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Conditional Strong Matching Preclusion of the Alternating Group Graph

Mohamad Abdallah
American University of Kuwait, mkmabdall@gmail.com

Eddie Cheng
Oakland University, echeng@oakland.edu

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Cover Page Footnote

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Abstract

The strong matching preclusion number of a graph is the minimum number of vertices and edges whose deletion results in a graph that has neither perfect matchings nor almost-perfect matchings. Park and Ihm introduced the problem of strong matching preclusion under the condition that no isolated vertex is created as a result of faults. In this paper, we find the conditional strong matching preclusion number for the n-dimensional alternating group graph AG_n.

1 Introduction

Given a graph $G = (V, E)$, a set M of pairwise nonadjacent edges is called matching. A *perfect matching* M in G is a matching such that every vertex in G is incident to exactly one edge in M. An *almost-perfect matching* in G is a set of edges such that every vertex in G, except one, is incident with exactly one edge in M, and the exceptional vertex is incident to none. If G has a perfect matching, then G has an even number of vertices; if G has an almost-perfect matching, then G has an odd number of vertices. We say that the graph G is *matchable* if it has either a perfect matching or an almost-perfect matching. Otherwise, it is called *unmatchable*.

A *matching preclusion set* of G is a set of edges whose deletion results in an unmatchable graph [3]. The matching preclusion number of G, denoted by $mp(G)$, is the minimum size of all possible matching preclusion sets of G. Any such optimal set is called an *optimal matching preclusion set*. If G is unmatchable, then $mp(G) = 0$. Brigham et al. [3] introduced the concept of matching preclusion as a measure of robustness in the event of edge failure in interconnection networks, as well as a theoretical connection to conditional connectivity.

A trivial case of matching preclusion occurs when all edges in G incident to a single vertex are deleted when G has even number of vertices, or when all edges in G incident to two particular vertices are deleted when G has an odd number of vertices. This case models a situation where link failures are concentrated at only a very few nodes of a communication network. When such case is unlikely to happen, Cheng et al. [10] introduced a useful notion called *conditional matching preclusion* which removes from consideration the case when the matching preclusion set produces a graph with an isolated vertex after the edge deletion. The *conditional matching preclusion number*, denoted $mp_1(G)$, is the minimum size of all conditional matching preclusion sets of G.

Park and Ihm [7] introduced the concept of strong matching preclusion where the matching preclusion set can contain vertices in additional to edges. This concept corresponds to the type of failure in a communication network which occurs through nodes and communication lines. The *strong matching preclusion set* of G is a set of vertices and/or edges whose deletion leads to an unmatchable graph. The *strong matching preclusion number* is the minimum size of strong matching preclusion sets in G. For the same reason Cheng et al. introduced the conditional matching preclusion, Park and Ihm [8] introduced the concept of *conditional strong matching preclusion* and discussed its fundamental properties for some classes of graphs and interconnection networks.

A popular class of interconnection networks is the class of alternating group graphs [5]. In this paper we find the *conditional strong matching preclusion number* of the alternating group graph AG_n, which is the minimum size of all conditional strong matching preclusion sets of AG_n.

2 Preliminaries

A trivial case of matching preclusion occurs when all edges in G incident to a single vertex are deleted when G has an even number of vertices, or when all edges in G incident to two particular vertices are deleted when G has an odd number of vertices. In this paper, our graphs will always have an even number of vertices. We call an optimal solution of the trivial case of matching preclusion a *trivial optimal matching preclusion set*. Let F be an optimal strong matching preclusion set of a graph $G = (V, E)$, and let $F = F^V \cup F^E$ where F^V consists of vertices in F and F^E consists of edges in F. We may assume that no element in F^E is incident to an element of F^V since F is optimal. In fact, if $f \in F^E$ is incident to $u \in F^V$, then $G - F = G - (F - \{f\})$. If F is an optimal strong matching preclusion set of G and $G - F$ has an isolated vertex, then F is a *basic optimal strong matching preclusion set*. Based on this definition, it is possible to have a basic optimal matching preclusion set F with $G - F$ odd and without almost-perfect matchings. We can further restrict this class by requiring that, in addition, $G - F$ must be even. Then F is called *optimal strong matching preclusion set*.

The following proposition considers the relationship between basic strong matching preclusion sets and trivial strong matching preclusion sets.

Proposition 2.1. *[4] Let G be a r-regular even graph with $r \geq 2$. Suppose that $smp(G) = r$. Then every basic optimal strong matching preclusion set is trivial.*

Let $F \subseteq V(G) \cup E(G)$, F is a *conditional strong matching preclusion set* of G if $G - F$ has neither a perfect matching nor an almost-perfect matching and no isolated vertices. The minimum cardinality of all such sets is denoted by $smp_1(G)$, and called the *conditional strong matching preclusion number* of G. In this paper, we assume G has no isolated vertices. If G is unmatchable, then $smp_1(G) = 0$. The following propositions follow directly from the fact that a matching preclusion set is a special case of a strong matching preclusion set consisting of edges only.

Proposition 2.2. *[4] Let G be a graph with an even number of vertices. Then $smp(G) \leq mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

Proposition 2.3. *[8] For every graph G for which all the four numbers, $mp(G)$, $mp_1(G)$, $smp(G)$, and $smp_1(G)$ are well defined, $smp(G) \leq smp_1(G) \leq mp_1(G)$ and $smp(G) \leq mp(G) \leq mp_1(G)$.*

Under the condition of no isolated vertices allowed after the deletion of edges and/or vertices, an easy way to build a conditional strong matching preclusion set in G is to try a fault set F that leaves after deletion a path (u, z, v) made of the three vertices u, z and v, where $\deg_{G-F}(u) = \deg_{G-F}(v) = 1$. If $G - F$ is even, then the resulting graph becomes unmatchable. Therefore we can build a candidate conditional strong matching preclusion set as follows. Let $N_G(\cdot)$ represents the set of neighboring vertices in G. Given a path (u, z, v) in a graph $G = (V, E)$, build a fault set, denoted F_{uzv}, in such a way that

1. F_{uzv} contains every vertex $w \in (N_G(u) \cap N_G(v)) - \{z\}$,
2. F_{uzv} contains the edge uv if $uv \in E(G)$,
3. for every vertex $w \in N_G(u) - N_G(v)$, F_{uzv} contains exactly one of w and uw,
4. for every vertex $w \in N_G(v) - N_G(u)$, F_{uzv} contains exactly one of w and vw.

The next fundamental proposition provides sufficient conditions to make F_{uzv} a conditional strong matching preclusion set.

Proposition 2.4. *[8] For an arbitrary path (u, z, v) in a graph G, F_{uzv} is a conditional strong matching preclusion set of G if*

1. *there is no isolated vertex in $G - F_{uzv}$, and*
2. *$G - F_{uzv}$ has an even number of vertices.*

The conditional strong matching preclusion set described in Proposition 2.4 is called *trivial* as it is one of the simplest ways of building a conditional strong matching preclusion set. The following proposition provides an upper bound for $smp_1(G)$.

Proposition 2.5. *[8] If there exists a trivial conditional strong matching preclusion set F_{uzv} for some path (u, z, v) in a graph G, then $smp_1(G) \leq deg_G(u) + deg_G(v) - 2 - g_G(u, v)$, where $g_G(u, v)$ is $|N(u) \cap N(v)|$ if $(u, v) \in E(G)$ or $|N(u) \cap N(v)| - 1$ otherwise.*

The alternating group graph was introduced by Jwo et al. [5] as an interconnection network topology for computing systems. Let A_n be the alternating group, that is, A_n is the set of even permutations of the set $\{1, 2, \ldots, n\}$. By [1], the size of A_n is $\frac{n!}{2}$ and the set $\Omega = \{(1\ 2\ i) \mid 3 \leq i \leq n\} \cup \{(1\ i\ 2) \mid 3 \leq i \leq n\}$ is a generator set for A_n. Define the alternating group graph $AG_n = (V_n, E_n)$ of dimension n as follows: $V_n = A_n$, the set of all even permutations, and $E_n = \{(p, q) \mid p, q \in A_n,\ q = p \cdot h,\ \text{ for } h \in \Omega\}$, where "$\cdot$" is the usual binary associative operator defined by $u \cdot v(x) = u(v(x))$. AG_n is a Cayley graph of the alternating group [6].

AG_n can be recursively built by using n copies of AG_{n-1}. Let H_i denote the induced subgraph of AG_n corresponding to the permutations $p \in A_n$ such that the last symbol of p is i. Note that instead of last position, we can consider H_i according to the jth position for any $3 \leq j \leq n$. We refer to this as a decomposition along the jth position. The following proposition is an easy and known result of the recursive structure of AG_n:

Proposition 2.6. *Let AG_n be the alternating group graph of dimension $n \geq 4$.*

(I) $|V_n| = \frac{n!}{2}$ *and* $|E_n| = \frac{(n-2)n!}{2}$.

(II) AG_n *is* $(2n - 4)$*-regular.*

(III) AG_n *consists of* n *vertex-disjoint subgraphs,* $H_1, H_2, \ldots, H_n$*, each isomorphic to* AG_{n-1}.

(IV) H_i *has* $(n - 1)!/2$ *vertices, and it is* $(2n - 6)$*-regular for all* i.

(V) *There are exactly* $(n-2)!$ *independent edges between* H_i *and* H_j *for all* $i \neq j$*. An edge between different* H_i*'s will be called a* cross edge*.*

(VI) *Each vertex in* H_i *has exactly two neighbors outside* H_i *(which are called its* outside neighbors*); these two outside neighbors are in different* H_k*'s, and there is an edge between them. Thus every vertex forms a triangle with its two outside neighbors.*

(VII) *For every different* i*,* j*, and* k*, there are exactly* $(n-3)!$ *vertices in* H_i *that have an outside neighbor in* H_j *and an outside neighbor in* H_k*.*

Proposition 2.7. *[12] Let* $u, v \in V_n$*. Then* $|N(u) \cap N(v)| \leq 2$.

Using the results from Proposition 2.5 and Proposition 2.7, we can establish an upper bound for the size of the conditional strong matching preclusion set of AG_n.

Proposition 2.8. *For* $n \geq 4$*,* $smp_1(AG_n) \leq 4n - 11$.

Lemma 2.1. *Suppose that a graph* G *has an almost-perfect matching that misses a vertex* v *that is not isolated, then there exists another almost-perfect matching in* G *that misses a vertex other than* v*.*

Proof. Let M be an almost-perfect matching in G that misses the vertex $v \in V(G)$. Since v is not isolated, then v is adjacent to some vertex $u \in V(G)$. The matching M saturates u, so let the edge $wu \in M$. Let $M' = (M - \{wu\}) \cup \{uv\}$, M' is an almost-perfect matching in G that misses the vertex w.

□

Lemma 2.2. *Let* $F \subseteq V_n$ *for* $n \geq 5$*, such that* $|F| = 4n - 12$*. If* $F \subseteq V(H_i)$ *for some* $i \in \{1, \ldots, n\}$*, then we can choose the decomposition via a different position so that at most* $4n - 15$ *of the vertices in* F *belong to* H_j *for some* $j \in \{1, \ldots, n\}$*.*

Proof. Without loss of generality we can assume that $F \subseteq V(H_n)$, then all the $4n - 12$ vertices end with n. We can look at these vertices as a $(4n - 12) \times n$ array, where each row represents the corresponding permutation of each vertex. The nth column contains the number n only. If the $(n-1)$st column contains at most $(4n - 15)$ numbers of the same number i for some $i \in \{1, \ldots, n-1\}$, then we can decompose along this position. Otherwise, this column contains at least $4n - 14$ i's for some $i \in \{1, 2, \ldots, n-1\}$. For notational convenience, we can assume that $i = n - 1$, so the entries of the $(n-1)$st column must be the number $(n-1)$ except for at most 2 rows. Repeating the same process, and assuming that we do not find the desired position to decompose along, then when we reach the 3rd column we will have at most $2(n-3)$ rows different from

$$\square\,\square\,3\;4\;5\;\ldots\;(n-1)\;n$$

so we get at least $(4n - 12) - 2(n - 3) = 2n - 6$ rows of the form

$$\square\,\square\,3\;4\;5\;\ldots\;(n-1)\;n$$

where the first two positions must be 1 and 2. For $n \geq 5$, the number of such vertices is at least 4, which is not possible. Then we should be able to find a column that contains no

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more than $4n-15$ numbers of the same number i, and therefore decomposing along the ith position guarantees that at most $4n-15$ vertices are in the new H_i.

□

Lemma 2.3. *Let $F \subseteq V_n \cup E_n$ for $n \geq 5$, such that $|F| = 4n-12$. If $F \subseteq V(H_i) \cup E(H_i)$ for some $i \in \{1, \ldots, n\}$, then we can choose the decomposition via a different position so that at most $4n-15$ of the elements in F belong to H_j for some $j \in \{1, \ldots, n\}$.*

Proof. Let $F = F^V \cup F^E$ be a subset of H_i for some $i \in \{1, 2, \ldots, n\}$, where F^V is the set of vertices in F and F^E is the set of edges in F. Let $F^V = \{u_1, u_2, \ldots, u_p\}$ and $F^E = \{x_1y_1, x_2y_2, \ldots, x_qy_q\}$. We have $p+q = 4n-12$. Consider the set $L = \{u_1, u_2, \ldots, u_p, x_1, x_2, \ldots, x_q\}$ in $V(H_1)$. By Proposition 2.2, We can decompose via a different position so that at most $4n-15$ elements of L belong to H_j for some $j \in \{1, \ldots, n\}$. Note that if some $x_i \in L$ is outside H_j and if $y_i \in V(H_j)$ then the faulty edge x_iy_i will be a cross edge and it is not a fault in H_j. Therefore we can conclude that at most $4n-15$ of the elements in F belong to H_j.

□

Theorem 2.4. *[4] Let $n \geq 4$. Then $smp(AG_n) = 2n-4$. Moreover every optimal strong matching preclusion set of AG_n is trivial.*

3 The Main Result

Theorem 3.1. *Let $n \geq 4$. Then $smp_1(AG_n) = 4n-11$.*

Proof. Let $F \subseteq V_n \cup E_n$ be a set of faults in AG_n such that $AG_n - F$ has no isolated vertices. $F = F^V \cup F^E$, where $F^V = V_n \cap F$ and $F^E = E_n \cap F$. We want to show that if $|F| \leq 4n-12$, then $AG_n - F$ has a perfect or an almost-perfect matching. Suppose $n = 4$. If $|F| \leq 4$ then, by Theorem 2.4, $AG_4 - F$ has a perfect matching or an almost-perfect matching and every matching preclusion set of size 4 is the set of vertices and/or edges that isolates a vertex. Note that when $n = 4$, $2n-4 = 4n-12 = 4$, so if we delete 4 faults without leaving an isolated vertex in $AG_n - F$, the graph $AG_n - F$ must contain a perfect or an almost-perfect matching. Then $smp_1(AG_4) \geq 5$. By Proposition 2.8, $smp_1(AG_4) \leq 5$, then we have $smp_1(AG_4) = 5$. Hence, the claim is true for $n = 4$.

We proceed by induction on n. Suppose that $smp_1(AG_{n-1}) = 4(n-1)-11 = 4n-15$, we want to show that $smp_1(AG_n) = 4n-11$. Let $F_i = F_i^V \cup F_i^E$ be the set of faults in H_i for $i = 1, \ldots, n$. We consider several cases depending on the faults' distribution in the H_i's.

Case 1. $|F_i| \leq 2n-7$ *for all* $i \in \{1, \ldots, n\}$. For $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, there are $(n-2)! - (4n-12)$ possible cross edges between $H_i - F_i$ and $H_j - F_j$. So when $n \geq 6$, there are at least 12 possible cross edges. Since H_i is $(2n-6)$-regular and $|F_i| \leq 2n-7$, then $H_i - F_i$ contains no isolated vertices, then by Theorem 2.4 (or by the induction hypothesis) there exists a perfect matching or an almost-perfect matching in every $H_i - F_i$ for $i = 1, \ldots, n$. For notational convenience, assume that $|F_1^V|, \ldots, |F_k^V|$ are odd for some $k \in \{1, \ldots, n\}$, and $|F_{k+1}^V|, \ldots, |F_n^V|$ are even. By the induction hypothesis, there exist almost-perfect matchings M_i in $H_i - F_i$ for $i = 1, \ldots, k$ and perfect matchings M_j in $H_j - F_j$ for $j = k+1, \ldots, n$. As mentioned above, there are at least 12 cross edges between any $H_i - F_i$ and $H_j - F_j$ for $i \neq j$. We want to choose a cross edge $e = uv$ in $AG_n - F$ between $H_i - F_i$ and $H_j - F_j$ where $u \in V(H_i - F_i)$ and $v \in V(H_j - F_j)$ so that $H_i - (F_i \cup \{u\})$ and $H_j - (F_j \cup \{v\})$ have no isolated vertices. In every $H_i - F_i$, there could be at most one such vertex u whose deletion produces an isolated vertex in $H_i - (F_i \cup \{u\})$, then there could be at most two cross edges that cannot be chosen. As a result, we can always find a cross edge $e = x_i x_j$ between $H_i - F_i$ and $H_j - F_j$ such that $H_i - (F_i \cup \{x_i\})$ and $H_j - (F_j \cup \{x_j\})$ have no isolated vertices. We consider two possibilities depending on the parity of k.

k is even. For $i = 1, \ldots, k-1$, pick cross edges $x_1x_2, x_3x_4, \ldots, x_{k-1}x_k \in E(AG_n - F)$ as described above, where $x_i \in V(H_i - F_i)$. Thus $H_i - (F_i \cup \{x_i\})$ contains an even number of vertices and has no isolated vertices. Since $|F_i \cup \{x_i\}| \leq 2n - 6$, then by Theorem 2.4 there exist perfect matchings M_i' in $H_i - (F_i \cup \{x_i\})$ for $i = 1, \ldots, k$. Therefore $\bigcup_{i=1}^{k} M_i' \cup \{x_1x_2, x_3x_4, \ldots, x_{k-1}x_k\} \cup \bigcup_{j=k+1}^{n} M_j$ is a perfect matching in $AG_n - F$.

k is odd. We use the same argument and notation as above. The set $\bigcup_{i=1}^{k-1} M_i' \cup \{x_1x_2, x_3x_4, \ldots, x_{k-2}x_{k-1}\} \cup \bigcup_{j=k}^{n} M_j$ is an almost-perfect matching in $AG_n - F$. See Figure 1.

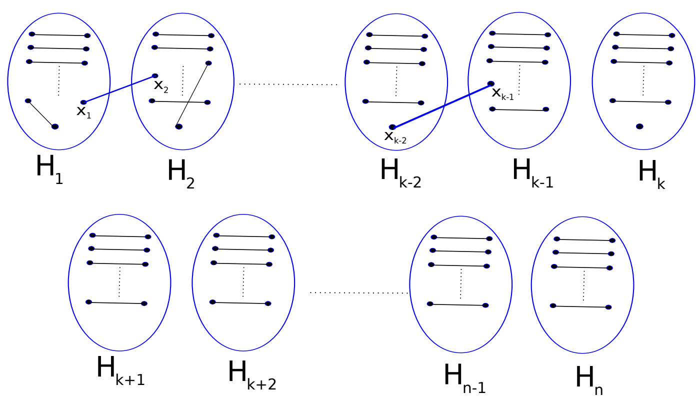

Figure 1: Case 1: $|F_i| \leq 2n - 7$ for $i \in \{1, \ldots, n\}$

Suppose $n = 5$. If there is an edge between every pair of $H_i - F_i$ and $H_j - F_j$ in $AG_n - F$, then we are done. But such a violation can only occur for one pair say $H_1 - F_1$ and $H_2 - F_2$. Thus the graph induced by $V(H_1) \cup V(H_2)$ has at least six faults. So there are only two faults not yet discovered. Now, the above construction works unless $k = 2$. (If $k \geq 3$, we can avoid pairing $H_1 - F_1$ and $H_2 - F_2$). Since there are only two faults not yet discovered and $n = 5$. We may assume that $|F_3| = 0$ and none of the cross edges between $H_1 - F_1$ and $H_3 - F_3$, and between $H_2 - F_2$ and $H_3 - F_3$ are faulty edges. Now choose x_1x_3 and x_2x_3' appropriately. Now $H_3 - \{x_3, x_3'\}$ has a perfect matching, and we can continue with the usual construction.

Case 2. $|F_1| = 2n - 6$ *and* $|F_i| \leq 2n - 7$ *for all* $i \neq 1$.

Case 2.1. $H_1 - F_1$ *has no isolated vertices.* By the induction hypothesis, there exists a perfect matching or an almost-perfect matching M_1 in $H_1 - F_1$, and by Theorem 2.4, every

$H_i - F_i$ contains a perfect or an almost-perfect matching M_i for $i \neq 1$. Then this case is similar to Case 1.

Case 2.2. *$H_1 - F_1$ has an isolated vertex $u \in V(H_1 - F_1)$.*

Case 2.2.1. *F_1 is not a matching preclusion set in H_1.* By Proposition 2.1, $|F_1^V|$ is odd. Then $H_1 - F_1$ has an almost-perfect matching M_1 missing the vertex u. Since $AG_n - F$ has no isolated vertices, then the vertex u must be adjacent to some outside neighbor u' in $AG_n - F$. Without loss of generality, assume that $u' \in V(H_2 - F_2)$. Let $F_2' = F_2 \cup \{u'\}$, we consider two cases depending on whether the resulting subgraph induced by $H_2 - F_2'$ has an isolated vertex or not.

Case 2.2.1.1. *$H_2 - F_2'$ has no isolated vertices.* Since $|F_2'| \leq 2n-6$, then by the induction hypothesis, $H_2 - F_2'$ has a perfect or an almost-perfect matching M_2. We consider two cases depending on whether M_2 is a perfect or an almost-perfect matching.

M_2 is a perfect matching. By the induction hypothesis, we can find perfect or almost-perfect matchings M_i in $H_i - F_i$ for $i \geq 3$. Since $|F_i| \leq 2n-7$ for $i \geq 3$, then we can proceed as in Case 1 and combine those matchings to get a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2))$. Then $M' \cup M_1 \cup M_2 \cup \{uu'\}$ is a perfect or an almost-perfect matching in $AG_n - F$.

M_2 is an almost-perfect matching. If $|F_2'| \leq 2n-7$, then we can proceed as in Case 1 to find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup \{u'\})$, and the set $M' \cup M_1 \cup \{uu'\}$ is a perfect or an almost-perfect matching in $AG_n - F$. If $|F_2'| = 2n-6$, then let $x \in V(H_2 - F_2')$ be the unsaturated vertex by M_2. Under this assumption, there is only one fault left outside the subgraph induced by $V(H_1) \cup V(H_2)$. Since x is not isolated in $H_2 - F_2'$, then by Proposition 2.1 there exists another almost-perfect matching in $H_2 - F_2'$ missing a vertex different from x. So without loss of generality we can assume that x has two outside neighbors in $AG_n - F$ or else we consider the other almost-perfect matching. Let x' be the outside neighbor of x, such that $x' \in V(H_j - F)$ for some $j \neq 1, 2$. Let $F_j' = F_j \cup \{x'\}$, then by the induction hypothesis there exists a perfect or an almost-perfect matching M_j in $H_j - F_j'$, and M_i in $H_i - F_i$ for all $i \neq 1, 2, j$. Since $|F_j'|$ and $|F_i|$ are less than or equal to $2n-7$, then we can use the construction of Case 1 to find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{u', x'\})$. Then the set $M' \cup M_1 \cup M_2 \cup \{uu', xx'\}$ is a perfect or an almost-perfect matching in $AG_n - F$.

Case 2.2.1.2. *$H_2 - F_2'$ has an isolated vertex.* Then we can choose the second outside neighbor of u, u'', which belongs to $V(H_k)$ for some $k \neq 1, 2$. If $u'' \in V(H_k - F_k)$ and $uu'' \in E(AG_n - F)$, then $H_k - (F_k \cup \{u'\})$ has no isolated vertices, and then we can proceed as in the previous case. Assume the worst case scenario when u'' is a faulty vertex or uu'' is a faulty edge. We

have $|F_2'| = 2n-6$ and u' is adjacent to a vertex $z \in V(H_2 - F_2)$ and all faults in H_2 are adjacent and/or incident to z as well. The vertex z has two outside neighbors in $AG_n - F$, let $z' \in V(H_k - F_k)$ be an outside neighbor of z for some $k \neq 1, 2$. We consider the parity of $|F_2^V|$.
$|F_2^V|$ *is even*. Then there exists a perfect matching M_2 in $H_2 - F_2$, and in this case the edge $u'z$ must belong to M_2. As in Case 1, we can find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{z'\})$. Then $M_1 \cup (M_2 - \{u'z\}) \cup \{uu', zz'\} \cup M'$ is a perfect or an almost-perfect matching in $AG_n - F$. See Figure 2.

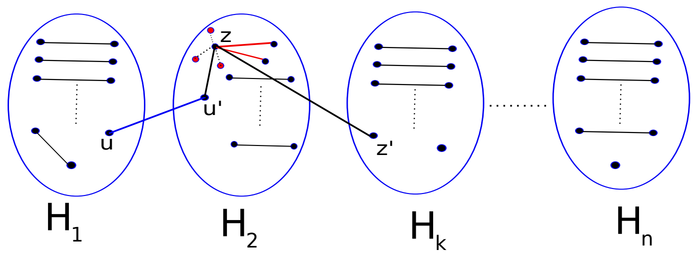

Figure 2: Case 2.2.1: $|F_2^V|$ is even

$|F_2^V|$ *is odd*. Then choose an appropriate vertex $x \in V(H_2 - F_2)$ such that x has an outside neighbor $x' \in V(H_p - F_p)$ for some $p \neq 1, 2$ with $xx' \in E(AG_n - F)$ and $H_2 - (F \cup \{x\})$ has no isolated vertices. Let $F_2'' = F_2 \cup \{x\}$, then $|F_2''^V|$ is even of size $2n-6$ and has no isolated vertices, then by Theorem 2.4 $H_2 - F_2''$ has a perfect matching M_2'. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_i in $H_i - (F_i \cup \{x', z'\})$ for $i \geq 3$. Since all the faults except one are in the subgraph induced by $V(H_1) \cup V(H_2)$, then $|F_i \cup \{x', z'\}| \leq 2n-7$ for $i \geq 3$, then we can use the construction of Case 1 to combine the M_i's and get a perfect or an almost-perfect matching M'' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{x', z'\})$. Therefore $M_1 \cup (M_2' - \{u'z\}) \cup \{uu', zz', xx'\} \cup M''$ is a perfect or an almost-perfect matching in $AG_n - F$. See Figure 3.

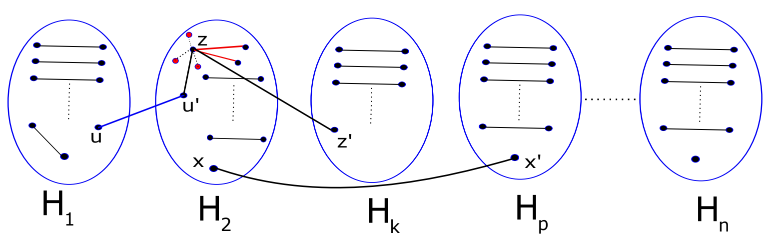

Figure 3: Case 2.2.1: $|F_2^V|$ is odd

Case 2.2.2. F_1 *is a matching preclusion set in* H_1. Then F_1 is a basic strong matching preclusion set. Since $smp(H_1) = 2n-6$, and H_1 is even and regular, then by Proposition 2.1 every basic optimal strong matching preclusion set is trivial, therefore $|F_1^V|$ is even.

Case 2.2.2.1. *Assume that* $F_1^V \neq \emptyset$. It is easy to find a vertex w in F_1 such that $H_1 - (F_1 - \{w\})$ has no isolated vertices. The vertex w must be adjacent to u in $H_1 - (F_1 - \{w\})$. Since u is isolated in H_1 and because $AG_n - F$ contains no isolated vertices, then u must have an outside neighbor u' in $AG_n - F$. Without loss of generality, assume that $u' \in V(H_2 - F_2)$. Let $v \in V(H_1 - F_1)$ such that $H_1 - (F_1 \cup \{v\})$ has no isolated vertices and v has an outside neighbor $v' \in V(H_p - F_p)$, with $vv' \in E(AG_n - F)$, for some $p \neq 1, 2$. Let $F_1' = (F_1 - \{w\}) \cup \{v\}$, then $|F_1'| = 2n-6$ and contains an even number of vertices. Moreover, $H_1 - F_1'$ has no isolated vertices, then by Theorem 2.4 we can find a perfect matching M_1 in $H_1 - F_1'$. This matching must contain the edge uw.

Assume that $H_2 - (F_2 \cup \{u'\})$ *has no isolated vertices.* Then by the induction hypothesis there exists a perfect or an almost-perfect matching M_i in $H_i - (F_i \cup \{u', v'\})$ for $i \geq 2$. As in Case 1, we can combine those matchings to get a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup \{u', v'\})$. Then the set $(M_1 - \{uw\}) \cup \{uu', vv'\} \cup M'$ is a perfect or an almost-perfect matching in $AG_n - F$.

Assume that $H_2 - (F_2 \cup \{u'\})$ *has an isolated vertex* z. Under this case we can choose the other outside neighbor of u, $u'' \in V(H_k)$, for some $k \neq 1, 2$. If u'' is a faulty vertex or uu'' is a faulty edge, then we can guarantee that the vertex z has two outside neighbors in $AG_n - F$. Let z' be an outside neighbor of z such that $z' \in V(H_k - F_k)$ with $zz' \in E(AG_n - F)$ for some $k \neq 1, 2$.

– If $|F_2^V|$ is even, then there exists a perfect matching M_2 in $H_2 - F_2$, and in this case the edge $u'z$ belongs to M_2. As in Case 1, we can find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{z', v'\})$. Then $M_1 \cup (M_2 - \{u'z\}) \cup \{uu', zz', vv'\} \cup M'$ is a perfect or an almost-perfect matching in $AG_n - F$. See Figure 4.

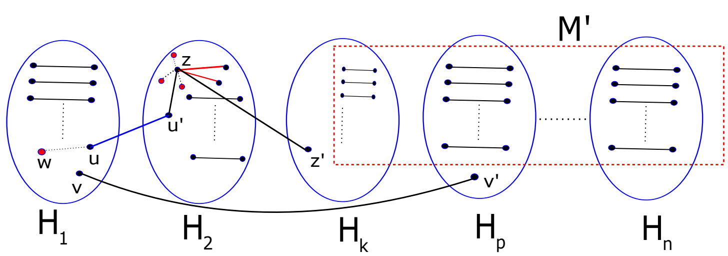

Figure 4: Case 2.2.2.1: $|F_2^V|$ is even

– If $|F_2^V|$ is odd, then choose a vertex $x \in V(H_2 - F_2)$ such that x has an outside neighbor $x' \in V(H_q - F_q)$ for some $q \neq 1, 2$ and $H_2 - (F_2 \cup \{x\})$ has no isolated vertices. Let $F_2'' = F_2 \cup \{x\}$, then $|F_2''^V|$ is even and of size $2n-6$ and has no isolated vertices, then by Theorem 2.4 $H_2 - F_2''$ has a perfect matching M_2'. As in Case 1, we can find a perfect or an almost-perfect matching M'' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{x', z', v'\})$. Then $M_1 \cup (M_2' - \{u'z\}) \cup \{uu', vv', zz', xx'\} \cup M''$ is a perfect or an almost-perfect matching in $AG_n - F$. See Figure 5.

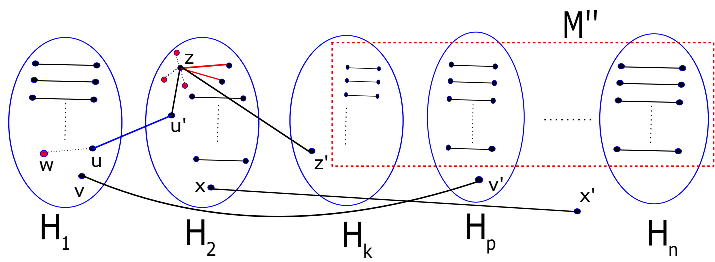

Figure 5: Case 2.2.2.1: $|F_2^V|$ is odd

Case 2.2.2.2. *Assume that* $F_1^V = \emptyset$. Then all the faults inside H_1 are edges. Then the vertex u is incident to $(2n-6)$ faulty edges in H_1. Deleting the vertex u from H_1 includes the deletion of all those $(2n-6)$ edges. Choose a vertex $v \in V(H_1 - \{u\})$ such that $H_1 - \{u, v\}$ has no isolated vertices and v has two outside neighbors in $AG_n - F$. By Theorem 2.4, there exists a perfect matching M_1 in $H_1 - \{u, v\}$. Then M_1 is a perfect matching in $H_1 - (F_1 \cup \{u, v\})$. The construction of the perfect or almost-perfect matching in $AG_n - F$ follows the same way as in Case 2.2.2.1.

Case 3. $|F_1| = 2n-6$ *and* $|F_2| = 2n-6$. Note that all the faults are in the subgraph induced by $V(H_1) \cup V(H_2)$.

Case 3.1. *Assume that* $H_1 - F_1$ *and* $H_2 - F_2$ *have no isolated vertices.* By the induction hypothesis, there exist perfect or almost-perfect matchings M_1 and M_2 in $H_1 - F_1$ and $H_2 - F_2$ respectively. If M_1 and/or M_2 are almost-perfect matchings, then at most two vertices will be missed by $M_1 \cup M_2$. Let $x \in V(H_1 - F_1)$ and $y \in V(H_2 - F_2)$ be these vertices and let $x' \in V(H_p - F_p)$ and $y' \in V(H_q - F_q)$, where $p, q \neq 1, 2$, be their corresponding outside neighbors. Since $|F_i| = 0$ for $i \geq 3$, then $|F_i \cup \{x', y'\}| \leq 2$ for $i \geq 3$, then we can use the construction of Case 1 to find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{x', y'\})$. Therefore $M_1 \cup M_2 \cup M' \cup \{xx', yy'\}$ is a perfect or an almost-perfect matching in $AG_n - F$.

Case 3.2. *Assume that* $H_1 - F_1$ *has an isolated vertex* u*, and* $H_2 - F_2$ *has no isolated vertices.* The vertex u must be adjacent to a vertex $u' \in V(H_k - F_k)$ for some $k \neq 1, 2$ such that $uu' \in E(AG_n - F)$. Since $|F_1| = 2n-6$ and $H_1 - F_1$ has an isolated vertex, then we will consider two cases depending on whether F_1 is a strong matching preclusion set in H_1 or not.

Case 3.2.1. *Assume that* F_1 *is not a matching preclusion set.* By Proposition 2.1, $|F_1^V|$ is odd. Then $H_1 - F_1$ has an almost-perfect matching M_1 missing the vertex u. By the induction hypothesis, $H_2 - F_2$ has a perfect or an almost-perfect matching M_2 and $H_i - (F_i \cup \{u'\})$ has a perfect or an almost-perfect matching

M_i for $i \geq 3$. Using the same idea of Case 1, we can find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{u'\})$.
If M_2 is a perfect matching, then $M_1 \cup M_2 \cup M' \cup \{uu'\}$ is a perfect or an almost-perfect matching in $AG_n - F$.
If M_2 is an almost-perfect matching, then M_2 misses a vertex $x \in V(H_2 - F)$ which has an outside neighbor x' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{u'\})$. Note that we are assuming that $x' \neq u'$, since if $x' = u'$, then by Proposition 2.1 we can find another almost-perfect matching in $H_2 - F$ which misses a vertex other than x and this vertex can not be adjacent to u'. By the induction hypothesis, $H_i - (F_i \cup \{u', x'\})$ has a perfect or an almost-perfect matching M_i' for $i \geq 3$. Using the same idea of Case 1, we can find a perfect or an almost-perfect matching M'' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{u', x'\})$. Thus the set $M_1 \cup M_2 \cup \{uu', xx'\} \cup M''$ is a perfect or an almost-perfect matching in $AG_n - F$.

Case 3.2.2. *Assume that F_1 is a matching preclusion set.* Then by Proposition 2.1, $|F_1^V|$ is even.

Case 3.2.2.1. *Assume $F_1^V \neq \emptyset$.* Let w be a faulty vertex in H_1 and chose $x \in V(H_1 - F_1)$ such that $H_1 - (F_1 \cup \{x\})$ has no isolated vertices. Since all the faults are in the subgraph induced by $V(H_1) \cup V(H_2)$, then the vertex x has at least one outside neighbor $x' \in V(H_k - F_k)$ with $xx' \in E(AG_n - F)$ for some $k \neq 1, 2$. Let $F_1' = (F_1 \cup \{x\}) - \{w\}$, the subgraph $H_1 - F_1'$ has no isolated vertices and $|F_1'| = 2n - 6$ and $|F_1'^V|$ is even, then by Theorem 2.4 there exists a perfect matching M_1 in $H_1 - F_1'$. This matching M_1 includes the edge uw. By the induction hypothesis, $H_2 - F$ has a perfect or an almost-perfect matching M_2.
Assume that M_2 is a perfect matching in $H_2 - F_2$. By the induction hypothesis, $H_i - (F_i \cup \{u', x'\})$ has a perfect or an almost-perfect matching M_i for $i \geq 3$, and since $|F_i \cup \{u', x'\}| \leq 2n - 7$, then we proceed as in Case 1 to find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{u', x'\})$. Thus the set $(M_1 - \{wv\}) \cup \{uu', xx'\} \cup M'$ is a perfect or an almost-perfect matching in $AG_n - F$.
Assume that M_2 is an almost-perfect matching, then M_2 misses a vertex $y \in V(H_2 - F_2)$ which has an outside neighbor y' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{u', x'\})$. Again, we are assuming that $y' \neq u'$, since if $y' = u'$, then by Proposition 2.1 we can find another almost-perfect matching in $H_2 - F_2$ which will miss a vertex other than y and this vertex can not be adjacent to u'. In addition, the choice of x is not unique in $H_1 - F_1$. By the induction hypothesis, $H_i - (F_i \cup \{u', x', y'\})$ has a perfect or an almost-perfect matching M_i' for $i \geq 3$. Using the same idea of Case 1, we can find a perfect or an almost-perfect matching M'' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{u', x', y'\})$. Thus the set $M_1 \cup M_2 \cup \{uu', xx', yy'\} \cup M''$ is a perfect or an almost-perfect

matching in $AG_n - F$.

Case 3.2.2.2. *Assume* $F_1^V = \emptyset$. Then u is incident to $(2n-6)$ faulty edges. Let e be a faulty edge in H_1 and $u, w \in V(H_1 - F_1)$ be its endpoints. The vertex w must have an outside neighbor $w' \in V(H_p - F_p)$ for some $p \neq 1, 2$. Note that it is possible to have $p = k$. By the induction hypothesis, $H_i - (F_i \cup \{u', w'\})$ has a perfect or an almost-perfect matching M_i for $i \geq 3$ and $H_2 - F_2$ has a perfect or an almost-perfect matching M_2. We can now proceed as in the case above to find a perfect or an almost-perfect matching in $AG_n - F$.

Case 3.3. *Assume that* $H_1 - F_1$ *has an isolated vertex* u*, and* $H_2 - F_2$ *has an isolated vertex* v*.* Then u must be incident to a cross edge that has its other endpoint in some $H_k - F_k$ where $k \neq 1, 2$, and v is incident to a cross edge that has its other endpoint in some $H_j - F_j$ where $j \neq 1, 2$. Let uu' and vv' be those edges, so $u' \in V(H_k - F_k)$ and $v' \in V(H_j - F_j)$. It is possible to have $k = j$ and $u' = v'$. If $u' = v'$, then by Proposition 2.6, $uv \in E(AG_n - F)$, and we can include this edge in the matching to saturate u and v.

Case 3.3.1. *If* F_1 *and* F_2 *are not matching preclusion sets in* H_1 *and* H_2 *respectively.* Then this case is similar to Case 3.2.1.

Case 3.3.2. *If* F_1 *is a matching preclusion set in* H_1 *but* F_2 *is not a matching preclusion set in* H_2*.* Then this case is similar to Case 3.2.2.

Case 3.3.3. *If* F_1 *and* F_2 *are matching preclusion sets in* H_1 *and* H_2 *respectively.* Then F_1 and F_2 are basic strong matching preclusion sets in H_1 and H_2 respectively. By Proposition 2.1, F_1 and F_2 are trivial, so $|F_1^V|$ and $|F_2^V|$ are even. As in Case 3.2.2, we can always find a matching M_1 in $H_1 - F_1$ missing the vertices $u, x \in V(H_1 - F_1)$, and a matching M_2 in $H_2 - F_2$ missing the vertices $v, y \in V(H_2 - F_2)$. Since the choice of the vertices x and y is not unique, we can choose those vertices so that their outside neighbors do not lie in the same H_i. Let u', x', v', y' be the outside neighbors of u, x, v, y respectively, such that x' and y' are in different H_i's. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_i in every $H_i - (F_i \cup \{u', x', v', y'\})$ for $i \geq 3$, and since $|F_i \cup \{u', x', v', y'\}| \leq 2n-7$ for $i \geq 3$, then we can proceed as in Case 1 and combine those matchings to get a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{u', x', v', y'\})$. Therefore, $M_1 \cup M_2 \cup M' \cup \{uu', xx', vv', yy'\}$ is a perfect or an almost-perfect matching in $AG_n - F$.

Case 4. $2n - 5 \leq |F_1| \leq 4n - 16$ *for* $n \geq 6$. When $n = 5$, the compound inequality is not valid. So we must consider the cases when $n = 5$ and $|F_1| = 5$ or $|F_1| = 6$. The case when $n = 5$ and $|F_1| = 5$ will be covered in Case 5. If $n = 5$ and $|F_1| = 6$ then, by Lemma 2.3, we can choose another decomposition via a different position so that $|F_1| \leq 5$.

Case 4.1. $H_1 - F_1$ *has no isolated vertices.* By the induction hypothesis, $H_1 - F_1$ contains a perfect or an almost-perfect matching M_1. We consider two cases depending on whether M_1 is a perfect or an almost-perfect matching.

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M_1 is a perfect matching. Since every $H_i - F_i$ contains no isolated vertices, then we can proceed as in Case 1 to find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - V(H_1)$. Then the set $M_1 \cup M'$ is a perfect or an almost-perfect matching in $AG_n - F$.

M_1 is an almost-perfect matching. Let $u \in V(H_1 - F_1)$ be the unsaturated vertex by M_1. If u has an outside neighbor u' in some $H_k - F_k$ where $H_k - (F_k \cup \{u'\})$ contains no isolated vertices, then we can also proceed as in Case 1 by finding a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup \{u'\})$, so $M_1 \cup M' \cup \{uu'\}$ is a perfect or an almost-perfect matching in $AG_n - F$. Assume that u has no such outside neighbor. Since $H_1 - F_1$ has no isolated vertices, then u is adjacent to some vertex v in $H_1 - F_1$ and v is saturated by M_1. Let $vw \in M_1$. The vertices u and w can have at most two common neighbors, so there are at least $4n - 14 - |F_1|$ vertices in $H_1 - F_1$ adjacent to either u or w. Let $\overline{F_1} = F - F_1$; we know that $|\overline{F_1}| + |F_1| = 4n - 12$, so there are at least $4n - 14 - (4n - 12 - |\overline{F_1}|) = |\overline{F_1}| + 2$ vertices in $H_1 - F_1$ adjacent to either u or w. Then we can guarantee that u or w are adjacent to at least $\lfloor \frac{|\overline{F_1}|+2}{2} \rfloor$ vertices x_k incident to edges $e_k = x_k y_k \in M_1$ for $k = 1, \ldots, \lfloor \frac{|\overline{F_1}|+2}{2} \rfloor$. We want to choose a vertex y_k such that y_k is not adjacent/incident to a fault outside H_1. Since each y_k has two outside neighbors, then we have at least $|\overline{F_1}| + 1$ choices for y_k; at most $|\overline{F_1}|$ can be at fault and no two outside neighbors can be in the same H_i. So we can find $y'_k \in H_k - F$, where $y_k y'_k \in E(AG_n - F)$, and satisfying the property $H_k - (F \cup \{y'_k\})$ has no isolated vertices. Thus we can use a similar construction to Case 1 to find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup V(H_2) \cup \{y'_k\})$. If $wx_k \in E(H_1 - F_1)$, then $M'_1 = M_1 \cup \{wx_k, uv\} - \{vw, x_k y_k\}$ is an almost-perfect matching in $H_1 - F_1$ missing y_k, then the set $M'_1 \cup M' \cup \{y_k y'_k\}$ is a perfect or an almost-perfect matching in $AG_n - F$. If $ux_k \in E(H_1 - F)$, then $M'_1 = M_1 \cup \{ux_k\} - \{x_k y_k\}$ is an almost-perfect matching in $H_1 - F$ missing y_k. Therefore, the set $M'_1 \cup M' \cup \{y_k y'_k\}$ is a perfect or an almost-perfect matching in $AG_n - F$.

Case 4.2. *$H_1 - F_1$ has an isolated vertex $u \in V(H_1 - F_1)$.* Let $N_{H_1}(u)$ be the set of neighbors of u in H_1.

Case 4.2.1. $N_{H_1}(u) \cap F_1^V \neq \emptyset$. Let $w \in N_{H_1}(u) \cap F_1^V$. We consider the parity of $|F_1^V|$.

Assume $|F_1^V|$ is odd. Let $F'_1 = F_1 - \{w\}$, so $|F'_1| \leq 4n - 17$ and $H_1 - F'_1$ has no isolated vertices, then by the induction hypothesis there exists a perfect matching M_1 in $H_1 - F'_1$. M_1 contains the edge uw. The vertex u must have an outside neighbor u' in $AG_n - F$, or else u becomes an isolated vertex in $AG_n - F$. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_i in $H_i - (F_i \cup \{u'\})$ for $i \geq 2$. Using the same idea as in Case 1, we can find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup \{u'\})$, then the set $(M_1 - \{uw\}) \cup \{uu'\} \cup M'$ is a perfect or an almost-perfect matching in $AG_n - F$.

Assume $|F_1^V|$ is even. Let $F'_1 = F_1 - \{w\}$, so $|F'_1| \leq 4n - 17$ and $H_1 - F'_1$ has no isolated vertices. There exists a vertex $x \in V(H_1 - F'_1)$ such that x has an outside neighbor $x' \in V(H_k - F_k)$ with $xx' \in E(AG_n - F)$ for some $k \neq 1$ and $H_1 - (F_1' \cup \{x\})$ has no isolated vertices. By the induction hypothesis, there exists a perfect matching M_1 in $H_1 - (F_1' \cup \{x\})$ that contains the edge uw, and there exists a perfect or an almost-perfect matching M_i in $H_i - (F_i \cup \{u', x'\})$ for $i \geq 2$. Using the same idea as in Case 1, we can find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup \{u', x'\})$, then the set $(M_1 - \{uw\}) \cup \{uu', xx'\} \cup M'$ is a perfect or an almost-perfect matching in $AG_n - F$.

Case 4.2.2. $N_{H_1}(u) \cap F_1^V = \emptyset$. Then u is incident to $(2n-6)$ faulty edges in H_1. Let f be one of those edges. We consider the parity of $|F_1^V|$.

Assume $|F_1^V|$ *is odd.* Let $F_1' = F_1 - f$, where f has endpoints u and v in $H_1 - F_1$. Consider the set $F_1'' = F_1' \cup \{u\}$. F_1'' has an even number of vertices, of size less than or equal to $4n - 16$ and $H_1 - F_1''$ has no isolated vertices. Then by the induction hypothesis, there exists a perfect matching M_1 in $H_1 - F_1''$. The vertex u must have an outside neighbor $u' \in V(H_k - F_k)$ for some $k \neq 1$ such that $uu' \in E(AG_n - F)$. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_i in $H_i - (F_i \cup \{u'\})$ for $i \geq 2$. We can assume that $|F_k \cup \{u'\}| \leq 2n - 7$, since if not then $|F_k| = 2n - 7$ and all the faults will be in the subgraph induced by $V(H_1) \cup V(H_k)$, so u can have another outside neighbor in $H_l - F_l$ for some $l \neq 1, k$. Then $|F_i \cup \{u'\}| \leq 2n - 7$ for $i \geq 2$ and hence we can apply the same idea as in Case 1 to construct a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup \{u'\})$. Then the set $M_1 \cup \{uu'\} \cup M'$ is a perfect or an almost-perfect matching in $AG_n - F$.

Assume $|F_1^V|$ *is even.* Let $F_1' = F_1 - f$. We can choose the edge $f = uw$ such that w has an outside neighbor $w' \in V(H_j - F_j)$ for some $j \neq 1$ with $ww' \in E(AG_n - F)$. The vertex u must have an outside neighbor $u' \in V(H_k - F_k)$ for some $k \neq 1$ such that $uu' \in E(AG_n - F)$. Note that there are $(2n-6)$ possible choices for f and there are at most $(2n-7)$ faults outside $H_1 - F_1$, so we can always choose w such that w has two outside neighbors in $AG_n - F$. Then we can assume that u' and w' are in different H_i, so we can claim that $|F_i \cup \{w', u'\}| \leq 2n - 7$. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_i in $H_i - (F_i \cup \{w', u'\})$ for $i \geq 2$. Using the same idea as in Case 1, we can find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup \{u', w'\})$. Therefore, the set $(M_1 - uw) \cup \{uu', ww'\} \cup M'$ is a perfect or an almost-perfect matching in $AG_n - F$.

Case 5. $|F_1| = 4n - 15$. Note that $H_1 - F_1$ can have at most one isolated vertex. We will assume that $H_1 - F_1$ has an isolated vertex $u \in V(H_1 - F_1)$. The case when $H_1 - F_1$ has no isolated vertices will be treated the same. Under this condition, there are three faults left outside H_1. Since $AG_n - F$ has no isolated vertices then there exists an outside neighbor $u' \in V(H_p - F_p)$ for some $p \neq 1$ with $uu' \in E(AG_n - F)$.

Case 5.1. $N_{H_1}(u) \cap F_1^V = \emptyset$. Then u is incident to $(2n-6)$ edges. Let $f = uv$ be a faulty edge in H_1 such that v has two outside neighbors in $AG_n - F$. Let $v' \in V(H_k - F)$, for some $k \neq 1, p$, be an outside neighbor of v. Since $|F_1 - f| = 4n - 16$ and

$H_1 - (F_1 - f)$ has no isolated vertices, then by the induction hypothesis there exists a perfect or an almost-perfect matching M_1 in $H_1 - (F_1 - f)$.

Assume that $f \notin M_1$. Then M_1 is a perfect or an almost-perfect matching in $H_1 - F_1$. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_i in $H_i - F_i$. We can combine these matchings to get a perfect or an almost-perfect matching M in $(AG_n - F)$.

Assume that $f \in M_1$. There are three faults outside H_1, we can claim that $|F_i \cup \{u'\}| \leq 3$ for every $i \geq 2$, because if not then u' is in the same H_i along with the other faults and this guarantees the existence of another outside neighbor of u in $AG_n - F$. Moreover, v has two outside neighbors in $AG_n - F$ so we can claim that $|F_i \cup \{u', v'\}| \leq 2n - 7$ for $i \geq 2$. Now as in Case 1 we can find a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup \{u', v'\})$. Therefore $M_1 \cup M' \cup \{uu', vv'\}$ is a perfect or an almost-perfect matching in $AG_n - F$.

Case 5.2. $N_{H_1}(u) \cap F_1^V \neq \emptyset$. Let $w \in N_{H_1}(u) \cap F_1^V$. By the induction hypothesis, there exists a perfect or an almost-perfect matching M_1 in $H_1 - (F_1 - \{w\})$. We consider the parity of $|F_1^V|$.

$|F_1^V|$ *is odd*. Then M_1 is a perfect matching containing the edge uw. As in the previous case, we can claim that $|F_i \cup \{u'\}| \leq 3$ for every $i \geq 2$, so by the induction hypothesis there exists a perfect or an almost-perfect matching M_i in $H_i - (F_i \cup \{u'\})$ for $i \geq 2$. We combine all the M_i's to get a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup \{u'\})$. Therefore $(M_1 - \{uw\}) \cup M' \cup \{uu'\}$ is a perfect or an almost-perfect matching in $AG_n - F$.

$|F_1^V|$ *is even*. Then M_1 is an almost-perfect matching in $H_1 - (F_1 - \{w\})$. Let $M_1' = M \cap E(H_1 - F)$. M_1' is a matching in $H_1 - F$ that misses two vertices, one of them is u and some other vertex x. Note that M_1' is an almost-perfect matching in the subgraph induced by $H_1 - (F_1 \cup \{u\})$, so we claim that x has outside neighbor(s) in $AG_n - F$, because if not then by Proposition 2.1 we can find another almost-perfect matching in $H_1 - (F_1 \cup \{u\})$ missing a vertex other than x. Let x' be the outside neighbor of x in $AG_n - F$. As in the previous case we can claim that $|F_i \cup \{x', u'\}| \leq 2n - 7$ for $i \geq 2$. By the induction hypothesis there exists a perfect or an almost-perfect matching M_i in $H_i - (F_i \cup \{x', u'\})$ for $i \geq 2$. We combine all the M_i's to get a perfect or an almost-perfect matching M' in the subgraph induced by $(AG_n - F) - (V(H_1) \cup \{x', u'\})$. Therefore $M_1' \cup M' \cup \{uu', xx'\}$ is a perfect or an almost-perfect matching in $AG_n - F$.

□

4 Conclusion

In this paper, we have studied the strong matching preclusion problem of the alternating group graph under the condition that no isolated vertex is created as a result of faulty edges and/or vertices. We proved that the conditional strong matching preclusion number of AG_n

is $4n - 11$. Classifying all types of optimal conditional strong matching preclusion sets of AG_n is left as a future research.

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