

PATH INTEGRALS AND THE SCALE ANOMALY

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the Faculty of the Department of Physics

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Chris Lin

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Chris Lin

APPROVED:

Dr. Carlos Ordóñez, Chairman
Dept. of Physics

Dr. Eric Bittner
Dept. of Chemistry

Dr. John Miller
Dept. of Physics

Dr. Arthur Weglein
Dept. of Physics

Dean, College of Natural Sciences and Mathematics

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Grace Yang drew Figure 2 of the thesis.

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Abstract

The theme of this thesis is scaling, units, and dimensional analysis, along with symmetry, and the failure of these concepts at the quantum level, along with implications of such a failure. Such a failure can be quantified by a quantity called the quantum anomaly. The term anomaly indicates deviation from expected behavior, where expected behavior is of course classical behavior, deviant behavior quantum behavior. In the introduction, we will give a short, simple, self-contained example explaining what is renormalization. This example will get to the heart of what we mean by the anomaly - the complete destruction of a system's symmetries due to quantum effects.

The applications of the anomaly are enormous, spanning several branches of physics, from atomic to condensed matter to particle to gravitational physics. For example, just within particle physics, the chiral anomaly is responsible for the decay rate of the Π^0 meson to two photons. The scale anomaly is responsible for the Yang-Mills mass gap in pure QCD and the formation of glueballs, and its calculation is an intermediate step in lattice QCD to calculate the QCD phase diagram - indeed, the anomaly is responsible for Λ_{QCD} itself, through dimensional transmutation.

However, one of the most exciting applications of anomalies has been realized only in this decade in the study of ultracold gases, where the measurement of various manifestations of the anomaly has only now become experimentally accessible to atomic physicists. In 2008, a set of universal thermodynamic

relations known as the Tan relations was published in a series of 3 back-to-back-to-back papers. In (2+1) dimensions, the Tan contact is merely the anomaly.

In this dissertation, we develop a novel framework for calculating anomalies using the path-integral and Fujikawa's determinant. In particular, we derive 4 results: the anomaly for a (3+1) relativistic Bose gas, the Tan-pressure relation for a (2+1) nonrelativistic Bose gas, a new derivation of the virial theorem, and the relationship between the Fujikawa determinant and the quantum effective potential using the background field method. Some unpublished results will also be discussed, and as how this all began, wildly speculative ideas end the dissertation.

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1 Outline

The introduction is multi-purpose. It provides a narrative, starting from Kepler in the 17th century and finding its way to the modern themes explored in the appendices. The introduction also serves as a guide, with information on where to go within the thesis for further information and details.

Since the thesis is about symmetry and the failure of symmetry due to quantum effects, we've included a section on symmetry and a section on quantum mechanics, sections 5 and 6, respectively. Of course, an exposition of quantum mechanics would fill an entire textbook, so section 6 is minimal. To make up for this, we provide a complete, self-contained pedagogical guide to symmetries and Noether's theorem, and the result, section 5, contains a lot more information than any single textbook. It should be mentioned that we use the term quantum mechanics and quantum field theory interchangeably: one can view quantum field theory as quantum mechanics applied to fields, or view quantum mechanics as quantum field theory in (0+1) dimensions.

The appendices represent papers already published, or are currently under review.

Section 7.1 is written at the undergraduate level, and comes from figuring out how to get undergraduates at this university to understand units. We will use it to explain setting $\hbar = c = 1$ in relativistic quantum mechanics and $\hbar = m = 1$ in nonrelativistic quantum mechanics, and with that information derive the Tan-

pressure relation. Although sections 5 and 7 are at an undergraduate level, we worked hard to be pedagogical, and hope that the resulting presentation exceeds anything that can be found out there, so that although the ideas in these sections are not novel, we hope the presentation can be.

2 Conventions

Many of these conventions will be explained in later sections where they are needed. However, they are all collected here for ease of reference. Most of these conventions are standard.

The notation used for the same function, but viewed in different coordinate systems, is standard in field theory, but evidently not in all of physics. Suppose you have a function $f(x, y) = x^2 + y^2$. If one writes this in polar coordinates as $f(r, \theta) = r^2$ then technically this is incorrect because f was defined as taking the first argument squared and adding it to the second argument squared, so that $f(r, \theta) = r^2 + \theta^2$. Therefore one needs to define a new function $g(x, y) = x^2$ that outputs the first argument squared and leaves the second alone, so that $g(r, \theta) = r^2$. However, it would be a shame to give an entirely different name to this second function, since it's obviously related to the first function. So we'll still call both functions f , but attach a superscript: $f^1 = f^1(x, y) = x^2 + y^2$ and $f^2 = f^2(x, y) = x^2$. By convention we call the first coordinate system the unprimed coordinate system, and the second coordinate system the primed coordinate system, so that $f(x, y) = x^2 + y^2$ and $f'(x, y) = x^2$. Of course context will tell us that the prime does not mean a derivative. The relationship between f and f' is that $f(x, y) = f'(x', y') = f'(r, \theta)$ or that $x^2 + y^2 = r^2$. $f'(r, \theta) = f(x, y)$, or more generally $f'(x', y') = f(x, y)$, can go with the mnemonic “new function at new coordinate equals old function at old coordinate,” which is true for functions that are “scalars” under the coordinate transformation. Note that (x, y)

and (r, θ) represent different coordinates, but correspond to the same physical point. As a concrete example, $(0, 1)$ in the Cartesian coordinate system refers to the same point as $(1, \frac{\pi}{2})$ in the polar coordinate system, but we will not write $(0, 1) = (1, \frac{\pi}{2})$ because the coordinates are not equal. When we take the difference of the two functions $\delta f = f' - f$, it will always be the difference at the same coordinate $\delta f = f' - f \equiv f'(x, y) - f(x, y)$, and not at the same point $\delta f = f' - f \neq f'(x', y') - f(x, y)$, where the latter is most often, but not always, zero (see mnemonic). This will be critical because in field theory, variations of fields are always done at the same coordinate, and in general for two different coordinate systems, the same coordinate represents different physical points. In general, language is used to speak about points, but mathematical manipulation and calculation considers coordinates. This can be replaced with the mnemonic “words use points, math uses coordinates.”

We use the Einstein convention where repeated indices are summed, e.g., $x^\mu x_\mu = \sum_{\mu=0}^D x^\mu x_\mu$, where D represents the number of spatial dimensions, and 0 represents the time dimension.

For a tensor with two indices, one upstairs and one downstairs, such as Θ_ν^μ , if it is not symmetric, then by default we will have $\Theta_\nu^\mu \equiv \Theta^\mu_\nu$. For tensors with more than two indices upstairs and downstairs, we will be more careful with the horizontal spacing of the indices.

The expression $x \rightarrow u$ acting on another expression simply means to replace all instances of x in that expression with u . This might be because $x = u$ so such a replacement is legitimate (though replacing \rightarrow with $=$ would be more clear in this instance), or it might be that $x \neq u$ and we just want to see how the expression transforms when we make the replacement anyways, which is convenient as it allows you to play with one side of an equation without constantly showing the required redefinitions of the other side to maintain the equality.

Calligraphic font will only be used for densities (although not all densities will be denoted by calligraphic fonts), so that \mathcal{L} is the Lagrangian density, \mathcal{E} is the energy density, \mathcal{A} is the anomaly density, and \mathcal{H} is the Hamiltonian density. Most of the time we'll leave off the word densities, so that we'll call \mathcal{L} the Lagrangian when strictly speaking it's the Lagrangian density.

In relativistic systems, we set $\hbar = c = k_B = 1$, and choose the signature of our metric $g^{\mu\nu}$ as $(+ - - -)$. We define $\epsilon_{0123} = 1$ so that $\epsilon^{0123} = -1$.

For nonrelativistic systems, we set $\hbar = m = k_B = 1$, $c = \infty$, and $\epsilon_{123} = \epsilon^{123} = 1$. A plane wave basis vector will have the form $\langle \vec{x} | \vec{k} \rangle = e^{i\vec{k} \cdot \vec{x}}$, so that $\langle \vec{k}' | \vec{k} \rangle = (2\pi)^D \delta^D(\vec{k}' - \vec{k})$, where D is the number of spatial dimensions. Since $|k\rangle$ is not normalized, the completeness relation is:

$$1 = \int d^D k \frac{|\vec{k}\rangle}{\sqrt{\langle \vec{k} | \vec{k} \rangle}} \frac{\langle \vec{k} |}{\sqrt{\langle \vec{k} | \vec{k} \rangle}} = \int d^D k \frac{|\vec{k}\rangle \langle \vec{k} |}{\langle \vec{k} | \vec{k} \rangle} = \int \frac{d^D k}{(2\pi)^D} |\vec{k}\rangle \langle \vec{k} | \equiv \int d^D \tilde{k} |\vec{k}\rangle \langle \vec{k} | \quad (1)$$

A coordinate x without any sub- or super-scripts indicates all the coordinates, so that $f(x) \equiv f(x^\mu) = f(x^0, x^1, \dots, x^D)$. We will use this convention when writing out all the indices only constitutes a distraction for the reader.

Under the simultaneous transformation with infinitesimal parameter ϵ :

$$\begin{aligned} x^\mu &\rightarrow x^\mu - \epsilon f^\mu(x^\eta) \\ \phi(x) &\rightarrow \phi(x) + \epsilon \delta\phi(x) \end{aligned} \quad (2)$$

the Noether current is taken to be:

$$j^\mu = \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right) - \mathcal{L} f^\mu \quad (3)$$

Note that this convention of the current has the opposite sign of the convention used for current algebra for Lie groups.

With this definition, upon quantization,

$$\begin{aligned} Q &\equiv \int d^D x j^0 \\ e^{iQ} \phi e^{-iQ} &= \phi + \delta\phi \end{aligned} \quad (4)$$

Strictly speaking, Q should be \hat{Q} and ϕ should be $\hat{\phi}$, to indicate these are now operators. However, unless there is any danger of confusion, we shall dispense

with the ornaments.

Partial differentiation is denoted by ∂ , while functional differentiation by δ , so that

$$\frac{\partial\phi(x)}{\partial\phi(y)} = \delta_{xy}^{D+1}, \quad \frac{\delta\phi(x)}{\delta\phi(y)} = \delta^{D+1}(x-y) \quad (5)$$

where δ_{xy}^{D+1} is Kronecker delta and $\delta^{D+1}(x-y)$ is Dirac delta.

3 Timeline

We initially set out to calculate the scale anomaly for the nonrelativistic $\delta^2(\vec{r})$ potential using path integrals with a technique borrowed from high-energy physics known as Fujikawa's method. Prior to our work, all attempts to use path integral techniques to solve for the anomaly in the context of nonrelativistic physics had resulted in failure. We had the choice of working in the 1st- or 2nd-quantized theory. The 1st-quantized theory is difficult because the interactions are nonpolynomial, and Fujikawa's method is at its core a field-theoretic technique, so it was not clear how to control the infinities for a nonpolynomial interaction, particularly for an interaction with the singular behavior of $\delta^2(\vec{r})$. So the 1st-quantized approach was abandoned. But even in the 2nd-quantized theory where the $\delta^2(\vec{r})$ potential becomes a polynomial interaction, it was not clear how to proceed. Indeed, even in the relativistic sector, there does not seem to exist a set procedure for deriving the anomaly using Fujikawa's method for non-quadratic interactions. The nonrelativistic sector is even worse: time and space are no longer symmetrical, and simultaneously controlling both the infinities in time and space added another layer of complexity absent in the relativistic sector. Ultimately these issues were overcome, resulting in the paper in section A.

At the same time, we were investigating the evaluation of the Jacobian for the nonrelativistic case, we developed a thermodynamic framework for the treatment of $SO(2, 1)$ anomalies using path integrals and scaling transformations [1], using the well-known relationship between quantum field theory and statistical mechan-

ics, and in particular the relationship between the path integral and the partition function. This clearly established the role of the Fujikawa Jacobian in describing anomalies for systems at finite temperature and density, generalizing the original work by Fujikawa at zero temperature and density. Shortly afterward, we exploited this framework to give a new general derivation of the virial theorem in the context of field theory, resulting in the paper in section B.

Although we borrowed Fujikawa's method from high-energy physics to solve problems in low-energy, we were able to give back: we took what we learned from applying Fujikawa's method to the nonrelativistic sector, in particular the importance of using a matrix-regulator, along with the thermodynamical relationship between field theory and statistical mechanics, and applied it towards solving high-energy problems, resulting in the papers of section C and D.

4 Introduction

In this introduction, we will give an overview of the various themes of this thesis, along with a road-map of where in the thesis to go for elaborations and more detailed information such as the mathematical articulation of these themes. We will also define much of the jargon through intuitive examples: a rigorous definition of the jargon will be reserved for other sections.

As mentioned in the abstract, the unifying theme of this thesis is scaling, units, and dimensional analysis, along with symmetry, and the failure of these concepts at the quantum level, along with the implications of such a failure. So let us start with classical scaling arguments.

Scaling arguments are used extensively in physics. As an example, consider a planet orbiting the sun. Newton's law reads:

$$-\left(\frac{GMm}{r^2}\right)\hat{e}_r = m\ddot{x} \quad (6)$$

Ignoring the constants, the scaling structure of (6) is of the form

$$\frac{1}{r^2} = \frac{d^2x}{dt^2} \quad (7)$$

If we make the replacement $x \rightarrow \ell x$ we get

$$\frac{1}{\ell^2 r^2} = \frac{\ell d^2x}{dt^2} \quad (8)$$

Now we can see that if we make the additional replacement $t \rightarrow \ell^{\frac{3}{2}}t$ in (8), we get back (7).

Therefore if we multiply the distance of the planet by ℓ , and the period by $\ell^{\frac{3}{2}}$, then this is also a solution of (7) and hence (6). This is Kepler's 3rd law, that $T \propto R^{\frac{3}{2}}$ or $T^2 \propto R^3$. Stated another way, if $x(t)$ is a solution of (6), then so is $\ell x(\ell^{-\frac{3}{2}}t)$.¹

We chose this example because all the transformations considered in this thesis will be a pair of simultaneous transformations of the form:

$$\begin{aligned} t &\rightarrow \ell^n t \\ f(t) &\rightarrow \ell^\xi f(\ell^{-n}t) \end{aligned} \tag{9}$$

This type of transformation involves scaling a coordinate t , and a function of this coordinate $f(t)$.² We will refer to these transformations as scale transformations or dilations (we will use these two terms interchangeably). The specific values of ℓ and ξ that we will use can be found in sections 5.5 and 5.6. Most of the time invariance of the equations of motion under a transformation leads to a conserved quantity: however, the transformation we gave for planetary motion does not lead to a conserved quantity via Noether's theorem, as demonstrated in comment three in section 5.4.3. Noether's theorem requires that the action remain invariant un-

¹Formally one can set $x(t) = \ell x'(t')$ and $t = \ell^{\frac{3}{2}}t'$ in (7), then $x'(t') = x(t)$ since the same equations have the same solution, so that $x(t) = \ell x'(t') = \ell x(t') = \ell x(\ell^{-\frac{3}{2}}t)$.

²In the example of the orbiting planet, $f(t) = x(t)$, $\eta = \frac{3}{2}$, and $\xi = 1$.

der the transformation (up to a surface term), which automatically ensures the equations of motion are invariant under the transformation, but the converse does not always hold true.

Scaling arguments are also used extensively in statistical mechanics. Consider a box of volume V filled with photons at temperature T . We wish to find the Helmholtz energy $F = F(T, V)$.³ The only variables in our theory are V , T , \hbar , c , and k_B , the latter three being the (reduced) Planck's constant, the speed of light, and Boltzmann's constant, respectively. T is called an intensive variable since it doesn't depend on the size of the system: if you double the size of the system, T remains the same. Both F and V are extensive variables: if you double the size of the system, F and V double. Therefore F must be proportional to V . Using dimensional analysis, the dependence of F on T and V must therefore be $F = n \left(\frac{k_B^4}{\hbar^3 c^3} \right) VT^4$. Dimensional analysis has given us everything except the pure number n . Using the thermodynamic relations $E = F + TS = F - T \frac{\partial F}{\partial T}$ and $P = -\frac{\partial F}{\partial V}$:

$$\begin{aligned} E - 3PV &= F - T \frac{\partial F}{\partial T} - 3 \left(-\frac{\partial F}{\partial V} \right) V \\ &= F - 4F + 3F = 0 \end{aligned} \tag{10}$$

We chose this example because, as we will see in section C, thermodynamically the anomaly for a relativistic system in 3+1 dimensions manifests itself as:

³Since photons are not conserved, $\mu = 0$, so by $\frac{\partial F}{\partial N} = \mu$, $F = F(T, V, N) = F(T, V)$.

$$\mathcal{A} = \frac{E - 3PV}{V} = \mathcal{E} - 3P \quad (11)$$

The anomaly for a nonrelativistic systems (section E) in 3+1 dimensions manifests itself as:

$$\mathcal{A} = \frac{2E - 3PV}{V} = 2\mathcal{E} - 3P \quad (12)$$

Therefore, based on the classical scaling arguments above, a box of photons has no anomaly. But is the zero value of the anomaly obtained from (10) for a box of photons under classical reasoning maintained under quantum scrutiny?

Many more such examples of classical scaling arguments could be told, and such arguments are useful for “back-of-the-envelope” calculations. Scaling arguments often work in physics because the problems considered only have a few scales. This is in contrast with architecture, where there are many scales, so that if one scale breaks, the structure still stands....

Well, so much for classical scaling arguments. We now turn to scaling arguments in the quantum theory.

We begin by investigating the consequences of scaling arguments on the energy spectrum of a system. Consider the Schrödinger equation for a free particle:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi \tag{13}$$

The goal is to find the energy E of the differential equation subject to the usual quantum constraints that are quite reasonable (finite solution at infinity, continuous and smooth). The only parameters of the theory are \hbar and m . However, there is just no possible way to form units of energy just out of \hbar and m . We can consider $\hbar^\ell m^\xi$, and try all possible values of ℓ and ξ , but we will never get a quantity with units of energy. This would be akin to trying to form units of mass out of length L and time T : $L^\ell T^\xi$, no matter what values ℓ and ξ are, will never get you a mass. In general, you need three independent units,⁴ from which all other units can be derived (in mechanics the standard choice is L , T , and M , and for example momentum is $M^1 L^1 T^{-1}$). Since we are unable to form units of energy out of the parameters of our system, we must therefore conclude that $E = 0$ or $E = \infty$, because these two numbers are special in that they are scale-invariant so require no units: zero centimeters equals zero meters equals zero light-years. And $\pm\infty$? - ditto. Actually, there is one more case: E could be a continuum between 0 and ∞ . Common sense says this is the case. So we have solved for the spectrum with dimensional analysis and common sense: $E = (0, \infty)$, where the units are MegaErgs, or whatever you please.

As another example, take the harmonic oscillator:

⁴See section (7.1) for a discussion of this.

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi \quad (14)$$

Solving for the spectrum requires knowledge of the classical orthogonal polynomials, or even more abstract, knowledge of ladder operators. Therefore, we will not solve it here. However, dimensional analysis says the only way to form units of energy out of \hbar , m , and ω , is through $\hbar^1 m^0 \omega^1 = \hbar\omega$. Therefore, the spectrum can be discrete with scale $\hbar\omega$. Indeed, the spectrum is $E_n = (n + 1/2)\hbar\omega$, $n = 0, 1, 2, \dots$

There is one more example before we get to a system with a quantum anomaly. Consider an infinite square-well with $V(x) = 0$ inside the well, and $V(x) = \infty$ outside the well. Our parameters are \hbar , m , and V . V already has units of energy - unfortunately, the values of V are 0 and ∞ , which can't provide a scale. Therefore we must conclude that the energy is 0, $\pm\infty$, or a continuum between those numbers. However, looking up the answer on Wikipedia, the spectrum is $E_n = \frac{\pi^2}{2} \left(\frac{\hbar^2}{mL^2} \right) n^2$. We had forgotten about the length of the well, L , which provides the scale in the problem.

A more systematic way of applying dimensional analysis to nonrelativistic quantum mechanics is discussed in section 7.2, but we will now turn our attention to a system with a quantum anomaly. But before we do, we should point out that 0 and ∞ are but two numbers on an uncountably infinite number line. But very small numbers and very large numbers have an approximate scale-invariance, so long as the scale transformations are reasonably small, e.g., $10^{\pm 10^{15}} \text{ cm} = 10^{\pm 10^{15}-5} \text{ km} \approx$

$10^{\pm 10^{15}}$ km. So a very large number like $10^{10^{15}}$ or a very small number like $10^{-10^{15}}$ needs no units, like their ∞ and 0 archetypes, so long as your scale transformations (e.g., cm \rightarrow km) are small. To illustrate this I will steal two quotations from a very good thermodynamics book [2]:

Ten percent or more of a complete stellar inventory consists of white dwarfs, just sitting there, radiating away the thermal (kinetic) energy of their carbon and oxygen nuclei from underneath very thin skins of hydrogen and helium. They will continue this uneventful course until the universe recontracts, their baryons decay, or they collapse to black holes by barrier penetration. (Likely time scales for these three outcomes are 10^{14} , 10^{33} and $10^{10^{76}}$ — years for the first two and for the third one it doesn't matter). Virginia Trimble, SLAC Beam Line **21**, 3 (fall, 1991).

It all works because Avogadro's number is closer to infinity than to ten. Ralph Baierlein, American Journal of Physics **46**, 1045 (1978).

Now onto the quantum anomaly and an explanation of renormalization. Consider the following system:

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi - \frac{\hbar^2 \lambda}{2m} \delta^2(\vec{r}) \psi = E\psi \quad (15)$$

Physically the system represents a two-dimensional, infinitely deep (but infinitesimally small) potential-well. Before we solve this, let us see what dimensional analysis has to say. The parameters in the theory are \hbar , m , and λ . However, λ has no dimensions: it is a pure number.⁵ Therefore, we conclude that the energy is $E = 0$ or $E = \pm\infty$, or a continuum in between those numbers. In particular, we are looking for a bound state, i.e., a particle trapped in the well, like an orbiting comet trapped in the sun's gravity, going in circles around it, never escaping. Therefore we conclude $E_b = 0$, as $E_b = -\infty$ would indicate that particle would emit infinite energy as it collapses to the center.

However, this is not the end of the story with this system. This system has been solved in many different ways by many different people [3, 4, 5, 6]. We will follow the procedure in [6], where we replace the potential $V(r) = -\frac{\hbar^2\lambda}{2m}\delta^2(\vec{r})$ with $V_R(r) = -\frac{\hbar^2\lambda}{2m}\frac{\theta(a-r)}{\pi a^2}$. V_R is more general than V , constituting a continuum of circular-well potentials parametrized by a such that:⁶

$$\lim_{a \rightarrow 0} V_R(r) = V(r) \tag{16}$$

The solution for the bound state energy with the potential V_R is:

$$E_b = \frac{-2\hbar^2 e^{-2\gamma}}{ma^2} e^{-\frac{4\pi}{\lambda}} \tag{17}$$

where $\gamma = .577\dots$ is the Euler-Mascheroni constant.

⁵ $\delta^2(\vec{r})$ has dimensions of L^{-2} , since $\int d^2\vec{r} \delta^2(\vec{r}) f(\vec{r}) = f(\vec{0})$.

⁶ $\int d^2\vec{r} \left(\frac{\theta(a-r)}{\pi a^2}\right) = 1$, and $\theta(a-r) = 0$ for $r > a$: this is $\delta^2(\vec{r})$.

The most immediate question is how were we able to form units of energy? That's simple: the introduction of the scale a by the replacement of V with V_R . The second question is more tricky: don't we have to send $a \rightarrow 0$ to describe the $\delta^2(r)$ potential? When $a \rightarrow 0$, (17) blows up, and we get $E_b = -\infty$, as expected by classical scaling argumentation, but absurd by physical argumentation (a bottomless pit to which you can rig a machine to extract infinite energy).

The theorist shrugs and says that the system can therefore not be physically realized in nature. The experimentalist goes out and measures E_b and gets the value $E_b = -7.45 \mu\text{eV}$ for a trapped hydrogen atom in that potential.

The theorist surmises that the experimentalist is joking that he's so good at experiment, that a is so tiny that it's pretty much a $\delta^2(\vec{r})$ potential - but there is no possible way he could have created a true $\delta^2(\vec{r})$ potential, or else the energy would be much higher, infinite if he truly achieved $\delta^2(\vec{r})$. Therefore the experimentalist must have created a well that was very deep and narrow, but not at the level of $\delta^2(\vec{r})$. And it is for this well that the value $E_b = -7.45 \mu\text{eV}$ was measured. The theorist then demands that the experimentalist tell him what is the size of a so that using (17) and the quoted value of E_b , he can calculate λ . However, the experimentalist insists that he created a genuine $\delta^2(\vec{r})$ potential so that there is no a (or equivalently, $a = 0$).

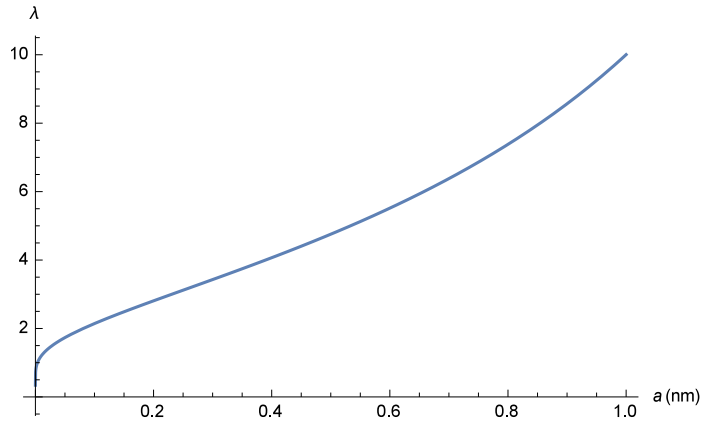


Figure 1: λ vs a for $E_b = -7.45 \mu\text{eV}$

The theorist ignores the experimentalist and guesses $a = 1$ nm, and using (17) with $E_b = -7.45 \mu\text{eV}$,⁷ arrives at $\lambda = 10$. However, he remembers the experimentalist scored some extra grant money to add juice to his machine, so maybe $a = .5$ nm, an even lower value, so that $\lambda = 4.76$. The theorist then decides to just do this for every a , and plots $\lambda = \lambda(a)$ such that $E_b = -7.45 \mu\text{eV}$, resulting in figure 1.

The theorist is now stuck since he doesn't know what the value of a is, so he sneaks into the experimentalist's lab to conduct some scattering experiments in the potential-well. Scattering happens when one shoots atoms at the potential with so much energy, that the atoms are only deflected by the well, and not trapped by it - like a comet moving so fast its trajectory is merely deflected by a large gravitational body, instead of being completely trapped in orbit around it. However, since the theorist is a novice as experiment, he is only able to shoot

⁷One can write $E_b = \frac{-2\hbar^2 e^{-2\gamma}}{ma^2} e^{-\frac{4\pi}{\lambda}} = \frac{-2(\hbar c)^2 e^{-2\gamma}}{(mc^2)a^2} e^{-\frac{4\pi}{\lambda}}$, use $\hbar c = 0.19732697 \text{ eV} \cdot \mu\text{m}$ and $m_H c^2 = 938.272046 \text{ MeV}$, and plug in E_b to get $\lambda = \lambda(a)$.

hydrogen atoms with low momentum. The probability that a particle of initial momentum \vec{q} scatters under V_R is given by:

$$f_q = \sqrt{\frac{2\pi}{q}} \left[\ln \left(\frac{\frac{q^2}{2m}}{\frac{2\hbar^2 e^{-2\gamma}}{ma^2} e^{\frac{-4\pi}{\lambda(a)}}} \right) - i\pi + O \left[\frac{qa}{\hbar} \right] \right]^{-1} \quad (18)$$

The theorist thinks now he's got the experimentalist: all he has to do is plug in every value of $(a, \lambda(a))$ from his graph in figure 1 into (18), and see which pair of values matches the scattering data f_q that the theorist secretly recorded. To the theorist's horror, every value reproduces the scattering data.

On closer inspection, the theorist notices that the denominator of the logarithm in (18) is exactly the expression for E_b in (17), and the graph of pairs of points $(a, \lambda(a))$ was determined precisely to give this value of E_b . The theorist is not skilled enough at experiment to observe $O \left[\frac{qa}{\hbar} \right]$ corrections.

Therefore, really (18) can be written as:

$$\begin{aligned} f_q &= \sqrt{\frac{2\pi}{q}} \left[\ln \left(\frac{\frac{q^2}{2m}}{|E_b|} \right) - i\pi + O \left[\frac{qa}{\hbar} \right] \right]^{-1} \\ &= \sqrt{\frac{2\pi}{q}} \left[\ln \left(\frac{\frac{q^2}{2m}}{|E_b|} \right) - i\pi + O \left[\frac{a}{\lambda_q} \right] \right]^{-1} \end{aligned} \quad (19)$$

where λ_q is the de Broglie wavelength of the hydrogen atom. The theorist pauses to muse over his fate. There is no possible way, given his limited experimental skills — which we call his ignorance — to figure out what λ and a are. They can be any point along trajectory that is figure 1. So to the theorist, a and λ are no

longer physical, observable values. They are parameters in his theory that have no physical meaning. E_b and f_q are the only physical values, and a and λ were only used as an intermediate step to relate the two physical quantities. If only the theorist were skilled enough to get the de Broglie wavelength λ_q of his hydrogen projectiles on the order of a nanometer could he discover the true physics beneath it all, since the $O\left[\frac{a}{\lambda_q}\right]$ term can select a pair $(a, \lambda(a))$. As it stands, the theorist can forget about figuring out if the experimentalist was pulling his leg about achieving a $\delta^2(\vec{r})$ potential - the theorist can't even tell if it's closer to 1 nm or .5 nm. The theorist is forced to admit that the experimentalist could be correct, that he achieved $a = 0$ and $E_b = -7.45 \mu\text{eV}$, the point at the origin of figure 1. The theorist decides to call it a day: he was unable to one-up the experimentalist, and besides, The Big Bang Theory television show is about to begin.

The process of choosing a model V_R is called regularization. The replacement of the unphysical parameters $(a, \lambda(a))$ that a theorist uses by physically measurable quantities (E_b, f_q) that an experimentalist measures is called renormalization. The transformation that takes $(a_1, \lambda(a_1))$ to $(a_2, \lambda(a_2))$ along the trajectory of figure 1 is called a renormalization group (RG) transformation, and the fact that a whole collection of points $(a_i, \lambda(a_i))$ gives the same physics is called invariance under the RG group. The fact that E_b is a physical scale in the problem that was nowhere to be found in (15) is called dimensional transmutation. Dimensional transmutation is responsible for the scale anomaly, the breaking of dilational symmetry. To see this, consider making the scaling $x \rightarrow \ell x$ in (15):

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial \ell^2 x^2} + \frac{\partial^2}{\partial \ell^2 y^2} \right) \psi - \frac{\hbar^2 \lambda}{2m} \delta^2(\ell \vec{r}) \psi = E \psi \quad (20)$$

It can be shown⁸ that $\delta^2(\ell \vec{r}) = \frac{1}{\ell^2} \delta^2(\vec{r})$. Therefore in order for (20) to give the same equation as (15), it must be true that $E = \frac{E}{\ell^2}$ (so that all the ℓ 's cancel on both sides), and if $E = \frac{E}{\ell^2}$ is true for any ℓ , $E = 0$ or $E \pm \infty$, i.e., E is invariant under change of scale. The existence of $E = E_b \neq 0$ on the RHS of (20) as a scale through dimensional transmutation breaks invariance under dilations. Indeed, the anomaly \mathcal{A} , a measure of the degree of symmetry-breaking, satisfies $\int d^2 \vec{x} \mathcal{A} = E_b$: this will be derived in section 6.1.2.

Let's pause to quickly go over the meaning of introducing a into the problem and having $\lambda = \lambda(a)$, for eventually it is a (dimensions of length) that got transmuted to E_b (dimensions of energy) which broke our symmetry, so that understanding the meaning of a will shed light into how E_b ultimately came about. The theorist's intuition was correct that the experimentalist could not possibly measure physics at $a = 0$. The smallest distances in physics are probed by the Large Hadron Collider (LHC), and while they can keep pushing towards understanding physics at the smallest of scales, they'll never reach $a = 0$ (at the time of this writing they are at $a = 1$ nanoångström). So physics below $a = 1$ nanoångström is unknown. So it makes perfectly good sense to choose a value of $(a, \lambda(a))$ such that a is just outside of the range of known physics for the area you're studying ($[0, a]$ is the region of our ignorance, and we say that a parametrizes our ignorance). It pays to

⁸ $\int d^2 \vec{r} \delta^2(\ell \vec{r}) f(\vec{r}) = \frac{1}{\ell^2} \int d^2(\ell^2 \vec{r}) \delta^2(\ell \vec{r}) f\left(\frac{\ell \vec{r}}{\ell}\right) = \frac{1}{\ell^2} \int d^2 \vec{r} \delta^2(\vec{r}) f\left(\frac{\vec{r}}{\ell}\right) = \frac{1}{\ell^2} f(0)$.

be conservative in this case, for the larger value of a we use, the smaller q needs to be before $O[qa]$ ⁹ terms grow and we admit our models need fixing. The growth of $O[qa]$ terms indicates that new physics is emerging, invalidating the choice of V_R with a certain a as an accurate model applicable to that scale: the model predicts its own demise through the increasing relevance of discarded $O[qa]$ terms. The flip side is that any V_R , at low enough q , will predict the same results, as then $O[qa]$ become too small to measure, and results only depend on E_b : the entire RG trajectory gives the same physics. As a consequence of this, our model can be completely wrong regarding the behavior at high q (indeed, different a have different $O[qa]$ terms: the exact value of a matters when we probe to this level of accuracy, and different a give different corrections), but it still predicts the correct results at low q when $O[qa]$ is undetectable: in other words, phenomena at the smallest of scales a , like quantum gravity, have no effect that we can observe if we can only probe the smallest of q . This is both a bane and a blessing: it ensures that what we don't know ($a < 1$ nanoångström) has no effect on what we know ($a > 1$ nanoångström), allowing us for example to understand classical mechanics without having to understand quantum mechanics, but at the same time this means the only way to probe what we don't know is through high q and, ultimately given the costs, we may never be able to probe beyond a certain point and there is no other way to learn about new physics: we would be stuck with a set of $(a, \lambda(a))$. In any case, speculation of the relationship between a and E_b for this specific $\delta^2(\vec{r})$ potential in terms of physics at unknown scales will be

⁹From this point on we write $O[\frac{qa}{\hbar}]$ as just $O[qa]$: for more information on this see section 7.2.

reserved for section 10. It should be noted that it would be incorrect to say that quantum gravity has no effect on our measurements at low q . It has no effect through $O[qa]$ terms due to the smallness of a (compared to our capabilities of producing high enough q), but it contributes to E_b , which is only accessible by experiment. We say that quantum gravity can only renormalize the couplings of our low-energy interactions, but can't provide an entirely different interaction because these interactions are suppressed by $O[qa]$. So when we measure the electric charge, that includes not only the electron interacting with its photon field, but also quantum gravitational effects: all these things together lead to the charge of the electron quoted in textbooks, and we are unable to disentangle how much of the charge is contributed by the photon and how much is due to quantum gravity, as the renormalized charge is determined by experiment where quantum gravity and electromagnetism are always on.¹⁰

Note this discussion about the meaning of renormalization is not something of only philosophical interest, but even has some utility. Evidently the $\delta^2(\vec{r})$ potential models any attractive potential when the de Broglie wavelength $\lambda_q = \frac{\hbar}{q}$ is large compared to the range a of the potential, because choosing smaller values of a has no effect on measurable quantities, since they contribute at order $O[\frac{qa}{\hbar}] = \frac{a}{\lambda_q}$. Or expressed another way, the de Broglie wavelength λ_q is too large to resolve the structure of a small range interaction a . The $\delta^2(\vec{r})$ potential is an example of a contact interaction, and all systems where the range of interaction

¹⁰Unlike the a of $\delta^2(\vec{r})$, for gravity we actually know the length λ_q such that $\frac{\lambda_q}{L_g} \approx 1$: this is $L_g = L_{\text{Planck}} = \sqrt{\frac{\hbar G}{c^3}} = 1.6162 \cdot 10^{-26}$ nm, where the scale G is Newton's gravitational constant.

a is small compared to all the other scales in the systems behave similarly - this is the essence of the Tan relations.

According to Noether's theorem, in classical physics, continuous symmetries of the action imply a conservation law (see section 5.4.2 for four different proofs of her theorem). The same thing essentially happens in quantum physics, except that the conservation law becomes an operator equation since the current is now an operator. However, due to the quantum anomaly, the symmetry is broken, and we would expect that the charge is no longer conserved. This is indeed correct. We show in section 6.2.3 that the modified conservation equation is:

$$\partial_\mu \langle 0 | j^\mu(x) | 0 \rangle = -i \left\langle 0 \left| \text{tr} \frac{\delta \delta \phi_k(x)}{\delta \phi_\ell(y)} \Big|_{y=x} \right| 0 \right\rangle \quad (21)$$

The RHS is called the anomaly. When the anomaly is zero, (21) is called the Ward identity, which is the quantum version of Noether's conservation equation.

Eqn. (21) for anomalies is derived from the path integral approach, which is discussed in section 6.2. Within the path-integral approach, the anomaly is simply a Jacobian, so that all anomalies reduce to the calculation of Jacobians. We calculate the Jacobian for a nonrelativistic (2+1) Bose gas in section A, for chargeless scalar electrodynamics in section C, and for a set of $O(N)$ scalar fields in section D.

The reason the conservation laws are modified by a Jacobian results from the fact that in quantum mechanics, we are interested in integrating the action $\int [d\phi(x)] e^{iS[\phi(x)]} =$

1, whereas in classical physics, we are interested in differentiating the action $\frac{\delta S[\phi(x)]}{\delta \phi(x)} = 0$. Symmetry transformations are effected through a change of variables, and although the action remains constant, the measure picks up a Jacobian:

$$\begin{aligned} \int [d\phi(x)] e^{iS[\phi(x)]} &= \int [d\phi'(x)] \left| \frac{\delta \phi(x)}{\delta \phi'(y)} \right| e^{iS[\phi'(x)]+\dots} \\ &= \int [d\phi'(x)] e^{i(S[\phi'(x)]-i \ln J)+\dots} \\ &= \int [d\phi(x)] e^{i(S[\phi(x)]-i \ln J)+\dots} \end{aligned} \tag{22}$$

where in the last line we relabeled the integration variables. Therefore the action is effectively no longer invariant, but rather $S[\phi(x)] \rightarrow S[\phi(x)] - i \ln J$. If the Jacobian $J = 1$, then $S[\phi(x)] \rightarrow S[\phi(x)]$ just as in the classical case, and the action effectively remains invariant. This gives the idea that the quantum effective action $\Gamma[\phi(x)]$ might contain information about the anomaly, since it already includes quantum effects. The quantum effective action is constructed so that the classical equations of motion $\frac{\delta \Gamma[\phi(x)]}{\delta \phi(x)} = 0$ gives the quantum result. This is discussed in section D.

By setting $J = 1$, we can also recover non-anomalous results. We do this in section B to derive the standard non-anomalous virial theorem and in section E to derive the Tan-pressure relation in 3 spatial dimensions where there is no anomaly.

5 Transformations and Symmetries

In this section, we carefully define what we mean by points, coordinates, transformations, functions, and symmetry. This will be the only section where we are pedantic about such things. In later sections, we will speak more loosely. Indeed, everything in this section is obvious by intuition - the only difficulty is in establishing notation where clarity is sometimes traded for convenience. Of course mathematicians are very pedantic about such things and if the notation in this section is confusing, one is free to adopt the notation one uses in the study of differential topology, where a clear distinction between points in a manifold and the mapping of the points to coordinate charts is established at the outset. The culmination of this section is several derivations of Noether's theorem, and the application of her theorem to systems with dilational symmetry.

Along the way, we'll provide plenty of concrete examples to use in the formulas, to further elucidate the notation. We'll also provide catchy mnemonic phrases to remember the formulas.

5.1 Coordinate Transformations

First, it's necessary to distinguish between points and coordinates. A point is a physical location such as the Kemah Boardwalk. A coordinate is a label for this point such as $(29.544, -95.022)$. Such labels depend on the coordinate system chosen (the example given above corresponds to a coordinate system whose origin is at the intersection of the equator and the prime meridian, with the positive di-

rections chosen to be north and east - this is the most common coordinate system in use, but it is not the only one), but to all observers with their varied range of coordinates, the Kemah Boardwalk is the Kemah Boardwalk.

With this definition in mind, lay out a coordinate axis x^μ to label points in your space. Then consider the mapping:

$$x'^\mu = x'^\mu(x^\nu) \tag{23}$$

This mapping can be interpreted in two different ways, both physically equivalent. Right now there is only one coordinate system and since x' and x are different coordinates, they correspond to different points in that one coordinate system. But we can also introduce a second coordinate system such that the (physical) point corresponding to x' in the second coordinate system is the same as the (physical) point described by x in the first coordinate system. The first interpretation (that there is only a single coordinate system) corresponds to physically moving objects from the point whose coordinate is x to the point whose coordinate is x' while keeping the observer still, while the second interpretation (introducing a second coordinate system) corresponds to keeping all objects still (i.e. keeping the same points) and moving the observer such that the coordinates of the objects change from x to x' . As far as the observer is concerned, there is no difference between these two scenarios, as the observer maintains his or her relative distance to the objects: e.g., moving all objects to the right is equivalent to moving the observer to the left.

We will call the mapping in (23) a coordinate transformation, since in both interpretations the coordinates change. Whether the points change (1st interpretation) or the observer changes (2nd interpretation) is immaterial, and we will freely make use of both interpretations.

Within the context of classical physics, the differences in interpretations has no perceptible effect on how one calculates. But in quantum mechanics, these two interpretations look different. Physically moving a point corresponds to physically changing a wavefunction $U|\psi\rangle$, whereas moving the observer corresponds to a change in the operator $U^\dagger O U$. Note that $|\psi\rangle$ is analogous to a point and not a coordinate. One can expand $|\psi\rangle$ as $\int dx |x\rangle \langle x|\psi\rangle = \int dx' |x'\rangle \langle x'|\psi\rangle$ of the two different coordinates, but $|\psi\rangle$ itself is independent of coordinate system, and hence is akin to a point.

5.2 Changes in Functions Induced by Coordinate Transformations

Suppose strewn across the lawn are hot coals, and $T = T(x, y)$ gives the temperature as a function of the coordinates in the coordinate system $\angle xy$. Consider the coordinate transformation

$$\begin{aligned} x' &= x'(x, y) \\ y' &= y'(x, y) \end{aligned} \tag{24}$$

This will induce a transformation on the function $T \rightarrow T'$. The goal is to find an expression for T' in terms of T .

In the first interpretation in section 5.1, the observer stays put but the coals move. Therefore, the temperature field changes from T to T' . The new temperature T' at the new point (x', y') is equal to the old temperature T at the old point (x, y) , i.e., $T'(x', y') = T(x, y)$. This is because the coal initially at (x, y) created the temperature $T(x, y)$, but this coal was then moved to (x', y') creating a temperature $T'(x', y') = T(x, y)$.

In the second interpretation in section 5.1, the coals stay put but the observer moves. Therefore the temperature T at the point corresponding to (x, y) is the same before and after the move. However, the moved observer uses the coordinate description (x', y') and T' , so that once again $T'(x', y') = T(x, y)$.

So both interpretations give the same result, as they must. We will now never mention both interpretations again. Instead, we will freely use whichever one feels most natural for what we're trying to do.

$$\begin{aligned}
 T'(x', y') &= T(x, y) \\
 T'(x', y') &= T((x', y')^{-1}) \\
 T'(x, y) &= T((x, y)^{-1})
 \end{aligned}
 \tag{25}$$

This is our main result. Under the coordinate change (24), the fields of physi-

cal quantities like temperature change too. The top line of (25) can be replaced with the mnemonic “new field at new coordinate equals old field at old coordinate.” Note that the bottom two lines are making comparisons of T and T' at the same coordinates, but in T the argument is inverted. This can be replaced with the mnemonic “coordinates go one way, arguments of functions go the other way.”

Let’s try a specific, concrete example. Consider a coordinate system in the $\angle xy$ plane. Now consider the coordinate transformation:

$$\begin{aligned}x' &= 2x \\y' &= 2y\end{aligned}\tag{26}$$

This corresponds to shrinking the observer by $1/2$ so that everything seems twice as big to him or her, i.e. compressing his or her coordinate axis.¹¹

The temperature field in the new coordinate system satisfies $T'(x', y') = T(x, y)$, so that the temperature of the primed coordinate system at the coordinate $(2,2)$ equals the temperature of the old coordinate system at $(1,1)$. From the third line of (25), $T'(x, y) = T\left(\frac{x}{2}, \frac{y}{2}\right)$, so that once again $T'(2, 2) = T\left(\frac{2}{2}, \frac{2}{2}\right) = T(1, 1)$.

However, suppose instead of the field $T(x, y)$, we have the field $H(x, y)$, which gives the height of a building on campus as a function of its location (x, y) . Then it stands to reason that the buildings themselves will get taller when viewed by a

¹¹In the first interpretation this would correspond to doubling the size of everything.

shrunk observer, so that

$$\begin{aligned} H'(x', y') &= 2H(x, y) \\ H'(x, y) &= 2H\left(\frac{x}{2}, \frac{y}{2}\right) \end{aligned} \tag{27}$$

Therefore, if $H(1, 1) = 100$ ft, then $H'(2, 2) = 2H(1, 1) = 200$ ft. If instead of $H(x, y)$, you had a function $V(x, y)$ which measured the volume of a puddle of water of unit radius centered at (x, y) , then under the dilation of (26), $V'(x', y') = 2^3V(x, y)$. In general, for dilations,

$$f'(x', y') = 2^{[f]_S} f(x, y) \tag{28}$$

where $[f]_S$ gives you the scaling dimensions of the field $f(x, y)$.¹²

The most general transformation considered in this thesis will be one on a set of coordinates x^μ and an induced linear transformation on a set of fields $\phi_i(x^\nu)$

$$\begin{aligned} x'^\mu &= x'^\mu(x^\nu) \\ \phi'_i(x') &= R_{ij}\phi_j(x) \\ \phi'_i(x) &= R_{ij}\phi_j(x^{-1}) \end{aligned} \tag{29}$$

This encompasses internal transformations where the coordinates don't change, in which case $x' = x$ and $x^{-1} = x$. As an example of (29), consider transformation under the Lorentz group, with $x'^\mu = \Lambda^\mu_\nu x^\nu$. Then $R_{ij} = R_{ij}(\Lambda)$, where $R_{ij}(\Lambda)$ is a

¹²The scaling dimension of f , denoted by $[f]_S$, can differ from its true dimension, denoted by $[f]$, if f contains dimensional parameters.

representation of the Lorentz group.

5.3 Symmetries

5.3.1 Symmetries of Functions

Using the notation in sections 5.1 and 5.2, the symmetry of a function $f(x)$ under transformation is expressed by its invariance under the transformation, i.e.,

$$f'(x) = f(x) \tag{30}$$

This is merely the intuitive statement that different observers, with different orientations, see the same thing relative to their orientations.

Let's take a concrete example. Suppose the temperature profile is $T(x, y) = x$, and consider rotating the coordinate system 90 degrees clockwise. Then that observer describes the temperature profile as $T'(x, y) = y$. According to (30), the temperature field is not symmetric. That is, if x is viewed as east and y is viewed as north, then the first observer sees no temperature variation to his north and the only temperature variations are to his east, while the second observer sees no temperature variations to her east and only temperature variations to her north. However, if $T(x, y) = x^2 + y^2$, then $T'(x, y) = x^2 + y^2$, and both observers see the same thing relative to their own orientations.

Let's take another concrete example, one that illustrates how clarity in jargon

can be sacrificed for convenience. Suppose the height of a building located at x is $H(x) = x$, i.e., buildings at ∞ are really tall and buildings at $-\infty$ are really subterranean. Now consider the transformation:

$$\begin{aligned}x' &= \lambda x \\H'(x') &= \lambda H(x) \\H'(x) &= \lambda H\left(\frac{x}{\lambda}\right)\end{aligned}\tag{31}$$

As explained in section 5.2, the primed coordinate system is shrunk by λ , so that what the unprimed coordinate system calls $x = 1$, the primed coordinate system sees as $x' = \lambda$. Moreover, if the height of a building as seen by the unshrunk observer is $H = 1$, then the height as seen by the shrunk observer is $H = \lambda$. Plugging in $H(x) = x$ into (31), one gets $H'(x) = x = H(x)$. Hence $H(x)$ is invariant by (30) under the coordinate transformation (31). So $H(x) = x$ is a weird function in that if you shrink the observer (but leave points alone, i.e., the environment), the shrunk observer cannot tell he was shrunk - he perceives everything the same relative to his surroundings as the unshrunk observer perceives relative to hers (this assumes buildings can only be distinguished by their height - if each building had a different design then the shrunk observer would notice that something is amiss). Note that all this is intuitively obvious - if you plot the graph $(x, H(x))$ for $H(x) = x$, and then scale the x and y axis by the same number say 200%, then you get the same graph which you can overlay on and completely cover over the original graph.¹³

¹³One would think that $H(x) = cx$ is the only function with the property $\lambda H\left(\frac{x}{\lambda}\right) = H(x)$. Just set $\lambda = x$ to get $xH(1) = H(x)$ and set $H(1) = c$ to get $H(x) = cx$. However, it turns out

However, this example is an example of inconsistent use of jargon, because it is often said that the function $H(x)$ is invariant under dilations $x \rightarrow \lambda x$ if it scales as $\frac{1}{\lambda}$. That is, we ignore that H itself has length, and only consider the behavior of H due to the coordinate change in its argument. Using this terminology, if a function scales as the negative of its dimension, then the function is symmetric under dilation.

Also, one has to be careful. One can rewrite the transformation (31) as

$$\begin{aligned} x' &= \frac{1}{\lambda}x \\ H'(x') &= \frac{1}{\lambda}H(x) \\ H'(x) &= \frac{1}{\lambda}H(\lambda x) \end{aligned} \tag{32}$$

Then using this terminology, if a function scales as its dimension (and not the negative of its dimension), then the function is symmetric under dilation. As an example, for the inverse-square potential $V(x) = \frac{1}{x^2}$, $V(\lambda x) = \frac{1}{\lambda^2} \frac{1}{x^2}$, and the potential has length dimensions of -2 (see section 7.2), so by (32) the inverse-square potential $V(x)$ is symmetric under dilations, and we say that it scales as $\frac{1}{\lambda^2}$. In the rest of this thesis, this will be the sense in which we use the term scale-invariance, i.e., when we change a function only through a rescaling of its arguments, if the function scales as its dimension, then we say that it is scale-invariant.

$H(x) = \frac{c}{\delta(x)}$ also satisfies this property.

One last possible point of confusion that's related to the above example. As mentioned in section 5.2, "coordinates go one way, arguments of functions go the other way." To use that mnemonic, one mustn't forget where x shows up in. For example, if you are talking about the function $H(x) = x$, then under the transformation (31), the x in H is to be replaced by $\frac{x}{\lambda}$, not λx , since the x is a function. However, if you are considering the transformation (32), then in $H(x) = x$, the x is to be replaced by λx . The same can be said about the replacement of x in $V(x) = \frac{1}{x^2}$. One can never get confused if one always writes both sides of $H(x) = x$ under the transformation (31) and always writes $H(x)$ instead of H : then clearly $H\left(\frac{x}{\lambda}\right) = \frac{x}{\lambda}$, whereas $H \rightarrow \frac{x}{\lambda}$ might give you some pause as you figure out if H is another coordinate, or a function, and whether they're using the transformations (31) or (32).

While on the subject of dilations, one can consider non-isotropic ones:

$$\begin{aligned}
 x' &= \lambda_H x \\
 f'(x') &= \lambda_V f(x) \\
 f'(x) &= \lambda_V f\left(\frac{x}{\lambda_H}\right)
 \end{aligned} \tag{33}$$

This scales a graph $(x, f(x))$ by stretching it by a factor of λ_H in the horizontal direction, and λ_V in the vertical direction. However, all dilations in this thesis will be isotropic in space. For relativistic dilations, it will also be isotropic in spacetime. For nonrelativistic dilations, we will stretch time and space differently, but they will be linked by the relation $\lambda_t = \lambda_x^2 = \lambda_y^2 = \lambda_z^2$.

As one can see, all these concepts are intuitive, but the jargon can cause confusion for those unfamiliar with it, since the jargon can be context-dependent. Unfortunately, the jargon has been so ingrained in the author that in the rest of the thesis he'll adhere to it, and hopes that at the very least sections 5.1, 5.2, 5.3.1, and 5.3.2 were spared from misleading statements due to unconscious adherence to the jargon.

Lastly, suppose that f in (30) only changes under a symmetry transformation via its argument. Then the statement that f is symmetric under the transformation can also be written as:

$$f(x') = f(x) \tag{34}$$

which intuitively states that the function is symmetric if it has the same value at all points related to each other by the symmetry transformation. This follows from (30) and the assumption that f only changes through its arguments.

5.3.2 Symmetries of the Action

The action $S = S[\phi_i, V, T] = \int_{VT} dx \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))$ is a functional of the fields $\phi_i(x)$ and a region of spacetime (V, T) . We will not consider actions that explicitly depend on the coordinates - the only space-time dependence will be through the fields $\phi_i(x)$.

Then from (34) the statement that the action is symmetric under the transforma-

tion

$$\begin{aligned}
x'^{\mu} &= x'^{\mu}(x^{\nu}) \\
\phi'_i(x') &= R_{ij}\phi_j(x) \\
\phi'_i(x) &= R_{ij}\phi_j(x^{-1})
\end{aligned} \tag{35}$$

is that

$$\begin{aligned}
S[\phi', V', T'] &= S[\phi, V, T] \\
\int_{V'T'} dx' \mathcal{L}(\phi'_i(x'), \partial'_\mu \phi'_i(x')) &= \int_{VT} dx \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))
\end{aligned} \tag{36}$$

It should be noted that the primes on the spacetime coordinates in the integrand on the LHS are dummy variables, and can be replaced by unprimed variables. However, we write them as primed to be able to better see this next step: on the LHS, pull back from (V', T') to (V, T) by making the substitution of variables $x'^{\mu} = x'^{\mu}(x^{\nu})$:

$$\begin{aligned}
S[\phi', V', T'] &= S[\phi, V, T] \\
\int_{VT} \left| \frac{\partial x'}{\partial x} \right| dx \mathcal{L}(R_{ij}\phi_j(x), \left(\frac{\partial x^\nu}{\partial x'^{\mu}} \right) \partial_\nu R_{ij}\phi_j(x)) &= \int_{VT} dx \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \\
\left| \frac{\partial x'}{\partial x} \right| \mathcal{L}(R_{ij}\phi_j(x), \left(\frac{\partial x^\nu}{\partial x'^{\mu}} \right) \partial_\nu R_{ij}\phi_j(x)) &= \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))
\end{aligned} \tag{37}$$

In equating the integrands, we made the assumption that the action is invariant under the symmetry transformation for any spacetime volume. Note that $\left(\frac{\partial x^\nu}{\partial x'^{\mu}} \right)$ can be calculated from knowing $x'^{\mu}(x^{\nu})$ and using $\left(\frac{\partial x^\nu}{\partial x'^{\mu}} \right) = \left(\frac{\partial x'^{\mu}}{\partial x^\nu} \right)^{-1}$. We have

succeeded in getting a condition on \mathcal{L} that determines symmetry rather than a condition on the action S . Note that (37) is intuitive - the Jacobian $J = \left| \frac{\partial x'}{\partial x} \right|$ is included because it is S , and not \mathcal{L} , that is required for the system to be invariant under symmetry transformation. The condition on \mathcal{L} indicated by (37) can be remembered by the mnemonic “Ignoring spacetime arguments, Lagrangian of new field and new derivative times Jacobian equals Lagrangian of old field and old derivative,” i.e., we ignore changes in the spacetime arguments of the fields, which are instead taken into account by J , and only compare changes in the discrete indices of the field and the discrete indices of its derivative. Let’s test (37) out on some concrete examples. In these examples, we will only consider the case where $R_{ij} \neq R_{ij}(x)$, thereby excluding gauge transformations.¹⁴ And of course, as always, we restrict ourselves to \mathcal{L} that do not depend explicitly on coordinates.

Translation

$x'^{\mu} = x^{\mu} + a^{\mu}$, so the Jacobian $J = 1$, $R_{ij} = \delta_{ij}$, and $\left(\frac{\partial x^{\nu}}{\partial x'^{\mu}} \right) = \delta_{\mu}^{\nu}$, so by (37), S is invariant.

Rotation

Assume $(R^{\dagger}R)_{ij} = \delta_{ij}$ and that the fields only appear in \mathcal{L} through the combination $\phi_i^* \phi_i$ or $\partial_{\mu} \phi_i^* \partial^{\mu} \phi_i$. For rotations $x'^{\mu} = \Lambda_{\nu}^{\mu} x^{\nu}$ where $\Lambda_{\eta}^{\mu} \Lambda_{\nu}^{\eta} = \delta_{\nu}^{\mu}$, which also implies $J = 1$. Then by either $(R^{\dagger}R)_{ij} = \delta_{ij}$ or $(R^{\dagger} \Lambda \Lambda^T R)_{ij\mu}^{\nu} = \delta_{ij} \delta_{\mu}^{\nu}$, according to (37), S is invariant.

Dilation

¹⁴For conservation laws, taking the symmetry to be global is sufficient.

Consider the Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g}{4!}\phi^4$ along with the dilation $x'^\mu = \lambda x^\mu$ and $\phi'(x') = \frac{1}{\lambda}\phi(x)$. This time let's use the mnemonic "Ignoring space-time arguments, Lagrangian of new field and new derivative times Jacobian equals Lagrangian of old field and old derivative." Then $\left(\frac{1}{2}\frac{\partial}{\partial(\lambda x^\mu)}\frac{\phi}{\lambda}\frac{\partial}{\partial(\lambda x_\mu)}\frac{\phi}{\lambda} - \frac{g}{4!}\left(\frac{\phi}{\lambda}\right)^4\right)\lambda^4 \stackrel{?}{=} \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g}{4!}\phi^4$? $\checkmark\checkmark\checkmark\checkmark$

Note that in the last example, \mathcal{L} has scaling dimension -4 , S has scaling dimension 0 , and \mathcal{H} has scaling dimension -1 : any one of these statements is a sign of scaling symmetry in the relativistic theory. More about this in section 5.6.

The statements made in this section are generalizable to transformations of the form:

$$\begin{aligned}x'^\mu &= x'^\mu(x^\nu) \\ \phi'_i(x') &= f_i(\phi_j(x), x^\mu) \\ \phi'_i(x) &= f_i(\phi_j(x^{-1}), x^{-1\mu})\end{aligned}\tag{38}$$

and with general Lagrangian $\mathcal{L} = \mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x), x^\mu)$. Then invariance of the action S implies that \mathcal{L} has the property

$$\left|\frac{\partial x'}{\partial x}\right| \mathcal{L}(f_i(\phi_j(x), x^\mu), \left(\frac{\partial x^\nu}{\partial x'^\mu}\right) \partial_\nu f_i(\phi_j(x), x^\mu), x'^\mu(x^\nu)) = \mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x), x^\mu)\tag{39}$$

We'll only need (38) in the context of 1st-quantized theories for transformations where $x'^\mu = x^\mu$ and \mathcal{L} doesn't depend on x^μ explicitly, in which case (39) becomes:

$$\mathcal{L}(f_i(\phi_j(x), x^\mu), \left(\frac{\partial x^\nu}{\partial x'^\mu}\right) \partial_\nu f_i(\phi_j(x), x^\mu)) = \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \quad (40)$$

In general, Noether's theorem requires that the action be either symmetric, or change at most by a surface term, under the symmetry transformation. Cases where the action changes by a surface term are sufficiently rare (occurring mainly in 1st-quantized theories) that we will not try to formulate Noether's theorem in the most general manner to include them: rather, we'll consider these rare cases separately. For now, let us just note that allowing the action to differ by a surface term implies that the condition for symmetry for case (40) can be extended to:

$$\mathcal{L}(f_i(\phi_j(x), x^\mu), \left(\frac{\partial x^\nu}{\partial x'^\mu}\right) \partial_\nu f_i(\phi_j(x), x^\mu)) = \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) + \partial_\mu \mathcal{K}^\mu \quad (41)$$

since the integral of a divergence is a surface term.

5.4 Noether's Theorem

We acknowledge the importance of Noether's theorem by offering several proofs of it. Noether's theorem states that any continuous symmetry in a system describable by an action principle has a conserved current. Moreover (and more importantly), Noether's theorem offers a formula for this conserved current.

5.4.1 Infinitesimal Transformations

The transformations we are considering are

$$\begin{aligned}
x'^{\mu} &= x'^{\mu}(x^{\nu}) \\
\phi'_i(x') &= R_{ij}\phi_j(x) \\
\phi'_i(x) &= R_{ij}\phi_j(x^{-1})
\end{aligned} \tag{42}$$

We consider the infinitesimal transformation $x'^{\mu} = x^{\mu} - \rho f^{\mu}(x^{\nu})$ and $R_{ij} = \delta_{ij} + \rho r_{ij}$ parametrized by the infinitesimal parameter ρ , so that (42) becomes:

$$\begin{aligned}
x'^{\mu} &= x^{\mu} - \rho f^{\mu}(x^{\nu}) \\
\phi'_i(x') &= \phi_i(x) + \rho r_{ij}\phi_j(x) \\
\phi'_i(x) &= \phi_i(x) + \rho r_{ij}\phi_j(x) + \rho f^{\nu}\partial_{\nu}\phi_i(x)
\end{aligned} \tag{43}$$

where we have thrown away $O[\rho^2]$ terms. We define $\rho\delta\phi_i(x) = \phi'_i(x) - \phi_i(x)$, the difference of the two fields at the same coordinate, so that:

$$\begin{aligned}
\delta\phi_i(x) &= r_{ij}\phi_j(x) + f^{\nu}\partial_{\nu}\phi_i(x) \\
\delta\partial_{\mu}\phi_i(x) &= \partial_{\mu}\delta\phi_i(x) = r_{ij}\partial_{\mu}\phi_j(x) + \partial_{\mu}(f^{\nu}\partial_{\nu}\phi_i(x))
\end{aligned} \tag{44}$$

Note that the ∂_{μ} commutes with δ , since δ is defined as the difference at the same coordinate.

5.4.2 Noether's Theorem

Proof 1

The first two proofs we will give of Noether's theorem doesn't involve the action S , but the Lagrangian \mathcal{L} . Suppose that, without using the equations of motion,

that under (44):

$$\frac{\partial \mathcal{L}}{\partial \phi_i(x)} \delta \phi_i(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \partial_\mu \phi_i(x) = \partial_\mu \mathcal{K}^\mu \quad (45)$$

Then, using the equations of motion:

$$\begin{aligned} \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \phi_i(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \partial_\mu \phi_i(x) &= \partial_\mu \mathcal{K}^\mu \\ \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \phi_i(x) - \mathcal{K}^\mu \right) &= 0 \end{aligned} \quad (46)$$

so that the current is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \phi_i(x) - \mathcal{K}^\mu \quad (47)$$

As an example, consider $\mathcal{L} = \frac{m}{2} \dot{x}^2 - V(x)$ with $f^0 = 1$, $r = 0$. Then $\rho \delta x(t) = \rho \dot{x}$, and without using the equations of motion:

$$\frac{\partial \mathcal{L}}{\partial x(t)} \delta x(t) + \frac{\partial \mathcal{L}}{\partial \dot{x}(t)} \delta \dot{x}(t) = -V' \dot{x} + m \dot{x} \ddot{x} = \frac{d\mathcal{L}}{dt} \quad (48)$$

So \mathcal{K}^0 has been identified with \mathcal{L} , and the conserved current by (47) is:

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{x}(t)} \delta x(t) - \mathcal{K}^0 = m \dot{x}^2 - \mathcal{L} = \frac{m}{2} \dot{x}^2 + V(x) \quad (49)$$

However, though simple, this doesn't identify \mathcal{K}^μ , which we had to calculate: it is only by considering S that we can get $\mathcal{K}^\mu = \mathcal{L} f^\mu$. (45) says that if under the symmetry transformation \mathcal{L} is invariant, or at most the divergence of a $(D + 1)$ vector, then there is a conserved current.

Proof 2

The second proof is essentially the first proof, except we'll insert a step before step (46):

$$\begin{aligned}
& \frac{\partial \mathcal{L}}{\partial \phi_i(x)} \delta \phi_i(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \partial_\mu \phi_i(x) = \partial_\mu \mathcal{K}^\mu \\
& \frac{\partial \mathcal{L}}{\partial \phi_i(x)} \delta \phi_i(x) - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \phi \right) = \partial_\mu \mathcal{K}^\mu \\
& - \left(\frac{\partial \mathcal{L}}{\partial \phi_i(x)} \delta \phi_i(x) - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \phi \right) = \partial_\mu j^\mu
\end{aligned} \tag{50}$$

where j^μ is as defined in (47).

We have shown that if $\delta \phi_i(x)$ is a symmetry transformation, then without using the equations of motion:

$$\begin{aligned}
& - \left(\text{Equation of Motion for } \phi_i(x) \right) \delta \phi_i(x) = \partial_\mu j^\mu \\
& - \left(\frac{\partial \mathcal{L}}{\partial \phi_i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \right) \delta \phi_i(x) = \partial_\mu j^\mu \\
& - \frac{\delta S}{\delta \phi_i(x)} \delta \phi_i(x) = \partial_\mu j^\mu
\end{aligned} \tag{51}$$

Then it's obvious that for fields that do obey the equations of motion, the LHS is zero, so that $0 = \partial_\mu j^\mu$. In (51), the left hand sides are all the same.

Let's try a concrete example, with $\mathcal{L} = \frac{m}{2} \dot{x}^2 - V(x)$, $\delta x(t) = \dot{x}(t)$, which has $j^0 = Q = \frac{m}{2} \dot{x}^2 + V(x)$:

$$-(-V'(x) - m\ddot{x})\dot{x} \stackrel{?}{=} \frac{d}{dt} \left(\frac{m}{2}\dot{x}^2 + V(x) \right) \quad (52)$$

(52) is indeed true, so that when we do use the equation of motion $-V'(x) - m\ddot{x} = 0$, Q on the RHS is conserved. This gives us confidence in the correctness of (51).

The formula $-\frac{\delta S}{\delta\phi_i(x)}\delta\phi_i(x) = \partial_\mu j^\mu$ from (51), since it does not use the classical equations of motion, will be useful in quantum physics, since $j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i(x)}\delta\phi_i(x) - \mathcal{K}^\mu$ is still the expression for the current in quantum physics. We will exploit this in section 6.2.3 to derive the Ward identities.

Proof 3

This proof utilizes the action, thereby capturing $\mathcal{K}^\mu = \mathcal{L}f^\mu$ for coordinate transformations. All the hard work was done in section 5.3.2. In that section, from the symmetry of the action, we derived the corresponding condition on \mathcal{L} :

$$\left| \frac{\partial x'}{\partial x} \right| \mathcal{L}(R_{ij}\phi_j(x), \left(\frac{\partial x^\nu}{\partial x'^\mu} \right) \partial_\nu R_{ij}\phi_j(x)) = \mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x)) \quad (53)$$

Plugging in (42), (43) and (44) gives to order ρ :

$$\begin{aligned} & \left| \frac{\partial x'}{\partial x} \right| \mathcal{L} \left(\phi_i(x) + \rho\delta\phi_i(x) - \rho f^\nu \partial_\nu\phi_i(x), \right. \\ & \left. \partial_\mu [\phi_i(x) + \rho\delta\phi_i(x) - \rho f^\nu \partial_\nu\phi_i(x)] + \rho\partial_\mu f^\nu \partial_\nu\phi_i(x) \right) = \mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x)) \quad (54) \end{aligned}$$

Since $|\frac{\partial x'}{\partial x}| = |\delta_\nu^\mu - \rho \partial_\nu f^\mu| = 1 - \rho \partial_\nu f^\nu + O[\rho^2]$, this equation to $O[\rho]$ becomes:

$$\begin{aligned} \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) - \rho \partial_\nu f^\nu \mathcal{L} + \rho \frac{\partial \mathcal{L}}{\partial \phi_i(x)} \left(\delta \phi_i(x) - f^\nu \partial_\nu \phi_i(x) \right) \\ + \rho \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \left(\partial_\mu \delta \phi_i(x) - f^\nu \partial_\nu \partial_\mu \phi_i(x) \right) = \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \end{aligned} \quad (55)$$

Since the x^μ dependence of \mathcal{L} is only through the fields, this becomes:

$$-\rho (\mathcal{L} f^\mu) + \rho \left(\frac{\partial \mathcal{L}}{\partial \phi_i(x)} \delta \phi_i(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \partial_\mu \phi_i(x) \right) = 0 \quad (56)$$

which is the same as (45) with $\mathcal{K}^\mu = \mathcal{L} f^\mu$. We have shown that, without using the equations of motion, a symmetry of the action implies (56), so following the rest of the steps after (45) in the first proof will complete this proof.

Proof 4

In this proof we make the infinitesimal parameter ρ spacetime dependent, $\rho = \rho(x)$.

$$\begin{aligned} \delta S &= \int_{VT} dx \left(\frac{\partial \mathcal{L}}{\partial \phi_i(x)} \rho(x) \delta \phi_i(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \partial_\mu (\rho(x) \phi_i(x)) \right) \\ &= \int_{VT} dx \rho(x) \left(\frac{\partial \mathcal{L}}{\partial \phi_i(x)} \delta \phi_i(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \partial_\mu \phi_i(x) \right) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \phi_i(x) \partial_\mu \rho(x) \end{aligned} \quad (57)$$

Using (45), this becomes

$$\delta S = \int_{VT} dx \rho(x) \partial_\mu \mathcal{K}^\mu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \phi_i(x) \partial_\mu \rho(x) \quad (58)$$

and integrating the second term by parts we get:

$$\begin{aligned} \delta S &= \int_{VT} dx \rho(x) \partial_\mu \left(\mathcal{K}^\mu - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \phi_i(x) \right) \\ &= - \int_{VT} dx \rho(x) \partial_\mu j^\mu(x) \end{aligned} \quad (59)$$

where (47) was used. The classical trajectory has the property $\delta S = 0$ for any $\delta \phi_i(x)$, and setting $\rho(x) = \delta^d(x - x')$ gives $\partial_\mu j^\mu(x') = 0$ for any point x' .

5.4.3 Comments

1) We first demonstrate Noether's theorem for Galilean boosts in a 1st-quantized theory, as it is a rare example of the action changing by a surface term, and leads to a conserved quantity that depends explicitly on time, a feature that the conserved quantity for dilational symmetry shares. The conserved quantity, the initial position of the center of mass of the system, is still conserved in time however.

We take

$$\mathcal{L} = \sum_i \frac{m_i}{2} \dot{x}_i^2 - \frac{1}{2} \sum_{i \neq j} V_{ij}(x_i - x_j) \quad (60)$$

An infinitesimal boost ν is given by:

$$\begin{aligned}
t' &= t \\
x'_i(t') &= x_i(t) - \nu t \\
x'_i(t) &= x_i(t) - \nu t
\end{aligned} \tag{61}$$

The potential is invariant under the boost, so

$$\delta\mathcal{L} = \sum_i m_i \dot{x}_i \delta\dot{x}_i = -\nu \frac{d}{dt} \sum_i m_i x_i \tag{62}$$

Therefore by (45) we identify $\mathcal{K}^0 = -\sum_i m_i x_i$ and the conserved charge is given by (47) as:

$$\begin{aligned}
Q &= -\sum_i m_i \dot{x}_i t + \sum_i m_i x_i \\
&= M \left(-\dot{X}_{\text{com}} t + X_{\text{com}} \right)
\end{aligned} \tag{63}$$

where $M = \sum_i m_i$. Since the center of mass is a free particle, it obeys the equation $X_{\text{com}}(t) = X_{\text{com}}(0) + \dot{X}_{\text{com}} t$, so that plugging into (63)

$$Q = M X_{\text{com}}(0) \tag{64}$$

The initial position of the center of mass is the same for all times t , so indeed Q given in (63), though explicitly dependent on time, is conserved for all times.

Integrating (62) shows that the action is not invariant but changes by a surface term:

$$\delta S = \int dt \delta \mathcal{L} = -\nu \sum_i m_i x_i \Big|_{t_1}^{t_2} = -\nu M \left(X_{\text{com}}(t_2) - X_{\text{com}}(t_1) \right) \quad (65)$$

2) We wish to emphasize that as far as visualization is concerned, there is significant difference between symmetry of the action versus symmetry of a function. The latter involves moving objects around and seeing if the resulting configuration is distinguishable from the initial configuration. The former involves moving objects around and seeing if the integral of the configuration (with new boundaries) is distinguishable from the integral of the initial configuration. In other words, it is not the configuration of objects that must be symmetrical for Noether's theorem to apply, but that the laws of physics are symmetrical, and by the laws of physics being symmetrical we mean that that the action changes by at most a surface term under the symmetry transformation. For example, if you translate your entire system to the right, then unless the distribution of your particles is homogeneous throughout space, then the new configuration of particles looks different: they are farther to the right. However, the action does not change.

3) As mentioned in comment **2)**, it is not enough that the transformed system obeys the same equations of motion for their to be a conserved quantity. The action must be the same, or differ by at most a surface term. For example, in the example of planetary motion in the introduction, we found that the transformation of Eqn. (9) with $\eta = \frac{3}{2}$ and $\xi = 1$ has the same equations of motion as the initial system. However, there is no conserved quantity corresponding to that transformation as the transformation doesn't preserve the action or changes it

at most by a surface term: the transformation multiplies the action by $\ell^{-1/2}$, so that the change in action is proportional to the action itself. If $S[x', T'] = S[x, T] + F(T)$, where $F(T)$ is a surface term independent of the trajectory $x'(t)$, then $x'(t)$ on the LHS is minimized by whatever trajectory $x_c(t)$ minimizes S on the RHS, so that $x'(t') = \ell x_c(t) \Rightarrow x'(t) = \ell x_c(\ell^{-\frac{3}{2}}t)$. Therefore $x'(t) = \ell x_c(\ell^{-\frac{3}{2}}t)$ minimizes $S[x', T']$, which represents the action for a trajectory whose initial and final points along with length of time are symmetry-transformed from their original values.¹⁵ The $f(T)$ doesn't matter in determining $x'(t)$ since $x_c(t)$ minimizes both $S[x, T]$ and $S[x, T] + F(T)$. But that relationship between actions does not hold for the planetary system which has instead: $S[x', T'] = \ell^{-1/2}S[x, T]$. Clearly $x_c(t)$, which minimizes the RHS, when symmetry-transformed, minimizes the LHS, so the transformed coordinate system sees the same equations of motion as the untransformed coordinate system. But the action isn't invariant up to a surface term, so Noether's theorem does not apply.

5.5 Nonrelativistic Dilations

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\vec{x}}^2 - V(\vec{x}) \quad (66)$$

and assume $V(\lambda\vec{x}) = \frac{1}{\lambda^2}V(\vec{x})$. Examples of such potentials are $V(\vec{x}) = \frac{1}{|\vec{x}|^2}$ and $V(\vec{x}) = \delta^2(\vec{x})$. Using the terminology just below Eqn. (32), we say that the

¹⁵In $S[x', T'] = S[x, T] + F(T)$, $x'(t)$ and $x(t)$ are put in the same action S , i.e., it is $S[x', T']$ and not $S'[x', T']$, so the equations of motion are the same.

potential scales as its dimension. Then under the transformation:

$$\begin{aligned}
t' &= e^{-2\rho t} = \lambda^2 t \\
x'(t') &= e^{-\rho} x(t) = \lambda x(t) \\
x'(t) &= e^{-\rho} x(e^{2\rho} t) = \lambda x(\lambda^{-2} t)
\end{aligned} \tag{67}$$

the action corresponding to (66) is invariant. To see this we will use the mnemonic just below Eqn. (37) to determine the condition for the action to be symmetric: “Ignoring spacetime arguments, Lagrangian of new field and new derivative times Jacobian equals Lagrangian of old field and old derivative.” Then

$$\begin{aligned}
\left(\frac{1}{2} \frac{d(\lambda \vec{x})}{d(\lambda^2 t)} \cdot \frac{d(\lambda \vec{x})}{d(\lambda^2 t)} - V(\lambda \vec{x}) \right) \lambda^2 &\stackrel{?}{=} \frac{1}{2} \dot{\vec{x}}^2 - V(\vec{x}) \\
\left(\frac{1}{2} \frac{d(\lambda \vec{x})}{d(\lambda^2 t)} \cdot \frac{d(\lambda \vec{x})}{d(\lambda^2 t)} - \frac{1}{\lambda^2} V(\vec{x}) \right) \lambda^2 &\stackrel{?}{=} \frac{1}{2} \dot{\vec{x}}^2 - V(\vec{x}) \checkmark
\end{aligned} \tag{68}$$

where we must remember that \vec{x} is a field and the spacetime argument is just the time argument t . So the conserved current is given by (47) as:

$$\begin{aligned}
Q = j^0 &= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} \delta \phi_i(x) - \mathcal{K}^0 \\
&= \dot{\vec{x}} \cdot \left(-\vec{x}(t) + 2t \vec{\dot{x}}(t) \right) - 2t \mathcal{L} \\
&= -\vec{p} \cdot \vec{x} + 2t \mathcal{H}
\end{aligned} \tag{69}$$

where $\mathcal{K}^0 = \mathcal{L} f^0$ was derived in (56), f^0 and $\delta x(t)$ were determined by setting ρ infinitesimal in (67), and $\mathcal{H} = \frac{1}{2} \dot{\vec{x}}^2 + V(\vec{x})$ is the Hamiltonian of the system.

The fact that Q is conserved can easily be tested for a free particle ($V(\lambda \vec{x}) =$

$\frac{1}{\lambda^2}V(\vec{x}) = 0$), where the equation of motion is $\vec{x}(t) = \vec{x}(0) + \dot{\vec{x}}t$, so that plugging this into (69) gives $Q = -\vec{p} \cdot \vec{x}(0)$. \vec{p} is conserved for all t for a free-particle, and $\vec{x}(0)$ is a constant (the initial position is always $\vec{x}(0)$ for all t).

Now consider the Lagrangian

$$\mathcal{L} = \psi^\dagger(x)(i\partial_t)\psi(x) - \frac{1}{2}\partial_i\psi^\dagger(x)\partial_i\psi(x) - \int d^Dy \psi^\dagger(x)\psi(x)V(x-y)\psi^\dagger(y)\psi(y) \quad (70)$$

under the transformation

$$\begin{aligned} x_i &= e^{-\rho}x_i = \lambda x_i \\ t' &= e^{-2\rho}t = \lambda^2 t \\ \psi'(x') &= e^{\frac{D}{2}\rho}\psi(x) = \lambda^{-\frac{D}{2}}\psi(x) \\ \psi'(x) &= e^{\frac{D}{2}\rho}\psi(e^\rho x_i, e^{2\rho}t) = \lambda^{-\frac{D}{2}}\psi(\lambda^{-1}x_i, \lambda^{-2}t) \\ \psi'^\dagger(x') &= e^{\frac{D}{2}\rho}\psi^\dagger(x) = \lambda^{-\frac{D}{2}}\psi^\dagger(x) \\ \psi'^\dagger(x) &= e^{\frac{D}{2}\rho}\psi^\dagger(e^\rho x_i, e^{2\rho}t) = \lambda^{-\frac{D}{2}}\psi^\dagger(\lambda^{-1}x_i, \lambda^{-2}t) \end{aligned} \quad (71)$$

The Lagrangian in (70) depends explicitly on coordinate via $V(x-y)$, so we will have to use (39) to determine whether it is invariant under the transformation of (71):

$$\begin{aligned}
& \left(\frac{\psi^\dagger(x)}{\lambda^{\frac{D}{2}}} \left(i \frac{\partial}{\partial(\lambda^2 t)} \right) \frac{\psi(x)}{\lambda^{\frac{D}{2}}} - \frac{1}{2} \frac{\partial}{\partial(\lambda x_i)} \frac{\psi^\dagger(x)}{\lambda^{\frac{D}{2}}} \frac{\partial}{\partial(\lambda x_i)} \frac{\psi(x)}{\lambda^{\frac{D}{2}}} - \right. \\
& \quad \left. \int d^D(\lambda y) \frac{\psi^\dagger(x)}{\lambda^{\frac{D}{2}}} \frac{\psi(x)}{\lambda^{\frac{D}{2}}} V(\lambda(x-y)) \frac{\psi^\dagger(y)}{\lambda^{\frac{D}{2}}} \frac{\psi(y)}{\lambda^{\frac{D}{2}}} \right) \lambda^{D+2} \\
& \stackrel{?}{=} \psi^\dagger(x) (i \partial_t) \psi(x) - \frac{1}{2} \partial_i \psi^\dagger(x) \partial_i \psi(x) - \int d^D y \psi^\dagger(x) \psi(x) V(x-y) \psi^\dagger(y) \psi(y)
\end{aligned} \tag{72}$$

which is true if and only if $V(\lambda(x-y)) = \frac{1}{\lambda^2} V(x-y)$. Assuming this is true of the potential, then the conserved charge, following the same procedure as in the 1st-quantized case of (69) is:

$$\begin{aligned}
j^0 &= i \psi^\dagger \left[\frac{D}{2} \psi + x_i \partial_i \psi \right] + 2t \mathcal{H} \\
\mathcal{H} &= \frac{1}{2} \partial_i \psi^\dagger(x) \partial_i \psi(x) + \int d^D y \psi^\dagger(x) \psi(x) V(x-y) \psi^\dagger(y) \psi(y)
\end{aligned} \tag{73}$$

Since complex conjugation commutes with differentiation ∂_t , we define a Hermitian charge $j^0 \rightarrow \frac{j^0 + j^{0\dagger}}{2}$, or alternatively we repeat the procedure using a Hermitian Lagrangian $\mathcal{L} \rightarrow \frac{\mathcal{L} + \mathcal{L}^\dagger}{2}$ instead of (70), to get:

$$\begin{aligned}
j^0 &= -x_i \left(\frac{i \partial_i \psi^\dagger \psi - i \psi^\dagger \partial_i \psi}{2} \right) + 2t \mathcal{H} \\
&= -x_i \mathcal{P}_i + 2t \mathcal{H}
\end{aligned} \tag{74}$$

where \mathcal{P}_i , the probability density current in quantum mechanics, is also the momentum corresponding to symmetry under spatial translations (one can check the sign by substituting $\psi = e^{ip_i x_i}$ into \mathcal{P}_i).

5.6 Relativistic Dilations

We could apply the methods we used for nonrelativistic dilations to relativistic ones. However, we will give a different method of determining whether a relativistic system is scale invariant. Unlike the nonrelativistic case, most relativistic systems do have scale invariance at the classical level, so long as we can ignore the mass of the particle, so a more general approach would be welcome. What we will prove is:

Theorem: If in a system of units where a length scale, \hbar , and c are our fundamental units (see section 7.3 for a review of choosing these units), then if all coupling constants have no length dimension, the theory is dilationally invariant, at least classically.

The theorem is intuitively obvious, and what follows is the simplest we could come up with for a proof. But first let's define the relativistic dilation:

$$\begin{aligned}
 x'^{\mu} &= e^{-\rho} x^{\mu} = \lambda x^{\mu} \Rightarrow \delta x^{\mu} = x^{\mu} \\
 \phi'_i(x') &= e^{-[\phi_i]\rho} \phi_i(x) \\
 \phi'_i(x) &= e^{-[\phi_i]\rho} \phi_i(e^{\rho} x) \Rightarrow \delta \phi_i(x) = -[\phi_i] \phi_i(x) + x^{\nu} \partial_{\nu} \phi_i(x) \\
 \partial_{\mu} \delta \phi_i(x) &= (-[\phi_i] + 1) \partial_{\mu} \phi_i(x) + x^{\nu} \partial_{\nu} \partial_{\mu} \phi_i(x)
 \end{aligned} \tag{75}$$

where $[\phi_i]$ gives the length dimension of the field ϕ_i in our system of units.

Now a single monomial term in \mathcal{L} can be written generically as:

$$\left(\prod_i \phi_i^{n_i} \right) \left(\prod_j (\partial_\mu \phi_j)^{m_j} \right) \quad (76)$$

times a dimensionless coupling. From dimensional analysis:

$$\sum_i \left(n_i [\phi_i] + m_i ([\phi_i] - 1) \right) = -D - 1 \quad (77)$$

since \mathcal{L} has length dimension $-D - 1$.

Now the variation of (76) under (75) gives:

$$\begin{aligned} \delta \mathcal{L} &= \sum_i \frac{\partial \mathcal{L}}{\partial \phi_i} \left[-[\phi_i] \phi_i(x) + x^\nu \partial_\nu \phi_i(x) \right] \\ &\quad + \sum_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \left[(-[\phi_i] + 1) \partial_\mu \phi_i(x) + x^\nu \partial_\nu \partial_\mu \phi_i(x) \right] \\ &= \sum_i \left[-n_i [\phi_i] \mathcal{L} + x^\nu \partial_\nu \phi_i(x) \frac{\partial \mathcal{L}}{\partial \phi_i} \right] + \sum_i \left[m_i (-[\phi_i] + 1) \mathcal{L} + x^\nu \partial_\nu \partial_\mu \phi_i(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \right] \\ &= (D + 1) \mathcal{L} + x^\nu \partial_\nu \mathcal{L} = \partial_\mu (\mathcal{L} x^\mu) \quad (78) \end{aligned}$$

where we used (77) in the last line. So we have shown that if \mathcal{L} has no dimensionful coupling, then without using the equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \phi_i} \delta \phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta \partial_\mu \phi_i = \partial_\mu (\mathcal{L} x^\mu) \quad (79)$$

so that by (45), $\mathcal{K}^\mu = \mathcal{L} x^\mu$, and the action is invariant by (56) and (75). The

current is given by (47) as:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \left[-[\phi_i] \phi_i(x) + x^\nu \partial_\nu \phi_i(x) \right] - \mathcal{L} x^\mu \quad (80)$$

6 Quantum Mechanics

We give a brief review of the quantum mechanics necessary to understand dilations. Although this work concerns only bosonic fields, it requires very little additional effort to treat fermion fields simultaneously, only requiring that a subscript be added to the commutator operation to indicate the use of the anticommutator instead. Therefore we include the fermion case.

6.1 Canonical Commutation Relations

Given fields ϕ_i with Lagrangian \mathcal{L} , the conjugate momentum is defined as:

$$\Pi_i(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i(x)} \quad (81)$$

To quantize the system, both $\phi_i(x)$ and $\Pi_i(x)$ are promoted to operators obeying the equal-time canonical (anti)commutation relation:

$$\begin{aligned} [\phi_i(\vec{x}, t), \Pi_j(\vec{y}, t)]_{\pm} &= i\delta^D(\vec{x} - \vec{y})\delta_{ij} \\ [\phi_i(\vec{x}, t), \phi_j(\vec{y}, t)]_{\pm} &= 0 \end{aligned} \quad (82)$$

where

$$\begin{aligned} [A, B]_+ &= AB - BA \quad (\text{bosons}) \\ [A, B]_- &= AB + BA \quad (\text{fermions}) \end{aligned} \quad (83)$$

It follows from (82) that

$$[\partial_{\vec{x}} \phi_i(\vec{x}, t), \phi_j(\vec{y}, t)]_{\pm} = [\phi_i(\vec{x}, t), \partial_{\vec{y}} \phi_j(\vec{y}, t)]_{\pm} = 0 \quad (84)$$

since the spatial derivative just gives the difference of ϕ at two different points at the same time, and the commutator of the two fields at the same time is zero.

Operators O obey the equation of motion:

$$\dot{O} = i[H, O] \quad (85)$$

where $H = \int d^D x \mathcal{H}(\phi_i(x), \Pi_i(x))$ is the Hamiltonian, and we use the convention that $[\dots, \dots]$ without a subscript indicates a commutator (and not anti-commutator) is being used, regardless if the fields are bosons or fermions.

We will frequently make use of the identity:

$$[AB, C] = A[B, C]_{\pm} \pm [A, C]_{\pm} B \quad (86)$$

i.e., $[AB, C] = ABC - CAB = A[B, C]_+ + [A, C]_+ B = A[B, C]_- - [A, C]_- B$. Obviously, it makes sense to choose $+$ for bosons and $-$ for fermions so that we can apply (82), but (86) is an equality and the choice of \pm doesn't depend on whether A , B , or C are boson or fermion fields.

6.1.1 Generators of Symmetries

In quantum mechanics the charge $Q(x^0)$ of Noether's theorem generates the symmetry transformation:

$$\begin{aligned}
e^{iQ(x^0)}\phi_i(x)e^{-iQ(x^0)} &= \phi'_i(x) \\
i[Q(x^0), \phi_i(x)] &= \delta\phi_i(x) \\
Q(x^0) &= \int d^Dx \left(\frac{\partial\mathcal{L}}{\partial\partial_0\phi_i(x)}\delta\phi_i(x) - \mathcal{L}f^0 \right)
\end{aligned} \tag{87}$$

It should be noted that $Q(x^0)$ does not have to be conserved classically for (87) to apply (in which case $\delta\phi_i(x)$ would just be a transformation, and not a symmetry transformation). The construction $\int d^Dx \left(\frac{\partial\mathcal{L}}{\partial\partial_0\phi_i(x)}\delta\phi_i(x) - \mathcal{L}f^0 \right)$ in and of itself generates the transformation $\delta\phi_i(x)$, regardless of what \mathcal{L} is and whether we attach the label Q to it. We will prove (87) with the only assumption being that $\delta\phi_i(x)$ depends on $\dot{\phi}(x)$ only through $\dot{\phi}(x)f^0$, i.e., $\delta\phi_i(x) = \dot{\phi}(x)f^0 + \dots$, where \dots does not contain a time-derivative, which is certainly true for transformations of the form (44). Note that $Q(x^0)$ and $\delta\phi_i(x)$ in (87) are at the same time. Therefore in the derivation that follows, all fields and conjugate momentum are at the same time (i.e., $x^0 = y^0$), so that (82) applies:

$$\begin{aligned}
\int d^Dy [\Pi_j(y)\delta\phi_j(y) - \mathcal{L}(y)f^0(y), \phi_i(x)] &= \int d^Dy \left(\pm [\Pi_j(y), \phi_i(x)]_{\pm} \delta\phi_j(y) \right. \\
&\quad \left. + \Pi_j(y)[\delta\phi_j(y), \phi_i(x)]_{\pm} - f^0[\mathcal{L}(y), \phi_i(x)] \right)
\end{aligned} \tag{88}$$

where we used (86) to get the first two terms on the RHS. The first term on the

RHS of (88) is $\mp i\delta\phi_i(x)$ using (82). Using that $[\delta\phi_j(y), \phi_i(x)]_{\pm} = [\dot{\phi}_j(y)f^0, \phi_i(x)]_{\pm}$ from (84), and that

$$\begin{aligned}\Pi_j(y)[\dot{\phi}_j(y)f^0, \phi_i(x)]_{\pm} &= [\Pi_j(y)\dot{\phi}_j(y)f^0, \phi_i(x)] \mp [\Pi_j(y)f^0, \phi_i(x)]_{\pm}\dot{\phi}_j(y) \\ &= [\Pi_j(y)\dot{\phi}_j(y)f^0, \phi_i(x)] + i\dot{\phi}_i(x)f^0\delta^D(\vec{x} - \vec{y})\end{aligned}$$

from (86), we get

$$\begin{aligned}\int d^D y [\Pi_j(y)\delta\phi_j(y) - \mathcal{L}(y)f^0(y), \phi_i(x)] &= \mp i\delta\phi_i(x) + i\dot{\phi}_i(x)f^0 \\ &+ \int d^D y \left([\Pi_j(y)\delta\phi_j(y), \phi_i(x)] - f^0[\mathcal{L}(y), \phi_i(x)] \right)\end{aligned}\quad (89)$$

In the last line, using $\Pi_j(y)\delta\phi_j(y) - \mathcal{L}(y)f^0 = \mathcal{H}(y)$ and (85), we get:

$$\begin{aligned}\int d^D y [\Pi_j(y)\delta\phi_j(y) - \mathcal{L}(y)f^0(y), \phi_i(x)] &= \mp i\delta\phi_i(x) \\ \therefore i[Q(x^0), \phi_i(x)] &= \pm\delta\phi_i(x)\end{aligned}\quad (90)$$

thereby proving (87). Evidently for fermions, $Q(x^0) = \int d^D x \left(\frac{\partial\mathcal{L}}{\partial\partial_0\phi_i(x)}\delta\phi_i(x) - \mathcal{L}f^0 \right)$ generates $-\delta\phi_i(x)$, opposite to bosons.

6.1.2 Virial Theorem

If we apply section (6.1.1) to section (5.6), we get that

$$Q = \int dV \left(\frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} [-[\phi_i] \phi_i(x) + x^\nu \partial_\nu \phi_i(x)] - \mathcal{L} x^0 \right)$$

generates the transformation $\delta \phi_i(x) = -[\phi_i] \phi_i(x) + x^\nu \partial_\nu \phi_i(x)$. We will restrict ourselves to spatial dilations so that:

$$\begin{aligned} Q_S &= \int dV \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} [-[\phi_i] \phi_i(x) + x^j \partial_j \phi_i(x)] \\ &= \int dV \Pi_i(x) [-[\phi_i] \phi_i(x) + x^j \partial_j \phi_i(x)] \end{aligned} \quad (91)$$

$$\delta \phi_i(x) = -[\phi_i] \phi_i(x) + x^j \partial_j \phi_i(x)$$

where Q_S is the virial. The charge density T_j^0 for spatial translations $\delta \phi_i(x) = \partial_j \phi_i(x)$ is given by:

$$\begin{aligned} T_j^0 &= \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \partial_j \phi_i(x) \\ &= \Pi_i(x) \partial_j \phi_i(x) \end{aligned} \quad (92)$$

so that the second term on the RHS of the second line of (91) is $x^j T_j^0$. In general the energy-momentum tensor T_ν^μ can be improved [7, 8] so that (91) can be expressed entirely as:

$$Q_S = \int dV x^j \Theta_j^0 \quad (93)$$

where Θ_ν^μ is the improved energy-momentum tensor. For bound systems, the average of the rate of change of Q_S over long periods of time vanishes, so that:

$$\begin{aligned}
\dot{Q}_S &= \int dV x^j \partial_0 \Theta_j^0 = 0 \\
&= - \int dV x^j \partial_k \Theta_j^k \\
&= \int dV \Theta_j^j = 0
\end{aligned} \tag{94}$$

where energy-momentum conservation $\partial_\mu \Theta_\nu^\mu = 0$ was used in the second line. Therefore using (94) we get:

$$\begin{aligned}
E &= \int dV \Theta_0^0 \\
&= \int dV \Theta_\mu^\mu
\end{aligned} \tag{95}$$

The addition of a confining pressure would require [9]:

$$E - DPV = \int dV \Theta_\mu^\mu \tag{96}$$

The improved energy-momentum tensor is related to the dilation current of (80) by

$$j^\mu = x^\nu \Theta_\nu^\mu \tag{97}$$

so that for a classically scale-invariant system

$$\begin{aligned}
\partial_\mu j^\mu &= \partial_\mu (x^\nu \Theta_\nu^\mu) \\
0 &= \Theta_\mu^\mu
\end{aligned} \tag{98}$$

implying by (95) that the bound state energy of such a system is zero. Energy-

momentum conservation $\partial_\mu \Theta_\nu^\mu = 0$ was once again used in deriving (98).

6.1.3 T-matrix for δ^2 -potential

We first briefly review how the T-matrix comes about. This review will be very informal. In a scattering experiment, we are uninterested in (and incapable of) calculating the trajectories of particles during the collision. We limit ourselves to answering the question what is the amplitude S_{fi} for observing the output state $|\psi'\rangle$ at $t_f = \infty$ when we input the state $|\psi\rangle$ at $t_i = -\infty$, or $S_{fi} = \langle\psi'|e^{-iH(t_f-t_i)}|\psi\rangle$. The states $|\psi\rangle$ and $\langle\psi'|$ are wavepackets built around the continuum scattering states of H , so they're almost eigenstates of H , but not quite, having a tiny bandwidth about some eigenvalue. Since the particles are widely separated at both $t_i = -\infty$ and $t_f = \infty$ and hence non-interacting, we can say that $|\psi\rangle$ and $\langle\psi'|$ are built around the scattering states of H_0 instead of H . So the scattering amplitude $\langle\psi'|e^{-iH(t_f-t_i)}|\psi\rangle$ has the fully interacting Hamiltonian H , but sandwiched between wavepackets built around the energy eigenstates of H_0 at time $t = \pm\infty$. Therefore the scattering amplitude is:

$$\begin{aligned}
S_{fi} &= \langle\psi'|e^{-iH(t_f-t_i)}|\psi\rangle \\
&= \langle\psi'|e^{-iH_0t_f}e^{iH_0t_f}e^{-iH(t_f-t_i)}e^{-iH_0t_i}e^{iH_0t_i}|\psi\rangle \\
&= {}_H\langle\psi'|e^{iH_0t_f}e^{-iH(t_f-t_i)}e^{-iH_0t_i}|\psi\rangle_H \\
&\equiv {}_H\langle\psi'|U(t_f, t_i)|\psi\rangle_H
\end{aligned} \tag{99}$$

where $|\psi\rangle_H$ is the state that $|\psi\rangle$ evolves into after time $|t_i|$, and ${}_H\langle\psi'|$ is the state that evolves into $\langle\psi'|$ after a time t_f , where the evolution is with the free

Hamiltonian H_0 . The subscript H on the states $|\dots\rangle_H$ is named after Heisenberg and has nothing to do with the fully interacting Hamiltonian H : pictorially they allow us to take our inputs and outputs as non-interacting wavepackets at the origin rather than non-interacting wavepackets at spatial infinity. Differentiating $U(t_f, t_i)$ of (99) w.r.t. t_f gives:

$$\begin{aligned}\dot{U} &= -iV_I(t)U \\ U(t_f, t_i) &= T \exp\left(-i \int_{t_i}^{t_f} dt V_I(t)\right) \\ V_I(t) &\equiv e^{iH_0 t}(H - H_0)e^{-iH_0 t} = e^{iH_0 t}V e^{-iH_0 t}\end{aligned}\tag{100}$$

At this point it should be mentioned that we tried to be informally formal by trying to justify using eigenstates of H_0 as our scattering states instead of the full H , but if we just assume we can do this then in (99) we can make the replacement $H(t_f - t_i) \rightarrow \int_{t_i}^{t_f} dt H(t)$ and (100) would still hold true, assuming H_0 is time-independent and that the explicit time-dependence of H is only through $V(t)$, i.e., $H = H_0 + V(t)$.¹⁶ In other words, (100) holds even when V depends explicitly on time.

$U(t_f, t_i)$ in (100) can be Taylor expanded in powers of V_I for small V_I , and assuming for now that $H \neq H(t)$ and that $|\psi\rangle_H$ and ${}_H\langle\psi'|$ are energy eigenstates of H_0 (at this point we reduce the bandwidth of the wave-packets to zero) then the time integration can be performed and the result is:

¹⁶Since $\delta^2(x)$ is localized, we can change the Hamiltonian to $H = H_0 + f(t)V$, where $f(t) = 0$ at $|t| > T$ and otherwise is equal to 1 except for a smooth transition region, for some large T . Then we can justify using the eigenstates of H_0 in place of the original H since these eigenstates adiabatically go to eigenstates of H at $|t| \approx T$ with probability 1.

$$\begin{aligned}
S_{fi} &= {}_H\langle\psi'|\psi\rangle_H - i(2\pi)\delta(E' - E)T_{fi} \\
-iT_{fi} &= {}_H\langle\psi'|T|\psi\rangle_H \\
T &= \sum_{n=0}^{\infty} V \left(\frac{1}{E - H_0 + i\epsilon} V \right)^n \\
&= V + V \left(\frac{1}{E - H_0 + i\epsilon} \right) T
\end{aligned} \tag{101}$$

The T -matrix can be interpreted in many ways. Within a Feynman diagram each n represents scattering $n + 1$ times, where the outgoing particles for one scattering event join together to be the incoming particles for another scattering event, with $(E - H_0 + i\epsilon)^{-1}$ propagating the particles from one V to another. However, we will view T as an effective potential whose tree-level/Born approximation gives the exact result. In the 2nd-quantized form this is the quantum effective potential.

Sandwiching (101) between momentum eigenstates $\langle p'|$ and $|p\rangle$

$$T(\vec{p}', \vec{p}) = V(\vec{p}', \vec{p}) + \int \frac{d^2k}{(2\pi)^2} V(\vec{p}', \vec{k}) \frac{1}{E - \frac{k^2}{2m} + i\epsilon} T(\vec{k}, \vec{p}) \tag{102}$$

For $V(\vec{X}) = \lambda\delta^2(\vec{X})$, $V(\vec{p}', \vec{p}) = \lambda\langle p'|\delta^2(\vec{X})|p\rangle = \lambda \int d^2\vec{x} e^{i(\vec{p}-\vec{p}')\cdot\vec{x}}\delta^2(\vec{x}) = \lambda$, so that

$$T(\vec{p}', \vec{p}) = \lambda + \lambda \int \frac{d^2k}{(2\pi)^2} \frac{1}{E - \frac{k^2}{2m} + i\epsilon} T(\vec{k}, \vec{p}) \tag{103}$$

Evidently, $T(\vec{p}', \vec{p})$ doesn't depend on \vec{p}' at all since the RHS has no such dependence, so $T(\vec{k}, \vec{p})$ doesn't depend on \vec{k} and can be brought out of the integral.

Indeed, evidently $T(\vec{p}', \vec{p}) = T(\vec{k}, \vec{p}) = T(E)$ [5].

$$\begin{aligned} T(E) &= \lambda + \lambda \int \frac{d^2k}{(2\pi)^2} \frac{1}{E - \frac{k^2}{2m} + i\epsilon} T(E) \\ \frac{1}{T(E)} &= \frac{1}{\lambda} - \int \frac{d^2k}{(2\pi)^2} \frac{1}{E - \frac{k^2}{2m} + i\epsilon} \end{aligned} \quad (104)$$

The integral diverges for large k , so we will use a hard momentum cutoff Λ :

$$\frac{1}{T(E)} = \frac{1}{\lambda} + \frac{m}{2\pi} \ln \left(-\frac{\Lambda^2}{2mE} + i\epsilon \right) \quad (105)$$

As in Eqn. (18) of the introduction, $\lambda = \lambda(\Lambda)$, such that (105) gives the correct result $T(E)$ for any value E . We can find the bound state energy E_b by noting that $\frac{1}{T(E_b)} = 0$:

$$\begin{aligned} 0 &= \frac{1}{\lambda(\Lambda)} + \frac{m}{2\pi} \ln \left(-\frac{\Lambda^2}{2mE_b} + i\epsilon \right) \\ E_b &= -\frac{\Lambda^2}{2m} e^{\frac{2\pi}{m\lambda(\Lambda)}} \end{aligned} \quad (106)$$

Moreover, we can eliminate the dependence of $T(E)$ on the unphysical parameters $(\Lambda, \lambda(\Lambda))$ by substituting the $\frac{1}{\lambda(\Lambda)}$ from the first line of (106) into (105):

$$\frac{1}{T(E)} = \frac{m}{2\pi} \left(\ln \left(\frac{|E_b|}{E} \right) + i\pi \right) \quad (107)$$

6.2 Path-Integral Formalism

The path-integral or functional-integral approach contains the same information as the canonical approach. However, the study of symmetries is more natural within the path-integral approach, since the path integral utilizes the action. Moreover,

the fields are treated as c-numbers rather than operators, so many of the results from section 5 can be directly used inside the path integral. We will not derive the path-integral from the canonical approach: instead, we will simply state without proof a number of well-known features of the path-integral.

6.2.1 The Path Integral

Time-ordered Green's functions $G^{(n)}(x_1, \dots, x_n)$ are given by the path integral:

$$\begin{aligned} G^{(n)}(x_1, \dots, x_n) &= \langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle \\ &= \int [d\phi(x)] e^{i\int d^d x \mathcal{L}(\phi(x), \partial_\mu \phi(x))} \phi(x_1)\dots\phi(x_n) \end{aligned} \quad (108)$$

where the measure $[d\phi(x)]$ is normalized so that $G^{(0)} = 1$, $|0\rangle$ is the ground state of the system, and \mathcal{L} is a slightly modified Lagrangian that has $\mathcal{H} \rightarrow \mathcal{H}(1 - i\epsilon)$.¹⁷ We will have no need to actually evaluate this functional integral. If one needed to evaluate this integral, one could put it on a lattice, or if the terms in \mathcal{L} higher than quadratic are small, one can evaluate the integral perturbatively by Taylor expanding those small terms and using the exponential of the quadratic piece as a probability density, in which case the problem reduces to the calculation of moments.

We will need an expression for time-ordered Green's functions of operators built from fields. If the operators contain no time-derivatives, then:

¹⁷In general the path integral without $i\epsilon$ gives the amplitude between two different wave-functional states, in which case we would have to be careful about normalization of the measure $[d\phi(x)]$ in actual calculations. The $i\epsilon$ prescription isolates the ground state.

$$\langle 0|TF[\phi(x_1)]\dots|0\rangle = \int [d\phi(x)] e^{i \int d^d x \mathcal{L}(\phi(x), \partial_\mu \phi(x))} F[\phi(x_1)]\dots \quad (109)$$

Some examples include $F[\phi(x_1)] = \phi^4(x_1)$, $F[\phi(x_1)] = \partial_i \phi(x_1)$. However, if $F[\phi(x_1)]$ involves a time-derivative, it is not true that:

$$\langle 0|T\dot{\phi}(x_1)\dots|0\rangle = \int [d\phi(x)] e^{i \int d^d x \mathcal{L}(\phi(x), \partial_\mu \phi(x))} \dot{\phi}(x_1)\dots \quad (110)$$

To see this, on the RHS of (110), one can make the replacement

$$\dot{\phi}(x_1) = \frac{1}{\epsilon} (\phi(x_1^0 + \epsilon, \vec{x}_1) - \phi(x_1^0, \vec{x}_1))$$

to get

$$\begin{aligned} \int [d\phi(x)] e^{i \int d^d x \mathcal{L}(\phi(x), \partial_\mu \phi(x))} \dot{\phi}(x_1)\dots &= \frac{d}{dx_1^0} \langle 0|T\phi(x_1)\dots|0\rangle \\ &\neq \langle 0|T\dot{\phi}(x_1)\dots|0\rangle \end{aligned} \quad (111)$$

$\frac{d}{dx_1^0} \langle 0|T\phi(x_1)\dots|0\rangle \neq \langle 0|T\dot{\phi}(x_1)\dots|0\rangle$ due to the time-dependence of $\langle 0|T\phi(x_1)\dots|0\rangle$ on time-ordering, in addition to the time-dependence of the field $\phi(x_1)$.

Therefore any path-integral derivation that results in a time-derivative in the integrand is to be taken out of the integrand before interpreting the integral as a Green's function, e.g.,

$$\int [d\phi(x)] e^{i \int d^d x \mathcal{L}(\phi(x), \partial_\mu \phi(x))} \partial_\mu j^\mu(x_1) = \partial_\mu^{x_1} \langle 0|Tj^\mu(x_1)|0\rangle \quad (112)$$

6.2.2 The Partition Function

As mentioned in the previous footnote, without the $i\epsilon$ prescription, the path integral, when properly normalized, is equal to:

$$\langle \phi''(t, \vec{x}) | e^{-iHt} | \phi'(0, \vec{x}) \rangle = \int_{\phi(0, \vec{x}) = \phi'(0, \vec{x})}^{\phi(t, \vec{x}) = \phi''(t, \vec{x})} [d\phi(x)] e^{i \int d^d x \mathcal{L}(\phi(x), \partial_\mu \phi(x))} \quad (113)$$

where the time integral in the action is only from t_1 to t_2 , and $|\phi'(t_1, \vec{x})\rangle$ are wave-functionals¹⁸

$$\hat{\phi}(\vec{x}) |\phi'(t, \vec{x})\rangle = \phi'(t, \vec{x}) |\phi'(t, \vec{x})\rangle \quad (114)$$

The wave-functionals are the generalization of the position eigenstates in quantum mechanics to the field eigenstates in quantum field theory. Indeed, quantum field theory can be done using wave-functionals along with the Schrödinger equation [10], instead of the canonical or path-integral approaches.

An equation for the partition function in terms of the path integral can be gotten from making the replacement $t \rightarrow -i\beta$ on the LHS of (113). When this is done:

$$\langle \phi''(-i\beta, \vec{x}) | e^{-\beta H} | \phi'(0, \vec{x}) \rangle = \int_{\phi(0, \vec{x}) = \phi'(0, \vec{x})}^{\phi(-i\beta, \vec{x}) = \phi''(-i\beta, \vec{x})} [d\phi(x)] e^{i \int d^d x \mathcal{L}(\phi(x), \partial_\mu \phi(x))} \quad (115)$$

¹⁸We are slightly abusing notation. The wave-functional state is $|\phi''(\vec{x})\rangle$ with no time-dependence, an eigenvalue of the field operator $\hat{\phi}(\vec{x})$. We put the time-label to indicate the system/state has the field value $\phi''(\vec{x})$ at time t . The QM analogy would be writing $\langle q_2 | e^{-iH(t_2-t_1)} | q_1 \rangle$ as $\langle q_2, t_2 | e^{-iH(t_2-t_1)} | q_1, t_1 \rangle$ to emphasize the times of the states. Of course ${}_H \langle q_2, t_2 | q_1, t_1 \rangle_H$ is unambiguous, but we want to avoid using Heisenberg states to emphasize the evolution operator e^{-iHt} .

where the time-integral of the action on the RHS is from 0 to $-i\beta$. Note that the replacement of t on the LHS of (113) corresponds to a Wick rotation by $\frac{\pi}{2}$ clockwise in the complex t -plane for the time-integral of the action on the RHS. We make the change of variables $t = -i\tau$ in the action integral to change the time-contour to be from 0 to β . The result is:

$$\begin{aligned}
\langle \phi''(-i\beta, \vec{x}) | e^{-\beta H} | \phi'(0, \vec{x}) \rangle &= \int_{\phi(0, \vec{x}) = \phi'(0, \vec{x})}^{\phi(-i\beta, \vec{x}) = \phi''(-i\beta, \vec{x})} [d\phi(x)] e^{-\int d^d x \mathcal{L}_E(\phi(x), \partial_\mu \phi(x))} \\
&= \int_{\phi(0, \vec{x}) = \phi'(0, \vec{x})}^{\phi(-i\beta, \vec{x}) = \phi''(-i\beta, \vec{x})} [d\phi(x)] e^{-S_E} \\
\mathcal{L}_E &= \mathcal{L}(\phi(x), i\partial_\tau \phi(x), \partial_j \phi(x))
\end{aligned} \tag{116}$$

where the Euclidean Lagrangian \mathcal{L}_E is gotten by replacing ∂_t in \mathcal{L} with $i\partial_\tau$. Finally, to get the partition function, we need to set $|\phi'(0, \vec{x})\rangle$ equal to $|\phi''(-i\beta, \vec{x})\rangle$ and take the trace. We can do this by setting periodic boundary conditions $\phi'(0, \vec{x}) = \phi''(-i\beta, \vec{x})$, so that the result for the partition function is:

$$\begin{aligned}
Z[\beta] &= \int_{\phi(0, \vec{x}) = \phi(\beta, \vec{x})} [d\phi(x)] e^{-\int d^d x \mathcal{L}_E(\phi(x), \partial_\mu \phi(x))} \\
&= \int_{\phi(0, \vec{x}) = \phi(\beta, \vec{x})} [d\phi(x)] e^{-S_E}
\end{aligned} \tag{117}$$

where we have removed the i in the labels on our integration variables $\phi(\text{label})$. For the grand canonical ensemble, we simply add $\mu \int d^d x j^0$ to $-S_E$ in the argument of the exponential of (117), where μ is the chemical potential associated with the conserved charge $\int d^D x j^0$.

As in the zero-temperature case of section 6.2.1, we will have no need to actually evaluate (117). Evaluating (117) perturbatively amounts to evaluating the Euclidean path-integral as in the zero-temperature case, except that frequency integrals are replaced by sums over discrete frequencies due to the periodic boundary conditions on $\phi(x)$ over the finite inverse-temperature interval $[0, \beta]$.

Also, for fermions, evidently the above argumentation fails, as anti-periodic boundary conditions are required [11]:

$$Z[\beta] = \int_{\phi(0, \vec{x}) = -\phi(\beta, \vec{x})} [d\phi(x)] e^{-S_E} \quad (118)$$

6.2.3 Fujikawa's Method

Consider the path-integral

$$\int [d\phi_i(x)] e^{iS[\phi_i(x)]} = 1 \quad (119)$$

Making the substitution $\phi_i(x) = \phi'_i(x) + \rho\delta\phi'_i(x)$:

$$\begin{aligned}
\int [d\phi_i(x)] e^{iS[\phi_i(x)]} &= \int [d\phi'_i(x)] \left| \frac{\delta\phi_k(x)}{\delta\phi'_\ell(y)} \right| e^{iS[\phi'_i(x) + \rho\delta\phi'_i(x)]} \\
&= \int [d\phi'_i(x)] \left| \delta^d(x-y) + \rho \frac{\delta\delta\phi'_k(x)}{\delta\phi'_\ell(y)} \right| e^{iS[\phi'_i(x)] + i\rho \int d^d x \frac{\delta S}{\delta\phi'_i(x)} \delta\phi'_i(x)} \\
&= \int [d\phi'_i(x)] e^{\rho \int d^d x \operatorname{tr} \frac{\delta\delta\phi'_k(x)}{\delta\phi'_\ell(y)} \Big|_{y=x}} e^{iS[\phi'_i(x)] + i\rho \int d^d x \frac{\delta S}{\delta\phi'_i(x)} \delta\phi'_i(x)} \\
&= \int [d\phi_i(x)] e^{\rho \int d^d x \operatorname{tr} \frac{\delta\delta\phi_k(x)}{\delta\phi_\ell(y)} \Big|_{y=x}} e^{iS[\phi_i(x)] + i\rho \int d^d x \frac{\delta S}{\delta\phi_i(x)} \delta\phi_i(x)}
\end{aligned} \tag{120}$$

where the identity $|A| = \det A = \exp(\operatorname{tr} \log A)$ was used, along with $\log(1 + \rho M) = \rho M + O[\rho^2]$. In the last line we relabeled the fields, which are dummy variables. Expanding to order $O[\rho]$:

$$0 = \int [d\phi_i(x)] e^{iS[\phi_i(x)]} \rho \int d^d x \left(\operatorname{tr} \frac{\delta\delta\phi_k(x)}{\delta\phi_\ell(y)} \Big|_{y=x} + i \frac{\delta S}{\delta\phi_i(x)} \delta\phi_i(x) \right) \tag{121}$$

Since this holds for any volume, we can ignore the spacetime integrals. Using (51) from section 5.4.2, then without using the equations of motion (121) becomes:

$$\begin{aligned}
0 &= \int [d\phi_i(x)] e^{iS[\phi_i(x)]} \left(\operatorname{tr} \frac{\delta\delta\phi_k(x)}{\delta\phi_\ell(y)} - i\partial_\mu j^\mu(x) \right) \\
\partial_\mu \langle 0 | j^\mu(x) | 0 \rangle &= -i \left\langle 0 \left| \operatorname{tr} \frac{\delta\delta\phi_k(x)}{\delta\phi_\ell(y)} \Big|_{y=x} \right| 0 \right\rangle
\end{aligned} \tag{122}$$

The trace acts over all internal indices that label the fields. The RHS of the second line in (122) is called the anomaly. When the anomaly is zero, (122) is called the Ward identity, which is the quantum version of Noether's conservation equation.

7 Units in Physics

In this section we briefly review what units are. We explain the choices for the fundamental scales we will use in both nonrelativistic and relativistic physics, and the meaning of setting certain physical quantities equal to one. Ultimately we will use this along with the path integral to derive the Tan-pressure relation.

7.1 What are Units?

Imagine you're a biologist on the savannah, and close to the horizon you see a rare giant penguin. You wish to measure the height of the penguin with your meter stick, but if you approach it then it might waddle away. But standing next to the penguin is a common savannah rabbit. You can imagine multiplying the rabbit and stacking them on top of each other until the stack equals the height of the penguin:

$$h_{\text{pen}} = 3h_{\text{bun}} \tag{123}$$

Since they're so common, the savannah rabbit's height is well-known: they're 3 feet with very little genetic variation. Therefore we can express the height of the penguin in two different ways:

quantity/multiple/number	scale/reference/standard/unit
3	bunnies
9	feet

From this example, we note two things: 1) a measurement is a comparison be-

tween what you want to measure (penguin) and some reference object or scale (bunnies, or some British monarch's foot) - there is no absolute measurement, one always needs another object to compare; 2) to specify a measurement, you have to give a pure number (e.g., 3) followed by the reference value (bunnies, or more accurately height of bunny h_{bun}). It should be clear that you can take any object as your reference value or scale, so there are an infinite number of units you can use. How do you know which scales to choose? It's clear that the experimentalist gets to use whatever is most convenient (bunnies, since using the meter stick would scare the penguin away, and also because the penguin and bunny are conveniently at the same distance from the observer, and therefore their heights are in proportion when viewed from this distance). When communicating with the locals one can use the common savannah rabbit since they're so common that everyone there knows what you mean when you use the rabbit as your unit of length. When communicating with Americans use feet. For scientific publication use meters. But without these considerations, one generally chooses scales so that the quantity/multiple/number is between .1 and 100. Another way of stating this is that one compares objects of similar size. It would be a mistake to express the penguin's height using the light-year: the penguin is practically zero compared to a light-year, and you end up with ridiculous numbers like .0000000000000000...light-years. Scientific notation mitigates some of this awkwardness, but numbers such as $1.6 * 10^{-19}$ Coulombs are still unfathomable - in any reasonable system of units we should be able to make the approximation $1.6 * 10^{-19} \approx 0$.¹⁹ Later we will see that it's best to use as scales those dimensionful

¹⁹Take a sheet of paper and tear it into ten pieces. Take one of those ten pieces and tear it



Figure 2: All measurement is a comparison.

parameters that appear in your equations: those parameters then have the value one when expressed in terms of those scales.

In SI units, we choose as our scales the kilogram, the meter, and the second. In analogy with the bunny and height where we make the replacement $h_{\text{bun}} \rightarrow$ bunnies, we also shorten $2 m_{\text{kilogram}}$ to 2 kilograms or 2 kg ($m_{\text{kilogram}} \rightarrow$ kilograms). But it's worth bearing in mind that if we say an object is 2 kilograms, what we really mean is $m_{\text{object}} = 2m_{\text{kilogram}}$, i.e., we are comparing the mass of the object to the mass of another object called the kilogram that is kept in some vault in Europe. For nonrelativistic quantum mechanics, we will use as our unit of angular momentum H the orbital angular momentum of the hydrogen atom in a p-state, and give this the abbreviation \hbar , just as we gave m_{kilogram} the abbreviation kg. So for example, we might express the angular momentum for the spin $1/2$ electron in the ground state as $H_{\text{electron}} = .5H_{\text{p-state}} = .5\hbar$

into another ten pieces. Repeat seventeen more times: anything that's left is an electron.

For everyday discourse social convention often dictates what scales we use, so we need not specify them. If we say the speed limit on I-45 is 65, then everyone understands we mean 65 mph. Or if we say the temperature has reached triple digits, then everyone understands Fahrenheit. This is mimicked in physics too, where if we express all our quantities as multiples of the standard SI units, then we can neglect writing down our units at all intermediate steps of a calculation, knowing that the units for the final answer will be the standard SI unit for that quantity. Let's restate this again and give concrete examples.

By convention in physics, there are three scales for which we need not specify units: the kilogram, the meter, and the second, which provide the reference objects for mass, length and time (M , L , T), respectively, in the SI system. That is, all quantities are expressed as multiples of these quantities. To see an example of this in a calculation, consider the calculation of the momentum of a charging elephant $m_{el} = 2000$ kg that runs a distance of $d_{el} = 4$ m in $t_{el} = 2$ s:

$$p_{el} = 2000 \left(\frac{4}{2} \right) = 4000 \quad (124)$$

We have neglected the units in the above calculation because convention dictates that we need not specify units if they are the SI units. To recover the dimension, we must know that momentum has dimensions equal to ML/T , and then replace each dimension with the chosen reference object or scale, so that:

$$p_{el} = 4000 \frac{\text{kg m}}{\text{s}} \quad (125)$$

We never have to write down our units in calculations so long as all quantities that go into the calculation are written as multiples of our chosen scales. We can always recover the units by writing down the dimensions of the quantity we are calculating as $M^\eta L^\xi T^\zeta$, and making the replacements $M \rightarrow \text{kg}$, $L \rightarrow \text{m}$, and $T \rightarrow \text{s}$. Now suppose we don't know the mass of our elephant, so leave it as a variable. Then:

$$p_{el} = m_{el} \left(\frac{4}{2} \right) = 2m_{el} \quad (126)$$

To recover the dimension, we know that momentum has dimension ML/T , and our formula already has a quantity that has a dimension M , so we just need L/T :

$$p_{el} = 2 \frac{m_{el} \text{ m}}{\text{s}} = 2 \frac{m_{el} L_{\text{met}}}{t_{\text{sec}}} \quad (127)$$

The above equation says that the momentum of a charging elephant that runs a distance of 4 meters in 2 seconds is the mass of the elephant, times the length of a meter, divided by a duration of time equal to a second, all times 2. We are free to express the mass of the elephant using any scale we want, e.g., kg or lb.

As another example, take a classical harmonic oscillator with mass 4 kg and spring constant 25 kg/s^2 :

$$\begin{aligned} m\ddot{x} &= -kx \\ 4m_{\text{kg}}\ddot{x} &= -25 \left(\frac{m_{\text{kg}}}{t_{\text{sec}}^2} \right) x \end{aligned} \quad (128)$$

We ignore the units to get:

$$\begin{aligned} 4\ddot{x} &= -25x \\ x &= C \sin\left(\frac{5}{2}t\right) \end{aligned} \tag{129}$$

and then restore the dimensions:

$$x = CL_{\text{met}} \sin\left(\frac{5}{2} \frac{t}{t_{\text{sec}}}\right) \tag{130}$$

Just as we ignore writing out the units m_{kg} , L_{met} , and t_{sec} in classical physics (as in the above calculations), we will be ignoring units like $H_{\text{p-state}} = \hbar$ and $V_{\text{light}} = c$ when doing quantum mechanics, only restoring them at the end of the calculation if we need to emphasize what units we are using.

7.2 Units in Nonrelativistic Quantum Field Theory

In nonrelativistic physics, instead of using MLT as our fundamental dimensions, we will choose MLH , i.e., replacing time with angular momentum. Our reference value for angular momentum H will be the orbital angular momentum of a hydrogen atom in the p-state, which is given by the abbreviation \hbar , just as the time interval whose duration is a second is given the abbreviation s . Our reference value for M will be the mass of the particle we're interested in. For our third dimension we will take length, but leave the unit for the length unspecified: in practice this means that anything that requires a length dimension we must leave as a free variable, like how m_{el} was used in (126). With these conventions we can

create a chart:

quantity	MLT	MLH	\hbar	m	L	units
m	M	M	0	1	0	m
x	L	L	0	0	1	L
t	T	ML^2H^{-1}	-1	1	2	mL^2/\hbar
p	MLT^{-1}	$L^{-1}H$	1	0	-1	L/\hbar
E	ML^2T^{-2}	$M^{-1}L^{-2}H^2$	2	-1	-2	\hbar^2/mL^2
\mathcal{E}	$ML^{2-D}T^{-2}$	$M^{-1}L^{-2-D}H^2$	2	-1	$-2 - D$	\hbar^2/mL^{2+D}
H	ML^2T^{-1}	H	1	0	0	\hbar

that tells us how to recover our units. The sixth column is particularly important that we create a new notation [...] that returns the value of the sixth column when a quantity is inserted into it:

$$[\text{quantity}] = \text{power of L} \quad (131)$$

Some examples:

$$\begin{aligned}
 [m_{el}] &= 0 \\
 [d_{el}] &= 1 \\
 [t_{el}] &= 2 \\
 [p_{el}] &= -1
 \end{aligned} \quad (132)$$

With our choice of units, the Schrödinger equation and canonical commutation relations are written as:

$$\begin{aligned}
i\dot{\psi} &= -\frac{1}{2}\nabla^2\psi + V\psi \\
[X_i, P_j] &= i\delta_{ij}
\end{aligned}
\tag{133}$$

where we do not need to write down units for \hbar and m , since these are our chosen fundamental scales. We say that we set $\hbar = m = 1$, by which we mean we chose a system of units such that when we write these quantities in those units, the quantity/multiple/number is equal to one.

7.3 Units in Relativistic Quantum Field Theory

In relativistic physics, instead of using MLT as our fundamental dimensions, we will choose HLV , i.e., replacing time with velocity and mass with angular momentum. Our reference value for angular momentum H will be the orbital angular momentum of a hydrogen atom in the p-state, which is given by the abbreviation \hbar . Our reference value for V will be the speed of light. For our third dimension we will take length, but leave the unit for the length unspecified: in practice this means that anything that requires a length dimension we must leave as a free variable. With these conventions we can create a chart:

quantity	MLT	HLV	\hbar	c	L	units
m	M	$HL^{-1}V^{-1}$	1	-1	-1	\hbar/cL
x	L	L	0	0	1	L
t	T	LV^{-1}	0	-1	1	L/c
p	MLT^{-1}	$L^{-1}H$	1	0	-1	\hbar/L
E	ML^2T^{-2}	$HL^{-1}V$	1	1	-1	$\hbar c/L$
\mathcal{E}	$ML^{2-D}T^{-2}$	$HL^{-1-D}V$	1	1	$-1 - D$	$\hbar c/L^{1+D}$
H	ML^2T^{-1}	H	1	0	0	\hbar

that tells us how to recover our units. The sixth column is particularly important that we create a new notation [...] that returns the value of the sixth column when a quantity is inserted into it:

$$[\text{quantity}] = \text{power of L} \quad (134)$$

Some examples:

$$\begin{aligned}
[m_{el}] &= -1 \\
[d_{el}] &= 1 \\
[t_{el}] &= 1 \\
[p_{el}] &= -1
\end{aligned} \quad (135)$$

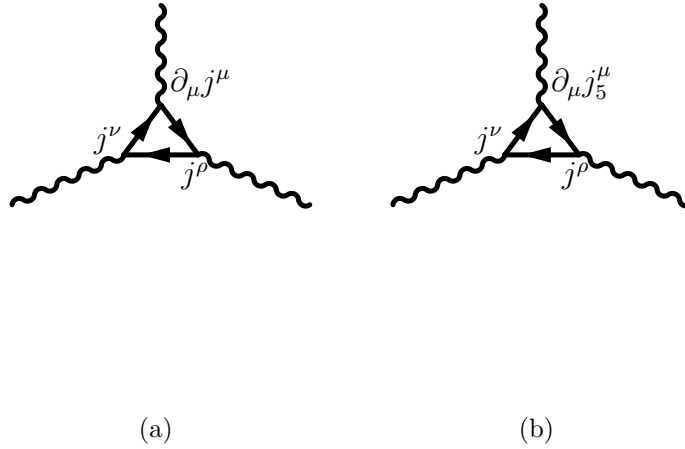


Figure 3: Triangle diagrams for the (a) gauge and (b) chiral anomalies. For each of the diagrams, there is another one (not shown) with the arrow in the fermion loop going counterclockwise.

8 Anomaly Measurement

8.1 Particle Physics

The origins of the anomaly can be traced to the Feynman diagrams in figure 3, where the diagrams live in a world where $\mathcal{L} = i\bar{\psi}\gamma^\mu (\partial_\mu + ieA_\mu)\psi$, where ψ is a quark field, A_μ is the photon field, and e is the charge of the quark, i.e., quantum electrodynamics with a massless quark.

If \mathcal{L} is viewed classically, the system has two continuous symmetries, with corresponding conserved currents given by Noether's theorem:

$$\begin{aligned}
\psi &\rightarrow e^{i\theta}\psi \\
\bar{\psi} &\rightarrow e^{-i\theta}\bar{\psi} \\
j^\mu &= \bar{\psi}\gamma^\mu\psi, \quad \partial_\mu j^\mu = 0
\end{aligned} \tag{136}$$

and

$$\begin{aligned}
\psi &\rightarrow e^{i\gamma_5\theta}\psi \\
\bar{\psi} &\rightarrow \bar{\psi}e^{i\gamma_5\theta} \\
j_5^\mu &= \bar{\psi}\gamma^\mu\gamma_5\psi, \quad \partial_\mu j_5^\mu = 0
\end{aligned} \tag{137}$$

Diagram 3(a) represents the calculation of

$$\partial_\mu \langle 0 | T j^\mu(x_1) j^\nu(x_2) j^\rho(x_3) | 0 \rangle \tag{138}$$

while 3(b) represents

$$\partial_\mu \langle 0 | \langle T j^{5\mu}(x_1) j^\nu(x_2) j^\rho(x_3) | 0 \rangle \tag{139}$$

where T is the time-ordering product and ∂_μ is w.r.t. x_1 .

Based on the classical result of eqns. (136) and (137), we would expect both diagrams to be zero. Of course quantum mechanics is different from classical mechanics, so a zero result in one does not necessarily imply a zero result in the other. However, for symmetries, it is usually the case that a quantity that is conserved in the classical system is also conserved when the system is quantized.

However, Adler, Bell, and Jackiw calculated the diagrams using the full rules of quantum mechanics, and found that they could not make both diagrams simultaneously zero. In general, quantities in quantum mechanics diverge, and to handle this one must alter the high energy behavior of the theory. The arbitrariness in this procedure allows one to fix the $\partial_\mu j^\mu$ diagram to zero, but necessarily then the $\partial_\mu j_5^\mu$ diagram is nonzero, i.e., an anomaly, as classically $\partial_\mu j_5^\mu = 0$. The $\partial_\mu j^\mu$ diagram is set to zero not just because electric charge is experimentally observed to be conserved, but also because gauge symmetry is important in quantum mechanics for the consistency of the theory (if the system doesn't have the global symmetry of eqn. (136), then it certainly doesn't have the gauge symmetry).

The triangle diagrams of figure 3 are important in the history of particle physics. The $\partial_\mu j_5^\mu$ diagram represents the decay of the neutral meson to two photons: $\Pi^0 \rightarrow \gamma + \gamma$. j^μ acting on the vacuum produces a photon. This can be seen by the interacting part of \mathcal{L} , which can be written as $-ej^\mu A_\mu$, so that A_μ or the photon serves as the source of the electromagnetic current j^μ . Since $j^\mu = \bar{\psi}\gamma^\mu\psi$, out of the photon source comes out a quark/anti-quark pair. $\partial_\mu j_5^\mu$ acting on the vacuum represents the Π^0 meson field, which is known as the partially conserved axial current hypothesis (PCAC). So the diagram represents one of the meson's quark/anti-quark constituents emitting a photon and then annihilating with the other anti-quark/quark.

If the classical symmetry $\partial_\mu j_5^\mu = 0$ held, then the diagram would be zero, so that the Π^0 meson could not decay electromagnetically to $\gamma + \gamma$. However, it does decay electromagnetically, and taking into account the anomaly, the experimentally measured decay rate matches the $\partial_\mu j_5^\mu$ diagram. Except the answer was off by a factor of 3. This can be fixed by assuming that there are three colors of quarks flowing in the triangle diagram, so that the calculation of the axial anomaly is important in the history of particle physics as it provided one of the first pieces of evidence that quarks have color, that quantum chromodynamics was indeed the correct description of the strong force.

The $\partial_\mu j^\mu$ diagram as mentioned is fatal to a theory if it contains an anomaly. When constructing new models such as a string theory or a grand unified theory, it is critical that this diagram is zero, which provides a non-trivial constraint on the theory. For vector theories such diagrams can be shown to vanish by charge conjugation symmetry (Furry's theorem), but the weak interaction is chiral (i.e., left-handed), so the vanishing of the diagram requires cancellations that restrict the particle content of the theory: indeed, the requirement that there is no $\partial_\mu j^\mu$ anomaly is achieved in the standard model by having the same number of quarks as leptons.

8.2 Atomic Physics

8.2.1 Contact Interactions

At very low temperatures the relative momentum of atoms are small, making the relative de Broglie wavelength λ_B much larger than the range of interaction r_0 of the potential, so that collisions are unable to resolve the structure of the potential [12]. In this case the potential can be approximated as a $\delta^D(\vec{r})$ potential. For example, in the Born approximation, for the scattering amplitude:

$$\langle k'|V|k\rangle = \int d^D x V(x) \psi_{k'}^*(x) \psi_k(x) \quad (140)$$

If the de Broglie wavelengths of $\psi_k(x)$ and $\psi_{k'}(x)$ are small, then over the range r_0 of the potential, $\psi_{k'}^*(x) \psi_k(x) \approx \psi_{k'}^*(0) \psi_k(0)$. Therefore:

$$\begin{aligned} \langle k'|V|k\rangle &= \int d^D x V(x) \psi_{k'}^*(x) \psi_k(x) \\ &= \int_{|x|<r_0} d^D x V(x) \psi_{k'}^*(x) \psi_k(x) \\ &= \psi_{k'}^*(0) \psi_k(0) \int_{|x|<r_0} d^D x V(x) \\ &= \psi_{k'}^*(0) \psi_k(0) a \\ &= \int d^D x [a\delta^D(x)] \psi_{k'}^*(x) \psi_k(x) \end{aligned} \quad (141)$$

so that $V(x) = a\delta^D(x)$ produces the same low-energy physics as the true potential [13], and is in fact indistinguishable from the true potential if one can only probe low energies. The argument generalizes to multiple scattering:

$$\begin{aligned}
\langle k'|VGV|k\rangle &= \int d^D x' d^D x \psi_{k'}^*(x') V(x') G(x, x') V(x) \psi_k(x) \\
&= \psi_{k'}^*(0) \psi_k(0) \int d^D x' d^D x V(x') G(x, x') V(x) \\
&= \psi_{k'}^*(0) \psi_k(0) a_1 \\
&= \int d^D x [a_1 \delta^D(x)] \psi_{k'}^*(x) \psi_k(x)
\end{aligned} \tag{142}$$

so that the T-matrix itself can be approximated as proportional to $\delta^D(x)$.

Using Feshbach resonances, experimentalists can tune the value of a , for example to $a = 0$ to create a non-interacting Bose-Einstein condensate [14].

8.2.2 2D Bose Gas

A 2D untrapped Bose gas with contact interaction can be described by the Lagrangian:

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi - g(\psi^\dagger \psi)^2 \tag{143}$$

where ψ , ψ^\dagger obey Bose statistics.

The grand potential Ω at zero temperature can be written as:

$$\begin{aligned}
\Omega[\mu] &= -\frac{1}{it} \ln Z \\
Z[\mu] &= \int [d\psi][d\psi^\dagger] e^{i \int d^2 x dt (\mathcal{L} + \mu \psi^\dagger \psi)} \\
P &= \frac{1}{itV} \ln Z
\end{aligned} \tag{144}$$

where we have expressed the grand partition function in real time with $t \rightarrow \infty$ instead of Wick rotating [15]. Since $T = 0$, the gas is a Bose-Einstein condensate. However, at any non-zero temperature, the gas is no longer a condensate. This can easily be seen for the non-interacting gas ($g = 0$):

$$n_{\text{excited}} = \int \frac{d^2k}{(2\pi)^2} \frac{1}{e^{\beta(\frac{k^2}{2} - \mu)} - 1} \quad (145)$$

where for $\mu = 0$ the integral suffers from an infrared divergence and if $\mu < 0$ then there can be no macroscopic occupation of the ground state, as:

$$\begin{aligned} N_0 &= \frac{1}{e^{\beta(\epsilon_0 - \mu)} - 1} \\ &= \frac{1}{e^{\beta(0 - \mu)} - 1} \\ &= -\frac{T}{\mu} \end{aligned} \quad (146)$$

where in the 2nd line we assumed $e^{\beta(0 - \mu)} \approx 1$ in order to maximize N_0 , and Taylor expanded the exponential. Since $\mu = \mu(\rho, T)$ there is no way to get a proportionality to N .

The result can be shown for the interacting case as well, agreeing with the Mermin-Wagner theorem, where infrared divergences in the propagation of Goldstone modes in $D \leq 2$ destabilize the system [16].

However, placed in a harmonic trap, the non-interacting 2D Bose gas can condense at finite temperature ($\omega = 1$):

$$\begin{aligned}
N_{\text{excited}} &= \int dn_x dn_y \frac{1}{e^{\beta((n_x+n_y+1)-\mu)} - 1} \\
&= \int dn_x dn_y \frac{1}{e^{\beta(n_x+n_y)} - 1}
\end{aligned} \tag{147}$$

where $\mu = 1 = \epsilon_0$. There is no longer an infrared divergence no matter the direction in which n_x and n_y go to zero, or more formally:

$$\begin{aligned}
N_{\text{excited}} &= \int dn_x dn_y \frac{1}{e^{\beta(n_x+n_y)} - 1} \\
&= \int_{-\infty}^{\infty} d\epsilon \int dn_x dn_y \frac{1}{e^{\beta(n_x+n_y)} - 1} \delta(\epsilon - n_x - n_y) \\
&= \int_{-\infty}^{\infty} d\epsilon \frac{1}{e^{\beta\epsilon} - 1} \int dn_x dn_y \delta(\epsilon - n_x - n_y) \\
&= \int_0^{\infty} d\epsilon \frac{1}{e^{\beta\epsilon} - 1} \int_0^{\epsilon} dn_x \\
&= \int_0^{\infty} d\epsilon \frac{\epsilon}{e^{\beta\epsilon} - 1} = \frac{1}{\omega^2} \int_0^{\infty} d\epsilon \frac{\epsilon}{e^{\beta\epsilon} - 1}
\end{aligned} \tag{148}$$

which has no divergence as $\epsilon \rightarrow 0$.

So far we have not taken into account the trap in our work, at least not in the two-dimensional plane of the gas. Experimentally, to confine a gas to two dimensions, ω_z is taken high enough compared to $k_B T$ to freeze out movement in the z -direction. So by representing the system with a 2D Lagrangian, we have automatically take ω_z to be high by default. However, presumably to cool the gas first requires confining the gas to a small region, so that a 2D ultra-cold gas untrapped in its plane is not experimentally feasible. However, the anomaly is due to the interaction of the gas molecules with each other, and not a property of the trap, so one should be able to understand the anomalous properties of a

gas in a trap by examination of the untrapped system: this will be discussed later.

Therefore taking all this into account, the system described by (143) and (144) is a Bose-Einstein condensate at absolute zero, but any finite temperature would destroy this. However, the anomaly for this system seems to occur in the excited states (with interactions, some of the atoms will be in excited states, even at absolute zero), whereas the condensate phase does not seem to exhibit the anomaly. So in spite of the fact that in any real experiment $T \neq 0$ and the gas is no longer a condensate, this does not imply the anomaly is destroyed at finite temperature, since it doesn't seem to be a property of the condensate phase, which we will now argue.

We evaluate $P[\mu] = \frac{1}{itV} \ln \int [d\psi][d\psi^\dagger] e^{i \int d^2x dt \left(\psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi - g(\psi^\dagger \psi)^2 + \mu \psi^\dagger \psi \right)}$ via a saddle-point expansion. The minimum of the argument of the exponential in the integrand is given by the Euler-Lagrangian equations and yields the Gross-Pitaevskii equation

$$\left(i\partial_t + \frac{\nabla^2}{2} \right) \psi - 2g(\psi^\dagger \psi)\psi + \mu\psi = 0 \quad (149)$$

In the simple model we are considering we want translational invariance of the ground state so we set derivative terms to zero and get $(\psi^\dagger \psi) = \frac{\mu}{2g}$, which gives:

$$\begin{aligned} P[\mu] &= \frac{1}{itV} \ln e^{itV \left(-g \left(\frac{\mu}{2g} \right)^2 + \mu \left(\frac{\mu}{2g} \right) \right)} \\ &= \frac{\mu^2}{4g} \end{aligned} \quad (150)$$

With this result one can in principle use thermodynamic identities to derive all other quantities. However, one can also just use quantum mechanics to argue them. The energy density of the ground state should equal μn_o , except this is overcounting by 2 in mean-field theory as the energy is due to pairwise interactions: $\frac{E_0}{V} = \mathcal{E}_0 = \frac{n_o \mu}{2}$. Moreover, the path integral gives $Z[\mu] = \langle 0 | e^{-i(H-\mu N)t} | 0 \rangle = e^{-i(E_0 - \mu N_0)t}$ which using (144) gives $P[\mu] = -\mathcal{E}_0 + \mu n_o$. Using these two equations to solve for \mathcal{E}_0 and n_o in terms of μ gives:

$$\begin{aligned} P[\mu] &= \frac{\mu^2}{4g} \\ n_o &= \frac{\mu}{2g} \\ \mathcal{E}_0 &= \frac{\mu^2}{4g} \end{aligned} \tag{151}$$

The ground state therefore has no anomaly, as $\mathcal{A} = 2\mathcal{E}_0 - 2P[\mu] = 0$.

However, when we go beyond the Gross-Pitaevskii equation to take into account quantum fluctuations, we do find the anomaly [17, 18].

$$\begin{aligned} P[\mu] &= \frac{\mu^2}{4g} - \frac{\mu^2}{16\pi} \left[1 + 2 \ln \frac{\mu}{M^2} \right] \\ n &= \frac{\mu}{2g} - \frac{\mu}{4\pi} \left[1 + \ln \frac{\mu}{M^2} \right] \\ \mathcal{E} &= \frac{\mu^2}{4g} - \frac{3\mu^2}{16\pi} \left[1 + \frac{2}{3} \ln \frac{\mu}{M^2} \right] \end{aligned} \tag{152}$$

All quantities in (152) are RG-invariant, with $g = g(M)$ and $\beta(g) = \frac{g^2}{\pi}$. The anomaly is given as $2\mathcal{E} - 2P = -\frac{n^2 g^2}{\pi}$.

In principle at least, the anomaly could be measured by measuring $P = P[n]$, calculating $\mathcal{E}(n)$ from this, and seeing if $2\mathcal{E} - 2P = 0$, as would be the case for a scale-invariant, non-anomalous system.

However, the anomaly is a quantum effect, an operator equation, which should still hold true at finite temperature, although when taking the thermal expectation value, perhaps it is washed out. In principle, the presence of a harmonic trap in the Hamiltonian can affect the operator equation, but the presence of the anomaly is unaffected. This can be seen within the path-integral approach or operator approach, but we show it within the operator approach, in 1st-quantized form. We will use this to get a well-known virial theorem for a Bose-Einstein condensate in a trap, and show how the anomaly would alter the equation.

The generator of the infinitesimal transformation $\delta x \rightarrow \rho x$, $\delta t \rightarrow 2\rho t$ is the operator $Q_D = 2tH - X \cdot P$, as can be seen by operating on a wave-function $\psi(\vec{x}, t)$:

$$\begin{aligned} \langle \vec{x} | e^{-i\rho Q_D} | \psi \rangle &= e^{-i\rho(2it\partial_t + ix_j\partial_j)} \psi(\vec{x}, t) \\ &= \psi(\vec{x}, t) + \rho(2t\partial_t + x_j\partial_j) \psi(\vec{x}, t) \end{aligned} \tag{153}$$

The commutator of H and Q_D is given by:

$$\begin{aligned}
[Q_D, H] &= [2tH - X \cdot P, H] \\
&= -[X \cdot P, \frac{P^2}{2} + V(X)] \\
&= -i \left(P^2 - X \cdot \frac{dV}{dX} \right) \\
&= -i \left(2H - \left[X \cdot \frac{dV}{dX} + 2V(X) \right] \right)
\end{aligned} \tag{154}$$

so that:

$$\begin{aligned}
\dot{Q}_D &= \frac{\partial Q_D}{\partial t} + i[H, Q_D] = 2H - \left(2H - \left[X \cdot \frac{dV}{dX} + 2V(X) \right] \right) \\
&= \left[X \cdot \frac{dV}{dX} + 2V(X) \right]
\end{aligned} \tag{155}$$

For a scale-invariant system H_s , $X \cdot \frac{dV_s}{dX} + 2V_s(X) = 0$. However, due to the anomaly, $\dot{Q}_D = A \neq 0$, which requires $i[H_s, Q_D] = -2H_s + A$ in eqn. (155), a modification of the algebra. Therefore for a scale-invariant system in a harmonic trap $H = H_s + H_{\text{osc}}$, and eqn. (154) with this modification gives:

$$\begin{aligned}
[Q_D, H] &= -i \left(2H - \left[X \cdot \frac{dV_{\text{osc}}}{dX} + 2V_{\text{osc}}(X) \right] - A \right) \\
&= -i(2H - 4V_{\text{osc}} - A)
\end{aligned} \tag{156}$$

The thermal average of the commutator $[Q_D, H]$ is proportional to

$$\langle [Q_D, H] \rangle \propto \text{Tr} [e^{-\beta(H-\mu N)} Q_D H] - \text{Tr} [e^{-\beta(H-\mu N)} H Q_D]$$

H commutes with $e^{-\beta(H-\mu N)}$, so the expectation value of this commutator is zero when using the cyclic property of the trace, hence:

$$\begin{aligned}
\langle [Q_D, H] \rangle &= 0 \\
2\langle H \rangle &= 4\langle V_{\text{osc}} \rangle + \langle A \rangle \\
2T + 2E_{\text{int}} &= 2E_{\text{osc}} + \langle A \rangle
\end{aligned}
\tag{157}$$

where we used eqn. (156) to replace the commutator in eqn. (157). This is the anomalous version of eqn. (44) in [19], if eqn. (44) were modified for 2D.

Similar relations exist for 2D fermions in a harmonic trap. For example, eqn. (157) holds for a 3D Fermi-gas at unitarity [20], and for a 2D Fermi-gas [21]. Indeed, for the 2D Fermi-gas, investigations of the frequency shift and damping rate of the breathing mode due to the anomaly has been studied [22].

9 Conclusion

In this thesis we derived four results, which are in the appendices. We calculated the scale anomaly for a non-relativistic interacting Bose gas in two spatial dimensions using Fujikawa's method, described by the Lagrangian

$$\begin{aligned}\mathcal{L} &= \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi \\ &\quad - \frac{1}{2} \int d^2\vec{y} \psi^\dagger(t, \vec{x}) \psi(t, \vec{x}) V(\vec{x} - \vec{y}) \psi^\dagger(t, \vec{y}) \psi(t, \vec{y}) \\ V(\vec{x} - \vec{y}) &= g \delta^2(\vec{x} - \vec{y}) \\ \mathcal{L} &= \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi - \frac{g}{2} (\psi^\dagger \psi)^2\end{aligned}$$

We derived the n -body virial theorem in D spatial dimensions using path integrals:

$$\begin{aligned}DPV &= 2T - \left\langle \frac{1}{n!} \int \left(\prod_i^n d^D \vec{x}_i \rho(\tau, \vec{x}_i) \right) \left[\vec{z}_{\text{COM}} \cdot \nabla_{\vec{z}_{\text{COM}}} \tilde{V}(\vec{z}_{\text{COM}}, \vec{z}_2, \dots, \vec{z}_n) \right] \right\rangle \\ &\quad - \left\langle \frac{1}{n!} \int \left(\prod_i^n d^D \vec{x}_i \rho(\tau, \vec{x}_i) \right) \left[\sum_{i=2}^n \vec{z}_i \cdot \nabla_{\vec{z}_i} \tilde{V}(\vec{z}_{\text{COM}}, \vec{z}_2, \dots, \vec{z}_n) \right] \right\rangle.\end{aligned}$$

We also considered the relativistic case, and compared the path integral method with other methods.

We gave an introduction to symmetries and the quantum mechanics relevant for our work, and briefly discussed some experimental aspects of anomalies. We have shown that the path-integral can be used to describe anomalies in non-relativistic physics.

10 Speculation

1. Relativistic (2+1) $\lambda\phi^4$ is super-renormalizable, meaning it's free of ultraviolet divergences. Its nonrelativistic limit is the $\lambda\delta^2(\vec{r})$ potential [12]. For a nonrelativistic system, the natural scale at which new physics appear occurs when $E \rightarrow mc^2$. Therefore it might be the case that $E_b \propto \lambda^n mc^2$ for some value of n if one integrates out the high-energy modes of the relativistic theory down to $\Lambda \rightarrow mc^2$.
2. The anomaly for the $\lambda\delta^2(\vec{r})$ potential is an exact, non-perturbative calculation. Moreover, it is not affected by spontaneous symmetry breaking. Therefore for the broken system Eqn. (24) of section E should still hold:

$$\begin{aligned}
 2\mathcal{E} - 2P &= -\frac{\lambda^2}{2\pi} \langle \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \rangle \\
 &= \frac{\lambda^2}{2\pi} \frac{\partial P}{\partial \lambda}
 \end{aligned}
 \tag{158}$$

At $T = 0$, thermodynamically $\mathcal{E} = -P + \mu \frac{\partial P}{\partial \mu}$. If one were to plug this into (158), one would get a differential equation in P that has the intriguing property that the RHS is proportional to one power of the coupling λ . In other words, say you know P to order λ^n . If you plug this into the RHS of (158), the RHS becomes proportional to λ^{n+1} . Then you have:

$$2 \left(-P + \mu \frac{\partial P}{\partial \mu} \right) - 2P = O[\lambda^{n+1}]
 \tag{159}$$

which gives a differential equation for P good to order λ^{n+1} . One might be able to exploit this to calculate P to higher orders in lieu of going one order higher in a diagrammatic calculation of P .

3. One could take relativistic $(2+1) \lambda\phi^4$, but perform a nonrelativistic scaling in which the speed of light c will scale as its nonrelativistic dimensions. Then take the nonrelativistic limit of this result.

11 Bibliography

- [1] C. R. Ordóñez, “Path-integral fujikawa’s approach to anomalous virial theorems and equations of state for systems with $so(2,1)$ symmetry,” arXiv:1503.01384, 2015.
- [2] D. Schroeder, An Introduction to Thermal Physics. Addison-Wesley, 1999.
- [3] R. Jackiw, “Delta function potentials in two-dimensional and three-dimensional quantum mechanics,” Beg Memorial Volume, vol. A60, 1991.
- [4] K. Huang, Quarks, Leptons and Gauge Fields. World Scientific, 1981.
- [5] D. R. Phillips, S. R. Beane, and T. D. Cohen, “Nonperturbative regularization and renormalization: Simple examples from nonrelativistic quantum mechanics,” Annals of Physics, vol. 263, no. 2, pp. 255 – 275, 1998.
- [6] G. N. J. Añaños, H. E. Camblong, C. Gorrichátegui, E. Hernández, and C. R. Ordóñez, “Anomalous commutator algebra for conformal quantum mechanics,” Phys. Rev. D, vol. 67, p. 045018, Feb 2003.
- [7] C. G. Callan, S. Coleman, and R. Jackiw, “A new improved energy-momentum tensor,” Annals of Physics, vol. 59, no. 1, pp. 42 – 73, 1970.
- [8] S. Coleman and R. Jackiw, “Why dilatation generators do not generate dilatations,” Annals of Physics, vol. 67, no. 2, pp. 552 – 598, 1971.
- [9] J. Gaiete, “The relativistic virial theorem and scale invariance,” Physics-Uspekhi, vol. 56, no. 9, p. 919, 2013.
- [10] B. Hatfield, Quantum Field Theory of Point Particles and Strings. Westview Press, 1998.
- [11] J. I. Kapusta and C. Gale, Finite-Temperature Field Theory: Principles and Applications. Cambridge University Press, 2006.
- [12] E. Braaten and H.-W. Hammer, “Universality in few-body systems with large scattering length,” Physics Reports, vol. 428, no. 56, pp. 259 – 390, 2006.
- [13] M. A. Luty, “Renormalization,” University of Maryland. January 2007.
- [14] S. Inouye, M. R. Andrews, et al., “Observation of feshbach resonances in a bose-einstein condensate,” Nature, vol. 392, pp. 151 – 154, 1998.

- [15] A. Schakel, Boulevard of Broken Symmetries: Effective Field Theories of Condensed Matter. World Scientific, 2008.
- [16] S. Coleman, “There are no goldstone bosons in two dimensions,” Communications in Mathematical Physics, vol. 31, no. 4, pp. 259–264, 1973.
- [17] G. Lozano, “Ground state energy for non-relativistic bosons coupled to chern-simons gauge fields,” Physics Letters B, vol. 283, no. 1, pp. 70 – 74, 1992.
- [18] T. Haugset and F. Ravndal, “Scale anomalies in nonrelativistic field theories in $2 + 1$ dimensions,” Phys. Rev. D, vol. 49, pp. 4299–4301, Apr 1994.
- [19] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, “Theory of bose-einstein condensation in trapped gases,” Rev. Mod. Phys., vol. 71, pp. 463–512, Apr 1999.
- [20] S. Tan, “Generalized virial theorem and pressure relation for a strongly correlated fermi gas,” Annals of Physics, vol. 323, no. 12, pp. 2987 – 2990, 2008.
- [21] J. Hofmann, “Quantum anomaly, universal relations, and breathing mode of a two-dimensional fermi gas,” Phys. Rev. Lett., vol. 108, p. 185303, May 2012.
- [22] C. Chafin and T. Schäfer, “Scale breaking and fluid dynamics in a dilute two-dimensional fermi gas,” Phys. Rev. A, vol. 88, p. 043636, Oct 2013.

A Path-Integral Derivation of Nonrelativistic Scale Anomaly

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Path-Integral Derivation of the Non-relativistic Scale Anomaly

Chris L. Lin and Carlos R. Ordóñez

Department of Physics, University of Houston, Houston, TX 77204-5005

In this paper we calculate the scale anomaly for a quantum field theoretic 2D non-relativistic Bose gas with contact interactions using Fujikawa's method, both in vacuum and in many-body systems. The use of path integrals for these problems is novel and motivated by a recently developed path-integral framework for addressing questions about scaling in these systems. A natural class of regulators is found that produces the correct value of the anomaly traditionally calculated via other methods, e.g., diagrammatically via the β function.

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1 Introduction

The use of Fujikawa's method in particle physics is well known and is now standard in textbooks [1]. It was originally developed to understand the chiral anomaly [2] but has since been extended to other cases, including the relativistic scale anomaly [3]. However, as far as we are aware, it has not been used before for non-relativistic physics. There are currently reasons to embark in such calculations. Non-relativistic anomalies have been studied since the seminal paper by R. Jackiw [4], mostly using canonical methods, not Fujikawa's¹. Interest in these anomalies

¹See [5, 6] and references therein.

has intensified in the study of ultracold 2D gases [7, 8, 9, 10, 11, 12, 13, 14, 15], with the work by J. Hofmann on anomalies of trapped 2D Fermion gases being of particular relevance [16]. Despite all this activity, there are still questions about anomalies and their impact in such systems that need to be answered [17]. A path-integral Fujikawa approach to study anomalies in systems with an $SO(2,1)$ classical symmetry, mainly in the context of 2D diluted gases, has been recently proposed in [18]. While this approach provides a nice picture of the structure of anomalies in many-body systems, the calculation of the Fujikawa Jacobian is crucial in order for this framework to also provide a practical scheme that will help us better understand the role of anomalies in lower-dimensional physics. We present here our first results of the Jacobian calculation for 2D complex fields with contact interactions in the case of constant background fields.

Within the path-integral formulation, anomalies result from the presence of Jacobians due to the non-invariance of the measure under symmetry transformations. These Jacobians are functional determinants and need to be regularized. For the chiral anomaly, all regulators lead to a finite result, whereas for the relativistic scale anomaly an infinite piece remains that is present even if the same regulator is used in the free theory, so this piece can be subtracted if the free theory is taken to be non-anomalous [19]. The non-relativistic scale anomaly is similar to the relativistic case in this respect. However, unlike the latter, space and time are treated on unequal footing in the former. Indeed, traditionally, for both the relativistic chiral and scale anomalies, one goes into Euclidean space where the Lagrangian

kinetic operator is Hermitian. In this Euclidean space one can work with functions of a single variable (the 4 momentum squared) that is positive semi-definite in all directions. In contrast, for the non-relativistic case the Lagrangian operator is Hermitian in real time (“Minkowski space”). Due to the asymmetry between space and time, one is stuck with ω and \vec{k}^2 rather than a single k^2 , making the task considerably more difficult, which may be a reason for why this problem has not been addressed before using Fujikawa’s method.

The structure of this paper is as follows: we give a brief introduction to Fujikawa’s method, after which we review the essential technical details for the system that will be considered here. We then proceed with the Jacobian calculation for zero and finite temperature. Conclusions and comments end the paper.

2 Fujikawa’s Derivation

The derivation of the anomaly via Fujikawa’s method presented here follows closely the path-integral derivation of the Ward identities, but now the Jacobian of the symmetry transformation is taken into account. Indeed, anomalies represent a breakdown of the Ward identities, and it is precisely the Jacobian that invalidates the identities. For simplicity we will demonstrate the derivation for a scalar field theory without sources: the generalization to other (multiple) fields is straightforward. With a change of variables given by $\phi'(x) = \phi(x) + \eta\delta\phi(x)$:

$$\begin{aligned}
\int [d\phi] e^{iS[\phi]} &= \int [d\phi'] \left| \frac{\delta\phi}{\delta\phi'} \right| e^{iS[\phi(\phi')]} \\
&= \int [d\phi'] \left| \delta^d(x-y) - \eta \frac{\delta\delta\phi'(x)}{\delta\phi'(y)} \right| e^{iS[\phi' - \eta\delta\phi']} \\
&= \int [d\phi] \left| \delta^d(x-y) - \eta \frac{\delta\delta\phi(x)}{\delta\phi(y)} \right| e^{iS[\phi - \eta\delta\phi]} \quad (1) \\
&= \int [d\phi] e^{-\eta \int d^d x \frac{\delta\delta\phi}{\delta\phi}} e^{iS[\phi]} e^{-i\eta \int d^d x \frac{\delta S}{\delta\phi} \delta\phi} \\
&= \int [d\phi] e^{iS[\phi]} \left(1 - \eta \int d^d x \frac{\delta\delta\phi}{\delta\phi} - i\eta \int d^d x \frac{\delta S}{\delta\phi} \delta\phi \right).
\end{aligned}$$

Since this holds for any volume V , it follows:

$$\left\langle \frac{\delta S}{\delta\phi} \delta\phi \right\rangle = i \left\langle \frac{\delta\delta\phi(x)}{\delta\phi(y)} \Big|_{y=x} \right\rangle. \quad (2)$$

Now $\frac{\delta S}{\delta\phi} \delta\phi = \frac{\partial\mathcal{L}}{\partial\phi} \delta\phi - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \delta\phi$. However, if $\delta\phi$ is a symmetry transformation, then $\frac{\partial\mathcal{L}}{\partial\phi} \delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \delta\partial_\mu\phi = \partial_\mu K^\mu$, so $\frac{\delta S}{\delta\phi} \delta\phi = -\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \delta\partial_\mu\phi + \partial_\mu K^\mu - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \delta\phi$ or $\frac{\delta S}{\delta\phi} \delta\phi = \partial_\mu \left(-\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \delta\phi + K^\mu \right) = -\partial_\mu j^\mu$.

So Fujikawa's method tells us that:

$$\langle \partial_\mu j^\mu \rangle = -i \left\langle \frac{\delta\delta\phi(x)}{\delta\phi(y)} \Big|_{y=x} \right\rangle. \quad (3)$$

Had we added a source term $\int d^d x J(x)\phi(x)$, the equation would read:

$$\langle \partial_\mu j^\mu \rangle - \langle J\delta\phi \rangle = -i \left\langle \frac{\delta\delta\phi(x)}{\delta\phi(y)} \Big|_{y=x} \right\rangle. \quad (4)$$

Differentiation w.r.t. to $J(x_i)$ n times and setting $J = 0$ would create contact

terms:

$$\begin{aligned} \langle \partial_\mu j^\mu(x) \phi(x_1) \dots \phi(x_n) \rangle + i \sum_{i=1}^n \langle \phi(x_1) \dots \delta\phi(x_i) \delta^d(x - x_i) \dots \phi(x_n) \rangle \\ = -i \left\langle \frac{\delta\delta\phi(x)}{\delta\phi(y)} \Big|_{y=x} \phi(x_1) \dots \phi(x_n) \right\rangle. \end{aligned} \quad (5)$$

Eq. (5) without the Jacobian contribution is the traditional Ward identity at zero temperature, in vacuum, presented in most textbooks [20]. In our case, we only need the Jacobian of the infinitesimal transformation by itself in order to compute the RHS of Eq. (3) and compare our results with the literature for both the zero-temperature and the finite-temperature case. For the latter, we will work within the framework of reference [18], for which a detailed calculation is mandatory.

3 Contact Interaction

The Schrödinger Lagrangian density for bosons with contact interaction in 2D is given by:

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi - \frac{g}{2} (\psi^\dagger \psi)^2, \quad (6)$$

which is the 2-body interaction with a $V(\vec{x} - \vec{y}) = g \delta^2(\vec{x} - \vec{y})$ potential:

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi - \frac{1}{2} \int d^2\vec{y} \psi^\dagger(t, \vec{x}) \psi(t, \vec{x}) V(\vec{x} - \vec{y}) \psi^\dagger(t, \vec{y}) \psi(t, \vec{y}). \quad (7)$$

The action corresponding to this Lagrangian is scale-invariant. This can be readily seen by noting that in $D = 2$, the coupling g has no dimensions in units of length (with $\hbar = m = 1$). Therefore Eq. (3) applies.

4 Scale Transformation

Under a non-relativistic dilation transformation [21]:

$$\begin{aligned}\vec{x}' &= \lambda\vec{x}, \\ t' &= \lambda^2 t, \\ \psi'(\vec{x}', t') &= \lambda^{-D/2} \psi(\vec{x}, t).\end{aligned}\tag{8}$$

Setting $\lambda = 1 + \eta$ for infinitesimal η :

$$\begin{aligned}\delta\vec{x} &= \eta\vec{x}, \\ \delta t &= 2\eta t, \\ \tilde{\delta}\psi &= \eta\theta\psi(t, \vec{x}) \equiv \eta\delta\psi, \\ \tilde{\delta}\psi^* &= \eta\theta\psi^*(t, \vec{x}) \equiv \eta\delta\psi^*, \\ \theta &\equiv \left(-\frac{D}{2} - \vec{x} \cdot \vec{\nabla} - 2t\partial_t\right).\end{aligned}\tag{9}$$

where $D = d - 1$ is the spatial dimension². In this paper we will set $D = 2$.

²We used $\tilde{\delta}\psi$ ($\tilde{\delta}\psi^*$) for the infinitesimal change in ψ (ψ^*), and set $\tilde{\delta}\psi = \eta\delta\psi$ to make the notation consistent with Eq. (2).

Therefore³:

$$\left. \frac{\delta\delta\psi(x)}{\delta\psi(y)} \right|_{y=x} = [\theta\delta(x_0 - y_0)\delta^2(\vec{x} - \vec{y})] \Big|_{y=x} = \left. \frac{\delta\delta\psi^*(x)}{\delta\psi^*(y)} \right|_{y=x}. \quad (10)$$

Note that unlike translations, for dilations both conventions,

$$\delta\psi = \eta \left(-1 - \vec{x} \cdot \vec{\nabla} - 2t\partial_t \right) \psi(t, \vec{x})$$

or

$$\delta\psi = \eta \left(1 + \vec{x} \cdot \vec{\nabla} + 2t\partial_t \right) \psi(t, \vec{x}),$$

leading to currents of opposite sign, are widely used. We've adopted the former, which leads to a dilation charge of [22]:

$$\begin{aligned} D &= \int d^2\vec{x} \vec{x} \cdot \vec{j} - 2tH, \\ \vec{j} &= -\frac{i}{2} \left(\psi^\dagger \vec{\nabla} \psi - \vec{\nabla} \psi^\dagger \psi \right). \end{aligned} \quad (11)$$

5 Fujikawa Calculation: Set Up

The generalization of the scalar case to our Lagrangian is straightforward:

³Sometimes we write $x = (x_0, \vec{x}) = (t, \vec{x})$ for notational convenience.

$$\begin{aligned}
\det \begin{pmatrix} \frac{\delta\psi(x)}{\delta\psi'(y)} & \frac{\delta\psi(x)}{\delta\psi'^*(y)} \\ \frac{\delta\psi^*(x)}{\delta\psi'(y)} & \frac{\delta\psi^*(x)}{\delta\psi'^*(y)} \end{pmatrix} &= \det \begin{pmatrix} \delta^3(x-y) - \eta\theta\delta^3(x-y) & 0 \\ 0 & \delta^3(x-y) - \eta\theta\delta^3(x-y) \end{pmatrix} \\
&= \exp \left(-\eta \int dt d^2\vec{x} \operatorname{tr} \begin{bmatrix} \theta\delta(x_0 - y_0)\delta^2(\vec{x} - \vec{y}) & 0 \\ 0 & \theta\delta(x_0 - y_0)\delta^2(\vec{x} - \vec{y}) \end{bmatrix} \Big|_{y=x} \right). \tag{12}
\end{aligned}$$

where we've used $\det A = e^{\hat{\operatorname{Tr}} \log A}$ ⁴. Comparison with Eq. (3) makes the generalization clear:

$$\langle \partial_\mu j^\mu \rangle = -i \operatorname{tr} \begin{bmatrix} \theta\delta^3(x-y) & 0 \\ 0 & \theta\delta^3(x-y) \end{bmatrix} \Big|_{y=x}. \tag{13}$$

This expression is singular so needs to be regularized. This is done by expanding $\delta^3(x-y)I_2$, where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, using the eigenbasis ϕ_n of a Hermitian operator M :

$$\delta^3(x-y)I_2 = \sum_n \phi_n(x_0, \vec{x}) \phi_n^\dagger(y_0, \vec{y}). \tag{14}$$

Inserting a regulator that's a function of M

$$\delta_R^3(x-y)I_2 = \sum_n R\left(\frac{M}{\Lambda^2}\right) \phi_n(x_0, \vec{x}) \phi_n^\dagger(y_0, \vec{y}), \tag{15}$$

with the property that $R(0) = 1$ so that at the end of the calculation we send

⁴ $\hat{\operatorname{Tr}}$ includes *both* functional and matrix indices; tr only refers to the 2×2 matrix indices in Eqs. (12) and (13).

$\Lambda \rightarrow \infty$ and $\lim_{\Lambda \rightarrow \infty} R\left(\frac{M}{\Lambda^2}\right) = 1$ ($[M] = [\Lambda^2]$). The idea is to choose R such that large eigenvalues are suppressed giving a convergent sum:

$$\delta_R^3(x - x)I_2 = \sum_n R\left(\frac{\lambda_n}{\Lambda^2}\right) \phi_n(x_0, \vec{x}) \phi_n^\dagger(x_0, \vec{x}). \quad (16)$$

Once the Hermitian operator M has been selected, then the sum over n in Eq. (15) gives:

$$\begin{aligned} \delta_R^3(x - y)I_2 &= \sum_n R\left(\frac{M}{\Lambda^2}\right) \phi_n(x_0, \vec{x}) \phi_n^\dagger(y_0, \vec{y}) \\ &= R\left(\frac{M}{\Lambda^2}\right) \delta^3(x - y)I_2, \end{aligned} \quad (17)$$

so that

$$\text{tr} [\delta_R^3(x - y)I_2] = \text{tr} \left[R\left(\frac{M}{\Lambda^2}\right) \delta^3(x - y)I_2 \right]. \quad (18)$$

For the class of regulators defined by Eqs. (19) and (20) in the next section - a consistent choice for the untrapped system - it will be shown that the non-trivial contribution to Eq. (18) will have an even integrand in *both* ω and \vec{k} (Eq. (28)). This means that the derivative terms in θ will give a null contribution and the only term that will survive is given by Eq. (18). To see this, one should take the space-time derivatives in Eq. (21), and then set $x = y$; these terms will give odd contributions to the integrand in (ω, \vec{k}) space when multiplied by the even terms of Eq. (28).

Therefore, Eq. (18) is the expression we aim to calculate.

6 Fujikawa Calculation: Mode Expansion

6.1 Zero Temperature

For our Hermitian matrix we will choose [23]:

$$M = \begin{pmatrix} i\partial_t + \frac{\nabla^2}{2} + \mu - 2g\psi^*\psi + i\epsilon & -g\psi^2 \\ -g\psi^{*2} & -i\partial_t + \frac{\nabla^2}{2} + \mu - 2g\psi^*\psi + i\epsilon \end{pmatrix} \quad (19)$$

where the fields in M are constant background fields, and $\frac{1}{2} \begin{pmatrix} \chi^\dagger & \chi \end{pmatrix} M \begin{pmatrix} \chi \\ \chi^\dagger \end{pmatrix}$ is the quadratic Lagrangian resulting from a saddle point expansion of the action about the background field ψ , and χ is the shift of the original field from ψ . We've included a chemical potential μ for many-body physics that explicitly breaks scale-invariance, but we can always set $\mu = 0$ and as we will demonstrate, the inclusion of μ has no effect on the anomaly.

For our regulating function R we will choose

$$R\left(\frac{M}{\Lambda^2}\right) = \left(1 \pm \frac{M}{\Lambda^2}\right)^{-1}, \quad (20)$$

which clearly satisfies $R(0) = 1$. Plugging in this regulator into Eq. (17) and Fourier expanding $\delta^3(x - y)$ gives:

$$\begin{aligned} \text{tr} [\delta_R^3(x-y)I_2] &= \int \frac{d\omega}{2\pi} \int \frac{d^2\vec{k}}{(2\pi)^2} \\ &\text{tr} \left(\begin{array}{cc} 1 \pm \frac{\omega - \frac{k^2}{2} + \mu - 2g\psi^*\psi + i\epsilon}{\Lambda^2} & \mp \frac{g\psi^2}{\Lambda^2} \\ \mp \frac{g\psi^{*2}}{\Lambda^2} & 1 \pm \frac{-\omega - \frac{k^2}{2} + \mu - 2g\psi^*\psi + i\epsilon}{\Lambda^2} \end{array} \right)^{-1} e^{-i\omega(x_0 - y_0) + i\vec{k} \cdot (\vec{x} - \vec{y})}. \end{aligned} \quad (21)$$

We will now take $(x_0, \vec{x}) = (y_0, \vec{y})$; will make a change of variables $\tilde{\omega} = \frac{\omega}{\Lambda^2}$ and $\tilde{k} = \frac{k}{\Lambda}$; and then replace the tildes since they are dummy indices (ω and \vec{k} are now dimensionless):

$$\begin{aligned} \text{tr} [\delta_R(0)I_2] &= \Lambda^4 \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} \\ &\text{tr} \left(\begin{array}{cc} 1 \pm \left(\omega - \frac{k^2}{2} + \frac{\mu - 2g\psi^*\psi + i\epsilon}{\Lambda^2} \right) & \mp \frac{g\psi^2}{\Lambda^2} \\ \mp \frac{g\psi^{*2}}{\Lambda^2} & 1 \pm \left(-\omega - \frac{k^2}{2} + \frac{\mu - 2g\psi^*\psi + i\epsilon}{\Lambda^2} \right) \end{array} \right)^{-1}. \end{aligned} \quad (22)$$

For notational convenience we will write the above expression as:

$$\text{tr} [\delta_R(0)I_2] = \pm \Lambda^4 \int \frac{d\omega}{2\pi} \int \frac{d^2\vec{k}}{(2\pi)^2} \text{tr} \left(\begin{array}{cc} \omega - \frac{k^2}{2} + A_{\pm} + i\epsilon & -\frac{g\psi^2}{\Lambda^2} \\ -\frac{g\psi^{*2}}{\Lambda^2} & -\omega - \frac{k^2}{2} + A_{\pm} + i\epsilon \end{array} \right)^{-1}, \quad (23)$$

with

$$A_{\pm} = \pm 1 + \frac{\mu - 2g\psi^*\psi}{\Lambda^2}. \quad (24)$$

To evaluate the inverse in Eq. (23), we will use the identity for matrix inverses $(D + B)^{-1} = D^{-1} - (D^{-1}B)D^{-1} + (D^{-1}B)(D^{-1}B)D^{-1} - \dots$ with

$$D_{\pm} = \begin{pmatrix} \omega - \frac{k^2}{2} + A_{\pm} + i\epsilon & 0 \\ 0 & -\omega - \frac{k^2}{2} + A_{\pm} + i\epsilon \end{pmatrix}, \quad (25)$$

$$B = \begin{pmatrix} 0 & -\frac{g\psi^2}{\Lambda^2} \\ -\frac{g\psi^{*2}}{\Lambda^2} & 0 \end{pmatrix}.$$

Note that the $i\epsilon$ makes D_{\pm} invertible.

So Eq. (22) becomes:

$$\text{tr} [\delta_R(0)I_2] = \pm\Lambda^4 \int \frac{d\omega}{2\pi} \int \frac{d^2\vec{k}}{(2\pi)^2} \text{tr} (D_{\pm}^{-1} - (D_{\pm}^{-1}B)D_{\pm}^{-1} + (D_{\pm}^{-1}B)(D_{\pm}^{-1}B)D_{\pm}^{-1}), \quad (26)$$

where we terminated the series at two powers of B , since each additional power of B produces a $\frac{1}{\Lambda^2}$ that the Λ^4 prefactor can't offset.

The first term in the series, D_{\pm}^{-1} , when doing the integral over ω , is independent of the coupling:

$$\begin{aligned}
& \pm \Lambda^4 \int \frac{d\omega}{2\pi} \int \frac{d^2 \vec{k}}{(2\pi)^2} \text{tr} \begin{pmatrix} \frac{1}{\omega - \frac{k^2}{2} + A_{\pm} + i\epsilon} & 0 \\ 0 & \frac{1}{-\omega - \frac{k^2}{2} + A_{\pm} + i\epsilon} \end{pmatrix} \\
&= \pm \Lambda^4 \int \frac{d\omega}{2\pi} \int \frac{d^2 \vec{k}}{(2\pi)^2} \begin{pmatrix} 2(\frac{k^2}{2} - A_{\pm} - i\epsilon) \\ \omega^2 - (\frac{k^2}{2} - A_{\pm} - i\epsilon)^2 \end{pmatrix} \\
&= \frac{\mp i}{2} \Lambda^4 \int \frac{d^2 \vec{k}}{(2\pi)^2}.
\end{aligned} \tag{27}$$

Therefore this term is also contained in the free-case, which we take to be anomaly-free. So we subtract this term when calculating the anomaly. The next term $(D_{\pm}^{-1}B)D_{\pm}^{-1}$ has no diagonal elements, so is traceless.

The only term to calculate is the $(D_{\pm}^{-1}B)(D_{\pm}^{-1}B)D_{\pm}^{-1}$ term which produces:

$$\begin{aligned}
\text{tr} [\delta_R(0)I_2] &= \pm \Lambda^4 \int \frac{d\omega}{2\pi} \int \frac{d^2 \vec{k}}{(2\pi)^2} \begin{pmatrix} -g\psi^2 \\ \Lambda^2 \end{pmatrix} \begin{pmatrix} -g\psi^{*2} \\ \Lambda^2 \end{pmatrix} \begin{pmatrix} -2(\frac{k^2}{2} - A_{\pm} - i\epsilon) \\ [\omega^2 - (\frac{k^2}{2} - A_{\pm} - i\epsilon)^2]^2 \end{pmatrix} \\
&= \pm g^2(\psi^*\psi)^2 \int \frac{d\omega}{2\pi} \int_0^{\infty} \frac{dk}{2\pi} \begin{pmatrix} -2k(\frac{k^2}{2} - A_{\pm} - i\epsilon) \\ [\omega^2 - (\frac{k^2}{2} - A_{\pm} - i\epsilon)^2]^2 \end{pmatrix}.
\end{aligned} \tag{28}$$

The integral over k is straightforward:

$$\text{tr} [\delta_R(0)I_2] = \pm \frac{g^2(\psi^*\psi)^2}{2\pi} \int \frac{d\omega}{2\pi} \begin{pmatrix} 1 \\ \omega^2 - (A_{\pm} + i\epsilon)^2 \end{pmatrix}. \tag{29}$$

Now A_{\pm} in Eq. (24) can be safely taken to ± 1 ($\Lambda \rightarrow \infty$). For both \pm cases, the result is the same:

$$\text{tr} [\delta_R(0)I_2] = i \frac{g^2(\psi^*\psi)^2}{4\pi}. \quad (30)$$

Plugging this into Eq. (13) gives:

$$\langle \partial_\mu j^\mu \rangle = - \frac{g^2(\psi^*\psi)^2}{4\pi}. \quad (31)$$

This can be compared with [24] (for the case of constant background fields) by making the replacement $g \rightarrow \frac{g}{2}$.

Because both $R\left(\frac{M}{\Lambda^2}\right) = \left(1 \pm \frac{M}{\Lambda^2}\right)^{-1}$ work as regulators, any linear combination such that their coefficients add to one works. For example:

$$\begin{aligned} R\left(\frac{M}{\Lambda^2}\right) &= \frac{1}{2} \left(1 + \frac{M}{\Lambda^2}\right)^{-1} + \frac{1}{2} \left(1 - \frac{M}{\Lambda^2}\right)^{-1} \\ &= \left(1 - \frac{M^2}{\Lambda^4}\right)^{-1}. \end{aligned} \quad (32)$$

We have also verified that the following regulators work:

$$R\left(\frac{M}{\Lambda^2}\right) = \left(1 \pm \frac{M}{\Lambda^2}\right)^{-2}. \quad (33)$$

6.2 Many-Body

Under the formalism developed in [18]⁵:

⁵The factor of βA in the denominator of Eq. (52) in [18] cancels the Euclidean version of the factor $\int_0^\beta d\tau \int d^2\vec{x}$ from Eq. (12) in this paper, since for constant background fields our class of regulators gives a constant value for $\text{tr}(\delta_R(0)I_2)$. The time and spatial derivatives in $\hat{\theta}_s$ and $\hat{\theta}$ in paper [18] give no contributions in this case (untrapped) as explained here. Notice Eq. (34)

$$2\mathcal{E} - 2P = \text{tr} [\delta_R(0)I_2]. \quad (34)$$

However, here the anomalous term is evaluated in Euclidean space using the finite temperature rules. Eq. (28) with plus chosen is:

$$\text{tr} [\delta_R(0)I_2] = \frac{g^2(\psi^*\psi)^2}{2\pi} \int \frac{d\omega}{2\pi} \left(\frac{1}{\omega^2 - (1 + i\epsilon)^2} \right). \quad (35)$$

In terms of the original dimensionful ω :

$$\begin{aligned} \text{tr} [\delta_R(0)I_2] &= \frac{g^2(\psi^\dagger\psi)^2}{2\pi} \Lambda^2 \int \frac{\Lambda^2 d\omega}{2\pi} \left(\frac{1}{(\Lambda^2\omega)^2 - (\Lambda^2 + i\epsilon)^2} \right) \\ &= \frac{g^2(\psi^\dagger\psi)^2}{2\pi} \Lambda^2 \int \frac{d\omega}{2\pi} \left(\frac{1}{\omega^2 - (\Lambda^2 + i\epsilon)^2} \right). \end{aligned} \quad (36)$$

When going to finite temperature, the difference is that we have $-\partial_\tau$ instead of $i\partial_t$. The effect is to replace ω with $i\omega$ in Eq. (35). That is, had we started directly in Euclidean space, we would still get Eq. (35), but with ω replaced by $i\omega$ stemming from $-\partial_\tau$ replacing $i\partial_t$ in our regulator. The second change is that the integral is a sum since the modes are discrete, with a β factor resulting from writing the delta function as $\delta^d(x - y) = \frac{1}{\beta} \sum_n \int \frac{d^D k}{(2\pi)^D} e^{-i\omega_n(x_0 - y_0)} e^{i\vec{k}\cdot(\vec{x} - \vec{y})}$. So a sum replaces the integral. So had we started directly in Euclidean space, we would have a sum over frequencies instead of an integral:

$$\text{tr} [\delta_R(0)I_2] = \frac{g^2(\psi^*\psi)^2}{2\pi} \frac{\Lambda^2}{\beta} \sum_n \left(\frac{1}{-\omega_n^2 - (\Lambda^2 + i\epsilon)^2} \right), \quad (37)$$

where $\omega_n = \frac{2\pi n}{\beta}$ for bosons. The $i\epsilon$ no longer matters and the summation is

is a 2×2 version of Eq. (52) in [18].

standard:

$$\text{tr} [\delta_R(0)I_2] = -\frac{g^2(\psi^*\psi)^2}{2\pi} \frac{\Lambda^2}{\beta} \left(\frac{\beta \coth\left(\frac{\beta\Lambda^2}{2}\right)}{2\Lambda^2} \right), \quad (38)$$

which in the limit of large Λ gives:

$$\text{tr} [\delta_R(0)I_2] = -\frac{g^2(\psi^*\psi)^2}{4\pi}. \quad (39)$$

So plugging this into Eq. (34)

$$2\mathcal{E} - 2P = -\frac{g^2(\psi^*\psi)^2}{4\pi}, \quad (40)$$

which agrees with [25] with $g \rightarrow 2g$. For the finite temperature case, there is some ambiguity in the continuation to Euclidean space that affects the sign, where A_+ leads to the correct sign, and A_- leads to the negative sign. We take the view that the zero-temperature limit must reproduce the vacuum result.

7 Conclusion

Fujikawa's path-integral method has been applied to the Schrödinger Lagrangian to describe anomalies for 2D non-relativistic, $SO(2,1)$ scale-invariant complex bosons with contact interactions. A class of natural regulators was identified that gives results consistent with those in the literature, obtained with other methods, in both zero and finite-temperature cases [24, 25]. This work was motivated by the recent formulation of Fujikawa's approach to analyze the anomaly structure for 2D

gases with $SO(2,1)$ classical symmetry (and other systems with such symmetry) [18], which is relevant in the study of ultracold 2D trapped gases [16]. It was important, therefore, that we made contact with established work using other techniques. Further work is needed for a deeper understanding of this method and its possible applications. In particular, heat kernel techniques will be used to investigate trapped systems. Work on these issues is in progress [26].

References

- [1] S. Weinberg, The Quantum Theory of Fields. Vol. 2. Cambridge University Press, 1996.
- [2] K. Fujikawa, “Path Integral Measure for Gauge Invariant Fermion Theories,” Phys.Rev.Lett., vol. 42, pp. 1195–1198, 1979.
- [3] K. Shizuya and H. Tsukahara, “Path Integral Formulation of Conformal Anomalies,” Z.Phys., vol. C31, pp. 553–556, 1986.
- [4] R. Jackiw, “Delta function potentials in two-dimensional and three-dimensional quantum mechanics,” Beg Memorial Volume, vol. A60, 1991.
- [5] H. E. Camblong and C. R. Ordenez, “Anomaly in conformal quantum mechanics: From molecular physics to black holes,” Phys.Rev., vol. D68, p. 125013, 2003.
- [6] S. Moroz, “Nonrelativistic scale anomaly, and composite operators with complex scaling dimensions,” Annals Phys., vol. 326, pp. 1368–1380, 2011.

- [7] M. Olshanii, H. Perrin, and V. Lorent, “Example of a quantum anomaly in the physics of ultracold gases,” Phys. Rev. Lett., vol. 105, p. 095302, Aug 2010.
- [8] M. Valiente, N. T. Zinner, and K. Mølmer, “Universal relations for the two-dimensional spin-1/2 fermi gas with contact interactions,” Phys. Rev. A, vol. 84, p. 063626, Dec 2011.
- [9] T. Fujiwara, Y. Igarashi, J. Kubo, and K. Maeda, “A New insight into BRST anomalies in string theory,” Nucl.Phys., vol. B391, pp. 211–228, 1993.
- [10] C. Fosco and R. Trincherio, “Anomalous Commutators and Functional Integration,” Phys.Rev., vol. D41, p. 1216, 1990.
- [11] M. Feld and et al Nature, vol. 480, p. 75, 2011.
- [12] D. Petrov, M. Holzmann, and G. Shlyapnikov, “Bose-Einstein Condensation in Quasi- D-2 Trapped Gases,” Phys.Rev.Lett., vol. 84, pp. 2551–2555, 2000.
- [13] S. K. Baur, B. Fröhlich, M. Feld, E. Vogt, D. Pertot, M. Koschorreck, and M. Köhl, “Radio-frequency spectra of feshbach molecules in quasi-two-dimensional geometries,” Phys. Rev. A, vol. 85, p. 061604, Jun 2012.
- [14] E. Vogt, M. Feld, B. Frohlich, D. Pertot, M. Koschorreck, et al., “Scale invariance and viscosity of a two-dimensional Fermi gas,” Phys.Rev.Lett., vol. 108, p. 070404, 2012.
- [15] C. Chafin and T. Schäfer, “Scale breaking and fluid dynamics in a dilute two-dimensional fermi gas,” Phys. Rev. A, vol. 88, p. 043636, Oct 2013.

- [16] J. Hofmann, “Quantum anomaly, universal relations, and breathing mode of a two-dimensional fermi gas,” Phys. Rev. Lett., vol. 108, p. 185303, May 2012.
- [17] J. Levinsen and M. Parish, “Strongly interacting two-dimensional fermi gases,” arXiv:1408.2737, 2014.
- [18] C. R. Ordóñez, “Path-integral fujikawa’s approach to anomalous virial theorems and equations of state for systems with $so(2,1)$ symmetry,” arXiv:1503.01384, 2015.
- [19] M. Umezawa, “Regularization of the Path Integral Measure for Anomalies,” Phys.Rev., vol. D39, p. 3672, 1989.
- [20] M. Srednicki, Quantum Field Theory. Cambridge University Press, 2007.
- [21] R. Jackiw, “Introducing scale symmetry,” Phys.Today, vol. 25N1, pp. 23–27, 1972.
- [22] C. Hagen, “Scale and conformal transformations in galilean-covariant field theory,” Phys.Rev., vol. D5, pp. 377–388, 1972.
- [23] A. Schakel, Boulevard of Broken Symmetries: Effective Field Theories of Condensed Matter. World Scientific, 2008.
- [24] O. Bergman, “Nonrelativistic field theoretic scale anomaly,” Phys.Rev., vol. D46, pp. 5474–5478, 1992.

- [25] T. Haugset and F. Ravndal, “Scale anomalies in nonrelativistic field theories in (2+1)-dimensions,” Phys.Rev., vol. D49, pp. 4299–4301, 1994.
- [26] C. L. Lin and C.R. Ordonez, “in progress.”

B Virial Theorem for Nonrelativistic Quantum Fields in D Spatial Dimensions

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Virial Theorem for Non-relativistic Quantum Fields in D Spatial Dimensions

Chris L. Lin and Carlos R. Ordóñez

Department of Physics, University of Houston, Houston, TX 77204-5005

The virial theorem for non-relativistic complex fields in D spatial dimensions and with arbitrary many-body potential is derived, using path-integral methods and scaling arguments recently developed to analyze quantum anomalies in low-dimensional systems. The potential appearance of a Jacobian J due to a change of variables in the path-integral expression for the partition function of the system is pointed out, although in order to make contact with the literature most of the analysis deals with the $J = 1$ case. The virial theorem is recast into a form that displays the effect of microscopic scales on the thermodynamics of the system. From the point of view of this paper the case usually considered, $J = 1$, is not natural, and the generalization to the case $J \neq 1$ is briefly presented.

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1 Introduction

The virial theorem has been proven using a variety of methods. Recently, a path-integral derivation of the virial theorem has been developed in the context of quantum anomalies in non-relativistic 2D systems, or more generally, systems

with $SO(2, 1)$ classical symmetry [1]. The path integral is most useful in isolating the anomaly contribution to the equation of state so obtained. This method is in fact quite general, and applicable for non-relativistic systems with an arbitrary 2-body potential $V(\vec{x}_1, \vec{x}_2)$ in D spatial dimensions, even when there are no quantum anomalies present. We present such derivation in this note, extending the original derivation using also diagrammatic analysis, and recasting the virial theorem into a general equation that relates macroscopic thermodynamics variables to the microscopic physics. As it will be shown, there is generically a Jacobian term J that may contribute to the virial theorem, regardless of the existence of a classical scaling symmetry. We will mainly concern ourselves here with the case $J = 1$ (which we term “non-anomalous”). Comments and conclusions end the note.

2 Virial Theorem

The work in [1] was based partly on the work by Toyoda et al. [2, 3, 4]. They postulated that spatial scalings¹

$$\begin{aligned}\vec{x}' &= \lambda\vec{x}, \\ \psi'(t, \vec{x}') &= \lambda^{-D/2}\psi(t, \vec{x}),\end{aligned}\tag{1}$$

¹Toyoda et al. introduced an auxiliary external potential that has the effect of confining the system to a volume V , and then, through a series of infinitesimal scalings and algebraic arguments derived what amounts to the equation of state, which they referred to as virial theorem. Unlike them, we’re not using an external potential but simply consider a system with a large volume V (so all the typical large-volume thermodynamical considerations apply), but like them, we’re also calling virial theorem the equation of state that will be derived in this paper.

leave the particle number density invariant:

$$d^D \vec{x} \psi^\dagger(t, \vec{x}) \psi(t, \vec{x}) = d^D \vec{x}' \psi^\dagger(t, \vec{x}') \psi'(t, \vec{x}'). \quad (2)$$

Let us consider a non-relativistic system whose microscopic physics is represented by a generic 2-body interaction²

$$\mathcal{L} = \psi^* \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi - \frac{1}{2} \int d^D \vec{y} \psi^*(t, \vec{x}) \psi(t, \vec{x}) V(\vec{x} - \vec{y}) \psi^*(t, \vec{y}) \psi(t, \vec{y}). \quad (3)$$

Giving our system a macroscopic volume V , temperature β^{-1} , and chemical potential μ , and going into imaginary time gives for the partition function:

$$Z[V, \beta, \mu] = \int [d\psi^*][d\psi] e^{-\int_0^\beta d\tau \int_V d^D \vec{x} \left[\psi^* \left(\partial_\tau - \frac{\nabla^2}{2} - \mu \right) \psi + \frac{1}{2} \int d^D \vec{y} \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) V(\vec{x} - \vec{y}) \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y}) \right]}. \quad (4)$$

Now consider a new system with the same temperature and chemical potential, but at volume $V' = \lambda^D V$:

$$Z[\lambda^D V, \beta, \mu] = \int [d\psi'^*][d\psi'] e^{-\int_0^\beta d\tau \int_{\lambda^D V} d^D \vec{x}' \left[\psi'^* \left(\partial_\tau - \frac{\nabla'^2}{2} - \mu \right) \psi' + \frac{1}{2} \int d^D \vec{y}' \psi'^*(\tau, \vec{x}') \psi(\tau, \vec{x}') V(\vec{x}' - \vec{y}') \psi'^*(\tau, \vec{y}') \psi'(\tau, \vec{y}') \right]}. \quad (5)$$

²In this paper we set $\hbar = m = 1$.

Substituting Eq. (1) into Eq. (5) gives:

$$Z[\lambda^D V, \beta, \mu] = \int [d\psi^*][d\psi] J e^{-\int_0^\beta d\tau \int_V d^D \vec{x} \left[\psi^* \left(\partial_\tau - \frac{1}{\lambda^2} \frac{\nabla^2}{2} - \mu \right) \psi + \frac{1}{2} \int d^D \vec{y} \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) V(\lambda(\vec{x} - \vec{y})) \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y}) \right]}, \quad (6)$$

where J is the Jacobian for the transformation $(\psi'^*, \psi') \rightarrow (\psi^*, \psi)$. As mentioned above, our emphasis will be in the non-anomalous case, and henceforth we assume $J = 1$ (see however comments and conclusions). Then $Z[\lambda^D V, \beta, \mu] \equiv Z^\lambda[V, \beta, \mu]$, where the superscript λ represents a microscopic system whose kinetic energy has a factor $\frac{1}{\lambda^2}$ and whose potential is $V(\lambda(\vec{x} - \vec{y}))$. Note that $Z^{\lambda=1}[V, \beta, \mu] = Z[V, \beta, \mu]$.

The pressures corresponding to $Z[\lambda^D V, \beta, \mu]$ and $Z[V, \beta, \mu]$ are equal, since the intensive variables μ and β^{-1} are the same, and they correspond to the same microscopic system. The argument we just made for the pressures being the same is valid in the thermodynamic limit, based on the principle that two intensive variables determine the third via an equation of state e.g., $P = \rho T$ for an ideal gas. However, in the next section we will also provide a diagrammatical proof that the two pressures are the same.

For now assume the pressures are equal. Then using $Z = e^{\beta PV}$, we get:

$$\begin{aligned}
e^{\beta PV'} - e^{\beta PV} &= Z[\lambda^D V, \beta, \mu] - Z[V, \beta, \mu], \\
\text{or } e^{\beta P \lambda^D V} - e^{\beta PV} &= Z^\lambda[V, \beta, \mu] - Z[V, \beta, \mu].
\end{aligned} \tag{7}$$

Following [1], we set $\lambda = 1 + \eta$ for infinitesimal η :

$$\begin{aligned}
e^{\beta PV} D\eta \beta PV &= Z^{\lambda=1}[V, \beta, \mu] + \partial_\lambda Z^\lambda[V, \beta, \mu] \Big|_{\lambda=1} \eta - Z[V, \beta, \mu] \\
&= \partial_\lambda Z^\lambda[V, \beta, \mu] \Big|_{\lambda=1} \eta \\
&= Z[V, \beta, \mu] \\
&\left\langle \int_0^\beta d\tau \int_V d^D x \left(-\psi^\dagger \nabla^2 \psi - \frac{1}{2} \int d^D \vec{y} \rho(\tau, \vec{y}) [(\vec{x} - \vec{y}) \cdot \nabla_{\vec{x}} V(\vec{x} - \vec{y})] \rho(\tau, \vec{x}) \right) \right\rangle \eta,
\end{aligned} \tag{8}$$

where we've defined $\rho(\tau, \vec{x}) \equiv \psi^\dagger(\tau, \vec{x})\psi(\tau, \vec{x})$. Cancelling the partition functions on both sides, noting that thermal expectation values for the fields at the same τ are independent of τ so that the τ integral pulls out a β , and denoting the kinetic energy as KE :

$$DPV = 2KE - \left\langle \frac{1}{2} \int d^D \vec{x} \int d^D \vec{y} \rho(\tau, \vec{y}) [(\vec{x} - \vec{y}) \cdot \nabla_{\vec{x}} V(\vec{x} - \vec{y})] \rho(\tau, \vec{x}) \right\rangle, \tag{9}$$

which is the virial theorem in D dimensions (Eqs. (3.30) and (2.6) in [3] and [4] respectively).

3 N-body

It is clear that this method can be generalized to the n-body case. Since by Eq. (2) the scaling transformation preserves $\int d^D \vec{x} \psi^\dagger(\tau, x) \psi(\tau, x) (\equiv \int d^D \vec{x} \rho(\tau, \vec{x}))$, an n-body term transforms as

$$\frac{1}{n!} \int \left(\prod_i^n d^D \vec{x}_i \rho(\tau, \vec{x}_i) \right) V(\vec{x}_1, \dots, \vec{x}_n) \rightarrow \frac{1}{n!} \int \left(\prod_i^n d^D \vec{x}_i \rho(\tau, \vec{x}_i) \right) V(\vec{x}'_1, \dots, \vec{x}'_n). \quad (10)$$

Setting $V(\vec{x}_1, \dots, \vec{x}_n) = \tilde{V}(\vec{z}_{\text{COM}}, \vec{z}_2, \dots, \vec{z}_n)$ where $\vec{z}_i \equiv \vec{x}_i - \vec{x}_1$ and \vec{z}_{COM} is the center of mass of the \vec{x}_i 's gives

$$\begin{aligned} DPV = 2KE - & \left\langle \frac{1}{n!} \int \left(\prod_i^n d^D \vec{x}_i \rho(\tau, \vec{x}_i) \right) \left[\vec{z}_{\text{COM}} \cdot \nabla_{\vec{z}_{\text{COM}}} \tilde{V}(\vec{z}_{\text{COM}}, \vec{z}_2, \dots, \vec{z}_n) \right] \right\rangle \\ & - \left\langle \frac{1}{n!} \int \left(\prod_i^n d^D \vec{x}_i \rho(\tau, \vec{x}_i) \right) \left[\sum_{i=2}^n \vec{z}_i \cdot \nabla_{\vec{z}_i} \tilde{V}(\vec{z}_{\text{COM}}, \vec{z}_2, \dots, \vec{z}_n) \right] \right\rangle. \end{aligned} \quad (11)$$

For translationally-invariant systems, we can ignore the potential term in the 1st line.

4 Diagrammatic Proof of $P=P'$

To prove diagrammatically that the pressure P' corresponding to $Z[\lambda^D V, \beta, \mu]$ is equal to the pressure P corresponding to $Z[V, \beta, \mu]$, it suffices to show that $\Omega[\lambda^D V, \beta, \mu] = \lambda^D \Omega[V, \beta, \mu]$, where Ω is the grand potential. By the cluster expansion

sion, Ω is given by the sum of connected vacuum graphs [5]. Using the Feynman rules, $\Omega[V, \beta, \mu] \propto \delta^D(0)\mathcal{M}(\beta, \mu)$, where $\delta^D(0)$ expresses conservation of momentum of the vacuum and $\mathcal{M}(\beta, \mu)$ is the Feynman amplitude³ which is independent of V , since \mathcal{M} contains expressions like $\frac{\Delta n_1 \dots \Delta n_D}{V} f\left(\frac{2\pi n_i}{L}\right)$ which in the continuum limit $\rightarrow \frac{d^D k}{(2\pi)^D} f(k_i)$ ⁴. Taking $\delta^D(0) \propto V$, it's clear that $\Omega[V, \beta, \mu] \propto V\mathcal{M}(\beta, \mu)$, so $\Omega[\lambda^D V, \beta, \mu] = \lambda^D \Omega[V, \beta, \mu]$ in the continuum limit.

Alternatively since $Z[\lambda^D V, \beta, \mu] = Z^\lambda[V, \beta, \mu]$, another way to show $P' = P$ is to show that the grand potential $\Omega^\lambda[V, \beta, \mu]$ of $Z^\lambda[V, \beta, \mu]$ is larger by a factor of λ^D than $\Omega[V, \beta, \mu]$. Then $\Omega^\lambda[V, \beta, \mu] = \Omega[\lambda^D V, \beta, \mu] = \lambda^D \Omega[V, \beta, \mu]$.

The grand potential Ω^λ is given by:

$$\Omega^\lambda = -\beta^{-1} \ln Z^\lambda[V, \beta, \mu]. \quad (12)$$

By the cluster expansion, Ω^λ is given by the sum of connected vacuum graphs. $Z^\lambda[V, \beta, \mu]$ and $Z[V, \beta, \mu]$ have the same macroscopic parameters and only differ in that Z^λ 's propagator is

$$\Delta^\lambda = \frac{1}{i\omega_n - \frac{k^2}{2\lambda^2} - \mu}, \quad (13)$$

³ \mathcal{M} is the T-matrix, and $\delta^D(0) = \int \frac{d^D x}{(2\pi)^D} e^{-i0*x} \propto V$.

⁴For finite volume, momenta are discrete and summed over: $k_i = \frac{2\pi n_i}{L}$. $\Delta n_1 \dots \Delta n_D$ is a box of unit volume surrounding the discrete lattice point n_i . In the limit of large L , $f\left(\frac{2\pi n_i}{L}\right)$ is assumed not to vary much, so any point within $\Delta n_1 \dots \Delta n_D$ not on the lattice would still contribute the same value of $f\left(\frac{2\pi n_i}{L}\right)$. Then $\sum_{n_i} \frac{1}{V} f\left(\frac{2\pi n_i}{L}\right) = \sum_{n_i} \frac{\Delta n_1 \dots \Delta n_D}{V} f\left(\frac{2\pi n_i}{L}\right) \rightarrow \int \frac{dn_1 \dots dn_D}{V} f\left(\frac{2\pi n_i}{L}\right) = \int \frac{d^D k}{(2\pi)^D} f(k_i)$.

and that the potential is

$$V^\lambda(\vec{x} - \vec{y}) = V(\lambda(\vec{x} - \vec{y})) \quad (14)$$

instead of $V(\vec{x} - \vec{y})$. Fourier transforming Eq. (14) gives the relationship:

$$V^\lambda\left(\vec{k}\right) = \frac{V\left(\frac{\vec{k}}{\lambda}\right)}{\lambda^D} \quad (15)$$

The Feynman rules for the theory say that each vertex contributes its Fourier transform $V^\lambda\left(\vec{k}\right)$, where \vec{k} is the momentum flowing through the vertex, and each propagator contributes Eq. (13). For vacuum graphs, all momenta \vec{k} in the vertices and propagators are integrated over in loop momenta $\int \frac{d^D k}{(2\pi)^D}$. Let us make the change of variables $\int \frac{d^D k}{(2\pi)^D} = \int \lambda^D \frac{d^D \tilde{k}}{(2\pi)^D \lambda^D} = \int \lambda^D \frac{d^D \tilde{k}}{(2\pi)^D}$ and relabel \tilde{k} as \vec{k} . This will cause $\Delta_\lambda(i\omega, \vec{k}) = \Delta\left(i\omega, \frac{\vec{k}}{\lambda}\right) \rightarrow \Delta(i\omega, \vec{k})$ and $V^\lambda\left(\vec{k}\right) = \frac{V\left(\frac{\vec{k}}{\lambda}\right)}{\lambda^D} \rightarrow \frac{V(\vec{k})}{\lambda^D}$ in the loop integrals.

Therefore, Ω^λ is the same as Ω , except for an overall scale factor of $\left(\frac{1}{\lambda^D}\right)^\nu (\lambda^D)^L$, where ν is the number of vertices and L is the number of loops. Topologically, for connected vacuum graphs of the 2-body potential, $L = \nu + 1$. So the overall scale factor becomes λ^D . Hence $\Omega^\lambda = \lambda^D \Omega$, and therefore $P' = P$.

This generalizes to translationally-invariant n-body potentials, and for spontaneous symmetry breaking. Suppose the interaction is of the form:

$$\int_{V'} \left(\prod_{i=1}^n d^D \vec{x}'_i \phi'^{m(i)}(\tau, \vec{x}'_i) \right) V(\vec{x}'_1, \dots, \vec{x}'_n) = \frac{\lambda^{Dn}}{\lambda^{\frac{DM}{2}}} \int_V \left(\prod_{i=1}^n d^D \vec{x}_i \phi^{m(i)}(\tau, \vec{x}_i) \right) V(\lambda \vec{x}_1, \dots, \lambda \vec{x}_n) \quad (16)$$

where $m(i)$ is the number of fields in the interaction with spatial coordinate \vec{x}_i , and $M = \sum_{i=1}^n m(i)$. For translationally-invariant potentials

$$V^\lambda = \frac{V\left(\frac{k}{\lambda}\right)}{\lambda^{D(n-1)}}. \quad (17)$$

So

$$\Omega^\lambda = \left(\frac{\lambda^{Dn}}{\lambda^{\frac{DM}{2}}} \frac{1}{\lambda^{D(n-1)}} \right)^\nu (\lambda^D)^L \Omega. \quad (18)$$

Since $L = \left(\frac{M}{2} - 1\right) \nu + 1$,⁵ this again gives:

$$\Omega^\lambda = \lambda^D \Omega. \quad (19)$$

For a diagram with a mixture of vertices of different types, $L = \sum_i \left(\frac{M_i}{2} - 1\right) \nu_i + 1$, where ν_i is the number of vertices of type i , and M_i is the number of lines coming out of each vertex:

⁵ M lines come out of each vertex, and each line coming out is 1/2 of an internal line, so $\frac{M\nu}{2} = I$ where I is the number of internal lines. The number of loops is the number of independent momenta, $L = I - \nu + 1$. So $L = \left(\frac{M}{2} - 1\right) \nu + 1$.

$$\begin{aligned}\Omega^\lambda &= \left[\prod_i \left(\frac{\lambda^{Dn_i}}{\lambda^{\frac{DM_i}{2}}} \frac{1}{\lambda^{D(n_i-1)}} \right)^{\nu_i} \right] (\lambda^D)^{\sum_i (\frac{M_i}{2}-1)\nu_i+1} \Omega \\ &= \lambda^D \Omega.\end{aligned}\tag{20}$$

5 Scale Equation

The virial equation, Eq. (9), can be recast into a different form that illustrates the effect of microscopic scales on the thermodynamics of a system. A simple way to see this is to write the potential as⁶:

$$V(|\vec{x} - \vec{y}|) = \frac{f\left(\frac{g_i}{|\vec{x} - \vec{y}|^{[g_i]}}\right)}{|\vec{x} - \vec{y}|^2}.\tag{21}$$

f is a dimensionless function whose arguments are the ratios of the couplings g_i of V to their length dimension $[g_i]$ expressed in units of $|\vec{x} - \vec{y}|$ ($\frac{\hbar^2}{m} \frac{1}{|\vec{x} - \vec{y}|^2}$ provides units of energy)⁷. Denoting $r = |\vec{x} - \vec{y}|$

$$\begin{aligned}r \frac{dV}{dr} &= -2V(r) + \frac{1}{r} \frac{df\left(\frac{g_i}{r^{[g_i]}}\right)}{dr} \\ &= -2V(r) - \frac{1}{r^2} [g_i] g_i \frac{\partial f\left(\frac{g_i}{r^{[g_i]}}\right)}{\partial g_i} \\ &= -2V(r) - [g_i] g_i \frac{\partial V}{\partial g_i}.\end{aligned}\tag{22}$$

where the chain rule was used in line 2. Substituting this into Eq. (9) gives

⁶We are now restricting ourselves to radial potentials.

⁷As an example, consider $V(|\vec{x} - \vec{y}|) = \frac{k}{2} |\vec{x} - \vec{y}|^2 + \lambda |\vec{x} - \vec{y}|$, where the coupling k has length dimension -4 and λ has length dimension -3. Then $f\left(\frac{k}{|\vec{x} - \vec{y}|^{[k]}, \frac{\lambda}{|\vec{x} - \vec{y}|^{[\lambda]}}}\right) = \frac{1}{2} \frac{k}{|\vec{x} - \vec{y}|^{-4}} + \frac{\lambda}{|\vec{x} - \vec{y}|^{-3}}$. The couplings k and λ provide the characteristic length scales.

$$\begin{aligned}
DPV &= 2KE + 2V - \left\langle \frac{1}{2} \int d^D \vec{x} \int d^D \vec{y} \rho(\tau, \vec{y}) \left(-[g_i] g_i \frac{\partial V}{\partial g_i} \right) \rho(\tau, \vec{x}) \right\rangle \\
&= 2E + \left\langle \frac{1}{2} \int d^D \vec{x} \int d^D \vec{y} \rho(\tau, \vec{y}) \left([g_i] g_i \frac{\partial V}{\partial g_i} \right) \rho(\tau, \vec{x}) \right\rangle.
\end{aligned} \tag{23}$$

Rearranging:

$$2E - DPV = - \left\langle \frac{1}{2} \int d^D \vec{x} \int d^D \vec{y} \rho(\tau, \vec{y}) \left([g_i] g_i \frac{\partial V}{\partial g_i} \right) \rho(\tau, \vec{x}) \right\rangle. \tag{24}$$

On the LHS of Eq. (24) are macroscopic thermodynamic variables. The RHS is a measure of the microscopic physics of the system. In particular, if the potential has no scales $[g_i] = 0$ and no anomalies (i.e., $J = 1$), you get 0 on the RHS, and Eq. (24) reduces to the equation of state for a non-relativistic scale-invariant system [6].

6 Conclusion and Comments

The goal of this paper has been to highlight certain features in the derivation of the virial theorem for non-relativistic systems, which display a potentially important omission due to the presence of the Jacobian needed in the path-integral derivation developed here. Indeed, while we set $J = 1$ at the outset in order to make contact with the literature (specifically, Toyoda's et al. work [2, 3, 4]), Eq. (6) shows that the natural procedure would be to not assume this and keep the contribution of the Jacobian, regardless of whether or not there is a classical scaling symmetry. Obviously, in the latter case, one has to keep the Jacobian in order to incorporate

the quantum anomaly as was shown in [1]. The formal mathematical steps in the general case presented here are the same as in that paper, and Eq. (24) would become

$$2E - DPV = - \left\langle \frac{1}{2} \int d^D \vec{x} \int d^D \vec{y} \rho(\tau, \vec{y}) \left([g_i] g_i \frac{\partial V}{\partial g_i} \right) \rho(\tau, \vec{x}) \right\rangle - \frac{1}{\beta} \hat{\text{Tr}} \left(\hat{\theta}_s \delta(\tau_x - \tau_y) \delta^D(\vec{x} - \vec{y}) I_2 \right), \quad (25)$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\hat{\theta}_s = - \left(1 + \vec{x} \cdot \vec{\nabla} \right)$, and we have also used the 2×2 matrix notation of [7] ($\hat{\text{Tr}}$ includes both a matrix and functional trace).

As with the work in [1] and [7], the key to assess the importance of the Jacobian term rests upon one's ability to compute its contribution in detail, which implies a careful regularization procedure, and possibly also renormalization. The actual details will depend of the type of potentials considered. An interesting direction is the relativistic generalization of these ideas. Work on this is currently in progress [8].

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References

- [1] C. R. Ordóñez, “Path-integral fujikawa’s approach to anomalous virial theorems and equations of state for systems with $so(2,1)$ symmetry,” [arXiv:1503.01384](https://arxiv.org/abs/1503.01384), 2015.
- [2] T. Toyoda, “Canonical generator of conformal transformations in nonrelativistic quantum many-body systems at finite temperatures,” Phys. Rev. A, vol. 48, pp. 3492–3498, Nov 1993.
- [3] T. Toyoda and K. ichi Takiuchi, “Quantum field theoretical reformulation of the virial theorem,” Physica A: Statistical Mechanics and its Applications, vol. 261, no. 34, pp. 471 – 481, 1998.
- [4] K. Takiuchi, M. Okada, H. Koizumi, K. Ito, and T. Toyoda, “Exact relations for two-dimensional electron gas spin correlation functions,” Physica E: Low-dimensional Systems and Nanostructures, vol. 6, no. 14, pp. 810 – 812, 2000.
- [5] S. Weinberg, The Quantum Theory of Fields. Vols. 1 and 2. Cambridge University Press, 2005.
- [6] F. Mancarella, G. Mussardo, and A. Trombettoni, “Energypressure relation for low-dimensional gases,” Nucl.Phys., vol. B887, pp. 216–245, 2014.

- [7] C. L. Lin and C. R. Ordóñez, “Path-integral derivation of the nonrelativistic scale anomaly,” Phys. Rev. D, vol. 91, p. 085023, Apr 2015.
- [8] C. L. Lin and C. R. Ordóñez, “In progress..”

C Path-Integral Approach to the Scale Anomaly at Finite Temperature

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Path-Integral Approach to the Scale Anomaly at Finite Temperature

Chris L. Lin and Carlos R. Ordóñez

Department of Physics, University of Houston, Houston, TX 77204-5005

We derive the relativistic thermodynamic scale equation using imaginary-time path integrals, with complex scalar field theory taken as a concrete example. We use Fujikawa's method to derive the scaling anomaly for this system using a matrix regulator. We make a general scaling argument to show how for anomalous systems, the β function of the vacuum theory can be derived from measurement of macroscopic thermodynamic parameters.

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1 Introduction

In a series of seminal papers by Callan, Coleman, and Jackiw [1, 2], it was noted that in general the trace of the Belifante stress-energy tensor θ_μ^μ for any renormalizable theory could be improved, so that classically for scale invariant systems (systems invariant under the conformal group),

$$\theta_\mu^\mu = 0. \tag{1}$$

This improved tensor has a number of desirable properties over the canonical tensor (the one derived from Noether's theorem) such as having finite matrix

elements in the quantum theory, and that the energy for bound states can be naturally expressed as the trace of this tensor. Shortly after these observations, it was noted that the same improvement program could be applied in the non-relativistic case [3], so that for classical scale invariant systems (systems invariant under the Schrodinger group):

$$2\theta^{00} - \sum_{i=1}^3 \theta^{ii} = 0, \quad (2)$$

where the 2 results from the fact that in non-relativistic theories time must scale as twice the power of space.¹

Eqs. (1) and (2) fail to consider the trace anomaly. In general, the trace of the stress-energy tensor taken between bound states gives the energy of the bound state:

$$E_b = \int dV \langle \theta_{\mu}^{\mu} \rangle, \quad (3)$$

which derives from the fact that the time-average of the field virial is zero for bound states [4]. With slight modification Eq. (3) holds in the non-relativistic case too (see [5] for a specific example). However, it is well-known that even though $\theta_{\mu}^{\mu} = 0$ for a classically scale-invariant system, which would imply bound states can only have zero energy,² the quantization procedure can destroy this relationship. When this happens this is called a scale anomaly, and is the mech-

¹The Schrödinger equation has only one derivative of time, and two of space, so for scale invariance time must scale as twice the power of space.

²This is also obvious from the fact that there are no scales to even form E_b .

anism that allows the bound state energy to differ from zero.

As an example, in QCD with massless quarks (or no quarks at all), the Lagrangian is classically scale-invariant so that $\theta_\mu^\mu = 0$. However, through the renormalization process, a scale appears as Λ_{QCD} . In general this makes $\langle\theta_\mu^\mu\rangle = \mathcal{A}$, where \mathcal{A} is the anomaly. The stress-energy tensor can then be further improved:

$$T^{\mu\nu} = \theta^{\mu\nu} + \frac{g^{\mu\nu}}{4} T_\eta^\eta, \quad (4)$$

so that $T^{\mu\nu}$ is no longer traceless. Then

$$E_b = \int dV \langle T^{00} \rangle = \int dV \langle \theta^{00} \rangle + \frac{E_b}{4}, \quad (5)$$

which implies that \mathcal{A} accounts for 1/4 of the energy of the hadron. This can explicitly be seen in the bag model where confinement of the quarks and gluons is the result of a cosmological constant term in the Lagrangian which contributes a positive energy and negative pressure $\Lambda g^{\mu\nu}$ to $\theta^{\mu\nu}$, which confines the system. Then from the tracelessness of $\theta^{\mu\nu}$, $\Lambda = \frac{1}{4} T_\mu^\mu$, so that confinement accounts for 1/4 of the hadron energy [4].

In this paper, we are interested in the thermal analogues of Eqns. (1) and (2). Both of these quantities are very important in their respective areas of physics. In the nonrelativistic sector, for an ultra-cold dilute gas, (2) would read:

$$2\mathcal{E} - 3P = -\frac{\hbar^2}{3m}\lambda \langle (\psi^\dagger(x)\psi(x))^2 \rangle. \quad (6)$$

The RHS is known as the Tan contact, and is extremely important in atomic physics. In terms of it, Tan derived a set of universal relations [6, 7, 8] that govern many relationships between the thermodynamics variables of the system and the behavior of the large momentum tails of correlation functions. These relationships hold even in the strongly interacting regime where perturbation theory becomes inadequate [9]. A field theoretic explanation of Tan's result was later developed in terms of the operator product expansion [10].

In QCD, the analog would be [11]:

$$\mathcal{E} - 3P = \sum_{i=1}^{n_f} m_i \langle \bar{\psi}_i \psi_i \rangle + \frac{2}{g}\beta(g)\frac{1}{4} \langle F_{\mu\nu}^a F^{\mu\nu a} \rangle. \quad (7)$$

In the low temperature regime where the coupling g is strong, the trace anomaly of the RHS is calculated by calculating the LHS of Eq. (7) using a lattice action. The goal is to calculate the QCD equation of state $P = P(T, \mu, V)$ rather than the anomaly itself. However, for technical reasons [12], $\mathcal{E} - 3P$ is important as an intermediate step in lattice QCD for calculating $P(T, \mu, V)$, where it is given by:

$$\mathcal{A} = \mathcal{E} - 3P = -\frac{T}{V} \frac{d \ln Z}{d \ln a}, \quad (8)$$

and plugging into Eq. (7) gives after using thermodynamic identities:

$$\frac{\partial}{\partial \ln T} \left(\frac{P}{T^4} \right) = \frac{\mathcal{A}}{T^4}, \quad (9)$$

which can then be integrated to get $P(T, \mu, V)$. a is the lattice spacing and Z is the partition function with lattice action.

In this paper, following the approach initiated in [13, 14, 15] for non-relativistic systems, we provide a continuum/non-lattice path-integral approach to deriving the thermodynamic trace equation $\mathcal{E} - 3P$, where anomalies naturally appear as a result of a change of variables of the path integral measure, the thermal analog of Fujikawa's method. This is in contrast to an operator approach, where one takes the thermal quantum statistical expectation values of both sides of Eqns. like (1) and (2), and identifying $\langle T^{00} \rangle = \mathcal{E}$ and $\langle T^{ii} \rangle = \mathcal{P}_H$, where \mathcal{P}_H is the hydrodynamic pressure [16]. Within this path-integral approach, no reference needs to be made about improvement of the stress-energy tensor, or the validity of equating the hydrodynamic pressure \mathcal{P}_H with the thermodynamic pressure P derived from the grand partition function, which is nontrivial, especially in the presence of anomalies [17, 18]. For concreteness, we will take as our system a complex scalar field theory, but the results can be extended for other systems. The Lagrangian is given by

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2 \quad (10)$$

and has a $U(1)$ symmetry

$$\begin{aligned}
\phi &\rightarrow e^{i\theta}\phi, \\
\phi^\dagger &\rightarrow e^{-i\theta}\phi^\dagger,
\end{aligned}
\tag{11}$$

leading to a conserved charge:

$$\begin{aligned}
j_0 &= i\phi^\dagger \overleftrightarrow{\partial}_0 \phi, \\
Q &= i \int d^3x \phi^\dagger \overleftrightarrow{\partial}_0 \phi.
\end{aligned}
\tag{12}$$

Under scale transformation:

$$\begin{aligned}
x'^\mu &= e^\rho x^\mu, \\
\phi'(x') &= e^{-\rho}\phi(x), \\
\phi'^\dagger(x') &= e^{-\rho}\phi^\dagger(x).
\end{aligned}
\tag{13}$$

2 Thermodynamic Dilation Equation

For a homogeneous system the grand potential $\Omega = \Omega(\beta, \mu, V)$ in the large volume limit equals $-PV$, so that the partition function is $Z = e^{-\beta\Omega} = e^{\beta PV}$, and can be expressed via a path integral:

$$Z = e^{\beta PV} = \sum_i \langle i | e^{-\beta(H - \mu Q)} | i \rangle = \int [d\phi][d\phi^*] e^{-S_E + \mu \int_0^\beta \int_V d^3x d\tau j_0}, \tag{14}$$

with³

³Due to the dependence of j_0 on conjugate momenta, when integrating out conjugate momenta to pass into the Lagrangian formulation of the path integral, \mathcal{L}_E acquires an additional $\mu^2 \phi^* \phi$ term: see [19].

$$\begin{aligned}
S_E &= \int_0^\beta \int_V d^3x d\tau \left(\partial_\mu \phi^* \partial_\mu \phi + (m^2 - \mu^2) \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2 \right), \\
j_0 &= -\phi^* \overleftrightarrow{\partial}_\tau \phi.
\end{aligned} \tag{15}$$

Now consider an infinitesimal “relativistic thermodynamic scaling”

$$\begin{aligned}
\beta' &= e^\rho \beta = \beta + \rho \beta = \beta + \delta\beta, \\
L'_i &= e^\rho L_i = L_i + \rho L_i = L_i + \delta L_i, \\
\mu' &= \mu.
\end{aligned} \tag{16}$$

where L_i is the length of the box in the i direction and ρ is a dimensionless infinitesimal parameter.

In the large volume limit it is assumed that $P(\beta, \mu, V) = P(\beta, \mu)$,⁴ so under the transformation of Eq. (16):

$$\begin{aligned}
\delta(\beta PV) &= (\delta\beta)PV + \beta(\delta P)V + \beta P(\delta V) \\
&= \rho \left(\beta PV + \beta \left(\frac{\partial P}{\partial \beta} \beta \right) V + \beta P(3V) \right).
\end{aligned} \tag{17}$$

Now using the identity $\beta V \frac{\partial P}{\partial \beta} = -PV - E + \mu Q$, we get

$$\delta(\beta PV) = \rho(-\beta E + \beta P(3V) + \beta \mu Q), \tag{18}$$

and therefore

⁴This can be shown via cluster decomposition: e.g., see [14].

$$\delta(e^{\beta PV}) = \delta(\beta PV) e^{\beta PV} = \rho\beta(-E + 3PV + \mu Q) e^{\beta PV}. \quad (19)$$

Eq. (19) represents the effect of the scaling in Eq. (16) on the LHS of Eq. (14). Now we analyze the effect of this scaling to the RHS of Eq. (14), the path integral part, from which anomalies originate, and eventually equate the two expressions.

The scaling in Eq. (16) represents a dilation of the system:

$$\begin{aligned} x'^{\mu} &= e^{\rho} x^{\mu}, \\ \phi'(x') &= e^{-\rho} \phi(x), \\ \phi'^*(x') &= e^{-\rho} \phi^*(x). \end{aligned} \quad (20)$$

The dilated system has

$$e^{\beta' P' V'} = \int [d\phi'] [d\phi'^*] e^{-S'_E + \mu \int_0^{\beta'} \int_{V'} d^D x' d\tau' j'_0}, \quad (21)$$

where

$$\begin{aligned} S'_E &= \int_0^{e^{\rho}\beta} \int_{e^{\rho}V} d^3 x' d\tau' \left(\partial'_\mu \phi'^* \partial'_\mu \phi' + (m^2 - \mu^2) \phi'^* \phi' + \frac{\lambda}{4} (\phi'^* \phi')^2 \right), \\ \mu \int_0^{\beta'} \int_{V'} d^3 x' d\tau' j'_0 &= \mu \int_0^{e^{\rho}\beta} \int_{e^{\rho}V} d^3 x' d\tau' \left(-\phi'^* \overset{\leftrightarrow}{\partial}'_\tau \phi' \right). \end{aligned} \quad (22)$$

To compare to the undilated system, we “pull back” to unprimed variables by substituting Eq. (20) into Eq. (21) and Eq. (22). Eq. (22) becomes:

$$\begin{aligned}
S'_E &= \int_0^{e^{\rho\beta}} \int_{e^{\rho V}} d^3x' d\tau' \left(\partial'_\mu \phi'^* \partial'_\mu \phi' + (m^2 - \mu^2) \phi'^* \phi' + \frac{\lambda}{4} (\phi'^* \phi')^2 \right) \\
&= \int_0^\beta \int_V e^{4\rho} d^3x d\tau \left(e^{-2\rho} \frac{\partial \phi^*}{\partial (e^\rho x_\mu)} \frac{\partial \phi}{\partial (e^\rho x_\mu)} + (m^2 - \mu^2) e^{-2\rho} \phi^* \phi + \frac{\lambda}{4} (e^{-2\rho} \phi^* \phi)^2 \right) \\
&= S_E + 2\rho \int_0^\beta \int_V d^3x d\tau (m^2 - \mu^2) \phi^* \phi.
\end{aligned} \tag{23}$$

Similarly:

$$\begin{aligned}
\mu \int_0^{\beta'} \int_{V'} d^3x' d\tau' j'_0 &= \mu \int_0^{e^{\rho\beta}} \int_{e^{\rho V}} d^3x' d\tau' \left(-\phi'^* \overset{\leftrightarrow}{\partial}'_\tau \phi' \right) \\
&= \mu \int_0^\beta \int_V d^3x d\tau j_0 + \rho\mu \int_0^\beta \int_V d^3x d\tau j_0.
\end{aligned} \tag{24}$$

Plugging in these expressions into Eq. (21):

$$e^{\beta' P' V'} = \int J[d\phi][d\phi^*] e^{-S_E + \mu \int_0^\beta \int_V d^3x d\tau j_0 - 2\rho \int_0^\beta \int_V d^3x d\tau (m^2 - \mu^2) \phi^* \phi + \rho\mu \int_0^\beta \int_V d^3x d\tau j_0}, \tag{25}$$

where J is the Jacobian of the transformation $(\phi', \phi'^*) \rightarrow (\phi, \phi^*)$. Expressing $J = 1 - \rho A$ and using Eq. (19):⁵

$$\begin{aligned}
\delta(e^{\beta P V}) &= \rho\beta(-E + 3PV + \mu Q) e^{\beta P V} \\
&= \rho \left(-A - 2 \left\langle \int_0^\beta \int_V d^3x d\tau (m^2 - \mu^2) \phi^\dagger \phi \right\rangle + \left\langle \mu \int_0^\beta \int_V d^3x d\tau j_0 \right\rangle \right) e^{\beta P V}.
\end{aligned} \tag{26}$$

⁵ $\langle F(\phi, \phi^\dagger) \rangle \equiv \frac{1}{Z} \int [d\phi][d\phi^*] F(\phi, \phi^*) e^{-S_E + \mu \int_0^\beta \int_V d^3x d\tau j_0}$.

The chemical potential terms drop out on both sides⁶ and we get:

$$\mathcal{E} - 3P = 2m^2 \langle \phi^\dagger \phi \rangle + \mathcal{A}, \quad (27)$$

where

$$\begin{aligned} J &= \left[\frac{\partial \phi' \partial \phi'^*}{\partial \phi \partial \phi^*} \right] = e^{\text{Tr} \log(I_2(\delta^4(x-y) + \rho(-1 - x_\mu \partial_\mu)\delta^4(x-y)))} \\ &= e^{\rho \int d^4x \text{tr} [(-1 - x_\mu \partial_\mu)\delta^4(x-y)I_2]} \Big|_{x=y} \\ &= 1 + \rho \int d^4x \text{tr} [(-1 - x_\mu \partial_\mu)\delta^4(x-y)I_2] \Big|_{x=y}, \end{aligned} \quad (28)$$

so that

$$\mathcal{A} = \text{tr} [(1 + x_\mu \partial_\mu)\delta^4(x-y)I_2] \Big|_{x=y}. \quad (29)$$

I_2 is the two dimensional identity matrix which results from having two fields, ϕ and ϕ^* .⁷ $\mathcal{A} = \frac{A}{\beta V}$ is the anomaly, a divergent quantity that requires regularization.

3 Fujikawa Calculation

In Euclidean space, $\mathcal{L}_E = \partial_\mu \phi^\dagger \partial_\mu \phi + m^2 \phi^\dagger \phi + \frac{\lambda}{4}(\phi^\dagger \phi)^2$. A saddle point expansion about a constant classical background ϕ produces the quadratic piece \mathcal{L}_2 :

⁶Using the identity $Q = \frac{\partial P}{\partial \mu}$ and Eq. (14), $Q = \frac{\partial P}{\partial \mu} = \langle j_0 \rangle + 2\mu \langle \phi^\dagger \phi \rangle$.

⁷Note that Tr in Eq. (28) refers to both discrete (2×2) and continuous variables, whereas tr in Eq. (29) refers to only (2×2).

$$\begin{aligned}
\mathcal{L}_2 &= \frac{1}{2} \begin{pmatrix} \eta^\dagger & \eta \end{pmatrix} \begin{pmatrix} -\partial^2 + m^2 + \lambda\phi^*\phi & \frac{\lambda}{2}\phi\phi \\ \frac{\lambda}{2}\phi^*\phi^* & -\partial^2 + m^2 + \lambda\phi^*\phi \end{pmatrix} \begin{pmatrix} \eta \\ \eta^\dagger \end{pmatrix} \\
&\equiv \frac{1}{2} \begin{pmatrix} \eta^\dagger & \eta \end{pmatrix} \begin{pmatrix} -\partial^2 + C & \frac{\lambda}{2}\phi\phi \\ \frac{\lambda}{2}\phi^*\phi^* & -\partial^2 + C \end{pmatrix} \begin{pmatrix} \eta \\ \eta^\dagger \end{pmatrix} \\
&\equiv \frac{1}{2} \begin{pmatrix} \eta^\dagger & \eta \end{pmatrix} M \begin{pmatrix} \eta \\ \eta^\dagger \end{pmatrix},
\end{aligned} \tag{30}$$

where $C = m^2 + \lambda\phi^*\phi$, η is the fluctuating field around ϕ , and M is a Hermitian matrix. Following Fujikawa [20], we use M , the bilinear matrix, as the Hermitian matrix that goes in our regulator⁸. Choose a regulator of the form $R = R\left(\frac{M}{\Lambda^2}\right)$ with the property that $R(0) = I_2$. The expression to be regulated is:

$$\mathcal{A} = \text{tr} \left(\begin{pmatrix} \theta\delta(x-y) & 0 \\ 0 & \theta\delta(x-y) \end{pmatrix} \right) \Big|_{x=y} \tag{31}$$

where $\theta = 1 + x_\mu\partial_\mu$, so that

$$\mathcal{A}_R = \text{tr} \left[R\left(\frac{M}{\Lambda^2}\right) \theta\delta(x-y)I_2 \right] \Big|_{x=y}. \tag{32}$$

This expression equals:

⁸e.g., for the chiral anomaly with $\mathcal{L} = \bar{\psi}i\mathcal{D}\psi$, the matrix $i\mathcal{D}$ is to be used as the argument of the regulator. M , the quadratic piece of the quantum action, naturally captures the 1-loop effects of interactions which are responsible for anomalies.

$$\begin{aligned}
\mathcal{A}_R &= \int \frac{d^4 k}{(2\pi)^4} \text{tr} R \begin{pmatrix} \frac{-\partial^2 + C}{\Lambda^2} & \frac{\lambda\phi\phi}{2\Lambda^2} \\ \frac{\lambda\phi^*\phi^*}{2\Lambda^2} & \frac{-\partial^2 + C}{\Lambda^2} \end{pmatrix} \theta e^{-ik(x-y)} \Big|_{x=y} \\
&= \int \frac{d^4 k}{(2\pi)^4} \text{tr} R \begin{pmatrix} \frac{k^2 + C}{\Lambda^2} & \frac{\lambda\phi\phi}{2\Lambda^2} \\ \frac{\lambda\phi^*\phi^*}{2\Lambda^2} & \frac{k^2 + C}{\Lambda^2} \end{pmatrix} (1 - ix_\mu k_\mu) \\
&= \Lambda^4 \int \frac{d^4 k}{(2\pi)^4} \text{tr} R \begin{pmatrix} k^2 + \frac{C}{\Lambda^2} & \frac{\lambda\phi\phi}{2\Lambda^2} \\ \frac{\lambda\phi^*\phi^*}{2\Lambda^2} & k^2 + \frac{C}{\Lambda^2} \end{pmatrix} (1 - i\Lambda x_\mu k_\mu) \\
&= \Lambda^4 \int \frac{d^4 k}{(2\pi)^4} \text{tr} R \begin{pmatrix} k^2 + \frac{C}{\Lambda^2} & \frac{\lambda\phi\phi}{2\Lambda^2} \\ \frac{\lambda\phi^*\phi^*}{2\Lambda^2} & k^2 + \frac{C}{\Lambda^2} \end{pmatrix},
\end{aligned} \tag{33}$$

where the k_μ term is odd so vanishes over the integral when multiplied by the even function $R(-k) = R(k) = f(k^2)$. Next we define:

$$\begin{aligned}
D &= k^2 I_2, \\
B &= \frac{1}{\Lambda^2} \begin{pmatrix} C & \frac{\lambda\phi\phi}{2} \\ \frac{\lambda\phi^*\phi^*}{2} & C \end{pmatrix},
\end{aligned} \tag{34}$$

so that the equation can be written succinctly:

$$\mathcal{A}_R = \Lambda^4 \int \frac{d^4 k}{(2\pi)^4} \text{tr} R(D + B). \tag{35}$$

We then Taylor expand about D (note that $[D, B] = 0$ so the Taylor expansion is valid):

$$\mathcal{A}_R = \Lambda^4 \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left(R(D) + R'(D)B + \frac{1}{2}R''(D)B^2 + \dots \right). \tag{36}$$

The first term is the same as in the non-interacting case, which is taken to be anomaly free [21], so we neglect it. The second term can be absorbed by a mass counter-term. Terms higher order than the third term fall faster than $\frac{1}{\Lambda^4}$ so the Λ^4 prefactor in Eq. (36) cannot keep them from going to zero. Only the 3rd term is independent of the cutoff. Therefore:

$$\begin{aligned}\mathcal{A}_R &= \Lambda^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{2} \text{tr} (R''(D)B^2) \\ &= \Lambda^4 \int \frac{k^2 dk^2}{16\pi^2} \frac{1}{2} \text{tr} (R''(D)B^2),\end{aligned}\tag{37}$$

where the solid angle $\Omega = 2\pi^2$ was used. Now

$$B^2 = \frac{1}{\Lambda^2} \begin{pmatrix} C^2 + \frac{\lambda^2(\phi^*\phi)^2}{4} & \lambda C\phi\phi \\ \lambda C\phi^*\phi^* & C^2 + \frac{\lambda^2(\phi^*\phi)^2}{4} \end{pmatrix} \equiv \frac{1}{\Lambda^2} \begin{pmatrix} B_1 & B_2 \\ B_2^* & B_1 \end{pmatrix},\tag{38}$$

and since $R(D)$ is diagonal, we can define:

$$R(D) = f(k^2)I_2.\tag{39}$$

Note that the derivative in Eq. (37) is w.r.t. k^2 . Therefore:

$$\begin{aligned}\mathcal{A}_R &= \Lambda^4 \int \frac{k^2 dk^2}{16\pi^2} \frac{1}{2} \text{tr} (R''(D)B^2) \\ &= B_1 \int \frac{k^2 dk^2}{16\pi^2} f''(k^2),\end{aligned}\tag{40}$$

where we have safely taken $\Lambda \rightarrow \infty$. Integrating by parts:

$$\begin{aligned}
\mathcal{A}_R &= \frac{B_1}{16\pi^2} [k^2 f'(k^2)] \Big|_0^\infty - \frac{B_1}{16\pi^2} \int dk^2 f'(k^2) \\
&= \frac{B_1}{16\pi^2} [k^2 f'(k^2)] \Big|_0^\infty - \frac{B_1}{16\pi^2} f(k^2) \Big|_0^\infty \\
&= \frac{B_1}{16\pi^2},
\end{aligned} \tag{41}$$

where we require

$$\begin{aligned}
f(0) &= 1 \\
f(\infty) &= 0 \\
[k^2 f'(k^2)] \Big|_0^\infty &= 0,
\end{aligned} \tag{42}$$

which are the same conditions on the regulator for the chiral case [22].

Plugging in B_1 from Eq. (38) into Eq. (41), we get:

$$\mathcal{A}_R = \frac{C^2 + \frac{\lambda^2(\phi^*\phi)^2}{4}}{16\pi^2} = \frac{5\lambda^2(\phi^*\phi)^2}{64\pi^2} + \frac{m^4}{16\pi^2} + \frac{\lambda m^2(\phi^*\phi)}{8\pi^2}. \tag{43}$$

The second term is independent of the coupling, and since the free theory is taken to be non-anomalous, we can subtract it. The third term can be absorbed into the mass term of Eq. (27), leaving only the 1st term as the anomaly [23]. Therefore

$$\mathcal{E} - 3P = \frac{5\lambda^2}{64\pi^2} \langle (\phi^\dagger \phi)^2 \rangle. \tag{44}$$

Note that the anomaly \mathcal{A}_R occurs inside the path integral, and

$$\frac{1}{Z} \int [d\phi d\phi^*] f(\phi, \phi^*) e^{-S_E + \dots} = \langle f(\phi, \phi^\dagger) \rangle,$$

so that in Eq. (44) there are expectation values. This replacement is valid up to 1-loop [23].

4 Dimensional Analysis for Relativistic Systems

In relativistic theories we set $\hbar = c = k_B = 1$. The units for all quantities can then be written as $\hbar^i c^j k_B^k L^\ell$, where L is a variable in the problem with units of length. Suppose the system has microscopic parameters g_k , which can be coupling constants or dimensionally transmuted quantities. We define $[g_k] = \ell$ as the power of L when g_k is written in units of $\hbar^i c^j k_B^k L^\ell$. So for example $[m] = [E] = -1$. The grand potential $\Omega = \Omega(\beta, \mu_i, V, g_i)$ has $[\Omega] = -1$ and can be written as:

$$\Omega(\beta, z_i, V, g_i) = V\beta^{-1-D} f(z_i, g_i\beta^{-[g_i]}), \quad (45)$$

where $f(z_i, g_i\beta^{-[g_i]})$ is a dimensionless function of dimensionless variables, z_i is the fugacity corresponding to μ_i ($z_i = e^{\beta\mu_i}$), and D is the number of spatial dimensions.⁹ Ω has this form because β and μ_i don't depend on the absolute size of the system (they are intensive variables). If one doubles the system keeping β and μ_i constant, then Ω , being an extensive quantity, should double. So Ω must be proportional to V .¹⁰ To make up for the remaining dimension ($[\Omega] = -1$), we are free to pull out one of the dimensionful arguments of Ω , and the rest of the arguments must be ratios with the argument we pulled out. We will pull out β .

⁹For example, if the coupling g_1 has dimensions of length, the corresponding dimensionless variable is $g_1\beta^{-1} = g_1T$ which is dimensionless. If the coupling g_2 has dimensions of energy, $g_2\beta^{-(-1)} = g_2\beta = \frac{g_2}{T}$.

¹⁰ $\Omega = -PV$, so Eq. (45) is consistent with the statement that $P(\beta, \mu, V) = P(\beta, \mu)$.

This is equivalent to choosing our scale as β and measuring all other quantities in units of β .

Take the derivative of Eq. (45) w.r.t. to β at constant fugacity z_i and volume V , and multiply times β :

$$\begin{aligned}
\beta \frac{\partial \Omega}{\partial \beta} \Big|_{z_i, V} &= (-1 - D) \Omega + V \beta^{-1-D} \beta \frac{\partial f(z_i, g_i \beta^{-[g_i]})}{\partial \beta} \Big|_{z_i} \\
&= (-1 - D) \Omega + V \beta^{-1-D} \beta \left[\sum_k \frac{-[g_k] g_k}{\beta} \frac{\partial f(z_i, g_i \beta^{-[g_i]})}{\partial g_k} \right] \Big|_{z_i} \\
&= (-1 - D) \Omega - \sum_k [g_k] g_k \frac{\partial \Omega}{\partial g_k}.
\end{aligned} \tag{46}$$

Now, we use the thermodynamic identity $E = \frac{\partial(\beta\Omega)}{\partial \beta} \Big|_{z_i, V} = \Omega + \beta \frac{\partial \Omega}{\partial \beta} \Big|_{z_i, V}$.

$$\begin{aligned}
E - DPV &= \left(\Omega + \beta \frac{\partial \Omega}{\partial \beta} \Big|_{z_i, V} \right) - DPV \\
&= \left(\Omega + (-1 - D) \Omega - \sum_k [g_k] g_k \frac{\partial \Omega}{\partial g_k} \right) - DPV \\
&= - \left(P + (-1 - D) P - \sum_k [g_k] g_k \frac{\partial P}{\partial g_k} \right) V - DPV \\
&= \sum_k [g_k] g_k \frac{\partial P}{\partial g_k} V \\
\mathcal{E} - DP &= \sum_k [g_k] g_k \frac{\partial P}{\partial g_k}.
\end{aligned} \tag{47}$$

where the derivatives are at constant β , μ , and V .

5 β Function

For a system that develops a microscopic scale M through dimensional transmutation via renormalization of the coupling constant:

$$\mathcal{E} - DP = [M]M \frac{d\lambda}{dM} \frac{\partial P}{\partial \lambda} = -M \frac{d\lambda}{dM} \frac{\partial P}{\partial \lambda} = -\beta(\lambda) \frac{\partial P}{\partial \lambda} = \beta(\lambda) \left\langle \frac{\partial \mathcal{H}_I}{\partial \lambda} \right\rangle, \quad (48)$$

since $\frac{\partial P}{\partial \lambda} = \frac{1}{\beta V} \frac{\partial}{\partial \lambda} \ln \left\{ \int [d\phi][d\phi^*] e^{-S_E + \mu \int d^D x d\tau j^o} \right\}$ pulls down the interaction term in the path integral, creating a thermal average.

Comparison of Eq. (10), Eq. (27), Eq. (44), and Eq. (48) gives:

$$\beta(\lambda) = \frac{5\lambda^2}{16\pi^2}, \quad (49)$$

as

$$\mathcal{E} - 3P = \frac{5\lambda^2}{64\pi^2} \langle (\phi^\dagger \phi)^2 \rangle = \beta(\lambda) \left\langle \frac{(\phi^\dagger \phi)^2}{4} \right\rangle \quad (50)$$

would give Eq. (49).

The β function of Eq. (49) can be gotten from setting $e = 0$ for the charge e in the calculation for the four-scalar vertex in scalar electrodynamics [24]. A diagram-

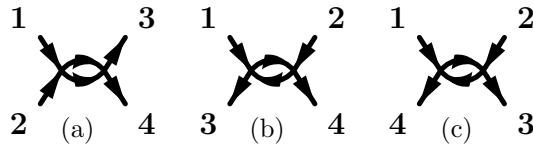


Figure 1: Diagrams contributing to the β function for complex scalar field theory. 1 and 2 refer to incoming particles, 3 and 4 to outgoing particles.

matic calculation requires the identification of 3 diagrams (see Fig 1). Diagram (a) contains a symmetry factor of $1/2$ due to the swapping of internal propagators. Modulo the symmetry factor, each diagram contributes the same amount to the β function, giving $1/2+1+1 = 5(1/2)$, or the first diagram's contribution multiplied by 5. The matrix M used for regularization automatically mixes the interactions, giving the factor of 5. Using the definition of the beta function $M \frac{d\lambda}{dM} = \frac{5\lambda^2}{16\pi^2}$ and setting the renormalization scale $M = T$, one can solve the differential equation for the coupling $\lambda(T) = \frac{16\pi^2}{5 \ln(\frac{\Lambda}{T})}$, where Λ is the Landau pole. As $\frac{T}{\Lambda} \rightarrow 0$ the coupling is small and the system behaves like a gas of noninteracting bosons, while as $T \rightarrow \Lambda$ the coupling blows up and perturbation theory fails.

6 Conclusions

In this paper we have extended to relativistic systems the path-integral approach to the study of quantum anomalies for many-body systems initiated in [13, 15, 14]. A notable difference is that in the relativistic case we have a very wide class of regulators characterized by the function $f(k^2)$ of Eq. (39), which other than satisfying Eq. (42), are of a very general nature. An interesting result of this paper is the extraction of the leading order result for the beta function for complex

fields, Eq. (49), obtained here by comparing Eqs. (10), (27), (44) and (48), without resorting to graphical methods [25, 24]. This result gives further support to the importance of Fujikawa's approach in the description of quantum anomalies for systems at finite temperature and density. We are currently pursuing further studies and extensions of this method, as well as applications to other systems with classical scale symmetry.

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References

- [1] C. G. Callan, S. Coleman, and R. Jackiw, "A new improved energy-momentum tensor," Annals of Physics, vol. 59, no. 1, pp. 42 – 73, 1970.
- [2] S. Coleman and R. Jackiw, "Why dilatation generators do not generate dilatations," Annals of Physics, vol. 67, no. 2, pp. 552 – 598, 1971.
- [3] C. Hagen, "Scale and conformal transformations in galilean-covariant field theory," Phys.Rev., vol. D5, pp. 377–388, 1972.
- [4] J. Gaiete, "The relativistic virial theorem and scale invariance," Physics-Uspekhi, vol. 56, no. 9, p. 919, 2013.

- [5] G. N. J. Añaños, H. E. Camblong, C. Gorrichátegui, E. Hernández, and C. R. Ordóñez, “Anomalous commutator algebra for conformal quantum mechanics,” Phys. Rev. D, vol. 67, p. 045018, Feb 2003.
- [6] S. Tan, “Energetics of a strongly correlated fermi gas,” Annals of Physics, vol. 323, no. 12, pp. 2952 – 2970, 2008.
- [7] S. Tan, “Large momentum part of a strongly correlated fermi gas,” Annals of Physics, vol. 323, no. 12, pp. 2971 – 2986, 2008.
- [8] S. Tan, “Generalized virial theorem and pressure relation for a strongly correlated fermi gas,” Annals of Physics, vol. 323, no. 12, pp. 2987 – 2990, 2008.
- [9] E. Braaten, “Universal relations for fermions with large scattering length,” in The BCS-BEC Crossover and the Unitary Fermi Gas (W. Zwerger, ed.), vol. 836 of Lecture Notes in Physics, pp. 193–231, Springer Berlin Heidelberg, 2012.
- [10] E. Braaten and L. Platter, “Exact relations for a strongly interacting fermi gas from the operator product expansion,” Phys. Rev. Lett., vol. 100, p. 205301, May 2008.
- [11] J. Preskill, “Quantum Chromodynamics, Lecture Notes for Physics Vol. 230abc.” California Institute of Technology, Pasadena CA, 1983.
- [12] C. DeTar and U. Heller, “Qcd thermodynamics from the lattice,” The European Physical Journal A, vol. 41, no. 3, pp. 405–437, 2009.

- [13] C. R. Ordóñez, “Path-integral Fujikawa’s approach to anomalous virial theorems and equations of state for systems with $so(2,1)$ symmetry,” [arXiv:1503.01384](https://arxiv.org/abs/1503.01384), 2015.
- [14] C. L. Lin and C. R. Ordóñez, “Virial Theorem for Non-relativistic Quantum Fields in D Spatial Dimensions,” Adv. High Energy Phys., vol. 2015, p. 796275, 2015.
- [15] C. L. Lin and C. R. Ordóñez, “Path-integral derivation of the nonrelativistic scale anomaly,” Phys. Rev. D, vol. 91, p. 085023, Apr 2015.
- [16] P. Jizba, “Hydrostatic pressure of the $o(n)$ φ^4 theory in the large N limit,” Phys. Rev. D, vol. 69, p. 085011, Apr 2004.
- [17] N. P. Landsman and C. G. van Weert, “Real- and imaginary-time field theory at finite temperature and density,” Phys.Rep, vol. 145, pp. 141–249, Jan. 1987.
- [18] I. Drummond, R. Horgan, P. Landshoff, and A. Rebhan, “{QCD} pressure and the trace anomaly,” Physics Letters B, vol. 460, no. 12, pp. 197 – 203, 1999.
- [19] J. I. Kapusta and C. Gale, Finite-Temperature Field Theory: Principles and Applications. Cambridge University Press, 2006.
- [20] K. Fujikawa, “Path Integral Measure for Gauge Invariant Fermion Theories,” Phys.Rev.Lett., vol. 42, pp. 1195–1198, 1979.

- [21] M. Umezawa, “Regularization of the Path Integral Measure for Anomalies,” Phys.Rev., vol. D39, p. 3672, 1989.
- [22] S. Weinberg, The Quantum Theory of Fields. Vol. 2. Cambridge University Press, 1996.
- [23] K. Shizuya and H. Tsukahara, “Path Integral Formulation of Conformal Anomalies,” Z.Phys., vol. C31, pp. 553–556, 1986.
- [24] M. Srednicki, Quantum Field Theory. Cambridge University Press, 2007.
- [25] S. Coleman and E. Weinberg, “Radiative corrections as the origin of spontaneous symmetry breaking,” Phys. Rev. D, vol. 7, pp. 1888–1910, Mar 1973.

D Relationship between Fujikawa's Method and the Background Field Method for the Scale Anomaly

This paper has yet to be published.

Relationship between Fujikawa's Method and the Background Field Method for the Scale Anomaly

Chris L. Lin and Carlos R. Ordóñez

Department of Physics, University of Houston, Houston, TX 77204-5005

We show the equivalence between Fujikawa's method for calculating the scale anomaly and the diagrammatic approach to calculating the effective potential via the background field method, for an $O(N)$ symmetric scalar field theory. Fujikawa's method leads to a sum of terms, each one superficially in one-to-one correspondence with a vacuum diagram of the 1-loop expansion. From the viewpoint of the classical action, the anomaly results in a breakdown of the Ward identities due to a scale-dependence of the couplings, whereas in terms of the effective action, the anomaly is the result of the breakdown of Noether's theorem due to explicit symmetry breaking terms of the effective potential.

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1 Introduction

Fujikawa showed that within the path-integral formalism, all anomalies are the result of non-invariance of the measure under symmetry transformations [1, 2, 3]. The resulting Jacobian then spoils the naive Ward identities. It is also known that the quantum effective action preserves the symmetries of the classical action, provided that the measure is non-invariant under the symmetry transformations [4].

Therefore there should be a relationship between Fujikawa's method and the non-invariant terms of the quantum effective action. We investigate this relationship in the context of an $O(N)$, $\lambda\phi^4$ theory, by comparing, term-by-term, the Taylor expansion of the Fujikawa determinant with all diagrams in the 1-loop expansion of the quantum effective potential.

The reason for embarking on this comparison is that a framework for applying Fujikawa's method to non-relativistic, classically scale-invariant systems was undertaken recently [5, 6, 7]. While the quantum effective action is a standard tool in non-relativistic physics (e.g., see [8, 9]), Fujikawa's method is not. Therefore a comparison of the two approaches, without a coupling to a gravitational background as is done for the relativistic case, might be helpful in a first approximation as a bridge between the two methods in the context of non-relativistic physics.

It is well-known that for the chiral anomaly, the choice of regulating function $f\left(\frac{\not{D}^2}{\Lambda^2}\right)$ one uses to regulate the Jacobian is pretty much arbitrary, except for a few conditions governing the behavior of f and its derivatives at 0 and ∞ that are quite reasonable [10]. The argument of the regulating function however is not arbitrary - one must choose the gauge invariant \not{D} . The anomaly calculated in this manner is both finite and exact.

For the scale anomaly, things aren't as clear. There is no symmetry that tells you what variable must go into the regulating function. Moreover, if one Taylor

expands the anomaly as one does in the chiral case, certain terms are infinite. If one ignores those terms, then one can recover the anomaly, but it is not exact, holding only to 1-loop order. One generally chooses the quadratic part of effective action for the argument since it characterizes 1-loop effects [11].

In this paper we attempt to explore the connection between certain terms in the effective potential when it is expanded by number of vertices and certain terms in the Jacobian of Fujikawa's method when it is Taylor expanded, thereby clarifying the statement that putting the quadratic part of the effective action in the regulating function captures the 1-loop effects. Also, we consider $O(N)$ as opposed to a single scalar field because despite the problems of Fujikawa's method for the case of the scale anomaly compared to the chiral anomaly, such as only capturing the 1-loop result, it still retains a universal quality in that it can capture the 1-loop result for any N .

In the next two sections, we give a quick review of Fujikawa's method and the background field method for calculating the effective action. In the fourth section we apply Fujikawa's method to calculate the anomaly and the β function of N scalar fields interacting via an $O(N)$ symmetric $\lambda\phi^4$ potential. In the fifth section we use the background field method to write an expression for the effective potential, organized by the number of vertices, and compare this result with the Taylor expansion resulting from Fujikawa's method to derive conditions on the Fujikawa regulator for the two approaches to give the same result. Finally, in the

sixth section we apply Noether's theorem to the effective action and compare it to anomalous scale-breaking of the classical action.

2 Fujikawa's Method

For simplicity we will demonstrate this method for a single scalar field: the generalization to multiple fields is straightforward. With a change of variables given by $\phi'(x) = \phi(x) + \epsilon\delta\phi(x)$:

$$\begin{aligned}
\int [d\phi] e^{iS[\phi]} &= \int [d\phi'] \left| \frac{\delta\phi}{\delta\phi'} \right| e^{iS[\phi(\phi')]} \\
&= \int [d\phi'] \left| \delta^d(x-y) - \epsilon \frac{\delta\delta\phi'(x)}{\delta\phi'(y)} \right| e^{iS[\phi' - \epsilon\delta\phi']} \\
&= \int [d\phi] \left| \delta^d(x-y) - \epsilon \frac{\delta\delta\phi(x)}{\delta\phi(y)} \right| e^{iS[\phi - \epsilon\delta\phi]} \quad (1) \\
&= \int [d\phi] e^{-\epsilon \int d^d x \frac{\delta\delta\phi}{\delta\phi}} e^{iS[\phi]} e^{-i\epsilon \int d^d x \frac{\delta S}{\delta\phi} \delta\phi} \\
&= \int [d\phi] e^{iS[\phi]} \left(1 - \epsilon \int d^d x \frac{\delta\delta\phi}{\delta\phi} - i\epsilon \int d^d x \frac{\delta S}{\delta\phi} \delta\phi \right).
\end{aligned}$$

Since this holds for any volume V , it follows:

$$\left\langle \frac{\delta S}{\delta\phi} \delta\phi \right\rangle = i \left\langle \frac{\delta\delta\phi(x)}{\delta\phi(y)} \Big|_{y=x} \right\rangle. \quad (2)$$

If $\phi \rightarrow \phi + \epsilon\delta\phi$ is a symmetry transformation, then $\frac{\delta S}{\delta\phi} \delta\phi = -\partial_\mu j^\mu$, so that Fujikawa's method tells us that:

$$\langle \partial_\mu j^\mu \rangle = -i \left\langle \frac{\delta\delta\phi(x)}{\delta\phi(y)} \Big|_{y=x} \right\rangle. \quad (3)$$

The transformation we're interested in are dilations for N scalar fields:

$$\begin{aligned}x'^{\mu} &= e^{-\rho} x^{\mu} \\ \phi_i(x') &= e^{\rho} \phi_i(x)\end{aligned}\tag{4}$$

so that the Jacobian is:

$$\begin{aligned}J &= \frac{\delta\phi_i(x)}{\delta\phi_j(y)} = (1 + x^{\mu}\partial_{\mu})\delta^4(x - y)I_n \\ &\equiv \theta\delta^4(x - y)I_n\end{aligned}\tag{5}$$

where I_n is the N -dimensional identity matrix and $\theta = (1 + x^{\mu}\partial_{\mu})$.

3 Background Field Method

We briefly review some facts about the effective action. The generation functional $W[J]$ for the connected correlation functions can be expressed via the path integral as:

$$e^{iW[J]} = \int [d\phi] e^{iS[\phi] + i\int J\phi}\tag{6}$$

The effective action is defined as the Legendre transform:

$$\begin{aligned}\Gamma[\phi_c] &= W[J(\phi_c)] - \int J(\phi_c)\phi_c \\ \phi_c &= \frac{\delta W}{\delta J} = \langle\phi\rangle_J\end{aligned}\tag{7}$$

$\Gamma[\phi_c]$ obeys the classical equations of motion:

$$\frac{\delta\Gamma}{\delta\phi_c} = -J \quad (8)$$

and can be expanded as:

$$\begin{aligned} \Gamma[\phi_c] &= \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n G_{1PI}^{(n)}(x_1, \dots, x_n) \phi_c(x_1) \dots \phi_c(x_n) \\ &= \int dx \left(-V_{\text{eff}}(\phi_c) + \frac{1}{2} Z(\phi_c) \partial_\mu \phi_c \partial^\mu \phi_c + \dots \right) \end{aligned} \quad (9)$$

which shows that $\Gamma[\phi_c]$ is the generating functional for the 1PI graphs and that the effective potential V_{eff} is the negative sum of all 1PI graphs with all external lines set to 0 momentum.

In the background field method¹, we define a new generating functional $\tilde{W}[J]$:

$$\begin{aligned} e^{i\tilde{W}[J]} &= \int [d\phi] e^{iS[\phi+\hat{\phi}] + i\int J\phi} = \int [d\phi] e^{iS[\phi] + i\int J(\phi-\hat{\phi})} \\ &= e^{iW[J]} e^{-iJ\hat{\phi}} \end{aligned} \quad (10)$$

Application of Eqn. (7) to $\tilde{W}[J]$ then gives the following relationships:

$$\begin{aligned} \tilde{W}[J] &= W[J] - J\hat{\phi} \\ \tilde{\phi}_c &= \phi_c - \hat{\phi} \\ \tilde{\Gamma}[\tilde{\phi}_c, \hat{\phi}] &= \Gamma[\tilde{\phi}_c + \hat{\phi}] \end{aligned} \quad (11)$$

Setting $\tilde{\phi}_c = 0$ for the effective action then gives us the result we'll need:

¹For a review of the background field method, see [12].

$$\Gamma[\hat{\phi}] = \tilde{\Gamma}[0, \hat{\phi}] \quad (12)$$

which states that to calculate the effective action $\Gamma[\hat{\phi}]$ associated with the classical action $S[\hat{\phi}]$, we need only calculate the 1PI vacuum graphs associated with the classical action $S[\phi + \hat{\phi}]$, i.e. the original action shifted by a background $\hat{\phi}$. In the following section we will relabel ϕ in $S[\phi + \hat{\phi}]$ as η .

4 Fujikawa Calculation

Consider the conformally invariant Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{\lambda}{4} (\phi_i \phi_i)^2 \quad (13)$$

where repeated indices are summed and $i = 1, 2, \dots, N$. The quadratic part of the action S expanded around the constant background fields $\hat{\phi}_i$ ($\phi_i = \hat{\phi}_i + \eta_i$) is given by:

$$\tilde{S}_2 = \frac{1}{2} \sum_{i,j=1}^N \int d^4x d^4y \frac{\delta^2 S}{\delta \phi_j(x) \delta \phi_i(y)} \eta_j(x) \eta_i(y) \quad (14)$$

which can be re-expressed in terms of the Lagrangian:

$$\tilde{S}_2 = \frac{1}{2} \sum_{i,j=1}^N \int d^4x \left(\frac{\partial^2 \mathcal{L}}{\partial \phi_i \partial \phi_j} \eta_i(x) \eta_j(x) + 2 \frac{\partial^2 \mathcal{L}}{\partial \phi_i \partial \partial_\mu \phi_j} \eta_i(x) \partial_\mu \eta_j(x) + \frac{\partial^2 \mathcal{L}}{\partial \partial_\nu \phi_i \partial \partial_\mu \phi_j} \partial_\nu \eta_i(x) \partial_\mu \eta_j(x) \right) \quad (15)$$

Plugging in Eqn. (13) into Eqn. (15) gives:

$$\begin{aligned} \tilde{S}_2 &= \frac{1}{2} \sum_{i,j=1}^N \int d^4x \left(\left[-2\lambda \hat{\phi}_i \hat{\phi}_j - \lambda (\hat{\phi}_k \hat{\phi}_k) \delta_{ij} \right] \eta_i(x) \eta_j(x) + \partial_\mu \eta_i(x) \partial^\mu \eta_j(x) \right) \\ &= \frac{1}{2} \sum_{i,j=1}^N \int d^4x \eta_i(x) (B_{ij} + D_{ij}) \eta_j(x) = \frac{1}{2} \sum_{i,j=1}^N \int d^4x \eta_i(x) M_{ij} \eta_j(x) \end{aligned} \quad (16)$$

where

$$D_{ij} = -\delta_{ij} \partial^2, \quad B_{ij} = \left[-2\lambda \hat{\phi}_i \hat{\phi}_j - \lambda (\hat{\phi}_k \hat{\phi}_k) \delta_{ij} \right] \quad (17)$$

We choose M_{ij} as the argument of our regulating matrix so that:

$$\mathcal{A} = \text{tr} \left[R \left(\frac{M}{\Lambda^2} \right) \theta \delta^4(x-y) I_n \right] \Big|_{x=y} \quad (18)$$

Going into Fourier space:

$$\begin{aligned}
\mathcal{A} &= \text{tr} \int \frac{d^4k}{(2\pi)^4} \left[R \left(\frac{M}{\Lambda^2} \right) \theta e^{ik \cdot (x-y)} I_n \right] \Big|_{x=y} \\
&= \text{tr} \int \frac{d^4k}{(2\pi)^4} \left[R \left(\frac{M}{\Lambda^2} \right) (1 + x_\mu k_\mu) I_n \right] \\
&= \Lambda^4 \text{tr} \int \frac{d^4k}{(2\pi)^4} \left[R \left(D + \frac{B}{\Lambda^2} \right) I_n \right]
\end{aligned} \tag{19}$$

where in the 2nd line y has been set equal to x and $D_{ij} = -\delta_{ij}\partial^2 \rightarrow \delta_{ij}k^2$. Since D_{ij} is even in k^2 , the $x_\mu k_\mu$ term vanishes upon integration. Since $[D, B] = 0$, $R(D + \frac{B}{\Lambda^2})$ admits a power series expansion about D :

$$\mathcal{A} = \Lambda^4 \text{tr} \int \frac{d^4k}{(2\pi)^4} \left[R(D) + R'(D) \frac{B}{\Lambda^2} + \frac{1}{2!} R''(D) \left(\frac{B}{\Lambda^2} \right)^2 + \dots \right] \tag{20}$$

Since D is diagonal, we can write $R^{(n)}(D) = f^{(n)}(k^2)I_n$ for some scalar function $f(k^2)$, so that Eqn. (20) becomes:

$$\begin{aligned}
\mathcal{A} &= \Lambda^4 N \int \frac{d^4k}{(2\pi)^4} f(k^2) + \Lambda^2 (\text{tr} B) \int \frac{d^4k}{(2\pi)^4} f'(k^2) + \frac{1}{2!} (\text{tr} B^2) \int \frac{d^4k}{(2\pi)^4} f''(k^2) + \dots \\
&= \Lambda^4 N \int \frac{d^4k}{(2\pi)^4} f(k^2) + \Lambda^2 (\text{tr} B) \int \frac{\Omega_3 dk^2}{2(2\pi)^4} k^2 f'(k^2) + \frac{1}{2!} (\text{tr} B^2) \int \frac{\Omega_3 dk^2}{2(2\pi)^4} k^2 f''(k^2) \\
&\quad + \sum_{n=3}^{\infty} \frac{1}{\Lambda^{(2n-4)}} \frac{1}{n!} (\text{tr} B^n) \int \frac{\Omega_3 dk^2}{2(2\pi)^4} k^2 f^{(n)}(k^2)
\end{aligned} \tag{21}$$

where $\Omega_3 = 2\pi^2$ is the solid angle. The minimum conditions on $f(k^2)$ required to produce the anomaly are:

$$f(0) = 1$$

$$f(\infty) = 0 \tag{22}$$

$$[k^2 f'(k^2)] \Big|_0^\infty = 0,$$

which are the same conditions for the chiral anomaly [10]. However, for simplicity we will specialize to $f(k^2) = e^{-k^2}$ which satisfies Eqn. (22) but in addition has the nice property that:

$$\int dk^2 k^2 f^{(n)}(k^2) = (-1)^n \tag{23}$$

so that plugging in this regulator into Eqn. (21) gives us:

$$\begin{aligned} \mathcal{A} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Lambda^{(2n-4)}} \frac{1}{n!} (\text{tr } B^n) \frac{\Omega_3}{2(2\pi)^4} \\ &= \Lambda^4 (\text{tr } B^0) \frac{\Omega_3}{2(2\pi)^4} - \Lambda^2 (\text{tr } B) \frac{\Omega_3}{2(2\pi)^4} + \frac{1}{2!} (\text{tr } B^2) \frac{\Omega_3}{2(2\pi)^4} \\ &\quad + \sum_{n=3}^{\infty} \frac{(-1)^n}{\Lambda^{(2n-4)}} \frac{1}{n!} (\text{tr } B^n) \frac{\Omega_3}{2(2\pi)^4} \end{aligned} \tag{24}$$

The first term in Eqn. (25) is independent of the coupling λ so would be present even in the free theory. Since the free theory is taken to be non-anomalous, we ignore this term [13]. The second term, proportional to Λ^2 is removed by mass renormalization: the precise meaning of this is discussed in the next section. The third term is the only remaining nonvanishing term in the $\Lambda \rightarrow \infty$ limit, and is independent of Λ . Evaluating $(\text{tr } B^2) = B_{ij} B_{ji}$ by substituting in B_{ij} from Eqn. (17) gives:

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2!} \left[\lambda^2 (N+8) (\hat{\phi}_k \hat{\phi}_k)^2 \right] \frac{\Omega_3}{2(2\pi)^4} \\
&= \frac{\lambda^2 (N+8)}{32\pi^2} (\hat{\phi}_k \hat{\phi}_k)^2 \\
&= \beta(\lambda) \frac{(\hat{\phi}_k \hat{\phi}_k)^2}{4} = \beta(\lambda) \frac{\partial \mathcal{H}_I}{\partial \lambda}
\end{aligned} \tag{25}$$

where $\beta(\lambda) = \frac{\lambda^2(N+8)}{8\pi^2}$ and \mathcal{H}_I is the interacting Hamiltonian.

5 Equivalence of Fujikawa With Background Field Calculation

We now apply the background field method to the Lagrangian in Eqn. (13). We make the shift $\phi_i(x) = \hat{\phi}_i + \eta_i(x)$ so that the $O(N)$ Lagrangian becomes:

$$\tilde{\mathcal{L}} = \frac{1}{2} \sum_{i,j=1}^N \int d^4x \eta_i(x) (D_{ij} + B_{ij}) \eta_j(x) + \mathcal{L}(\hat{\phi}_i, \partial_\mu \hat{\phi}_i) + \mathcal{L}_T + \mathcal{L}_I \tag{26}$$

In the above expression, $\mathcal{L}(\hat{\phi}_i, \partial_\mu \hat{\phi}_i)$ is the original $O(N)$ Lagrangian with the background field substituted for ϕ . This term has no dependence on η and contributes to the 1PI vacuum graphs at tree-level (i.e., w.r.t. the η field this term is like a cosmological constant). \mathcal{L}_T are terms that contain only one η field: these produce tadpole diagrams which are reducible, so \mathcal{L}_T can be neglected in calculation of 1PI graphs. \mathcal{L}_I are terms involving η^3 and η^4 interactions. For 1PI vacuum graphs, these interactions contribute beginning at the 2-loop level, and hence can be ignored for a 1-loop calculation (see Fig. 1).



Figure 1: Lowest-loop 1PI vacuum graphs with 3 and 4 vertices.

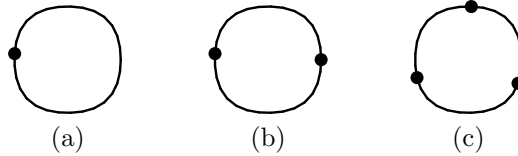


Figure 2: 1-loop 1PI vacuum graphs with 1, 2, and 3 vertices.

So the Lagrangian we will use to calculate the 1PI vacuum graphs at 1-loop is:

$$\tilde{\mathcal{L}} = \frac{1}{2} \sum_{i,j=1}^N \int d^4x \eta_i(x) D_{ij} \eta_j(x) + \frac{1}{2} \sum_{i,j=1}^N \int d^4x \eta_i(x) B_{ij} \eta_j(x) \quad (27)$$

Since the background field $\hat{\phi}_i$ (contained in B_{ij} of Eqn. (17)) is constant and the Lagrangian is only quadratic in η , we could sum all the 1-loop vacuum graphs at once by calculating the determinant $D_{ij} + B_{ij}$ [14]. However, instead we choose as the propagator D_{ij}^{-1} , and treat interaction B_{ij} as an interaction vertex that joins two propagators, and categorize the loops by the number of vertices B_{ij} which corresponds to twice the number of background fields $\hat{\phi}$ (see Fig. 2). We do this to match the result of Eqn. (24) from Fujikawa's method, which is an expansion in powers of B_{ij} .

The Feynman rules are straightforward. For each vertex we write iB_{ij} , as the $1/2$ in Eqn. (27) accounts for swapping connections of the two propagators to which each vertex connects. For each propagator we write iD_{ij}^{-1} , where the $1/2$ takes

care of which end of the propagator connects to a vertex. An overall symmetry factor is required that depends on the number of vertices B_{ij} . This symmetry factor is $\frac{1}{2n}$ where n is the number of vertices: the 2 is due to reflection symmetry and n to cyclic permutation of the vertices.

For an n -vertex diagram:

$$-iV_{\text{eff}}^n = \frac{1}{2n} \int \frac{id^4k}{(2\pi)^4} \left(\frac{i}{-k^2} \right)^n \text{tr} [(iB)^n] = \frac{i}{2n} \frac{\Omega_3}{(2\pi)^4} \text{tr} B^n \left(\int_0^\Lambda dk \frac{k^3}{k^{2n}} \right) \quad (28)$$

where a Wick rotation was performed. The anomaly in Fujikawa's method was given in Eqn. (24) as $\mathcal{A} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \frac{\Omega_3}{(2\pi)^4} (\text{tr} B^n) \Lambda^{4-2n}$. Following the renormalization group analysis of [15], we apply the operator $\frac{\partial}{\partial \ln \Lambda} = \Lambda \frac{\partial}{\partial \Lambda}$ to Eqn. (28). Then from the fundamental theorem of calculus $\Lambda \frac{\partial}{\partial \Lambda} \int_0^\Lambda \frac{k^3}{k^{2n}} = \Lambda^{4-2n}$, we get the result that:

$$-\frac{\partial}{\partial \ln \Lambda} V_{\text{eff}} = \sum_{n=0}^{\infty} \frac{1}{2n} \frac{\Omega_3}{(2\pi)^4} (\text{tr} B^n) \Lambda^{4-2n} \quad (29)$$

Only for $n = 2$ does this match the anomaly given by Fujikawa's method. Indeed, it is impossible to construct a regulator in Fujikawa's method that exactly produces Eqn. (29). However, the terms for $n \geq 3$ vanish in the limit $\Lambda \rightarrow \infty$. Since diagrams for which $n \geq 3$ are convergent, they do not contribute to the anomaly, and in Fujikawa's method they correspond to the vanishing $n \geq 3$ terms in the Taylor expansion. The anomaly is contained entirely in Fig. 2(b). The quadratic

divergence in Fig. 2(a) is a well-known artifact of cutoff regularization and can be avoided by dimensional regularization, where the loop integral is zero [16]. However, Fujikawa's method does not work with dimensional regularization since in $d - 2\epsilon$ dimensions, the δ -function is zero [17]. Within the context of dimensional regularization, the anomaly arises from the fact that $\lambda\phi^4$ in $d - 2\epsilon$ dimensions is not conformally invariant [18] rather than through the noninvariance of the path integral measure.

This can readily be seen by calculating the effective potential. The effective potential is given by summing across all n of Eqn. (28):

$$V_{\text{eff}} = - \sum_{n=1}^{\infty} \frac{1}{2n} \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2} \right)^n \text{tr } B^n \quad (30)$$

One can swap the integral with the summation: this avoids the need for an IR regulator, as the summation results in a log which is IR-free. However, we are interested in the contribution of each n -vertex diagram – therefore we introduce a fictitious mass m to regulate the theory in the IR, and a cutoff Λ to regulate the theory in the UV:

$$\begin{aligned} -V_{\text{eff}} &= \sum_{n=1}^{\infty} \frac{1}{2n} \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m^2} \right)^n \text{tr } B^n \\ &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m^2} \right) \text{tr } B + \frac{1}{4} \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m^2} \right)^2 \text{tr } B^2 \\ &\quad + \sum_{n=3}^{\infty} \frac{1}{2n} \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m^2} \right)^n \text{tr } B^n \end{aligned} \quad (31)$$

The integrals are standard, and the result in the $m^2 \rightarrow 0$ limit is:

$$\begin{aligned}
\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m^2} \right) \text{tr } B &= -\frac{\Lambda^2}{32\pi^2} \text{tr } B \\
\frac{1}{4} \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m^2} \right)^2 \text{tr } B^2 &= \frac{1}{64\pi^2} [1 - \log(\Lambda^2/m^2)] \text{tr } B^2 \\
\sum_{n=3}^{\infty} \frac{1}{2n} \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + m^2} \right)^n \text{tr } B^n &= \frac{1}{128\pi^2} \text{tr} \left[-3B^2 + 2B^2 \log \left(\frac{-B}{m^2} \right) \right]
\end{aligned} \tag{32}$$

One can see that diagrams with $n \geq 3$ are independent of Λ , and that $-\frac{\partial}{\partial \ln \Lambda}$ acting on $n = 2$ produces the anomaly. Both $\text{tr } B = -\lambda(N+2)\phi_k\phi_k$ and $\text{tr } B^2 = \lambda^2(N+8)(\phi_k\phi_k)^2$ are of the form of the original Lagrangian, so can be cancelled by counter-terms. Adding all the terms in Eqn. (32) gives:

$$V_{\text{eff}} = -\frac{\Lambda^2}{32\pi^2} \text{tr } B - \frac{\text{tr } B^2}{128\pi^2} + \frac{1}{64\pi^2} \text{tr} \left[B^2 \log \left(\frac{-B}{\Lambda^2} \right) \right] \tag{33}$$

The result is independent of m^2 as it should be. The $n \geq 3$ terms have produced a nonpolynomial log interaction, and the $n = 2$ term has provided the scale for this interaction.

6 Noether's Theorem and Dimensional Transmutation

The field ϕ_c obeys the classical equations of motion Eqn. (8), with the effective action $\Gamma[\phi_c]$ replacing the classical one $S[\phi_c]$. Therefore, Noether's theorem, which

is based on the classical EOM, would apply if $\Gamma[\phi_c]$ has symmetry. In general the quantum corrections will create terms in $\Gamma[\phi_c]$ that explicitly break scale symmetry. Classically the measure of symmetry-breaking is $\sum_{i=1}^N \frac{\partial V_{\text{eff}}}{\partial \phi_{ic}} \phi_{ic} - 4V_{\text{eff}}$, which gives zero for the classically scale-invariant tree-level contribution $V = \frac{\lambda}{4}(\phi_{ic}\phi_{ic})^2$ to the effective potential. Specializing to $N = 1$ the effective potential Eqn. (33) reads:

$$V_{\text{eff}} = \frac{\lambda\phi_c^4}{4} + \frac{9\lambda^2\phi_c^4}{64\pi^2} \left(\ln \left(\frac{3\lambda\phi_c^2}{\Lambda^2} \right) - \frac{1}{2} \right) \quad (34)$$

Applying $\sum_{i=1}^N \frac{\partial V_{\text{eff}}}{\partial \phi_{ic}} \phi_{ic} - 4V_{\text{eff}}$ to Eqn. (34), we get:

$$\mathcal{A} = \frac{9\lambda^2\phi_c^4}{32\pi^2} \quad (35)$$

in agreement with Eqn. (25). From the viewpoint of classical physics, a term like $\phi_c^4 \ln M^2$ is scale-invariant, acting like a ϕ_c^4 potential. It is $\phi_c^4 \ln \phi_c^2$ term that breaks scale-invariance. Both terms are related since dimensional transmutation of the $n = 2$ graph provides the scale for the $n \geq 3$ graphs which generate nonpolynomial interactions.

7 Conclusion

The scale anomaly, and anomalies in general, are the result of the failure to maintain classical symmetry upon quantization. One cannot regularize the system in a way to preserve all the symmetries of the theory. The absence of dimensionful

parameters in the action is sufficient for the classical theory to be scale invariant. However, the introduction of a dimensionful parameter through regularization can provide a scale to support non-invariant ϕ^{2n} interactions with $n \geq 3$ in the $O(N)$ quantum theory. Fujikawa's method is equivalent to the 1-loop calculation of the anomaly in the effective potential.

We plan to investigate these connections and apply these methods for the non-relativistic case to study questions of interest to atomic physicists, in particular to the field of ultra-cold atoms, where unlike the case in particle physics, the manifestations of the scale anomaly in these systems have only now been accessible to experimentalists in this decade.

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References

- [1] K. Fujikawa, "Path-integral measure for gauge-invariant fermion theories," Phys. Rev. Lett., vol. 42, pp. 1195–1198, Apr 1979.
- [2] K. Fujikawa, "Energy-momentum tensor in quantum field theory," Phys. Rev. D, vol. 23, pp. 2262–2275, May 1981.

- [3] K. Fujikawa and H. Suzuki, Path Integrals and Quantum Anomalies. Oxford Univ. Pr., 2014.
- [4] M. Srednicki, Quantum Field Theory. Cambridge University Press, 2007.
- [5] C. R. Ordóñez, “Path-integral fujikawa’s approach to anomalous virial theorems and equations of state for systems with $so(2,1)$ symmetry,” arXiv:1503.01384, 2015.
- [6] C. L. Lin and C. R. Ordóñez, “Path-integral derivation of the nonrelativistic scale anomaly,” Phys. Rev. D, vol. 91, p. 085023, Apr 2015.
- [7] C. L. Lin and C. R. Ordóñez, “Virial Theorem for Non-relativistic Quantum Fields in D Spatial Dimensions,” Adv. High Energy Phys., vol. 2015, p. 796275, 2015.
- [8] E. Braaten and A. Nieto, “Quantum corrections to the energy density of a homogeneous bose gas,” The European Physical Journal B - Condensed Matter and Complex Systems, vol. 11, no. 1, pp. 143–159, 1999.
- [9] J. Andersen, “Ground state pressure and energy density of an interacting homogeneous bose gas in two dimensions,” The European Physical Journal B - Condensed Matter and Complex Systems, vol. 28, no. 4, pp. 389–396, 2002.
- [10] S. Weinberg, The Quantum Theory of Fields. Vol. 2. Cambridge University Press, 1996.

- [11] K. Shizuya and H. Tsukahara, “Path Integral Formulation of Conformal Anomalies,” Z.Phys., vol. C31, pp. 553–556, 1986.
- [12] L. F. Abbott, “Introduction to the Background Field Method,” Acta Phys. Polon., vol. B13, p. 33, 1982.
- [13] M. Umezawa, “Regularization of the Path Integral Measure for Anomalies,” Phys.Rev., vol. D39, p. 3672, 1989.
- [14] R. Jackiw, “Functional evaluation of the effective potential,” Phys. Rev., vol. D9, p. 1686, 1974.
- [15] S. R. Coleman and E. J. Weinberg, “Radiative Corrections as the Origin of Spontaneous Symmetry Breaking,” Phys. Rev., vol. D7, pp. 1888–1910, 1973.
- [16] K. A. Meissner and H. Nicolai, “Effective action, conformal anomaly and the issue of quadratic divergences,” Physics Letters B, vol. 660, no. 3, pp. 260 – 266, 2008.
- [17] L. Brown, Quantum Field Theory. Cambridge Univ. Pr., 1992.
- [18] J. Hofmann, “Quantum anomaly, universal relations, and breathing mode of a two-dimensional fermi gas,” Phys. Rev. Lett., vol. 108, p. 185303, May 2012.

E The Tan-Pressure Relation

This paper has yet to be published.

Dilational Symmetry-Breaking in Thermodynamics

Chris L. Lin and Carlos R. Ordóñez

Department of Physics, University of Houston, Houston, TX 77204-5005

Using thermodynamic relations and dimensional analysis we derive a general formula for the thermodynamical trace $2\mathcal{E} - DP$ for non-relativistic systems and $\mathcal{E} - DP$ for relativistic systems, where D is the number of spatial dimensions, in terms of the microscopic scales of the system within the grand canonical ensemble. We demonstrate the formula for a variety of cases, including anomalous systems which develop scales through dimensional transmutation. Using this relation, we make explicit the connection between dimensional analysis and the virial theorem. This paper is focused mainly on the non-relativistic aspects of this relation.

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1 Introduction

The quantity $2\mathcal{E} - DP$ for non-relativistic systems, or $\mathcal{E} - DP$ for relativistic systems, where \mathcal{E} is the thermal energy density, D the number of spatial dimensions, and P the pressure, plays an important role in physics. This quantity is the thermal analog of the trace of the improved stress-energy tensor which is a measure of dilational symmetry-breaking and which plays a central role in the renormalization group [1].

In non-relativistic physics, $2\mathcal{E} - DP$ can be used as a measure of deviations of real gases from ideal ones. Traditionally, such deviations are measured by giving the two systems the same value for two of their thermodynamic variables, and taking the difference between them for a third. For ideal gases, and in general non-anomalous scale-invariant systems, $2\mathcal{E} - DP = 0$. Therefore at constant pressure and volume, one can define $(2\mathcal{E}_{\text{real}} - DP) = (2\mathcal{E}_{\text{real}} - DP) - (2\mathcal{E}_{\text{ideal}} - DP) = (2\mathcal{E}_{\text{real}} - 2\mathcal{E}_{\text{ideal}}) \equiv 2\mathcal{E}_{\text{res}}$, so that $(2\mathcal{E}_{\text{real}} - DP)$ equals twice the residual internal energy characterizing the departure of the system from ideal [2]. In other words, for any system, $2\mathcal{E} - DP$ is equal to the difference in its energy from any non-anomalous scale-invariant system's energy at the same V and P .

For ultracold gases interacting via contact interaction, $2\mathcal{E} - DP$ is proportional to the Tan contact $\lambda^2 \langle \psi_{\uparrow}^{\dagger} \psi_{\downarrow}^{\dagger} \psi_{\downarrow} \psi_{\uparrow} \rangle$ [3]. Many universal relations depending only on the contact exist, independent of the exact details of the experimental setup [4].

For systems that are scale invariant at the level of the classical action, a non-zero value of $2\mathcal{E} - DP$ signifies a quantum anomaly, so that $2\mathcal{E} - DP$ measures quantum anomalies. Previously, it was shown that even in anomalous non-relativistic systems, $2\mathcal{E} - DP$ can be expressed as a functional determinant via use of Fujikawa's path integral methods [5, 6]. Therefore, one can potentially extract information about $\beta(C)$ and hence obtain information from or even solve the scattering problem by extracting information from the thermodynamic problem.

In this paper we derive a simple expression for $2\mathcal{E} - DP$ from dimensional analysis and thermodynamics, independent of quantum mechanics or field theory and independent of Noether's theorem and canonical commutation relations. In this paper we will use units where $\hbar = m = k_B = 1$. The units for all quantities can then be written as $\hbar^i m^j k_B^k L^\ell = L^\ell$, where L is a variable in the problem with units of length. We will define $[g_k] = \ell$, and call ℓ the dimensions of the variable g_k .

With this convention, $2\mathcal{E} - DP = \sum_k [g_k] g_k \frac{\partial P}{\partial g_k}$, where $\mathcal{E} = \frac{E}{V}$ is the thermal energy per unit volume, P is the pressure, and D is the number of spatial dimensions. In this formula g_k are the microscopic parameters of the theory, and $[g_k]$ are the dimensions of these parameters. The derivatives w.r.t. microscopic parameters are taken at constant temperature β^{-1} , volume V , and chemical potential for each species μ_i . The LHS is written in terms of pure macroscopic thermodynamic variables, while the RHS contains derivatives purely on the microscopic parameters. Such an equation can be seen as connecting thermodynamics on the LHS (variables characterizing the macrostate) and statistical mechanics on the RHS (microscopic variables that are system dependent). In particular, for a theory in which all the couplings are dimensionless (in the sense that they have no length dimension as defined above), $[g_k] = 0$, and one might expect the system to be scale invariant with $2\mathcal{E} - DP = 0$. However, for such systems, we show $2\mathcal{E} - DP = -\beta(C) \frac{\partial P}{\partial C}$. The microscopic parameters g_k of a system usually appear in its Hamiltonian as coupling constants, except in the case of dimensional transmutation. The latter leads to a new microscopic scale appearing in the pressure P , and in the litera-

ture this called a quantum anomaly. Therefore $2\mathcal{E} - DP$ is also a measure of the anomaly for scale-invariant systems.

The relativistic generalization is $\mathcal{E} - DP = \sum_k [g_k] g_k \frac{\partial P}{\partial g_k}$. It was shown [7] that the trace of the improved stress-energy tensor in relativistic $\lambda\phi^4$ has the property $\theta^{00} - \sum_i \theta^{ii} = m^2\phi^2$, where the mass term represents a dilational symmetry-breaking term. Identifying θ^{00} as \mathcal{E} and $\sum_i \theta^{ii} = DP_H$, where P_H is the hydrodynamic pressure, one derives the thermal analog. θ^{ii} is equal to the hydrodynamic pressure [8]: however, in equilibrium, the thermodynamic pressure P equals P_H via the virial theorem (although anomalies can complicate matters [9]). Therefore deriving this expression requires an improvement of the stress-energy tensor, and an identification of field variables with thermodynamic variables.

We avoid the complications of having to construct the improved stress-energy tensor, or having to work in the context of field theory, by working directly within thermodynamics. We show the consistency of the equation for a variety of cases, with and without anomalies, and then we show that starting from $2\mathcal{E} - DP = \sum_k [g_k] g_k \frac{\partial P}{\partial g_k}$, one can derive the virial theorem, further illustrating the robustness of the expression and showing the relationship between scaling and the virial theorem. The relativistic case is also considered.

2 Finite-Temperature

For ease of presentation, take the independent, dimensionful microscopic parameters g_k of your theory, and form new parameters E_k with dimensions of energy, and rewrite the pressure in terms of these new variables¹. The grand potential $\Omega = \Omega(\beta, \mu_i, V, E_i)$ for a homogeneous system in D -spatial dimensions must have the form

$$\Omega(\beta, z_i, V, E_i) = V\beta^{-1-\frac{D}{2}} f(z_i, \beta E_i), \quad (1)$$

where $f(z_i, \beta E_i)$ is a dimensionless function of dimensionless variables, and z_i is the fugacity corresponding to μ_i . The reason it must have this form is because β and μ_i don't depend on the absolute size of the system (they are intensive variables). If you double the system keeping β and μ_i constant, then Ω , being an extensive quantity, should double. So Ω must be proportional to V . To make up for the remaining dimension ($[\Omega] = -2$), we are free to pull out one of the dimensionful arguments of Ω , and the rest of the arguments must be ratios with the argument we pulled out. We will pull out β . This is equivalent to choosing our scale as β and measuring all other quantities in units of β .

Now take the derivative of eqn. (1) w.r.t. to β at constant fugacity z and volume V , and multiply times β :

¹e.g. if you have a scattering length a , replace it with the variable $E_k = 1/a^2$.

$$\begin{aligned}
\beta \frac{\partial \Omega}{\partial \beta} \Big|_{z_i, V} &= \left(-1 - \frac{D}{2} \right) \Omega + V \beta^{-1 - \frac{D}{2}} \beta \frac{\partial f(z_i, \beta E_i)}{\partial \beta} \Big|_{z_i} \\
&= \left(-1 - \frac{D}{2} \right) \Omega + V \beta^{-1 - \frac{D}{2}} \beta \left[\sum_k \frac{E_k}{\beta} \frac{\partial f(z_i, \beta E_i)}{\partial E_k} \right] \Big|_{z_i} \\
&= \left(-1 - \frac{D}{2} \right) \Omega + \sum_k E_k \frac{\partial \Omega}{\partial E_k}.
\end{aligned} \tag{2}$$

Now, we use the thermodynamic identity $E = \frac{\partial(\beta\Omega)}{\partial\beta} \Big|_{z_i, V} = \Omega + \beta \frac{\partial\Omega}{\partial\beta} \Big|_{z_i, V}$.

$$\begin{aligned}
2E - DPV &= 2 \left(\Omega + \beta \frac{\partial \Omega}{\partial \beta} \Big|_{z_i, V} \right) - DPV \\
&= 2 \left(\Omega + \left(-1 - \frac{D}{2} \right) \Omega + \sum_k E_k \frac{\partial \Omega}{\partial E_k} \right) - DPV \\
&= -2 \left(P + \left(-1 - \frac{D}{2} \right) P + \sum_k E_k \frac{\partial P}{\partial E_k} \right) V - DPV \\
&= -2 \sum_k E_k \frac{\partial P}{\partial E_k} V \\
2\mathcal{E} - DP &= -2 \sum_k E_k \frac{\partial P}{\partial E_k}.
\end{aligned} \tag{3}$$

3 0-Temperature

For 0-temperature, we lose β as a scale. Instead we use μ_1 , where μ_1 is the chemical potential for one of the particles:

$$\Omega = V \mu_1^{1+D/2} f \left(\frac{\mu_1}{E_i}, \frac{\mu_1}{\mu_{j \neq 1}} \right), \tag{4}$$

Calculating the number of particles:

$$\begin{aligned}
N_1 &= -\left. \frac{\partial \Omega}{\partial \mu_1} \right|_{V, \mu_{j \neq 1}} \\
&= -(1 + D/2) \frac{\Omega}{\mu_1} - V \mu_1^{1+D/2} \frac{\partial f\left(\frac{\mu_1}{E_j}, \frac{\mu_1}{\mu_{j \neq 1}}\right)}{\partial \mu_1} \\
&= -(1 + D/2) \frac{\Omega}{\mu_1} \\
&\quad - V \mu_1^{1+D/2} \left[-\sum_k \frac{E_k}{\mu_1} \frac{\partial f\left(\frac{\mu_1}{E_j}, \frac{\mu_1}{\mu_{j \neq 1}}\right)}{\partial E_k} - \sum_{\ell \neq 1} \frac{\mu_{\ell \neq 1}}{\mu_1} \frac{\partial f\left(\frac{\mu_1}{E_j}, \frac{\mu_1}{\mu_{j \neq 1}}\right)}{\partial \mu_{\ell \neq 1}} \right] \quad (5) \\
&= -(1 + D/2) \frac{\Omega}{\mu_1} + \sum_k \frac{E_k}{\mu_1} \frac{\partial}{\partial E_k} \Omega + \sum_{\ell \neq 1} \frac{\mu_{\ell \neq 1}}{\mu_1} \frac{\partial}{\partial \mu_{\ell \neq 1}} \Omega \\
N_1 \mu_1 &= -(1 + D/2) \Omega + \sum_k E_k \frac{\partial}{\partial E_k} \Omega - \sum_{\ell \neq 1} N_{\ell \neq 1} \mu_{\ell \neq 1} \\
\sum_i N_i \mu_i &= -(1 + D/2) \Omega + \sum_k E_k \frac{\partial}{\partial E_k} \Omega
\end{aligned}$$

The energy E of the system at zero temperature is given by $E = \sum_i N_i \mu_i - PV$.

Therefore

$$\begin{aligned}
2E - DPV &= 2 \left(\sum_i N_i \mu_i - PV \right) - DPV = 2 \sum_i N_i \mu_i - (D+2)PV \\
&= 2 \left(-(1 + D/2)\Omega + \sum_k E_k \frac{\partial}{\partial E_k} \Omega \right) - (D+2)PV \\
&= 2 \left(-(1 + D/2)(-PV) + \sum_k E_k \frac{\partial}{\partial E_k} (-PV) \right) - (D+2)PV \quad (6) \\
&= -2V \sum_k E_k \frac{\partial}{\partial E_k} P \\
2\mathcal{E} - DP &= -2 \sum_k E_k \frac{\partial P}{\partial E_k}.
\end{aligned}$$

4 Arbitrary Scale

In general, so long as your theory has microscopic parameters g_i that have dimensions of length (and not necessarily energy or L^{-2} , then by forming appropriate dimensionless variables $x_i = \beta^{-\frac{[g_i]}{2}} g_i$ for the argument of $\Omega(\beta, z, V, g_i) = V\beta^{-1-\frac{D}{2}} f(z, \beta^{-\frac{[g_i]}{2}} g_i)$, then one gets:

$$2\mathcal{E} - DP = \sum_k [g_k] g_k \frac{\partial P}{\partial g_k}. \quad (7)$$

Alternatively, one can note that $E_k = g_k^{-\frac{2}{[g_k]}}$, and apply the chain rule to eqn. (3) to get eqn. (7).

5 Relativistic Systems

In relativistic theories, $\hbar = c = k_B = 1$, and mass attains a dimension equal to $1/L$. The units for all quantities can then be written as $\hbar^i c^j k_B^k L^\ell = L^\ell$, and we define the dimensions of the parameter g_k as $[g_k] = \ell$. The grand potential Ω has $[\Omega] = -1$ rather than the NR case $[\Omega] = -2$, and can be written as:

$$\Omega(\beta, z_i, V, E_i) = V\beta^{-1-D} f(z_i, \beta E_i). \quad (8)$$

Following more or less the same steps as before one derives:

$$\mathcal{E} - DP = \sum_k [g_k] g_k \frac{\partial P}{\partial g_k}, \quad (9)$$

where again as in the nonrelativistic case, the derivatives are taken w.r.t. constant β^{-1} , V , and μ_i .

6 Examples

6.1 No anomalies, no dimensionful parameters

The free gas in any dimension has no dimensionful parameters. Hence by eqn. (7):

$$2\mathcal{E} - DP = 0, \quad (10)$$

as can be verified using $E = \frac{D}{2}NKT$ and $P = \frac{NKT}{V}$.

6.2 No anomalies, dimensionful parameters

For a contact-interaction Bose gas at 0-T (i.e. $\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi - \frac{g}{2} (\psi^\dagger \psi)^2$), in odd dimensions $D = 2n + 1$ (perfectly finite in dimensional regularization, no anomalies), one can make the following 1-loop calculation [10]

$$\Omega = \left(-\frac{1}{2} \frac{\mu^2}{g} - L_D \mu^{\frac{D}{2}+1} \right) V, \quad (11)$$

where Ω is the grand potential, L_D is a pure number that depends on dimension. We will verify eqn. (7) by computing the LHS involving macroscopic thermodynamic parameters by using thermodynamic relations on eqn. (11). Then we will calculate the LHS of eqn. (7) by differentiation w.r.t. microscopic parameters of eqn. (11), and compare the two results.

For the LHS, the following thermodynamic identities will be used, true for any system:

$$\begin{aligned} \Omega &= -PV, \\ \Omega &= E - TS - \mu N \Rightarrow E = \Omega + \mu N \quad (T=0), \\ N &= -\frac{\partial \Omega}{\partial \mu}. \end{aligned} \quad (12)$$

Calculating N for eqn. (11) using eqn. (12):

$$N = \left(\frac{\mu}{g} + L_D \left(\frac{D}{2} + 1 \right) \mu^{\frac{D}{2}} \right) V. \quad (13)$$

Therefore:

$$\begin{aligned}
2E - DPV &= 2(\mu N - PV) - DPV = 2\mu N - (D + 2)PV \\
&= 2\mu \left(\frac{\mu}{g} + L_D \left(\frac{D}{2} + 1 \right) \mu^{\frac{D}{2}} \right) V + (D + 2) \left(-\frac{1}{2} \frac{\mu^2}{g} - L_D \mu^{\frac{D}{2}+1} \right) V \\
&= \left(\left[1 - \frac{D}{2} \right] \frac{\mu^2}{g} \right) V \\
2\mathcal{E} - DP &= \left(\left[1 - \frac{D}{2} \right] \frac{\mu^2}{g} \right).
\end{aligned} \tag{14}$$

Now make the same calculation but using the microscopic scales. Since we restrict ourselves to $D = 2n + 1$, there is no renormalization scale as everything is perfectly finite, a feature peculiar to odd dimensions. However, there is a microscopic length scale associated with the coupling g , where $[g] = D - 2$:

$$\frac{\partial P}{\partial g} [g]g = \frac{-\partial \left(\frac{\Omega}{V} \right)}{\partial g} (D - 2)g = \left(\left[1 - \frac{D}{2} \right] \frac{\mu^2}{g} \right). \tag{15}$$

For fermions in 3-dimensions interacting via contact interactions $\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi - 4\pi a \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow$, $[a] = 1$:

$$2\mathcal{E} - 3P = [a]a \frac{\partial P}{\partial a} \tag{16}$$

Now $\beta PV = \ln \int [d\psi d\psi^\dagger] e^{-\int_0^\beta \int_V d\tau d^2x (\mathcal{L}_0 + 4\pi a \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow)}$ so that differentiating the path integral w.r.t. a :

$$[a]a \frac{\partial P}{\partial a} = -4\pi a \langle \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \rangle. \tag{17}$$

Plugging into (16), we get Tan's pressure relation:

$$2\mathcal{E} - 3P = -4\pi a \langle \psi_{\uparrow}^{\dagger} \psi_{\downarrow}^{\dagger} \psi_{\downarrow} \psi_{\uparrow} \rangle = -\frac{C}{4\pi a}, \quad (18)$$

where $C = (4\pi a)^2 \langle \psi_{\uparrow}^{\dagger} \psi_{\downarrow}^{\dagger} \psi_{\downarrow} \psi_{\uparrow} \rangle$ is the Tan contact [3].

6.3 Anomalies, no dimensionful parameters

A Fermi-gas in $D = 2$ has no dimensionful parameters in the Lagrangian, $\mathcal{L} = \psi^{\dagger} \left(i\partial_t + \frac{\nabla^2}{2} \right) \psi - C \psi_{\uparrow}^{\dagger} \psi_{\downarrow}^{\dagger} \psi_{\downarrow} \psi_{\uparrow}$, $[C] = 0$. Nevertheless, the system develops a bound state via dimensional transmutation. Using cutoff regularization, the T-matrix is [11]:

$$\frac{1}{T(E)} = \frac{1}{C} - \frac{1}{4\pi} \ln \left(\frac{E}{\Lambda^2} \right). \quad (19)$$

The bound state is special since $T(E)$ blows up there, so that $\frac{1}{T(E_b)} = 0$. Therefore plugging in $E = E_b$ into eqn. (19) gives:

$$\frac{1}{C} = \frac{1}{4\pi} \ln \left(\frac{E_b}{\Lambda^2} \right). \quad (20)$$

Taking the derivative w.r.t. E_b on both sides of eqn. (20):

$$\begin{aligned} -\frac{\frac{dC}{dE_b}}{C^2} &= \frac{1}{4\pi} \frac{1}{E_b} \\ \frac{dC}{dE_b} &= -\frac{C^2}{4\pi} \frac{1}{E_b}. \end{aligned} \quad (21)$$

Therefore:

$$-2E_b \frac{\partial P}{\partial E_b} = -2E_b \frac{dC}{dE_b} \frac{\partial P}{\partial C} = \frac{C^2}{2\pi} \frac{\partial P}{\partial C}. \quad (22)$$

Now $\beta PV = \ln \int [d\psi d\psi^\dagger] e^{-\int_0^\beta \int_V d\tau d^2x (\mathcal{L}_0 + C\psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow)}$ so that differentiating the path integral w.r.t. C :

$$\frac{\partial P}{\partial C} = -\langle \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \rangle. \quad (23)$$

Plugging this result into eqn. (22) and using $2\mathcal{E} - DP = -2E_b \frac{\partial P}{\partial E_b}$:

$$2\mathcal{E} - 2P = -\frac{C^2}{2\pi} \langle \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \rangle, \quad (24)$$

agreeing with [12]. The coupling is bare, but the RHS is finite, and both sides are RG-invariant.

In our example, for eqn. (7), the microscopic parameter is the bound-state energy. If you have a pressure written in term of bare parameters and cutoff $P = P(C, \Lambda)$ or renormalized with scale μ , $P = P(C_R, \mu)$, then it is not correct to regard Λ or μ as a microscopic parameter with dimensions of momentum (L^{-1}), because $\frac{dP}{d\Lambda} = \frac{dP}{d\mu} = 0$, so that there is in fact no dependence on these parameters. For our particular example, from eqn. (20), it is true that $2E_b \frac{dC}{dE_b} = -\Lambda \frac{dC}{d\Lambda} = -\beta(C)$ where $\beta(C)$ is the beta function of the theory, so that eqn. (22) into our eqn. (7) would give:

$$2\mathcal{E} - DP = -\beta(C) \frac{\partial P}{\partial C}, \quad (25)$$

and comparison with eqn. (24) allows us to read off $\beta(C) = \frac{C^2}{2\pi}$.

7 Connection with Virial Theorem

In previous work [13] we derived the virial theorem via path integrals, and then used the virial theorem to derive eqn. (7). One can also work backwards from eqn. (7) to derive the virial theorem by following the argument backwards. We reproduce the argument here. For a two-body potential $U = \frac{1}{2} \int d^D x d^D y \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) V(\vec{x} - \vec{y}) \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y})$:

$$\begin{aligned}
2E - DPV &= V \sum_k [g_k] g_k \frac{\partial P}{\partial g_k} \\
&= \sum_k [g_k] g_k \frac{1}{\beta} \partial_{g_k} \ln \int [d\psi d\psi^\dagger] e^{-\int_0^\beta \int_V d\tau d^D x (\mathcal{L}_0 + \frac{1}{2} \int d^D \vec{y} \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) V(\vec{x} - \vec{y}) \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y}))} \\
&= \sum_k [g_k] g_k \left(\frac{-1}{2} \right) \left\langle \int_V \int_V d\tau d^D x d^D y \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) \frac{\partial V}{\partial g_k} \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y}) \right\rangle.
\end{aligned} \tag{26}$$

Denoting $r = |\vec{x} - \vec{y}|$, one can show that $-\sum_k [g_k] g_k \frac{\partial V}{\partial g_k} = r \frac{dV}{dr} + 2V$ (see appendix).

Plugging this into eqn. (26) gives:

$$\begin{aligned}
2E - DPV &= \frac{1}{2} \left\langle \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) r \frac{dV}{dr} \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y}) \right\rangle + 2 \langle U \rangle \\
DPV &= 2KE - \\
&\quad \frac{1}{2} \left\langle \int d^D x d^D y \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) [(\vec{x} - \vec{y}) \cdot \nabla_{\vec{x}} V(\vec{x} - \vec{y})] \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y}) \right\rangle,
\end{aligned} \tag{27}$$

which is the virial theorem [14].

8 Conclusion

We have derived an expression for $2\mathcal{E} - DP$ using only dimensional arguments, valid for classical and quantum systems, for use in the grand canonical ensemble. We worked directly within the framework of thermodynamics, not having to improve the stress-energy tensor and invoke hydrodynamics, but instead working directly with thermodynamic variables. In the case of quantum systems, since the microscopic scales appear as coupling constants, or in the case of dimensional transmutation appear via the coupling constants, $\sum_k [g_k] g_k \frac{\partial P}{\partial g_k}$ manifests itself as thermal expectation values of the operators multiplying the coupling constants in the system's Hamiltonian, which is manifest in the path integral formalism. Finally, with the help of the path integral, we've shown how dimensional analysis leads to the virial theorem.

9 Appendix

The potential $V(r)$ has dimensions $[V] = -2$, so can generically be written:

$$V(r) = \frac{f\left(\frac{g_i}{r^{[g_i]}}\right)}{r^2}. \quad (28)$$

f is a dimensionless function whose arguments are the ratios of the couplings g_i of $V(r)$ to their length dimension $[g_i]$ expressed in units of r .

$$\begin{aligned} r \frac{dV}{dr} &= -2V(r) + \frac{1}{r} \frac{df\left(\frac{g_i}{r^{[g_i]}}\right)}{dr} \\ &= -2V(r) - \frac{1}{r^2} \sum_i [g_i] g_i \frac{\partial f\left(\frac{g_i}{r^{[g_i]}}\right)}{\partial g_i} \\ &= -2V(r) - \sum_i [g_i] g_i \frac{\partial V}{\partial g_i}. \end{aligned} \quad (29)$$

References

- [1] L. Brown, Quantum field theory. Cambridge Univ. Pr., 1992.
- [2] F. Mancarella, G. Mussardo, and A. Trombettoni, “Energy-pressure relation for low-dimensional gases,” Nucl.Phys., vol. B887, pp. 216–245, 2014.
- [3] S. Tan, “Generalized virial theorem and pressure relation for a strongly correlated fermi gas,” Annals of Physics, vol. 323, no. 12, pp. 2987 – 2990, 2008.
- [4] S. Tan, “Large momentum part of a strongly correlated fermi gas,” Annals of Physics, vol. 323, no. 12, pp. 2971 – 2986, 2008.

- [5] C. R. Ordóñez, “Path-integral fujikawa’s approach to anomalous virial theorems and equations of state for systems with $so(2,1)$ symmetry,” [arXiv:1503.01384](https://arxiv.org/abs/1503.01384), 2015.
- [6] C. L. Lin and C. R. Ordóñez, “Path-integral derivation of the nonrelativistic scale anomaly,” Phys. Rev. D, vol. 91, p. 085023, Apr 2015.
- [7] S. Coleman and R. Jackiw, “Why dilatation generators do not generate dilatations,” Annals of Physics, vol. 67, no. 2, pp. 552 – 598, 1971.
- [8] P. Jizba, “Hydrostatic pressure of the $o(n)$ φ^4 theory in the large N limit,” Phys. Rev. D, vol. 69, p. 085011, Apr 2004.
- [9] N. P. Landsman and C. G. van Weert, “Real- and imaginary-time field theory at finite temperature and density,” physrep, vol. 145, pp. 141–249, Jan. 1987.
- [10] A. Schakel, Boulevard of Broken Symmetries: Effective Field Theories of Condensed Matter. World Scientific, 2008.
- [11] D. R. Phillips, S. R. Beane, and T. D. Cohen, “Nonperturbative regularization and renormalization: Simple examples from nonrelativistic quantum mechanics,” Annals of Physics, vol. 263, no. 2, pp. 255 – 275, 1998.
- [12] J. Hofmann, “Quantum anomaly, universal relations, and breathing mode of a two-dimensional fermi gas,” Phys. Rev. Lett., vol. 108, p. 185303, May 2012.

- [13] C. L. Lin and C. R. Ordóñez, “Virial Theorem for Non-relativistic Quantum Fields in D Spatial Dimensions,” Adv. High Energy Phys., vol. 2015, p. 796275, 2015.
- [14] T. Toyoda and K. ichi Takiuchi, “Quantum field theoretical reformulation of the virial theorem,” Physica A: Statistical Mechanics and its Applications, vol. 261, no. 34, pp. 471 – 481, 1998.