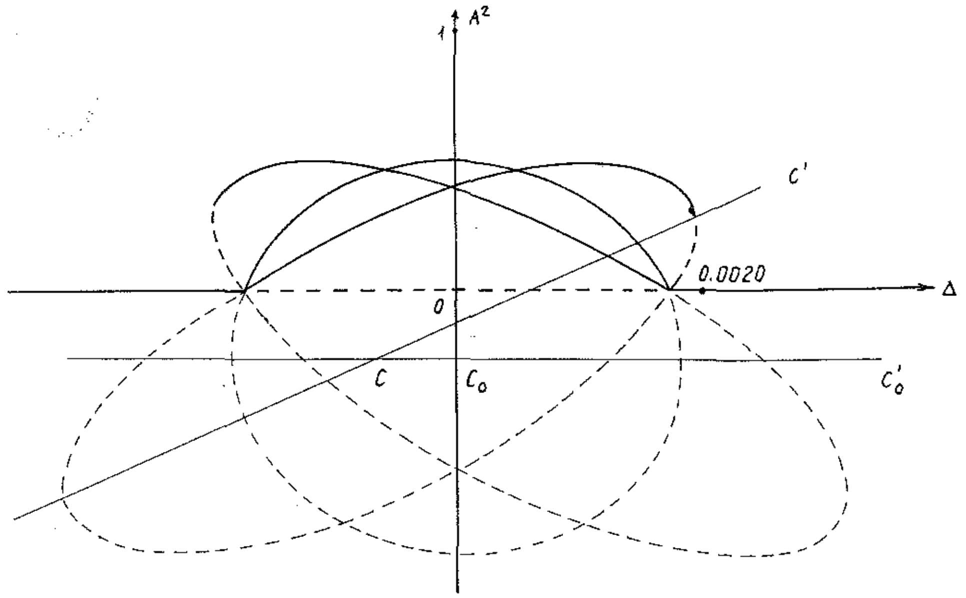

Fig. 2

§3. WEAK SELF-SUSTAINED SYSTEM UNDER PARAMETRIC EXCITATION

For the case $f = f_2$ (parametric excitation), we have:

$$A_1 = 0, \quad B_1 = 0, \quad u_1 = \frac{1}{\omega^2}\Big\{pa\cos\theta - \frac{pa}{3}\cos(2\psi - \theta)\Big\} \tag{3.1}$$

and:

$$\begin{aligned} \dot{a} &= -\frac{\varepsilon^2 a}{2\omega}\Big\{\omega\big(\frac{3}{4}k\omega^2 a^2 - \lambda\big) + \frac{p^2}{\omega^2}\sin 2\theta\Big\} \\ a\dot{\theta} &= -\frac{\varepsilon a}{2\omega}\Big\{\Big[\big(\Delta + \frac{2p^2}{3\omega^2}\big) - \frac{3\gamma}{4}a^2\Big] + \frac{p^2}{\omega^2}\cos 2\theta\Big\} \end{aligned} \tag{3.2}$$

The stability condition of the equilibrium regime are:

$$\mathrm{Re}\Big\{-\lambda \pm \sqrt{p^4 - \big(\Delta + \frac{2p^2}{3}\big)^2}\Big\} > 0 \tag{3.3}$$

The relationship (1.9) is of the form:

$$W = W_2(A_0^2, \Delta) - (k^2 + \gamma^2)(A_0^2)^2 - 2\Big[kh + \gamma\big(\Delta + \frac{2p^2}{3}\big)\Big]A_0^2 + h^2 + \big(\Delta + \frac{2p^2}{3}\big)^2 - (p^2)^2 = 0 \tag{3.4}$$

or

$$A_0^2 = \frac{kh}{k^2 + \gamma^2} + \frac{\gamma\big(\Delta + \frac{2p^2}{3}\big)}{k^2 + \gamma^2} \pm \frac{\sqrt{D_2}}{k^2 + \gamma^2} \tag{3.5}$$

where

$$D_2 = (k^2 + \gamma^2)(p^2)^2 - \Big[k\big(\Delta + \frac{2p^2}{3}\big) - \gamma h\Big]^2 \geq 0 \tag{3.6}$$

in the interval

$$\left|\Delta + \frac{2p^2}{3}\right| \leq \frac{\gamma\lambda}{k} + \frac{p^2}{k}\sqrt{k^2+\gamma^2} \tag{3.7}$$

The first stability condition (1.12a) of the stationary oscillation is given by the same inequality (1.13a).

The second stability condition (1.12b) gives:

$$\frac{\partial W_2}{\partial (A_0^2)} > 0 \quad \text{or} \quad A_0^2 > \frac{k\lambda}{k^2+\gamma^2} + \frac{\gamma(\Delta + \frac{2p^2}{3})}{k^2+\gamma^2} \tag{3.8}$$

Comparing (3.3)-(3.7) and (2.4)-(2.10) we find that for $k = 0.0024$; $\lambda = 0.0012$; $p^2 = 0.0018$ and $\gamma = 0$, $\gamma = \pm 0.0024$ the resonance curves are given in the same figure 1 with a little modification: the center C_0 deplaces on the left, its new abscissa is $\delta = -\frac{2p^2}{3}$

CONCLUSION

We have examined an oscillating system subjected simultaneously to weak (of order ε^2) positive friction force and to less weak (of order ε) external or parametric excitations in the resonance cases. The results obtained show that these force and excitations renforce their actions together so that the oscillation of large amplitude can be observed. The resonance curves are the ellipes whose centers correspond to the oscillatory regime of the pure self-sustained system. In the case of external excitation, the quadratic non-linearity (of order ε) is necessary.

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HỆ TỰ CHẤN YẾU DƯỚI TÁC ĐỘNG CỦA NHỮNG KÍCH ĐỘNG ÍT YẾU HƠN

Bài báo khảo sát hai trường hợp tương tác trong hệ dao động á tuyến giữa những kích động khác cấp: kích động tự chấn ở cấp ε^2 với kích động cưỡng bức hoặc thông số ở cấp ε không ở tình trạng cộng hưởng chủ yếu (tần số tương ứng lân cận gấp đôi hoặc bằng tần số riêng). Kết quả cho thấy các kích động đã cộng tác dụng: các đường cộng hưởng là những enlip với tâm tương ứng mức biên độ của hệ thuần tự chấn.