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Fractional Radon Transform and its Convolution Theorem

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Abstract

Fractional Radon transform which is symbolized with the notation $\mathcal{R}_\alpha$, it is different to the classical Radon transform. The shift property of fractional Radon transform is controlled by the fractional order α. Rotation of the input object at angle ζ will rotate the fractional Radon transform at that angle ζ, thus, the fractional Radon transform is rotation invariant. The fractional Fourier transform, with respect to λ, of the fractional Radon transform of an object is the central slice at angle ζ of the n-dimensional fractional Fourier transform of this object. In this paper we explain the mathematical formation of fractional Radon transform and established a convolution theorem for the fractional Radon transform.

Keywords: Radon transform; Fractional Radon transform; Fourier transform; Fractional Fourier transform.

1. Introduction

The Radon Transform (RT) [1] is a useful mathematical technique in optics and signal-processing systems and is related to the classical Fourier Transform. In 1996, using the Fourier slice theorem Zalevsky and Mendlovic were the first to define the mathematics of two-dimensional fractional Radon transform (FrRT) [2].

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They originated its fundamental properties such as linearity, shift, rotation invariance, and Fourier slice theorem. The fractional Radon transform is a promising mathematical tool which has many applications in the fields of optics as well as in pattern-recognition. The minimization of mean-square error obtained after filtering non-stationary signal is directly linked to the fractional Radon transform [3]. After that, the n-dimensional fractional Radon transform, and its inversion formula was derived [4]. The novel of (FrRT) is based both on fractional Fourier transform and classical Radon transform, which combines both in a reasonable method. The fractional Fourier transform (FrFT) was introduced in mathematics literature [5], as a generalization of the conventional Fourier transform (FT). However, the fractional Fourier transform had brought much attention in scientific community when Mendlovic and Ozaktas physically interpreted it with relation to quadratic graded-index media [6]. The fractional Fourier transform offers new application for quantum optics, signal processing, patter-recognition and swept-frequency filters [7-10]. Recently, image reconstruction in MRI was proposed [11]. The product and convolution theorems for the fractional (FrFT) have been derived through various approaches in the literature [12-13] .

2. Results and Discussions

2.1 Mathematical formalism of fractional Radon transform

The n-dimensional Fourier transform of $f(\eta)$ can be written as

$$\hat{f}(\mathrm{k}) = \mathcal{F}_n f = \int f(\eta) e^{-2i\pi \mathrm{k}\cdot\eta} d\eta, \qquad (1)$$

Its inverse transform is

$$f(\eta) = \mathcal{F}_n^{-1}\hat{f} = \int \hat{f}(\mathrm{k})\, e^{2i\pi \mathrm{k}\cdot\eta} d\mathrm{k}. \qquad (2)$$

To connect the Fourier transform with the Radon transform, equation (1) may be rewritten in the form

$$\hat{f}(\mathrm{k}) = \int_{-\infty}^{\infty} dt \int f(\eta) e^{-2i\pi t} \delta(t - \mathrm{k}\cdot\eta) d\eta, \quad (3)$$

Letting $k = s\zeta$ and $t = s\lambda$, with t real and ζ a unit vector in $\mathbb{R}^n$, then

$$\hat{f}(s\zeta) = |s| \int_{-\infty}^{\infty} d\lambda \int f(\eta)\, e^{-2i\pi s\lambda} \delta(s\lambda - s\zeta\cdot\eta) d\eta,$$

$$= \int_{-\infty}^{\infty} d\lambda e^{-2i\pi s\lambda} \int f(\eta)\, \delta(\tau - \zeta\cdot\eta) d\eta \qquad (4)$$

The integration over the $d\eta$ on the right-hand side is just the classical Radon transform f, and is denoted as

$$\check{f}(\lambda,\zeta) = \int f(\eta)\delta(\tau - \zeta \cdot \eta)d\eta \qquad (5)$$

It's after

$$\hat{f}(s\zeta) = \int_{-\infty}^{\infty} \check{f}(\lambda,\zeta)e^{-2i\pi s\lambda}d\lambda, \qquad (6)$$

Using the definition of the inverse FT (2) with $n = 1$ the inverse of (6) can be written as

$$\check{f}(\lambda,\zeta) = \int_{-\infty}^{\infty} \hat{f}(s\zeta)e^{2i\pi s\lambda}ds. \qquad (7)$$

Combing the result of Equations (1) and (7) explicitly

$$\check{f}(\lambda,\zeta) = \int_{-\infty}^{\infty}\int f(\eta)e^{-2i\pi s\zeta\cdot\eta}e^{-2i\pi s\lambda}d\eta ds. \qquad (8)$$

Its inverse is

$$f(\eta) = \int_{-\infty}^{\infty}\int \check{f}(\lambda,\zeta)e^{2i\pi \mathrm{k}\cdot\eta}e^{-2i\pi s\lambda}d\mathrm{k}d\lambda. \qquad (9)$$

On the other hand, the n-dimensional FrFT of $f(\eta)$ is defined in [5],[10] explicitly,

$$\hat{f}_\alpha(\mathrm{k}) = \mathcal{F}_{\alpha,\mathrm{k}}[f] = (C_\alpha)^n \int f(\eta)\exp\left(\frac{i(\eta^2+\mathrm{k}^2)}{2\tan\alpha} - \frac{i\mathrm{k}\cdot\eta}{\sin\alpha}\right)d\eta, \quad (10)$$

Where α is fractional order of FrFT,

$$C_\alpha = \left[\frac{e^{i\alpha}}{2\pi i\ \sin\alpha}\right]^{1/2} \qquad (11)$$

One can easily see that, if $\alpha = \frac{\pi}{2}$ equation (11) reduces to the classical FT. comparing with equation (7) and (8) we perform a 1-dimensional inverse FrFT for $\mathcal{F}_\alpha(s\zeta)$ in s-space,

$$\check{f}_\alpha(\lambda,\zeta) = [c_\alpha]^{1-n}\Im_{-\alpha,t}[\mathcal{F}_\alpha(s\zeta)] = [c_\alpha]^{1-n}c_{-\alpha}\int_{-\infty}^{\infty}\mathcal{F}_\alpha(s\zeta)\exp\left(-\frac{i(\lambda^2+s^2)}{2\tan\alpha} + \frac{i\lambda s}{\sin\alpha}\right)d\lambda, \qquad (12)$$

Where $\mathcal{F}_\alpha(s\zeta)$ is the fractional Fourier transform of $f(\eta)$ in η-space, Inserting equation (10) into (12) we have

$$\breve{f}_\alpha(\lambda,\zeta) = [c_\alpha]^{1-n} c_{-\alpha}(c_\alpha)^n \int_{-\infty}^{\infty}\int f(\eta)\ \exp\left(-\frac{i(\lambda^2+s^2)}{2\tan\alpha}+\frac{i\tau s}{\sin\alpha}+\frac{i(\eta^2+\lambda^2)}{2\tan\alpha}-\frac{is\zeta\cdot\eta}{\sin\alpha}\right)d\eta d\lambda,$$

$$\breve{f}_\alpha(\lambda,\zeta) = \int f(\eta)\exp\left(\frac{i(\eta^2-\lambda^2)}{2\tan\alpha}\right)\delta(\lambda-\zeta\cdot\eta)d\eta, \qquad (13)$$

Where $\breve{f}_\alpha(\lambda,\zeta)$ is the fractional Radon transform of $f(\eta)$. It is easily find, when $\alpha=\frac{\pi}{2}$, equation (13) reduced to the conventional Radon transform.

3. Fractional Convolution theorem

A new fractional convolution structure of two functions define by (Prasad & Manna and his colleagues 2013) explicitly,

$$f(\eta) = (g^\alpha * h)(\xi) = \int g(\xi)h(\eta-\xi)\exp\left(\frac{-i(\eta^2-\xi^2)}{2\tan\alpha}\right)d\xi \quad (14)$$

Where α denotes the fractional convolution operator.

Where $\breve{g}_\alpha$ and $\breve{h}_\alpha$ denote the FrRT of g, h respectively,

Then,

$\breve{f}_\alpha = \mathrm{A}_\alpha[\breve{g}^\alpha * \breve{h}]$

Proof: from the definition of the FrRT, we have,

$$\breve{f}_\alpha(\lambda,\zeta) = \int f(\eta)\delta(\lambda-\zeta\cdot\eta)\exp\left(\frac{i(\eta^2-\lambda^2)}{2\tan\alpha}\right)d\eta, \qquad (16)$$

Substituting equation (15) in (16)

$$\breve{f}_\alpha(\lambda,\zeta) = \int\int g(\xi)h(\eta-\xi)\exp\left(\frac{-i(\eta^2-\xi^2)}{2\tan\alpha}\right)\exp\left(\frac{i(\eta^2-\lambda^2)}{2\tan\alpha}\right)\delta(\lambda-\zeta\cdot\eta)d\xi d\eta \ (17)$$

Some simplification,

$$\breve{f}_\alpha(\lambda,\zeta) = \int g(\xi)\,d\xi\int h(\eta-\xi)\exp\left(\frac{i(\xi^2-\lambda^2)}{2\tan\alpha}\right)\delta(\lambda-\zeta\cdot\eta)d\eta, \qquad (18)$$

Inserting a new variable $\tau = \eta - \xi \Rightarrow \eta = \xi + \tau \Rightarrow d\eta = d\tau$

$$= \int g(\xi)\, d\xi \int h(\tau) \exp\left(\frac{i\left((\eta-\tau)^2 - \lambda^2\right)}{2\tan\alpha}\right) \delta(\lambda - \zeta(\xi + \tau)) d\tau, \qquad (19)$$

Multiplying and dividing (19) by $A_\alpha = \exp\left(\frac{(\eta^2 - 2\eta\tau)}{2\tan\alpha}\right)$ and rewriting, result in

$$= \exp\left(\frac{(\eta^2 - 2\eta\tau)}{2\tan\alpha}\right) \int g(\xi)\, d\xi \int h(\tau) \exp\left(\frac{i(\tau^2 - \lambda^2)}{2\tan\alpha}\right) \delta(\lambda - \zeta\cdot\xi - \zeta\cdot\tau) d\tau, (20)$$

The $d\tau$ integral is the fractional Radon transform of the function h using the shift property

$$= A_\alpha \int \breve{h}_\alpha(\lambda - \zeta\cdot\xi, \zeta) g(\xi) d\xi, \qquad (21)$$

Let us insert a new integration $d\rho$ with kernel $\exp\left(\frac{i(\xi^2 - \lambda^2)}{2\tan\alpha}\right)\delta(\rho - \zeta\cdot\eta)$

$$= A_\alpha \int \breve{h}_\alpha(\lambda - \zeta\cdot\xi, \zeta) \int \exp\left(\frac{i(\xi^2 - \lambda^2)}{2\tan\alpha}\right) \delta(\rho - \zeta\cdot\eta)\; g(\xi) d\xi, \qquad (22)$$

$$= A_\alpha \int \breve{g}_\alpha(\rho, \zeta) \breve{h}_\alpha(\lambda - \zeta\cdot\xi, \zeta) d\rho \qquad (23)$$

Thus

$\breve{f}_\alpha = A_\alpha\left[\breve{g}^\alpha * \breve{h}\right]$

This is a new approach for fractional convolution theorem under fractional Radon transform. Here one can easily that when $\alpha = \frac{\pi}{2}$, equation (23) reduced to classical convolution theorem.

4. Conclusion

We have derived the mathematical formulations of fractional Radon transform and established the convolution theorem for fractional Radon transform with its proof , which will play a significant role in signal processing and optics.

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