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GENERALIZATION OF THE NUALART–PECCATI CRITERION

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The celebrated Nualart–Peccati criterion [*Ann. Probab.* **33** (2005) 177–193] ensures the convergence in distribution toward a standard Gaussian random variable N of a given sequence $\{X_n\}_{n\ge 1}$ of multiple Wiener–Itô integrals of fixed order, if $\mathbb{E}[X_n^2] \to 1$ and $\mathbb{E}[X_n^4] \to \mathbb{E}[N^4] = 3$. Since its appearance in 2005, the natural question of ascertaining which other moments can replace the fourth moment in the above criterion has remained entirely open. Based on the technique recently introduced in [*J. Funct. Anal.* **266** (2014) 2341–2359], we settle this problem and establish that the convergence of *any even moment*, greater than four, to the corresponding moment of the standard Gaussian distribution, guarantees the central convergence. As a by-product, we provide many new moment inequalities for multiple Wiener–Itô integrals. For instance, if X is a normalized multiple Wiener–Itô integral of order greater than one,

$$\forall k \ge 2, \qquad \mathbb{E}[X^{2k}] > \mathbb{E}[N^{2k}] = (2k-1)!!.$$

1. Introduction and summary of the main results. Let $\{B_t\}_{t\ge 0}$ be a standard Brownian motion and p be an integer greater than 1. For any deterministic and symmetric function $f \in L^2(\mathbb{R}_+^p, \lambda_p)$ (λ_p stands for p-dimensional Lebesgue measure), let $I_p(f)$ be the pth multiple Wiener–Itô integral of f with respect to $\{B_t\}_{t\ge 0}$ (see [24] for a precise definition). The vector space spanned by all the multiple integrals of order p is called the pth Wiener chaos. A fundamental result of stochastic calculus, customarily called the *Wiener–Itô decomposition*, asserts that any square integrable functional of $\{B_t\}_{t\ge 0}$ can be uniquely expanded as an orthogonal sum of multiple Wiener–Itô integrals. As such, the study of the properties of multiple Wiener–Itô integrals becomes a central topic of research in modern stochastic analysis and a great part of the so-called *Malliavin calculus* (see, e.g., [19, 24]) relies on it.

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The following result, nowadays known as the *fourth moment theorem*, yields an effective criterion of central convergence for a given sequence of multiple Wiener–Itô integrals of a fixed order.

THEOREM 1.1 (Nualart–Peccati [26]). *Let $p \geq 2$ and f_n be a sequence of symmetric elements of $L^2(\mathbb{R}_+^p, \lambda_p)$. Assume $X_n = I_p(f_n)$ verifies $\mathbb{E}[X_n^2] \to 1$. Then, as $n \to \infty$,*

$$X_n \overset{law}{\to} N \sim \mathcal{N}(0,1) \quad \textit{if and only if} \quad \mathbb{E}\big[X_n^4\big] \to \mathbb{E}\big[N^4\big] = 3.$$

The main goal of this article is to show that the above theorem is a particular case of a more general phenomenon. This is the content of the next theorem.

THEOREM 1.2. *Under the assumptions of Theorem 1.1, for any integer $k \geq 2$, as $n \to \infty$,*

$$X_n \overset{law}{\to} N \sim \mathcal{N}(0,1) \quad \textit{if and only if} \quad \mathbb{E}\big[X_n^{2k}\big] \to \mathbb{E}\big[N^{2k}\big] = (2k-1)!!,$$

where the double factorial is defined by $(2k-1)!! = \prod_{i=1}^k (2i-1)$.

The discovery of the fourth moment theorem by Nualart and Peccati (see [26]) is arguably a major breakthrough in the field of Gaussian approximation in the Wiener space. It resulted in a drastic simplification of the so-called *method of moments* (consisting in checking the convergence of *all* the moments) which was so far the only alternative to establish a central limit theorem. We refer to Breuer and Major [8], Chambers and Slud [10], Surgailis [31] and [13, 16] for a nonexhaustive exposition of some results provided by the method of moments. The first proof of Theorem 1.1 relies on tools from stochastic analysis (namely the Dambis, Dubins and Schwartz's theorem; see, e.g., [29], Chapter V). Later on, in the seminal paper [25], Nualart and Ortiz-Latorre discovered a fundamental link between the central convergence of a sequence of elements of a fixed Wiener chaos and the convergence to a constant of the norms of their Malliavin derivatives. The role played by the Malliavin derivative in the fourth moment theorem was later on confirmed in the landmark paper [18]. There, Nourdin and Peccati combined Stein's method and Malliavin calculus to provide a new proof of Theorem 1.1 culminating in sharp estimates for various distances. As an illustrative example of such estimates, they could prove the following quantitative version of Theorem 1.1.

THEOREM 1.3 (Nourdin–Peccati [18]). *Let $p \geq 2$. Assume that $X = I_p(f)$ where f is a symmetric element of $L^2(\mathbb{R}_+^p, \lambda_p)$ such that $\mathbb{E}[X^2] = 1$. Then,*

$$d_{\mathrm{TV}}\big(X, \mathcal{N}(0,1)\big) \leq \tfrac{2}{\sqrt{3}}\sqrt{\mathbb{E}\big[X^4\big] - 3}, \tag{1.1}$$

where d_{TV} stands for the total variation distance.

This innovative approach, combining Malliavin calculus and Stein's method, gave a new impetus to the well studied field of normal approximation within Gaussian spaces. Indeed, it resulted in spectacular improvements of many classical results previously obtained by the method of moments. For an exposition of this fertile line of research, one can consult the book [19], the surveys [11, 27] and the following frequently updated webpage which aims at referencing all the articles dealing with the so-called *Malliavin–Stein method*: https://sites.google.com/site/malliavinstein/home. Finally, we mention that Theorem 1.3 has been generalized in various directions such as the optimality of the rate of convergence [20], multivariate settings [22, 28], free probability settings [14] and general homogeneous sums [21].

Unfortunately, if the quantitative aspects of the latter approach are now quite well understood, the heavily combinatorial nature of the proofs remains a major stumbling block in tackling the following central questions:

(A) What are the target distributions for which a moment criterion similar to (1.1) is available?

(B) What are the moment conditions ensuring the central convergence?

(C) What are the special properties of Wiener chaos playing a role in the fourth moment phenomenon?

Indeed, most of the aforementioned proofs of the fourth moment theorem make crucial use of the product formula for multiple Wiener–Itô integrals together with some properties of the underlying Fock space. Such an approach becomes already inextricable when one tries to make explicit the Wiener–Itô decomposition of the 6th power of a multiple integral. As such, writing explicitly the Wiener–Itô decompositions of the successive powers of a given multiple Wiener–Itô integral, which is the core of the previous strategies, seems totally hopeless for our purpose. Inspired by the remarkable intuition that the fourth moment phenomenon could also be explained by the spectral properties of Markov operators, Ledoux produced a new proof of Theorem 1.1 (see [15]). His approach, exclusively based on the study of some differential operators such as the Ornstein–Uhlenbeck generator and the iterated gradients, and avoids completely the product formula for multiple integrals. Unfortunately, due to an inappropriate definition of the chaos of a Markov operator, this attempt became rather involved and could not produce any of the expected generalizations of the fourth moment criterion. Later on, in the same spirit as [15] (i.e., exploiting spectral theory of Markov operators and Gamma-calculus), the authors of [3] could produce a very simple and fully transparent proof of the fourth moment theorem, henceforth bringing both, a complete answer to question (C), and some generalizations of the criterion for other Markov operators than Ornstein–Uhlenbeck. Roughly speaking, the technique used in [3] consists in exploiting the stability of chaoses under the product operation to provide some suitable spectral inequalities, which, after elementary computations, become moment inequalities.

In particular, the latter approach does not need any of the combinatorics computations required by the product formula. The present article fully generalizes this idea and builds a complete methodology enabling to provide a wide range of inequalities for polynomial functionals of a Gaussian field, hence for Wiener chaoses as well. In particular, it leads to a partial answer to question (B). Combining the formalism and the ideas of [3] together with some fine deterministic properties of Hermite polynomials, we could prove Theorem 1.2, which is our main achievement. Interestingly, we could also prove the following quantitative version which extends the celebrated estimate (1.1) to all even moments. Indeed, taking $k = 2$ in the theorem below gives back the bound in (1.1).

THEOREM 1.4. *Under the assumptions of Theorem 1.3, for all $k \geq 2$, we have the following general quantitative bound*:

$$d_{\mathrm{TV}}\big(X, \mathcal{N}(0,1)\big) \leq C_k \sqrt{\frac{\mathbb{E}[X^{2k}]}{(2k-1)!!} - 1}, \tag{1.2}$$

where the constant $C_k = \frac{4}{\sqrt{2k(k-1)\int_0^1 ((1+t^2)/2)^{k-2}\,dt}}$.

For the sake of clarity, we stated so far our main results in the more familiar context of the Wiener space. Nevertheless, throughout the whole article, instead of the Wiener–Itô multiple integrals, we shall consider a more general concept of eigenfunctions of a diffusive Markov operator. We refer the reader to Section 2.1 for a precise exposition of our assumptions. We also stress that this gain of generality enables us to give central limit criteria in situations far beyond the scope of the usual criteria holding in the Wiener chaoses.

2. The setup.

2.1. *The general setup and assumptions* (a)–(b)–(c). One possible way to study the properties of the elements of the Wiener space can be through the exploration of the spectral properties of the so-called *Ornstein–Uhlenbeck operator*. In order to situate more precisely our purpose, we will restate below its main properties.

The Wiener space is the space $L^2(E, \mu)$ where $E = \mathbb{R}^{\mathbb{N}}$ and μ is the standard Gaussian measure on $\mathbb{R}^{\mathbb{N}}$. The Ornstein–Uhlenbeck operator is the unbounded, symmetric, negative operator $\mathbf{L}$ acting on some dense domain of $L^2(E, \mu)$ and defined (on the set of smooth enough cylindric functionals Φ) by

$$\mathbf{L}[\Phi] = \Delta\Phi - \vec{x} \cdot \vec{\nabla}\Phi$$

(where Δ is the usual Laplacian and $\vec{\nabla}$ is the gradient vector). We shall denote its domain by $\mathcal{D}(\mathbf{L})$ and for a general $X \in \mathcal{D}(\mathbf{L})$, $\mathbf{L}[X]$ is defined via standard closure operations. The associated *carré-du-champ* operator is the symmetric, positive, bilinear form defined by

$$\Gamma[\Phi, \Psi] = \vec{\nabla}\Phi \cdot \vec{\nabla}\Psi.$$

Below, we summarize the fundamental properties of the Ornstein–Uhlenbeck operator $\mathbf{L}$.

(a) *Diffusion*: For any $\mathcal{C}_b^2$ function $\phi : \mathbb{R} \to \mathbb{R}$, any $X \in \mathcal{D}(\mathbf{L})$, it holds that $\phi(X) \in \mathcal{D}(\mathbf{L})$ and

$$\mathbf{L}\big[\phi(X)\big] = \phi'(X)\mathbf{L}[X] + \phi''(X)\Gamma[X, X]. \tag{2.1}$$

Note that, by taking $\phi = 1 \in \mathcal{C}_b^2$, we get $\mathbf{L}[1] = 0$ which is the Markov property. Equivalently, Γ is a derivation in the sense that

$$\Gamma\big[\phi(X), X\big] = \phi'(X)\Gamma[X, X].$$

(b) *Spectral decomposition*: The operator $-\mathbf{L}$ is diagonalizable on $L^2(E, \mu)$ with $\mathbf{sp}(-\mathbf{L}) = \mathbb{N}$, that is to say

$$L^2(E, \mu) = \bigoplus_{i=0}^{\infty} \mathbf{Ker}(\mathbf{L} + i\,\mathbf{Id}).$$

(c) *Spectral stability*: For any pair of eigenfunctions (X, Y) of the operator $-\mathbf{L}$ associated with eigenvalues (p_1, p_2),

$$XY \in \bigoplus_{i \le p_1 + p_2} \mathbf{Ker}(\mathbf{L} + i\,\mathbf{Id}). \tag{2.2}$$

We refer to [7] for a precise exposition as well as all the domain and integrability assumptions. Actually, these three properties are the only one we will use. Thus, we naturally define the following class of structures for which our results will hold.

DEFINITION 2.1. A (a)–(b)–(c) structure is a triplet $(E, \mu, \mathbf{L})$, with an associated "carré-du-champ" operator Γ, where:

- (E, μ) is a probability space,
- $\mathbf{L}$ is a symmetric unbounded operator defined on some dense domain of $L^2(E, \mu)$,
- Γ is defined by

$$2\Gamma[X, Y] = \mathbf{L}[XY] - X\mathbf{L}[Y] - Y\mathbf{L}[X], \tag{2.3}$$

such that the aforementioned properties (a), (b) and (c) hold. In this context, we will sometimes write $\Gamma[X]$ to denote $\Gamma[X, X]$ and $\mathbb{E}$ for the integration against μ.

Property (a) is important regarding functional calculus. For instance, we will use several times the following *integration by parts formula*: for any X, Y in $\mathcal{D}(\mathbf{L})$ and $\phi \in \mathcal{C}_b^2$:

$$\mathbb{E}[\phi'(X)\Gamma[X, Y]] = -\mathbb{E}[\phi(X)\mathbf{L}[Y]] = -\mathbb{E}[Y\mathbf{L}[\phi(X)]]. \tag{2.4}$$

Property (b) allows to use spectral theory. Actually, we stress that our results extend under the weaker assumption that $\mathbf{sp}(-\mathbf{L}) \subset \mathbb{R}_+$ is simply discrete. However, we stick to the assumption $\mathbf{sp}(-\mathbf{L}) = \mathbb{N}$ since it encompasses the most common cases (Wiener space and Laguerre space). The reader interested in relaxing this spectral assumption can consult [3] where the spectrum is only assumed to be discrete.

Property (c) is our main assumption, which will allow us to obtain fundamental spectral inequalities. A simple induction on (2.2) shows that, for any $X \in \mathbf{Ker}(\mathbf{L} + p\mathbf{Id})$ and any polynomial P of degree m, we have

$$P(X) \in \bigoplus_{i \le mp} \mathbf{Ker}(\mathbf{L} + i\mathbf{Id}). \tag{2.5}$$

For further details on our setup, we refer to [5, 6]. We also refer to Section 2.2 for many other examples.

REMARK 2.1. We remark that under the assumptions (a)–(b)–(c), the eigenspaces are hypercontractive (see [5] for sufficient conditions), that is, for any integer M, we have that

$$\bigoplus_{i \le M} \mathbf{Ker}(\mathbf{L} + i\mathbf{Id}) \subseteq \bigcap_{p \ge 1} L^p(E, \mu). \tag{2.6}$$

Next, by using the open mapping theorem, we see that the embedding (2.6) is continuous, that is, there exists a constant $C(M, k)$ such that for any $X \in \bigoplus_{i \le M} \mathbf{Ker}(\mathbf{L} + i\mathbf{Id})$:

$$\mathbb{E}(X^{2k}) \le C(M, k)\mathbb{E}(X^2)^k. \tag{2.7}$$

We close this subsection with an useful lemma which will be used several times in the sequel. This lemma is proved in [23], Lemma 2.4, in the Wiener structure but can be easily adapted to our framework by taking into account the Remark 2.1.

LEMMA 2.1. *Let $\{X_n\}_{n \ge 1}$ be a sequence of random variables living in a finite sum of eigenspaces* ($X_n \in \bigoplus_{i \le M} \mathbf{Ker}(\mathbf{L} + i\mathbf{Id}), \forall n \ge 1$) *of a Markov generator* $\mathbf{L}$ *such that our assumptions* (a), (b) *and* (c) *hold. Assume that the sequence* $\{X_n\}_{n \ge 1}$ *converges in distribution as n tends to infinity. Then*

$$\sup_{n \ge 1} \mathbb{E}(|X_n|^r) < \infty \qquad \forall r \ge 1.$$

2.2. *Examples of structures fulfilling assumptions* (a)–(b)–(c). We refer to the article [3] for a proof of the validity of the assumptions (a)–(b)–(c) in the cases of the Wiener and Laguerre structures. We now show how the validity of the assumptions (a)–(b)–(c) is preserved by the elementary operations of tensorization and superposition of structures. This simple fact will allow us to produce many structures in which our results hold.

Tensorization. Let $(E_1, \mu_1, \mathbf{L}_1)$ and $(E_2, \mu_2, \mathbf{L}_2)$ be two Markov triplets fulfilling assumptions (a)–(b)–(c). On the product space $E_1 \times E_2$ with measure $\mu_1 \otimes \mu_2$, we define the following operator $\mathbf{L}_3$. For $\Psi : E_1 \times E_2 \to \mathbb{R}$, we set $\Psi_x(y) = \Psi_y(x) = \Psi(x, y)$, and we define

$$\mathbf{L}_3[\Psi](x, y) = \mathbf{L}_1[\Psi_y](x) + \mathbf{L}_2[\Psi_x](y). \tag{2.8}$$

In (2.8), $\mathbf{L}_3$ is defined on the set of maps Ψ such that:

(1) μ_2-a.s., $\Psi_y \in \mathrm{Dom}(\mathbf{L}_1)$ and μ_1-a.s., $\Psi_x \in \mathrm{Dom}(\mathbf{L}_2)$,
(2)
$$\int_{E_1 \times E_2} \big(\mathbf{L}_3[\Psi](x, y)\big)^2 \, d\mu_1 \, d\mu_2 < \infty.$$

We claim that the triplet $(E_1 \times E_2, \mu_1 \otimes \mu_2, \mathbf{L}_3)$ verifies assumptions (a)–(b)–(c). First, it is well known that this procedure preserves assumption (a); see, for instance, [7]. Assumption (b) is also preserved by tensorization taking into account that

$$\begin{aligned}\mathbf{Ker}(\mathbf{L}_3 + k\mathbf{Id}) = \mathrm{Vect}\big\{\phi(x)\psi(y) | \phi \in \mathbf{Ker}(\mathbf{L}_1 + k_1\mathbf{Id}),\\ \psi \in \mathbf{Ker}(\mathbf{L}_2 + k_2\mathbf{Id}), k_1 + k_2 = k\big\}.\end{aligned} \tag{2.9}$$

Finally, we check assumption (c) for $\mathbf{L}_3$. Let $\Psi_1 = \phi_1(x)\psi_1(y)$ with $\phi_1 \in \mathbf{Ker}(\mathbf{L} + k_1\mathbf{Id})$ and $\psi_1 \in \mathbf{Ker}(\mathbf{L} + k_2\mathbf{Id})$, and let $\Psi_2 = \phi_2(x)\psi_2(y)$ where $\phi_2 \in \mathbf{Ker}(\mathbf{L} + k_3\mathbf{Id})$ and $\psi_2 \in \mathbf{Ker}(\mathbf{L} + k_4\mathbf{Id})$. By applying (c) to $\phi_1\phi_2$ and $\psi_1\psi_2$ together with equation (2.9), we infer that

$$\phi_1(x)\phi_2(x)\psi_1(y)\psi_2(y) \in \bigoplus_{i \le k_1+k_2+k_3+k_4} \mathbf{Ker}(\mathbf{L}_3 + i\mathbf{Id}).$$

Hence, using bilinearity, we see that assumption (c) also holds for operator $\mathbf{L}_3$.

Superposition. As before, we are given a Markov triplet $(E, \mathbf{L}, \mu)$ satisfying assumptions (a)–(b)–(c). The superposition procedure consists in adding an independent noise to $(E, \mathbf{L}, \mu)$. To do so, we consider a generic probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which will induce the noise on $(E, \mathbf{L}, \mu)$. We define on the set $E \times \Omega$ equipped with the product probability measure $\mu \otimes \mathbb{P}$:

$$\mathrm{Dom}(\mathbf{L}_\Omega) = \left\{\Psi(x, \omega) \middle| \Psi_\omega \in \mathrm{Dom}(\mathbf{L}), \int_\Omega \mathbb{E}_\mu\big[(\mathbf{L}\Psi_\omega)^2\big] \, d\mathbb{P} < \infty\right\}, \tag{2.10}$$

$$\mathbf{L}_\Omega[\Psi](x, \omega) := \mathbf{L}[\Psi_\omega](x) \qquad \forall \Psi \in \mathrm{Dom}(\mathbf{L}_\Omega). \tag{2.11}$$

Preservation of assumption (a) is a well-known consequence of the superposition procedure. We refer to [7] where superposition/product/semidirect product of Markov triplets (i.e., Dirichlet forms) are studied to provide ways of constructing Dirichlet forms. To check assumption (b), we are given $\Psi(x,\omega) \in L^2(\mu \otimes \mathbb{P})$. By assumption (b) on the space $L^2(E,\mu)$, we get

$$\Psi_\omega(x) = \sum_{k=1}^{\infty} f_{k,\omega}(x), \qquad f_{k,\omega} \in \mathbf{Ker}(\mathbf{L} + k\mathbf{Id}). \tag{2.12}$$

Besides,

$$\int_\Omega \mathbb{E}_\mu\big[\Psi(x,\omega)^2\big]\,d\mathbb{P} = \sum_{k=1}^{\infty} \int_\Omega \mathbb{E}_\mu\big[f_k(x,\omega)^2\big]\,d\mathbb{P} < \infty.$$

This ensures that $\mathbb{P}$-a.s., $f_{k,\omega} \in \mathbf{Ker}(\mathbf{L} + k\mathbf{Id})$ and that $f_k \in \mathrm{Dom}(\mathbf{L}_\Omega)$. Finally, one can see that

$$\mathbf{Ker}(\mathbf{L}_\Omega + k\mathbf{Id}) = \big\{\Psi(x,\omega) \in \mathrm{Dom}(\mathbf{L}_\Omega) | \mathbb{P}\text{-a.s. } \Psi_\omega \in \mathbf{Ker}(\mathbf{L} + k\mathbf{Id})\big\}. \tag{2.13}$$

We infer that $f_k \in \mathbf{Ker}(\mathbf{L}_\Omega + k\mathbf{Id})$ which achieves the proof of (b). Strictly speaking, assumption (c) is not necessarily preserved because we need integrability on the product of two eigenfunctions of $\mathbf{L}_\Omega$. This integrability, unlike in the tensorization procedure is not automatically fulfilled in the superposition procedure. Fortunately, under some slight additional assumption, (c) holds for $\mathbf{L}_\Omega$. More precisely, we have for all $X(x,\omega) \in \mathbf{Ker}(\mathbf{L}_\Omega + k_1\mathbf{Id})$ and $Y(x,\omega) \in \mathbf{Ker}(\mathbf{L}_\Omega + k_2\mathbf{Id})$ such that $XY \in L^2(\mu \otimes \mathbb{P})$:

$$XY \in \bigoplus_{i \le k_1 + k_2} \mathbf{Ker}(\mathbf{L}_\Omega + i\mathbf{Id}). \tag{2.14}$$

REMARK 2.2. One can consult the reference [6], page 515, to see that the two aforementioned operations are a particular case of the so-called wrapped product of symmetric diffusive operators.

2.3. *Some auxiliary results*. To be self-contained, we restate here two well-known facts about Stein's method applied to eigenfunctions of a diffusive Markov operator. For more details, the reader can consult, for instance, [15] or the survey [12].

THEOREM 2.1 ([15]). *Let* $\mathbf{L}$ *be a Markov diffusive operator satisfying the assumptions* (a)–(b) *of Section* 2.1, *and* X *be in* $\mathbf{Ker}(\mathbf{L} + p\mathbf{Id})$ *such that* $\mathbb{E}[X^2] = 1$. *Then*

$$d_{\mathrm{TV}}\big(X, \mathcal{N}(0,1)\big) \le \frac{2}{p}\sqrt{\mathrm{Var}\big(\Gamma[X]\big)}.$$

As a matter of fact, for a given sequence $\{X_n\}_{n\ge1}$ *in* $\mathbf{Ker}(\mathbf{L}+p\mathbf{Id})$ *such that* $\mathbb{E}[X_n^2]\to 1$:

$$\Gamma[X_n]\overset{L^2}{\to} p \quad\Rightarrow\quad X_n\overset{law}{\to}\mathcal{N}(0,1).$$

REMARK 2.3. In [15], Proposition 2, given a sequence $\{X_n\}_{n\ge1}$ in $\mathbf{Ker}(\mathbf{L}+p\mathbf{Id})$ with $\mathbb{E}[X_n^2]\to\theta$, it is shown that

$$\operatorname{Var}\bigl(\Gamma[X_n]-pX_n\bigr)\to 0 \quad\Rightarrow\quad X_n+\theta\overset{\text{law}}{\to}\gamma(\theta),$$

where $\gamma(\theta)$ stands for the gamma distribution of parameter θ. This fact will be used only in the proof of Theorem 3.2.

Furthermore, we restate below the fourth moment theorem under the assumptions (a)–(b)–(c). Actually, it can be proved under the weaker assumption that, for any eigenfunction $X\in\mathbf{Ker}(\mathbf{L}+p\mathbf{Id})$, we have

$$X^2\in\bigoplus_{k\le 2p}\mathbf{Ker}(\mathbf{L}+k\mathbf{Id}),$$

which in fact is a very particular case of the assumption (c). The stronger assumption (c) will allow us to establish analogous statements for higher moments.

THEOREM 2.2 ([12, 15]). *Let* $\mathbf{L}$ *be a Markov diffusive operator satisfying the assumptions* (a)–(b)–(c) *and* $X\in\mathbf{Ker}(\mathbf{L}+p\mathbf{Id})$ *with* $\mathbb{E}[X^2]=1$. *Then*

$$d_{\mathrm{TV}}\bigl(X,\mathcal{N}(0,1)\bigr)\le\frac{2}{p}\sqrt{\operatorname{Var}\bigl(\Gamma[X]\bigr)}\le\frac{2}{\sqrt{3}}\sqrt{\mathbb{E}[X^4]-3}. \tag{2.15}$$

Thus, for a given sequence $\{X_n\}_{n\ge1}$ *in* $\mathbf{Ker}(\mathbf{L}+p\mathbf{Id})$ *such that* $\mathbb{E}[X_n^2]\to 1$ *and* $\mathbb{E}[X_n^4]\to 3$, *we have*

$$X_n\overset{law}{\to}\mathcal{N}(0,1).$$

3. Algebraic framework. The aforementioned assumptions (a)–(b)–(c) on the Markov generator $\mathbf{L}$ can be suitably used to build an algebraic framework in order to study properties of eigenfunctions of the generator $\mathbf{L}$. Throughout this section, we shall use these assumptions in a natural way in order to introduce a family of bilinear, symmetric and positive forms $\mathcal{M}_k$. The fundamental assumption (2.5) is the crucial element yielding the positivity of the bilinear forms $\mathcal{M}_k$.

Let $\mathbb{R}_k[T]$ stand for the ring of all polynomials of T of degree at most k over $\mathbb{R}$. Let X be an eigenfunction of the generator $\mathbf{L}$ with eigenvalue $-p$, that is, $-\mathbf{L}X=pX$. We consider the following map:

$$\mathcal{M}_k:\begin{cases}\mathbb{R}_k[T]\times\mathbb{R}_k[T]\longrightarrow\mathbb{R},\\ (P,Q)\longmapsto\mathbb{E}\bigl[Q(X)(\mathbf{L}+kp\mathbf{Id})P(X)\bigr].\end{cases}$$

REMARK 3.1. Notice that the mapping $\mathcal{M}_k$ strongly depends on the eigenfunction X. We also remark that thanks to Remark 2.1, $\mathcal{M}_k$ is well defined.

The following theorem is the cornerstone of our approach.

THEOREM 3.1. *The mapping $\mathcal{M}_k$ is bilinear, symmetric and nonnegative. Moreover, its matrix representation over the canonical basis $\{1, T, T^2, \dots, T^k\}$ is given by $p\mathbf{M}_k$ where*

$$\mathbf{M}_k = \left(\left(k - \frac{ij}{i+j-1}\right)\mathbb{E}[X^{i+j}]\right)_{0\le i,j\le k} \tag{3.1}$$

with the convention that $\frac{ij}{i+j-1} = 0$ for $(i,j) = (0,1)$ or $(1,0)$.

PROOF. Expectation is a linear operator, so the bilinearity property follows. Symmetry proceeds from the symmetry of the diffusive generator $\mathbf{L}$. To prove positivity of the matrix $\mathbf{M}_k$, using the fundamental assumption (2.5) we obtain that for any polynomial P of degree $\le k$,

$$P(X) \in \bigoplus_{i\le kp} \mathbf{Ker}(\mathbf{L} + i\mathbf{Id}).$$

Therefore, denoting by $J_i : L^2(E,\mu) \to \mathbf{Ker}(\mathbf{L} + i\mathbf{Id})$ the orthogonal projections,

$$\begin{aligned}
\mathbb{E}\big[\big((\mathbf{L} + kp\mathbf{Id})P(X)\big)^2\big] &= \mathbb{E}\big[\mathbf{L}P(X)(\mathbf{L} + kp\mathbf{Id})P(X)\big] \\
&\quad + kp\mathbb{E}\big[P(X)(\mathbf{L} + kp\mathbf{Id})P(X)\big] \\
&= \sum_{i=0}^{kp}(-i)(kp-i)\mathbb{E}\big[J_i^2\big(P(X)\big)\big] \\
&\quad + kp\mathbb{E}\big[P(X)(\mathbf{L} + kp\mathbf{Id})P(X)\big] \\
&\le kp\mathcal{M}_k(P,P).
\end{aligned} \tag{3.2}$$

Hence, $\mathcal{M}_k$ is a positive form. To complete the proof, notice that the (i,j)-component of the matrix $\mathbf{M}_k$ is given by $\mathbb{E}[X^j(\mathbf{L} + kp\mathbf{Id})X^i]$. So, using the diffusive property of the generator $\mathbf{L}$, we obtain

$$\begin{aligned}
X^j(\mathbf{L} + kp\mathbf{Id})X^i &= i(i-1)X^{i+j-2}\Gamma(X) + p(k-i)X^{i+j} \\
&= \frac{i(i-1)}{i+j-1}\Gamma\big(X^{i+j-1}, X\big) + p(k-i)X^{i+j}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{M}_k(X^i, X^j) &= \frac{i(i-1)}{i+j-1}\mathbb{E}\big[\Gamma\big(X^{i+j-1}, X\big)\big] + p(k-i)\mathbb{E}\big[X^{i+j}\big] \\
&= p\frac{i(i-1)}{i+j-1}\mathbb{E}\big[X^{i+j}\big] + p(k-i)\mathbb{E}\big[X^{i+j}\big]
\end{aligned}$$

$$= p\left(\frac{i(i-1)+(k-i)(i+j-1)}{i+j-1}\right)\mathbb{E}[X^{i+j}]$$
$$= p\left(k - \frac{ij}{i+j-1}\right)\mathbb{E}[X^{i+j}]. \qquad \square$$

REMARK 3.2. In Theorem 3.1, we only stated the positivity of the family of quadratic forms $\mathcal{M}_k$. However, it is worth mentioning that, thanks to the inequality (3.2), each quadratic form $\mathcal{M}_k$ dominates the nonnegative quadratic form

$$P \longmapsto \mathbb{E}\big[\big\{(\mathbf{L} + kp\mathbf{Id})P(X)\big\}^2\big].$$

COROLLARY 3.1. *For any eigenfunction X of the generator $\mathbf{L}$ with eigenvalue $-p$, that is, $-\mathbf{L}(X) = pX$:*

(i) *All the eigenvalues of matrix $\mathbf{M}_k$ are nonnegative.*
(ii) *All the lth leading principal minor of the matrix $\mathbf{M}_k$ are nonnegative for $l \le k$.*

PROOF. The proof follows directly from standard linear algebra (see, e.g., [30]). $\square$

The moments matrix $\mathbf{M}_k$ can help one to give nontrivial moment inequalities, sometimes sharper than the existing estimates so far, involving the moments of the eigenfunctions of a generator $\mathbf{L}$. Here is an application where we sharpen the standard fourth moment inequality $\mathbb{E}[X^4] \ge 3\mathbb{E}[X^2]^2$. We mention that the next theorem unifies the two well-known criteria of convergence in law (i.e., Gaussian and Gamma approximation) for a sequence of random variables inside a fixed Wiener chaos; see [17, 26].

THEOREM 3.2. *If X is a nonzero eigenfunction of generator $\mathbf{L}$, then*

$$\frac{\mathbb{E}[X^4]}{3} - \mathbb{E}[X^2]^2 \ge \frac{\mathbb{E}[X^3]^2}{2\mathbb{E}[X^2]}. \tag{3.3}$$

Moreover, assume that $X_n \in \mathbf{Ker}(\mathbf{L} + p\mathbf{Id})$ for each $n \ge 1$ and

$$\frac{\mathbb{E}[X_n^4]}{3} - \mathbb{E}[X_n^2]^2 - \frac{\mathbb{E}[X^3]^2}{2\mathbb{E}[X^2]} \to 0. \tag{3.4}$$

Then all the adherence values in distribution of the sequence $\{X_n\}_{n\ge1}$ is either a Gaussian or a scaling of a centered Gamma random variable.

PROOF. The moments matrix $\mathbf{M}_2$ associated to X is given by

$$\mathbf{M}_2(X) = \begin{pmatrix} 2 & 0 & 2\mathbb{E}[X^2] \\ 0 & \mathbb{E}[X^2] & \mathbb{E}[X^3] \\ 2\mathbb{E}[X^2] & \mathbb{E}[X^3] & \frac{2}{3}\mathbb{E}[X^4] \end{pmatrix}. \tag{3.5}$$

Using Corollary 3.1, we infer that

$$\det(\mathbf{M}_2) = 4\mathbb{E}[X^2]\left\{\frac{\mathbb{E}[X^4]}{3} - \mathbb{E}[X^2]^2\right\} - 2\mathbb{E}[X^3]^2 \geq 0,$$

which immediately implies (3.3). Up to extracting a subsequence, we may assume that $X_n \to X_\infty$ in distribution. We further assume that $X_\infty \neq 0$. Assumption (3.4) entails that

$$\det \mathbf{M}_2(X_n) \to 0.$$

Let $V_n = (\frac{2}{3}\mathbb{E}[X_n^4]\mathbb{E}[X_n^2] - \mathbb{E}[X_n^3]^2, 2\mathbb{E}[X_n^2]\mathbb{E}[X_n^3], -2\mathbb{E}[X_n^2]^2)$ be the first line of the adjugate matrix of $\mathbf{M}_2(X_n)$. Since X_n converges in distribution, we have $V_n \to V_\infty = (a, b, c)$. We set $P(X) = cX^2 + bX + a$. As a result, we have:

$$\mathcal{M}_2(X_n)(P, P) \to 0. \tag{3.6}$$

Using Remark 3.2, we see that

$$\mathbb{E}\big[\{(\mathbf{L} + 2p\mathbf{Id})P(X_n)\}^2\big] \to 0.$$

Next,

$$\begin{aligned}(\mathbf{L} + 2p\mathbf{Id})P(X_n) &= c(\mathbf{L} + 2p\mathbf{Id})X_n^2 + bpX_n + 2ap \\ &= 2c\Gamma(X_n) + bpX_n + 2ap.\end{aligned}$$

We notice that $c \neq 0$ since $X_\infty \neq 0$. Now two possible cases can happen.

Case (1): If $\mathbb{E}[X_n^3] \to 0$, then $b = 0$. Hence, we have $\mathbb{E}[\Gamma(X_n) + \frac{ap}{c}]^2 \to 0$ and therefore the sequence $\{X_n\}_{n\geq 1}$ converges toward a Gaussian random variable. See Theorem 2.1.

Case (2): If $\mathbb{E}[X_n^3] \not\to 0$, then $b \neq 0$. Hence, we have $\mathbb{E}[\Gamma(X_n) + \frac{bp}{2c}X_n + \frac{ap}{c}]^2 \to 0$. We set $X_n = \lambda Y_n$ and we may choose λ in such way that

$$\mathrm{Var}\big(\Gamma(Y_n) - pY_n\big) \to 0.$$

This enables us to use the content of the Remark 2.3 and assert that $Y_n + \mathbb{E}[Y_n^2]$ converges in distribution toward a gamma random variable. Hence, X_n converges in distribution toward a scaling of a centered gamma law. $\square$

It is clear that Theorem 3.2 also gives back the fourth moment theorem in (a)–(b)–(c) structures from the fact that a random variable G satisfying $\mathbb{E}[G^2] = 1$, $\mathbb{E}[G^3] = 0$ and $\mathbb{E}[G^4] = 3$ cannot have a gamma distribution.

The following proposition states a nontrivial inequality between the second, fourth and sixth moments of eigenfunctions of $\mathbf{L}$.

PROPOSITION 3.1. *If X is an eigenfunction of $\mathbf{L}$, then*

$$\mathbb{E}[X^4]^2 \leq \tfrac{3}{5}\mathbb{E}[X^6]\mathbb{E}[X^2]. \tag{3.7}$$

REMARK 3.3. Notice that this inequality is an equality when the distribution of X is Gaussian.

PROOF OF PROPOSITION 3.1. The moments matrix $\mathbf{M}_3$ associated to X has the form

$$\mathbf{M}_3 = \begin{pmatrix} 3 & \star & 3\mathbb{E}[X^2] & \star \\ \star & 2\mathbb{E}[X^2] & \star & 2\mathbb{E}[X^4] \\ 3\mathbb{E}[X^2] & \star & \frac{5}{3}\mathbb{E}[X^4] & \star \\ \star & 2\mathbb{E}[X^4] & \star & \frac{6}{5}\mathbb{E}[X^6] \end{pmatrix}. \tag{3.8}$$

Since this matrix is positive, we have in particular

$$\begin{vmatrix} 2\mathbb{E}[X^2] & 2\mathbb{E}[X^4] \\ 2\mathbb{E}[X^4] & \frac{6}{5}\mathbb{E}[X^6] \end{vmatrix} \geq 0,$$

which gives the claimed inequality. □

Using Proposition 3.1, we can already prove the following sixth moment theorem, that is, Theorem 1.2 in the case $k = 3$. Note that we will get back this result when we will prove our main result (Section 4.1).

COROLLARY 3.2. *A sequence $\{X_n\}_{n\geq 1}$ such that $X_n \in \mathbf{Ker}(\mathbf{L} + p\mathbf{Id})$ for each $n \geq 1$, converges in distribution toward the standard Gaussian law if and only if $\mathbb{E}[X_n^2] \to 1$ and $\mathbb{E}[X_n^6] \to 15$.*

PROOF. By Proposition (3.1), for $X \in \mathbf{Ker}(\mathbf{L} + p\mathbf{Id})$, we have

$$\mathbb{E}[X^6] \geq \frac{5}{3}\frac{\mathbb{E}[X^4]^2}{\mathbb{E}[X^2]} \geq \frac{5}{3}\frac{(3\mathbb{E}[X^2]^2)^2}{\mathbb{E}[X^2]} = 15\mathbb{E}[X^2]^3.$$

Therefore, for the sequence $\{X_n\}_{n\geq 1}$ in $\mathbf{Ker}(\mathbf{L} + p\mathbf{Id})$, if $\mathbb{E}[X_n^2] \to 1$ and $\mathbb{E}[X_n^6] \to 15$, then from the previous chain of inequalities, we deduce that $\mathbb{E}[X_n^4] \to 3$. Hence, the sequence $\{X_n\}_{n\geq 1}$ converges in distribution toward $\mathcal{N}(0, 1)$ according to Theorem 2.2. □

4. New central limit theorems. In this section, we will establish our main criteria for central convergence. In a first subsection, we will first focus on the main theorem of the paper, the so-called *even moment criterion*. In a second subsection, we will give additional criteria of central convergence. As before, we work under assumptions (a)–(b)–(c) stated in Section 2.

4.1. *The even moment criterion.* We state below our main result. Note that Theorem 1.2 is a particular case of Theorem 4.1, by simply choosing $\mathbf{L}$ to be the Ornstein–Uhlenbeck generator.

THEOREM 4.1. *Let* $\mathbf{L}$ *be a Markov operator satisfying* (a)–(b)–(c), $p \geq 1$ *be an eigenvalue of* $-\mathbf{L}$, *and* $\{X_n\}_{n\geq 1}$ *a sequence of elements in* $\mathbf{Ker}(\mathbf{L} + p\mathbf{Id})$ *for all* $n \geq 1$, *such that* $\lim_{n\to\infty} \mathbb{E}[X_n^2] = 1$. *Then, for any integer* $k \geq 2$, *as* $n \to \infty$, *we have*

$$X_n \overset{law}{\to} \mathcal{N}(0,1) \quad \text{if and only if} \quad \mathbb{E}[X_n^{2k}] \to \mathbb{E}[N^{2k}] = (2k-1)!!. \tag{4.1}$$

The proof of Theorem 4.1 is rather lengthy; it is thus divided in three steps which are detailed below.

SKETCH OF THE PROOF. *Step* (1): We find a family $\mathscr{P} = \{W_k | k \geq 2\}$ of real polynomials which satisfies the two following properties:

(i) $\mathbb{E}[W_k(X_n)] \geq 0, \forall k \geq 2, \forall n \geq 1$,

(ii) $X_n \overset{\text{law}}{\to} \mathcal{N}(0,1)$ if and only if $\mathbb{E}[W_2(X_n)] \to 0$, as $n \to \infty$.

Step (2): In the second step, we construct a polynomial T_k such that, under the assumptions of Theorem 4.1, we have

$$T_k = \sum_{i=2}^{k} \alpha_{i,k} W_i \qquad \text{such that for all } i, \alpha_{i,k} > 0,$$

and

$$\mathbb{E}[T_k(X_n)] \to 0 \qquad \text{as } n \to \infty.$$

Step (3): In the last step, using the fact that $\alpha_{i,k} > 0$ and property (i) of step (1), we obtain that $\mathbb{E}[W_2(X_n)] \to 0$. Finally, using property (ii) of step (1), we complete the proof. □

PROOF OF THEOREM 4.1. The "if" part is a simple consequence of Lemma 2.1. For the "only if" part, we go into the details of the three aforementioned steps.

Step (1): First, we introduce the suitable family $\mathscr{P}$ of polynomials. To this end, we denote by $\{H_k\}_{k\geq 0}$ the family of Hermite polynomials defined by the recursive relation

$$H_0(x) = 1, \qquad H_1(x) = x, \qquad H_{k+1}(x) = xH_k(x) - kH_{k-1}(x). \tag{4.2}$$

For any $k \geq 2$, we define the polynomial W_k as

$$W_k(x) = (2k-1)\left(x \int_0^x H_k(t)H_{k-2}(t)\,dt - H_k(x)H_{k-2}(x)\right), \tag{4.3}$$

and the family $\mathscr{P}$ as

$$\mathscr{P} = \left\{ P \,\middle|\, P(x) = \sum_{k=2}^{m} \alpha_k W_k(x); m \geq 2, \alpha_k \geq 0, 2 \leq k \leq m \right\}. \tag{4.4}$$

The family $\mathscr{P}$ encodes interesting properties of central convergence which are the content of the two next lemmas. Below, Lemma 4.1 will provide the answer to property (i) of step (1).

LEMMA 4.1. *Let* $\mathbf{L}$ *be a general Markov generator satisfying assumptions* (a)–(b)–(c) *in Section* 2, *and let* P *be a polynomial belonging to* $\mathscr{P}$. *Then*:

(1) *If* $N \sim \mathcal{N}(0,1)$, $\mathbb{E}[P(N)] = 0$.
(2) *If* X *is an eigenvalue of* $\mathbf{L}$, $\mathbb{E}[P(X)] \geq 0$.

PROOF. It is enough to prove that $E[W_k(X)] \geq 0$ and $E[W_k(N)] = 0$. Using the diffusive property (2.1), the fact that $-\mathbf{L}X = pX$ and the recursive property of Hermite polynomials, we obtain that

$$\begin{aligned}(\mathbf{L} + kp\mathbf{Id})H_k(X) &= H_k''(X)\Gamma(X) + H_k'(X)\mathbf{L}(X) + kpH_k(X)\\ &= H_k''(X)\Gamma(X) - pXH_k'(X) + kpH_k(X)\\ &= H_k''(X)\big(\Gamma(X) - p\big)\\ &= k(k-1)H_{k-2}(X)\big(\Gamma(X) - p\big).\end{aligned} \tag{4.5}$$

Therefore,

$$\begin{aligned}\mathcal{M}_k(H_k) &= \mathbb{E}\big[H_k(X)(\mathbf{L} + kp\mathbf{Id})H_k(X)\big]\\ &= k(k-1)\mathbb{E}\big[H_k(X)H_{k-2}(X)\big(\Gamma(X) - p\big)\big].\end{aligned} \tag{4.6}$$

Next, by the integration by parts formula (2.4), we have

$$\begin{aligned}&\mathbb{E}\big[H_k(X)H_{k-2}(X)\big(\Gamma(X) - p\big)\big]\\ &\quad = \mathbb{E}\left[\Gamma\left(\int_0^X H_k(t)H_{k-2}(t)\,dt, X\right)\right] - p\mathbb{E}\big[H_k(X)H_{k-2}(X)\big]\\ &\quad = p\mathbb{E}\left[X\int_0^X H_k(t)H_{k-2}(t)\,dt - H_k(X)H_{k-2}(X)\right]\\ &\quad = \frac{p}{2k-1}\mathbb{E}\big[W_k(X)\big].\end{aligned} \tag{4.7}$$

Hence,

$$\mathcal{M}_k(H_k) = \frac{pk(k-1)}{2k-1}\mathbb{E}\big[W_k(X)\big],$$

and the inequality $\mathbb{E}[W_k(X)] \geq 0$ follows from the positivity of the bilinear form $\mathcal{M}_k$. Finally, choosing $\mathbf{L}$ to be the Ornstein–Uhlenbeck generator and $X = N$ a standard Gaussian random variable living in the first Wiener chaos (i.e., $p = 1$) with variance 1, then $\Gamma(N) = p = 1$ and computation (4.7) shows that $\mathbb{E}[W_k(N)] = 0$ for every $k \geq 2$. Hence, $\mathbb{E}[P(N)] = 0$ for every $P \in \mathscr{P}$. $\square$

The next lemma is central in the proof of the even moment Theorem 4.1. In fact, the next lemma will provide answer to property (ii) of step (1).

LEMMA 4.2. *Assume that* $\mathbf{L}$ *be a general Markov generator satisfying assumptions* (a)–(b)–(c) *of Section* 2. *Let* $p \geq 1$ *and* $\{X_n\}_{n\geq 1}$ *a sequence of elements in* $\mathbf{Ker}(\mathbf{L} + p\mathbf{Id})$ *for all* $n \geq 1$. *Let* $P = \sum_{k=2}^{m} \alpha_k W_k \in \mathscr{P}$ *such that* $\alpha_2 \neq 0$. *Then, as* $n \to \infty$, *we have*

$$X_n \xrightarrow{law} \mathcal{N}(0,1) \quad \textit{if and only if} \quad \mathbb{E}\big[P(X_n)\big] \to \mathbb{E}\big[P(N)\big] = 0.$$

PROOF. In virtue of Lemma 4.1,

$$\begin{aligned} \mathbb{E}\big[P(X_n)\big] &= \sum_{k=2}^{m} \alpha_k \mathbb{E}\big[W_k(X_n)\big] \\ &\geq \alpha_2 \mathbb{E}\big[W_2(X_n)\big] \\ &= \alpha_2\big(\mathbb{E}\big[X_n^4\big] - 6\mathbb{E}\big[X_n^2\big] + 3\big). \end{aligned}$$

This leads to

$$0 \leq \mathbb{E}\big[X_n^4\big] - 6\mathbb{E}\big[X_n^2\big] + 3 \leq \frac{1}{\alpha_2}\mathbb{E}\big[P(X_n)\big].$$

By assumption, $\mathbb{E}[P(X_n)] \to 0$, so $\mathbb{E}[X_n^4] - 6\mathbb{E}[X_n^2] + 3 \to 0$. On the other hand,

$$\mathbb{E}\big[X_n^4\big] - 6\mathbb{E}\big[X_n^2\big] + 3 = \mathbb{E}\big[X_n^4\big] - 3\mathbb{E}\big[X_n^2\big]^2 + 3\big(\mathbb{E}\big[X_n^2\big] - 1\big)^2.$$

Thus, we obtain that $\mathbb{E}[X_n^2] \to 1$ and $\mathbb{E}[X_n^4] \to 3$, and we can use Theorem 2.2 to conclude. □

Step (2): This step consists in finding a suitable polynomial $T_k \in \mathscr{P}$ of the form

$$T_k(x) = x^{2k} - \alpha_k x^2 + \beta_k, \qquad \alpha_k, \beta_k \in \mathbb{R}. \tag{4.8}$$

To find such a polynomial, notice that according to step (1), the function $\phi_k : x \mapsto \mathbb{E}[T_k(xN)]$ must be positive and vanish at $x = 1$. Hence, we must have $\phi_k(1) = \phi_k'(1) = 0$. This leads us to the following system of equations:

$$\begin{cases} (2k-1)!! - \alpha_k + \beta_k = 0, \\ 2k(2k-1)!! - 2\alpha_k = 0. \end{cases}$$

Therefore, the coefficients α_k and β_k are necessarily given by

$$\alpha_k = k(2k-1)!! \quad \text{and} \quad \beta_k = (k-1)(2k-1)!!.$$

It remains to check that the corresponding polynomial $T_k(x) = x^{2k} - k(2k-1)!!x^2 + (k-1)(2k-1)!! \in \mathscr{P}$. To this end, one needs to show that T_k can be expanded over the basis $\{W_k\}_{k\geq 2}$ with positive coefficients. We answer to this by the affirmative with the next proposition, which also provides an explicit formula for the coefficients.

PROPOSITION 4.1. *Let $k \geq 2$, and $T_k(x) = x^{2k} - k(2k-1)!!x^2 + (k-1)(2k-1)!!$. Then*

$$T_k(x) = \sum_{i=2}^{k} \alpha_{i,k} W_i(x), \tag{4.9}$$

where

$$\alpha_{i,k} = \frac{(2k-1)!!}{2^{i-1}(2i-1)(i-2)!}\binom{k}{i}\int_0^1 (1-u)^{-1/2}u^{i-2}\left(1-\frac{u}{2}\right)^{k-i} du.$$

In particular, $T_k \in \mathscr{P}$ and $\alpha_{2,k} > 0$ for all $k \geq 1$.

The proof of this proposition is rather involved and can be found in the Appendix.

Step (3): Let $p \geq 1$. Assume that $\{X_n\}_{n\geq 1}$ is a sequence of elements of $\mathbf{Ker}(\mathbf{L} + p\mathbf{Id})$ for all $n \geq 1$ such that $\lim_{n\to\infty} \mathbb{E}[X_n^2] = 1$. We further assume that $\mathbb{E}[X_n^{2k}] \to (2k-1)!!$. Using step (2), we have

$$\begin{aligned}\mathbb{E}[T_k(X_n)] &= \mathbb{E}[X_n^{2k}] - k(2k-1)!!\mathbb{E}[X_n^2] + (k-1)(2k-1)!! \\ &\to 0.\end{aligned}$$

To finish the proof, by step (2), we know that $T_k \in \mathscr{P}$ and $c_{2,k} > 0$. Thus, Lemma 4.2 applies and one gets the desired conclusion. $\square$

We end this section with the following result containing a quantitative version of the Theorem 4.1. We remark that item (1) of Theorem 4.2 contains Theorem 1.2 in the Introduction by assuming $\mathbf{L}$ to be Ornstein–Uhlenbeck operator.

THEOREM 4.2. *Let $\mathbf{L}$ be a Markov operator satisfying assumptions* (a)–(b)–(c) *of Section* 2. *Let $p \geq 1$ and X be an eigenfunction of $\mathbf{L}$ with eigenvalue p such that $\mathbb{E}[X^2] = 1$. Assume that $k \geq 2$. Then*

(1) *We have the following general quantitative bound*:

$$d_{\mathrm{TV}}(X, \mathcal{N}(0,1)) \leq C_k \sqrt{\frac{\mathbb{E}[X^{2k}]}{(2k-1)!!} - 1}, \tag{4.10}$$

where the constant $C_k = \frac{4}{\sqrt{2k(k-1)\int_0^1 ((1+t^2)/2)^{k-2}\,dt}}$.

(2) *The moment estimate $\mathbb{E}[X^{2k}] \geq \mathbb{E}[N^{2k}] = (2k-1)!!$ holds.*

PROOF. Taking into account Remark 3.2, for any polynomial $P = \sum_{k=2}^m \alpha_k W_k$ in family $\mathscr{P}$, we obtain that

$$\mathbb{E}[P(X)] \geq \frac{1}{p^2}\sum_{k=2}^{m}(2k-1)(k-1)\alpha_k \mathbb{E}[H_{k-2}(X)^2(\Gamma(X) - p)^2].$$

By applying the latter bound to $P = T_k$ and using Proposition 4.1, we infer that

$$\begin{aligned}\mathbb{E}\big[T_k(X)\big] &\geq \frac{1}{p^2}\sum_{i=2}^{m}(2i-1)(i-1)\alpha_{i,k}\mathbb{E}\big[H_{i-2}(X)^2\big(\Gamma(X)-p\big)^2\big]\\ &\geq \frac{3\alpha_{2,k}}{p^2}\mathbb{E}\big[\big(\Gamma(X)-p\big)^2\big].\end{aligned}$$

On the other hand, Proposition 4.1 shows that

$$\alpha_{2,k} = \frac{(2k-1)!!}{6}\binom{k}{2}\int_0^1 (1-u)^{-1/2}\left(1-\frac{u}{2}\right)^{k-2} du.$$

This leads us to

$$\begin{aligned}\mathbb{E}\big[X_n^{2k}\big] - (2k-1)!! &\geq \left(\frac{(2k-1)!!}{4}k(k-1)\int_0^1 \frac{1}{\sqrt{1-u}}\left(1-\frac{u}{2}\right)^{k-2} du\right)\\ &\quad\times \mathbb{E}\left[\left(\frac{\Gamma(X_n)}{p}-1\right)^2\right].\end{aligned} \tag{4.11}$$

Now, the desired inequality follows from Theorem 2.2 and identity $\int_0^1 \frac{1}{\sqrt{1-u}}(1-\frac{u}{2})^{k-2}\,du = 2\int_0^1 (\frac{1+t^2}{2})^{k-2}\,dt$. We stress that with taking $k=2$ in (4.10), we recover the well-known bound (see, e.g., [18, 20]):

$$d_{\mathrm{TV}}\big(X_n, \mathcal{N}(0,1)\big) \leq \frac{2}{\sqrt{3}}\sqrt{\mathbb{E}\big[X_n^4\big]-3}.$$

The second item (2) easily follows from the fact that $\mathbb{E}[T_k(X)] \geq 0$. When $\mathbb{E}[X^2] \neq 1$, using the normalized random variable $\tilde{X} = \frac{X}{\sqrt{\mathbb{E}[X^2]}}$, we obtain the inequality $\mathbb{E}[X^{2k}] \geq \mathbb{E}^k[X^2]\mathbb{E}[N^{2k}]$ for all $k \geq 1$. $\square$

REMARK 4.1. The statement (2) of Theorem 4.2 does not hold for any kind of Markov operators. Below, we present a simple counterexample. Let U denote a uniform random variable on the interval $(-1,1)$. Set $X = U^2 - \frac{1}{3}$. Then X belongs to the second Wiener chaos of the Jacobi structure (see [3], Section 4) with parameters $\alpha = \beta = 1$. Besides, $\mathbb{E}[X^2] = \frac{4}{45}$. Then it is straightforward to check that the inequality $\mathbb{E}[X^{2k}] \geq \mathbb{E}[N^{2k}]\mathbb{E}^k[X^2]$ in the item (2) of Theorem 4.2 does not hold even for $k=2$. This is mainly because the assumption (c) fails in this setup. Roughly speaking, the spectrum of Jacobi operators has a quadratic growth whereas our assumption suggests a linear growth.

REMARK 4.2. Here, we give a concrete application of Theorem 4.2 in some situation where the usual criteria in the Wiener space fail. Let $\nu \geq 1$ be an integer number. Assume that $\{Q_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables having chi-squared distribution with ν degrees of freedom. We are also given $\{N_n\}_{n\geq 1}$ an independent sequence of i.i.d. standard Gaussian random variables. As a result, $\{S_n\}_{n\geq 1} = \{N_n \times \sqrt{\frac{\nu}{Q_n}}\}_{n\geq 1}$ is a sequence of i.i.d. Student random variables with ν degrees of freedom. Now, set

$$X = \sum_{1=i_1<i_2<\cdots<i_p}^{\infty} \alpha(i_1, \dots, i_p) S_{i_1} \cdots S_{i_p},$$

such that $\mathbb{E}[X^2] = 1$. Relying on the superposition procedure (see Section 2.2) and Theorem 4.2, if $\nu > 2k$, it can be shown that

$$d_{\mathrm{TV}}\big(X, \mathcal{N}(0,1)\big) \leq C_k \sqrt{\frac{\mathbb{E}[X^{2k}]}{(2k-1)!!} - 1}. \tag{4.12}$$

In addition, since X does not have moments of all orders, X does not belong to any Wiener chaos and therefore the estimate (4.12) is strictly beyond existing moments-based total-variation estimates on Wiener space.

4.2. *Other polynomial criteria for central convergence.* In the previous section, in order to prove the even moment theorem, we use heavily the fourth moment Theorem 2.2. The reason is that in the decomposition of T_k over the basis $\{W_k\}_{k\geq 2}$, the coefficient α_2 in front of W_2 is strictly positive. It is then natural to consider the cases where $\alpha_2 = 0$, which turns out to be more delicate. The main result of this section is the following.

THEOREM 4.3. *Let* $\mathbf{L}$ *be a general Markov generator satisfying assumptions* (a)–(b)–(c) *in Section* 2. *Assume that* $\{X_n\}_{n\geq 1}$ *is a sequence of eigenfunctions of* $\mathbf{L}$ *with eigenvalue* $-p$, *that is*, $-\mathbf{L}X_n = pX_n$ *for each* n. *We suppose that* $P = \sum_{k=2}^{m} \alpha_k W_k$ *is a nonzero polynomial belonging to the family* $\mathscr{P}$, *such that as* $n \to \infty$, *we have*

$$\mathbb{E}\big[P(X_n)\big] \to \mathbb{E}\big[P(N)\big] = 0. \tag{4.13}$$

Then, as $n \to \infty$, *the two following statements hold*:

(1) *If there exist at least two indices* $2 < i < j$ *such that* $\alpha_i \alpha_j > 0$ *and* i *or* j *is even, then*

$$X_n \overset{law}{\to} \mathcal{N}(0,1).$$

(2) *If there exist at least two indices* $2 < i < j$ *such that* $\alpha_i \alpha_j > 0$ *and both* i *and* j *are odd integers, then each accumulation point of sequence* $\{X_n\}_{n\geq 1}$ *in distribution is in the form*

$$\alpha \mathcal{N}(0,1) + (1-\alpha)\delta_0$$

for some $\alpha \in [0,1]$.

PROOF. We will consider each case separately.

Case (1): Let us notice that there exist $A > 0$ and $B \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $P(x) \geq Ax^2 + B$. Then $Ax^2 < P(x) - B$. By assumption, $\mathbb{E}[P(X_n)] \to 0$, so $\mathbb{E}[P(X_n) - B]$ is bounded and $\mathbb{E}[X_n^2]$ is bounded as well. Hence, by Lemma 2.1, the sequence $\{X_n\}_{n\geq 1}$ is bounded in $L^p(E, \mu)$ for each $p \geq 1$. Since $\Gamma(X_n) = \frac{1}{2}(\mathbf{L} + 2p\mathbf{Id})[X_n^2]$, and because of the fact that $\mathbf{L}$ is a continuous operator when its domain is restricted to a finite sum of eigenspaces of $\mathbf{L}$, $\Gamma(X_n)$ is also bounded in any $L^p(E, \mu)$. Finally, up to extracting a subsequence, we may assume that the sequence of random vectors $\{(X_n, \Gamma(X_n))\}_{n\geq 1}$ converges in distribution toward a random vector (U, V). As a consequence of Remark 3.2, we have

$$\mathbb{E}\bigl[H_{i-2}(X_n)^2\bigl(\Gamma[X_n] - p\bigr)^2\bigr] \to 0,$$
$$\mathbb{E}\bigl[H_{j-2}(X_n)^2\bigl(\Gamma[X_n] - p\bigr)^2\bigr] \to 0.$$

Recalling that $\{(X_n, \Gamma(X_n))\}_{n\geq 1}$ converges in distribution toward (U, V), we infer that almost surely

$$H_{i-2}(U)(V - p) = H_{j-2}(U)(V - p) = 0. \tag{4.14}$$

Thus, on the set $\{V \neq p\}$, we have $H_{i-2}(U) = H_{j-2}(U) = 0$. But the roots of two Hermite polynomials of different orders are distinct if at least one of the orders is even. By assumption, either $i - 2$ or $j - 2$ is even, and we conclude that $\mathbb{P}(V \neq p) = 0$. This proves that any accumulation point (in distribution) of the sequence $\{\Gamma(X_n)\}_{n\geq 1}$ is p, and, as a consequence, the sequence $\Gamma(X_n)$ converges to p in L^2. Now, we can conclude by using Theorem 2.1.

Case (2): Following the same line of reasoning as in case (1), we obtain:

$$H_{i-2}(U)(V - p) = H_{j-2}(U)(V - p) = 0, \qquad \text{a.s.}$$

On the set $\{V \neq p\}$, we have $H_{i-2}(U) = H_{j-2}(U) = 0$. But the roots of two Hermite polynomials with odd orders only coincide at 0. This implies $U(V - p) = 0$ almost surely. Now, let ϕ be any test function. Using the integration by parts formula (2.4) with $Y = X_n$ and $X = \phi(X_n)$ and letting $n \to +\infty$, one leads to

$$\mathbb{E}\bigl[\phi'(U)V\bigr] = p\mathbb{E}\bigl(U\phi(U)\bigr]. \tag{4.15}$$

Splitting the expectations in (4.15) into the disjoint sets $\{V = p\}$ and $\{V \neq p\}$, we obtain

$$p\mathbb{E}\bigl[\bigl(\phi'(U) - U\phi(U)\bigr)\mathbb{1}_{\{V=p\}}\bigr] + \phi'(0)\mathbb{E}[V\mathbb{1}_{\{V\neq p\}}] = 0. \tag{4.16}$$

Take $\phi(x) = e^{i\xi x}$. Then (4.16) reads

$$pi\xi\mathbb{E}\bigl[e^{i\xi U}\mathbb{1}_{\{V=p\}}\bigr] - p\mathbb{E}\bigl[Ue^{i\xi U}\mathbb{1}_{\{V=p\}}\bigr] + i\xi\mathbb{E}[V\mathbb{1}_{\{V\neq p\}}] = 0.$$

Setting $f(\xi) = \mathbb{E}[e^{i\xi U}\mathbb{1}_{\{V=p\}}]$, we obtain that

$$p\xi f(\xi) + pf'(\xi) + \xi\mathbb{E}[V\mathbb{1}_{\{V\neq p\}}] = 0,$$
$$f(\xi) = \left(\mathbb{P}(V = p) - \frac{1}{p}\mathbb{E}[V]\right) + \frac{1}{p}\mathbb{E}[V]e^{-\xi^2/2}.$$

It is straightforward to deduce from above equations that the characteristic function of random variable U is given by

$$\mathbb{E}\big[e^{i\xi U}\big] = \mathbb{P}(V \neq p) + f(\xi)$$
$$= \left(1 - \frac{1}{p}\mathbb{E}[V]\right) + \frac{1}{p}\mathbb{E}[V]e^{-\xi^2/2}. \qquad \square$$

Although case (2) in Theorem 4.3 seems less interesting than case (1), we point out that a Dirac mass at zero may appear naturally under assumptions (a)–(b)–(c). Here is a simple example of this phenomenon.

EXAMPLE 4.1. Set $E = \mathbb{R}^2$ and $\mu = \mathcal{N}(0, 1) \otimes (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)$. Define

$$\mathbf{L}[\phi](x, y) = y\left(\frac{\partial^2\phi}{\partial x^2} - x\frac{\partial\phi}{\partial x}\right). \tag{4.17}$$

One can check that $\mathbf{L}$ fulfills assumptions (a)–(b)–(c) in Section 2. Consider the sequence

$$X_n(x, y) = xy \in \mathbf{Ker}(\mathbf{L} + \mathbf{Id}), \qquad n \geq 1.$$

Then $X_n \sim \frac{1}{2}\mathcal{N}(0, 1) + \frac{1}{2}\delta_0$ for each $n \geq 1$. Moreover, $\mathbb{E}[W_3(X_n)] = \mathbb{E}[W_5(X_n)] = 0$. As a matter of fact, the conclusions of Theorem 4.3 are sharp when applied to $P = W_3 + W_5 \in \mathscr{P}$.

However, we show that in the particular setting of the Wiener space, that is, when $\mathbf{L}$ is the Ornstein–Uhlenbeck operator, the case (2) of Theorem 4.3 cannot take place. Furthermore, condition (4.13) will be a necessary and sufficient condition for central convergence. To this end, we need the following lemma, which has an interest on its own.

LEMMA 4.3. *Let $\{U_n\}_{n\geq 1}$ and $\{V_n\}_{n\geq 1}$ be two bounded sequences such that for some integer $M > 0$, we have*

$$U_n, V_n \in \bigoplus_{i=0}^{M} \mathbf{Ker}(\mathbf{L} + i\mathbf{Id}) \qquad \forall n \in \mathbb{N}.$$

If $\mathbb{E}[U_n^2 V_n^2] \to 0$ as n tends to infinity, then $\mathbb{E}[U_n^2]\mathbb{E}[V_n^2] \to 0$ as n tends to infinity.

We will make use of the next theorem, due to Carbery–Wright, restated here for convenience. More precisely, we will apply it to Gaussian distribution, which is log-concave.

THEOREM 4.4 ([9], Carbery–Wright). *Assume that μ is a log-concave probability measure on $\mathbb{R}^m$. Then there exists an absolute constant $c > 0$ (independent of m and μ) such that for any polynomial $Q : \mathbb{R}^m \to \mathbb{R}$ of degree at most k and any $\alpha > 0$, the following estimate holds*:

$$\left(\int Q^2 \, d\mu\right)^{1/(2k)} \times \mu\{x \in \mathbb{R}^m : |Q(x)| \le \alpha\} \le ck\alpha^{1/k}. \tag{4.18}$$

PROOF OF LEMMA 4.3. Let us denote $E = \mathbb{R}^{\mathbb{N}}$, $\mu = \mathcal{N}(0,1)^{\otimes \mathbb{N}}$ and let $\mathbf{L}$ be the Ornstein–Uhlenbeck generator. We assume that $\mathbb{E}[U_n^2]$ does not converge to zero. Up to extracting a subsequence, we can suppose that $\mathbb{E}[U_n^2] > \theta > 0$ for each $n \ge 1$. Following the method of [23], page 659, inequality (3.21), we can approximate in $L^2(E, \mu)$ the random variable U_n by polynomials of degree M. Hence, applying the Carbery–Wright inequality for the approximating sequence, and taking the limit, we obtain

$$\mu\{x \in E : |U_n(x)| \le \alpha\} \le \frac{cM\alpha^{1/M}}{\theta^{1/2M}} \le K\alpha^{1/M}, \tag{4.19}$$

with $K = \frac{cM}{\theta^{1/2M}}$. Next, we have the following inequalities:

$$\begin{aligned}
\mathbb{E}[V_n^2] &= \mathbb{E}\left[V_n^2 \frac{U_n^2}{U_n^2} \mathbf{1}_{\{|U_n| > \alpha\}}\right] + \mathbb{E}[V_n^2 \mathbf{1}_{\{|U_n| \le \alpha\}}] \\
&\le \frac{1}{\alpha^2}\mathbb{E}[U_n^2 V_n^2] + \sqrt{\mathbb{E}[V_n^4]}\sqrt{\mu\{x \in E : |U_n(x)| \le \alpha\}} \\
&\le \frac{1}{\alpha^2}\mathbb{E}[U_n^2 V_n^2] + CK\alpha^{1/2M},
\end{aligned}$$

where K is the constant from the Carbery–Wright inequality and C is such that $\sup_{n \ge 1} \mathbb{E}[V_n^4] \le C^2$. Note that constant C exists by hypercontractivity (see Remark 2.1). We immediately deduce that

$$\limsup_{n \to \infty} \mathbb{E}[V_n^2] \le CK\alpha^{1/2M},$$

which is valid for any $\alpha > 0$. Let $\alpha \to 0$ to achieve the proof. $\square$

THEOREM 4.5. *Let $\mathbf{L}$ stand for the Ornstein–Uhlenbeck operator and let $\{X_n\}_{n \ge 1}$ be a sequence of elements of $\mathbf{Ker}(\mathbf{L} + p\mathbf{Id})$ with variance bounded from below by some positive constant. Then, for any nonzero polynomial $P \in \mathscr{P}$, as $n \to \infty$, we have*

$$X_n \xrightarrow{law} \mathcal{N}(0,1) \quad \textit{if and only if} \quad \mathbb{E}[P(X_n)] \to 0.$$

PROOF. Although in Theorem 4.5 we assume that $\mathbf{L}$ is the Ornstein–Uhlenbeck generator, we stress that the proof works in the Laguerre structure or any tensor products of Laguerre and Wiener structures. The "if" part is straightforward by using the continuous mapping theorem. To show the "only if" part, we take a nonzero polynomial $P \in \mathscr{P}$ of the form

$$P(x) = \sum_{k=2}^{m} \alpha_k W_k(x),$$

with $\alpha_m > 0$. Thanks to Remark 3.2, as $n \to \infty$, we know that

$$\mathbb{E}\big[H_{m-2}(X_n)^2\big(\Gamma(X_n) - p\big)^2\big] \to 0. \tag{4.20}$$

Let $Z_{m-2} = \{t_1, t_2, \dots, t_{m-2}\}$ be the set of the (real) roots of the Hermite polynomial H_{m-2}. Then, as $n \to \infty$, we have

$$\mathbb{E}\left[\left(\prod_{k=1}^{m-2}(X_n - t_k)^2\right)\big(\Gamma(X_n) - p\big)^2\right] \to 0.$$

From the fact that $\Gamma(X_n) = \frac{1}{2}(\mathbf{L} + p\mathbf{Id})(X_n^2)$ together with fundamental assumption (2.5) (which holds in the Wiener structure), we deduce that $H_{m-2}(X_n)$ and $\Gamma(X_n) - p$ are both finitely expanded over the eigenspaces of the generator $\mathbf{L}$. Besides, repeating the same argument as in the proof of Theorem 4.3, we can show that the sequence $\{X_n\}_{n\geq 1}$ is bounded in $L^2(E, \mu)$, as well as $\{\Gamma(X_n) - p\}_{n\geq 1}$. Thus, from Lemma 4.3, as $n \to \infty$, we obtain

$$\left(\prod_{k=1}^{m-2}\mathbb{E}\big[(X_n - t_k)^2\big]\right)\mathbb{E}\big[\big(\Gamma(X_n) - p\big)^2\big] \to 0.$$

Since $\mathbb{E}[(X_n - t_k)^2] \geq \mathrm{Var}(X_n)$ is bounded from below by assumption, we conclude that $\Gamma(X_n) \to p$ in $L^2(E)$. Hence, using Theorem 2.1, we obtain that the sequence $\{X_n\}_{n\geq 1}$ converges in distribution toward $\mathcal{N}(0, 1)$. □

5. Conjectures. The main motivation of this article is to provide an answer to the question (B) stated in the Introduction. We have shown that the convergence of any even moment guarantees the central convergence of a normalized sequence (i.e., $\mathbb{E}[X_n^2] \to 1$) living inside $\mathbf{Ker}(\mathbf{L} + p\mathbf{Id})$. In the latter criterion, we have dealt with normalized sequences because it seems more natural from the probabilistic point of view. However, one could also try to replace this assumption by the convergence of another even moment. Indeed, our framework could provide a wider class of polynomial conditions ensuring central convergence, namely through the family $\mathscr{P}$. Then it is natural to check whether the family $\mathscr{P}$ is rich enough to produce other pair of even moments ensuring a criterion for central convergence. To be more precise, assume that for some pair (k, l) ($k < l$) of positive integers, we have $\mathbb{E}[X_n^{2k}] \to \mathbb{E}[N^{2k}]$ and $\mathbb{E}[X_n^{2l}] \to \mathbb{E}[N^{2l}]$, we want to know if this implies a central convergence. Our method would consist in deducing the existence of a nontrivial polynomial $T_{k,l} \in \mathscr{P}$ such that $\mathbb{E}[T_{k,l}(X_n)] \to 0$. Natural candidates are polynomials of the form

$$T_{k,l}(x) = x^{2l} + \alpha x^{2k} + \beta,$$

where $\alpha, \beta \in \mathbb{R}$. Using the same arguments as in the step (2) of the proof of Theorem 4.1, one can show that the condition $P \in \mathscr{P}$ entails necessarily that $\alpha = \frac{l(2l-1)!!}{k(2k-1)!!}$ and $\beta = (\frac{l}{k} - 1)(2k-1)!!$. Then the question becomes: does the polynomial $T_{k,l}$ belong to family $\mathscr{P}$?

We exhibit the decomposition of $T_{k,l}$ for each pair of integers in the set $\Theta = \{(2,3); (2,4); (2,5); (3,4); (3,5)\}$:

$$T_{2,3}(x) = x^6 - \frac{15}{2}x^4 + \frac{15}{2} = W_3(x) + \frac{\mathbf{5}}{\mathbf{2}}W_2(x),$$

$$T_{2,4}(x) = x^8 - 70x^4 + 105 = W_4(x) + \frac{\mathbf{84}}{\mathbf{5}}W_3(x) + \mathbf{28}W_2(x),$$

$$\begin{aligned} T_{2,5}(x) &= x^{10} - \frac{1575}{2}x^4 + \frac{2835}{2} \\ &= W_5(x) + \frac{\mathbf{180}}{\mathbf{7}}W_4(x) + \mathbf{234}W_3(x) + \frac{\mathbf{585}}{\mathbf{2}}W_2(x), \end{aligned}$$

$$T_{3,4}(x) = x^8 - \frac{28}{3}x^6 + 35 = W_4(x) + \frac{\mathbf{112}}{\mathbf{5}}W_3(x) + \frac{\mathbf{14}}{\mathbf{3}}W_2(x),$$

$$T_{3,5}(x) = x^{10} - 105x^6 + 630 = W_5(x) + \frac{\mathbf{180}}{\mathbf{7}}W_4(x) + \mathbf{129}W_3(x) + \mathbf{30}W_2(x).$$

The coefficients of each decomposition are positive, thus, for each pair $(k,l) \in \Theta$, the convergence of the $2k$th and $2l$th moments entails the central convergence. Naturally, we are tempted to formulate the following conjecture.

CONJECTURE 1. Let $k, l \geq 2$ be two different positive integers. For any sequence $\{X_n\}_{n\geq 1}$ of eigenfunctions in the same eigenspace of a Markov generator $\mathbf{L}$ satisfying assumptions (a)–(b)–(c), as $n \to \infty$, the following statements are equivalent:

(i) $X_n \xrightarrow{law} N \sim \mathcal{N}(0,1)$.
(ii) $\mathbb{E}[X_n^{2k}] \to \mathbb{E}[N^{2k}]$ and $\mathbb{E}[X_n^{2l}] \to \mathbb{E}[N^{2l}]$.

Unfortunately, we could not prove it since $T_{4,5}$ does not belong to family $\mathscr{P}$:

$$T_{4,5}(x) = x^{10} - \frac{45}{4}x^8 + \frac{945}{4} = W_5(x) + \frac{405}{28}W_4(x) + W_3(x) - \frac{\mathbf{45}}{\mathbf{2}}W_2(x).$$

We insist on the fact that the above conjecture might be true nonetheless.

Another perspective of our algebraic framework is to provide nontrivial moments inequalities for the eigenfunctions of the Markov operator $\mathbf{L}$ satisfying suitable assumptions. The special role of the fourth cumulant κ_4 in normal approximation for a sequence living inside a fixed eigenspace is now well understood and it is known that $\kappa_4(X) \geq 0$. In a recent preprint, the authors of [4] observed the prominent role of κ_6 for studying convergence in distribution toward $N_1 \times N_2$, where N_1 and N_2 are two independent $\mathcal{N}(0,1)$ random variables, of a given sequence in a fixed Wiener chaos. The computations suggest that κ_6 could be greater than the variance of some differential operator (analogous to $\mathrm{Var}(\Gamma[X,X])$ in the case of normal approximation). However, the techniques presented in [4] could not provide the positivity of the sixth cumulant. We recall that

$$\kappa_6(X) = \mathbb{E}[X^6] - 15\mathbb{E}[X^2]\mathbb{E}[X^4] - 10\mathbb{E}[X^3]^2 + 30\mathbb{E}[X^2]^3.$$

Computations show that the least eigenvalue of the moment matrix $\mathbf{M}_3(X)$ is always bigger than $\kappa_6(X)$. Therefore, our method does not give results precise enough, to insure the positivity of the sixth cumulant. However, we know that $\kappa_6(X) \geq 0$ in the two first Wiener chaoses. Moreover, using Proposition 3.1, we could prove the following partial criterion.

PROPOSITION 5.1. *Let X be a multiple Wiener–Itô integral of odd order such that $\mathbb{E}[X^2] = 1$. If $\kappa_4(X) \geq 3$, then $\kappa_6(X) \geq 0$.*

These two facts lead us to formulate the following conjecture.

CONJECTURE 2. For any multiple Wiener–Itô integral X of order $p \geq 2$, we have $\kappa_6(X) > 0$.

APPENDIX

We give here a proof of Proposition 4.1. In the following, w stands for the density of the standard Gaussian distribution over $\mathbb{R}$. Let us begin by stating a lemma on elementary computations on Hermite polynomials.

LEMMA A.1. *Let $l, m, n \in \mathbb{N}$. Then*

$$\int_{\mathbb{R}} x^{2m} H_{2n}(x) w(x)\, dx = \frac{(2m)!}{2^{m-n}(m-n)!} \tag{A.1}$$

and

$$\int_{\mathbb{R}} H_l(x) H_m(x) H_n(x) w(x)\, dx = \frac{l!m!n!}{((-l+m+n)/2)!((l-m+n)/2)!((l+m-n)/2)!}, \tag{A.2}$$

with the convention that $\frac{1}{p!} = 0$ if $p \notin \mathbb{N}$.

PROOF. We first focus on (A.1). Recall that $e^{-x^2/2}H_n(x) = (-1)^n \times \frac{d^n}{dx^n}(e^{-x^2/2})$. Performing $2n$ integrations by parts (with $n \le m$), we obtain

$$\begin{aligned}\int_{\mathbb{R}} x^{2m} H_{2n}(x) w(x)\,dx &= \int_{\mathbb{R}} \frac{d^{2n}}{dx^{2n}}(x^{2m}) w(x)\,dx \\ &= \frac{(2m)!}{(2(m-n))!}\int_{\mathbb{R}} x^{2(m-n)} w(x)\,dx \\ &= \frac{(2m)!(2(m-n)-1)!!}{(2(m-n))!} \\ &= \frac{(2m)!}{2^{m-n}(m-n)!}.\end{aligned}$$

If $m > n$, the formula follows from our convention. Now, (A.2) is a mere consequence of the product formula for Hermite polynomials, which states that (see, e.g., Theorem 6.8.1 in [2])

$$H_n(x)H_m(x) = \sum_{k=0}^{\min(n,m)} \binom{n}{k}\binom{m}{k} k! H_{n+m-2k}(x),$$

for all positive integers n, m. Indeed, integrating last equation against $H_l w$, and using the orthogonality of Hermite polynomials with respect to w, we obtain the desired result. $\square$

Now, let us prove Proposition 4.1.

PROOF OF PROPOSITION 4.1. To make the notation less cluttered, we set $\beta_k = (k-1)(2k-1)!!$ and $\alpha_k = k(2k-1)!!$. Since W_p is an even polynomial and $\deg(W_p) = 2p$, there exists a unique expansion of the form

$$x^{2k} - \alpha_k x^2 + \beta_k = \sum_{p=2}^{k} c_{p,k} W_p(x) + ax^2 + b. \tag{A.3}$$

Recall that the coefficients α_k and β_k are chosen in such a way that $\phi(t) = \mathbb{E}[t^{2k}N^{2k} - \alpha_k t^2 N^2 + \beta_k]$ satisfies $\phi(1) = \phi'(1) = 0$. Coming back to Lemma 4.1, for each $p \ge 2$ the two following conditions hold:

$$\begin{cases} \mathbb{E}[W_p(N)] = 0, & \\ \forall x \in \mathbb{R}, & \psi_p(x) = \mathbb{E}[W_p(xN)] \ge 0. \end{cases}$$

Thus, ψ_p reaches its minimum at $x = 1$ and we have $\psi_p(1) = \psi_p'(1) = 0$. Setting

$$\psi(x) = \mathbb{E}\left[\sum_{p=2}^{k} c_{p,k} W_p(xN)\right] = \sum_{p=2}^{k} c_{p,k}\psi_p(x),$$

we must also have $\psi(1) = \psi'(1) = 0$. Plugging the above conditions on ϕ and ψ into (A.3) implies that, if $\delta(x) = \mathbb{E}[ax^2N^2 + b] = ax^2 + b$, then $\delta(1) = \delta'(1) = 0$. Hence, $a + b = 0$ and $2a = 0$ so $a = b = 0$. Define the (even) polynomial $Q_k(x) = \sum_{p=2}^{k} c_{p,k}(2p-1)H_p(x)H_{p-2}(x)$. Using the definition of W_p and (A.3), we see that Q_k is solution of the polynomial equation

$$x\int_0^x Q_k(t)\,dt - Q_k(x) = x^{2k} - \alpha_k x^2 + \beta_k. \tag{A.4}$$

In the following lemma, we solve the above equation.

LEMMA A.2. *Equation* (A.4) *has a unique even polynomial solution of degree* $2k-2$, *which is*

$$Q_k(x) = \sum_{p=2}^{k} c_{p,k}(2p-1)H_p(x)H_{p-2}(x) = -\beta_k + \sum_{p=1}^{k-1} \frac{(2k-1)!!}{(2p-1)!!}x^{2p}. \tag{A.5}$$

PROOF. Let Φ be the linear operator from $\mathbb{R}[X]$ to $\mathbb{R}[X]$ defined by $\Phi(P)(X) = X\int_0^X P(t)\,dt - P(X)$. Assume that $\Phi(P) = \Phi(Q)$, then $\Delta(X) = \int_0^X (P(t) - Q(t))\,dt$ satisfies the differential equation $xy(x) - y'(x) = 0$. Thus, there exists $C > 0$ such that $\Delta(X) = Ce^{X^2/2}$. But Δ is a polynomial function so $C = 0$. This implies that $P - Q$ is a constant polynomial. By setting $x = 0$ in equation (A.4), we get that $Q_k(0) = -\beta_k$. Now, set

$$R_k(X) = -\beta_k + \sum_{p=1}^{k-1} \frac{(2k-1)!!}{(2p-1)!!}X^{2p},$$

we also have $R_k(0) = -\beta_k$. As a result, one is left to show that $\Phi(R_k) = \Phi(Q_k)$. Indeed,

$$\begin{aligned}
\Phi(R_k) &= -\beta_k(X^2 - 1) + \sum_{p=1}^{k-1} \frac{(2k-1)!!}{(2p-1)!!}\left(\frac{1}{2p+1}X^{2p+2} - X^{2p}\right) \\
&= -\beta_k(X^2 - 1) + (2k-1)!!\sum_{p=1}^{k-1} \frac{1}{(2p+1)!!}X^{2p+2} \\
&\quad - (2k-1)!!\sum_{p=1}^{k-1} \frac{1}{(2p-1)!!}X^{2p} \\
&= -\beta_k(X^2 - 1) + X^{2k} - (2k-1)!!X^2 \\
&= X^{2k} - \alpha_k X^2 + \beta_k \\
&= \Phi(Q_k).
\end{aligned}$$

□

Integrating (A.5) against $H_{2n}w$ over $\mathbb{R}$ for each $1 \le n \le k-1$ and using Lemma A.1 shows that $\{c_{p,k}\}_{2\le p\le k}$ is the solution of the following triangular array:

$$\sum_{p=n+1}^{k} c_{p,k}(2p-1)\frac{(p-2)!p!(2n)!}{(n+1)!(n-1)!(p-n-1)!}$$
$$= \sum_{p=n+1}^{k} \frac{(2k-1)!!(2p-2)!}{(2p-3)!!2^{p-n-1}(p-n-1)!} \qquad \forall n \in [1, k-1],$$

which can be equivalently stated as

$$\text{(A.6)}\quad \forall n \in [1, k-1], \qquad \sum_{p=n}^{k-1} \frac{a_{p,k}}{(p-n)!} = \frac{2^n(n+1)!(n-1)!}{(2n)!}\sum_{p=n}^{k-1}\frac{p!}{(p-n)!},$$

by denoting, for all $1 \le p \le k-1$,

$$\text{(A.7)}\qquad a_{p,k} = \frac{(2p+1)(p-1)!(p+1)!}{(2k-1)!!}c_{p+1,k}.$$

In order to solve (A.6), we introduce the polynomial functions

$$f(x) = -k + \sum_{p=0}^{k-1} x^p, \qquad g(x) = \sum_{p=1}^{k-1}\frac{a_{p,k}}{p!}x^p.$$

Remark that, in terms of the functions f and g, (A.6) reads

$$\forall n \in [1, k-1], \qquad g^{(n)}(1) = \frac{2^n(n+1)!(n-1)!}{(2n)!}f^{(n)}(1).$$

The multiplication formula for the Gamma function and a classic property of the beta function (see, e.g., [1], formulas (6.1.20) and (6.2.2)) imply

$$\begin{aligned}\frac{2^n(n+1)!(n-1)!}{(2n)!} &= 2^n\frac{\Gamma(n+2)\Gamma(n)}{\Gamma(2n+1)}\\ &= \frac{\Gamma(n+2)}{2^n\Gamma(n+1)}\cdot\frac{\Gamma(1/2)\Gamma(n)}{\Gamma(n+1/2)}\\ &= \frac{n+1}{2^n}\int_0^1 u^{n-1}(1-u)^{-1/2}\,du.\end{aligned}$$

Thus, $\forall x \in (1/4, 3/4)$,

$$\begin{aligned}g(1-2x) - g(1) &= \sum_{n=1}^{k-1}\frac{g^{(n)}(1)}{n!}(-1)^n 2^n x^n\\ &= \sum_{n=1}^{k-1}\frac{f^{(n)}(1)}{n!}(n+1)\int_0^1 u^{n-1}(1-u)^{-1/2}\,du(-1)^n x^n\end{aligned}$$

$$= \int_0^1 (1-u)^{-1/2} \sum_{n=1}^{k-1} \frac{f^{(n)}(1)}{n!} (n+1)(-1)^n u^{n-1} x^n \, du$$

$$= \int_0^1 (1-u)^{-1/2} u^{-1} \frac{d}{du}\bigl(uf(1-ux)\bigr)\, du.$$

Since $f(x) = -k + \frac{1-x^k}{1-x}$,

$$\frac{d}{du}\bigl[uf(1-ux)\bigr] = \frac{d}{du}\left[-ku + \frac{1-(1-ux)^k}{x}\right] = k\bigl((1-ux)^{k-1} - 1\bigr),$$

so that, $\forall x \in (1/4, 3/4)$,

$$g(1-2x) - g(1) = k \int_0^1 (1-u)^{-1/2} u^{-1} \bigl((1-ux)^{k-1} - 1\bigr)\, du.$$

Derive last equation to obtain that $\forall p \in [1, k-1]$, $\forall x \in (1/4, 3/4)$,

$$2^p g^{(p)}(1-2x) = \frac{k!}{(k-1-p)!} \int_0^1 (1-u)^{-1/2} u^{p-1} (1-ux)^{k-1-p} \, du.$$

Note that we used Lebesgue's derivation theorem, which applies since

$$\sup_{x \in (1/4,3/4)} \bigl|(1-u)^{-1/2} u^{p-1} (1-ux)^{k-p-1}\bigr| \le (1-u)^{-1/2} u^{p-1} \left(1 - \frac{u}{4}\right)^{k-p-1},$$

and the upper bound in the last equation is in $L^1((0,1))$ as a function of u. Finally, for all $1 \le p \le k-1$,

$$a_{p,k} = g^{(p)}(0) = 2^{-p} \frac{k!}{(k-p-1)!} \int_0^1 (1-u)^{-1/2} u^{p-1} \left(1 - \frac{u}{2}\right)^{k-p-1} du,$$

and we can use (A.7) to conclude. $\square$

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