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ON SYMMETRIES IN THE THEORY OF SINGULAR PERTURBATIONS

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ABSTRACT. For a nonnegative self-adjoint operator A_0 acting on a Hilbert space $\mathfrak{H}$ singular perturbations of the form $A_0 + V$ are studied under some additional requirements for the associated selfadjoint realizations. Here V takes values in $\mathfrak{H}_{-2}(A_0)$, outside the original Hilbert space $\mathfrak{H}$, and is typically determined by a collection of certain unbounded functionals. To restrict the selection of self-adjoint realizations for the formal expression $A + V$ a class of admissible operators is introduced. Further symmetry requirements are expressed by using a notion of $p(t)$-homogeneous operators, a concept which is defined here by means of a one-parameter family of unitary operators $\mathfrak{U}$, which is closed under taking adjoints. A related requirement of $\xi(t)$-invariance for the unbounded functionals appearing in singular perturbations is also studied. This gives an abstract framework to study singular perturbations with symmetries and it allows to incorporate physically meaningful restrictions for the corresponding self-adjoint realizations. The results are applied for the investigation of singular perturbations of the Schrödinger operator in $L_2(\mathbb{R}^3)$ assuming $\xi(t)$-invariance with respect to scaling transformations in $\mathbb{R}^3$.

1. INTRODUCTION

Let A_0 be a nonnegative self-adjoint (in general unbounded) operator acting on a Hilbert space $\mathfrak{H}$ and let

$$\mathfrak{H}_2(A_0) \subset \mathfrak{H}_1(A_0) \subset \mathfrak{H} \subset \mathfrak{H}_{-1}(A_0) \subset \mathfrak{H}_{-2}(A_0)$$

be the standard scale of Hilbert spaces associated with A_0. More precisely, this means that

$$\mathfrak{H}_k(A_0) = \mathcal{D}(A_0^{k/2}), \quad k = 1, 2, \tag{1.1}$$

equipped with the norm $\|u\|_k = \|(A_0 + I)^{k/2}u\|$. The conjugated spaces $\mathfrak{H}_{-k}(A_0)$ can be defined as the completions of $\mathfrak{H}$ with respect to the norms

$$\|u\|_{-k} = \|(A_0 + I)^{-k/2}u\|, \quad u \in \mathfrak{H}. \tag{1.2}$$

By (1.2), the resolvent operator $(A_0 + I)^{-1}$ can be continuously extended to an isometric mapping $(\mathbb{A}_0 + I)^{-1}$ from $\mathfrak{H}_{-2}(A_0)$ onto $\mathfrak{H}$ and, hence, the relation

$$< \psi, u >= ((A_0 + I)u, (\mathbb{A}_0 + I)^{-1}\psi), \quad u \in \mathfrak{H}_2(A_0) \tag{1.3}$$

enables one to identify the elements $\psi \in \mathfrak{H}_{-2}(A_0)$ as continuous linear functionals on $\mathfrak{H}_2(A_0)$.

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Consider the formal expression

$$A_0 + \sum_{i,j=1}^{n} b_{ij} < \psi_j, \cdot > \psi_i, \quad b_{ij} \in \mathbb{C}, \quad n \in \mathbb{N}, \tag{1.4}$$

where elements ψ_j ($1 \leq j \leq n$) form a linearly independent system in $\mathfrak{H}_{-2}(A_0)$. In what follows it is supposed that the linear span $\mathcal{X}$ of $\{\psi_j\}_{j=1}^n$ satisfies the condition $\mathcal{X} \cap \mathfrak{H} = \{0\}$, i.e., elements ψ_j are $\mathfrak{H}$-independent. In this case, the perturbation $V = \sum_{i,j=1}^n b_{ij} < \psi_j, \cdot > \psi_i$ is said to be singular and the formula

$$A_{\text{sym}} = A_0 \upharpoonright \mathcal{D}(A_{\text{sym}}), \quad \mathcal{D}(A_{\text{sym}}) = \{ u \in \mathcal{D}(A_0) : < \psi_j, u >= 0, \ 1 \leq j \leq n \} \tag{1.5}$$

determines a closed symmetric densely defined operator in $\mathfrak{H}$.

From the point of view of the theory of singular perturbations, cf. e.g. [4], [5], [26], each intermediate extension A of A_{sym}, i.e., $A_{\text{sym}} \subset A \subset A_{\text{sym}}^*$, can be viewed to be singularly perturbed with respect to A_0 and, in general, such an A can be regarded as an operator-realization of the formal expression (1.4) in $\mathfrak{H}$. In this context, the natural question arises whether and how one could establish a physically meaningful correspondence between the parameters b_{ij} of the singular potential V and the intermediate extensions of A_{sym}.

The investigation of this problem is one of the main aims of the present paper. In particular, the Albeverio – Kurasov approach [5], [6] is augmented and combined with the boundary triplets technique [19], [22], [31], cf. Section 2. The approach used in [5], [6] allows one to involve parameters b_{ij} of the singular perturbation in the description of the corresponding operator realization of (1.4), while boundary triplets provide a convenient tool for some further investigation of such operators. This leads to simple descriptions for the associated operator realizations (Theorem 2.3) without the standard assumption of orthonormality of ψ_j or the requirement of the matrix $\mathbf{B} = (b_{ij})_{i,j=1}^n$ to be an invertible, see e.g. [5, Theorem 3.1.2].

Recall that in the Albeverio-Kurasov approach a regularization

$$\mathbb{A}_{\mathbf{R}} := \mathbb{A}_0 + \sum_{i,j=1}^{n} b_{ij} < \psi_j^{\text{ex}}, \cdot > \psi_i, \tag{1.6}$$

for (1.4) is constructed such that $\mathbb{A}_{\mathbf{R}}$ is well defined as an operator from $\mathcal{D}(A_{\text{sym}}^*)$ to $\mathfrak{H}_{-2}(A_0)$. The corresponding operator realization A of (1.4) is then determined by the formula

$$A = \mathbb{A}_{\mathbf{R}} \upharpoonright \mathcal{D}(A), \quad \mathcal{D}(A) = \{ f \in \mathcal{D}(A_{\text{sym}}^*) : \mathbb{A}_{\mathbf{R}} f \in \mathfrak{H} \}. \tag{1.7}$$

A principal point here is the construction of the extended functionals $< \psi_j^{\text{ex}}, \cdot >$ ($j = 1, \ldots, n$) defined on $\mathcal{D}(A_{\text{sym}}^*)$. These functionals are uniquely determined by the choice of a Hermitian matrix $\mathbf{R} = (r_{jp})_{j,p=1}^n$. There are certain natural requirements for the choice of $\mathbf{R}$ induced by the fact that any functional $< \psi, \cdot >$ where $\psi \in \mathcal{X} \cap \mathfrak{H}_{-1}(A_0)$ admits a natural continuation onto $\mathfrak{H}_1(A_0) \cap \mathcal{D}(A_{\text{sym}}^*)$ (for further details, see [5] and Section 3 below). In order to preserve these natural continuations of $< \psi, \cdot >$ ($\psi \in \mathcal{X} \cap \mathfrak{H}_{-1}(A_0)$), the concept of admissible matrices $\mathbf{R}$ for the regularization of (1.4) has been introduced in [6].

In Section 3, the notion of *admissible operators* for the regularization of (1.4) is defined, which is convenient for applications, and the set of all admissible matrices is described via admissible operators. Geometric characterizations for the set of admissible operators are established (see Theorem 3.4, Theorem 3.6) involving a connection to the Friedrichs extension A_F of $A_{\rm sym}$. Also it is shown that there exists a nonnegative admissible operator for the regularization of (1.4) if and only if the Friedrichs and the Krein-von Neumann extension of $A_{\rm sym}$ are transversal (Theorem 3.9). It should be noted that the selection of a certain "basic operator" for the regularization of a formal expression $A_0 + V$, where V is, in general, an infinite dimensional singular perturbation, plays also a key role in the approach suggested recently by Arlinskii and Tsekanovskii [12] for determining self-adjoint realizations of A_0+V. Observe, that for fixed parameters b_{ij} of the singular perturbation in (1.4) the corresponding operator realization A depends on the choice of an admissible matrix $\mathbf{R}$ or, what is equivalent (see (3.3)), of an admissible operator $\widetilde{A}$. If the singular perturbation V in (1.4) is form-bounded (i.e., $\mathcal{X} \subset \mathfrak{H}_{-1}(A_0)$), then the admissible operator is determined uniquely and it coincides with the Friedrichs extension of $A_{\rm sym}$ (cf. Corollary 3.7). So, in this case, the formulas (1.6), (1.7) define a unique operator realization for (1.4) with the parameters b_{ij} fixed. Otherwise, $\mathcal{X} \not\subset \mathfrak{H}_{-1}(A_0)$ and then one needs to impose some extra assumptions in order to select a unique admissible operator for the regularization of (1.4).

It is well known, see e.g. [3], [6], [8], [15], [16], [27], [36], that various symmetry properties of the unperturbed operator A_0 and its singular perturbation V play an important role in applications to quantum mechanics. For this reason, it is natural to impose additional symmetry conditions for the choice of a unique admissible operator in order to ensure a physically meaningful correspondence between the parameters b_{ij} of the singular perturbation and the corresponding operator realization.

To study this problem in an abstract framework, one needs to define the notion of symmetry for the unperturbed operator A_0 and for the singular elements ψ_j in (1.4). Generalizing the ideas suggested in [35], [28], and [5], the required definitions will be formulated here as follows:

Let $\mathfrak{T}$ be some subset of the real line $\mathbb{R}$ and let $\mathfrak{U} = \{U_t\}_{t\in\mathfrak{T}}$ be a one-parameter family of unitary operators acting on $\mathfrak{H}$ with the following additional property:

$$U_t \in \mathfrak{U} \iff U_t^* \in \mathfrak{U} \tag{1.8}$$

Definition 1.1. *A linear operator A acting in $\mathfrak{H}$ is said to be $p(t)$-homogeneous with respect to $\mathfrak{U}$ if there exists a real function $p(t)$ defined on $\mathfrak{T}$, such that*

$$U_t A = p(t) A U_t, \quad \forall t \in \mathfrak{T}. \tag{1.9}$$

In other words, the set $\mathfrak{U}$ determines the structure of a symmetry and the property of A to be $p(t)$-homogeneous with respect to $\mathfrak{U}$ means that A possesses a certain symmetry with respect to $\mathfrak{U}$.

Definition 1.2. *A singular element $\psi \in \mathfrak{H}_{-2}(A_0) \setminus \mathfrak{H}$ is said to be $\xi(t)$-invariant with respect to $\mathfrak{U}$ if there exists a real function $\xi(t)$ defined on $\mathfrak{T}$, such that*

$$\mathbb{U}_t \psi = \xi(t)\psi, \quad \forall t \in \mathfrak{T}. \tag{1.10}$$

Here $\mathbb{U}_t$ stands for the continuation of U_t onto $\mathfrak{H}_{-2}(A_0)$, see Section 4 for details. The condition of $\xi_j(t)$-invariance of ψ_j is equivalent to the relation (see (4.11))

$$\xi_j(t) < \psi_j, u > = < \psi_j, U_t^* u >, \quad \forall u \in \mathfrak{H}_2(A_0), \quad \forall t \in \mathfrak{T}.$$

In Section 4, it is shown that the preservation of these properties for the extended functionals $< \psi_j^{\rm ex}, \cdot >$ in (1.6) is equivalent to the $p(t)$-homogeneity of the corresponding admissible operator $\widetilde{A}$ (see Theorem 4.6). Consequently, in the case where the unperturbed operator A_0 is $p(t)$-homogeneous and the singular elements ψ_j are $\xi_j(t)$-invariant, the natural requirement of $\xi_j(t)$-invariance for the extended functionals $<\psi_j^{\rm ex}, \cdot>$ gives the possibility to select a unique admissible operator $\widetilde{A}$ by imposing the additional requirement of $p(t)$-homogeneity of $\widetilde{A}$. In Section 4 this problem is studied in detail.

It turns out that the existence of $p(t)$-homogeneous admissible operators for the regularization of (1.4) is closely related to the transversality of the Friedrichs and the Krein-von Neumann extension of $A_{\rm sym}$ (cf. Theorem 3.9, Theorem 4.9). Furthermore, the construction of an admissible operator $\widetilde{A}$ in Theorem 4.10 immediately implies that $\widetilde{A}$ is an extremal extension of $A_{\rm sym}$, see Definition 3.10 for details. It should be noted that extremality is a physically reasonable concept. For example, only the operators which are extremal in this sense determine a free evolution in the Lax–Phillips scattering theory [30], [33].

In Section 5, the properties of self-adjoint operator realizations of (1.4) are studied under the assumptions that the operator A_0 and the singular elements ψ_j in (1.4), respectively, are $p(t)$-homogeneous and $\xi_j(t)$-invariant with respect to a family $\mathfrak{U}$ and an admissible operator $\widetilde{A}$ for the regularization of (1.4) is chosen to be $p(t)$-homogeneous.

In Section 6, the results obtained in the earlier sections are applied for the investigation of finite rank singular perturbations of the Schrödinger operator $-\Delta$ assuming the $\xi(t)$-invariance with respect to scaling transformations in $\mathbb{R}^3$. The choice of $\mathfrak{U}$ as the set of scaling transformations is inspired here by the fact that Shrödinger operators with regular potentials homogeneous with respect to scaling transformations have a lot of interesting properties, see e.g. [17], which appear due the homogeneity of potentials. The results of Section 6 show that the $\xi(t)$-invariance of singular potentials with respect to scaling transformations also ensures specific properties for the corresponding self-adjoint operator realizations of (1.4). It is emphasized that this condition of symmetry makes it possible to get simple solutions to many non-trivial problems (like description of nonnegative operator realizations, spectral properties, completeness of the wave operators, explicit form of the scattering matrix, and so on).

Throughout the paper $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\ker A$ denote the domain, the range, and the null-space of a linear operator A, respectively. $A \restriction \mathcal{D}$ stands for the restriction of A to the set $\mathcal{D}$. The transpose (of a matrix or a vector) is denoted by $B^\top$, $v^\top$.

2. Operator realizations of finite rank singular perturbations

Consider the heuristic expression (1.4) involving the singular perturbation term $V = \sum_{i,j=1}^n b_{ij} < \psi_j, \cdot > \psi_i$. Following [5], [6] first some regularization (1.6) of (1.4) is constructed as an operator from $\mathcal{D}(A_{\rm sym}^*)$ to the scale space $\mathfrak{H}_{-2}(A_0)$ and then the corresponding operator realization A of (1.4) is defined by (1.7) as an operator in $\mathfrak{H}$.

To clarify the meaning of $\mathbb{A}_0$ and $\psi_j^{\rm ex}$ in (1.6), observe that $\mathbb{A}_0$ stands for the continuation of A_0 as a bounded linear operator acting from $\mathfrak{H}$ into $\mathfrak{H}_{-2}(A_0)$. Using the extended resolvent in (1.3) this continuation of A_0 can be determined also by the formula

$$\mathbb{A}_0 f := [(\mathbb{A}_0 + I)^{-1}]^{-1} f - f, \quad \forall f \in \mathfrak{H}. \tag{2.1}$$

The linear functionals $< \psi_j^{\rm ex}, \cdot >$ are extensions of the functionals $< \psi_j, \cdot >$ onto $\mathcal{D}(A^*_{\rm sym})$. Using the well-known relation

$$\mathcal{D}(A^*_{\rm sym}) = \mathcal{D}(A_0)\dot{+}\mathcal{H}, \quad \text{where} \quad \mathcal{H} = \ker\,(A^*_{\rm sym} + I), \tag{2.2}$$

one concludes that $< \psi_j, \cdot >$ can be extended onto $\mathcal{D}(A^*_{\rm sym})$ by fixing their values on $\mathcal{H}$. It follows from (1.2), (1.3), and (1.5) that the vectors

$$h_j = (\mathbb{A}_0 + I)^{-1}\psi_j, \quad j = 1, \dots, n, \tag{2.3}$$

form a basis of the defect subspace $\mathcal{H} = \ker\,(A^*_{\rm sym} + I)$ of $A_{\rm sym}$. Hence, $< \psi_j^{\rm ex}, \cdot >$, $1 \le j \le n$, are well-defined by the formula

$$< \psi_j^{\rm ex}, f > := < \psi_j, u > + \sum_{p=1}^{n} \alpha_p r_{jp} \tag{2.4}$$

for all elements $f = u + \sum_{p=1}^n \alpha_p h_p \in \mathcal{D}(A^*_{\rm sym})$ $(u \in \mathcal{D}(A_0),\ \alpha_p \in \mathbb{C})$, when the entries

$$r_{jp} := < \psi_j, (\mathbb{A}_0 + I)^{-1}\psi_p > = < \psi_j, h_p >$$

of the matrix $\mathbf{R} = (r_{jp})_{j,p=1}^n$ are determined.

If all $\psi_j \in \mathfrak{H}_{-1}(A_0)$, then r_{jp} are well defined and $\mathbf{R}$ is a Hermitian matrix (see [5]). Otherwise, the matrix $\mathbf{R}$ is not uniquely determined. In what follows, it is assumed that $\mathbf{R}$ *is already chosen as a Hermitian*[1] *matrix.* The problem of an appropriate choice of $\mathbf{R}$ will be discussed in Section 3.

In order to describe an operator realization A of (1.4) in terms of parameters b_{ij} of the singular perturbation V, the method of boundary triplets (see [22], [31], [19], and the references therein) is now incorporated.

Definition 2.1 ([22]). *A triplet (N, Γ_0, Γ_1), where N is an auxiliary Hilbert space and Γ_0, Γ_1 are linear mappings of $\mathcal{D}(A^*_{\rm sym})$ into N, is called a boundary triplet of $A^*_{\rm sym}$ if*

$$(A^*_{\rm sym} f, g) - (f, A^*_{\rm sym} g) = (\Gamma_1 f, \Gamma_0 g)_N - (\Gamma_0 f, \Gamma_1 g)_N, \qquad f, g \in \mathcal{D}(A^*_{\rm sym})$$

and the mapping $(\Gamma_0, \Gamma_1) : \mathcal{D}(A^*_{\rm sym}) \to N \oplus N$ is surjective.

The next two results (Lemma 2.2 and Theorem 2.3) were proved in [7]. For the convenience of the reader some principal steps of their proofs are repeated.

Lemma 2.2. *The triplet $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$, where the linear operators $\Gamma_i : \mathcal{D}(A^*_{\rm sym}) \to \mathbb{C}^n$ are defined by the formulas*

$$\Gamma_0 f = \begin{pmatrix} < \psi_1^{\rm ex}, f > \\ \vdots \\ < \psi_n^{\rm ex}, f > \end{pmatrix}, \qquad \Gamma_1 f = -\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \tag{2.5}$$

where $f = u + \sum_{j=1} \alpha_j h_j \in \mathcal{D}(A^*_{\rm sym})$ $(u \in \mathcal{D}(A_0),\ \alpha_j \in \mathbb{C})$ and $< \psi_j^{\rm ex}, f >$ is defined by (2.4), forms a boundary triplet for $A^*_{\rm sym}$.

[1] the requirement of Hermiticity arises from the natural assumption that an operator realization of (1.4) obtained via its regularization is self-adjoint if the singular perturbation V is symmetric, see Theorem 2.3 for details.

Proof. Using (1.3), (2.2), and (2.3) it is easy to verify with straightforward calculations that the mappings

$$\widehat{\Gamma}_0 f = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \qquad \widehat{\Gamma}_1 f = \begin{pmatrix} < \psi_1, u > \\ \vdots \\ < \psi_n, u > \end{pmatrix}, \qquad f = u + \sum_{j=1} \alpha_j h_j, \tag{2.6}$$

satisfy the conditions of Definition 2.1. Thus, $(\mathbb{C}^n, \widehat{\Gamma}_0, \widehat{\Gamma}_1)$ is a boundary triplet for $A^*_{\rm sym}$.

It follows from (2.4), (2.5), and (2.6) that

$$\Gamma_0 f = \widehat{\Gamma}_1 f + \mathbf{R}\widehat{\Gamma}_0 f, \qquad \Gamma_1 f = -\widehat{\Gamma}_0 f, \qquad f \in \mathcal{D}(A^*_{\rm sym}). \tag{2.7}$$

These relations between Γ_i and $\widehat{\Gamma}_i$, and the fact that $(\mathbb{C}^n, \widehat{\Gamma}_0, \widehat{\Gamma}_1)$ is a boundary triplet for $A^*_{\rm sym}$, imply that $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$ also is a boundary triplet for $A^*_{\rm sym}$. □

Observe, that using [19] it is easy to see that the Weyl functions $M(z)$ and $\widehat{M}(z)$ associated with the boundary triplets (2.5) and (2.6), respectively, are connected via the linear fractional transform

$$M(z) = -(R + \widehat{M}(z))^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Theorem 2.3. *The operator realization A of (1.4) is an intermediate extension of $A_{\rm sym}$ which coincides with the operator*

$$A_{\mathbf{B}} = A^*_{\rm sym} \restriction_{\mathcal{D}(A_{\mathbf{B}})}, \quad \mathcal{D}(A_{\mathbf{B}}) = \{ f \in \mathcal{D}(A^*_{\rm sym}) : \mathbf{B}\Gamma_0 f = \Gamma_1 f \}, \tag{2.8}$$

where Γ_i are defined by (2.5) and $\mathbf{B} = (b_{ij})^n_{i,j=1}$ is the coefficient matrix of the singular perturbation $V = \sum^n_{i,j=1} b_{ij} < \psi_j, \cdot > \psi_i$ in (1.4).

If V is symmetric, i.e., $< Vu, v > = < u, Vv >$ $(u, v \in \mathfrak{H}_2(A_0))$, then the corresponding operator realization $A_{\mathbf{B}}$ becomes self-adjoint.

Proof. It follows from (2.1) that

$$\mathbb{A}_0 h_j = \psi_j - h_j \tag{2.9}$$

for all h_j defined by (2.3). Rewriting $f \in \mathcal{D}(A^*_{\rm sym})$ in the form $f = u + \sum_{i=1} \alpha_i h_i$, where $u \in \mathcal{D}(A_0)$, $h_i \in \mathcal{H}$, $\alpha_i \in \mathbb{C}$, and using (1.6), (2.5), and (2.9) leads to

$$\begin{aligned} \mathbb{A}_{\mathbf{R}} f &= A_0 u - \sum_{i=1}^n \alpha_i h_i + \sum_{i,j=1}^n b_{ij} < \psi_j^{\rm ex}, f > \psi_i + \sum_{i=1}^n \alpha_i \psi_i \\ &= A^*_{\rm sym} f + (\psi_1, \ldots, \psi_n)[\mathbf{B}\Gamma_0 f - \Gamma_1 f]. \end{aligned}$$

This equality and (1.7) show that $f \in \mathcal{D}(A)$ if and only if $\mathbf{B}\Gamma_0 f - \Gamma_1 f = 0$. Therefore, the operator realization A of (1.4) is an intermediate extension of $A_{\rm sym}$ and A coincides with the operator $A_{\mathbf{B}}$ defined by (2.8).

To complete the proof, it suffices to observe that V is symmetric if and only if the corresponding matrix of coefficients $\mathbf{B} = (b_{ij})^n_{i,j=1}$ is Hermitian, i.e., $\mathbf{B} = \overline{\mathbf{B}}^\top$. In this case, the formula (2.8) immediately implies the self-adjointness of $A_{\mathbf{B}}$ (see [22]). Theorem 2.3 is proved. □

Remark 2.4. Another approach, also involving the use of boundary triplets, to determine self-adjoint operator realizations of finite rank singular perturbations of the form $A_0 + G\alpha G^*$, where G is an injective linear mapping from $\mathbb{C}^n$ to $\mathfrak{H}_{-k}(A_0)$ was presented in [18, Section 4].

3. Admissible matrices and admissible operators

There are certain natural requirements for the determination of the entries r_{jp} of the matrix $\mathbf{R}$ in (2.4). Indeed, if the subsapce

$$\mathcal{X} = \mathrm{span}\,\{\, \psi_j \,:\, j = 1, \dots, n \,\} \tag{3.1}$$

has a nonzero intersection with $\mathfrak{H}_{-1}(A_0)$, then for any $\psi \in \mathcal{X} \cap \mathfrak{H}_{-1}(A_0)$, the corresponding element $h = (\mathbb{A}_0 + I)^{-1}\psi$ belongs to $\mathfrak{H}_1(A_0)$ and, hence, the functional $< \psi, \cdot >$ defined by (1.3) admits the following extension by continuity onto $\mathfrak{H}_1(A_0)$:

$$< \psi, f >= ((A_0 + I)^{1/2} f, (A_0 + I)^{1/2} h), \qquad \forall f \in \mathfrak{H}_1(A_0).$$

In order to preserve such natural extensions of $< \psi, \cdot >$ onto $\mathcal{D}(A^*_{\mathrm{sym}}) \cap \mathfrak{H}_1(A_0)$ in the definition (2.4), the concept of admissible matrices $\mathbf{R}$ as introduced in [6] is used.

Definition 3.1. *A Hermitian matrix $\mathbf{R} = (r_{jp})^n_{j,p=1}$ is called admissible for the regularization $\mathbb{A}_{\mathbf{R}}$ of (1.4) if its entries r_{jp} are chosen in such a way that if a singular element $\psi = c_1\psi_1 + \cdots + c_n\psi_n$ belongs to $\mathfrak{H}_{-1}(A_0)$, then for all $f \in \mathcal{D}(A^*_{\mathrm{sym}}) \cap \mathfrak{H}_1(A_0)$*

$$< \psi^{\mathrm{ex}}, f >= \sum_{j=1}^{n} c_j < \psi_j^{\mathrm{ex}}, f >= ((A_0 + I)^{1/2} f, (A_0 + I)^{1/2} h), \tag{3.2}$$

where $< \psi_j^{\mathrm{ex}}, f >$ are defined by (2.4) and $h = (\mathbb{A}_0 + I)^{-1}\psi$.

It is convenient to describe the set of admissible matrices in terms of a certain associated operators. In fact, it follows from Lemma 2.2, relations (2.7), and the general theory of boundary triplets [19], [31] that the operator

$$\widetilde{A} := A^*_{\mathrm{sym}} \restriction_{\mathcal{D}(\widetilde{A})}, \quad \mathcal{D}(\widetilde{A}) = \ker\, \Gamma_0 = \{\, f \in \mathcal{D}(A^*_{\mathrm{sym}}) \,:\, -\mathbf{R}\widehat{\Gamma}_0 f = \widehat{\Gamma}_1 f \,\}, \tag{3.3}$$

is a self-adjoint extension of A_{sym} and the choice of an admissible matrix $\mathbf{R}$ in (2.4) *is equivalent* to the choice of $\widetilde{A}$ defined by (3.3).

Definition 3.2. *An operator $\widetilde{A}$ is called admissible for the regularization of (1.4) if $\widetilde{A}$ is defined by (3.3) with an admissible matrix $\mathbf{R}$.*

Since $\mathbf{R}$ is Hermitian, Definition 3.2 implies that $\widetilde{A}$ is a self-adjoint extension of A_{sym}. In general, an admissible operator $\widetilde{A}$ need not be nonnegative. It is nonnegative if and only if

$$(A_F + I)^{-1} \le (\widetilde{A} + I)^{-1} \le (A_N + I)^{-1}, \tag{3.4}$$

where A_F is the Friedrichs extension and A_N is the Krein-von Neumann extension of A_{sym} (see e.g. [29], [24] and the references therein).

The next lemma gives some useful facts concerning the (unperturbed) nonnegative self-adjoint operator A_0 and its relation to the Friedrichs extension A_F of A_{sym}. They can be considered to be well known from the extension theory of nonnegative operators, therefore details for the present formulations with their proofs are left to the reader; see e.g. [9], [20], [24], [25], [29], [31].

Lemma 3.3. *Let $C = (A_0 + I)^{-1} - (A_F + I)^{-1}$ and let $S_0 = A_0 \cap A_F$. Moreover, denote $\mathcal{H} = \ker\,(A^*_{\mathrm{sym}} + I)$ and $\mathcal{H}' = \ker\,(S_0^* + I)$. Then:*

(i) $\overline{\mathcal{R}(C)} = \mathcal{H}'$;

(ii) $\ker C = \operatorname{ran}(S_0+I) = \operatorname{ran}(A_{\rm sym}+I)\oplus\mathcal{H}''$, *where* $\mathcal{H}''=\mathcal{H}\ominus\mathcal{H}'$;
(iii) $\mathcal{R}(C^{1/2}) = \operatorname{dom} A_0^{1/2}\cap\mathcal{H} = \operatorname{dom} A_0^{1/2}\cap\mathcal{H}'$;
(iv) $\mathcal{D}(A_0^{1/2}) = \mathcal{D}(A_F^{1/2})\dot{+}\mathcal{R}(C^{1/2})$.

Using the spaces introduced in (1.1) one can rewrite the decomposition in part (iv) of Lemma 3.3 as follows:

$$\mathfrak{H}_1(A_0) = \mathcal{D}\oplus_1\mathcal{R}(C^{1/2}), \tag{3.5}$$

where $\mathcal{D}\,(=\mathcal{D}(A_F^{1/2}))$ stands for the completion of $\mathcal{D}(A_{\rm sym})$ in the Hilbert space $\mathfrak{H}_1(A_0)$ and $\oplus_1$ denotes the orthogonal sum in $\mathfrak{H}_1(A_0)$.

The set of all admissible operators can now be characterized as follows.

Theorem 3.4. *A self-adjoint extension $\widetilde{A}$ of $A_{\rm sym}$ is an admissible operator for the regularization of (1.4) if and only if $\widetilde{A}$ is transversal to A_0 (i.e., $\mathcal{D}(A_0)+\mathcal{D}(\widetilde{A}) = \mathcal{D}(A^*_{\rm sym})$) and*

$$\mathcal{D}(\widetilde{A})\cap\mathfrak{H}_1(A_0)\subset\mathcal{D}(A_F), \tag{3.6}$$

where A_F is the Friedrichs extension of $A_{\rm sym}$.

Proof. Assume that the self-adjoint extension $\widetilde{A}$ of $A_{\rm sym}$ is transversal to A_0 and satisfies the condition (3.6). In view of (2.6) $\mathcal{D}(A_0)=\ker\widehat{\Gamma}_0$. Therefore transversality of $\widetilde{A}$ and A_0 is equivalent to the representation of $\mathcal{D}(\widetilde{A})$ in the form (3.3) with an $n\times n$ Hermitian matrix $\mathbf{R}$ (here $A_{\rm sym}$ has finite defect numbers (n,n)), cf. [19], [20, Proposition 1.4]. By Lemma 3.3, see also (3.5), one can write

$$\mathfrak{H}_1(A_0)=\mathcal{D}\oplus_1\mathcal{H}',\quad \mathcal{H}'=\mathcal{H}\cap\mathfrak{H}_1(A_0)=(\mathbb{A}_0+I)^{-1}[\mathcal{X}\cap\mathfrak{H}_{-1}(A_0)], \tag{3.7}$$

where $\mathcal{X}$ is as in (3.1). Since

$$\mathcal{D}(A_F)=\mathcal{D}\cap\mathcal{D}(A^*_{\rm sym}), \tag{3.8}$$

equality (3.7) shows that the condition (3.6) is equivalent to the relation

$$((A_0+I)^{1/2}\tilde{f},(A_0+I)^{1/2}h)=0,\quad \forall\tilde{f}\in\mathcal{D}(\widetilde{A})\cap\mathfrak{H}_1(A_0),\quad \forall h\in\mathcal{H}'. \tag{3.9}$$

Now it is shown that $\mathbf{R}$ is an admissible matrix in the sense of Definition 3.1 by verifying (3.2) for all $\psi\in\mathcal{X}\cap\mathfrak{H}_{-1}(A_0)$. Observe, that the mapping Γ_0 defined in Lemma 2.2, see also (2.7), determines the extended functionals $<\psi_j^{\rm ex},f>$ in (2.4).

The transversality of $\widetilde{A}$ and A_0 yields the following decomposition for the elements $f\in\mathcal{D}(A^*_{\rm sym})$:

$$f=\tilde{f}+u, \tag{3.10}$$

where $\tilde{f}\in\mathcal{D}(\widetilde{A})$ and $u\in\mathcal{D}(A_0)$ are uniquely determined modulo $\mathcal{D}(A_{\rm sym})$. If $\psi=\sum_{j=1}^n c_j\psi_j\in\mathfrak{H}_{-1}(A_0)$, then by (3.7) $h=(\mathbb{A}_0+I)^{-1}\psi\in\mathcal{H}'$. Now with $f\in\mathcal{D}(A^*_{\rm sym})\cap\mathfrak{H}_1(A_0)$ decomposed as in (3.10) one obtains:

$$\begin{aligned}<\psi^{\rm ex},f> &= \sum_{j=1}^n c_j<\psi_j^{\rm ex},f>=\mathbf{c}\Gamma_0 f\stackrel{(3.10)}{=}\mathbf{c}\Gamma_0(\tilde{f}+u)\\ &\stackrel{(2.7)}{=}\mathbf{c}(\widehat{\Gamma}_1+\mathbf{R}\widehat{\Gamma}_0)u=\mathbf{c}\widehat{\Gamma}_1 u\stackrel{(2.6)}{=}<\psi,u>\stackrel{(1.3)}{=}((A_0+I)u,h)\end{aligned} \tag{3.11}$$

where $\mathbf{c}:=(c_1,\ldots,c_n)$. On the other hand, it follows from (3.9) that

$$((A_0+I)^{1/2}f,(A_0+I)^{1/2}h)=((A_0+I)^{1/2}(\tilde{f}+u),(A_0+I)^{1/2}h)=((A_0+I)u,h),$$

which combined with (3.11) proves (3.2). Thus, $\mathbf{R}$ is an admissible matrix and $\widetilde{A}$ is an admissible operator.

Conversely, assume that $\widetilde{A}$ is an admissible operator. Then the relation (3.3) ensures the transversality of $\widetilde{A}$ and A_0 and $\mathbf{R}$ determines the extended functionals $<\psi_j^{\rm ex}, f>$ via (2.4). Reasoning as in (3.11)) it is seen that (3.2) implies

$$0 = ((A_0+I)^{1/2}f, (A_0+I)^{1/2}h) - <\psi^{\rm ex}, f> = ((A_0+I)^{1/2}\tilde{f}, (A_0+I)^{1/2}h)$$

for all $f \in \mathcal{D}(A^*_{\rm sym}) \cap \mathfrak{H}_1(A_0)$ and $h \in \mathcal{H}'$. Thus, the relation (3.9) and, equivalently, the relation (3.6) is satisfied. Theorem 3.4 is proved. $\square$

For some further study of admissible operators the following lemma is needed.

Lemma 3.5. *Let $\widetilde{\mathcal{H}}$ be a subspace of $\mathcal{H} = \ker\,(A^*_{\rm sym}+I)$. Then the symmetric operator*

$$S = A_F \restriction_{\mathcal{D}(S)}, \quad \mathcal{D}(S) = (A_F+I)^{-1}[\mathcal{R}(A_{\rm sym}+I) \oplus \widetilde{\mathcal{H}}] \tag{3.12}$$

satisfies the relations

$$\mathcal{D}(S) \cap \mathcal{D}(A_0) = \mathcal{D}(A_{\rm sym}) \quad \text{and} \quad \mathcal{D}(S) + \mathcal{D}(A_0) = \mathcal{D}(A_F) \dot{+} \mathcal{H}' \tag{3.13}$$

if and only if

$$\dim \widetilde{\mathcal{H}} = \dim \mathcal{H}' \quad \text{and} \quad \widetilde{\mathcal{H}} \cap \mathcal{H}'' = \{0\}, \tag{3.14}$$

where $\mathcal{H}' = \mathcal{H} \cap \mathfrak{H}_1(A_0)$ and $\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}'$. In this case, the domain of S admits the following description:

$$\mathcal{D}(S) = \mathcal{D}(A_{\rm sym}) \dot{+} \{\, u + h' : \, h' \in \mathcal{H}', \quad u = u(h')\,\}, \tag{3.15}$$

where $u = u(h') \in \mathcal{D}(A_0)$ can be (uniquely) determined from $h' \in \mathcal{H}'$; in particular, u satisfies the relation

$$((A_0+I)u, \widetilde{h}^\perp) = <\psi, u> = 0, \quad \forall \widetilde{h}^\perp \in \mathcal{H} \ominus \widetilde{\mathcal{H}}, \quad \psi = (\mathbb{A}_0+I)\widetilde{h}^\perp. \tag{3.16}$$

Proof. Denote $S_0 := A_F \cap A_0$. By Lemma 3.3

$$\mathcal{D}(S_0) = (A_0+I)^{-1}[\mathcal{R}(A_{\rm sym}+I) \oplus \mathcal{H}''] = (A_F+I)^{-1}[\mathcal{R}(A_{\rm sym}+I) \oplus \mathcal{H}''], \tag{3.17}$$

where $\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}'$. Comparing (3.12) and (3.17), one concludes that

$$\mathcal{D}(S) \cap \mathcal{D}(A_0) = \mathcal{D}(S) \cap \mathcal{D}(S_0) = (A_F+I)^{-1}[\mathcal{R}(A_{\rm sym}+I) \oplus (\widetilde{\mathcal{H}} \cap \mathcal{H}'')].$$

Thus,

$$\mathcal{D}(S) \cap \mathcal{D}(A_0) = \mathcal{D}(A_{\rm sym}) \iff \widetilde{\mathcal{H}} \cap \mathcal{H}'' = \{0\}.$$

The relations (3.12) and (3.17) also show that

$$\mathcal{D}(S) + \mathcal{D}(A_0) = (A_F+I)^{-1}[\mathcal{R}(A_{\rm sym}+I) \oplus (\widetilde{\mathcal{H}} \dot{+} \mathcal{H}'')] + (A_0+I)^{-1}\mathcal{H}'. \tag{3.18}$$

Here $(A_0+I)^{-1}\mathcal{H}'$ can be represented as

$$(A_0+I)^{-1}\mathcal{H}' = \{\, (A_F+I)^{-1}h' + Ch' : \, h' \in \mathcal{H}'\,\}, \tag{3.19}$$

where $C = (A_0+I)^{-1} - (A_F+I)^{-1}$. It follows from Lemma 3.3 that

$$\mathcal{R}(C) = \mathcal{H}', \quad \ker C = \operatorname{ran}(A_{\rm sym}+I) \oplus \mathcal{H}''. \tag{3.20}$$

Relations (3.18), (3.19), and (3.20) show that the second identity in (3.13) holds if and only if an arbitrary element $h' \in \mathcal{H}'$ admits the representation $h' = \widetilde{h} + h''$, $\widetilde{h}(\neq 0) \in \widetilde{\mathcal{H}}$, $h'' \in \mathcal{H}''$. Since $\widetilde{\mathcal{H}} \cap \mathcal{H}'' = \{0\}$, this representation is possible only in the case where $\dim \widetilde{\mathcal{H}} = \dim \mathcal{H}'$.

The definition (3.12) shows that $\mathcal{D}(S) = \mathcal{D}(A_{\rm sym}) \dot{+} (A_F + I)^{-1}\widetilde{\mathcal{H}}$, where

$$(A_F + I)^{-1}\widetilde{\mathcal{H}} = \{\, (A_0 + I)^{-1}\widetilde{h} - C\widetilde{h} : \ \widetilde{h} \in \widetilde{\mathcal{H}}\,\}.$$

Since $\widetilde{\mathcal{H}}$ satisfies (3.14), it follows from (3.20) that $C\widetilde{\mathcal{H}} = \mathcal{H}'$. Now, setting $u = (A_0 + I)^{-1}\widetilde{h}$ and $h' = -C\widetilde{h}$, one obtains (3.15) and (3.16). Note that the preimage $\widetilde{h} = C^{-1}h' \in \widetilde{\mathcal{H}}$, and therefore also u, is uniquely determined by $h' \in \mathcal{H}'$, □

The next theorem gives a description of all admissible operators.

Theorem 3.6. *Let $\widetilde{A}$ be a self-adjoint extension of $A_{\rm sym}$ and let the symmetric operator $S := \widetilde{A} \cap A_F$ be represented as in* (3.12) *with some subspace $\widetilde{\mathcal{H}}$ of $\mathcal{H}$. Then the following statements are equivalent:*

(i) *$\widetilde{A}$ is an admissible operator for the regularization of* (1.4)*;*
(ii) *$\widetilde{A}$ is a self-adjoint extension of S transversal to the Friedrichs extension S_F of S and the subspace $\widetilde{\mathcal{H}}$ satisfies the conditions in* (3.14).

Proof. Let $\widetilde{A}$ be an admissible operator. Since $\widetilde{A}$ and A_0 are transversal, one has

$$\mathcal{D}(\widetilde{A}) \cap \mathcal{D}(A_0) = \mathcal{D}(A_{\rm sym}), \quad \mathcal{D}(\widetilde{A}) + \mathcal{D}(A_0) = \mathcal{D}(A_F) \dot{+} \mathcal{H} = \mathcal{D}(A^*_{\rm sym}). \tag{3.21}$$

The condition (3.6) is equivalent to

$$\mathcal{D}(\widetilde{A}) \cap \mathfrak{H}_1(A_0) = \mathcal{D}(\widetilde{A}) \cap \mathcal{D}(A_F) = \mathcal{D}(\widetilde{A} \cap A_F).$$

Thus, intersecting all parts of (3.21) with $\mathfrak{H}_1(A_0)$ one concludes that the relations (3.13) are true for $S = \widetilde{A} \cap A_F$. By Lemma 3.5, the subspace $\widetilde{\mathcal{H}}$ satisfies (3.14). Furthermore, since the Friedrichs extension S_F of S coincides with A_F, one gets $\mathcal{D}(S_F) \cap \mathcal{D}(\widetilde{A}) = \mathcal{D}(A_F) \cap \mathcal{D}(\widetilde{A}) = \mathcal{D}(S)$. This implies the transversality of S_F and $\widetilde{A}$. The implication (i) $\Rightarrow$ (ii) is proved.

Now, assume that (ii) is satisfied. Since $S \supset A_{\rm sym}$, the operator $\widetilde{A}$ is a self-adjoint extension of $A_{\rm sym}$. It follows from (3.12) that $\ker\,(S^* + I) = \mathcal{H} \ominus \widetilde{\mathcal{H}}$ and hence, $\mathcal{D}(S^*) = \mathcal{D}(S_F) + \ker\,(S^* + I) = \mathcal{D}(A_F) \dot{+} (\mathcal{H} \ominus \widetilde{\mathcal{H}})$. On the other hand, the transversality of S_F and $\widetilde{A}$ gives $\mathcal{D}(S^*) = \mathcal{D}(A_F) + \mathcal{D}(\widetilde{A})$. Therefore, $\mathcal{D}(A_F) + \mathcal{D}(\widetilde{A}) = \mathcal{D}(A_F) \dot{+} (\mathcal{H} \ominus \widetilde{\mathcal{H}})$. This equality together with the second relation in (3.13) yields

$$\begin{aligned} \mathcal{D}(A_0) + \mathcal{D}(\widetilde{A}) &= \mathcal{D}(S) + \mathcal{D}(A_0) + \mathcal{D}(\widetilde{A}) \\ &= (\mathcal{D}(A_F) \dot{+} \mathcal{H}') + \mathcal{D}(\widetilde{A}) \\ &= \mathcal{D}(A_F) \dot{+} \mathcal{H}' \dot{+} (\mathcal{H} \ominus \widetilde{\mathcal{H}}). \end{aligned} \tag{3.22}$$

The conditions (3.14) imply that $\mathcal{H}' \dot{+} (\mathcal{H} \ominus \widetilde{\mathcal{H}}) = \mathcal{H}$. Hence, (3.22) shows that $\mathcal{D}(A_0) + \mathcal{D}(\widetilde{A}) = \mathcal{D}(A_F) \dot{+} \mathcal{H} = \mathcal{D}(A^*_{\rm sym})$, i.e., $\widetilde{A}$ and A_0 are transversal. Furthermore, by Lemma 3.3, see also (3.8), $\mathcal{D}(A_F) \dot{+} \mathcal{H}' = \mathfrak{H}_1(A_0) \cap \mathcal{D}(A^*_{\rm sym})$. Now, employing the second relation in (3.13) one obtains

$$\mathcal{D}(\widetilde{A}) \cap \mathfrak{H}_1(A_0) = \mathcal{D}(\widetilde{A}) \cap (\mathcal{D}(S) + \mathcal{D}(A_0)) = \mathcal{D}(S) \dot{+} \mathcal{D}(A_{\rm sym}) = \mathcal{D}(S) \subset \mathcal{D}(A_F).$$

According to Theorem 3.4 this means that $\widetilde{A}$ is an admissible operator for the regularization of (1.4). Thus, the implication (ii) $\Rightarrow$ (i) is proved. □

Corollary 3.7. *If all the singular elements ψ_j in the formal expression (1.4) belong to $\mathfrak{H}_{-1}(A_0)$, then there exists a unique admissible operator for the regularization of (1.4) and it coincides with the Friedrichs extension A_F of $A_{\rm sym}$.*

Proof. Assume that $\psi_j \in \mathfrak{H}_{-1}(A_0)$ for all $j = 1, \dots, n$. Then $\mathcal{D}(A^*_{\rm sym}) \subset \mathfrak{H}_1(A_0)$ and $\mathcal{H}' = \mathcal{H}$. Let $\widetilde{A}$ be an admissible operator for the regularization of (1.4) and let $S = \widetilde{A} \cap A_F$. By Theorem 3.6 the corresponding subspace $\widetilde{\mathcal{H}}$ satisfies (3.14) in Lemma 3.5, so that $\widetilde{\mathcal{H}} = \mathcal{H}$. Now (3.12) gives $S = A_F$ and since $S = \widetilde{A} \cap A_F$, one concludes that $\widetilde{A} = A_F$. This completes the proof. □

Corollary 3.8. *If all the elements ψ_j in* (1.4) *are $\mathfrak{H}_{-1}(A_0)$-independent (i.e. $\mathcal{X} \cap \mathfrak{H}_{-1}(A_0) = \{0\}$), then every self-adjoint extension $\widetilde{A}$ of $A_{\rm sym}$ transversal to A_0 is admissible for the regularization of (1.4).*

Proof. The condition of $\mathfrak{H}_{-1}(A_0)$-independency means that $\mathcal{H}' = \{0\}$. In this case, only the zero subspace $\widetilde{\mathcal{H}} = \{0\}$ can satisfy (3.14). The corresponding operator S coincides with $A_{\rm sym}$. Moreover, since $\mathcal{H}' = \{0\}$, Lemma 3.3 shows that $S_F = A_F = A_0$. Thus, by Theorem 3.6, $\widetilde{A}$ is admissible if and only if it is transversal to A_0. □

The properties of admissible operators for the regularization of (1.4) is closely related to the transversality of the Friedrichs and the Krein-von Neumann extensions of $A_{\rm sym}$.

Theorem 3.9. *There exists a nonnegative admissible operator $\widetilde{A}$ for the regularization of (1.4) if and only if the Friedrichs extension A_F and the Krein-von Neumann extension A_N of $A_{\rm sym}$ are transversal.*

Proof. Let $\widetilde{A}$ be a nonnegative admissible operator. Then $\widetilde{A}$ is a nonnegative extension of $A_{\rm sym}$ and therefore $(\widetilde{A}+I)^{-1}$ satisfies the inequalities (3.4). Recall that transversality of self-adjoint extensions $\widetilde{A}_1$ and $\widetilde{A}_2$ of $A_{\rm sym}$ is equivalent to

$$[(\widetilde{A}_1 + I)^{-1} - (\widetilde{A}_2 + I)^{-1}]\mathcal{H} = \mathcal{H}, \tag{3.23}$$

(see e.g. [19]). Hence, if A_F and A_N are not transversal then $(A_F + I)^{-1}h = (A_N + I)^{-1}h$ for some nonzero $h \in \mathcal{H}$. Then nonnegativity of $\widetilde{A}$ and A_0 yields $(\widetilde{A}+I)^{-1}h = (A_0 + I)^{-1}h$ due to (3.4) (with similar inequalities for A_0), so that

$$[(\widetilde{A}+I)^{-1} - (A_0 + I)^{-1}]\mathcal{H} \subset \mathcal{H} \ominus < h >,$$

and by (3.23) $\widetilde{A}$ and A_0 cannot be transversal. This is a contradiction to the admissibility of $\widetilde{A}$. Thus A_F and A_N are transversal.

To prove the converse statement assume that A_F and A_N are transversal. Let $\widetilde{\mathcal{H}}$ be a subspace of $\mathcal{H}$, which satisfies (3.14) and let the symmetric operator S be defined by (3.12) in Lemma 3.5. Moreover, let $\widetilde{A}$ be the Krein-von Neumann extension of S. Clearly, $\widetilde{A}$ is a nonnegative self-adjoint extension of $A_{\rm sym}$. It remains to prove that the operator $\widetilde{A}$ is admissible for the regularization of (1.4). To see this, observe that the Friedrichs extension of S coincides with A_F. Then it follows from [11, Proposition 7.2] that the Friedrichs extension $S_F = A_F$ and the Krein-von Neumann extension $\widetilde{A}$ of S are transversal with respect to S. Therefore, by Theorem 3.6, $\widetilde{A}$ is an admissible operator. □

Observe that S in Theorem 3.9 is a restriction of the Friedrichs extension A_F of $A_{\rm sym}$. Since the admissible operator $\widetilde{A}$ constructed in Theorem 3.9 is the Krein-von Neumann extension of S it is a consequence of [11, Theorem 6.4] that $\widetilde{A}$ is an extremal extension of $A_{\rm sym}$ in the sense of the following definition

Definition 3.10. *[[10], [11]] A self-adjoint extension $\widetilde{A}$ of $A_{\rm sym}$ is called extremal if it is nonnegative and satisfies the condition*

$$\inf_{u\in\mathcal{D}(A_{\rm sym})}(\widetilde{A}(f-u),f-u)=0 \quad \text{for all} \quad f\in\mathcal{D}(\widetilde{A}).$$

Theorem 3.11. *Let the Friedrichs extension A_F and the Krein-von Neumann extension A_N of $A_{\rm sym}$ be transversal, and let S be defined by* (3.12) *and* (3.14). *Then among all self-adjoint extensions of S there exists a unique extremal admissible operator $\widetilde{A}$ for the regularization of (1.4).*

Proof. In view of Theorem 3.9, it suffices to show that the Krein-von Neumann extension $\widetilde{A}$ of S is the only extremal extension of $A_{\rm sym}$ which is admissible for the regularization of (1.4).

To prove this assume that $\widehat{A}$ is extremal and admissible. Then by [11, Theorem 6.4] $\widehat{A}$ as an extremal extension of $A_{\rm sym}$ is the Krein-von Neumann extension of the symmetric operator $\widehat{S}:=\widehat{A}\cap A_F$. Moreover, by Theorem 3.6 the admissibility of $\widehat{A}$ means that $\widehat{S}$ is determined via (3.12) where the corresponding subspace $\widehat{\mathcal{H}}$ satisfies (3.14).

Since $\widehat{A}$ is an extension of S, one has $S\subseteq\widehat{S}$ or, equivalently, $\widetilde{\mathcal{H}}\subseteq\widehat{\mathcal{H}}$, where the subspaces $\widetilde{\mathcal{H}}$ and $\widehat{\mathcal{H}}$ correspond to S and $\widehat{S}$ in (3.12). Now the first equality in (3.14) forces that $\widetilde{\mathcal{H}}=\widehat{\mathcal{H}}$ and hence $S=\widehat{S}$. Therefore, $\widehat{A}=\widetilde{A}$ and this completes the proof. □

Remark 3.12. The selection of a self-adjoint operator $\widetilde{A}$ transversal to the initial one A_0 (but without additional condition of admissibility, see (3.6)) is also a key point of the approach used in [12] to the determination of self-adjoint realizations of a formal expression A_0+V, where a singular perturbation V is assumed to be (in general) an unbounded self-adjoint operator $V:\mathfrak{H}_2(A_0)\to\mathfrak{H}_{-2}(A_0)$ such that $\ker V$ is dense in $\mathfrak{H}$. In this case, the regularization of A_0+V takes the form $A_{\mathcal{P},V}=\mathbb{A}_0+V\mathcal{P}$ and it is well defined on the domain

$$\mathcal{D}(A_{\mathcal{P},V})=\{\,f\in\mathcal{D}(A^*_{\rm sym}):\,\mathcal{P}f\in\mathcal{D}(V)\,\},$$

where $\mathcal{P}$ is the skew projection onto $\mathfrak{H}_2(A_0)$ in $\mathcal{D}(A^*_{\rm sym})$ that is uniquely determined by the choice of $\widetilde{A}$.

4. Singular perturbations with symmetries and uniqueness of admissible operators

According to (2.4) and (3.3) the regularization $\mathbb{A}_{\mathbf{R}}$ of (1.4) depends on the choice of an admissible operator $\widetilde{A}$. Apart from the case of form bounded singular perturbations, admissible operators are not determined uniquely, cf. Theorem 3.6. However, the uniqueness can be attained by imposing some extra assumptions motivated by the specific nature of the underlying physical problem.

In typical cases (see, e.q. [5], [6]), where the original operator A_0 and its singular perturbation $V=\sum_{i,j=1}^n b_{ij}<\psi_j,\cdot>\psi_i$ possess some symmetry properties with respect to a certain family of unitary operators $\mathfrak{U}$, the preservation of initial symmetries of ψ_j for the extended functionals $< \psi_j^{\rm ex}, \cdot >$ enables one to determine a unique admissible operator $\widetilde{A}$. In this section, we study this problem in an abstract framework.

4.1. **Preliminaries.** First some general facts concerning $p(t)$-homogeneous operators are given. Let the operator A in $\mathfrak{H}$ be $p(t)$-homogeneous with respect to a one-parameter family $\mathfrak{U} = \{U_t\}_{t\in\mathfrak{T}}$ of unitary operators acting on $\mathfrak{H}$, cf. Definition 1.1. The equality

$$(Au, u) = (U_t Au, U_t u) = p(t)(AU_t u, U_t u), \quad u \in \mathcal{D}(A),$$

shows that if $A \neq 0$ is symmetric (nonnegative), then $p(t) \in \mathbb{R} \setminus \{0\}$ (respectively $p(t) > 0$). Moreover, (1.8) and (1.9) imply that if $A \neq 0$ then

$$p(t)p(g(t)) = 1, \quad \forall t \in \mathfrak{T}, \tag{4.1}$$

where the function of conjugation $g(t) : \mathfrak{T} \to \mathfrak{T}$ is uniquely determined by the formula

$$U_{g(t)} = U_t^*, \quad t \in \mathfrak{T}. \tag{4.2}$$

If A is densely defined then the adjoint of A is a densely defined operator, which is $p(t)$-homogeneous, too.

Lemma 4.1. *Let A be a closed densely defined $p(t)$-homogeneous operator with respect to a family $\mathfrak{U} = \{U_t\}_{t\in\mathfrak{T}}$ (cf. Definition 1.1). Then also its adjoint A^* is $p(t)$-homogeneous with respect to $\mathfrak{U}$ and moreover for all $t \in \mathfrak{T}$ and all $z \in \mathbb{C}$,*

$$U_t(\ker\,(A - zI)) = \ker\,\,(p(t)A - zI)\,. \tag{4.3}$$

In particular, $\ker A$ *(as well as* $\ker A^*$*) is a reducing subspace for every U_t, $t \in \mathfrak{T}$.*

Proof. Since A is $p(t)$-homogeneous one has $U_t A = p(t)AU_t$ for all $t \in \mathfrak{T}$. As a unitary operator U_t is bounded with bounded inverse, and therefore, the previous equality is equivalent to

$$A^*U_t^* = p(t)U_t^*A^* \quad \iff \quad U_tA^* = p(t)A^*U_t, \qquad \forall t \in \mathfrak{T},$$

which means that A^* is $p(t)$-homogeneous with respect to $\mathfrak{U}$.

The assertion (4.3) is immediate from the $p(t)$-homogeneity of A: if $Ah = zh$ then $zU_th = U_tAh = p(t)AU_th$. Hence, $U_t(\ker\,(A - zI)) \subset \ker\,\,(p(t)A - zI)$ and if $A \neq 0$ the reverse inclusion is obtained by using (4.1). For $A = 0$ the equality (4.3) is trivial.

The last assertion follows from (4.3) with $z = 0$ and the assumption (1.8) concerning the family $\mathfrak{U}$. □

In the case that A is symmetric the formula (4.3) in Lemma 4.1 shows how the unitary operators U_t, $t \in \mathfrak{T}$, transform the defect subspaces $\ker\,(A^* - zI)$ of A.

Corollary 4.2. *Let A in Lemma 4.1 be nonnegative and $p(t)$-homogeneous with respect to the family $\mathfrak{U} = \{U_t\}_{t\in\mathfrak{T}}$, and let A_0 be a nonnegative selfadjoint extension of A. Then*

$$(p(t)A_0 + I)(A_0 + I)^{-1}U_t(\ker\,(A^* + I)) = \ker\,(A^* + I).$$

Proof. By Lemma 4.1 the adjoint A^* of A is also $p(t)$-homogeneous and (4.3) implies that

$$U_t(\ker\ (A^*+I)) = \ker\ \left(A^* + \frac{1}{p(t)}\,I\right).$$

Moreover, the equality

$$(p(t)A_0+I)(A_0+I)^{-1}\ker\ \left(A^* + \frac{1}{p(t)}\,I\right) = \ker\ (A^*_{\rm sym}+I)$$

is always satisfied for a nonnegative self-adjoint extensions A_0 of A. $\square$

Let the operator A_0 in (1.4) be $p(t)$-homogeneous with respect to a one-parameter family $\mathfrak{U} = \{U_t\}_{t\in\mathfrak{T}}$ of unitary operators acting on $\mathfrak{H}$ (see Definition 1.1). Define a family of self-adjoint operators on $\mathfrak{H}$ by

$$G_t = (p(t)A_0+I)(A_0+I)^{-1}, \quad t\in\mathfrak{T}. \tag{4.4}$$

Clearly, G_t is positive and bounded with bounded inverse for all $t\in\mathfrak{T}$. Moreover, it follows from (1.9) and (4.1) that

$$(A_0+I)^{-1}U_t = U_t(p(g(t))A_0+I)^{-1} \tag{4.5}$$

and

$$G_tU_t = U_tG^{-1}_{g(t)} = (G_{g(t)}U_{g(t)})^{-1}. \tag{4.6}$$

The definition of the norm on $\mathfrak{H}_{-2}(A_0)$ given in (1.2) and the identity

$$(A_0+I)^{-1}U_t = G_tU_t(A_0+I)^{-1}$$

show that for all $g\in\mathfrak{H}$

$$\|U_tg\|_{-2} \le \|G_t\|\,\|g\|_{-2}.$$

Hence, the operators U_t can be continuously extended to bounded operators $\mathbb{U}_t$ in $\mathfrak{H}_{-2}(A_0)$ and, furthermore,

$$(\mathbb{A}_0+I)^{-1}\mathbb{U}_t\psi = G_tU_t(\mathbb{A}_0+I)^{-1}\psi \tag{4.7}$$

for all $\psi\in\mathfrak{H}_{-2}(A_0)$ and $t\in\mathfrak{T}$. The equality (4.2) shows that $\mathbb{U}_t$ has a bounded inverse which satisfies $\mathbb{U}_t^{-1} = \mathbb{U}_{g(t)}$. The operator $\mathbb{U}_t$ can be characterized also as the dual mapping (adjoint) of $U_{g(t)}$ with respect to the form defined in (1.3). In fact, using (1.3), (1.9), (4.2), and (4.7), it is seen that the action of the functional $<\mathbb{U}_t\psi,\cdot>$ on the elements $u\in\mathfrak{H}_2(A_0)$ is determined by the formula

$$\begin{aligned} <\mathbb{U}_t\psi,u> &= ((A_0+I)u, G_tU_th) = (U_{g(t)}(p(t)A_0+I)u,h) \\ &= ((A_0+I)U_{g(t)}u,h) = <\psi, U_{g(t)}u>, \end{aligned} \tag{4.8}$$

where $h = (\mathbb{A}_0+I)^{-1}\psi$.

Now consider a singular element $\psi\in\mathfrak{H}_{-2}(A_0)$, cf. (1.4). The assumption that ψ is $\xi(t)$-invariant with respect to $\mathfrak{U}$, i.e. $\mathbb{U}_t\psi = \xi(t)\psi$ for all $t\in\mathfrak{T}$ (see Definition 1.2), implies some relations between the functions $\xi(t)$, $p(t)$, and $g(t)$.

Proposition 4.3. *Let the operator A_0 in (1.4) be $p(t)$-homogeneous with respect to the family $\mathfrak{U}$ and let $\psi\in\mathfrak{H}_{-2}(A_0)\setminus\mathfrak{H}$ be $\xi(t)$-invariant with respect to $\mathfrak{U}$. Then for all $t\in\mathfrak{T}$ one has*

$$\xi(t)\xi(g(t)) = 1 \tag{4.9}$$

and, moreover,

$$|\xi(t)| = 1 \ \textit{if}\ p(t)=1 \quad \textit{and} \quad \min\{1,p(t)\} < |\xi(t)| < \max\{1,p(t)\} \ \textit{if}\ p(t)\neq 1.$$

Proof. It follows from (1.10) and (4.7) that $\psi \in \mathfrak{H}_{-2}(A_0) \setminus \mathfrak{H}$ is $\xi(t)$-invariant with respect to $\mathfrak{U}$ if and only if

$$G_t U_t h = \xi(t) h, \quad \forall t \in \mathfrak{T}, \tag{4.10}$$

where $h = (\mathbb{A}_0 + I)^{-1}\psi$. This together with (4.6) implies that

$$h = (G_{g(t)}U_{g(t)})(G_t U_t)h = \xi(t) G_{g(t)} U_{g(t)} h = \xi(t)\xi(g(t))h,$$

which proves (4.9). Moreover, (4.10) shows that $|\xi(t)|\|h\| = \|G_t U_t h\|$. In particular, if $p(t) = 1$, then $G_t = I$ and $|\xi(t)|\|h\| = \|U_t h\| = \|h\|$ and, hence, $|\xi(t)| = 1$.

In the case where $p(t) \neq 1$ the formula for G_t in (4.4) with an evident reasoning leads to the following estimates

$$\alpha(t)\|h\| = \alpha(t)\|U_t h\| < \|G_t U_t h\| < \beta(t)\|U_t h\| = \beta(t)\|h\|,$$

where $\alpha(t) = \min\{1, p(t)\}$ and $\beta(t) = \max\{1, p(t)\}$. This completes the proof. □

4.2. **$p(t)$-homogeneous self-adjoint extensions of $A_{\rm sym}$.** Let the operator A_0 be $p(t)$-homogeneous with respect to the family $\mathfrak{U}$. In what follows all the singular elements ψ_j ($j = 1, \dots, n$) appearing in (1.4) are assumed to be $\xi_j(t)$-*invariant* with respect to $\mathfrak{U}$. In view of (1.10) and (4.8) the $\xi_j(t)$-invariance of ψ_j is equivalent to

$$\xi_j(t) < \psi_j, u > = < \psi_j, U_{g(t)} u >, \quad \forall u \in \mathfrak{H}_2(A_0), \quad \forall t \in \mathfrak{T}, \tag{4.11}$$

where the linear functionals $< \psi_j, \cdot >$ are defined by (1.3). This implies the following basic result.

Lemma 4.4. *Let A_0 be $p(t)$-homogeneous and let ψ_j be $\xi_j(t)$-invariant with respect to $\mathfrak{U}$, $j = 1, \dots, n$. Then the symmetric operator $A_{\rm sym}$ defined by (1.5) and its adjoint $A^*_{\rm sym}$ are also $p(t)$-homogeneous with respect to $\mathfrak{U}$.*

Proof. It follows from (1.5) and (4.8) that

$$< \psi_j, U_t u > = < \mathbb{U}_{g(t)} \psi_j, u > = \xi_j(g(t)) < \psi_j, u > = 0$$

for every $u \in \mathcal{D}(A_{\rm sym})$. Thus $U_t : \mathcal{D}(A_{\rm sym}) \to \mathcal{D}(A_{\rm sym})$ and hence by (1.9) $A_{\rm sym}$ is $p(t)$-homogeneous: $U_t A_{\rm sym} = p(t) A_{\rm sym} U_t$. By Lemma 4.1 also the adjoint $A^*_{\rm sym}$ is $p(t)$-homogeneous with respect to $\mathfrak{U}$. □

If the assumptions in Lemma 4.4 are satisfied, the defect subspace ker $(A^*_{\rm sym} + I)$ of $A_{\rm sym}$ is invariant under $G_t U_t$, see Corollary 4.2.

For the next result recall that if A is a nonnegative operator (or in general a nonnegative relation) in a Hilbert space $\mathfrak{H}$, then the Friedrichs extension A_F and the Krein-von Neumann extension A_N of A can be characterized as follows (see [9] for the densely defined case and [23], [24], [25] for the general case):

If $\{f, f'\} \in A^*$, then $\{f, f'\} \in A_F$ if and only if

$$\inf \left\{ \|f - h\|^2 + (f' - h', f - h) \, : \, \{h, h'\} \in A \right\} = 0. \tag{4.12}$$

If $\{f, f'\} \in A^*$, then $\{f, f'\} \in A_N$ if and only if

$$\inf \left\{ \|f' - h'\|^2 + (f' - h', f - h) \, : \, \{h, h'\} \in A \right\} = 0. \tag{4.13}$$

Lemma 4.5. *Let $A_{\rm sym}$ be $p(t)$-homogeneous with respect to $\mathfrak{U}$. Then the Friedrichs extension A_F and the Krein-von Neumann extension A_N of $A_{\rm sym}$ in (1.5) are also $p(t)$-homogeneous with respect to $\mathfrak{U}$. Moreover, $U_t(\mathcal{D}(A_F^{1/2})) \subset \mathcal{D}(A_F^{1/2})$ and $U_t(\mathcal{R}(A_N^{1/2})) \subset \mathcal{R}(A_N^{1/2})$ for all $t \in \mathfrak{T}$.*

Proof. By Lemma 4.1 $A^*_{\rm sym}$ is $p(t)$-homogeneous with respect to $\mathfrak{U}$. Hence, in view of (1.8) and (1.9), a self-adjoint extension $\widetilde{A}$ of $A_{\rm sym}$ is $p(t)$-homogeneous with respect to $\mathfrak{U}$ if and only if

$$U_t : \mathcal{D}(\widetilde{A}) \to \mathcal{D}(\widetilde{A}), \quad \forall t \in \mathfrak{T}. \tag{4.14}$$

To prove that A_F is $p(t)$-homogeneous with respect to $\mathfrak{U}$, assume that $f \in \mathcal{D}(A_F)$. Then $g = U_t f \in \mathcal{D}(A^*_{\rm sym})$ and there is a sequence $h_n \in \mathcal{D}(A_{\rm sym})$ attaining the infimum in (4.12). Then $U_t h_n \in \mathcal{D}(A_{\rm sym})$, $U_t h_n \to U_t f = g$, and

$$\left(A^*_{\rm sym}U_t f - A_{\rm sym}U_t h_n, U_t f - U_t h_n\right) = (p(g(t))\left(A^*_{\rm sym}f - A_{\rm sym}h_n, f - h_n\right) \to 0, \tag{4.15}$$

so that $g \in \mathcal{D}(A_F)$ by (4.12). Therefore, $U_t(\mathcal{D}(A_F)) \subset \mathcal{D}(A_F)$ and A_F is $p(t)$-homogeneous with respect to $\mathfrak{U}$.

To prove the $p(t)$-homogeneity of A_N assume that $f \in \mathcal{D}(A_N)$. Then again $g = U_t f \in \mathcal{D}(A^*_{\rm sym})$ and there is a sequence $h_n \in \mathcal{D}(A_{\rm sym})$ attaining the infimum in (4.13). In particular, $A_{\rm sym}h_n \to A^*_{\rm sym}f$, $U_t h_n \in \mathcal{D}(A_{\rm sym})$, and

$$A_{\rm sym}U_t h_n = p(g(t))U_t A_{\rm sym}h_n \to p(g(t))U_t A^*_{\rm sym}f = A^*_{\rm sym}U_t f = A^*_{\rm sym}g.$$

Moreover, (4.15) is satisfied. Therefore, (4.13) shows that $g \in \mathcal{D}(A_N)$. This proves that $U_t(\mathcal{D}(A_N)) \subset \mathcal{D}(A_N)$ and thus A_N is $p(t)$-homogeneous with respect to $\mathfrak{U}$.

Finally, recall that the domain $\mathcal{D} = \mathcal{D}(A_F^{1/2})$, see (3.7), can be characterized as the set of vectors $f \in \mathfrak{H}$ satisfying

$$h_n \to f, \quad (A_{\rm sym}(h_n - h_m), h_n - h_m) \to 0, \quad m, n \to \infty,$$

and the range $\mathcal{R}(A_N^{1/2})$ as the set of vectors $g \in \mathfrak{H}$ satisfying

$$A_{\rm sym}h_n \to g, \quad (A_{\rm sym}(h_n - h_m), h_n - h_m) \to 0, \quad m, n \to \infty,$$

with $h_n \in \mathcal{D}(A_{\rm sym})$. The last statement is clear from these characterizations using similar arguments as above with the sequence h_n. This completess the proof. □

According to (4.11) the $\xi_j(t)$-invariance of ψ_j can be described with the aid of the linear functionals $< \psi_j, \cdot >$ in (1.3). The next theorem shows that the preservation of the $\xi_j(t)$-invariance for the extended functionals $< \psi_j^{\rm ex}, \cdot >$ defined by (2.4) is closely related to the existence of $p(t)$-homogeneous self-adjoint extensions of $A_{\rm sym}$ transversal to A_0.

Theorem 4.6. *Let A_0 be $p(t)$-homogeneous, let $\psi_1, \dots, \psi_n$ be $\xi_j(t)$-invariant with respect to $\mathfrak{U}$, and let $< \psi_j^{\rm ex}, f >$ be defined by (2.4). Then the relations*

$$\xi_j(t) < \psi_j^{\rm ex}, f > = < \psi_j^{\rm ex}, U_{g(t)}f >, \quad 1 \le j \le n, \quad \forall t \in \mathfrak{T}, \tag{4.16}$$

are satisfied for all $f \in \mathcal{D}(A^*_{\rm sym})$ if and only if the corresponding self-adjoint operator $\widetilde{A}$ defined by (3.3) *is $p(t)$-homogeneous with respect to $\mathfrak{U}$.*

Proof. Denote

$$\mathbf{\Xi(t)} = \begin{pmatrix} \xi_1(t) & 0 & \dots & 0 \\ 0 & \xi_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \xi_n(t) \end{pmatrix}. \tag{4.17}$$

Then $\det \boldsymbol{\Xi}(\mathbf{t}) \neq 0$, $t \in \mathfrak{T}$, by Proposition 4.3, since ψ_i is $\xi_j(t)$-invariant with respect to $\mathfrak{U}$. By using (2.5) in Lemma 2.2 the conditions (4.16) can be rewritten as follows:

$$\Xi(t)\Gamma_0 f = \Gamma_0 U_{g(t)} f, \quad \forall f \in \mathcal{D}(A^*_{\rm sym}), \quad \forall t \in \mathfrak{T}. \tag{4.18}$$

Since $\mathcal{D}(\widetilde{A}) = \ker \Gamma_0$, (4.18) immediately implies that $U_t(\mathcal{D}(\widetilde{A})) \subset \mathcal{D}(\widetilde{A})$, cf. (4.2). Thus the relations (4.16) ensure the $p(t)$-homogeneity of $\widetilde{A}$ with respect to $\mathfrak{U}$.

Conversely, assume that $\widetilde{A}$ is $p(t)$-homogeneous with respect to $\mathfrak{U}$. According to (3.3), (4.2), and (4.14) this is equivalent to

$$-\mathbf{R}\widehat{\Gamma}_0 U_{g(t)} f = \widehat{\Gamma}_1 U_{g(t)} f, \quad \forall f \in \mathcal{D}(\widetilde{A}), \quad \forall t \in \mathfrak{T}. \tag{4.19}$$

Using (4.4), (4.9), and (4.10) it is seen that

$$\begin{aligned} U_{g(t)} h_j &= p(t) G_{g(t)} U_{g(t)} h_j + (I - p(t) G_{g(t)}) U_{g(t)} h_j \\ &= \frac{p(t)}{\xi_j(t)} h_j + (1 - p(t))(A_0 + I)^{-1} U_{g(t)} h_j, \end{aligned} \tag{4.20}$$

where $h_j = (\mathbb{A}_0 + I)^{-1}\psi_j$, $j = 1, \dots, n$. This expression and relations (2.6), (4.8) yield the following equalities for all $f = u + \sum_{j=1} \alpha_j h_j \in \mathcal{D}(A^*_{\rm sym})$ and $t \in \mathfrak{T}$:

$$\widehat{\Gamma}_0 U_{g(t)} f = p(t)\boldsymbol{\Xi}(\mathbf{t})^{-1}\widehat{\Gamma}_0 f, \quad \widehat{\Gamma}_1 U_{g(t)} f = \boldsymbol{\Xi}(\mathbf{t})\widehat{\Gamma}_1 f + (1 - p(t))\mathbf{G}^\top(\mathbf{t})\widehat{\Gamma}_0 f, \tag{4.21}$$

where $\mathbf{G}(\mathbf{t}) = ((h_i, U_t h_j))_{i,j=1}^n$. Now with $f \in \mathcal{D}(\widetilde{A})$ substituting these expressions into (4.19), using (3.3), and taking into account that $\widehat{\Gamma}_0(\mathcal{D}(\widetilde{A})) = \mathbb{C}^n$, one concludes that the $p(t)$-homogeneity of $\widetilde{A}$ is equivalent to the matrix equality

$$\boldsymbol{\Xi}(\mathbf{t})\mathbf{R} - p(t)\mathbf{R}\boldsymbol{\Xi}(\mathbf{t})^{-1} = (1 - p(t))\mathbf{G}^\top(\mathbf{t}), \quad \forall t \in \mathfrak{T}. \tag{4.22}$$

Finally, employing (2.7) and (4.21) it is easy to see that equality (4.22) is equivalent to (4.18). Therefore, the extended functionals $< \psi_j^{\rm ex}, \cdot >$ satisfy the relations (4.16). Theorem 4.6 is proved. □

Remark 4.7. In the particular case where $p(t) = t^\beta$ and $\xi(t) = t^\theta$ with $\beta, \theta \in \mathbb{R}$, another condition for the preservation of $\xi(t)$-invariance for $< \psi_j^{\rm ex}, \cdot >$ has been obtained in [5, Lemma 1.3.2].

By Theorem 4.6 the existence of extended functionals $< \psi_j^{\rm ex}, \cdot >$ for which the $\xi_j(t)$-invariance properties (4.16) are satisfied is equivalent to the existence of a $p(t)$-homogeneous self-adjoint extension $\widetilde{A}$ of $A_{\rm sym}$ transversal to A_0. Such type extensions can easy be described with the aid of the relation (4.22). Indeed, the proof of Theorem 4.6 shows that (4.22) is equivalent to the $p(t)$-homogeneity of $\widetilde{A}$. By rewriting (4.22) componentwise as follows

$$\beta_{ij}(t) r_{ij} = (1 - p(t))(h_j, U_t h_i), \quad \beta_{ij}(t) = \left(\xi_i(t) - \frac{p(t)}{\xi_j(t)}\right), \quad 1 \le i, j \le n, \tag{4.23}$$

one concludes that $\widetilde{A}$ is a $p(t)$-homogeneous self-adjoint extension of $A_{\rm sym}$ transversal to A_0 if and only if $\widetilde{A}$ is defined by (3.3) and the entries r_{ij} of $\mathbf{R}$ in (3.3) satisfy (4.23) for all $t \in \mathfrak{T}$.

In the case that $p(x) \equiv 1$, the right-hand side of (4.23) vanishes and (4.23) reduces to $\beta_{ij}(t) r_{ij} = 0$, $1 \le i, j \le n$. Moreover, by Proposition 4.3 $\beta_{ii}(t) \equiv 0$ and, therefore, the entries r_{ii} cannot be uniquely determined from (4.23). This implies the existence of infinitely many 1-homogeneous self-adjoint extensions of $A_{\rm sym}$ transversal to A_0.

Example 4.8. Let $\alpha > 0$ and let $\widetilde{A}$ be defined by

$$\widetilde{A}_\alpha = A^*_{\rm sym} \upharpoonright \mathcal{D}(\widetilde{A}_\alpha), \quad \mathcal{D}(\widetilde{A}_\alpha) = \mathcal{D}(A_{\rm sym}) \dot{+} \ker\,(A^*_{\rm sym} + \alpha I).$$

Then for all $\alpha > 0$, $\widetilde{A}_\alpha$ is a 1-homogeneous self-adjoint extensions of $A_{\rm sym}$ transversal to A_0.

In the case that $p(t) \not\equiv 1$, the next theorem shows that transversality of A_F and A_N is *a necessary condition* for the existence of $p(t)$-homogeneous self-adjoint extensions of $A_{\rm sym}$ transversal to A_0.

Theorem 4.9. *If $p(t) \neq 1$ at least for one point $t \in \mathfrak{T}$, and the Friedrichs A_F and the Krein-von Neumann A_N extensions of $A_{\rm sym}$ are not transversal, then $p(t)$-homogeneous self-adjoint extensions of $A_{\rm sym}$ transversal to A_0 do not exist.*

Proof. Assume that $\widetilde{A}$ is a self-adjoint $p(t)$-homogeneous extension of $A_{\rm sym}$ and that $p(t_0) \neq 1$ for $t_0 \in \mathfrak{T}$. It follows from (1.9) that

$$U_{t_0}(\widetilde{A} - \lambda I) = p(t_0)\left(\widetilde{A} - \frac{\lambda}{p(t_0)} I\right) U_{t_0}, \quad \lambda \in \mathbb{R}.$$

Now, if λ is a negative eigenvalue of $\widetilde{A}$, then the infinite series of negative numbers $\lambda/p(t_0)^n$, $n \in \mathbb{N}$, also are eigenvalues of $\widetilde{A}$, and this contradicts the assumption of finite defect indices of $A_{\rm sym}$. Hence, $\widetilde{A}$ is a nonnegative extension of $A_{\rm sym}$. Now it follows from Theorem 3.9 that, in the case where A_F and A_N are not transversal, $p(t)$-homogeneous self-adjoint extensions of $A_{\rm sym}$ transversal to A_0 do not exist. $\square$

4.3. **Uniqueness of $p(t)$-homogeneous admissible operators.** In this subsection the class of admissible operators is studied in further detail in the case that the additional condition of $\xi(t)$-invariance or $p(t)$-homogeneity is imposed for the regularization of the singular perturbations determined by (1.4).

The requirement of $\xi_j(t)$-invariance (4.16) for the extended functionals $< \psi_j^{\rm ex}, \cdot >$ is in a good agreement with the fact that certain elements ψ_j in (1.4) may belong to $\mathfrak{H}_{-1}(A_0)$ and, hence, the corresponding functionals $< \psi_j, \cdot >$ admit natural extensions (3.2) by continuity onto $\mathfrak{H}_1(A_0) \cap \mathcal{D}(A^*_{\rm sym})$. In particular, if the linear span $\mathcal{X}$ of $\{\psi_j\}_{j=1}^n$ belongs to $\mathfrak{H}_{-1}(A_0)$, then the extended functionals $< \psi_j^{\rm ex}, \cdot >$ are determined by continuity onto $\mathcal{D}(A^*_{\rm sym})$ and they automatically possess the property of $\xi_j(t)$-invariance (4.16), since $U_t \upharpoonright_{\mathcal{D}(A_0)}$ can be extended by continuity onto $\mathfrak{H}_1(A_0)$. In this case, the set of admissible operators consists of a unique element which coincides with the Friedrichs extension A_F (see Corollary 3.7) and this admissible operator is $p(t)$-homogeneous.

In the case that $\mathcal{X} \cap \mathfrak{H}_{-1}(A_0) \neq \mathcal{X}$ admissible operators for the regularization of (1.4) are not determined uniquely. In this subsection, it is shown that the natural assumption of $\xi_j(t)$-invariance for the extended functionals $< \psi_j^{\rm ex}, \cdot >$ in the regularization (1.6) of (1.4) allows one to select, in many cases, a unique admissible operator $\widetilde{A}$.

By Theorem 4.6 the $\xi_j(t)$-invariance of $< \psi_j^{\rm ex}, \cdot >$ is equivalent to the $p(t)$-homogeneity of the corresponding operator $\widetilde{A}$ defined by (3.3). Therefore, equivalently, the requirement of $p(t)$-homogeneity imposed on the set of admissible operators can be used to select a unique admissible operator $\widetilde{A}$.

Theorem 4.10. *Let A_F and A_N be transversal, let the operator S defined in (3.12) be $p(t)$-homogeneous for some choice of $\widetilde{\mathcal{H}}$ satisfying conditions* (3.14)*, and, assume that for every $\beta_{ij}(t)$ in (4.23), there exists at least one point $t_{ij} \in \mathfrak{T}$ such that $\beta_{ij}(t_{ij}) \neq 0$. Then there exists a unique $p(t)$-homogeneous admissible operator for the regularization of (1.4).*

Proof. Let $\widetilde{A}$ be the Krein-von Neumann extension of S. The second part of the proof of Theorem 3.9 shows that $\widetilde{A}$ is admissible operator. By Lemma 4.5, $\widetilde{A}$ is $p(t)$-homogeneous. Its uniqueness follows from the fact that condition $\beta_{ij}(t_{ij}) \neq 0$ ensures in view of (4.23) the uniqueness of $p(t)$-homogeneous self-adjoint extensions of $A_{\rm sym}$ transversal to A_0. □

The next statement contains some conditions for the $p(t)$-homogeneity of S which are convenient for applications.

Proposition 4.11. *Let A_0 and $A_{\rm sym}$ be $p(t)$-homogeneous with respect to $\mathfrak{U}$ and let the symmetric operator S be defined by (3.12), where $\widetilde{\mathcal{H}}$ satisfies (3.14) and let $\mathcal{Y} = (\mathbb{A}_0 + I)(\mathcal{H} \ominus \widetilde{\mathcal{H}})$. Then:*

(i) *S is $p(t)$-homogeneous if and only if $\mathcal{Y}$ is invariant under $\mathbb{U}_t$, $t \in \mathfrak{T}$, and*

$$(h', U_t\widetilde{h}^{\perp}) = 0, \quad \forall h' \in \mathcal{H}', \quad \forall \widetilde{h}^{\perp} \in \mathcal{H} \ominus \widetilde{\mathcal{H}}, \quad \forall t \in \mathfrak{T}_0 = \{\, t \in \mathfrak{T} : \ p(t) \neq 1\}. \tag{4.24}$$

(ii) *If G_tU_t, $t \in \mathfrak{T}$, is self-adjoint, then S with $\widetilde{\mathcal{H}} = \mathcal{H}'$ is $p(t)$-homogeneous if and only if* (4.24) *holds.*

(iii) *If $\mathcal{Y}$ is a linear span of some $\xi_j(t)$-invariant singular elements ψ_j in (1.4), then S is $p(t)$-homogeneous if and only if* (4.24) *holds.*

(iv) *S is $p(t)$-homogeneous if the singular elements ψ_j in (1.4) form an $\mathfrak{H}_{-1}(A_0)$-independent system.*

Proof. (i) By Lemmas ,4.1, 4.5 $A^*_{\rm sym}$ and A_F are $p(t)$-homogeneous with respect to $\mathfrak{U}$. By the definition (3.12) $S \subset A_F$ and hence $p(t)$-homogeneity of S is equivalent to the relation $U_{g(t)}(\mathcal{D}(S)) \subset \mathcal{D}(S)$ for all $t \in \mathfrak{T}$.

The definition (3.12) shows that ker $(S^* + I) = \mathcal{H} \ominus \widetilde{\mathcal{H}}$. Hence, if S is $p(t)$-homogeneous with respect to $\mathfrak{U}$ then $G_tU_t(\mathcal{H} \ominus \widetilde{\mathcal{H}}) = \mathcal{H} \ominus \widetilde{\mathcal{H}}$ by Corollary 4.2. According to (4.7) the subspace $\mathcal{H} \ominus \widetilde{\mathcal{H}}$ is invariant under G_tU_t if and only if $\mathcal{Y} = (\mathbb{A}_0 + I)(\mathcal{H} \ominus \widetilde{\mathcal{H}})$ is invariant under the operator $\mathbb{U}_t$, $t \in \mathfrak{T}$. Thus, if S is $p(t)$-homogeneous with respect to $\mathfrak{U}$ then $\mathcal{Y}$ is invariant under $\mathbb{U}_t$, $t \in \mathfrak{T}$.

Now let $f = h' + u \in \mathcal{D}(S)$ be decomposed as in Lemma 3.5, see (3.15), (3.16). The definition of S in (3.12) implies that

$$U_{g(t)}f \in \mathcal{D}(S) \iff ((A_F + I)U_{g(t)}f, \widetilde{h}^{\perp}) = 0, \quad \forall \widetilde{h}^{\perp} \in \mathcal{H} \ominus \widetilde{\mathcal{H}}. \tag{4.25}$$

It follows from (4.20) that

$$(A_F + I)U_{g(t)}f = (A^*_{\rm sym} + I)U_{g(t)}f = (1 - p(t))U_{g(t)}h' + (A_0 + I)U_{g(t)}u.$$

By taking (4.8) into account one obtains

$$\begin{aligned} ((A_F + I)U_{g(t)}f, \widetilde{h}^{\perp}) &= (1 - p(t))(U_{g(t)}h', \widetilde{h}^{\perp}) + ((A_0 + I)U_{g(t)}u, \widetilde{h}^{\perp}) \\ &= (1 - p(t))(h', U_t\widetilde{h}^{\perp}) + < \mathbb{U}_t\psi, u > . \end{aligned} \tag{4.26}$$

If $\mathcal{Y}$ is invariant under $\mathbb{U}_t$, $t \in \mathfrak{T}$, then $< \mathbb{U}_t\psi, u >= 0$ for all $f = h' + u \in \mathcal{D}(S)$. Now (4.25) and (4.26) show that $U_{g(t)}f \in \mathcal{D}(S)$ if and only if the condition (4.24) is satisfied. Therefore, S is $p(t)$-homogeneous if and only if $\mathcal{Y}$ is invariant under $\mathbb{U}_t$ and (4.24) holds.

(ii) Since A_0 and A_F are $p(t)$-homogeneous, the symmetric restriction $S_0 := A_F \cap A_0$ and its adjoint S_0^* are also $p(t)$-homogeneous, see Lemma 4.1. It follows from (3.17) that $f \in \mathcal{D}(S_0)$ if and only if $f \in \mathcal{D}(A_0)$ and

$$((A_0 + I)f, h') = 0, \quad \forall h' \in \mathcal{H}' = \mathcal{H} \cap \mathfrak{H}_1(A_0).$$

Hence, ker $(S_0^* + I) = \mathcal{H}'$ and $G_t U_t \mathcal{H}' = \mathcal{H}'$ for all $t \in \mathfrak{T}$ by Corollary 4.2. Similarly $G_t U_t \mathcal{H} = \mathcal{H}$ for all $t \in \mathfrak{T}$, since $A_{\rm sym}$ is $p(t)$-homogeneous. Therefore, if $G_t U_t$ is self-adjoint, then $\mathcal{H}$ and $\mathcal{H}'$ are reducing subspaces for the operators $G_t U_t$ and consequently $G_t U_t \mathcal{H}'' \subset \mathcal{H}''$ is satisfied for all $t \in \mathfrak{T}$. Then, according to (4.7), $\mathcal{Y} = (\mathbb{A}_0 + I)\mathcal{H}''$ is invariant under $\mathbb{U}_t$. Now the claim follows from part (i) with $\widetilde{\mathcal{H}} = \mathcal{H}'$ and $\mathcal{H} \ominus \widetilde{\mathcal{H}} = \mathcal{H}''$.

(iii) If $\mathcal{Y}$ has a basis formed by some $\xi_j(t)$-invariant singular elements ψ_j, then $\mathcal{Y}$ is invariant under $\mathbb{U}_t$, see (1.10). So, the statement is reduced to (i).

(iv) The assumption implies that $\mathcal{H}' = \{0\}$. Hence by Lemma 3.12 $S = A_{\rm sym}$ and S is $p(t)$-homogeneous. □

Example 4.12. *A general zero-range potential in* $\mathbb{R}$.

A one-dimensional Schrödinger operator corresponding to a general zero-range potential at the point $x = 0$ can be given by the expression

$$A_0 + b_{11} < \delta, \cdot > \delta(x) + b_{12} < \delta', \cdot > \delta(x) + b_{21} < \delta, \cdot > \delta'(x) + b_{22} < \delta', \cdot > \delta'(x),$$

where $A_0 = -d^2/dx^2$ $(\mathcal{D}(A_0) = W_2^2(\mathbb{R}))$ acts in $\mathfrak{H} = L_2(\mathbb{R})$, $\delta'(x)$ is the derivative of the Dirac δ-function (with support at 0).

In this case, $A_{\rm sym} = -d^2/dx^2 \restriction \{u(x) \in W_2^2(\mathbb{R}) : u(0) = u'(0) = 0\}$ and the corresponding Friedrichs and Krein-von Neumann extensions are transversal (see, e.g., [11]). The functions

$$(\mathbb{A}_0 + I)^{-1}\psi_1 = h'(x) = \frac{1}{2}\begin{cases} e^{-x}, & x > 0 \\ e^{x}, & x < 0 \end{cases},$$

$$(\mathbb{A}_0 + I)^{-1}\psi_2 = h''(x) = \frac{1}{2}\begin{cases} -e^{-x}, & x > 0 \\ e^{x}, & x < 0 \end{cases},$$

where $\psi_1 = \delta(x)$ and $\psi_2 = \delta'(x)$, form an orthogonal basis of $\mathcal{H} = \ker\,(A^*_{\rm sym} + I)$ such that $\mathcal{H}' =< h'(x) >$ and $\mathcal{H}'' =< h''(x) >$.

Define $\mathfrak{U} = \{U_t\}_{t \in [0,\infty)}$ as a collection of *the space parity operator* $U_0 f(x) = f(-x)$ $(f(x) \in L_2(\mathbb{R}))$ and the set of *scaling transformations* $U_t f(x) = \sqrt{t} f(tx)$, $t > 0$. In this case, A_0 is $p(t)$-homogeneous with respect to $\mathfrak{U}$, where $p(0) = 1$ and $p(t) = t^{-2}$ if $t > 0$. The elements ψ_j $(j = 1, 2)$ are $\xi_j(t)$-invariant, where $\xi_1(0) = 1$, $\xi_1(t) = t^{-1/2}$ $(t > 0)$ and $\xi_2(0) = -1$, $\xi_2(t) = t^{-3/2}$ $(t > 0)$. Furthermore, for such a choice of $\mathfrak{U}$, $\mathfrak{T}_0 = \{\, t \in [0, \infty) : \ p(t) \neq 1\} = (0, \infty)$ and

$$(h', U_t h'') = t^{1/2} \int_{-\infty}^{\infty} h'(x)\overline{h''(tx)}dx = 0, \qquad \forall t \in \mathfrak{T}_0.$$

Let us put $\widetilde{\mathcal{H}} = \mathcal{H}'$. Then $\mathcal{Y} = (\mathbb{A}_0 + I)\mathcal{H}'' =< \psi_2 >$ and part (iii) of Proposition 4.11 implies that the operator S defined by (3.12) is $p(t)$-homogeneous. Calculating $\beta_{ij}(t)$ in (4.23) for $\xi_1(t)$, $\xi_2(t)$, and $p(t)$ as given above, it is easy to see that $\beta_{ij}(0) \neq 0$ if $i \neq j$ and $\beta_{ii}(t) \neq 0$ for all $t > 0$. In this case, by Theorem 4.10 there exists a unique $p(t)$-homogeneous admissible operator $\widetilde{A}$ for the regularization of the one-dimensional Schrödinger operator with general zero-range potential.

To identify $\widetilde{A}$ it suffices to determine the entries r_{ij} of the corresponding admissible matrix $\mathbf{R}$ with the aid of (4.23):

For $t=0$, (4.23) takes the form $\begin{pmatrix} 0 & 2r_{12} \\ -2r_{21} & 0 \end{pmatrix} = 0$ and, hence, $r_{12}=r_{21}=0$. On the other hand, for $t>0$ calculating both sides of (4.23) leads to

$$t^{-1/2}(1-t)\begin{pmatrix} r_{11} & 0 \\ 0 & -r_{22} \end{pmatrix} = (1-t^{-2})\begin{pmatrix} \frac{\sqrt{t}}{8(1+t)} & 0 \\ 0 & \frac{\sqrt{t}}{8(1+t)} \end{pmatrix}$$

and thus $r_{11}=8, r_{22}=-8$. Substituting the coefficients r_{ij} in (2.4) results in the well-known extensions of $\delta(x)$ and $\delta'(x)$ onto $\mathcal{D}(A^*_{\mathrm{sym}}) = W_2^2(\mathbb{R}\backslash\{0\})$ (see [5]):

$$< \delta_{\mathrm{ex}}, f >= \frac{f(+0)+f(-0)}{2}, \qquad < \delta'_{\mathrm{ex}}, f >= -\frac{f'(+0)+f'(-0)}{2}.$$

The corresponding admissible operator $\widetilde{A}$ is the restriction of $-d^2/dx^2$ to

$$\mathcal{D}(\widetilde{A}) = \left\{ f(x) \in W_2^2(\mathbb{R}\backslash\{0\}) : \, -f(-0) = f(+0), \quad -f'(-0) = f'(+0) \right\}.$$

4.4. **The case of rank one singular perturbations.** In the case of rank one singular perturbations $A_0 + b < \psi, \cdot > \psi$ with A_0 $p(t)$-homogeneous and ψ $\xi(t)$-invariant, the condition for the existence of a unique $p(t)$-homogeneous admissible operator turns out to be particularly simple.

Proposition 4.13. *If $p(t_0) \neq \xi^2(t_0)$ at least for one point $t_0 \in \mathfrak{T}$ and the operators A_F and A_N do not coincide, then there exists a unique $p(t)$-homogeneous admissible operator $\widetilde{A}$ for the regularization of $A_0 + < \psi, \cdot > \psi$. Furthermore, if $\psi \in \mathfrak{H}_{-1}(A_0)$ then one has $\widetilde{A} = A_F$ and $A_0 = A_N$, and if $\psi \in \mathfrak{H}_{-2}(A_0) \setminus \mathfrak{H}_{-1}(A_0)$ then $\widetilde{A} = A_N$ and $A_0 = A_F$.*

Proof. If $p(t_0) \neq \xi^2(t_0)$, then $\beta(t_0) \neq 0$ in (4.23). This means that there exists only one $p(t)$-homogeneous self-adjoint extension of A_{sym} that is different from A_0. By Lemma 4.5, the extensions A_F and A_N are $p(t)$-homogeneous. Hence, one of them is $\widetilde{A}$ and the other one coincides with A_0.

If $\psi \in \mathfrak{H}_{-1}(A_0)$, then by Corollary 3.7 $\widetilde{A}$ coincides with A_F and consequently $A_0 = A_N$. If $\psi \in \mathfrak{H}_{-2}(A_0) \setminus \mathfrak{H}_{-1}(A_0)$, then (3.7) shows that $A_0 = A_F$ and, hence, $\widetilde{A} = A_N$. Proposition 4.13 is proved. □

Example 4.14. *A point interaction in $\mathbb{R}^n$ $(n = 1,2,3)$.*

Consider the following singular rank one perturbation

$$-\Delta + b < \delta, \cdot > \delta(x),$$

where $\delta(x)$ is the Dirac δ-function with support at 0 and $A_0 = -\Delta$ ($\mathcal{D}(A_0) = W_2^2(\mathbb{R}^n)$ is the Laplace operator in $\mathfrak{H} = L_2(\mathbb{R}^n)$.

The operator A_0 is t^{-2}-homogeneous with respect to the set of scaling transformations $\mathfrak{U} = \{U_t\}_{t\in(0,\infty)}$ in $L_2(\mathbb{R}^n)$, where $U_t f(x) = t^{n/2} f(tx)$. Moreover, the singular element $\psi = \delta$ is $t^{-n/2}$-invariant (cf. [5]).

If $n=1$, then $\delta(x) \in \mathfrak{H}_{-1}(A_0) = W_2^{-1}(\mathbb{R})$, and by Proposition 4.13 the operator A_0 is the Krein-von Neumann extension of

$$A_{\mathrm{sym}} = -d^2/dx^2 \restriction \{ u(x) \in W_2^2(\mathbb{R}) : \, u(0) = 0 \}. \tag{4.27}$$

The Friedrichs extension $\widetilde{A}$ has the domain

$$\mathcal{D}(\widetilde{A}) = \{ u(x) \in W_2^2(\mathbb{R} \setminus \{0\}) \cap W_2^1(\mathbb{R}) : \ u(0) = 0 \}.$$

If $n = 2$, then $p(t) = t^{-2} = \xi^2(t)$ for all $t > 0$ and hence Proposition 4.13 cannot be applied. In fact, in this case the Friedrichs extension A_F and the Krein-von Neumann extension A_N of $A_{\rm sym}$ in (4.27) are equal and they coincide with $-\Delta$. Theorem 4.9 shows that t^{-2}-homogeneous self-adjoint extensions of $A_{\rm sym}$ different from $A_0 = -\Delta$ do not exist.

If $n = 3$, then $\delta(x) \in W_2^{-2}(\mathbb{R}^3) \setminus W_2^{-1}(\mathbb{R}^3)$, and by Proposition 4.13 A_0 is the Friedrichs extension of

$$A_{\rm sym} = -\Delta \restriction \{ u(x) \in W_2^2(\mathbb{R}^3) : \ u(0) = 0 \}.$$

The Krein-von Neumann extension $\widetilde{A}$ of $A_{\rm sym}$ has the form:

$$\widetilde{A}f(x) = -\Delta u(x) - u(0)\frac{e^{-|x|}}{|x|}, \ \mathcal{D}(\widetilde{A}) = \{ f = u(x) + u(0)\frac{e^{-|x|}}{|x|} : \ u \in W_2^2(\mathbb{R}^3) \}.$$

and it coincides with the unique t^{-2}-homogeneous admissible operator. Another description of the Krein-von Neumann extension of $A_{\rm sym}$ obtained with the aid of the Fourier transformation can be founded in [13].

Using the following functionals introduced in [34]

$$\Phi_0(f) = \lim_{|x|\to 0} \mid x \mid f(x), \quad \Phi_1(f) = \lim_{|x|\to 0} \left(f(x) - \frac{\Phi_0(f)}{\mid x \mid} \right), \quad \forall f(x) \in \mathcal{D}(A_{\rm sym}^*),$$

one can rewrite the domains of the Friedrichs extension A_0 and Krein-von Neumann extension $\widetilde{A}$ of $A_{\rm sym}$ as follows:

$$\mathcal{D}(A_0) = \ \{ f \in \mathcal{D}(A_{\rm sym}^*) : \ \Phi_0(f) = 0 \}, \quad \mathcal{D}(\widetilde{A}) = \{ f \in \mathcal{D}(A_{\rm sym}^*) : \ \Phi_0(f) = \Phi_1(f) \}.$$

5. Operator-realizations in the case of singular perturbations with symmetries

In this section, self-adjoint operator realizations $A_{\mathbf{B}}$ of (1.4) given by formula (2.8) are studied under the condition that the unperturbed operator A_0 and the singular elements ψ_j in (1.4) are, respectively, $p(t)$-homogeneous and $\xi_j(t)$-invariant with respect to $\mathfrak{U}$. Moreover, it is assumed that the admissible operator $\widetilde{A}$ for the regularization of (1.4) is chosen to be $p(t)$-homogeneous.

5.1. Special classes of operator realizations.

Since the unperturbed operator A_0 and singular elements ψ_j in (1.4) possess different symmetries with respect to $\mathfrak{U}$, the operator realizations $A_{\mathbf{B}}$ of (1.4) preserving the initial symmetry of A_0 (i.e., the property of $p(t)$-homogeneity) can be interpreted as "transparent" with respect to singular perturbations considered in (1.4).

Theorem 5.1. *The operator $A_{\mathbf{B}}$ defined by (2.8) is $p(t)$-homogeneous if and only if the relations*

$$\xi_i(t)\xi_j(t) = p(t), \qquad \forall t \in \mathfrak{T}$$

hold for all indices $1 \leq i, j \leq n$ corresponding to non-zero entries b_{ij} of $\mathbf{B}$.

Proof. By Lemma 4.4, the operator $A^*_{\rm sym}$ is $p(t)$-homogeneous. Therefore, $A_{\mathbf{B}}$ is $p(t)$-homogeneous if and only if $U_{g(t)} : \mathcal{D}(A_{\mathbf{B}}) \to \mathcal{D}(A_{\mathbf{B}}), \quad \forall t \in \mathfrak{T}$. By (2.8), this relation can be rewritten as

$$\mathbf{B}\Gamma_0 U_{g(t)} f = \Gamma_1 U_{g(t)} f, \quad \forall t \in \mathfrak{T}, \quad \forall f \in \mathcal{D}(A_{\mathbf{B}}). \tag{5.1}$$

Since the admissible operator $\widetilde{A}$ is $p(t)$-homogeneous, the boundary operator Γ_0 satisfies (4.18) (see Theorem 4.6). Hence, $\mathbf{B}\Gamma_0 U_{g(t)} f = \mathbf{B}\Xi(t)\Gamma_0 f$. On the other hand, relations (2.7), (2.8), and (4.21) lead to $\Gamma_1 U_{g(t)} f = p(t)\Xi(\mathbf{t})^{-1}\mathbf{B}\Gamma_0 f$. The last two equalities and (5.1) show that the $p(t)$-homogeneity of $A_{\mathbf{B}}$ is equivalent to the matrix equality $\Xi(\mathbf{t})\mathbf{B}\Xi(\mathbf{t}) = p(t)\mathbf{B}$, $t \in \mathfrak{T}$. Rewriting this componentwise, one obtains the equalities $\xi_i(t)\xi_j(t)b_{ij} = p(t)b_{ij}$, $1 \le i,j \le n$. $\square$

Corollary 5.2. *If there exists a point $t_0 \in \mathfrak{T}$ such that $p(t_0) \neq 1$ and relations $\xi_i(t_0)\xi_j(t_0) = p(t_0)$ hold for all indices $1 \le i,j \le n$ corresponding to non-zero entries b_{ij} of $\mathbf{B}$, then $A_{\mathbf{B}}$ is a nonnegative operator.*

Proof. If the matrix $\mathbf{B}$ satisfies the conditions above, then $A_{\mathbf{B}}$ is $p(t)$-homogeneous with respect to the family $\mathfrak{U}_0 := \{\, U_t \in \mathfrak{U} : t \in \{t_0, g(t_0)\}\,\}$. Now, to complete the proof, it suffices to repeat the arguments of Theorem 4.9. $\square$

5.2. **The Weyl function and the resolvent formula.** Let $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$ be the boundary triplet of $A^*_{\rm sym}$ constructed in Lemma 2.2 and let $\widetilde{A}$ be a self-adjoint extension of $A_{\rm sym}$ defined by (3.3).

The γ-field $\gamma(z)$ and the Weyl function $\mathbf{M}(\mathbf{z})$ associated with the boundary triplet $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$ are defined by

$$\gamma(z) = (\Gamma_0 \upharpoonright \mathcal{H}_z)^{-1}, \qquad \mathbf{M}(\mathbf{z}) = \Gamma_1\gamma(z), \quad z \in \rho(\widetilde{A}). \tag{5.2}$$

Here $\mathcal{H}_z = \ker\,(A^*_{\rm sym} - zI)$, $z \in \mathbb{C}$, denote the defect subspaces of $A_{\rm sym}$. The mappings Γ_i are defined by (2.5) and $\mathbf{M}(\mathbf{z})$ is an $n \times n$-matrix function.

Theorem 5.3. *The operator $\widetilde{A}$ is $p(t)$-homogeneous with respect to $\mathfrak{U}$ if and only if for at least one point $z = z_0 \in \mathbb{C}\setminus\mathbb{R}$ (and then for all non-real points z) the Weyl function $\mathbf{M}(\mathbf{z})$ satisfies the relation*

$$p(t)\mathbf{M}(\mathbf{z}) = \Xi(\mathbf{t})\mathbf{M}(\mathbf{p}(\mathbf{t})\mathbf{z})\Xi(\mathbf{t}), \quad \forall t \in \mathfrak{T}, \tag{5.3}$$

where $\Xi(\mathbf{t})$ is defined by (4.17).

Proof. Let $f_z \in \mathcal{H}_z$, $z \in \mathbb{C}$. Then Lemma 4.1 and relation (4.1) imply

$$U_{g(t)} f_z \in \ker\,(A^*_{\rm sym} - \frac{z}{p(g(t))}I) = \ker\,(A^*_{\rm sym} - p(t)zI). \tag{5.4}$$

Furthermore, it follows from (2.7) and the proof Theorem 4.6 that the equality

$$\Gamma_1 U_{g(t)} f = p(t)\Xi(\mathbf{t})^{-1}\Gamma_1 f \tag{5.5}$$

is satisfied for all $f \in \mathcal{D}(A^*_{\rm sym})$. Putting $f = f_z \in \mathcal{H}_z$ in (5.5), recalling (5.4), and taking into account that $\mathbf{M}(\mathbf{z})\Gamma_0 f_z = \Gamma_1 f_z$, $z \in \mathbb{C}$ (see (5.2)), one can rewrite (5.5) as follows:

$$\mathbf{M}(\mathbf{p}(\mathbf{t})\mathbf{z})\Gamma_0 U_{g(t)} f_z = p(t)\Xi(\mathbf{t})^{-1}\mathbf{M}(\mathbf{z})\Gamma_0 f_z. \tag{5.6}$$

If the identity (5.3) holds for some non-real $z = z_0$, then (5.6) implies that

$$\Gamma_0 U_{g(t)} f = \Xi(\mathbf{t})\Gamma_0 f \tag{5.7}$$

for all $f = f_{z_0} \in \mathcal{H}_{z_0}$. Since $\mathbf{M}^*(\mathbf{z}) = \mathbf{M}(\overline{\mathbf{z}})$ [19] and hence, (5.3) holds for $\overline{z}_0$, the relation (5.7) is also true for $f = f_{\overline{z}_0} \in \mathcal{H}_{\overline{z}_0}$. Moreover, since ψ_j are $\xi_j(t)$-invariant, equalities (4.16) are satisfied for all $f \in \mathcal{D}(A_{\rm sym})$. This means that (5.7) holds for $f \in \mathcal{D}(A_{\rm sym})$. Consequently (5.7) is true on the domain $\mathcal{D}(A^*_{\rm sym}) = \mathcal{D}(A_{\rm sym})\dot{+}\mathcal{H}_{z_0}\dot{+}\mathcal{H}_{\overline{z}_0}$. By Theorem 4.6 this yields the $p(t)$-homogeneity of $\widetilde{A}$.

Conversely, assume that $\widetilde{A}$ is $p(t)$-homogeneous. In this case, (5.7) holds for all $f \in \mathcal{D}(A^*_{\rm sym})$ (see the proof of Theorem 4.6). But then, for all non-real z and all $f_z \in \mathcal{H}_z$,

$$\begin{aligned}\mathbf{M}(\mathbf{p}(\mathbf{t})\mathbf{z})\Xi(\mathbf{t})\Gamma_0 f_z &\stackrel{(5.7)}{=} \mathbf{M}(\mathbf{p}(\mathbf{t})\mathbf{z})\Gamma_0 U_{g(t)} f_z \stackrel{(5.4)}{=} \Gamma_1 U_{g(t)} f_z \\ &\stackrel{(5.5)}{=} p(t)\Xi(\mathbf{t})^{-1}\Gamma_1 f_z = p(t)\Xi(\mathbf{t})^{-1}\mathbf{M}(\mathbf{z})\Gamma_0 f_z\end{aligned}$$

that justifies (5.3). Theorem 5.3 is proved. □

Let $A_{\mathbf{B}}$ be a self-adjoint realization of (1.4) defined by (2.8). Then the resolvents of $A_{\mathbf{B}}$ and $\widetilde{A}$ are connected via Krein's formula

$$(A_{\mathbf{B}} - zI)^{-1} = (\widetilde{A} - zI)^{-1} + \gamma(z)(\mathbf{B} - \mathbf{M}(\mathbf{z}))^{-1}\gamma(\overline{z})^*, \quad z \in \rho(A_{\mathbf{B}}) \cap \rho(\widetilde{A}). \tag{5.8}$$

Moreover, $z \in \rho(A_{\mathbf{B}})$ if and only if $\det(\mathbf{B} - \mathbf{M}(\mathbf{z})) = 0$, cf. [19].

The explicit form of $\mathbf{M}(\mathbf{z})$ can be found as follows. By (2.7) it is easy to see that the Weyl functions $\mathbf{M}(\mathbf{z})$ and $\widehat{\mathbf{M}}(\mathbf{z})$ associated with the boundary triplets (2.5) and (2.6), respectively, are connected via the linear fractional transform

$$\mathbf{M}(\mathbf{z}) = -(\mathbf{R} + \widehat{\mathbf{M}}(\mathbf{z}))^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{5.9}$$

The boundary triplet (2.6) is one of the most used boundary triplets and the corresponding Weyl function $\widehat{\mathbf{M}}(\mathbf{z})$ is studied well. In particular, if the singular elements ψ_j in (1.4) form an orthonormal system in $\mathfrak{H}_{-2}$, then (see [19, Remark 4])

$$\widehat{\mathbf{M}}(\mathbf{z}) = (z+1)P_{\mathcal{H}}[I + (z+1)(A_0 - zI)^{-1}]P_{\mathcal{H}}.$$

By combining this relation with (5.9) one gets an explicit form for $\mathbf{M}(\mathbf{z})$.

6. Singular perturbations of the Schrödinger operator $\xi(t)$-invariant with respect to scaling transformations in $\mathbb{R}^3$

In this section we study spectral and scattering properties of operator realizations of the formal expression

$$-\Delta + \sum_{i,j=1}^{n} b_{ij} < \psi_j, \cdot > \psi_i, \qquad b_{ij} \in \mathbb{C}, \quad b_{ij} = \overline{b_{ji}}, \qquad n \in \mathbb{N}, \tag{6.1}$$

where elements $\psi_j \in W_2^{-2}(\mathbb{R}^3) \setminus L_2(\mathbb{R}^3)$ are $\xi_j(t)$-invariant with respect to the set of scaling transformations $\mathfrak{U} = \{U_t\}_{t\in(0,\infty)}$ ($U_t f(x) = t^{3/2} f(tx)$) in $L_2(\mathbb{R}^3)$ and the linear span $\mathcal{X}$ of $\{\psi_j\}_{j=1}^n$ satisfies the condition $\mathcal{X} \cap L_2(\mathbb{R}^3) = \{0\}$.

In the case of scaling transformations, it is easy to verify that the free Schrödinger operator $A_0 = -\Delta$, $(D(\Delta) = W_2^2(\mathbb{R}^3))$ is t^{-2}-homogeneous and the function of conjugation $g(t)$ (see (4.2)) has the form $g(t) = 1/t$ (see [5]).

The next theorem gives a description of all continuous functions $\xi(t)$ for which there exists at least one $\xi(t)$-invariant singular element $\psi \in W_2^{-2}(\mathbb{R}^3)$.

Theorem 6.1. *For a continuous function $\xi(t)$ defined on $(0,\infty)$ there exists at least one singular element $\psi \in W_2^{-2}(\mathbb{R}^3)\setminus L_2(\mathbb{R}^3)$ which is $\xi(t)$-invariant with respect to $\mathfrak{U}$ if and only if $\xi(t) = t^{-\alpha}$, where $0 < \alpha < 2$.*

Proof. Assume that for a given continuous function $\xi(t)$ there exists a $\xi(t)$-invariant element $\psi \in W_2^{-2}(\mathbb{R}^3)\setminus L_2(\mathbb{R}^3)$. Since $U_tU_p = U_pU_t = U_{tp}$ $(p>0, t>0)$, equality (1.10) yields that $\xi(t)\xi(p) = \xi(tp)$. This relation is possible only if $\xi(t) = 0$ or $\xi(t) = t^{-\alpha}$ $(\alpha \in \mathbb{R})$ (see, e.g. [21]). Furthermore, Proposition 4.3 enables one to restrict the set of possible functions $\xi(t)$ as follows: $\xi(t) = t^{-\alpha}$, where $0 < \alpha < 2$.

To complete the proof of Theorem 6.1 it suffices to construct $t^{-\alpha}$-invariant singular elements for any $0 < \alpha < 2$.

Fix $n(w) \in L_2(S^2)$, where $L_2(S^2)$ is a Hilbert space of functions sguare-integrable on the unit sphere S^2 in $\mathbb{R}^3$ and determine the functional $\frac{\hat{n}(w)}{|y|^{3/2-\alpha}} \in W_2^{-2}(\mathbb{R}^3)$ by the formula

$$(6.2)\quad <\frac{\hat{n}(w)}{|y|^{3/2-\alpha}}, u(x)> = \int_{\mathbb{R}^3} \frac{n(w)}{|y|^{3/2-\alpha}(|y|^2+1)}(|y|^2+1)\hat{u}(y)dy \ (y = |y|w \in \mathbb{R}^3);$$

here $\hat{u}(y) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{ix\cdot y}u(x)dx$ is the Fourier transformation of $u(x) \in W_2^2(\mathbb{R}^3)$.

It is easy to verify that

(6.3)
$$(U_{g(t)}u)^{\wedge}(y) = (U_{1/t}u)^{\wedge}(y) = \frac{1}{(2\pi t)^{3/2}}\int_{\mathbb{R}^3} e^{iy\cdot x}u(x/t)dx = U_t\hat{u}(y) = t^{3/2}\hat{u}(ty).$$

Using (6.2) and (6.3), one obtains

$$<\frac{\hat{n}(w)}{|y|^{3/2-\alpha}}, U_{g(t)}u> = t^{-\alpha} <\frac{\hat{n}(w)}{|y|^{3/2-\alpha}}, u>, \quad \forall u \in W_2^2(\mathbb{R}^3).$$

By (4.11) this means that the functional $\psi = \frac{\hat{n}(w)}{|y|^{3/2-\alpha}}$ is $t^{-\alpha}$-invariant with respect to scaling transformations $\mathfrak{U}$. Theorem 6.1 is proved. □

A more detailed study of functionals that are $t^{-\alpha}$-invariant with respect to scaling transformations leads to the conclusion that the set $\mathcal{L}_\alpha$ of all $t^{-\alpha}$-invariant singular elements $\psi \in W_2^{-2}(\mathbb{R}^3)\setminus L_2(\mathbb{R}^3)$ coincides with the infinite dimensional subspace

$$\mathcal{L}_\alpha = \left\{ \psi = \frac{\hat{n}(w)}{|y|^{3/2-\alpha}} : n(w) \in L_2(S^2) \right\} \quad \text{of} \quad W_2^{-2}(\mathbb{R}^3).$$

If all elements ψ_j in (6.1) belong to $\mathcal{L}_\alpha$ for a fixed α $(0<\alpha<2)$, i.e., if all ψ_j are $t^{-\alpha}$-invariant, then the singular perturbation $V = \sum_{i,j=1}^n b_{ij} < \psi_j, \cdot > \psi_i$ also possesses a certain symmetry with respect to $\mathfrak{U}$. Indeed, by means of (4.8), (1.10), and Proposition 4.3, it is easy to see that

$$(6.4)\qquad \mathbb{U}_tVu = \xi^2(t)VU_tu = t^{-2\alpha}VU_tu, \quad \forall u \in \mathfrak{H}_2(A_0), \quad t>0.$$

In this case, the singular potential V possesses a certain homogeneity property ($t^{-2\alpha}$-homogeneity) with respect to scaling transformations.

Note that the delta function $\delta(x)$ belongs to $\mathcal{L}_{3/2}$ and the elements of this space admit a particularly simple description as $\hat{n}(w)$, where $n(w)$ runs through $L_2(S^2)$.

In particular, if one chooses $n(w)$ in the form of spherical harmonic of zero order $Y_0(w) = 1/(2\pi)^{3/2}$, then $\overset{\wedge}{Y_0(w)} = \delta(x)$.

For this reason, the expression (6.1) where all $\psi_j \in \mathcal{L}_{3/2}$ can be considered as a generalization of the classical one-point interaction $-\Delta + b < \delta, \cdot > \delta$ in $\mathbb{R}^3$.

In what follows all singular elements ψ_j in (6.1) are assumed to be $t^{-3/2}$-invariant with respect to scaling transformations, i.e., $\psi_j = \overset{\wedge}{n_j(w)}$ $(n_j(w) \in L_2(S^2))$. The symmetric operator $A_{\text{sym}} = -\Delta_{\text{sym}}$ in (1.5) associated with (6.1) takes the form (6.5)

$$-\Delta_{\text{sym}} = -\Delta \restriction_{\mathcal{D}(\Delta_{\text{sym}})}, \ \mathcal{D}(\Delta_{\text{sym}}) = \{\, u(x) \in W_2^2(\mathbb{R}^3) : < \psi_j, u >= 0, \ 1 \leq j \leq n \,\}.$$

Comparing (1.3) and (6.2), one concludes that the functions h_j in (2.3)) have the form

$$h_j(x) = \overline{\left(\frac{n_j(w)}{|y|^2+1}\right)^{\wedge}(x)} = \left(\frac{\overline{n_j(w)}}{|y|^2+1}\right)^{\vee}(x), \tag{6.6}$$

where the symbol $^\vee$ denotes the inverse Fourier transformation.

It is easy to verify that $h_j(x) \in L_2(\mathbb{R}^3) \setminus W_2^1(\mathbb{R}^3)$ and hence, all the elements ψ_j in (6.1) are $W_2^{-1}(\mathbb{R}^3)$-independent. By Corollary 3.8 this means that any self-adjoint extension $-\widetilde{\Delta}$ of $-\Delta_{\text{sym}}$ transversal to $-\Delta$ is admissible for the regularization of (6.1). Imposing the additional restriction of t^{-2}-homogeneity of $-\widetilde{\Delta}$ with respect to scaling transformations allows one to select a unique admissible operator.

Proposition 6.2. *The Krein-von Neumann extension of $-\Delta_{\text{sym}}$ is the unique t^{-2}-homogeneous admissible operator for the regularization of (6.1)*

Proof. According to [33, Theorem 3.1], the Friedrichs and the Krein-von Neumann extensions of the symmetric operator

$$-\Delta_{\min} = -\Delta \restriction_{\mathcal{D}(\Delta_{\min})}, \ \mathcal{D}(\Delta_{\min}) = \{\, u(x) \in W_2^2(\mathbb{R}^3) : < \psi, u >= 0, \ \psi \in \mathcal{L}_{3/2} \,\}$$

are transversal and the Friedrichs extension of $-\Delta_{\min}$ coincides with $-\Delta$. Since $-\Delta_{\text{sym}}$ defined by (6.5) is, simultaneously, an extension of $-\Delta_{\min}$ and a restriction of $-\Delta$, Proposition 7.2 in [11] implies that the Friedrichs extension $-\Delta$ and the Krein-von Neumann extension $-\Delta_N$ of $-\Delta_{\text{sym}}$ are also transversal.

Since the singular elements ψ_j in (6.1) are $W_2^{-1}(\mathbb{R}^3)$-independent, the operator S defined by (3.12) coincides with $-\Delta_{\text{sym}}$ and Corollary 3.8 shows that the operator $-\Delta_N$ is admissible for the regularization of (6.1). By Lemma 4.5 this operator is t^{-2}-homogeneous with respect to the scaling transformations. Moreover, in our case, the coefficients $\beta_{ij}(t)$ in (4.23) have the form

$$\beta_{ij}(t) = t^{-1/2}(t^{-1} - 1) \tag{6.7}$$

and hence, $\beta_{ij}(t) \neq 0$ $(\forall t > 0, t \neq 1)$. This fact and (4.23) ensures the uniqueness of $-\Delta_N$ as a t^{-2}-homogeneous admissible operator for the regularization of (6.1). □

To describe the admissible operator $-\Delta_N$ one needs to determine the entries r_{ij} of the corresponding admissible matrix $\mathbf{R}$ in (3.3). To do this, it suffices to calculate the scalar products $(h_j, U_t h_i)$ in (4.23).

It follows from (6.3) and (6.6) that

$$\overline{U_t h_i(x)} = U_t \left(\frac{n_i(w)}{|y|^2+1}\right)^{\wedge}(x) = \left(U_{1/t}\frac{n_i(w)}{|y|^2+1}\right)^{\wedge}(x) = t^{1/2}\left(\frac{n_i(w)}{|y|^2+t^2}\right)^{\wedge}(x).$$

Hence,

$$(h_j, U_t h_i) = t^{1/2} \int_{\mathbb{R}^3} \frac{n_i(w)\overline{n_j(w)}}{(|y|^2+t^2)(|y|^2+1)}\, dy = \frac{\pi}{2}\frac{t^{1/2}}{1+t}(n_i, n_j)_{L_2(S^2)},$$

where $(n_i, n_j)_{L_2(S^2)} = \int_{S^2} n_i(w)\overline{n_j(w)}dw$ is the scalar product in $L_2(S^2)$. Substituting this expression into (4.23) and taking (6.7) into account, one gets $\mathbf{R} = \|r_{ij}\|_{i,j=1}^n$, where $r_{ij} = -\frac{\pi}{2}(n_i, n_j)_{L_2(S^2)}$. Furthermore, it follows from (1.3) and (6.2) (for $\alpha = 3/2$) that

$$r_{ij} = -\frac{\pi}{2}(n_i, n_j)_{L_2(S^2)} = -2(\overset{\wedge}{n_i}(w), \overset{\wedge}{n_j}(w))_{W_2^{-2}(\mathbb{R}^3)} = -2(\psi_i, \psi_j)_{W_2^{-2}(\mathbb{R}^3)}.$$

Hence, $\mathbf{R} = -2((\psi_i, \psi_j)_{W_2^{-2}(\mathbb{R}^3)})_{i,j=1}^n$, where $\psi_j = \overset{\wedge}{n_j}(w)$ are singular elements in (6.1). In particular, if ψ_j form an orthonormal system in $W_2^{-2}(\mathbb{R}^3)$, then $\mathbf{R} = -2\mathbf{E}$, where $\mathbf{E}$ stands for the identity matrix.

The following statement is a direct consequence of Theorem 2.3 and [32, Theorem 3] if one takes into account that $-\Delta$ is the Friedrichs extension of $-\Delta_{\text{sym}}$ and the Krein-von Neumann extension $-\Delta_N$ is defined by $\mathbf{R} = -2\mathbf{E}$ in (3.3).

Proposition 6.3. *Let the singular elements $\psi_j = \overset{\wedge}{n_j}(w)$ in (6.1) form an orthonormal system in $W_2^{-2}(\mathbb{R}^3)$. Then the self-adjoint operator realization $A_{\mathbf{B}} = -\Delta_{\mathbf{B}}$ of (6.1) defined by (2.8) is nonnegative if and only if* $\det(2\mathbf{B} - \mathbf{E}) \neq 0$ *and*

$$0 \leq \mathbf{B}(2\mathbf{B} - \mathbf{E})^{-1} \leq \frac{1}{2}\mathbf{E}.$$

Remark 6.4. A description of all nonnegative self-adjoint operator realizations of (1.4) presented in [32] is based on the extremal properties (3.4) of the Friedrichs and the Krein-von Neumann extensions and the specific form (see (2.5), (2.7)) of the boundary operators Γ_i. A general approach to the description of all nonnegative self-adjoint extensions of a symmetric operator has been proposed recently in [13].

Since here the singular potential V is t^{-3}-homogeneous with respect to scaling transformations (cf. (6.4) with $\alpha = 3/2$), it is natural to expect that the corresponding self-adjoint realizations $-\Delta_{\mathbf{B}}$ of (6.1) possess specific spectral and scattering properties, which appear due to the homogeneity of singular perturbations.

Theorem 6.5. *For any self-adjoint operator realization $A_{\mathbf{B}} = -\Delta_{\mathbf{B}}$ of (6.1) defined by (2.8), the following statements are true:*

(i) *the point spectrum $\sigma_p(-\Delta_{\mathbf{B}})$ has empty intersection with $\mathbb{R}_+$;*
(ii) *the spectrum of $-\Delta_{\mathbf{B}}$ consists of the set $[0, \infty)$ of absolutely continuous spectrum and at most n (counting multiplicities) negative eigenvalues;*
(iii) *if $-\Delta_{\mathbf{B}}$ is nonnegative, then the wave operators $W_\pm = \lim_{t\to\pm\infty} e^{-it\Delta_{\mathbf{B}}}e^{i\Delta t}$ exist and are unitary operators in $L_2(\mathbb{R}^3)$;*
(iv) *if $-\Delta_{\mathbf{B}}$ is nonnegative and the singular elements $\psi_j = \overset{\wedge}{n_j}(w)$ in (6.1) form an orthonormal system in $W_2^{-2}(\mathbb{R}^3)$, then the S-matrix*

$$\mathbb{S}_{(-\Delta_{\mathbf{B}}, -\Delta)} = FW_+^* W_- F^{-1}$$

(F is the Fourier transformation in $L_2(\mathbb{R}^3)$) of the Schrödinger equation $iu_t = -\Delta_{\mathbf{B}} u$ coincides with the multiplication operator by the boundary

value[2] $\mathbb{S}_{(-\Delta_{\mathbf{B}},-\Delta)}(\delta)$ *of the contractive operator-valued function*

$$\mathbb{S}_{(-\Delta_{\mathbf{B}},-\Delta)}(\lambda) = (\mathbf{E} - 2i\lambda\mathbf{B})(\mathbf{E} + 2i\lambda\mathbf{B})^{-1}, \quad \lambda \in \mathbb{C}_+$$

analytic in the upper half-plane $\mathbb{C}_+$.

Proof. Statements (i)–(iii) follow from [33, Theorem 3.3]. Since the Friedrichs extension of $-\Delta_{\mathrm{sym}}$ coincides with $-\Delta$, the explicit form of $\mathbb{S}_{(-\Delta_{\mathbf{B}},-\Delta)}(\lambda)$ is a direct consequence of [32, Section 4]. □

Remark 6.6. In [32] the expression for $\mathbb{S}_{(-\Delta_{\mathbf{B}},-\Delta)}(\lambda)$ in terms of the coefficients b_{ij} of the singular perturbation in (6.1) was obtained by using the Lax–Phillips method in the scattering theory. Another description of $\mathbb{S}_{(-\Delta_{\mathbf{B}},-\Delta)}(\lambda)$ in terms of parameters in the Krein's resolvent formula was obtained in [1]. In this case, the stationary approach in the scattering theory has been used.

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