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I. Existence of Eigenvalues for Integral Equations; II. A Collocation Method for Boundary Value Problems

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EXISTENCE OF EIGENVALUES FOR INTEGRAL EQUATIONS - RUSSELL

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I. EXISTENCE OF EIGENVALUES FOR INTEGRAL EQUATIONS;

II. A COLLOCATION METHOD FOR BOUNDARY VALUE PROBLEMS

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I. EXISTENCE OF EIGENVALUES FOR INTEGRAL EQUATIONS;

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BY

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B.S., University of New Mexico, 1967

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DISSERTATION

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ABSTRACT OF DISSERTATION

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Abstract

I. The existence of eigenvalues is shown for certain classes of integral equations with continuous kernels. A number of interesting and useful results are thereby treated in a unified and relatively elementary way. The simplicity of these new proofs make the results accessible to introductory courses on the theory of integral equations.

II. Collocation with piecewise polynomial functions is developed as a method for solving two-point boundary value problems. Convergence is shown for a general class of linear problems and a rather broad class of nonlinear problems. Some computational examples are presented to illustrate the wide applicability and efficiency of the procedure.

TABLE OF CONTENTS

I. EXISTENCE OF EIGENVALUES OF INTEGRAL EQUATIONS

1. Introduction. The concept of eigenvalues λ and eigenfunctions $\phi(x) \not\equiv 0$, solutions of

$$\lambda\phi(x) = \int_0^1 K(x,y)\phi(y)dy \,, \tag{1.1}$$

is central to the theory of integral equations. Nonetheless introductory treatments of the subject rarely show the existence of eigenvalues for other than Hermitian kernels. Even more advanced treatments such as [5] are limited to kernels rather similar to Hermitian kernels. But this still leaves many problems where the existence of eigenvalues is of interest because of their physical significance. Examples are the continuous kernels arising in the theory of lasers which we mention later. For some continuous kernels (1.1) has no nonzero eigenvalues, a simple example being $K(x,y) = \sin 2\pi x \cos 2\pi y$. However, there are useful sufficient conditions that do assure eigenvalues exist. The major results [6, pp. 1550-1552] for continuous kernels assert the existence of a non-zero eigenvalue if

(i) $K(x,y)$ is positive or

(ii) the trace of some iterated kernel of order three or higher is non-zero or

(iii) $K(x,y)$ satisfies a Hölder condition in either variable with exponent $\alpha > {}^1/2$ and the trace of some iterated kernel is non-zero [2,8,9]. We believe the reason these results are not often established is that they depend on relatively deep results of the Fredholm theory and on an understanding of the genus of entire functions, which is material not usually developed in courses today. The favored approaches to integral equations appear to be the determinant-free ones such as Schmidt's dissection procedure and the theory of compact operators. They are simpler than Fredholm's theory and say a great deal about eigenvalues and functions <u>if they exist</u> but do not actually show existence. Complementary existence results such as the ones mentioned obviously make these theories considerably more interesting and significant. The main objective of this part of the thesis is to derive existence and related results for eigenvalues in a way accessible to introductory courses on integral equations; the proofs are new and are of some interest in themselves.

A natural approach to the eigenvalue problem (1.1) would be to replace the integral by a Riemann sum

$$\lambda\phi(x) \doteq \frac{1}{n} \sum_{j=1}^{n} K(x,{}^j/n)\phi({}^j/n) \qquad (1.2)$$

or proceeding further, approximate (1.1) by a matrix problem

$$\lambda\phi({}^i/n) \doteq \frac{1}{n} \sum_{j=1}^{n} K({}^i/n,{}^j/n)\phi({}^j/n) \qquad i = 1,\ldots,n \, . \qquad (1.3)$$

Indeed, one of Hilbert's approaches to the eigenvalue problem for symmetric kernels proceeds in just this way. Results about the eigenvalues of matrices are relatively easy to obtain and it seems reasonable to attempt to establish the existence of eigenvalues of (1.1) from an approximating matrix problem. We shall do this and establish all three sufficient conditions mentioned above. Along with (i) the fundamental results of positive kernel theory are recovered. With the exception of tools from advanced calculus like Arzelà's theorem, all the theorems we use are found in intermediate-level matrix theory texts such as [1,3]. It is worth remarking that the result (ii) is sharp [8, p. 80] so that it is perhaps surprising our simple method suffices.

2. Preliminaries. We collect here a few definitions and those non-trivial results of matrix theory which we shall draw upon.

The p^{th} iterated kernel is defined recursively by

$$K^1(x,y) \equiv K(x,y)$$
$$K^p(x,y) = \int_0^1 K^{p-1}(x,z)K(z,y)dz \ .$$

By the trace of K we mean

$$\sigma_1 = \text{trace } K = \text{tr } K = \int_0^1 K(x,x)dx$$

and the higher order traces are

$$\sigma_p = \text{tr } K^p = \int_0^1 K^p(x,x)dx .$$

The $n \times n$ matrix $A = (a_{ij})$ is said to be positive if $a_{ij} > 0$ for all i,j. It is relatively simple to establish the Perron theory of positive matrices which, among other things, says A has a real eigenvalue $\rho > 0$ which is larger in magnitude than any other eigenvalue, there is an eigenvector v associated with ρ which has positive components, and

$$\min_i \sum_{j=1}^n a_{ij} \leq \rho \leq \max_i \sum_{j=1}^n a_{ij} . \tag{2.1}$$

The Euclidean, or Schur, norm of an $n \times n$ complex-valued matrix A is defined to be

$$\|A\|_E = (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2} .$$

If $\{\lambda_i\}$ are the eigenvalues of A, Schur's inequality states

$$\sum_{i=1}^n |\lambda_i|^2 \leq \|A\|_E^2 .$$

The trace of a matrix A is

$$\text{tr } A = \sum_{i=1}^n a_{ii} ,$$

and it is an elementary result that

$$\mathrm{tr}\, A^p = \sum_{i=1}^{n} \lambda_i^p . \qquad (2.2)$$

If the columns of A are $a^1, a^2, \ldots, a^n$, Hadamard's inequality is

$$|\det A| \leq \|a^1\| \cdot \|a^2\| \cdots \|a^n\|$$

where the Euclidean norm is used.

Except when specified to the contrary, $\omega(\delta)$ denotes a modulus of uniform continuity for a kernel $K(x,y)$ continuous on $0 \leq x, y \leq 1$ and M denotes

$$M = \max_{0 \leq x, y \leq 1} |K(x,y)| .$$

3. Existence Theorems. Positive kernels will be studied first. This is because the theory of positive matrices gives us a number of tools unavailable in the general case. Our proof, however, is general in nature and the succeeding results will just require more attention to certain details. It is convenient and interesting to go on to establish most of Jentzsch's results [9] for positive kernels, and we shall do so.

Theorem 1. Let the function $K(x,y)$ be continuous and positive for $0 \leq x, y \leq 1$. Then the integral equation

$$\lambda\phi(x) = \int_0^1 K(x,y)\phi(y)dy \ , \qquad 0 \leq x \leq 1$$

has an eigenvalue λ_1 which is positive, simple, larger in modulus than any other eigenvalue, and has an associated eigenfunction $\phi(x) > 0$ for $0 \leq x \leq 1$.

Proof. With (1.3) to motivate us, consider the matrix problem

$$\lambda^{(n)} g_i^{(n)} = \frac{1}{n} \sum_{j=1}^{n} K(^i/n, ^j/n) g_j^{(n)}, \qquad 1 \leq i \leq n \ , \tag{3.1}$$

and henceforth denote the matrix $K^{(n)} = (\frac{1}{n} K(^i/n, ^j/n))$. From the theory of positive matrices there is a real eigenvalue $\lambda_1^{(n)}$ which is largest in magnitude and it has a positive eigenvector $(\phi_i^{(n)})$. For reasons that will be clear later, normalize this vector so that

$$\frac{1}{n} \sum_{i=1}^{n} \phi_i^{(n)} = 1 \ . \tag{3.2}$$

Equation (1.2) suggests defining an approximate eigenfunction $\phi^{(n)}(x)$ by

$$\lambda_1^{(n)} \phi^{(n)}(x) = \frac{1}{n} \sum_{j=1}^{n} K(x, ^j/n) \phi_j^{(n)} \ . \tag{3.3}$$

Notice $\phi^{(n)}(^i/n) = \phi_i^{(n)}$ for $1 \leq i \leq n$.

Our aim is to show a subsequence of the $\lambda_1^{(n)}$ and the $\phi^{(n)}(x)$ converge to an eigenvalue λ_1 and eigenfunction $\phi(x)$ respectively. To prevent the definition (3.3) from collapsing in the limit, we shall need to show the eigenvalues bounded away from zero. Now the inequality (2.1) obviously implies

$$0 < m = \min_{0 \leq x,y \leq 1} K(x,y) \leq \lambda_1^{(n)} \leq M \tag{3.4}$$

independently of n. Because all $\lambda_1^{(n)}$ lie in $[m,M]$ we can extract a convergent subsequence n', $\lambda_1^{(n')} \to \lambda_1$ and $0 < m \leq \lambda_1 \leq M$.

Now we need a convergent subsequence of the $\phi^{(n')}(x)$. Uniform boundedness of this set follows directly from (3.2), (3.3), (3.4) which imply

$$0 < \frac{m}{M} \leq \phi^{(n')}(x) \leq \frac{M}{m}, \qquad 0 \leq x \leq 1 . \tag{3.5}$$

Moreover,

$$|\phi^{(n')}(x)-\phi^{(n')}(y)| = \frac{1}{n'\lambda_1^{(n')}} \left|\sum_{j=1}^{n'} [K(x,{}^j/n')-K(y,{}^j/n')]\phi_j^{(n')}\right|$$

$$\leq \frac{\omega(|x-y|)}{\lambda_1^{(n')}} \leq \frac{\omega(\delta)}{m}, \quad |x-y| \leq \delta . \tag{3.6}$$

This inequality holds for all n' so the set $\{\phi^{(n')}(x)\}$ is equicontinuous. Arzelà's theorem guarantees the existence of a

subsequence n'' such that $\phi^{(n'')}(x)$ converges uniformly to a continuous limit function $\phi(x)$. This limit function is non-trivial because of (3.5) but we shall see a more generally applicable reason in a moment.

To justify a passage to the limit from matrix to integral equation we need a lemma on the closeness of approximation.

Lemma 1. Let $f(x) \in C[0,1]$ and let $\omega(\delta)$ be a modulus of uniform continuity for $f(x)$. Then

$$\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{j=1}^{n} f(^j/n) \right| \leq \omega(^1/n) .$$

Proof.

$$\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{j=1}^{n} f(^j/n) \right| \leq \sum_{j=1}^{n} \left| \int_{\frac{j-1}{n}}^{^j/n} [f(x)-f(^j/n)]dx \right|$$

$$\leq \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{^j/n} \omega(^1/n)dx = \omega(^1/n) . \quad \blacksquare$$

If $\omega(\delta)$ is a modulus of continuity for $K(x,y)$, then from (3.5) and (3.6) we can easily show

$$\max_{|y-z| \leq \delta} |K(x,y)\phi^{(n')}(y)-K(x,z)\phi^{(n')}(z)| \leq \frac{2M}{m} \omega(\delta)$$

for all n'. Then Lemma 1 implies

$$\left|\int_0^1 K(x,y)\phi^{(n'')}(y)dy - \frac{1}{n''}\sum_{j=1}^{n''} K(x,{}^j/n'')\phi^{(n'')}({}^j/n'')\right| \leq \frac{2M}{m}\,\omega({}^1/n'') .$$

Combining this with (3.3) shows

$$\lambda_1^{(n'')}\phi^{(n'')}(x) - \int_0^1 K(x,y)\phi^{(n'')}(y)dy = \epsilon(n'') \tag{3.7}$$

where

$$|\epsilon(n'')| \leq \frac{2M}{m}\,\omega({}^1/n'')$$

uniformly in x. If now we let $n'' \to \infty$ in (3.7) and realize the order of limit and integration may be interchanged because of the uniformity of convergence, we have proved the existence of the positive eigenvalue λ_1 and its positive eigenfunction $\phi(x)$.

We already know $\phi(x)$ is non-trivial but the following argument to the same effect will generalize. Apply Lemma 1 to $\phi^{(n'')}(x)$ using the modulus of continuity of (3.6) and the normalization (3.2). Then

$$\left|\int_0^1 \phi^{(n'')}(x)dx - \frac{1}{n''}\sum_{j=1}^{n''} \phi^{(n'')}({}^j/n'')\right| = \left|\int_0^1 \phi^{(n'')}(x)dx - 1\right| \leq \frac{1}{m}\,\omega(\tfrac{1}{n''}). \tag{3.8}$$

Passing to the limit shows

$$\int_0^1 \phi^{(n'')}(x)dx \to \int_0^1 \phi(x)dx = 1 .$$

Our remaining task is to show λ_1 is simple and larger in modulus than any other eigenvalue. Applying the above procedure to the transposed kernel $K(y,x)$ shows that λ_1 is an eigenvalue of $K(y,x)$ with a positive eigenfunction $\phi_1'(x)$. This is because the matrix problem (3.1) for $K(y,x)$ has the matrix $K^{(n)T}$ and we know the matrices $K^{(n)}$, $K^{(n)T}$ have the same eigenvalues. Choosing then $\lambda_1^{(n')}$ as before we are led to the assertion. By definition

$$\lambda_1 \phi_1'(x) = \int_0^1 K(y,x)\phi_1'(y)dy \; .$$

If λ_k is any other eigenvalue of $K(x,y)$, $\lambda_k \neq \lambda_1$, and $\phi_k(x)$ its eigenfunction, then

$$\lambda_1 \int_0^1 \phi_1'(x)\phi_k(x)dx = \int_0^1 \int_0^1 K(y,x)\phi_1'(y)\phi_k(x)dydx$$

$$= \lambda_k \int_0^1 \phi_1'(y)\phi_k(y)dy$$

on interchanging the order of integration. This equation says $\phi_1'(y)$ is orthogonal to $\phi_k(y)$, hence $\phi_k(y)$ is not of one sign. As a consequence

$$|\lambda_k| \; |\phi_k(x)| < \int_0^1 K(x,y)|\phi_k(y)|dy, \qquad 0 \leq x \leq 1 \tag{3.9}$$

which then implies

$$\lambda_1 \int_0^1 \phi_1^*(y)|\phi_k(y)|dy = \int_0^1 \int_0^1 K(x,y)\phi_1^*(x)|\phi_k(y)|dxdy$$

$$> |\lambda_k| \int_0^1 \phi_1^*(x)|\phi_k(x)|dx \qquad (3.10)$$

and we conclude that

$$\lambda_1 > |\lambda_k| .$$

To see λ_1 is simple let us suppose there is another eigenfunction $\psi(x)$ and

$$\int_0^1 |\psi(x)|dx = \int_0^1 \phi(x)dx = 1 .$$

There are two cases. First, suppose $\psi(x)$ is not of one sign. Then (3.9), (3.10) hold with $|\phi_k(x)|$ replaced by $|\psi(x)|$, λ_k replaced by λ_1 and we get the contradiction $\lambda_1 > |\lambda_1|$. If $\psi(x)$ is of one sign, then $\phi(x) - |\psi(x)|$ must be a non-trivial eigenfunction corresponding to λ_1. By the previous argument it too must be of one sign, hence identically zero because of the normalization. The contradiction shows λ_1 is simple. ∎

The inequality (3.4) gives a bound on λ_1. A direct application of (2.1) to $K(x,y)$ or $K(y,x)$ gives sharper bounds.

Corollary 1. Let

$$A(x) = \int_0^1 K(x,y)dy, \qquad B(y) = \int_0^1 K(x,y)dx .$$

Then

$$\min_{0 \leq x \leq 1} A(x) \leq \lambda_1 \leq \max_{0 \leq x \leq 1} A(x)$$

$$\min_{0 \leq y \leq 1} B(y) \leq \lambda_1 \leq \max_{0 \leq y \leq 1} B(y) .$$

Proof. Use (2.1) directly as in (3.4) and pass to the limit via Lemma 1. ∎

An interesting and useful corollary is the comparison result which follows.

Corollary 2. If $K_1(x,y)$, $K_2(x,y)$ satisfy the hypotheses of the theorem and

$$K_1(x,y) \geq K_2(x,y), \qquad K_1(x,y) \not\equiv K_2(x,y) ,$$

then their largest eigenvalues satisfy $\lambda^{(1)} > \lambda^{(2)}$.

Proof. From the theorem we know there are positive eigenfunctions $\phi(x)$, corresponding to $\lambda^{(1)}$ for $K_1(x,y)$, and $\psi(x)$, corresponding to $\lambda^{(2)}$ for the transposed kernel $K_2(y,x)$. The inequality

$$\lambda^{(1)} \int_0^1 \phi(x)\psi(x)dx = \int_0^1 \int_0^1 K_1(x,y)\phi(y)\psi(x)dxdy$$

$$> \int_0^1 \int_0^1 K_2(x,y)\phi(y)\psi(x)dxdy$$

$$= \lambda^{(2)} \int_0^1 \phi(x)\psi(x)dx$$

gives the result. ∎

A crucial step in the proof of this theorem was showing the sequence of eigenvalues $\{\lambda_1^{(n)}\}$ of the matrix problems had a positive lower bound to keep the definition (3.3) from degenerating. If we can establish this on other grounds, examination of the proof gives hope it will be more generally valid. This is the case.

Theorem 2. If $K(x,y)$ is continuous (not necessarily real-valued) on $0 \leq x, y \leq 1$, and if $\sigma_p = \text{tr } K^p \neq 0$ for some $p \geq 3$, then there exists a non-zero eigenvalue of the integral equation (1.1).

Proof. Once again consider the matrix problem (3.1). Let $\lambda_1^{(n)}$ be an eigenvalue of maximum modulus for each n. Corresponding to the normalization (3.2) we now require

$$\frac{1}{n} \sum_{j=1}^{n} |\phi_j^{(n)}| = 1 \qquad (3.11)$$

of the eigenvector.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

It is easy to obtain uniform upper bounds on the $\{|\lambda_1^{(n)}|\}$ by classical matrix inequalities. For example Schur's inequality gives

$$|\lambda_1^{(n)}| \leq \left(\sum_{i,j=1}^{n} \left| \frac{1}{n} K(i/n, j/n) \right|^2 \right)^{1/2} \leq M .$$

As before define $\phi^{(n)}(x)$ by (3.3). To see the $\{|\lambda_1^{(n)}|\}$ are bounded below by a positive constant suppose the contrary. That is, suppose there is a sequence n' such that $|\lambda_1^{(n')}| \to 0$ as $n' \to \infty$. The elementary result (2.2) then guarantees

$$|tr(K^{(n')})^p| = \left| \sum_{j=1}^{n'} (\lambda_j^{(n')})^p \right|$$

$$\leq \sum_{j=1}^{n'} |\lambda_j^{(n')}|^p \leq |\lambda_1^{(n')}|^{p-2} \sum_{j=1}^{n'} |\lambda_j^{(n')}|^2 .$$

Schur's inequality says

$$\sum_{j=1}^{n'} |\lambda_j^{(n')}|^2 \leq \|K^{(n')}\|_E^2 \leq M^2 ,$$

independently of n'. Accordingly we find that for all $p \geq 3$,

$$tr(K^{(n')})^p \to 0 \text{ as } n' \to \infty .$$

It is natural to expect

$$tr(K^{(n')})^p \to tr\, K^p = \sigma_p$$

which leads to a contradiction, since by hypothesis some $\sigma_p \neq 0$ with $p \geq 3$. This expectation is justified by

Lemma 2. If $K(x,y)$ is continuous on $0 \leq x,\ y \leq 1$, then

$$\operatorname{tr}(K^{(n)})^p \to \operatorname{tr} K^p = \int_0^1 K^p(x,x)dx \quad \text{as} \quad n \to \infty .$$

Proof.

$$\left| \int_0^1 K^p(x_1,x_1)dx_1 - \operatorname{tr}(K^{(n)})^p \right|$$

$$= \left| \int_0^1 \int_0^1 \cdots \int_0^1 K(x_1,x_2)K(x_2,x_3) \cdots K(x_p,x_1)dx_1dx_2 \cdots dx_p \right.$$

$$\left. - \frac{1}{n^p} \sum_{k_1,k_2,\ldots,k_p=1}^{n} K(\tfrac{k_1}{n},\tfrac{k_2}{n})K(\tfrac{k_2}{n},\tfrac{k_3}{n}) \cdots K(\tfrac{k_p}{n},\tfrac{k_1}{n}) \right|$$

$$\leq \sum_{k_1,k_2,\ldots,k_p=1}^{n} \left| \int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}} \cdots \int_{\frac{k_p-1}{n}}^{\frac{k_p}{n}} [K(x_1,x_2)\cdots K(x_p,x_1) \right.$$

$$\left. - K(\tfrac{k_1}{n},\tfrac{k_2}{n}) \cdots K(\tfrac{k_p}{n},\tfrac{k_1}{n})]dx_1 \cdots dx_p \right|$$

$$\leq n^p \cdot \frac{1}{n^p} \cdot p[M^{p-1}\omega(\tfrac{1}{n})] = pM^{p-1}\omega(\tfrac{1}{n}) .$$

Let $n \to \infty$ in this inequality to complete the proof. ∎

Combining these results we see there is a constant $m > 0$ for which $M \geq |\lambda_1^{(n)}| \geq m > 0$ for all n. Thus we may extract a

subsequence n' such that $\lambda_1^{(n')} \to \lambda$ and $M \geq |\lambda| \geq m > 0$. As in (3.5), (3.6) one may now show

(i) $\{\phi^{(n')}(x)\}$ is uniformly bounded with $|\phi^{(n')}(x)| \leq \frac{M}{m}$, $0 \leq x \leq 1$,

and

(ii) $\{\phi^{(n')}(x)\}$ is equicontinuous with $|\phi^{(n')}(x)-\phi^{(n')}(y)| \leq \frac{\omega(\delta)}{m}$ for $|x-y| \leq \delta$.

Thus Arzelà's theorem allows us to extract a uniformly convergent subsequence $\phi^{(n'')}(x) \to \phi(x)$. Arguing as in (3.8) but now with the normalization (3.11) we have

$$\int_0^1 |\phi^{(n'')}(x)|dx \to \int_0^1 |\phi(x)|dx = 1$$

so that the limit function is non-trivial.

Now passage from the matrix to integral equation is just as before when we remark $\frac{2M}{m}\omega(\delta)$ is a modulus of uniform continuity for $K(x,y)\phi^{(n'')}(y)$ considered as a function of y. ■

This theorem is known to be sharp but the condition is rather inconvenient to test for a given kernel and we seek some simpler sufficient conditions. Naturally we would expect a kernel being Hermitian suffices for this theorem, and it does as we now show.

Corollary 1. If $K(x,y) \not\equiv 0$ is a continuous Hermitian kernel, it has a non-zero eigenvalue.

Proof. It is easy to see

$$K^2(x,y) = \int_0^1 K(x,z)K(z,y)dz$$

is also continuous and Hermitian. Moreover

$$K^2(x,x) = \int_0^1 K(x,z)\overline{K(x,z)}dz = \int_0^1 |K(x,z)|^2 dz > 0$$

for some x since K is non-trivial. But then

$$\sigma_4 = \int_0^1 K^4(x,x)dx = \int_0^1 \int_0^1 K^2(x,z)K^2(z,x)dzdx$$

$$= \int_0^1 \int_0^1 |K^2(x,z)|^2 dzdx > 0 ,$$

and the result follows from the theorem.

It is also easy to complement Theorem 1 with a relatively simple sufficient condition when $K(x,y)$ is non-negative.

Corollary 2. If $K(x,y)$ is a continuous kernel with $K(x,y) \geq 0$ for $0 \leq x, y \leq 1$, and if $\mathrm{tr}\, K^p \not\equiv 0$ for some $p \geq 1$, then it has a non-zero eigenvalue.

Proof. Because of the theorem only the cases $p = 1,2$ are open. If $\sigma_1 \not\equiv 0$, there is some $K(x,x) \not\equiv 0$ and accordingly an $\epsilon > 0$ such that $K(y,z) \geq m > 0$ for $|x-y| \leq \epsilon$, $|x-z| \leq \epsilon$.

But then $\sigma_3 \geq m(2\varepsilon)^3 > 0$ and the theorem guarantees an eigenvalue. Similarly if $\sigma_2 \neq 0$, there are x, $\varepsilon > 0$ such that $K^2(y,z) \geq m > 0$, for $|x-y| \leq \varepsilon$, $|x-z| \leq \varepsilon$, and $\sigma_4 \geq m(2\varepsilon)^2 > 0$. ∎

The next theorem is another effort to find simple but powerful sufficient conditions. It has seen good use recently in showing the integral equations of laser theory have eigenvalues. Our proof will draw on some understanding of determinants and is rather less obvious than the preceding theorems. This is largely because of the nature of the sufficient condition itself, as Fredholm chose a condition suitable to his theory of integral equations. Nevertheless our approach requires no knowledge of the Fredholm theory whatsoever.

Theorem 3. If $K(x,y)$ satisfies a Hölder condition in either variable with exponent $\alpha > {}^1/2$ and if $\operatorname{tr} K^p \neq 0$ for some $p \geq 1$, then (1.1) has at least one non-zero eigenvalue.

Proof. First we shall establish the proposition

(P) If $K(x,y)$ satisfies a Hölder condition in either variable with exponent $\alpha > {}^1/2$, it is not possible for $\sigma_1 \neq 0$ and $\sigma_p = 0$ for all $p > 1$.

To show this we shall consider as in [5, p. 114]

$$u_n = \int_0^1 \int_0^1 \cdots \int_0^1 K\binom{x_1,\ldots,x_n}{x_1,\ldots,x_n} dx_1 dx_2 \cdots dx_n, \qquad n \geq 1$$

with $u_0 = 1$ and

$$K\begin{pmatrix} x_1,\ldots,x_n \\ x_1,\ldots,x_n \end{pmatrix} = \begin{vmatrix} K(x_1,x_1) & K(x_1,x_2) & \cdots & K(x_1,x_n) \\ K(x_2,x_1) & K(x_2,x_2) & \cdots & K(x_2,x_n) \\ \vdots & & & \\ K(x_n,x_1) & K(x_n,x_2) & \cdots & K(x_n,x_n) \end{vmatrix} . \tag{3.12}$$

The u_n will be bounded using the Hölder condition and Hadamard's inequality in a way discovered by Fredholm [4] and used later by Cochran [2] and Hochstadt [7]. We wish to emphasize that the origin and significance of this quantity to the Fredholm theory is completely immaterial to our development. As far as we are concerned it is just a quantity which can be easily related to the traces and which can be bounded with the Hölder condition.

It will be clear that the Hölder condition can hold for either variable, but to be specific let

$$|K(x,y_1)-K(x,y_2)| \leq L|y_1-y_2|^{\alpha} . \tag{3.13}$$

Subtracting the second column of the determinant from the first, the third from the second, ... , the <u>nth</u> from the (n-1)st, (3.12) can be rewritten as

$$
\begin{vmatrix}
K(x_1,x_1) - K(x_1,x_2) & K(x_1,x_2) - K(x_1,x_3) & \cdots & K(x_1,x_n) \\
K(x_2,x_1) - K(x_2,x_2) & K(x_2,x_2) - K(x_2,x_3) & \cdots & K(x_2,x_n) \\
\vdots & \vdots & & \vdots \\
K(x_n,x_1) - K(x_n,x_2) & K(x_n,x_2) - K(x_n,x_3) & \cdots & K(x_n,x_n)
\end{vmatrix} .
$$

Then using the Hölder inequality (3.13) and Hadamard's inequality we find

$$
\left| K\begin{pmatrix} x_1,\ldots,x_n \\ x_1,\ldots,x_n \end{pmatrix} \right| \leq n^{n/2}\, C^n |(x_1-x_2)(x_2-x_3)\cdots(x_{n-1}-x_n)|^\alpha
$$

for a suitable constant C. The determinant being symmetric in the x_i we may assume

$$
1 \geq x_1 > x_2 > \cdots > x_n \geq 0 .
$$

It is then easy to prove by induction that

$$
|(x_1-x_2)(x_2-x_3)\cdots(x_{n-1}-x_n)| \leq (n-1)^{1-n} .
$$

Combining these inequalities we find

$$
|u_n| \leq C^n\, n^{n/2} (n-1)^{\alpha(1-n)} . \tag{3.14}
$$

On the other hand, from the definition of the determinant

(3.12)

$$K\binom{x_1,\ldots,x_n}{x_1,\ldots,x_n} = \sum_{\pi} s(\pi)K(x_1,x_{j_1})K(x_2,x_{j_2})\cdots K(x_n,x_{j_n})$$

where the summation extends over all $n!$ permutations $\pi(1,\ldots,n)$ and $s(\pi)$ is the sign of the permutation. For each product extract the component of the form $K(x_1,x_i)\cdots K(x_r,x_1)$, and sum all the terms with this common factor. These sums comprise all combinations of one element from each of the remaining rows and columns (with the appropriate sign), or equivalently they are the determinants of the matrices formed by removing rows and columns $1, i,\ldots,r$. Thus we get [5, p. 113]

$$K\binom{x_1,\ldots,x_n}{x_1,\ldots,x_n} = K(x_1,x_1)K\binom{x_2,\ldots,x_n}{x_2,\ldots,x_n}$$

$$+ \sum (-1)^{k+1}K(x_1,x_i)\cdots K(x_r,x_1)K\binom{x_1',x_2',\ldots,x_{n-k}'}{x_1',x_2',\ldots,x_{n-k}'} \tag{3.15}$$

where $x_1,x_i,\ldots,x_r$ represent k of the variables and $x_1',\ldots,x_{n-k}'$ are the remaining $n-k$. Suppose now the proposition (P) is false, so that $\sigma_1 \neq 0$ but $\sigma_p = 0$ for all $p > 1$. On carrying out the integrations of (3.15) to get u_n we find

$$u_n = \sigma_1 u_{n-1} - (n-1)\sigma_2 u_{n-2} + (n-1)(n-2)\sigma_3 u_{n-3} + \cdots$$

$$+ (-1)^{p+1}(n-1)(n-2)\cdots(n-p+1)\sigma_p u_{n-p}$$

$$+ \cdots + (-1)^{n+1}(n-1)!\, \sigma_n u_0$$

$$= \sigma_1 u_{n-1} .$$

This implies $u_n = \sigma_1^n$ which if $\alpha > 1/2$ contradicts the bound (3.14) no matter what C is, and the contradiction establishes (P).

The theorem itself requires several cases. It is implied by Theorem 2 when $\sigma_p \neq 0$ for some $p \geq 3$. If $\sigma_2 = 0$ and $\sigma_1 \neq 0$, the proposition (P) guarantees some $\sigma_p \neq 0$ for $p \geq 3$ and the result again follows from Theorem 2. If $\sigma_2 \neq 0$, consider the kernel

$$G(x,y) = K^2(x,y) .$$

Now $G(x,y)$ satisfies

$$|G(x,y_1)-G(x,y_2)| = \left| \int_0^1 K(x,z)K(z,y_1)dz - \int_0^1 K(x,z)K(z,y_2)dz \right|$$

$$\leq ML\, |y_1-y_2|^\alpha .$$

Since we can apply proposition (P) to $G(x,y)$ we conclude

$$\mathrm{tr}\, G^p = \mathrm{tr}\, K^{2p} = \sigma_{2p} \neq 0$$

for some $p > 1$ and this last case is settled by reference to Theorem 2. ∎

A recent application of this theorem is to the kernel

$$K(x,y) = e^{iH(x-y)^2}$$

which arises in the consideration of a laser resonant cavity involving rectangular reflecting surfaces which are mirror images of each other. The quantity H is a parameter so that the use of Theorem 3 is obvious. The theorem also applies to laser kernels associated with other practical reflector configurations. It was only recently realized that the existence of the eigenvalues could be established this way. The interesting papers [2, 7, 10] discuss these applications in some detail.

II. A Collocation Method for Boundary Value Problems

1. Introduction. The approximation of solutions of boundary value problems has recently been an area of much interest and intense activity for numerical analysts. One important approach used by a number of Russian authors is collocation with polynomials. We continue the study of collocation but use piecewise polynomial functions. (This is done for special cases in [10] and [1, p. 52].) With this approach better convergence in a simpler setting is obtained. Piecewise polynomials turn out to be much better computationally, too. The resulting linear systems are easy to arrive at and involve band matrices instead of the dense matrices when polynomials are used. Piecewise polynomial functions are also more adaptable to special problems. For example, we prove a quite unique result which states that if a solution to the differential equation has discontinuities in its derivatives, a high rate of convergence can be achieved when these points are used for collocation. Applications to quite general nonlinear problems are then developed which, although powerful, are by no means complete. Computational examples, some rather difficult, are presented to illustrate the theoretical results.

Our conclusion is that collocation with piecewise polynomials is a very satisfactory way to solve boundary value problems. Among the virtues of our collocation procedure in the linear case are its generality and ease in application. In addition, it is relatively

simple to vary the order of the method or to adapt the mesh to a particular solution. This is in marked contrast to virtually any other method. The author shares the opinion of L.F. Shampine* that the only methods which can generally compete with this procedure in solving a single mth order linear problem are extrapolation to the limit [12] and perhaps the parallel shooting technique [7; 13, p.61].

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* Opinion expressed to the author in a private communication.

2. The Linear Problem. We shall first be concerned with the linear differential equation

$$L[u] = u^{(m)}(s) + \sum_{k=0}^{m-1} e_k(s)u^{(k)}(s) = f(s), \quad (a \leq s \leq b) \tag{2.1}$$

subject to the m linearly independent, homogeneous boundary conditions

$$\sum_{k=0}^{m-1} [\alpha_{ik}u^{(k)}(a)+\beta_{ik}u^{(k)}(b)] = 0, \qquad (\alpha_{ik}, \beta_{ik} = \text{const}; \quad 1 \leq i \leq m) . \tag{2.2}$$

There is no loss of generality in assuming the boundary conditions (2.2) are homogeneous. For suppose we seek a solution $v(s)$ of (2.1) subject to the nonhomogeneous boundary conditions

$$\sum_{k=0}^{m-1} [\alpha_{ik}v^{(k)}(a)+\beta_{ik}v^{(k)}(b)] = \eta_i , \qquad (\alpha_{ik}, \beta_{ik}, \eta_i = \text{const}; 1 \leq i \leq m) . \tag{2.2'}$$

Let $q(s)$ be <u>any</u> function in $C^{(m)}[a,b]$ which satisfies (2.2'). For example, a polynomial could be constructed for this purpose. Then on setting $v(s) = q(s) + u(s)$, the problem is conveniently transformed to solving

$$L[u] = g(s) = f(s) - L[q(s)]$$

subject to the homogeneous boundary conditions (2.2).

The collocation method will be used to approximate solutions of (2.1), (2.2). Although the method itself is not very precisely

defined, it basically involves forming an approximate solution as a linear combination of a convenient set of functions, the coefficients of which are determined by requiring the combination to satisfy the differential equation at certain points (collocation at these points).

Polynomial functions, a reasonable choice for the basis functions, were used by Karpilovskaya [11, p.581] and more recently by Shindler [22,23] and Vainniko [27]. In their analyses the differential equation is transformed to an operator equation. The collocation condition turns out to be equivalent to a projection of the operator equation into a finite dimensional subspace, where the relevant projection operators represent interpolation by polynomials. Karpilovskaya and Shindler considered these operators as mappings from C into C. A classical result of Natanson [18, p. 91] shows the projection operators cannot then be uniformly bounded. However, in the special case of interpolation at the Gauss, Legendre, or Chebyshev points, the rate of growth of the norms of these operators is known and the collocation method was shown to converge. Vainniko's treatment included these results as special cases. The operators were considered as mappings from C into L^2_ρ, where $L^2_\rho = L^2_\rho[a,b]$ is the space of functions square summable on the interval [a,b] with weight function $\rho(s)$. Since the Erdös-Turán theorem implies the uniform boundedness of the relevant projection operators, the proofs were simplified and the rate of convergence guaranteed by the theory was improved.

In the development of numerical analysis, it has generally turned out that using piecewise polynomial functions leads to better convergence

results and simpler proofs than using polynomials, especially when dealing with boundary value problems. Using piecewise polynomial functions the resulting projection operators from C into C have uniformly bounded norms even if one allows much greater flexibility in the selection of collocating points than in the polynomial case. For these reasons we are led to consider collocating with piecewise polynomial functions. It will develop that in fact the proofs do turn out to be simpler and do give stronger results and that for a suitable choice of a basis the matrices which arise are band, as opposed to the dense matrices when polynomials are used, which is of great computational significance.

It is now convenient to dispose of some mathematical preliminaries. First we recall one of Jackson's theorems [9, p.123] which shows the rate of convergence of polynomials to a continuous function in the minimax sense. Throughout we use for the norm of a continuous function $v(s)$

$$\|v\|_\infty = \max_{a \leq s \leq b} |v(s)| .$$

We shall just write $\|v\|$ in what follows unless $\|v\|_\infty$ is needed for clarity.

<u>Theorem 1</u>. Let $E_n(v)$ denote the error of the best uniform approximation to $v \in C = C[a,b]$ by a polynomial of degree n (more precisely, less than or equal to n). Then

$$E_n(v) \leq 6\,\omega\left(\frac{b-a}{2n}\right) , \tag{2.3a}$$

where $\omega(\delta)$ is a modulus of continuity for the function $v(s)$. If $v \in C^{(1)}$ then

$$E_n(v) \leq \frac{3(b-a)}{n} \|v'\| . \tag{2.3b}$$

Lastly, if $v \in C^{(p)}$ and $n \geq p-1 \geq 1$ then

$$E_n(v) \leq \frac{6^p(p-1)^{p-1}}{(p-1)!\; n^p} \cdot p(b-a)^p \, \|v^{(p)}\| \quad . \tag{2.3c}$$

If π_n is any partition of $[a,b]$ into n subintervals $[s_i,s_{i+1}]$, $0 \leq i \leq n-1$, such that

$$a = s_0 < s_1 < \cdots < s_n = b \; , \tag{2.4a}$$

then we define

$$h(\pi_n) = \max_{0 \leq i \leq n-1} (s_{i+1}-s_i) \; .$$

Definition: A function $v(s)$ is in the family $L(\pi_n,k,m)$ if $v(s)$ is a polynomial of degree (at most) k on each subinterval of π_n and $v \in C^{(m)}$. The subfamily $L'(\pi_n,k,m)$ consists of all functions in $L(\pi_n,k,m)$ which satisfy the boundary conditions (2.2).

It is natural to require collocation on π_n. To determine the coefficients of the piecewise polynomials, it will also be necessary to collocate on additional points. We use similarly placed points in each subinterval of π_n. More specifically, assume a partition of $[0,1]$ is given,

$$0 = t_0 < t_1 < \cdots < t_d = 1 \; . \tag{2.5}$$

Define the mapping $\xi_i(t) = s_i + t(s_{i+1}-s_i)$ from $[0,1]$ onto $[s_i,s_{i+1}]$ for each i. Labelling $s_{i,j} = \xi_i(t_j)$, we arrive at a set of $nd + 1$ points

$$s_i = s_{i,0} < s_{i,1} < \dots < s_{i,d} = s_{i+1} \quad (0 \leq i \leq n-1) \tag{2.4b}$$

which we denote by $\mathcal{S}_n$. While this uniform partitioning of the subintervals of π_n is not a necessity, it will be very convenient.

We might hope that an element of $L'(\pi_n,m+d,m)$ could be found which satisfies (2.1) on the points of $\mathcal{S}_n$, i.e., a piecewise polynomial function approximating the solution of (2.1), (2.2) found by collocating on $\mathcal{S}_n$. In a natural representation the number of coefficients that determines such a piecewise polynomial function is $n(m+d+1)$, and this number of equations arises from the collocation procedure sketched. This leads us to suspect the problem is well-posed, as we shall prove for sufficiently fine meshes.

Theorem 2. Suppose that the coefficients $e_k(s)$ and free term $f(s)$ of (2.1) are at least continuous on $[a,b]$ and that (2.1), (2.2) has a unique solution $u(s)$. Also suppose that the problem $u^{(m)} = 0$ with boundary conditions (2.2) has a unique solution. Let a partition (2.5) of $[0,1]$ and a sequence of partitions $\pi_n (n=1,2,3,\dots)$ of $[a,b]$ such that $\lim_{n\to\infty} h(\pi_n) = 0$ be given. Using (2.4a), (2.4b), form the set of points $\mathcal{S}_n$ $(n=1,2,3,\dots)$. Then for all sufficiently large n there is a unique $u_n(s) \in L'(\pi_n,m+d,m)$ which

collocates on $\mathcal{S}_n$. Moreover, $u_n(s)$ and its derivatives up to and including order m converge uniformly to $u(s)$ and its derivatives of corresponding orders. The convergence satisfies

$$\|u_n^{(k)}-u^{(k)}\| \leq \Omega\, F_n(u^{(m)}) \qquad (0 \leq k \leq m) \tag{2.6}$$

where Ω is a constant independent of n and the free term $f(s)$, and $F_n(u^{(m)})$ is the error of the best uniform approximation to $u^{(m)}(s)$ by an element of $L(\pi_n,d,0)$.

Proof: It is well-known [17, p. 29] that the hypotheses imply the existence of the Green's function $G(s,t)$ for the problem $u^{(m)} = 0$ and (2.2). The continuity conditions and boundary conditions imply that if the mth derivative of $u_n(s)$ is specified, then the approximate solution is uniquely specified via the Green's function. So if we can show that $v_n(s) = u_n^{(m)}(s)$ in $L(\pi_n,d,0)$ is uniquely defined, the same is true of $u_n(s)$ in $L'(\pi_n,m+d,m)$. More generally

$$u_n^{(k)}(s) = \int_a^b \frac{\partial^k G(s,t)}{\partial s^k}\, v_n(t)dt, \quad (0 \leq k \leq m-1)\ . \tag{2.7}$$

The collocation condition when expressed in terms of $v_n(s)$ takes the form

$$v_n(s') + \sum_{k=0}^{m-1} e_k(s') \int_a^b \frac{\partial^k G(s',t)}{\partial s^k}\, v_n(t)dt - f(s') = 0 \tag{2.8}$$

for all $s' \in \mathcal{S}_n$.

Define a linear projection P_n which maps each continuous function into its Lagrange interpolating polynomial of degree d on each subinterval of π_n, where the nodes of the n polynomials are given in (2.4b). Notice that P_n projects the space C onto $L^{(m)}(\pi_n, m+d, m) = L(\pi_n, d, 0)$ since the s_i are points of interpolation. The collocation on $\mathcal{S}_n$ given by (2.8) can now be written in operator form as

$$P_n(v_n + Kv_n) = P_n f ,$$

where K is an integral operator with kernel

$$K(s,t) = \sum_{k=0}^{m-1} e_k(s) \frac{\partial^k G(s,t)}{\partial s^k} .$$

Because of the uniqueness of the interpolation polynomial of degree d on each subinterval, $P_n v_n = v_n$, so (2.8) simplifies further to

$$v_n + P_n K v_n = P_n f . \qquad (2.8')$$

This operator equation is thus equivalent to determining $v_n(s)$ by collocating on $\mathcal{S}_n$. Continuity of $v_n(s)$ follows since the joints s_i are among the points of interpolation. Thus if (2.8') can be shown to have a unique solution, then it is apparent from the relation (2.7) that an approximate solution $u_n(s) \in L'(\pi_n, m+d, m)$ is uniquely defined.

Lemma: Under the hypotheses of Theorem 2, the operators P_n converge strongly to the identity operator $I: C \to C$.

Proof: Let $g \in C$. For any n consider an interval $[s_i, s_{i+1}]$ in π_n subdivided as in (2.4b). From (2.3a) of Jackson's theorem,

$$\sup_{s_i \leq s \leq s_{i+1}} |g(s)-p_d^*(s)| \leq 6\,\omega\left(\frac{s_{i+1}-s_i}{2d}\right),$$

where $p_d^*(s)$ is the best uniform approximation to $g(s)$ by a polynomial of degree d on $[s_i, s_{i+1}]$. Thus, if $\mathcal{L}_g(s)$ is Lagrange's form of the polynomial interpolating $g(s)$ at the nodes (2.4b) of $[s_i, s_{i+1}]$, then

$$\max_{s_i \leq s \leq s_{i+1}} |g(s)-\mathcal{L}_g(s)|$$

$$\leq \max_{s_i \leq s \leq s_{i+1}} |g(s)-p_d^*(s)| + \max_{s_i \leq s \leq s_{i+1}} |p_d^*(s)-\mathcal{L}_g(s)|$$

$$\leq 6\omega\left(\frac{s_{i+1}-s_i}{2d}\right) + \max_{s_i \leq s \leq s_{i+1}} \left| \sum_{j=0}^{d} [p_d^*(s_{ij}) - g(s_{ij})]\ell_j(s)\right|$$

$$\leq 6\omega\left(\frac{s_{i+1}-s_i}{2d}\right)\left[1 + \Lambda_i^n\right]$$

$$\leq 6\omega\left(\frac{h(\pi_n)}{2d}\right)\left[1 + \Lambda_i^n\right], \quad (0 \leq i \leq n-1), \tag{2.9}$$

where the Lebesgue constant

$$\Lambda_i^n = \max_{s_i \le s \le s_{i+1}} \sum_{j=0}^{d} |\ell_j(s)|$$

and

$$\ell_j(s) = \prod_{\substack{r=0 \\ r \neq j}}^{d} \left(\frac{s-s_{i,r}}{s_{i,j}-s_{i,r}} \right) .$$

The relationship between (2.4b) and (2.5) shows easily that

$$\ell_j(s) = \ell_j^*(t) = \prod_{\substack{r=0 \\ r \neq j}}^{d} \left(\frac{t-t_r}{t_j-t_r} \right)$$

for $t = \dfrac{s-s_i}{s_{i+1}-s_i}$ and $0 \le i \le n$.

This shows

$$\Lambda_i^n = \max_{0 \le t \le 1} \sum_{j=0}^{d} |\ell_j^*(t)| = M \qquad (2.10)$$

independent of i and n .

Recalling the definition of $P_n g$, this equality and (2.9) imply

$$\|g-P_n g\| \le 6(1+M)\omega\left(\frac{h(\pi_n)}{2d}\right) \to 0 \quad \text{as} \quad n \to \infty .$$

Since $g \in C$ was aribitary, the lemma is proved. ■

It follows directly that the norms of the operators P_n are uniformly bounded, because using (2.10) for any $g \in C$ implies

$$\|P_n g\| \leq \max_{0 \leq i \leq n-1} \max_{s_i \leq s \leq s_{i+1}} \left| \sum_{j=0}^{d} g(s_{ij}) \ell_j(s) \right|$$

$$\leq \max_{a \leq s \leq b} |g(s)| \cdot \max_{0 \leq i \leq n-1} \Lambda_i^n = \|g\| \cdot M$$

which in turn implies that

$$\|P_n\| \leq M \qquad (n=1,2,3,\ldots) . \tag{2.11}$$

From the continuity of the coefficients $e_k(s)$ and known properties of the Green's function [17, p. 31], $K(s,t)$ is continuous except possibly for $s = t$, where it has a jump discontinuity if $e_{m-1}(t) \neq 0$. By familiar arguments one can then show that K is a completely continuous operator mapping C into C. This complete continuity of K and the strong convergence of the operators P_n imply that the operators $P_n K$ in C converge strongly to K, i.e.,

$$\|P_n K - K\| \to 0 \quad \text{as} \quad n \to \infty . \tag{2.12}$$

There is by assumption a unique continuous solution $v(s) = u^{(m)}(s)$ of the equation

$$v + Kv = f \tag{2.1'}$$

for any $f \in C$, hence the bounded linear operator $(I+K)^{-1}$ exists. From the strong convergence of the completely continuous operators $P_n K$ to K stated in (2.12), we may conclude that for all sufficiently large n the bounded operators $(I+P_n K)^{-1}$ exist and are in fact uniformly bounded, say

$$\|(I+P_n K)^{-1}\| \leq \sigma \qquad (n=n_0, n_0+1, \ldots) \ . \tag{2.13}$$

Therefore, equation (2.8') is uniquely solvable for all sufficiently large n (say for $n \geq n_0$). Equivalently, a unique element of $L'(\pi_n, m+d, m)$ satisfying (2.8) exists. We have demonstrated that the collocation procedure being used is well-defined.

Lastly, the rate of convergence is to be bounded. Subtracting (2.8') from

$$(I+P_n K)v = P_n(I+K)v + (v-P_n v)$$

$$= P_n f + (v-P_n v)$$

we obtain

$$(I+P_n K)(v-v_n) = v - P_n v \ . \tag{2.14}$$

Then (2.13) and (2.14) imply that

$$\|v-v_n\| \leq \|(I-P_n K)^{-1}\| \ \|v-P_n v\| \leq \sigma \|v-P_n v\| \qquad (n \geq n_0) \ .$$

Let $z_n(s)$ be any element of $L(\pi_n, d, 0)$. From (2.11)

$$\|v_n - v\| \leq \sigma\|v - P_n v\|$$

$$= \sigma\|(v-z_n) - P_n(v-z_n)\|$$

$$\leq \sigma(\|v-z_n\| + \|P_n(v-z_n)\|)$$

$$\leq \sigma(1+M)\|v-z_n\| \quad , \quad (n \geq n_0) \; .$$

Since $z_n(s) \in L(\pi_n, d, 0)$ was arbitrary,

$$\|v_n - v\| \leq \sigma(1+M)F_n(u^{(m)}) \; . \tag{2.15}$$

This shows that the <u>mth</u> derivatives converge. For convergence of the lower order derivatives, (2.7) implies the relation

$$u_n^{(k)}(s) - u^{(k)}(s) = \int_a^b \frac{\partial^k G(s,t)}{\partial s^k} [v_n(t) - v(t)]dt \; , \quad (0 \leq k \leq m-1; \; n \geq n_0) \; .$$

Denoting

$$M_k = \max_{a \leq s \leq b} \int_a^b \left|\frac{\partial^k G(s,t)}{\partial s^k}\right| dt \; ,$$

the inequality

$$\|u_n^{(k)} - u^{(k)}\| \leq M_k\|v_n - v\| \quad (0 \leq k \leq m-1; \; n \geq n_0)$$

follows immediately. Combining this with (2.15) we conclude that there is a constant Ω satisfying (2.6), and the proof is complete. ∎

We are now interested in the order of the bound $F_n(u^{(m)})$ in (2.6). The following lemma gives us a useful bound.

Lemma: Let $F_n(v)$ denote the minimax error in approximating $v \in C$ by an element of $L(\pi_n,d,0)$. Then

$$F_n(v) \leq 18\omega\left(\frac{h(\pi_n)}{2d}\right) . \tag{2.16a}$$

If $v \in C^{(1)}$ then

$$F_n(v) \leq \frac{9h(\pi_n)}{d} \|v'\|_C . \tag{2.16b}$$

Lastly, if $v \in C^{(p)}$ and $d \geq p-1 \geq 1$ then

$$F_n(v) \leq \frac{3 \cdot 6^p (p-1)^{p-1}}{(p-1)!d^p} \cdot p[h(\pi_n)]^p \|v^p\|_C . \tag{2.16c}$$

Proof: Suppose there are polynomials $p_i(s)$ of degree d on $[s_i, s_{i+1}]$ such that

$$\max_{s_i \leq s \leq s_{i+1}} |v(s)-p_i(s)| \leq \epsilon \qquad (0 \leq i \leq n-1;\ n \geq 1) .$$

These bounds are not directly applicable to $F_n(v)$ in (2.6) because elements of $L(\pi_n,d,0)$ must be continuous. We form a suitable function by adding a linear function $\lambda_i(s)$ to $p_i(s)$ on every other subinterval of π_n, so that

$$z_n(s) = \begin{cases} p_i(s) \text{ on } [s_i, s_{i+1}] & , \ i \text{ even} \\ p_i(s) + \lambda_i(s) \text{ on } [s_i, s_{i+1}] & , \ i \text{ odd} \end{cases}$$

is in $L(\pi_n, d, 0)$. Since

$$|v(s)-p_i(s)-\lambda_i(s)| \leq |v(s)-p_i(s)| + |\lambda_i(s)|$$

$$\leq 3\,\epsilon \quad (s_i \leq s \leq s_{i+1}\ ,\ i \text{ odd})\ ,$$

it follows that

$$\|v-z_n\| \leq 3\epsilon\ .$$

The lemma is now immediate from (2.3a), (2.3b) and (2.3c) in the previously stated theorem of Jackson. ■

The following corollary is a direct consequence of the lemma.

Corollary: Assume that the hypotheses of Theorem 2 are satisfied. Suppose also that the solution $u(s)$ of (2.1), (2.2) is $C^{(p)}$ on every subinterval of π_n $(n = 1,2,\ldots)$, where $u^{(p)}(s)$ is defined at the joints by right and left hand limits if necessary. Then for $d \geq p - m - 1 \geq 0$, as $n \to \infty$

$$\|u_n^{(k)}-u^{(k)}\| = O([h(\pi_n)]^{p-m}),\ (0 \leq k \leq m)\ . \qquad (2.17)$$

Proof: We have $v(s) = u^{(m)}(s) \in C^{(p-m)}$ on each subinterval of

$\pi_n (n = 1,2,\ldots)$. It is easily shown that (2.16c) is valid under this weakened hypothesis, so the result follows from (2.6). ∎

We point out that a finite number of jump discontinuities in $u^{(p)}(s)$ are permitted if we place joints there. In this case we have the remarkable result that convergence is unaffected. One might know where discontinuities are on physical grounds, for example in beam problems where there are discontinuities only at the supports, or on examining jumps in the derivatives of the coefficients $e_k(s)$, $f(s)$.

3. Higher Order Convergence. From the last theorem we see that a lack of continuous derivatives reduces the order of convergence that can be guaranteed, although we have just observed that a finite number of jump discontinuities have no effect provided we know where they are. We now ask if improved convergence can be shown when $u^{(p)}(s)$ is continuous, but $u^{(p+1)}(s)$ has a finite number of discontinuities and we do not know where they are. In this case no action by the analyst is necessary, as we are just noting a situation when a higher rate of convergence is guaranteed. We have learned of some recent work of Swartz and Varga being prepared for publication which has results of this general nature, but otherwise our results seem unique.

<u>Theorem 3</u>. Suppose that the hypotheses of Theorem 2 are satisfied, the <u>pth</u> derivative of $u(s)$ is continuous, and its $(p+1)$<u>st</u> derivative has discontinuities at a finite number of points (say at N points). Then for $d \geq p - m \geq 0$,

$$\|u_n^{(k)}-u^{(k)}\|_{\infty} = O([h(\pi_n)]^{p-m+1/2}) , \quad (0 \leq k \leq m-1) \tag{3.1}$$

as $n \to \infty$.

Proof: It will be necessary to consider the operators as acting in different spaces than those of Theorem 2. Let the operator K now be a mapping from $L_2 = L_2[a,b]$ into C. The boundedness of K follows directly from Cauchy's inequality, for if $v \in L_2$ then

$$\|Kv\|_\infty = \max_{a \le s \le b} \left| \int_a^b K(s,t)v(t)dt \right|$$

$$\le \left[\int_a^b |K(s,t)|^2 dt \right]^{1/2} \|v\|_2$$

where

$$\|v\|_2 = \left[\int_a^b |v(t)|^2 dt \right]^{1/2} .$$

From the continuity properties of $K(s,t)$, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_a^b |K(s_1,t)-K(s_2,t)|^2 dt < \varepsilon^2$$

for all $s_1, s_2 \in [a,b]$ satisfying $|s_1-s_2| < \delta$. As a consequence,

$$|Kv(s_1)-Kv(s_2)| = \left| \int_a^b (K(s_1,t)-K(s_2,t))v(t)dt \right|$$

$$\le \left[\int_a^b |K(s_1,t)-K(s_2,t)|^2 dt \right]^{1/2} \|v\|_2$$

$$< \varepsilon \|v\|_2$$

for $|s_1-s_2| < \delta$. Thus any set of uniformly bounded functions is mapped into an equicontinuous family. Since this family is compact in C, the operator K is completely continuous.

The operators P_n $(n=1,2,3,...)$ are uniformly bounded as mappings from C into C, so they are also uniformly bounded when

considered as mappings from C into L_2. Likewise the operators P_n converge strongly to the inclusion operator P from C into L_2. Then the complete continuity of $K:L_2 \to C$ implies that in L_2 the operators satisfy

$$\|P_nK-PK\| \to 0 \quad \text{as} \quad n \to \infty .$$

Equation (2.1') has a solution $v(s)$ unique in L_2 where we let $PK = K$. So the bounded operator $(I+PK)^{-1}$ from L_2 into L_2 exists, and the same arguments as in Theorem 2 imply $(I+P_nK)^{-1}: L_2 \to L_2$ exist and are uniformly bounded for all sufficiently large n, say

$$\|(I+P_nK)^{-1}\| \leq \sigma' \quad (n \geq n_0) . \tag{2.13'}$$

In this manner we can show as before that the approximate solutions from collocation satisfy

$$\|v_n-v\|_2 \leq \sigma'(1+M')F_n'(u^{(m)}) \qquad (n \geq n_0) , \tag{2.15'}$$

where now M' is the uniform bound on the operators $P_n: C \to L_2$ and $F_n'(u^{(m)})$ is the error of the best approximation in the L_2 sense to $u^{(m)}(s)$ by an element of $L(\pi_n,d,0)$.

We now desire a bound for this error $F_n'(u^{(m)})$. Let $p_i(s)$ be the best uniform approximation to $v(s) = u^{(m)}(s)$ on $[s_i, s_{i+1}]$ by a polynomial of degree d. Then form $z_n(s)$ by the construction in the previous lemma. It follows easily that

$$\|z_n - v\|_2^2 = \int_a^b |z_n(t) - v(t)|^2 dt$$

$$= \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} |z_n(t) - v(t)|^2 dt$$

$$= \Sigma_1 \int_{s_i}^{s_{i+1}} |z_n(t) - v(t)|^2 dt +$$

$$+ \Sigma_2 \int_{s_i}^{s_{i+1}} |z_n(t) - v(t)|^2 dt .$$

The first sum consists of all subintervals containing discontinuities of $v^{(p-m+1)}(s)$ and the subintervals adjacent to them. The second sum consists of the remaining subintervals. Using (2.16c) we obtain

$$\|z_n - v\|_2^2 \leq \Sigma_1 \int_{s_i}^{s_{i+1}} M_1 h_i^{2(p-m)} dt + \Sigma_2 \int_{s_i}^{s_{i+1}} M_2 h_i^{2(p-m+1)} dt$$

$$\leq 3NM_1 \, [h(\pi_n)]^{2(p-m)+1} + (b-a) \, M_2 [h(\pi_n)]^{2(p-m+1)}$$

$$\leq M_3 [h(\pi_n)]^{2(p-m)+1}$$

where $M_i (1 \leq i \leq 3)$ are constants.

Combining the last inequality with (2.15') we conclude that

$$\|v_n - v\|_2 = O([h(\pi_n)]^{p-m+\frac{1}{2}}).$$

The result (3.1) will now follow because for any $s' \in [a,b]$,

$$|u_n^{(k)}(s')-u^{(k)}(s')| = \left| \int_a^b \frac{\partial^k G(s',t)}{\partial s^k} [v_n(t)-v(t)]dt \right|$$

$$\leq \left[\int_a^b \left|\frac{\partial^k G(s',t)}{\partial s^k}\right|^2 dt\right]^{1/2} \left[\int_a^b |v_n(t)-v(t)|^2 dt\right]^{1/2}$$

$$\leq Q_k \|v_n - v\|_2 , \qquad (Q_k = \text{const}; \;\; 0 \leq k \leq m-1). \quad \blacksquare$$

4. The Nonlinear Problem. We now extend our development of the collocation method to the nonlinear ordinary differential equation

$$u^{(m)}(s) = f(s,u,u',\dots,u^{(m-1)}), \quad (a \leq s \leq b) \tag{4.1}$$

subject to the linear, homogeneous boundary conditions

$$\sum_{k=0}^{m-1} [\alpha_{ik}u^{(k)}(a)+\beta_{ik}u^{(k)}(b)] = 0, \quad (\alpha_{ik},\beta_{ik} = \text{const}; \; 1 \leq i \leq m). \tag{4.2}$$

Unlike the linear case, a solution of (4.1), (4.2) is not necessarily unique. For this reason we only consider the collocation procedure when applied to a sufficiently small neighborhood of an isolated solution. A paper of Vainniko [28] treated the nonlinear problem by using polynomials in a way quite similar to his treatment of the linear case. In particular, the boundary value problem (4.1), (4.2) was reformulated as an operator equation in a Banach space. The collocation condition is again equivalent to a projection of the operator equation into a finite dimensional subspace. However, the problem is now complicated by the fact that these operator equations are nonlinear. Krasnosel'skii [14] has developed a general theory of approximate solutions of operator equations by projection methods, and Vainniko based his treatment on these results. We only need to alter Vainniko's analysis to deal with piecewise polynomial functions. This is more appropriately done in different spaces and is basically simpler. We first state a few fundamental definitions and give the main theorem used by Vainniko.

Suppose that T is a nonlinear, completely continuous operator defined on a Banach space B. The operator T is Frechet differentiable and $T'(v)$ is its Frechet derivative at the point v if

$$Tw - Tv = T'(v)(w-v) + \eta(w,v) , \tag{4.3}$$

where η is an operator such that

$$\frac{\|\eta(w,v)\|}{\| w-v \|} \to 0 \quad \text{as} \quad w \to v .$$

In addition, we say that the operator T is continuously differentiable at the point v if it is differentiable at each point of a neighborhood of the point v and the linear operator T' satisfies

$$\|T'(w)-T'(v)\| \to 0 \quad \text{as} \quad w \to v.$$

When possible we use notation consistent with that of the linear theory. The linear case is obviously included in the following analysis, so the statement of the theorem is perhaps more easily understood by interpreting it for linear equations.

<u>Theorem 4</u>. Suppose that in a Banach space B the operator equation

$$w = Tw \tag{4.4}$$

has a solution v. A sufficient condition that v be an isolated solution of (4.4) in some sphere $\|w-v\| \leq \sigma$ $(\sigma > 0)$, and have a non-zero index (see [14, p. 90]) is that K be differentiable at the point v and the homogeneous equation

$$w - PK'(v)w = 0 \tag{4.5}$$

have only the trivial solution $w = 0$. Suppose the operators T and $\{T_n\}$ can be represented in the form

$$T = PK, \quad T_n = P_n K ,$$

where K is a nonlinear, completely continuous operator mapping B into another Banach space B', and P and $\{P_n\}$ are continuous linear operators taking B' into B. Suppose further that the sequence of operators P_n converges to the operator P. Then the equation

$$w = T_n w \tag{4.6}$$

has a solution v_n satisfying $\|v_n - v\| \leq \sigma$ for all sufficiently large n, $v_n \to v$ as $n \to \infty$, and the rate of convergence is bounded by the inequality

$$\|v_n - v\| \leq M'\|(P_n - P)Kv\| \quad (M' = \text{const};\ n \geq n_0).$$

Moreover, if the operator K is continuously differentiable at the point v, then for all sufficiently large n the solution v_n of (4.6) is unique in the sphere $\|v_n - v\| \leq \sigma$.

Applying the collocation method to approximate a solution of (4.1), (4.2) proceeds in basically the same way as in the linear

problem (2.1), (2.2). Given a partition π_n of $[a,b]$, the set of points $\mathcal{S}_n$ is constructed from (2.4a), (2.4b). Then an approximate solution $u_n(s)$ in $L'(\pi_n,m+d,m)$ is sought which satisfies (4.1) at all of the points of $\mathcal{S}_n$, i.e., which collocates on $\mathcal{S}_n$. To find such a function requires solving a nonlinear system of equations, which must be shown to have solutions. In contrast to the linear case, a solution to the continuous problem may not be unique. Accordingly we must restrict our attention to a neighborhood of an isolated solution of the continuous problem and show that the nonlinear finite difference equations have unique solutions in this neighborhood for sufficiently fine meshes. This will require either an assumption or sufficient condition to guarantee an isolated solution of the differential equation--an intrinsic distinction from the linear case.

Theorem 5. Suppose that $u(s)$ is a solution of the boundary value problem (4.1), (4.2), that the functions $f(s,z_0,z_1,\ldots,z_{m-1})$ and $\partial f(s,z_0,z_1,\ldots,z_{m-1})/\partial z_k$ $(0 \leq k \leq m-1)$ are defined and continuous for

$$a \leq s \leq b, \quad |z_k-u^{(k)}(s)| \leq \delta \qquad (0 \leq k \leq m-1;\ \delta > 0)\ , \qquad (4.8)$$

and that the homogeneous equation $y^{(m)} = 0$ subject to the boundary conditions (4.2) has only the trivial solution. If the linear homogeneous equation

$$y^{(m)}(s) - \sum_{k=0}^{m-1} \frac{\partial f(s,u,u',\dots,u^{(m-1)})}{\partial z_k} y^{(k)}(s) = 0 \qquad (4.9)$$

subject to (4.2) has only the trivial solution, then this is <u>sufficient</u> to guarantee that there exists a $\sigma > 0$ such that $u(s)$ is the unique solution of (4.1), (4.2) in the sphere $\|w-u^{(m)}\| \leq \sigma$.

Assume a partition (2.5) of $[0,1]$ and a sequence of partitions π_n $(n=1,2,3,\dots)$ of $[a,b]$ satisfying

$$\lim_{n\to\infty} h(\pi_n) = 0$$

are given. Form the set of points $\mathcal{S}_n$ $(n=1,2,3,\dots)$ from (2.4a), (2.4b). Then for all sufficiently large n there is a unique element $u_n(s)$ of $L'(\pi_n,m+d,m)$ satisfying (4.1) at each point of $\mathcal{S}_n$ and $\|u_n^{(m)}-u^{(m)}\| \leq \sigma$. The approximate solutions $u_n(s)$ and their derivatives through order m tend uniformly to $u(s)$ and its derivatives of corresponding orders. The rate of convergence is bounded by

$$\|u_n^{(k)}-u^{(k)}\| \leq \Theta\, F_n(u^{(m)}) \qquad (0 \leq k \leq m-1) \qquad (4.10)$$

where Θ is a constant independent of n and $F_n(u^{(m)})$ is the error of the best uniform approximation to $u^{(m)}(s)$ in $L(\pi_n,d,0)$.

Proof: We shall first transform (4.1), (4.2) to an operator equation of the form (4.4) in the Banach space C and transform the collocation equations to the form (4.6). Then upon verifying the other conditions of Theorem 4, we shall use it to prove the theorem.

Let $G(s,t)$ be the Green's function for the homogeneous equation $y^{(m)} = 0$ subject to the boundary conditions (4.2). Properties of this Green's function which were stated in section 2 imply that the operators

$$G_k w(s) = \int_a^b \frac{\partial^k G(s,t)}{\partial s^k} w(t)dt \qquad (0 \leq k \leq m-1;\ G_0 = G) \tag{4.11}$$

mapping C into C are completely continuous. As in the linear case denote $v(s) = u^{(m)}(s)$, so that $u^{(k)}(s) = G_k v(s)$, $(0 \leq k \leq m-1)$. Since the mappings G_k are bounded, there exists a $\delta_1 > 0$ such that if $\|w-v\| \leq \delta_1$ then $z_k(s) = G_k w(s)$, $(0 \leq k \leq m-1)$, lies in the region (4.8). Hence on the sphere $\|w-v\| \leq \delta_1$ we can define a nonlinear operator K mapping C into C by

$$Kw \equiv f\left(s, \int_a^b G(s,t)w(t)dt, \int_a^b \frac{\partial G(s,t)}{\partial s} w(t)dt, \ldots, \int_a^b \frac{\partial^{m-1} G(s,t)}{\partial s^{m-1}} w(t)dt\right)$$

$$= f(s, G_0 w, G_1 w, \ldots, G_{m-1} w) \, . \tag{4.12}$$

In fact since the operators G_k are completely continuous and $f(s,z_0,z_1,\ldots,z_{m-1})$ is continuous in the region (4.8), the operator K is completely continuous in the sphere $\|w-v\| \leq \delta_1$.

If P denotes the identity operator from C into C, then (4.12) implies that the operator equation

$$w = PKw \tag{4.4'}$$

in the Banach space C is equivalent to (4.1), (4.2). Furthermore, the solutions of these equations are connected by the relationships $v = u^{(m)}$ and $u = Gv$.

Approximating a solution of (4.1), (4.2) by the collocation method is equivalent to finding an element $u_n^{(m)}(s)$ of $L(\pi_n,d,0)$ which satisfies

$$u_n^{(m)}(s') = f(s',G_0u_n^{(m)}(s'),G_1u_n^{(m)}(s'),\ldots,G_{m-1}u_n^{(m)}(s')),s' \in \mathcal{S}_n\ , \tag{4.13}$$

and then determining the approximate solution $u_n(s)$ in $L'(\pi_n,m+d,m)$ via the Green's function. To represent this in operator form, define as before the linear operator P_n as the mapping which takes a continuous function $v(s)$ into the unique function in $L(\pi_n,d,0)$ that interpolates $v(s)$ on the points (2.4b) of each sub-interval of π_n. Using the definition of the operator K from (4.12) and denoting $v_n = u_n^{(m)}$, the collocation equation above may be expressed in C as the operator equation.

$$P_n v_n = P_n K v_n \; .$$

But $P_n v_n = v_n$, so this simplifies to

$$v_n = P_n K v_n \; . \tag{4.6'}$$

Upon letting $B = B' = C$ and denoting $T = PK$, $T_n = P_n K$, equations (4.4'), (4.6') become (4.4), (4.6). A previous lemma implies that the continuous linear operators P_n converge strongly in C to the identity operator P. By hypothesis equation (4.9) subject to (4.2) has only the trivial solution. Thus if we show that equations (4.9), (4.2) are equivalent to (4.5), then Theorem 4 will give us convergence of the approximate solutions using the collocation method and the bound (4.7) on the rate of convergence.

The partial derivatives of $f(s, z_0, z_1, \ldots, z_{m-1})$ are continuous in the region (4.8) by assumption. So with definitions (4.3) and (4.12) we can show that the operator K is differentiable in the region $\|w-v\| \leq \delta_1$ and

$$K'(w) = \sum_{k=0}^{m-1} \frac{\partial f(s, G_0 w, G_1 w, \ldots, G_{m-1} w)}{\partial z_k} \, G_k \; .$$

Also K is clearly continuously differentiable. This relation implies that equations (4.9), (4.2) in operator form are just (4.5). Therefore, (4.5) has only the trivial solution, and the point v is an isolated solution of (4.4') with a non-zero index in a sufficiently

small sphere, say $\|w-v\| \leq \sigma$ $(0 < \sigma \leq \delta)$. We have shown all of Theorem 4 is applicable to our collocation procedure.

For all sufficiently large n (say $n \geq n_0$) the collocation equation (4.13) has a unique solution $v_n(s) = u_n^{(m)}(s)$ in $L(\pi_n, d, 0)$ satisfying $\|v_n - v\| \leq \sigma$. Moreover, equation (4.7) gives the bound

$$\|v_n - v\| \leq M'\|(P_n - P)Kv\|$$

$$= M'\|P_n v - v\| \quad , \quad (n \geq n_0)$$

where the constant M' is independent of n. If $z_n(s)$ is any element of $L(\pi_n, d, 0)$, then $P_n z_n = z_n$ so that

$$\|v_n - v\| \leq M'\|P_n v - v - P_n z_n + z_n\|$$

$$\leq M'[\|P_n(v - z_n)\| + \|v - z_n\|]$$

$$\leq M'(1+M)\|v - z_n\| \quad ,$$

where M is the uniform bound on $\|P_n\|$ given in (2.11). This last inequality implies

$$\|v_n - v\| \leq M'(1+M)F_n(v).$$

Finally, by using the boundedness of the operators G_k $(0 \leq k \leq m-1)$ and relation (2.7), the inequality (4.10) follows by the same arguments as those in the linear case. ∎

The current level of development of numerical methods for nonlinear boundary value problems permits a complete solution only for relatively simple nonlinear problems. By a complete solution we mean a difference scheme that is formulated in a computable way, a process guaranteed to solve the resulting finite set of nonlinear equations for the coefficients of an approximate solution, and a proof that the approximate solutions converge to a solution of the continuous nonlinear problem. We achieve all except the second requirement for quite realistic problems. Indeed they are much more general than those for which a complete treatment is available. There are other results of the type we give, notably those of Urabe [25,26], and numerical experience with real problems has shown no particular difficulty in solving the nonlinear equations.

5. Computational Aspects. We must now deal with the more practical aspects of the collocation method. Specifically, suppose we wish to compute an approximate solution of the boundary value problem (2.1), (2.2). To do this we need an element of $L'(\pi_n, m+d, m)$ which collocates on the points of $\mathcal{S}_n$. Consequently, a convenient representation of such a function is required.

Suppose a partition π_n of $[a,b]$ of the form (2.4a) is given and the points (2.4b) are determined from a partition (2.5) of $[0,1]$. Perhaps the most natural representation for a function $u_n(s)$ in $L'(\pi_n, m+d, m)$ is of the form

$$u_n(s) = x_i^1(s-s_{i+1})^{m+d} + x_i^2(s-s_{i+1})^{m+d-1} + \cdots + x_i^{m+d}(s-s_{i+1}) + x_i^{m+d+1} \tag{5.1}$$

for $s \in [s_i, s_{i+1}]$, $(0 \leq i \leq n-1)$.

Then $u_n(s)$ is uniquely determined by the $n(m+d+1)$ coefficients x_i^j, $(1 \leq j \leq m+d+1,\ 0 \leq i \leq n-1)$. These coefficients are calculated from the m boundary conditions, the $nd+1$ collocation conditions (on $\mathcal{S}_n$), and the $(m+1)(n-1)$ continuity conditions. Arrange these equations in the order they arise when moving from left to right. That is, the first equations are those from the boundary conditions at $a = s_0$, then the collocation condition on the points of $\mathcal{S}_n$ in $[s_0, s_1]$, then the continuity conditions at s_1, then the collocation condition on the points of $\mathcal{S}_n$ in $(s_1, s_2]$, etc. If the boundary conditions are separated, the resulting matrix is block tridiagonal

with block size $(m+d+1) \times (m+d+1)$. It is in fact a band matrix with band width $2(m+d)+1$, simplifying considerably the computation of the solution of a large system of equations.

The continuity conditions undergo no changes in different problems, and one might expect that a proper choice of a basis for $L'(\pi_n, m+d, m)$ would have them "built in." Such bases have been worked out for certain cases. These representations which we mention below are given in [5]. Suppose that we are concerned with approximate solutions from the spaces $L'(\pi_n, 2m+1, m)$, $(n = 1,2,3,\ldots)$, the so-called smooth Hermite spaces $H^{(m+1)}(\pi_n)$ which result when $d = m + 1$. We may think of an element of $H^{(m+1)}(\pi_n)$ in the following way: At each joint s_i $(0 \leq i \leq n)$, associate the parameters x_k^i $(0 \leq k \leq m)$. Then in each subinterval $[s_i, s_{i+1}]$ there is a unique polynomial $v_i(s)$ of degree $2m + 1$ satisfying

$$v_i^{(k)}(s_i) = x_k^i, \quad v_i^{(k)}(s_{i+1}) = x_k^{i+1} \qquad (0 \leq k \leq m) . \tag{5.2}$$

An approximate solution $u_n(s)$ defined on $[s_i, s_{i+1}]$ by $v_i(s)$ $(0 \leq i \leq n)$ is uniquely determined by these $(n+1)(m+1)$ coefficients. To calculate these coefficients, a convenient basis for $H^{(m+1)}(\pi_n)$ is

$$\{z_{i,k}(s, m, \pi_n): \ 0 \leq i \leq n, \ 0 \leq k \leq m\} ,$$

where

$$z_{i,k}^{(\ell)}(s_j,m,\pi_n) = \delta_{ij}\delta_{\ell k} \quad (0 \leq \ell \leq m, \quad 0 \leq j \leq n) . \qquad (5.3)$$

Each $z_{i,k}(s,m,\pi_n)$ has its support in the interval $[s_{i-1},s_{i+1}]$. By using this approach, the continuity conditions are automatically satisfied. The $(n+1)(m+1)$ unknown coefficients can be efficiently determined by ordering the equations as before (as they arise when moving from left to right). With separated boundary conditions, the matrix from the resulting linear system is block tridiagonal with block size $(m+1) \times (m+1)$. Also, this band matrix has band width $2[\frac{3}{2} m] + 1$. The case of the general Hermite spaces arises if $d \geq m + 1$. When these are treated in a similar fashion, there are $nd + m + 1$ resulting coefficients to determine. Lastly, we mention the spline spaces, $Sp^{(\frac{m+2}{2})}(\pi_n)$, which arise when m is even and $d = 1$. A basis can be formed whose elements have their support contained in $m+2$ adjacent subintervals. The matrix problem to be solved has dimension $n + m + 1$. One computational difficulty of these bases must be stressed, however. While the collocation conditions are of a very simple nature at the joints $s_i (0 \leq i \leq n)$, they become somewhat unruly at the other points of $\mathcal{S}_n$ when we use the given bases which have the continuity conditions automatically satisfied. The matrix problem being solved does depend on the differential operator, so there seems to be no reason for choosing one basis over another on grounds of conditioning. However, we have seen difficulties with the bases with continuity conditions built in. Residual correction may be a good idea for such cases. A separate effect is the increased

difficulty in accurately forming the matrix using the smooth bases. But in any event the standard bases with continuity built in, while desirable in the sense that band widths of the matrices are decreased, can be rather unsatisfactory.

Since they are easily solved with Gaussian elimination, the band matrices obtained from our use of piecewise polynomials are much more economical than the dense matrices from using polynomials. We must pivot to guarantee stability, but with partial pivoting the elimination can be arranged to preserve the band structure. Although this increases the band width, the storage is still very economical. The code is made more efficient by testing for zero multipliers, thereby utilizing the block tridiagonal nature of the systems.

We have seen in our analysis that ideally the Lebesgue constant (2.10) would be minimized. So a computational consideration is that of the choice of the partition (2.5) of [0,1]. It is known that the best order for these constants is asymptotically achieved with the Chebyshev points, but a best choice of points for the finite case is an open question. It has been demonstrated, however, that for practical computations the "expanded" Chebyshev points give a smaller Lebesgue constant than from the Chebyshev points [16]. Furthermore, the endpoints of the interval are always elements of the "expanded" Chebyshev points, a necessity for our purposes.

6. Illustrative Computations. All computational examples were carried out on an IBM 360 in single precision (which is approximately seven decimal digits) unless otherwise stated. Uniformly spaced points were used for collocation in all except the third example. In each case the approximate solutions were piecewise cubic or quintic polynomial functions computed by using the representation (5.1). In the first example we compare results from this approach to those from using the basis (5.3), where the continuity conditions are automatically satisfied. In the case of a second order differential equation the approximation spaces $L(\pi_n, m+d, m)$ are the cubic spline space $S_p^{(2)}(\pi_n)$ and the smooth Hermite space $H^{(3)}(\pi_n)$. In some instances numerical results from the paper of Ciarlet, Schultz, and Varga [5] are presented by way of comparison. They will be marked with an asterisk. The notation $1.4 \times 10^{-3} = 1.4(-3)$ is used for convenience.

Problem 1. First we consider the almost trivial second order problem

$$u''(s) - 4u(s) = 4 \cosh 1, \quad u(0) = u(1) = 0 . \tag{6.1}$$

This problem has the unique solution

$$u(s) = \cosh (2s-1) - \cosh 1. \tag{6.2}$$

The homogeneous problem $u'' = 0$ with the boundary conditions in (6.1) obviously has a Green's function, and Theorem 2 is applicable. It turns out that cubic splines with either our collocation method or the variational procedure of [5] yield approximate solutions to (6.1) that are $O(h^2)$.

However, the constant in the error bound can be expected to be smaller for the variational method, as the results below show.

$$\|u_n - u\|$$

h	$Sp^{(2)}$ - Variational*	$Sp^{(2)}$ - Collocation	$H^{(3)}$ - Collocation
1/5	4.23(-5)	5.23(-3)	3.46(-6)
1/7	1.71(-5)	2.63(-3)	2.21(-6)
1/9	5.80(-6)	1.58(-3)	2.15(-6)

Comparison is made between the $O(h^4)$ computations with $H^{(3)}$ and some discrete methods having the same order of convergence, too. These methods are Collatz's Mehrstellenverfahren [6 , p. 164], the Bramble and Hubbard five-point scheme [3], and Numerov's scheme [15 , p. 70].

$$\|u_n - u\|$$

h	Collatz*	Bramble-Hubbard *	Numerov
1/5	2.56(-5)	2.06(-3)	3.88(-5)
1/10	1.65(-6)	1.64(-4)	4.83(-6)

We now illustrate the computational difficulties that can arise when using the basis with continuity built in, given in (5.2), (5.3). The solution (6.2) and its derivatives are approximated in $H^{(3)}$ using both single and double precision.

Single Precision

h	$\|u_n-u\|$	$\|u_n'-u'\|$	$\|u_n'' - u''\|$
1/5	1.29(-5)	5.05(-5)	3.86(-4)
1/7	3.28(-6)	4.39(-5)	1.00(-3)
1/9	2.06(-5)	8.87(-5)	1.13(-3)

Double Precision

h	$\|u_n-u\|$	$\|u_n'-u'\|$	$\|u_n''-u''\|$
1/5	1.49(-6)	7.22(-5)	7.14(-5)
1/7	3.90(-7)	1.88(-6)	1.92(-5)
1/9	1.43(-7).	6.87(-7)	7.18(-6)

The roundoff errors from using the representation (5.1), however, are more reasonable than those using (5.2), (5.3), since the number of accurate digits are basically equal to the number in single precision arithmetic.

Problem 2. The next example is a "real" problem,

$$u''(s) + (3 \cot s + 2 \tan s)\, u'(s) + .7\, u(s) = 0,$$

$$u(30) = 0,\ u(60) = 5\ . \tag{6.3}$$

This differential equation arises when considering the stress distribution in a spherical membrane having normal and tangential loads. The unique solution to (6.3) undergoes a steep rise from $u(30) = 0$ to $u(31) \doteq 277$. The physical application also requires a good approximation to $u'(s)$. Considerable effort has been unsuccessful in attempting to solve this problem by a shooting method on a CDC 6600. We use collocation with 51 joints to solve (6.3) and compare our results to those using central differences with 3000 mesh points and double precision [19]. It is easy to show Theorem 2 applies to this slightly more general problem. Rather than solve the problem with homogeneous boundary conditions by the technique in section 2, the inhomogeneous boundary conditions are just incorporated directly in the matrix problem. The agreement below is sufficiently good that we believe the $H^{(3)}$ solution is accurate.

s	Differences	$Sp^{(2)}$	$H^{(3)}$
35.01	1.71437(2)	1.71383(2)	1.71436(2)
40.01	8.89492(1)	8.89450(1)	8.89487(1)
45.01	4.40578(1)	4.40560(1)	4.40576(1)
50.01	2.12367(1)	2.12360(1)	2.12367(1)
55.01	1.01900(1)	1.01898(1)	1.01900(1)

Problem 3. Consider another linear second order problem

$$\begin{aligned} &\varepsilon u''(s) - (2-s^2)u(s) = -1, \\ &u(-1) = u(1) = 0, \quad 0 < \varepsilon << 1. \end{aligned} \tag{6.4}$$

This singular perturbation problem is given as an illustration in a paper on geophysical dynamics [4]. There exists a unique solution to (6.4) symmetric about 0 and having a boundary layer at 1 of width $O(\sqrt{\varepsilon})$. The paper states that an asymptotic solution is

$$u(s) \sim \frac{1}{2-s^2} - \exp\left(-\frac{(1+s)}{\sqrt{\varepsilon}}\right) - \exp\left(-\frac{(1-s)}{\sqrt{\varepsilon}}\right). \tag{6.5}$$

Letting $\varepsilon = 1.(-8)$, we compare an approximate solution from using collocation in $H^{(3)}$ with 49 joints to the solution (6.5). The joints were placed symmetrically about the origin on an ad hoc basis, with those in [0,1] placed at .0,.1,.3,.5,.7,.8,.9, .95,.97,.98,.985,.99,.9925,.995,.996,.997,.9975,.998,.9985,.999, .9995,.9997,.9999, and 1.0. The answers are symmetric; the maximum errors over several subintervals are shown below.

Interval	Max Difference
[0.0,0.8]	9.54(-7)
[0.8,0.999]	4.17(-6)
[0.999,0.9995]	5.84(-5)
[0.9995,1.0]	5.24(-4)

The last interval contains the boundary layer.

Problem 4. We turn now to a third order differential equation,

$$u'''(s) + \frac{1}{s} u''(s) - \frac{1}{s^2} u'(s) = \frac{1}{s} ,$$

$$u''(1) + .3u'(1) = u''(2) + \frac{.3}{2} u'(2) = u(2) = 0. \tag{6.6}$$

This boundary value problem which describes the symmetrical bending of a laterally loaded circular plate [24 , p.53] has the unique solution

$$u(s) = \frac{s^2}{4}\left(\ln \frac{s}{2} - 1\right) - \frac{s^2}{8}\left(\frac{.7}{1.3} + \frac{2}{3} \ln 2\right)$$

$$- \frac{2.6}{2.1} \ln 2 \ln \frac{s}{2} + \frac{3.3}{2.6} + \frac{\ln 2}{3} . \tag{6.7}$$

Note that variational methods could not be applied to (6.6), at least in its present form, because it is not self-adjoint. Using $C^{(3)}$ piecewise quintics, we expect the convergence of collocation to (6.7) and its derivatives to be $O(h^3)$.

h	$\|u_n-u\|$	$\|u_n' - u'\|$	$\|u_n'' - u''\|$	$\|u_n''' - u'''\|$
1/5	3.81(-5)	7.44(-5)	2.58(-4)	3.76(-3)
1/7	1.14(-5)	2.00(-5)	7.84(-5)	1.58(-3)
1/9	7.63(-6)	8.58(-6)	3.22(-5)	8.05(-4)

Problem 5. The highest order equation that we consider is fourth order. The problem

$$u^{(iv)}(s) = (s^4 + 14s^3 + 49s^2 + 32s - 12)\, e^s \ ,$$
$$u(0) = u'(0) = u(1) = u'(1) = 0 \ , \qquad (6.8)$$

corresponds to the bending of a thin beam clamped at both ends. The unique solution of (6.8) is

$$u(s) = s^2(1-s)^2\, e^s \ . \qquad (6.9)$$

Approximations to (6.9) using the variational method with functions in $H^{(2)}$ and the collocation method with $C^{(4)}$ piecewise quintics are compared, as are the first derivative approximations.

	$H^{(2)}$- Variational*		$L'(\pi_n,5,4)$ - Collocation	
h	$\|u_n-u\|$	$\|u'_n-u'\|$	$\|u_n-u\|$	$\|u'_n-u\|$
1/5	6.95(-4)	1.09(-2)	5.42(-3)	1.78(-2)
1/7	1.98(-4)	4.33(-3)	2.75(-3)	9.09(-3)
1/9	7.61(-5)	2.13(-3)	1.66(-3)	5.48(-3)

Problem 6. Our last linear example is the second order problem

$$u''(s) + su'(s) - u(s) = se^s - |s|(6 - 12s + 2s^2 - 3s^3) \ ,$$
$$u(-1) = e^{-1} - 2, \ u(1) = e. \qquad (6.10)$$

The unique solution to (6.10) is

$$u(s) = \begin{cases} e^s - s^3 + s^4 & , \quad s \geq 0 \\ e^s + s^3 - s^4 & , \quad s \leq 0 \end{cases} \tag{6.11}$$

Although the derivatives of $u(s)$ have jump discontinuities at the origin, the corollary to Theorem 2 implies that this will not hinder convergence of our collocation procedure if 0 is a joint. For example, we expect approximate solutions in $H^{(3)}$ and their derivatives to converge to the solution (6.11) and its corresponding derivatives at the rate $O(h^4)$ in such cases. But when 0 is not a joint, the convergence should be slower. These expectations are born out in the following double precision calculations.

h	$\|u_n - u\|$	$\|u'_n - u'\|$	$\|u''_n - u''\|$
1/4	3.40(-6)	1.00(-5)	6.54(-5)
1/8	2.17(-7)	6.18(-7)	4.47(-6)
1/12	4.29(-8)	1.22(-7)	9.07(-7)
1/16	1.36(-8)	3.85(-8)	2.91(-7)
1/3	3.88(-3)	3.20(-2)	4.95(-1)
1/9	1.61(-4)	3.10(-3)	1.66(-1)
1/15	3.56(-5)	1.07(-3)	1.00(-1)

Problem 7. Now we turn to a nonlinear problem,

$$u''(s) = \exp(u(s)) \ , \ u(0) = u(1) = 0. \tag{6.12}$$

The negative function

$$u(s) = \ln 2 + 2\ln\left[c \sec\left(\frac{c(s - \frac{1}{2})}{2}\right)\right], \tag{6.13}$$

where $c \doteq 1.336056$,

is the unique solution of (6.12). So Theorem 5 applies to our example, and we can approximate a solution to (6.13) using collocation in $H^{(3)}$. This approximation is found by an iterative procedure which requires the solution of a linear problem for each iterate:

$$y''_{n+1}(s) - y_{n+1}(s) = \exp y_n(s) - y_n(s),$$

$$y_{n+1}(0) = y_{n+1}(1) = 0 ,$$

where $y_0(s) \equiv 0$. The scheme is known to converge for this problem [20]. To determine a criterion for stopping the iterations, denote the coefficients of $y_n(s)$ and $y_{n+1}(s)$ given in (5.1) by y_i^j and $\bar{y}_i^j$ $(0 \leq j \leq 5,\ 0 \leq i \leq n\text{-}1)$, respectively. If we stop iterating when

$$\max_{\substack{0 \leq i \leq n-1 \\ 0 \leq j \leq 5}} \left\{h^{5-j}|y_i^j - \bar{y}_i^j|\right\} < \beta , \tag{6.14}$$

then (5.1) guarantees that $\|y_n - y_{n+1}\| < 5\,\beta$.

So we incorporate this criterion in our computer problem with $\beta = .5(-6)$ and take the latest iterate as the approximate solution of the nonlinear problem when this test is passed.

	$H^{(2)}$ - Variational*	$H^{(3)}$ - Collocation		
h	$\|u_n-u\|$	$\|u_n-u\|$	$\|u_n'-u'\|$	$\|u_n''-u''\|$
1/3	1.19(-5)	1.11(-6)	3.58(-6)	2.83(-5)
1/4	4.48(-6)	1.13(-6)	1.31(-6)	9.83(-6)
1/5	3.69(-6)	1.27(-6)	6.56(-7)	4.59(-6)

Problem 8. Lastly, we consider a problem both physically and theoretically interesting. This nonlinear problem is Bratu's equation,

$$u''(s) = -\exp(u(s)) \,,$$
$$u(0) = u(1) = 0 \,. \tag{6.15}$$

There are two solutions to (6.15), both of which are positive and symmetric about .5, and one is strictly greater than the other [8]. It is shown in [2] that the smaller solution lies in [0,c), where $c = \ln 1.2$, and it has

$$v_1(s) = \frac{\cos(\frac{1}{2} - s)}{\cos\frac{1}{2}} - 1$$

as a lower bound and

$$v_2(s) = \left[\frac{\cos\left(\frac{1}{2}e^{\frac{c}{2}} - te^{\frac{c}{2}}\right)}{\left(\frac{1}{2}e^{\frac{c}{2}}\right)}\right] e^{-c}$$

as an upper bound. The linear differential equation

$$y''(s) + e^{u(s)}y(s) = 0,$$

$$y(0) = y(1) = 0$$

which corresponds to (4.9) has only the trivial solution when $u(s)$ is the smaller solution of (6.15) (see [2, p. 31]). Consequently, this solution has a non-zero index, and Theorem 5 can be applied. We again use collocation in the space $H^{(3)}$. It is known that with the iteration scheme $y_0(s) \equiv 0$,

$$y''_{n+1}(s) = -\exp(y_n(s)), \qquad y_{n+1}(0) = y_{n+1}(1) = 0,$$

the successive iterates monotonically increase to this solution [21]. Using the criterion (6.14) with $\beta = 1.0(-6)$ as a stopping rule, the following results were obtained.

$H^{(3)}$ Collocation

s	Lower Bound	10 joints	15 joints	Upper Bound
1/15	3.4171(-2)	3.4078(-2)	3.4221(-2)	3.4867(-2)
2/15	6,3748(-2)	6.3794(-2)	6.3992(-2)	6.5107(-2)
1/5	8.8599(-2)	8.8824(-2)	8.9027(-2)	9.0556(-2)
4/15	1.0861(-1)	1.0900(-1)	1.0921(-1)	1.1108(-1)
1/3	1.2370(-1)	1.2422(-1)	1.2443(-1)	1.2657(-1)
2/5	1.3380(-1)	1.3406(-1)	1.3462(-1)	1.3694(-1)
7/15	1.3386(-1)	1.3951(-1)	1.3973(-1)	1.4214(-1)

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